

# Continuous Optimization: Assignment 1

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## Exercise 1

(a)

**Claim:**  $\lim_{k \rightarrow \infty} x^{(k)} = \frac{1}{30}$

**Proof:**

Let  $\epsilon > 0$  be given. Choose  $N = \max\{\frac{1}{90\epsilon}, 8\}$ . Assume  $n > N$ . We have

$$n > N \Rightarrow n > 9 > \sqrt[3]{600} \Rightarrow n^3 > 600 \Rightarrow 5n^3 > 3000 \Rightarrow 10n^3 - 5n^3 > 3000 \Rightarrow 3000 + 5n^3 < 10n^3$$

and obviously

$$900n^4 > 150n^3 + 900n^4$$

To check the validity of the limit we need to show  $|x^{(n)} - x^*| < \epsilon$  where  $x^* = \frac{1}{30}$ .

$$\begin{aligned} \left| \frac{n^4 - 100}{5n^3 + 30k^4} - \frac{1}{30} \right| &= \left| \frac{30n^4 - 3000 - 5n^3 - 30k^4}{150n^3 + 900n^4} \right| \\ &= \left| \frac{-3000 - 5n^3}{150n^3 + 900n^4} \right| \\ &= \frac{3000 + 5n^3}{150n^3 + 900n^4} \\ &< \frac{10n^3}{900n^4} = \frac{1}{90n} && \text{(by the inequalities above)} \\ &< \frac{1}{90N} \\ &< \frac{1}{\frac{90}{90\epsilon}} = \epsilon \end{aligned}$$

(b)

We have the following accumulation points:

$$\begin{aligned} x^{(8n+1)} &= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \quad x^{(8n+2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x^{(8n+3)} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \quad x^{(8n+4)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ x^{(8n+5)} &= \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \quad x^{(8n+6)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad x^{(8n+7)} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \quad x^{(8n)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \in \mathbb{N} \end{aligned}$$

We can prove by showing that  $x^{(k)}$  is not a cauchy sequence thus does not converges (Proposition A.5, Lecture Script) i.e.  $\exists \epsilon > 0$  such that  $\forall N \in \mathbb{N}, \exists n, m > N$  such that  $\|x^{(n)} - x^{(m)}\| \geq \epsilon$ .

**Proof:**

Let  $\epsilon = 1$ , for all  $N \in \mathbb{N}$ , choose  $n = 8N$  and  $m = 8N + 4$ . We have

$$\|x^{(8N)} - x^{(8N+4)}\| = \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\| = 2 \geq 1 = \epsilon$$

## Exercise 2

(a)

The interior of a set  $C$  is defined as the union of all open sets contained in  $C$ . The closure of a set  $C$  is the set  $C$  together with all of its limit points.

- (i) Let  $C = \mathbb{R}$ . We have  $\text{int}(C) = \mathbb{R}$  and  $\text{cl}(C) = \mathbb{R}$ . It is obvious that  $\text{int}(\text{cl}(C)) = \text{int}(C) = \mathbb{R}$ .
- (ii) Let  $C = \mathbb{Q}$ . We have  $\text{int}(C) = \emptyset$  and  $\text{cl}(C) = \mathbb{R}$ . This is because every open interval in  $\mathbb{R}$  contains irrational numbers. Thus  $\text{int}(\text{cl}(C)) = \text{int}(\mathbb{R}) = \mathbb{R} \neq \text{int}(C) = \emptyset$ .

(b)

Consider function  $f(x) = \sqrt{\|x\|_1}$ ,  $x \in \mathbb{R}^2$  has sublevel sets  $\{x \in \mathbb{R}^2 \mid f(x) \leq \alpha\}$ . The function can also be written as  $f(x) = \sqrt{|x_1| + |x_2|}$ ,  $x = (x_1, x_2)$ .

Pick two elements  $x, y$  from the sublevel set of  $\alpha$  such that  $f(x) \leq \alpha$  and  $f(y) \leq \alpha$ . We have

$$f(x) = \sqrt{|x_1| + |x_2|} \leq \alpha \Rightarrow |x_1| + |x_2| \leq \alpha^2 \text{ and similarly } |y_1| + |y_2| \leq \alpha^2$$

Now take a point  $z := \lambda x + (1 - \lambda)y$ ,  $\lambda \in [0, 1]$ . Also  $z_1 = \lambda x_1 + (1 - \lambda)y_1$ ,  $z_2 = \lambda x_2 + (1 - \lambda)y_2$

$$\begin{aligned} |z_1| + |z_2| &= |\lambda x_1 + (1 - \lambda)y_1| + |\lambda x_2 + (1 - \lambda)y_2| \\ &\leq \lambda|x_1| + (1 - \lambda)|y_1| + \lambda|x_2| + (1 - \lambda)|y_2| \\ &\leq \lambda(|x_1| + |x_2|) + (1 - \lambda)(|y_1| + |y_2|) \\ &\leq \lambda\alpha^2 + (1 - \lambda)\alpha^2 = \alpha^2 \end{aligned}$$

thus  $\{x \in \mathbb{R}^2 \mid f(x) \leq \alpha\}$  is a convex set.

