Exercises

Continuous Optimization

Prof. Dr. Peter Ochs

www.mop.uni-saarland.de/teaching/OPT24

— Summer Term 2024 —

— Assignment 1 —

Exercise 1. [2 + 2 = 4 Points]

(a) Show that the sequence $(x^{(k)})_{k\in\mathbb{N}}$ given by

$$x^{(k)} = \frac{k^4 - 100}{k(5k^2 + 30k^3)}$$

converges as $k \to \infty$ and determine the limit point $\lim_{k\to\infty} x^{(k)}$.

(b) Let $x^0 = (1, 0)^T$, $\theta = \frac{\pi}{4}$, $R \in \mathbb{R}^{2\times 2}$ be a rotation matrix given by

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Show that the sequence $(x^{(k)})_{k\in\mathbb{N}}$ given by

$$x^{(k)} = R^k x^0$$

does not converge as $k \to \infty$. Determine all possible distinct cluster points.

Exercise 2. [3 + 2 + 3 + 2 = 10 points]

Answer the following questions.

(a) Provide an example of a set $C \subseteq \mathbb{R}$ such that

(i) $\mathrm{int}(\mathrm{cl}(C)) = \mathrm{int}(C)$.

(ii) $\mathrm{int}(cl(C)) \neq \mathrm{int}(C)$.

where int denotes the interior of a set and cl denotes the closure of a set. Provide justification for your examples.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for all $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R}^2 \ : \ f(x) \leq \alpha\}$ is convex. Then, is it true that f must be a convex function ? You can justify through an example.

(c) Derive an expression for projecting a vector $v \in \mathbb{R}^n$ onto the nullspace of a matrix $A \in \mathbb{R}^{n\times n}$. Recall, that null-space of a matrix is defined as

$$\mathcal{N}(A) := \{x \in \mathbb{R}^n;\ Ax = 0\}.$$

In your derivation, do not resort to any optimization techniques, use only linear algebra and properties of subspaces associated with a matrix.

(d) Consider a function $f\colon E \to \mathbb{R}$ where $E \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, show that the following sets are equal

(i) $\{x \in E;\ f(x) \geq c\} = \bigcap_{k=1}^{\infty} \{x \in E; f(x) > c - \frac{1}{k}\}$.

(ii) $\{x \in E;\ f(x) > c\} = \bigcup_{k=1}^{\infty} \{x \in E; f(x) \geq c + \frac{1}{k}\}$.

Exercise 3. [2 + 3 + 3 = 8 points]
The goal of this exercise is to get familiarized with basic tools from multi-variable calculus which lies at the foundation of mathematical optimization theory.

Suppose $E \subset \mathbb{R}^n$ is an open set and $f\colon E \to \mathbb{R}^m$, and $x \in E$. Then, we say that f is differentiable at $x \in E$ if there exists a linear transformation $A\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|}{|h|} = 0\,, \tag{1}$$

where $\|.\|$ denotes the standard Euclidean norm and we write $Df(x) = A$. Assume the standard canonical basis for $\mathbb{R}^n$ and $\mathbb{R}^m$ and denote the components of f as $f_1, \ldots, f_m$. Complete the following tasks.

(i) Show that the linear transformation A that satisfies definition (1) is unique, that is, if (1) holds for $A = A_1$ and $A = A_2$, then $A_1 = A_2$.

(ii) Show that if f admits a derivative at $x \in E$ as in (1), it implies continuity of f at x.

(iii) Derive an expression for the derivative (linear transformation) $Df(x)$ in terms of partial derivatives of the component functions $f_1, \ldots, f_m$.

Exercise 4. [18 Points]
Consider the function $f\colon \mathbb{R}^N \to \mathbb{R}$ given by

$$f(u) = \frac{1}{2}\sum_{i=1}^{N}(u_i - c_i)^2 + \frac{\mu}{2}\sum_{i=1}^{N-1}(u_{i+1} - u_i)^2\,,$$

where $c \in \mathbb{R}^N$ and $\mu > 0$ and $N = 5$.

(a) Write the function f in compact matrix–vector notation.
(*Use matrix-vector products, addition of vectors, and the Euclidean norm* $\|\cdot\|$.)

(b) Compute the gradient of f.

