Continuous Optimization: Assignment 4

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Exercise 1

The objective function can be rewritten as:

$$f(x) = \langle x, Qx \rangle$$

where

$$Q = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

We can obtain τ_0 by exact line search i.e. solving the following equation:

$$\begin{aligned} \frac{\partial f(x_0 + \tau_0 k_0)}{\partial \tau_0} &= 0 \\ \langle d_0, \nabla f(x_0 + \tau_0 d_0) \rangle &= 0 \\ \langle d_0, Q(x_0 + \tau_0 d_0) \rangle &= 0 \\ \langle d_0, Qx_0 \rangle + \tau_0 \langle d_0, Qd_0 \rangle &= 0 \\ \tau_0 &= \frac{-\langle d_0, Qx_0 \rangle}{\langle d_0, Qd_0 \rangle} = \frac{3}{4} \end{aligned}$$

also

$$x_1 = x_0 + \tau_0 d_0 = \begin{pmatrix} -\frac{1}{4} \\ -1 \end{pmatrix}$$

Note $Q \in \mathbb{S}_{++}(2)$ which can be shown by calculating the eigenvalues of Q.

$$\begin{aligned} \det(Q - \lambda I) &= 0 \\ \det \begin{pmatrix} 4 - \lambda & -1 \\ -1 & 1 - \lambda \end{pmatrix} &= 0 \\ (4 - \lambda)(1 - \lambda) - 1 &= 0 \\ \lambda &= \frac{5 \pm \sqrt{13}}{2} > 0 \end{aligned}$$

We can find an optimal solution by using the conjugate direction method given two Q-conjugate directions d_0 and d_1 i.e. $\langle d_0, Qd_1 \rangle = 0$. Let $d_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ we have

$$\begin{aligned} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 0 \\ \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} &= 0 \\ 4a &= b \end{aligned}$$

d_1 has unit length i.e. $a^2 + b^2 = 1$. Therefore, $a = \frac{1}{\sqrt{17}}$ and $b = \frac{4}{\sqrt{17}}$. We have $d_1 = \begin{pmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{pmatrix}$.

Exercise 2

(λ, v) is an eigen pair of $M^{-1}A$ i.e.

$$\begin{aligned}
M^{-1}Av &= \lambda v \\
MM^{-1}Av &= \lambda M v \\
IAv &= \lambda M v \\
Av &= \lambda E E^{\top} v \\
E^{-1}Av &= \lambda E^{-1} E E^{\top} v \\
E^{-1}Av &= \lambda I E^{\top} v \\
E^{-1}Av &= \lambda E^{\top} v
\end{aligned}$$

Let $\hat{v} = E^{\top} v$ which $v = E^{-\top}\hat{v}$. Subsititue v in the above equation, we have

$$\begin{aligned}
E^{-1}AE^{-\top}\hat{v} &= \lambda E^{\top} E^{-\top}\hat{v} \\
E^{-1}AE^{-\top}\hat{v} &= \lambda I \hat{v} \\
E^{-1}AE^{-\top}\hat{v} &= \lambda \hat{v}
\end{aligned}$$

Therefore, $(\lambda, \hat{v})$ is an eigen pair of $E^{-1}AE^{-\top}$.

Exercise 3

Proof by Induction
Base Case: $n = 1$
$d^{(0)} = -r^{(0)}$ thus $\operatorname{span}(d^{(0)}) = \operatorname{span}(r^{(0)})$
Induction Hypothesis:
Now suppose the statement is true for $n = k$ i.e. $\operatorname{span}(d^{(0)}, \ldots, d^{(k-1)}) = \operatorname{span}(r^{(0)}, \ldots, r^{(k-1)})$
Inductive Step:
We want to show that the statement is true for $n = k + 1$. We know that $d^{(k)} = -r^{(k)} + \frac{\langle r^{(k)}, r^{(k)}\rangle}{\langle r^{(k-1)}, r^{(k-1)}\rangle} d^{(k-1)}$
Take $v \in \operatorname{span}(d^{(0)}, \ldots, d^{(k)})$ then we have $v = \sum_{i=0}^{k} c_i d^{(i)}$ for some c_i.

$$\begin{aligned}
v &= \sum_{i=0}^{k} c_i d^{(i)} \\
&= \sum_{i=0}^{k-1} c_i d^{(i)} + c_k d^{(k)} \\
&= \sum_{i=0}^{k-1} c_i d^{(i)} + c_k \left(-r^{(k)} + \frac{\langle r^{(k)}, r^{(k)}\rangle}{\langle r^{(k-1)}, r^{(k-1)}\rangle} d^{(k-1)} \right) \\
&= \sum_{i=0}^{k-2} c_i d^{(i)} + \left(c_{k-1} + \frac{\langle r^{(k)}, r^{(k)}\rangle}{\langle r^{(k-1)}, r^{(k-1)}\rangle} \right) d^{(k-1)} - c_k r^{(k)}
\end{aligned}$$

The first two term can be seen as a linear combination of $d^{(0)}, \ldots, d^{(k-1)}$ which is the same as a linear combination of $r^{(0)}, \ldots, r^{(k-1)}$ by the induction hypothesis. Together with the last term we have that v is a linear combination of $r^{(0)}, \ldots, r^{(k)}$.