Continuous Optimization: Assignment 8

Due on June 18, 2024

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Exercise 1

The constraint set C is defined as

$$C := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1,\, x_2 \geq 0,\; x_1 x_2 = 0 \right\}$$

which are the points lie on the axes of the first quadrant shown in Figure 1.

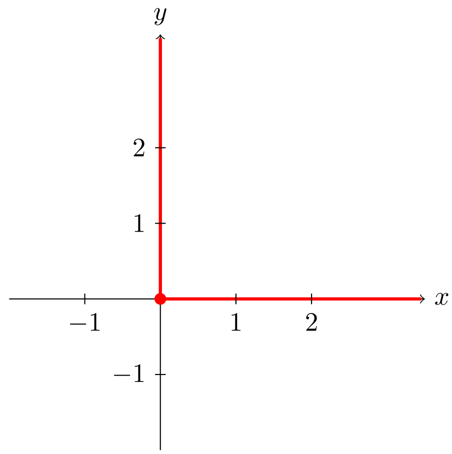

Figure 1: Illustration of the constraint set C marked in red.

Tangent Cone

The tangent cone of a set C at a point $\bar{x} \in C$ is the closure of the set of all feasible directions of C at $\bar{x}$. Apparently, the vectors on the axes pointing at positive directions are the feasible directions of C at $\bar{x}$ and also the closure of such set is itself.

Thus, the tangent cone to C at $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

$$T_C(\bar{x}) = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t_1,\, t_2 \geq 0,\; t_1 t_2 = 0 \right\}$$

as shown in Figure 2.

Normal Cone

The normal cone of a set C at a point $\bar{x} \in C$ is the set of all vectors v s.t. $\langle v, x - \bar{x} \rangle \leq 0, \forall x \in C$. Another way to visualize this is that v must form acute angles between all feasible directions of C at $\bar{x}$.

Apparently, the normal cone to C at $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

$$N_C(\bar{x}) = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; n_1,\, n_2 \leq 0 \right\}$$

as shown in Figure 3.

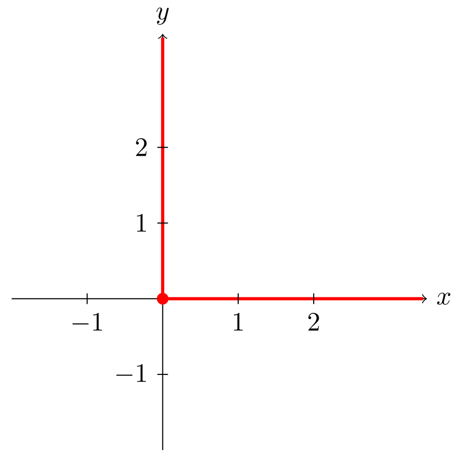

Figure 2: Illustration of the tangent cone of C at $\bar{x}$ marked in red.

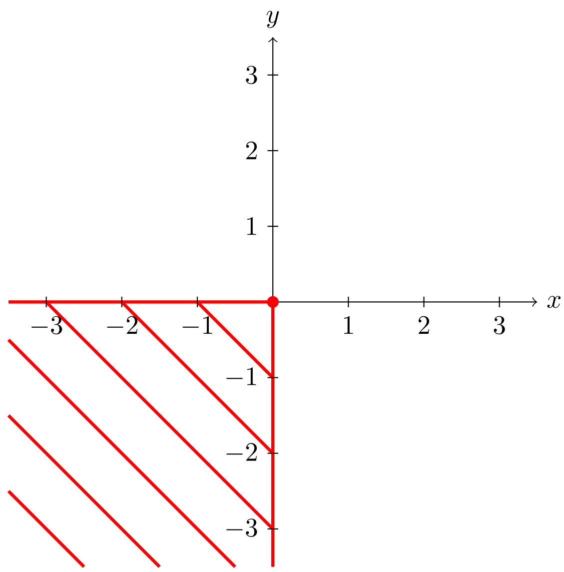

Figure 3: Illustration of the normal cone of C at $\bar{x}$ marked in red.

