Continuous Optimization: Assignment 8

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Exercise 1

The constraint set C is defined as

$$C := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1,\, x_2 \geq 0,\; x_1 x_2 = 0 \right\}$$

which are the points lie on the axes of the first quadrant shown in Figure 1.

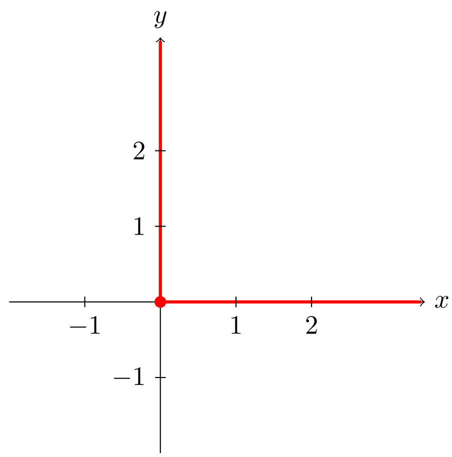

Figure 1: Illustration of the constraint set C marked in red.

Tangent Cone

The tangent cone of a set C at a point $\bar{x} \in C$ is the closure of the set of all feasible directions of C at $\bar{x}$. Apparently, the vectors on the axes pointing at positive directions are the feasible directions of C at $\bar{x}$ and also the closure of such set is itself.
Thus, the tangent cone to C at $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

$$T_C(\bar{x}) = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t_1,\, t_2 \geq 0,\; t_1 t_2 = 0 \right\}$$

as shown in Figure 2.

Normal Cone

The normal cone of a set C at a point $\bar{x} \in C$ is the set of all vectors v s.t. $\langle v, x - \bar{x} \rangle \leq 0, \forall x \in C$. Another way to visualize this is that v must form acute angles between all feasible directions of C at $\bar{x}$.
Apparently, the normal cone to C at $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

$$N_C(\bar{x}) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; n_1,\, n_2 \leq 0 \right\}$$

as shown in Figure 3.

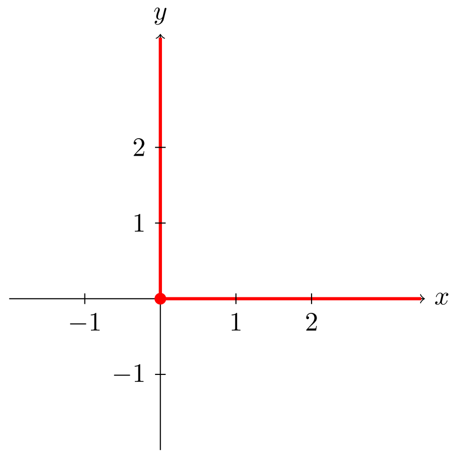

Figure 2: Illustration of the tangent cone of C at $\bar{x}$ marked in red.

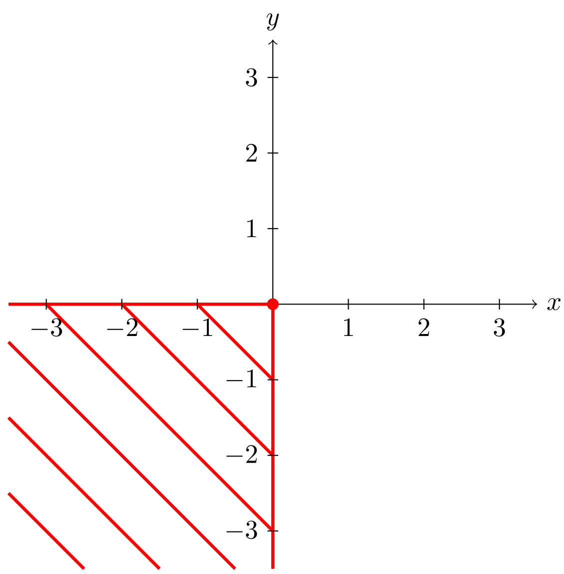

Figure 3: Illustration of the normal cone of C at $\bar{x}$ marked in red.

