

Continuous Optimization: Assignment 5

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Exercise 1

Consider quadratic function $f(x) = \frac{1}{2}x^\top Qx + b^\top x$ where Q is a symmetric positive definite matrix. Using Newton's method we obtain

$$\begin{aligned} x^{(1)} &= x^{(0)} - (\nabla^2 f(x^{(0)}))^{-1} \nabla f(x^{(0)}) \\ &= x^{(0)} - Q^{-1}(Qx^{(0)} + b) \\ &= x^{(0)} - Q^{-1}Qx^{(0)} + Q^{-1}b \\ &= x^{(0)} - x^{(0)} + Q^{-1}b \\ &= Q^{-1}b \end{aligned}$$

$\nabla f(x^{(1)}) = -QQ^{-1}b + b = 0 \Rightarrow x^{(1)}$ is the global minimizer

Exercise 2

(a)

$Q \in \mathbb{S}_{++}(n)$ with eigenvectors (v_1, \dots, v_n) . We want to show $v_i^\top Qv_j = 0$ for any $i \neq j$. We know $Q = U\Lambda U^\top$ where

$$U = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

U is an orthogonal matrix i.e. $UU^\top = I$.

For any $i \neq j$, we have

$$\begin{aligned} v_i^\top Qv_j &= v_i^\top U\Lambda U^\top v_j \\ &= v_i^\top \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}^\top v_j \\ &= (0 \quad \cdots \quad v_i^\top v_i \quad \cdots \quad 0) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ v_j^\top v_j \\ \vdots \\ 0 \end{pmatrix} \\ &= (0 \quad \cdots \quad \lambda_i v_i^\top v_i \quad \cdots \quad 0) \begin{pmatrix} 0 \\ \vdots \\ v_j^\top v_j \\ \vdots \\ 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

(b)

$\{d^{(0)}, \dots, d^{(k)}\}$ are conjugate directions w.r.t. Q . We have $d_i^\top Qd_j = 0$ for any $i \neq j$.

Proof by Contradiction:

Assume that $\{d^{(0)}, \dots, d^{(k)}\}$ is not a linearly independent set. Then there exists a non-zero vector d_j such that d_j can be expressed as a linear combination of some other vectors in the set i.e. $d_j = \sum_{i \neq j} \delta_i d_i$. Take d_m that is part of the linear combination. We have

$$\begin{aligned}
 d_j^\top Q d_m &= \left(\sum_{i \neq j} \delta_i d_i \right)^\top Q d_m \\
 &= \delta_m d_m^\top Q d_m + \left(\sum_{i \neq j, i \neq m} \delta_i d_i \right)^\top Q d_m \\
 &= \delta_m d_m^\top Q d_m && \text{Property of Conjugate Directions} \\
 &\neq 0 && Q \in \mathbb{S}_{++}(n)
 \end{aligned}$$

which is a contradiction.

(c)

For all $i = 0, \dots, k-1$, we have

$$\begin{aligned}
 \langle d^{(i)}, \nabla f(x^{(k)}) \rangle &= \langle d^{(i)}, Qx^{(k)} - b \rangle \\
 &= \langle Qx^{(k)}, d^{(i)} \rangle - \langle b, d^{(i)} \rangle \\
 &= \langle Q \left(x^{(i+1)} + \sum_{j=i+1}^{k-1} \tau_j d^{(j)} \right), d^{(i)} \rangle - \langle b, d^{(i)} \rangle \\
 &= \langle Qx^{(i+1)}, d^{(i)} \rangle + \sum_{j=i+1}^{k-1} \langle \tau_j d^{(j)}, Qd^{(i)} \rangle - \langle b, d^{(i)} \rangle \\
 &= \langle Qx^{(i+1)}, d^{(i)} \rangle - \langle b, d^{(i)} \rangle && \text{Property of Conjugate Directions} \\
 &= \langle Qx^{(i+1)} - b, d^{(i)} \rangle \\
 &= \langle \nabla f(x^{(i+1)}), d^{(i)} \rangle \\
 &= 0 && \text{Exact line search optimal condition}
 \end{aligned}$$