Now we need to show that  $f$  is not convex. Let  $x = (0, 0)$ ,  $y = (1, 0)$  and  $\lambda = \frac{1}{2}$ . We have  $f(x) = 0$ ,  $f(y) = 1$  and

$$f(\lambda x + (1 - \lambda)y) = \sqrt{\frac{1}{2}} > \lambda f(x) + (1 - \frac{1}{2})f(y) = \frac{1}{2}$$

Check here for visualization of the sublevel sets.

(c)

Let  $v_{N_A}$  be  $v$ 's projection onto the null space and  $\lambda \in \mathbb{R}^m$ . We know that  $v$  can be decompose into

$$\begin{aligned} v &= A^\top \lambda + v_{N_A} \Rightarrow Av = A(A^\top \lambda + v_{N_A}) \\ &\Rightarrow Av = AA^\top \lambda + 0 \\ &\Rightarrow Av = AA^\top \lambda \\ &\Rightarrow (AA^\top)^{-1}Av = \lambda \end{aligned}$$

Replace  $\lambda$  in the original equation we get

$$\begin{aligned} v &= A^\top (AA^\top)^{-1}Av + v_{N_A} \Rightarrow v - A^\top (AA^\top)^{-1}Av = v_{N_A} \\ &\Rightarrow (I - A^\top (AA^\top)^{-1}A)v = v_{N_A} \end{aligned}$$

(d)

To show equivalence of two sets  $A$  and  $B$ , we need to show both  $A \subseteq B$  and  $B \subseteq A$ .

- (i) Show  $\{x \in E \mid f(x) \geq c\} \subseteq \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > c - \frac{1}{k}\}$

Take  $v \in \{x \in E \mid f(x) \geq c\}$  we have

$$f(v) \geq c > c - \frac{1}{k} \text{ for all } k \geq 1$$

This shows that  $v \in \{x \in E \mid f(x) > c - \frac{1}{k}\}$  for all  $k \geq 1$ , i.e.  $v \in \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > c - \frac{1}{k}\}$ .

Show  $\bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > c - \frac{1}{k}\} \supseteq \{x \in E \mid f(x) \geq c\}$

**Proof by contradiction:**

Take  $v \in \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > c - \frac{1}{k}\}$ . Assume that  $v \notin \{x \in E \mid f(x) \geq c\}$  we have

$$f(v) < c \Rightarrow c - f(v) > 0$$

$$\Rightarrow \exists k \geq 1 \text{ such that } c - f(v) - \frac{1}{k} \geq 0$$

$$\Rightarrow c - \frac{1}{k} \geq f(v)$$

Contradiction

(ii) Show  $\{x \in E \mid f(x) > c\} \subseteq \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + \frac{1}{k}\}$

Take  $v \in \{x \in E \mid f(x) > c\}$ . Let  $f(v) = c + \frac{1}{k}$ .

We can always pick a  $k'$  such that  $k' \geq k$  so that  $f(v) = c + \frac{1}{k} \geq c + \frac{1}{k'}$ .

This shows that  $\exists k' > 1$  such that  $v \in \{x \in E \mid f(x) \geq c + \frac{1}{k'}\}$ .

Show  $\{x \in E \mid f(x) > c\} \supseteq \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + \frac{1}{k}\}$

Take  $v \in \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + \frac{1}{k}\}$ .

$$f(v) \geq c + \frac{1}{k} > c$$

## Exercise 3

(i)

The assumption of  $h$  is a vector with same entry is incorrect which leads to a wrong result. Please see the correct answer below.

Suppose each the entry of  $h$  is  $t$ , we have  $\|h\| = \sqrt{nt^2}$ . To make the derivation cleaner, we represent