Exercise 2

The constraint set C is defined as

$$C := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; |x_1| + |x_2| \leq 1 \right\}$$

as shown in Figure 4. There are three cases of where the point $\bar{x}$ can land in C:

i. $\bar{x}$ is in the interior of C

ii. $\bar{x}$ is on the edges of C

iii. $\bar{x}$ is on the vertices C

Exercise 2 continued on next page...

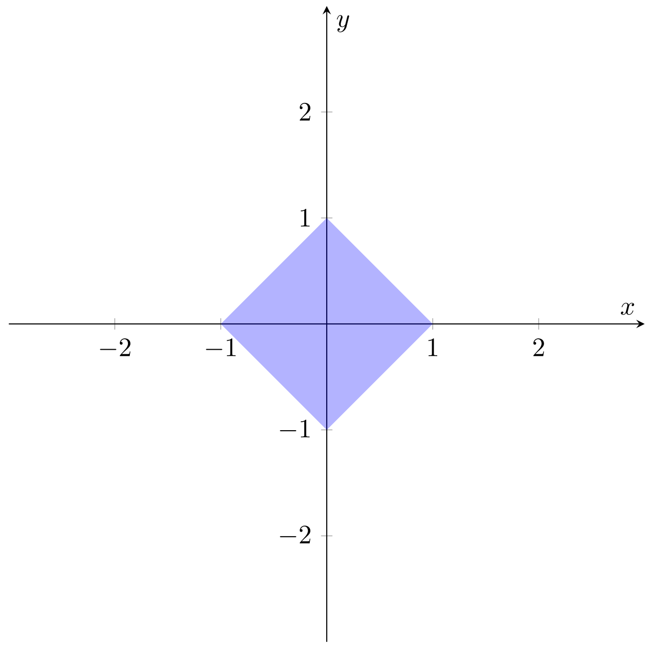

Figure 4: Illustration of the constraint set C as the area covered in blue.

(i) $\bar{x}$ is in the interior of C

As shown in Figure 5, $\bar{x}$ is in the interior of set C. Every vector pointing out of $\bar{x}$ in any direction is a feasible direction of C at $\bar{x}$. Thus, the tangent cone of C at $\bar{x}$ is the entire $\mathbb{R}^2$ and the normal cone of C at $\bar{x}$ is $\{0\}$.

(ii) $\bar{x}$ is on the edges of C

As shown in Figure 6, $\bar{x}$ is on the edges of set C. The feasible directions of C at $\bar{x}$ are the vectors pointing into C. Thus, the tangent cone of C at $\bar{x}$ is the area under the line of the edge where $\bar{x}$ lies. In case of Figure 6,

$$T_C(\bar{x}) = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t_1 + t_2 \leq 1 \right\}$$

The normal cone of C at $\bar{x}$ is the ray starts from $\bar{x}$ and points out of C and is perpendicular to the edge In our case, it can be expressed as

$$N_C(\bar{x}) = \left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + n \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \;\middle|\; n \in \mathbb{R}^2,\, n \geq 0 \right\}$$

(iii) $\bar{x}$ is on the vertices of C

As shown in Figure 7, $\bar{x}$ is on the vertices of set C. The feasible directions of C at $\bar{x}$ are the vectors pointing into C. Thus, the tangent cone of C at $\bar{x}$ is the area under the two lines of the two edges coincides at $\bar{x}$ In case of Figure 7,

$$T_C(\bar{x}) = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t_1 + t_2 \leq 1 \text{ and } t_1 - t_2 \leq 1 \right\}$$

