Continuous Optimization: Assignment 12

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Exercise 1

1

(i) Not convex. Consider $p = 3$ we have $f(x) = x^3$ and $f''(x) = 6x \leq 0$ when $x \leq 0$. **Note:** The function is convex when p is even and not convex when p is odd.

(ii) Convex. $f''(x) = x^{-2} \geq 0$

(iii) Convex. $f''(x) = \alpha^2 e^{\alpha x} \geq 0$

(iv) Convex. $f''(x) = \frac{1}{(1-x)} \geq 0$ when $x \in (0, 1)$

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(i) Convex.
Take $x, y \in C$.
Consider $||(1-\lambda)x + \lambda y||_2 \overset{Cauchy-Schwarz}{\leq} ||(1-\lambda)x||_2 + ||\lambda y||_2 = (1-\lambda)||x||_2 + \lambda||y||_2 \leq 1$.

(ii) Not convex. Consider $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda = \frac{1}{2}$.

Clearly $||(1-\lambda)x + \lambda y||_2 = \frac{\sqrt{2}}{2} \neq 1$.

(iii) Not convex. Consider $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $\lambda = \frac{1}{2}$.

Clearly $||(1-\lambda)x + \lambda y||_\infty = || \begin{pmatrix} 0 \\ 0 \end{pmatrix} ||_\infty = 0 \neq 1$.

(iv) Convex. Similar to (i).
Take $x, y \in C$:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

s.t.

$$|x_1| + |x_2| \leq 1 \text{ and } |y_1| + |y_2| \leq 1$$

Consider $z = (1-\lambda)x + \lambda y$ where $\lambda \in [0, 1]$:

$$\begin{aligned} ||z||_1 &= |(1-\lambda)x_1 + \lambda y_1| + |(1-\lambda)x_2 + \lambda y_2| \\ &\leq (1-\lambda)|x_1| + \lambda|y_1| + (1-\lambda)|x_2| + \lambda|y_2| \\ &= (1-\lambda)\underbrace{(|x_1| + |x_2|)}_{\leq 1} + \lambda\underbrace{(|y_1| + |y_2|)}_{\leq 1} \quad \text{interpolate between 2 numbers less and equal to 1} \\ &\leq 1 \end{aligned}$$

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(i) We have

$$\nabla f(x, y) = \begin{pmatrix} 4x^3 - 1 \\ 4y^3 - 1 \end{pmatrix}$$

and

$$\nabla^2 f(x, y) = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

Exercise 1 continued on next page...

which is positive semi-definite everywhere implies the function is convex thus coercive. **Note:** This is not the definition of coercive. To prove coerciveness, we need to show that the function goes to infinity as the norm of the input goes to infinity. On the other hand, to prove non-coerciveness, we need to find a sequence of inputs such that the norms of the inputs tend to infinity but the function values do not.

$$x^4 + y^4 - x - y = \underbrace{(x^4 + y^4)}_{\substack{\to\infty \\ \sqrt{x^2+y^2}\to\infty}}(1 - \underbrace{\frac{x+y}{x^4+y^4}}_{\substack{\to 0 \\ \sqrt{x^2+y^2}\to\infty}})$$

(ii) Consider the sequence $(x^{(k)}, y^{(k)}) := (k, -k)$. Clearly $\left\|(x^{(k)}, y^{(k)})\right\|_2 = \sqrt{2k^2} \to \infty$ when $k \to \infty$. However, $f(x^{(k)}, y^{(k)}) = ak - bk + c = (a-b)k + c \to c$ when $k \to \infty$ if $a = b$ or $(a-b)k + c \to \infty^-$ if $a \leq b$.

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$$T_C(\begin{pmatrix}1 & 1\end{pmatrix}^\top) = \left\{\begin{pmatrix}x_1 \\ x_2\end{pmatrix} \in \mathbb{R}^2 \mid x_1 \leq 1, x_2 \leq 1\right\}$$
$$N_C(\begin{pmatrix}1 & 1\end{pmatrix}^\top) = \left\{\begin{pmatrix}x_1 \\ x_2\end{pmatrix} \in \mathbb{R}^2 \mid x_1 \geq 1, x_2 \geq 1\right\}$$

Note: Both the tangent and normal cones are set of directions which have angles and magnitudes. In case of tangent and normal cones, their magnitudes don't matter. So, the correct answer should be:

$$T_C(\begin{pmatrix}1 & 1\end{pmatrix}^\top) = \left\{\begin{pmatrix}x_1 \\ x_2\end{pmatrix} \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 \leq 0\right\}$$
$$N_C(\begin{pmatrix}1 & 1\end{pmatrix}^\top) = \left\{\begin{pmatrix}x_1 \\ x_2\end{pmatrix} \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\right\}$$

Exercise 2

(A)

In steepest descent method, we have $x^{(k+1)} = x^{(k)} - \alpha\nabla f(x^{(k)})$. We can approximate $f(x^{(k+1)})$ with its second order Taylor expansion around $x^{(k)}$.

