Continuous Optimization: Assignment 3

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Honglu Ma **Hiroyasu Akada** **Mathivathana Ayyappan**

Exercise 1

1. $f(x) - f(\bar{x}) = \int_0^1 \langle \nabla f(x + t(x - \bar{x}), x - \bar{x}) \rangle \, dt$
   should be $f(x) - f(\bar{x}) = \int_0^1 \langle \nabla f(\bar{x} + t(x - \bar{x}), x - \bar{x}) \rangle \, dt$

2. (continue from the previous equation) $\ldots | \int_0^1 \langle \nabla f(\bar{x} + t(x - \bar{x}) - \nabla f(\bar{x}), x - \bar{x}) \rangle \, dt |$
   should drop the absolute value sign

3. The correct equation consists the previous three errors should be

$$
\begin{aligned}
& f(x) - f(\bar{x}) - \langle \nabla f(\bar{x}), x - \bar{x} \rangle \\
&= \int_0^1 \langle \nabla f(\bar{x} + t(x - \bar{x}), x - \bar{x}) - \nabla f(\bar{x}), x - \bar{x} \rangle \, dt \\
&\leq \int_0^1 ||\nabla f(\bar{x} + t(x - \bar{x})) - \nabla f(\bar{x})|| \, ||x - \bar{x}|| \, dt && \text{Cauchy Schwarz Inequality}
\end{aligned}
$$

4. Continue with previous derivation we have

$$
\begin{aligned}
& \int_0^1 ||\nabla f(\bar{x} + t(x - \bar{x})) - \nabla f(\bar{x})|| \, ||x - \bar{x}|| \, dt \\
&\leq \int_0^1 L||\bar{x} + t(x - \bar{x}) - \bar{x}|| \, ||x - \bar{x}|| \, dt && \text{Lipschitz Continuous Gradient} \\
&= \int_0^1 L||t(x - \bar{x})|| \, ||x - \bar{x}|| \, dt \\
&= L||x - \bar{x}||^2 \int_0^1 t \, dt \\
&= L||x - \bar{x}||^2 (\frac{t^2}{2}|_0^1) \\
&= \frac{L}{2}||x - \bar{x}||^2
\end{aligned}
$$

   where in the original proof, there is an error at $\ldots \int_0^1 \frac{Lt}{2} \ldots$

5-10. There are serveral errors in the last part of the proof. In addition to adding a half to Lt. The derivation also does not follow the correct steps of integration. Item 3 and 4 above show the correct derivation and conclude the proof.

Exercise 2

(a)

Since $\mathbf{Q}$ is a square matrix, we can write $\mathbf{Q} = \mathbf{U}\Lambda\mathbf{U}^\top$ where $\mathbf{U}$ is an orthogonal matrix and Λ is a diagonal matrix with the eigenvalues of $\mathbf{Q}$ i.e. λ_i on the diagonal. Then, we have

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{Q}\mathbf{x} \rangle &= \langle \mathbf{x}, \mathbf{U}\Lambda\mathbf{U}^\top \mathbf{x} \rangle \\
&= \mathbf{x}^\top \mathbf{U}\Lambda\mathbf{U}^\top \mathbf{x} \\
&= (\mathbf{U}^\top \mathbf{x})^\top \Lambda \mathbf{U}^\top \mathbf{x} \\
&= (\mathbf{U}\mathbf{x})^\top \Lambda (\mathbf{U}\mathbf{x}) && \mathbf{U} = \mathbf{U}^\top \\
&= \sum_{i=1}^{n} \lambda_i ((\mathbf{U}\mathbf{x})_i)^2 \\
&\leq \sum_{i=1}^{n} \lambda_{\max}(\mathbf{Q}) ((\mathbf{U}\mathbf{x})_i)^2 && \lambda_i \leq \lambda_{\max}(\mathbf{Q}) \\
&= \lambda_{\max}(\mathbf{Q}) \sum_{i=1}^{n} ((\mathbf{U}\mathbf{x})_i)^2 \\
&= \lambda_{\max}(\mathbf{Q}) (\mathbf{U}\mathbf{x})^\top \mathbf{U}\mathbf{x} \\
&= \lambda_{\max}(\mathbf{Q}) \mathbf{x}^\top \mathbf{U}^\top \mathbf{U}\mathbf{x} \\
&= \lambda_{\max}(\mathbf{Q}) \mathbf{x}^\top \mathbf{x} && \mathbf{U}^\top \mathbf{U} = I \\
&= \lambda_{\max}(\mathbf{Q}) ||\mathbf{x}||^2
\end{aligned}$$

