Continuous Optimization: Assignment 10

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Honglu Ma   **Hiroyasu Akada**   **Mathivathana Ayyappan**

Exercise 1

We can reformulate the problem as follows:

$$\min_{x \in \mathbb{R}^n} \langle c, x \rangle \quad \text{s.t.} \quad \forall i \in \{1, \ldots, m\} : b_i - \langle a_i, x \rangle = 0 \quad \text{and} \quad \forall j \in \{1, \ldots, n\} : -x_j \leq 0$$

where a_i is the i-th row of A, b_i is the i-th element of b and x_j is the j-th element of x.

We can name the objective function as $f(x) = \langle c, x \rangle$, the equality constraints as $f_i(x) = b_i - \langle a_i, x \rangle$, $\forall i \in \{1, \ldots, m\}$ and the inequality constraints as $g_j(x) = -x_j$, $\forall j \in \{1, \ldots, n\}$ such that we have a problem that fits the general form provided in Corollary 15.19, namely:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad f_i(x) = 0, i \in \mathcal{E} \quad \text{and} \quad g_j(x) \leq 0, j \in \mathcal{I}$$

By Corollary 15.19, we know that at optimal $x^\star$, we have

$$\nabla f(x^\star) + \sum_{i \in \mathcal{E}} \lambda_i \nabla f_i(x^\star) + \sum_{j \in \mathcal{A}(x^\star)} \mu_j \nabla g_j(x^\star) = 0$$

$$c - \sum_{i=1}^{m} \lambda_i a_i - \sum_{j \in \mathcal{A}(x^\star)} \mu_j e_j = 0$$

$$c - A^\top \lambda - \mu = 0$$

where

$$\mu = \begin{cases} 0 & \text{if } x_j^\star > 0 \\ \mu_j > 0 & \text{if } x_j^\star = 0 \end{cases}$$

observe that $\mu \geq 0$ which is the fouth KKT condition and $\mu_j x_j^\star = 0$, $\forall j \in \{1, \ldots, n\}$ which is the fifth KKT condition a.k.a the complementary condition. Also by reformulate the equation we derivate at optimal, we have the first KKT condition:

$$c = A^\top \lambda + \mu$$

Exercise 2

Note that

$$\text{tr}(B^\top X) = \sum_{i=1}^{n} \langle B_{i,}^\top, X_{,i} \rangle$$

where $B_{i,}^\top$ denotes the i-th row of $B^\top$ and $X_{,i}$ denotes the i-th column of X which can also just be seen as a sum of dot products between each column of B and X i.e. $\sum_{i=1}^{n} \langle b_i, x_i \rangle$ where b_i is the i-th column of B and x_i is the i-th column of X.

To find the minimum of a sum is the same as finding the minimum of each term in the sum i.e. instead of 1 objective function $f(x) = \sum_{i=1}^{n} \langle b_i, x_i \rangle$, we have n objective functions $f_i(x) = \langle b_i, x_i \rangle$ with same constraints on x_i as such

$$\min_{x_i \in \mathbb{R}^n} \langle b_i, x_i \rangle \quad \text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = 1 \quad \text{and} \quad \forall j \in \{1, \ldots, n\} : x_{ij} \geq 0$$

for all $i \in \{1, \ldots, n\}$

The Lagrangian for the i-th objective function is (to simplify notation, we will drop the i subscript from now on)

$$L(x, \lambda, \mu) = \langle b, x \rangle - \lambda \left(\sum_{j=1}^{n} x_j - 1 \right) - \sum_{j=1}^{n} \mu_j x_j$$

$$= \langle b, x \rangle - \lambda \left(\sum_{j=1}^{n} x_j - 1 \right) - \langle \mu, x \rangle$$

where

$$\mu = \begin{cases} 0 & \text{if } x_j > 0 \\ \mu_j > 0 & \text{if } x_j = 0 \end{cases}$$

We take the derivative of the Lagrangian with respect to x, λ and μ respectively:

$$\nabla_x L(x, \lambda, \mu) = b - \lambda \mathbb{1} - \mu = 0 \Leftrightarrow \mu_j = b_j - \lambda, \ \forall j \in \{1, \ldots, n\}$$

$$\nabla_\lambda L(x, \lambda, \mu) = - \left(\sum_{j=1}^{n} x_j - 1 \right) = 0 \Leftrightarrow \sum_{j=1}^{n} x_j = 1$$

$$\nabla_\mu L(x, \lambda, \mu) = -x \leq 0 \Leftrightarrow x \geq 0$$

$$\mu \geq 0$$

$$\mu_j \cdot x_j = 0, \ \forall j \in \{1, \ldots, n\}$$

We have such system of equations ($n + 1$ unknowns and $n + 1$ equations)

$$\mu_j \cdot x_j = 0$$

$$\sum_{j=1}^{n} x_j = 1$$

Solve for each column in X and we have the solution to the original problem.