$$\begin{aligned} f(x^{(k+1)}) &\approx f(x^{(k)}) + \nabla f(x^{(k)})^\top(x^{(k+1)} - x^{(k)}) + \frac{1}{2}(x^{(k+1)} - x^{(k)})^\top\nabla^2 f(x^{(k)})(x^{(k+1)} - x^{(k)}) \\ &= f(x^{(k)}) - \alpha\nabla f(x^{(k)})^\top\nabla f(x^{(k)}) + \frac{\alpha^2}{2}\nabla f(x^{(k)})^\top\nabla^2 f(x^{(k)})\nabla f(x^{(k)}) =: g(\alpha) \end{aligned}$$

We can find the optimal α by solving $\frac{dg}{d\alpha} = 0$.

$$\begin{aligned} \frac{dg}{d\alpha} &= -\nabla f(x^{(k)})^\top\nabla f(x^{(k)}) + \alpha\nabla f(x^{(k)})^\top\nabla^2 f(x^{(k)})\nabla f(x^{(k)}) = 0 \\ \nabla f(x^{(k)})^\top\nabla f(x^{(k)}) &= \nabla f(x^{(k)})^\top(\alpha\nabla^2 f(x^{(k)}))\nabla f(x^{(k)}) \end{aligned}$$

Exercise 2 continued on next page...

By choosing $\alpha = (\nabla^2 f(x^{(k)}))^{-1}$, we can cancel the affect of rescaling of the Hessian matrix. Thus we arrive at the Newton's method: $x^{(k+1)} = x^{(k)} - (\nabla^2 f(x^{(k)}))^{-1}\nabla f(x^{(k)})$.

Corrected Answer (Direction)

For a quadratic function $f(x) = \frac{1}{2}x^\top Qx + b^\top x$. Suppose Q is symmetric and positive definite. We can decompose Q as $Q = LL^\top$ where L is a lower triangular matrix. Let $y = L^\top x$, we reformulate the optimization problem to solve for y:

$$\min_y \frac{1}{2}y^\top y + b^\top L^{-\top} y$$

Perform steepest descent on y:

$$y^{(k+1)} = y^{(k)} - \alpha\nabla f(y^{(k)}) = y^{(k)} - \alpha(y^{(k)} + L^{-1}b)$$

(B)

Step 0 Let $x^{(0)} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} d^{(0)} = r^{(0)} &= -\nabla f(x^{(0)}) \\ &= -b - Qx^{(0)} \\ &= -\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \tau_0 &= \frac{\langle r^{(0)}, r^{(0)} \rangle}{\langle d^{(0)}, Qd^{(0)} \rangle} \\ &= \frac{1}{2} \end{aligned}$$

Step 1

$$\begin{aligned} x^{(1)} &= x^{(0)} + \tau_0 d^{(0)} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ r^{(1)} &= r^{(0)} + \tau_0 Qd^{(0)} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 & 3 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \\ \beta_1 &= \frac{\langle r^{(1)}, r^{(1)} \rangle}{\langle r^{(0)}, r^{(0)} \rangle} \\ &= \frac{17}{4} \\ d^{(1)} &= -r^{(1)} + \beta_1 d^{(0)} \\ &= \begin{pmatrix} \frac{11}{4} \\ -\frac{7}{4} \end{pmatrix} \end{aligned}$$

Exercise 3

See Appendix A: Handwritten Solution for Exercise 3

Exercise 4

(A)

The optimality condition is

$$\begin{aligned} \nabla f(x^\star) + \lambda^\star \nabla g(x^\star) + \langle \mu^\star, \nabla h(x^\star) \rangle = 0 \\ \Rightarrow -(\alpha + x^\star)^{-1} + \lambda^\star x^\star - \mu^\star = 0 \\ x^\star \geq 0 \\ \mu^\star \geq 0 \\ \langle \mu^\star, x^\star \rangle = 0 \end{aligned}$$

where $f(x) = \sum_{i=1}^n -\log(\alpha + x_i)$, $g(x) = \sum_{i=1}^n x_i - 1$ and $h(x) = -x$. Note $(v^{-1})_i = \frac{1}{v_i}$ for $v \in \mathbb{R}^n$.

(B)

In each time step k, we first compute a point $\tilde{x}^{(k)} = x^{(k)} - \alpha \nabla f(x^{(k)})$. Then we project $\tilde{x}^{(k)}$ onto the feasible set $C = \{x \in \mathbb{R}^n \mid \alpha^\top x = \beta\}$. We know that C is a hyperplane and the projection has closed form:

$$\hat{x}^{(k)} = \mathrm{proj}_C(\tilde{x}^{(k)}) = \tilde{x}^{(k)} - \frac{\alpha^\top \tilde{x}^{(k)} - \beta}{\|\alpha\|_2^2} \alpha$$

Then we can update compute the next time step by

$$x^{(k+1)} = x^{(k)} + \tau_k (\hat{x}^{(k)} - x^{(k)})$$

with τ_k that satisfies the Armijo condition.