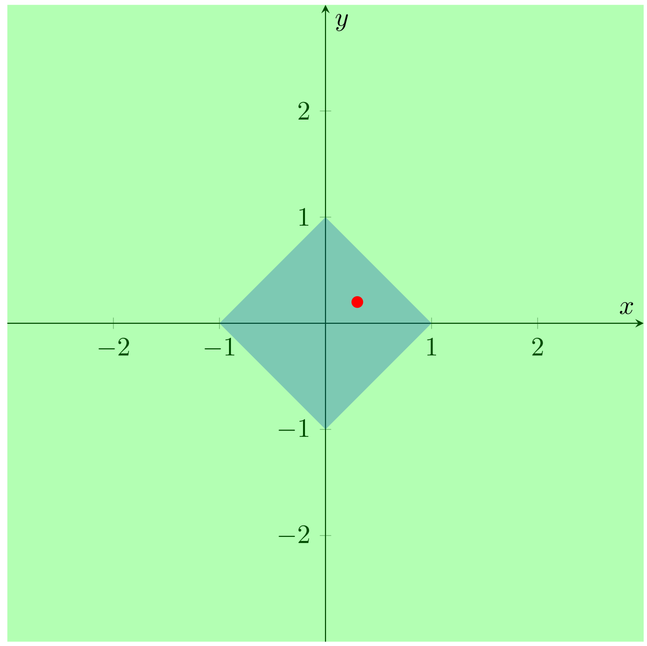

Figure 5: Illustration of $\bar{x}$ in the interior of C marked in red, the tangent cone in green which is the entire $\mathbb{R}^2$.

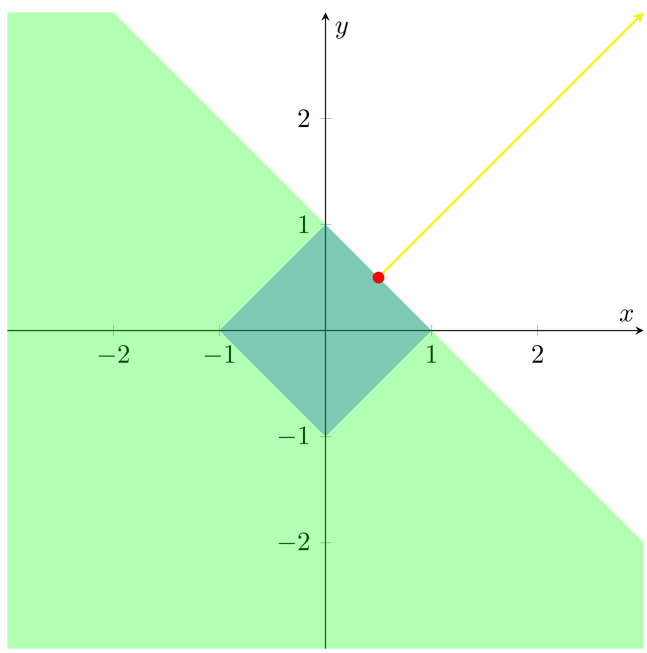

Figure 6: Illustration of $\bar{x}$ on the edges of C marked in red, the tangent cone in green, and the normal cone in yellow.

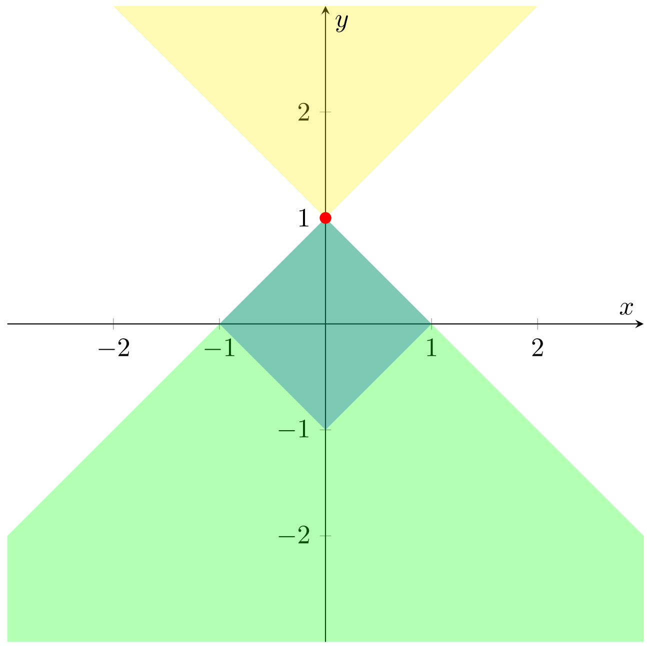

Figure 7: Illustration of $\bar{x}$ on the vertices of C marked in red, the tangent cone in green, and the normal cone in yellow.

