Exercises

Continuous Optimization

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www.mop.uni-saarland.de/teaching/OPT24

— Summer Term 2024 —

— Assignment 11 —

Exercise 1. [20 points]

Consider the linear program of the form

$$\min_{x\in\mathbb{R}^n} \langle c, x\rangle\,, \quad s.t.\ Ax = b,\, Gx \leq h\,,$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $G \in \mathbb{R}^{r\times n}$ and $h \in \mathbb{R}^r$. In this exercise we derive the dual for this problem in two ways.

(1) Transform the problem into canonical form and state the dual problem. (8 points)

(2) Follow the following steps which uses the view of Lagrangian duality. (12 points)

Denote the feasible set as $C = \big\{x \in \mathbb{R}^n : Ax = b,\ Gx \leq h\big\}$ and let $L(x, u, v) = c^\top x + \langle u, Ax - b\rangle + \langle v, Gx - h\rangle$. For simplicity we assume that the linear program has a solution with finite value.

(a) Show that for $v \geq 0$, we have that

$$f^\star = \min_{x\in C}\langle c, x\rangle \geq \min_{x\in C} L(x, u, v) \geq \min_{x\in\mathbb{R}^n} L(x, u, v) =: g(u, v)\,,$$

which defines the dual function g(u,v).

(b) Note that part (a) shows that $g(u, v)$ is a lower bound on $f^\star$. The dual problem aims to find the best lower bound on $f^\star$. That is, the goal is to solve

$$\max_{u,v} g(u, v)\,, \quad s.t.\ v \geq 0.$$

In order to write the dual, your task is to find $g(u, v)$ by solving

$$\min_{x\in\mathbb{R}^n} L(x, u, v)\,,$$

where we can restrict (u, v) such that $g(u, v)$ is finite ($> -\infty$). *Hint:* Note that $L(x, u, v)$ is an affine function and such a function is lower bounded by $-\infty$. As a result $g(u, v) = -\infty$. Thus, in order to avoid this, you must provide an additional constraint and write the complete dual problem.

Exercise 2. [20 points]
In this exercise we solve the image segmentation problem (lecture slides 18) by solving the following optimization problem,

$$\min_{x \in \mathbb{R}^N} \langle c, x \rangle + \sum_{i=1}^{2N} |(Dx)_i|, \text{ s.t. } \forall i,\ x_i \in [-1, 1]$$

where c and D are defined on the slides.

(a) Reformulate this problem as a linear program and convert the linear program into its standard form. (10 points)

(b) Use the Scipy linear programming package (scipy.optimize.linprog) to solve this optimization problem using the simplex method as well as interior point method. (10 points)

(c) Redefine $c \in \mathbb{R}^N$ based on the given image such that segmentation improves.

Submission Instructions: This assignment sheet comprises the theoretical and programming parts.

- **Theoretical Part:** Write down your solutions clearly on a paper, scan them and convert them into a file named *theory(Name).pdf* where Name indicates the name of student submitting the assignment. Take good care of the ordering of the pages in this file. You are also welcome to submit the solutions prepared with LaTeX or some digital handwriting tool. You must write the full names of all the students in your group on the top of first page.
- **Programming part:** Submit your solution for the programming exercise with the filename `ex12(solution).py` where Name is the name of the student who submits the assignment on behalf of the group. You can only use python3.
- **Submission Folder:** Create a folder with the name *MatA_MatB_MatC* where MatA, MatB and MatC are the matriculation number (Matrikelnummer) of all the students in your group; depending on the number of people in the group. For example, if there are three students in a group with matriculation numbers 123456, 789012 and 345678 respectively, then the folder should be named: *123456_789012_345678*.
- **Submission:** Add all the relevant files to your submission folder and compress the folder into *123456_789012_345678.zip* file and upload it on the link provided on Moodle.
- **Deadline:** The submission deadline is 09.07.2024, 2:00 p.m. (always Tuesday 2 p.m.) via Moodle.