Exercises

Continuous Optimization

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— Summer Term 2024 —

— Assignment 9 —

Exercise 1. [5 points]

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest and farthest to the point $(2, 4, 2)$.
Hint: Pose the problem as an equality constrained problem where objective function is the distance function from $(2, 4, 2)$ to a point (x, y, z) lying on the sphere.

Exercise 2. [5 points]

Use Lagrange multipliers to find least square solution for an under determined system of linear equations.

$$\min_{x\in\mathbb{R}^n} \|x\|^2 \,, \quad s.t.\ Ax = b\,.$$

where $A \in \mathbb{R}^{m\times n}$, $m < n$.

Exercise 3. [7 points]

Let $\tau > 0$ and $c, \bar{x} \in \mathbb{R}^n$. Solve the following convex minimization problem using Corollary 15.18 in the lecture notes:

$$\min_{x\in\mathbb{R}^n} \langle c, x\rangle + \frac{1}{2\tau}\|x - \bar{x}\|^2 \,, \quad s.t.\ \sum_{i=1}^{n} x_i = 1\,.$$

Exercise 4. [3 + 3 + 2 = 8 points]

Consider a symmetric $n \times n$ matrix Q. Define

$$\lambda_1 = \min_{\|x\|^2=1} \langle x, Qx\rangle \quad \text{and} \quad v_1 \in \operatorname*{argmin}_{\|x\|^2=1} \langle x, Qx\rangle \,,$$

and for $k = 0, \dots, n-1$:

$$\lambda_{k+1} = \min_{\substack{\|x\|^2=1 \\ \langle v_i,x\rangle=0,\ i=1,\dots,k}} \langle x, Qx\rangle \quad \text{and} \quad v_{k+1} \in \operatorname*{argmin}_{\substack{\|x\|^2=1 \\ \langle v_i,x\rangle=0,\ i=1,\dots,k}} \langle x, Qx\rangle \,.$$

(a) Show that

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n\,.$$

(b) Show that the vectors $v_1, \dots, v_n$ are linearly independent.

(c) Interpret $\lambda_1, \dots, \lambda_n$ as Lagrange multipliers, and show that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of Q, while $v_1, \dots, v_n$ are the corresponding eigenvectors.

Exercise 5. [5 + 10 = 15 points]

Your tasks are the following:

(a) Let $a \in \mathbb{R}^n, b \in \mathbb{R}$. Consider the following set:

$$C = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}\,.$$

The projection of the point $\bar{x} \in \mathbb{R}^n$ onto the set C is given by

$$P_C(\bar{x}) := \operatorname*{argmin}_{x \in C} \frac{1}{2}\|x - \bar{x}\|^2\,.$$

Your task is to calculate the expression for $P_C(\bar{x})$ and justify that it is a singleton set.

(b) We consider the binary classification problem, where the goal is to separate the input data into two different classes (or categories). We consider one of the simplest setting here, where we assume that input data represented as real vectors can be separated with a hyperplane, where each side of the hyperplane is associated with a class (or category).

You are provided with the training data $(x_i, y_i)_{i=1,\dots,m}$, where $x_i \in \mathbb{R}^n$ is the input data represented as a vector and $y_i \in \{-1, 1\}$ is the class label, for any $i \in \{1, \dots, m\}$. As stated earlier, we assume that the data is linearly separable in the sense that there exists $w \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that the following conditions hold:

$$\langle w, x_i \rangle + b \geq 0 \text{ if } y_i = 1\,, \text{ and } \langle w, x_i \rangle + b \leq 0 \text{ if } y_i = -1 \quad \forall i \in \{1, \dots, m\}\,.$$

Set of all such w, b is given by

$$W := \{(w, b) \in \mathbb{R}^n \times \mathbb{R} : \langle w, x_i \rangle + b \geq 0 \text{ if } y_i = 1\,, \text{ and } \langle w, x_i \rangle + b \leq 0 \text{ if } y_i = -1\ \forall i \in \{1, \dots, m\}\}\,.$$

The goal is to find a point in W, which can in turn linearly separate the data. Denote the quantities $\tilde{x}_i = \begin{pmatrix} x_i \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$ and $\tilde{w} = \begin{pmatrix} w \\ b \end{pmatrix} \in \mathbb{R}^{n+1}$, using which we rewrite W as following

$$W = \bigcap_{i=1}^{m} C_i\,,$$

where for $i \in \{1, \dots, m\}$ we define

$$C_i := \left\{\tilde{w} \in \mathbb{R}^{n+1} : \langle \tilde{w}, \tilde{x}_i \rangle \geq 0 \text{ if } y_i = 1\,, \text{ and } \langle \tilde{w}, \tilde{x}_i \rangle \leq 0 \text{ if } y_i = -1\right\}\,.$$

Furthermore, C_i can be rewritten as

$$C_i := \left\{\tilde{w} \in \mathbb{R}^{n+1} : y_i \langle \tilde{w}, \tilde{x}_i \rangle \geq 0\right\}\,.$$

In order to find a point in W, first note that C_i is a closed convex set, for any $i \in \{1, \dots, m\}$.

Feasibility problems involve finding points in intersection of sets. Finding a point in W is also a feasibility problem, as W is represented as intersection of sets. We know that the data is linearly separable, thus $W \neq \emptyset$. One of the popular algorithms to solve such feasibility problem, when the sets involved are closed and convex is Projection Onto Convex Sets (POCS) algorithm.

Consider the initialization $w^{(0)} \in \mathbb{R}^n$, the update step of POCS for $k \geq 0$ is given by

$$w^{(k+1)} = P_{C_m}(P_{C_{m-1}} \dots (P_{C_1}(w^{(k)})))\,.$$

Your task is to fill in the TODOs in `ex09_03_pocs.py` to implement the POCS update step, using the result from (a).

Submission Instructions: This assignment sheet comprises the theoretical and programming parts.

- **Theoretical Part:** Write down your solutions clearly on a paper, scan them and convert them into a file named *theory(Name).pdf* where Name indicates the student submitting the assignment on behalf of the group. Take good care of the ordering of the pages in this file. You are also welcome to submit the solutions prepared with LaTeX or some digital handwriting tool. You must write the full names of all the students in your group on the top of first page.
- **Programming part:** Submit your solution for the programming exercise with the filename `ex09_03_pocs.py` where Name is the name of the student who submits the assignment on behalf of the group. You can only use python3.
- **Submission Folder:** Create a folder with the name *MatA_MatB_MatC* where MatA, MatB and MatC are the matriculation number (Matrikelnummer) of all the students in your group; depending on the number of people in the group. For example, if there are three students in a group with matriculation numbers 123456, 789012 and 345678 respectively, then the folder should be named: *123456_789012_345678*.
- **Submission:** Add all the relevant files to your submission folder and compress the folder into *123456_789012_345678.zip* file and upload it on the link provided on Moodle.
- **Deadline:** The submission deadline is 25.06.2024, 2:00 p.m. (always Tuesday 2 p.m.) via Moodle.