and the normal cone of C at $\bar{x}$ is the area above C where the inverse direction of the two edges surrounds. as shown in Figure 7,

Exercise 3

We are given the set C defined as

$$C = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;:\; F(x_1, x_2) \in D \right\}$$

where

$$F(x_1, x_2) = \begin{pmatrix} x_1^2 + 3x_2^2 + 2x_2 \\ 3x_1^4 + 2x_2^3 + x_1 + x_2 \end{pmatrix}$$

and

$$D := \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2 \;:\; z_1^2 + z_2^2 \leq 1 \right\}$$

Given $\bar{x} := \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$ s.t. $4\bar{x}_1(6\bar{x}_2^2 + 1) > (6\bar{x}_2 + 2)(12\bar{x}_1^3 + 1)$, we want to find $N_C(\bar{x})$.
The question we should ask is whether $\bar{x} \in C$ and if so, does the result of transformation $F(\bar{x})$ lies on the boundary of the unit circle or it is in the interior of the unit circle?
Take the point $\bar{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as an example. One can verify that $\bar{x}$ satisfies the inequality condition but it is not in C because the $F(\bar{x}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ which is not in D.

Exercise 3 continued on next page...

Let's assume that indeed, $\bar{x} \in C$ and there will be two cases to consider:

i. $\bar{z} = F(\bar{x})$ is in the interior of D

ii. $\bar{z} = F(\bar{x})$ is on the boundary of D

By Theorem 15.15 (change of coordinates), we know that for a point $\bar{x} \in C$ and $\bar{z} = F(\bar{x}) \in D$ we can calculate the normal cone of C at $\bar{x}$ as

$$N_C(\bar{x}) = DF(\bar{x})^T N_D(\bar{z})$$

where $DF(\bar{x})$ is the Jacobian matrix of F at $\bar{x}$ and $N_D(\bar{z})$ is the normal cone of D at $\bar{z}$ and $DF(\bar{x})$ is required to have full rank.
The Jacobian of F is given by

$$DF(x) = \begin{pmatrix} 2x_1 & 6x_2 + 2 \\ 12x_1^3 + 1 & 6x_2^2 + 1 \end{pmatrix}$$

(i) $\bar{z} = F(\bar{x})$ is in the interior of D

If $\bar{z}$ is in the interior of D, then the normal cone of D at $\bar{z}$ is $\{0\}$. Thus,

$$N_C(\bar{x}) = DF(\bar{x})^T \cdot \{0\} = \{0\}$$

(ii) $\bar{z} = F(\bar{x})$ is on the boundary of D

If $\bar{z}$ is on the boundary of D, then the normal cone of D at $\bar{z}$ is

$$N_D(\bar{z}) = \{n\bar{z} \;:\; n \in \mathbb{R}\,,\, n \geq 0\}$$

Thus, the normal cone of C at $\bar{x}$ is

$$\begin{aligned} N_C(\bar{x}) &= n \begin{pmatrix} 2x_1 & 6x_2 + 2 \\ 12x_1^3 + 1 & 6x_2^2 + 1 \end{pmatrix}^\top \begin{pmatrix} x_1^2 + 3x_2^2 + 2x_2 \\ 3x_1^4 + 2x_2^3 + x_1 + x_2 \end{pmatrix} \\ &= n \begin{pmatrix} 2x_1 & 12x_1^3 + 1 \\ 6x_2 + 2 & 6x_2^2 + 1 \end{pmatrix} \begin{pmatrix} x_1^2 + 3x_2^2 + 2x_2 \\ 3x_1^4 + 2x_2^3 + x_1 + x_2 \end{pmatrix} \end{aligned}$$

where $n \in \mathbb{R}$ and $n \geq 0$.