Appendix A: Handwritten Solution for Exercise 3

$f(x_1, x_2, x_3) = x_1 \sin(x_2) + x_3 e^{x_1 x_2}$

$x_1 = x_1$

$x_2 = x_2$

$x_3 = x_3$

$x_4 = \sin(x_2)$

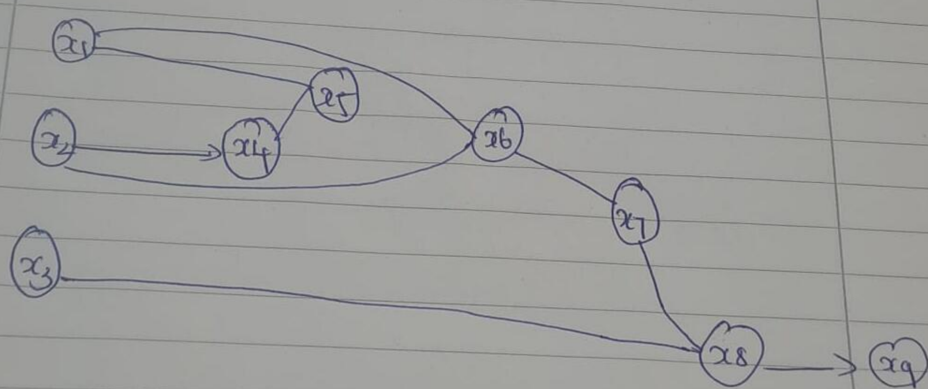
$x_5 = x_1 x_4$

$x_6 = x_1 x_2$

$x_7 \Rightarrow e^{x_6}$

$x_8 = x_3 x_7$

$x_9 = x_8 + x_5$



derivates

$$\frac{\delta \dot{x}_1}{\delta x_1} = 1$$

$$\frac{\delta x_2}{\delta x_2} = 1$$

$$\frac{\delta x_3}{\delta x_3} = 1$$

$$\frac{\delta x_4}{\delta x_2} = \cos(x_2)$$

$$\frac{\delta x_5}{\delta x_1} = x_4$$

$$\frac{\delta x_5}{\delta x_4} = x_1$$

$$\frac{\delta x_6}{\delta x_1} = x_2$$

$$\frac{\delta x_6}{\delta x_2} = x_1$$

$$\frac{\delta x_7}{\delta x_6} = e^{x_6}$$

$$\frac{\delta x_8}{\delta 7} = x_3$$

$$\frac{\delta x_8}{\delta x_3} = x_7$$

$$\frac{\delta x_9}{\delta x_8} = 1$$

$$\frac{\delta x_9}{\delta x_5} = 1.$$

Forward mode

$x_1 = \tilde{x}_1$

$x_2 = \tilde{x}_2$

$x_3 = \tilde{x}_3$

$x_4 = \sin(\tilde{x}_2)$

$x_5 = \tilde{x}_1 \sin(\tilde{x}_2)$

$x_6 = \tilde{x}_1 \tilde{x}_2$

$x_7 \Rightarrow e^{\tilde{x}_1 \tilde{x}_2}$

$x_8 \Rightarrow \tilde{x}_3 e^{\tilde{x}_1 \tilde{x}_2}$

$x_9 \Rightarrow \tilde{x}_1 \sin(\tilde{x}_2) + \tilde{x}_3 e^{\tilde{x}_1 \tilde{x}_2}$

$\dot{x}_1 \Rightarrow 1$

$\dot{x}_2 = 1$

$\dot{x}_3 = 1$

$\dot{x}_4 = \cos(x_2)(\dot{x}_2)$

$\dot{x}_5 \Rightarrow \dot{x}_1 \sin(x_2) + \sin(x_2) x_1$

$\dot{x}_6 \Rightarrow \dot{x}_1 x_2 + x_2 \dot{x}_1$

$\dot{x}_7 \Rightarrow e^{(x_1 x_2)} (\dot{x}_1)(\dot{x}_2)$

$\dot{x}_8 = \dot{x}_3 e^{x_1 x_2} + (e^{x_1 x_2})(\dot{x}_1 \dot{x}_2)$ x_3

$x_9 \Rightarrow \dot{x}_8 + \dot{x}_5$

$\Rightarrow$

$\tilde{x_9} = \frac{\delta 8}{x_9} = 1.$

$\tilde{x_8} \Rightarrow \frac{\delta 8}{x_9} \cdot \frac{\delta 9}{\delta 8} \Rightarrow 1.1 = 1.$

$\tilde{x_7} \Rightarrow \frac{\delta 9}{\delta 8} \cdot \frac{\delta 8}{\delta 7} = 1 \cdot \tilde{x}_3 = x_3$

$\tilde{x}_6 \Rightarrow \frac{\delta 8}{\delta 7} \cdot \frac{\delta 7}{\delta 6} \Rightarrow \tilde{x}_3 e^{\tilde{x}_6} \Rightarrow \tilde{x}_3 e^{\tilde{x}_1 \tilde{x}_2}$

$\tilde{x}_1 \Rightarrow \frac{\delta 7}{\delta 6} \cdot \frac{\delta 6}{\delta x_1} = \tilde{x}_3 e^{\tilde{x}_6} \tilde{x}_2 \Rightarrow \tilde{x}_3 x_2 e^{x_6}$

$\Rightarrow x_3 x_2 e^{x_1 x_2}$