# Continuous Optimization: Assignment 3

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# Exercise 1

# Exercise 2

## (a)

Since $Q$ is a square matrix, we can write $Q = U\Lambda U^T$ where $U$ is an orthogonal matrix and $\Lambda$ is a diagonal matrix with the eigenvalues of $Q$ i.e. $\lambda_i$ on the diagonal. Then, we have

$$\begin{aligned}
\langle x, Qx\rangle &= \langle x, U\Lambda U^\top x\rangle \\
&= x^\top U\Lambda U^\top x \\
&= (U^\top x)^\top \Lambda U^\top x \\
&= (Ux)^\top \Lambda (Ux) && U = U^\top \\
&= \sum_{i=1}^n \lambda_i (Ux)_i^2 \\
&\leq \sum_{i=1}^n \lambda_{\max}(Q)(Ux)_i^2 && \lambda_i \leq \lambda_{\max}(Q) \\
&= \lambda_{\max}(Q)\sum_{i=1}^n (Ux)_i^2 \\
&= \lambda_{\max}(Q)(Ux)^\top Ux \\
&= \lambda_{\max}(Q)x^\top U^\top Ux \\
&= \lambda_{\max}(Q)x^\top x && U^\top U = I \\
&= \lambda_{\max}(Q)||x||^2
\end{aligned}$$

Similar derivation can be shown for the smallest eigenvalue: $\langle x, Qx\rangle \geq \lambda_{\min}(Q)||x||^2$.

## (b)

Suppose $\lambda$ is an eigenvalue of $Q$ with eigenvector $v$. Then, we have

$$\begin{aligned}
Qv = \lambda v &\Rightarrow \tau Qv = \tau\lambda v \\
&\Rightarrow Iv - \tau Qv = Iv - \tau\lambda v \\
&\Rightarrow (I - \tau Q)v = (I - \tau \mathrm{diag}(\lambda))v
\end{aligned}$$

$I - \tau\mathrm{diag}(\lambda)$ is a matrix with same diagonal entries $1 - \tau\lambda$

$$\Rightarrow (I - \tau Q)v = (1 - \tau\lambda)v$$

Above shows that $1 - \tau\lambda$ is an eigenvalue of $I - \tau Q$.

$$\begin{aligned}
&\Rightarrow (I - \tau Q)(I - \tau Q)v = (I - \tau Q)(1 - \tau\lambda)v \\
&\Rightarrow (I - \tau Q)(I - \tau Q)v = (1 - \tau\lambda)(I - \tau Q)v \\
&\Rightarrow (I - \tau Q)(I - \tau Q)v = (1 - \tau\lambda)(1 - \tau\lambda)v \\
&\Rightarrow (I - \tau Q)^2 v = (1 - \tau\lambda)^2 v
\end{aligned}$$

Exercise 2 continued on next page...

Thus $(1 - \tau\lambda)^2$ is an eigenvalue of $(I - \tau Q)^2$ for each eigenvalue $\lambda$ of $Q$.