Continuous Optimization: Assignment 3

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Exercise 1

Exercise 2

(a)

Since Q is a square matrix, we can write $Q = U\Lambda U^T$ where U is an orthogonal matrix and Λ is a diagonal matrix with the eigenvalues of Q i.e. λ_i on the diagonal. Then, we have

$$\begin{aligned}
\langle x, Qx\rangle &= \langle x, U\Lambda U^\top x\rangle \\
&= x^\top U\Lambda U^\top x \\
&= (U^\top x)^\top \Lambda U^\top x \\
&= (Ux)^\top \Lambda (Ux) && U = U^\top \\
&= \sum_{i=1}^n \lambda_i (Ux)_i^2 \\
&\leq \sum_{i=1}^n \lambda_{\max}(Q)(Ux)_i^2 && \lambda_i \leq \lambda_{\max}(Q) \\
&= \lambda_{\max}(Q)\sum_{i=1}^n (Ux)_i^2 \\
&= \lambda_{\max}(Q)(Ux)^\top Ux \\
&= \lambda_{\max}(Q)x^\top U^\top Ux \\
&= \lambda_{\max}(Q)x^\top x && U^\top U = I \\
&= \lambda_{\max}(Q)||x||^2
\end{aligned}$$

Similar derivation can be shown for the smallest eigenvalue: $\langle x, Qx\rangle \geq \lambda_{\min}(Q)||x||^2$.

(b)

We can rewrite $I - \tau Q$ as follows:

$$\begin{aligned}
I - \tau Q &= I - \tau U\Lambda U^\top \\
&= UU^\top - \tau U\Lambda U^\top \\
&= (U - \tau U\Lambda)U^\top \\
&= (UI - \tau U\Lambda)U^\top \\
&= U(I - \tau\Lambda)U^\top
\end{aligned}$$

thus we have

$$\begin{aligned}
(I - \tau Q)^2 &= U(I - \tau\Lambda)U^\top U(I - \tau\Lambda)U^\top \\
&= U(I - \tau\Lambda)^2 U^\top
\end{aligned}$$

where $(I - \tau\Lambda)^2$ consists of $(1 - \tau\lambda_i)^2$ on the diagonal.