Continuous Optimization: Assignment 3

Due on May 14, 2024

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Exercise 1

Exercise 2

(a)

Since Q is a square matrix, we can write $Q = U\Lambda U^T$ where U is an orthogonal matrix and Λ is a diagonal matrix with the eigenvalues of Q i.e. λ_i on the diagonal. Then, we have

$$\begin{aligned}
\langle x, Qx\rangle &= \langle x, U\Lambda U^\top x\rangle \\
&= x^\top U\Lambda U^\top x \\
&= (U^\top x)^\top \Lambda U^\top x \\
&= (Ux)^\top \Lambda (Ux) && U = U^\top \\
&= \sum_{i=1}^{n} \lambda_i (Ux)_i^2 \\
&\leq \sum_{i=1}^{n} \lambda_{\max}(Q)(Ux)_i^2 && \lambda_i \leq \lambda_{\max}(Q) \\
&= \lambda_{\max}(Q) \sum_{i=1}^{n} (Ux)_i^2 \\
&= \lambda_{\max}(Q)(Ux)^\top Ux \\
&= \lambda_{\max}(Q)x^\top U^\top Ux \\
&= \lambda_{\max}(Q)x^\top x && U^\top U = I \\
&= \lambda_{\max}(Q)||x||^2
\end{aligned}$$

Similar derivation can be shown for the smallest eigenvalue: $\langle x, Qx\rangle \geq \lambda_{\min}(Q)||x||^2$.

(b)

Suppose λ is an eigenvalue of Q with eigenvector v. Then, we have

$$\begin{aligned}
Qv = \lambda v &\Rightarrow \tau Qv = \tau\lambda v \\
&\Rightarrow Iv - \tau Qv = Iv - \tau\lambda v \\
&\Rightarrow (I - \tau Q)v = (I - \tau\mathrm{diag}(\lambda))v
\end{aligned}$$

$I - \tau\mathrm{diag}(\lambda)$ is a matrix with same diagonal entries $1 - \tau\lambda$

$$\Rightarrow (I - \tau Q)v = (1 - \tau\lambda)v$$

Above shows that $1 - \tau\lambda$ is an eigenvalue of $I - \tau Q$.

$$\begin{aligned}
&\Rightarrow (I - \tau Q)(I - \tau Q)v = (I - \tau Q)(1 - \tau\lambda)v \\
&\Rightarrow (I - \tau Q)(I - \tau Q)v = (1 - \tau\lambda)(I - \tau Q)v \\
&\Rightarrow (I - \tau Q)(I - \tau Q)v = (1 - \tau\lambda)(1 - \tau\lambda)v \\
&\Rightarrow (I - \tau Q)^2 v = (1 - \tau\lambda)^2 v
\end{aligned}$$

Exercise 2 continued on next page. . .

Thus $(1-\tau\lambda)^2$ is an eigenvalue of $(I-\tau Q)^2$ for each eigenvalue λ of Q.

Exercise 3

(a)

Let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto the column space of $\mathbf{A}$ i.e. $\mathbf{p} \in R(\mathbf{A})$. We know that $\exists\, \mathbf{e}, \mathbf{e} \in N(\mathbf{A}^\top)$ s.t. $\mathbf{v} = \mathbf{p} + \mathbf{e}$.

$$
\begin{aligned}
\mathbf{e} &= \mathbf{v} - \mathbf{p} \\
\mathbf{e} &= \mathbf{v} - \mathbf{A}\hat{\mathbf{x}} && \text{for some } \hat{\mathbf{x}} \in \mathbb{R}^m \\
0 &= \mathbf{A}^\top(\mathbf{v} - \mathbf{A}\hat{\mathbf{x}}) \\
0 &= \mathbf{A}^\top\mathbf{v} - \mathbf{A}^\top\mathbf{A}\hat{\mathbf{x}}
\end{aligned}
$$