Exercise 2

The constraint set C is defined as

$$C := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; |x_1| + |x_2| \leq 1 \right\}$$

as shown in Figure 4. There are three cases of where the point $\bar{x}$ can land in C:

i. $\bar{x}$ is in the interior of C

ii. $\bar{x}$ is on the edges of C

iii. $\bar{x}$ is on the vertices C

Exercise 2 continued on next page...

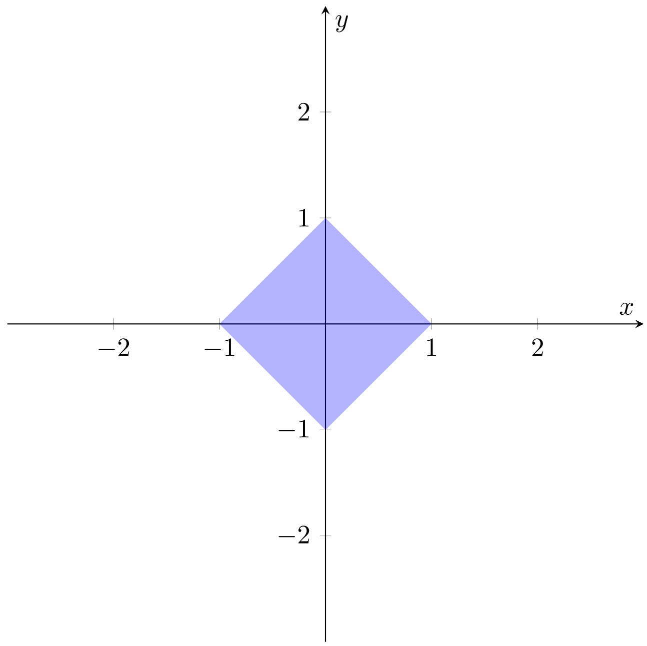

Figure 4: Illustration of the constraint set C as the area covered in blue.

(i) $\bar{x}$ is in the interior of C

As shown in Figure 5, $\bar{x}$ is in the interior of set C. Every vector pointing out of $\bar{x}$ in any direction is a feasible direction of C at $\bar{x}$. Thus, the tangent cone of C at $\bar{x}$ is the entire $\mathbb{R}^2$ and the normal cone of C at $\bar{x}$ is $\{0\}$.

(ii) $\bar{x}$ is on the edges of C

As shown in Figure 6, $\bar{x}$ is on the edges of set C. The feasible directions of C at $\bar{x}$ are the vectors pointing into C. Thus, the tangent cone of C at $\bar{x}$ is the area under the line of the edge where $\bar{x}$ lies. In case of Figure 6,

$$T_C(\bar{x}) = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t_1 + t_2 \leq 1 \right\}$$

The normal cone of C at $\bar{x}$ is the ray starts from $\bar{x}$ and points out of C and is perpendicular to the edge In our case, it can be expressed as

$$N_C(\bar{x}) = \left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + n \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \;\middle|\; n \in \mathbb{R}^2,\, n \geq 0 \right\}$$

(iii) $\bar{x}$ is on the vertices of C

As shown in Figure 7, $\bar{x}$ is on the vertices of set C. The feasible directions of C at $\bar{x}$ are the vectors pointing into C. Thus, the tangent cone of C at $\bar{x}$ is the area under the two lines of the two edges coincides at $\bar{x}$ In case of Figure 7,

$$T_C(\bar{x}) = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t_1 + t_2 \leq 1 \text{ and } t_1 - t_2 \leq 1 \right\}$$

and the normal cone of C at $\bar{x}$ is the area above C where the inverse direction of the two edges surrounds. as shown in Figure 7,