Similar derivation can be shown for the smallest eigenvalue: $\langle \mathbf{x}, \mathbf{Q}\mathbf{x} \rangle \geq \lambda_{\min}(\mathbf{Q})||\mathbf{x}||^2$.

(b)

Suppose λ is an eigenvalue of $\mathbf{Q}$ with eigenvector $\mathbf{v}$. Then, we have

$$\begin{aligned}
\mathbf{Q}\mathbf{v} = \lambda\mathbf{v} &\Rightarrow \tau\mathbf{Q}\mathbf{v} = \tau\lambda\mathbf{v} \\
&\Rightarrow \mathbf{I}\mathbf{v} - \tau\mathbf{Q}\mathbf{v} = \mathbf{I}\mathbf{v} - \tau\lambda\mathbf{v} \\
&\Rightarrow (\mathbf{I} - \tau\mathbf{Q})\mathbf{v} = (\mathbf{I} - \tau\mathrm{diag}(\lambda))\mathbf{v}
\end{aligned}$$

$\mathbf{I} - \tau\mathrm{diag}(\lambda)$ is a matrix with same diagonal entries $1 - \tau\lambda$

$$\Rightarrow (\mathbf{I} - \tau\mathbf{Q})\mathbf{v} = (1 - \tau\lambda)\mathbf{v}$$

Above shows that $1 - \tau\lambda$ is an eigenvalue of $\mathbf{I} - \tau\mathbf{Q}$.

$$\begin{aligned}
&\Rightarrow (\mathbf{I} - \tau\mathbf{Q})(\mathbf{I} - \tau\mathbf{Q})\mathbf{v} = (\mathbf{I} - \tau\mathbf{Q})(1 - \tau\lambda)\mathbf{v} \\
&\Rightarrow (\mathbf{I} - \tau\mathbf{Q})(\mathbf{I} - \tau\mathbf{Q})\mathbf{v} = (1 - \tau\lambda)(\mathbf{I} - \tau\mathbf{Q})\mathbf{v} \\
&\Rightarrow (\mathbf{I} - \tau\mathbf{Q})(\mathbf{I} - \tau\mathbf{Q})\mathbf{v} = (1 - \tau\lambda)(1 - \tau\lambda)\mathbf{v} \\
&\Rightarrow (\mathbf{I} - \tau\mathbf{Q})^2\mathbf{v} = (1 - \tau\lambda)^2\mathbf{v}
\end{aligned}$$

Thus $(1 - \tau\lambda)^2$ is an eigenvalue of $(\mathbf{I} - \tau\mathbf{Q})^2$ for each eigenvalue λ of $\mathbf{Q}$.

Exercise 3

(a)

Projection of $\mathbf{v}$ onto the space spanned by the columns of $\mathbf{A}$ means that we want to find a point in the column space of $\mathbf{A}$ that is closest to $\mathbf{v}$ i.e. for vector $\mathbf{p}$ in the column space we want the distance between $\mathbf{v}$ and $\mathbf{p}$ to be as small as possible

$$\operatorname*{argmin}_p \operatorname{dist}(\mathbf{v}, \mathbf{p})$$

which is equivalent to

$$\operatorname*{argmin}_p ||\mathbf{v} - \mathbf{p}||$$

or

$$\operatorname*{argmin}_p \sqrt{\langle \mathbf{v} - \mathbf{p}, \mathbf{v} - \mathbf{p} \rangle}$$