(c) Express the gradient of f in matrix–vector notation.

(d) Derive a closed-form expression for the solution u of $\nabla f(u) = 0$.
(*Matrix inversion need not be computed.*)

(e) Argue that the solution of $\nabla f(u) = 0$ exists and is unique.

(f) Solve the practical exercise in `ex01.py`.

Submission Instructions: This assignment sheet comprises the theoretical and programming parts.

- **Theoretical Part:** Write down your solutions clearly on a paper, scan them and convert them into a file named *theory.pdf*. Take good care of the ordering of the pages in this file. You are also welcome to submit the solutions prepared with LaTeX or some digital handwriting tool. You must write the full names of all the students in your group on the top of first page.
- **Programming part:** This assignment sheet also contains a programming exercise in the form of ex01.py file. Submit your solution for the programming exercise with the filename ex01solution.py. You can only use python3.
- **Submission Folder:** Create a folder with the name *MatA_MatB_MatC* where MatA, MatB and MatC are the matriculation number (Matrikelnummer) of all the students in your group; depending on the number of people in the group. For example, if there are three students in a group with matriculation numbers 123456, 789012 and 345678 respectively, then the folder should be named: *123456_789012_345678*.
- **Submission:** Add the two files, namely, *theorey.pdf* and `ex01solution.py` to your submission folder and compress the folder into *123456_789012_345678.zip* file and upload it on the link provided on Moodle.
- **Deadline:** The submission deadline is 30.04.2023, 2:00 p.m. (always Tuesday 2 p.m.) via Moodle.

Guidelines for solving problems: Finding the solution of a problem is only the first step. The presentation of the solution is of equal importance. In the beginning, this requires some extra work, however, once you have collected some experience, exposing your solution adequately will become an automatic skill. A clear exposition streamlines your thoughts, hence, gives an even deeper understanding, and careless mistakes will be less likely.

- Embed your solution in sentences, which clearly state your thoughts.
- State/mention your strategy to solve the problem, for example, “We verify the definition of a convex set.”, “We work towards a contradiction.”, “The following calculation verifies the assumptions of Theorem X.”.
- If you use equality “=” or equivalence “⇔” statements in your solution, make sure that it is clear what transformations or properties are used. For example, “$A \overset{(i)}{=} B$ where (i) holds thanks to Theorem X”, “[...] thanks to the property that [...]”.
- Clarify that statement B follows from statement A. For example: “We have observed that A holds, and *therefore*, B must also hold.”. Such logical connections are important for others to understand your solution.
- Note that, for statements A and B, the symbol “$A \Rightarrow B$” is a mathematical statement, which means “whenever A holds, then also B holds”. However, it does not imply that A holds. Therefore, it should not be used as a replacement for the words “therefore”, “hence”, “thus”, ... If you insist to abbreviate these words, you may use “$\rightsquigarrow$”. (I don’t like the symbol “$\therefore$”.)
- The symbol “&” is not a mathematical symbol for “and”. The right symbol would be “$\wedge$”, for example “$a > 0 \wedge a < 0$”. However, I prefer and suggest to write “and”. Moreover, I do not recommend to use “&” to replace “and” in normal text.
- Whenever you tend to use the word “obvious” or “trivial”, you should be careful. You should convince yourself with clear arguments that the fact is indeed true. Try to write down the proof for the “obvious” statement. It turns out that these are often not so obvious, or at least sometimes technically difficult to verify.

Problem Operators: How to solve a problem?

no keyword	If the exercise is a statement, for example, “property A implies property B”, then this has the same meaning as “Show that the following statement holds: property A implies property B”.
“show”, “derive”	Requires a logic chain of arguments to derive the desired result. Clearly justify each step. Explain what you are doing.
“draw”, “visualize”	This requires accurate drawing without using a ruler or circle. For coordinate systems, a scale must be fixed.
“compute”, “find the solution”	This requires a chain of arguments/computations to arrive at the solution. Each step must be clearly justified. Stating the solution only *is not sufficient.*
“reformulate”	Derive an equivalent representation. This includes also a proof why the intermediate steps are equivalent. Stating the reformulation only *is not sufficient.*
“state”, “write down”	This does not require a derivation or justification of the result.