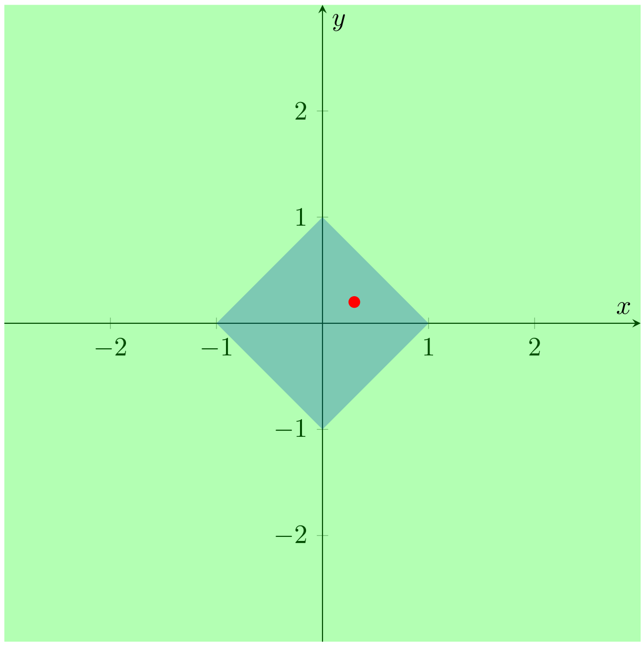

Figure 5: Illustration of $\bar{x}$ in the interior of C marked in red, the tangent cone in green which is the entire $\mathbb{R}^2$.

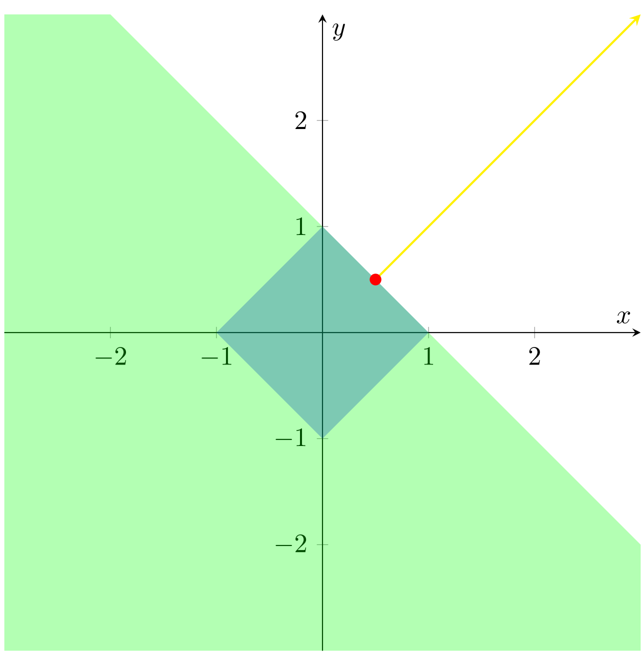

Figure 6: Illustration of $\bar{x}$ on the edges of C marked in red, the tangent cone in green, and the normal cone in yellow.

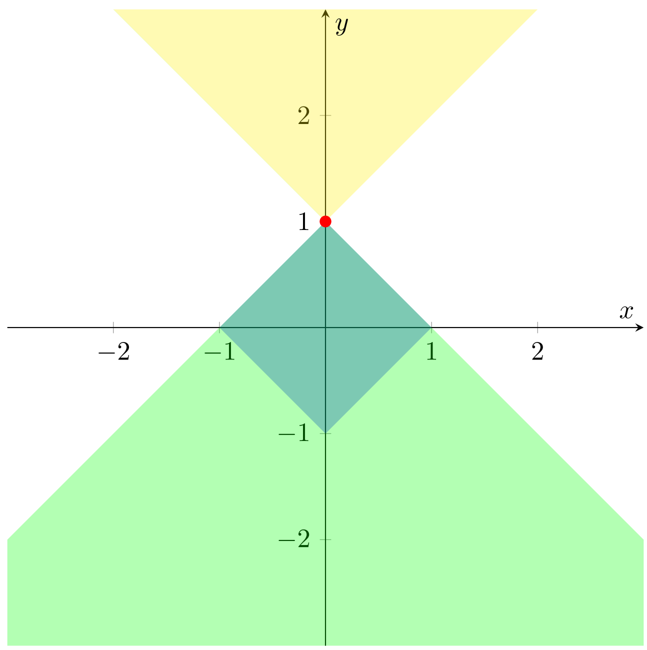

Figure 7: Illustration of $\bar{x}$ on the vertices of C marked in red, the tangent cone in green, and the normal cone in yellow.