(b)

In case of $m = 1$, the subspace spanned by $\mathbf{u}$ is just a line $c\mathbf{u}$ for some constant $c \in \mathbb{R}$. We have

$$\operatorname*{argmin}_c \sqrt{\langle \mathbf{v} - c\mathbf{u}, \mathbf{v} - c\mathbf{u} \rangle}$$

which by definition of the inner product, is equivalent to

$$\operatorname*{argmin}_c (\mathbf{v} - c\mathbf{u})^\top \mathbf{Q} (\mathbf{v} - c\mathbf{u})$$

To find minimum, we take derivative of function

$$f(c) = (\mathbf{v} - c\mathbf{u})^\top \mathbf{Q} (\mathbf{v} - c\mathbf{u}) = \mathbf{v}^\top \mathbf{Q} \mathbf{v} - 2(\mathbf{v}^\top \mathbf{Q} \mathbf{u})c + (\mathbf{u}^\top \mathbf{Q} \mathbf{u})c^2$$

We have

$$f'(c) = 2(\mathbf{u}^\top \mathbf{Q} \mathbf{u})c - 2(\mathbf{v}^\top \mathbf{Q} \mathbf{u})$$

and the second derivative

$$f''(c) = 2(\mathbf{u}^\top \mathbf{Q} \mathbf{u}) \geq 0$$

which indicate function f is convex now we just have to solve $f'(c) = 0$

$$2(\mathbf{u}^\top \mathbf{Q} \mathbf{u})c - 2(\mathbf{v}^\top \mathbf{Q} \mathbf{u}) = 0$$
$$c = \frac{\mathbf{v}^\top \mathbf{Q} \mathbf{u}}{\mathbf{u}^\top \mathbf{Q} \mathbf{u}}$$

and the projection $\mathbf{p} = \frac{\mathbf{v}^\top \mathbf{Q} \mathbf{u}}{\mathbf{u}^\top \mathbf{Q} \mathbf{u}} \mathbf{u}$

(c)

In case of $m > 1$, we have a vector $\hat{\mathbf{c}} \in \mathbb{R}^m$ defined by

$$\hat{\mathbf{c}} := \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

Exercise 3 continued on next page...

where the projection $\mathbf{p} = \mathbf{A}\hat{\mathbf{c}}$. We have

$$\operatorname*{argmin}_{\hat{\mathbf{c}}} \, (\mathbf{v} - \mathbf{A}\hat{\mathbf{c}})^\top \mathbf{Q}(\mathbf{v} - \mathbf{A}\hat{\mathbf{c}})$$

To find the minimum, similarly, we define a function

$$g(\hat{\mathbf{c}}) = \mathbf{v}^\top \mathbf{Q}\mathbf{v} - 2\mathbf{v}^\top \mathbf{Q}\mathbf{A}\hat{\mathbf{c}} + \hat{\mathbf{c}}^\top \mathbf{A}^\top \mathbf{Q}\mathbf{A}\hat{\mathbf{c}}$$

We calculate its gradient

$$\nabla g(\hat{\mathbf{c}}) = -2\mathbf{v}^\top \mathbf{Q}\mathbf{A} + 2\mathbf{A}^\top \mathbf{Q}\mathbf{A}\hat{\mathbf{c}}$$

and its Hessian

$$\nabla^2 g(\hat{\mathbf{c}}) = 2\mathbf{A}^\top \mathbf{Q}\mathbf{A} = \mathbf{H}$$

We can show that $\mathbf{H}$ is positive semi-definite. Suppose λ is an eigenvalue of $\mathbf{H}$ and with corresponding eigenvector $\mathbf{e}$

$$\begin{aligned}
\mathbf{H}\mathbf{e} &= \lambda\mathbf{e} \\
2\mathbf{A}^\top \mathbf{Q}\mathbf{A}\mathbf{e} &= \lambda\mathbf{e} \\
2\mathbf{e}^\top \mathbf{A}^\top \mathbf{Q}\mathbf{A}\mathbf{e} &= \lambda\mathbf{e}^\top \mathbf{e} \\
\lambda = \frac{2(\mathbf{A}\mathbf{e})^\top \mathbf{Q}\mathbf{A}\mathbf{e}}{\mathbf{e}^\top \mathbf{e}} &\geq 0
\end{aligned}$$

Thus we can get the minimizer $\hat{\mathbf{c}}$ by solving $\nabla g(\hat{\mathbf{c}}) = 0$

$$\begin{aligned}
-2\mathbf{v}^\top \mathbf{Q}\mathbf{A} + 2\mathbf{A}^\top \mathbf{Q}\mathbf{A}\hat{\mathbf{c}} &= 0 \\
\mathbf{A}^\top \mathbf{Q}\mathbf{A}\hat{\mathbf{c}} &= \mathbf{v}^\top \mathbf{Q}\mathbf{A}
\end{aligned}$$

Exercise 4

(a)

Define $\tilde{A}$ and $\tilde{f}$ accordingly:

$$\tilde{A} = \begin{pmatrix} A \\ \sqrt{\mu}D \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Note the 0 in $\tilde{f}$ denotes a zero vector in $\mathbb{R}^{2N}$

where $\tilde{A} \in \mathbb{R}^{3N\times N}$ and $\tilde{f} \in \mathbb{R}^{3N}$.
We can verify that the reconstruction is equivalent to the original problem.

$$\begin{aligned}
\frac{1}{2}||\tilde{A}u - \tilde{f}||^2 &= \frac{1}{2}(||\tilde{A}u||^2 - 2(\tilde{A}u)^\top \tilde{f} + ||\tilde{f}||^2) \\
&= \frac{1}{2}(||Au||^2 + ||\sqrt{\mu}Du||^2 - 2(Au)^\top f + ||f||^2 + 0) \\
&= \frac{1}{2}(||Au||^2 + \mu||Du||^2 - 2(Au)^\top f + ||f||^2) \\
&= \frac{1}{2}(||Au||^2 - 2(Au)^\top f + ||f||^2) + \frac{\mu}{2}||Du||^2 \\
&= \frac{1}{2}||Au - f||^2 + \frac{\mu}{2}||Du||^2
\end{aligned}$$

Exercise 4 continued on next page. . .

(b)

We have the following objective function:

$$
\begin{aligned}
L(u) &= \frac{1}{2}||\tilde{A}u - \tilde{f}||^2 \\
&= \frac{1}{2}||\tilde{A}u||^2 - \tilde{f}^\top \tilde{A}u + \frac{1}{2}||\tilde{f}||^2 \\
&= \frac{1}{2}\langle u, \tilde{A}^\top \tilde{A}u\rangle + \langle -A^\top f, u\rangle + \frac{1}{2}||f||^2
\end{aligned}
$$

with corresponding standard form $L(u) = \frac{1}{2}\langle u, Qu\rangle + \langle b, u\rangle + c$ where $Q = \tilde{A}^\top \tilde{A} = A^\top A + \mu D^\top D$, $b = -A^\top f$ and $c = \frac{1}{2}||f||^2$ and its gradient:

$$
\nabla L(u) = Qu + b = (A^\top A + \mu D^\top D)u - A^\top f
$$

We can find the optimal step size for exact line search by solving the following minimization problem:

$$
\min_{\tau_k > 0} L(u^{(k)} - \tau_k \nabla L(u^{(k)}))
$$

where the derivation is similar to Exercise 1, Assignment 2. In the end, we get:

$$
\tau_k = \frac{\nabla L(u^{(k)})^\top \nabla L(u^{(k)})}{\nabla L(u^{(k)})^\top Q \nabla L(u^{(k)})}
$$