$f' = f(x + h)$  and  $f = f(x)$ . We can rewrite the equation in the form of vector entry terms:

$$\begin{aligned}
& \lim_{t \rightarrow 0} \frac{\sqrt{\sum_i^n (f'_i - f_i - t \sum_j^m A_{ij})^2}}{\sqrt{nt^2}} = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \sqrt{\frac{\sum_i^n (f'_i - f_i - t \sum_j^m A_{ij})^2}{nt^2}} = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \frac{\sum_i^n (f'_i - f_i - t \sum_j^m A_{ij})^2}{nt^2} = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \left( \sum_i^n \frac{(f'_i - f_i - t \sum_j^m A_{ij})^2}{nt^2} \right) = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \left( \sum_i^n \frac{(f'_i - f_i)^2 - 2t(f'_i - f_i) \sum_j^m A_{ij} + t^2(\sum_j^m A_{ij})^2}{nt^2} \right) = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \left( \sum_i^n \frac{(f'_i - f_i)^2}{nt^2} - \frac{2(f'_i - f_i) \sum_j^m A_{ij}}{nt} + \frac{(\sum_j^m A_{ij})^2}{n} \right) = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \left( \sum_i^n \frac{(f'_i - f_i)^2}{nt^2} \right) - \lim_{t \rightarrow 0} \left( \sum_i^n \frac{2(f'_i - f_i) \sum_j^m A_{ij}}{nt} \right) + \lim_{t \rightarrow 0} \left( \sum_i^n \frac{(\sum_j^m A_{ij})^2}{n} \right) = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \left( \frac{1}{nt^2} \sum_i^n (f'_i - f_i)^2 \right) - \lim_{t \rightarrow 0} \left( \frac{2}{nt} \sum_i^n (f'_i - f_i) \sum_j^m A_{ij} \right) + \frac{1}{n} \sum_i^n (\sum_j^m A_{ij})^2 = 0
\end{aligned}$$

**Proof:** Assume that  $\exists A_1, A_2 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - A_1 h\|}{\|h\|} = 0 \text{ and } \lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - A_2 h\|}{\|h\|} = 0$$

With the previous derivation, we can write the two equations as such:

$$\lim_{t \rightarrow 0} \left( \frac{1}{nt^2} \sum_i^n (f'_i - f_i)^2 \right) - \lim_{t \rightarrow 0} \left( \frac{2}{nt} \sum_i^n (f'_i - f_i) \sum_j^m A_{1ij} \right) + \frac{1}{n} \sum_i^n (\sum_j^m A_{1ij})^2 = 0$$

and

$$\lim_{t \rightarrow 0} \left( \frac{1}{nt^2} \sum_i^n (f'_i - f_i)^2 \right) - \lim_{t \rightarrow 0} \left( \frac{2}{nt} \sum_i^n (f'_i - f_i) \sum_j^m A_{2ij} \right) + \frac{1}{n} \sum_i^n (\sum_j^m A_{2ij})^2 = 0$$

Subtracting the two equations:

$$\begin{aligned}
 & \lim_{t \rightarrow 0} \left( \frac{2}{nt} \sum_i^n (f'_i - f_i) \left( \sum_j^m A_{1ij} - \sum_j^m A_{2ij} \right) \right) + \frac{1}{n} \sum_i^n \left( \sum_j^m A_{1ij} \right)^2 - \frac{1}{n} \sum_i^n \left( \sum_j^m A_{2ij} \right)^2 = 0 \\
 & \Rightarrow \lim_{t \rightarrow 0} \left( \sum_i^n \frac{2(f'_i - f_i)}{nt} \left( \sum_j^m A_{1ij} - \sum_j^m A_{2ij} \right) \right) + \frac{1}{n} \sum_i^n \left( \sum_j^m A_{1ij} \right)^2 - \frac{1}{n} \sum_i^n \left( \sum_j^m A_{2ij} \right)^2 = 0 \\
 & \Rightarrow \sum_i^n \lim_{t \rightarrow 0} \left( \frac{2(f'_i - f_i)}{nt} \left( \sum_j^m A_{1ij} - \sum_j^m A_{2ij} \right) \right) + \frac{1}{n} \sum_i^n \left( \sum_j^m A_{1ij} \right)^2 - \frac{1}{n} \sum_i^n \left( \sum_j^m A_{2ij} \right)^2 = 0 \\
 & \Rightarrow \sum_i^n \left( \left( \sum_j^m A_{1ij} - \sum_j^m A_{2ij} \right) \lim_{t \rightarrow 0} \frac{2(f'_i - f_i)}{nt} \right) + \frac{1}{n} \sum_i^n \left( \sum_j^m A_{1ij} \right)^2 - \frac{1}{n} \sum_i^n \left( \sum_j^m A_{2ij} \right)^2 = 0
 \end{aligned}$$

Observe term  $\sum_i^n \left( \left( \sum_j^m A_{1ij} - \sum_j^m A_{2ij} \right) \lim_{t \rightarrow 0} \frac{2(f'_i - f_i)}{nt} \right)$ . As  $t \rightarrow 0$ ,  $f'_i - f_i \rightarrow 0$ , we cannot determine  $\lim_{t \rightarrow 0} \frac{2(f'_i - f_i)}{nt}$ . To make each term of the sum equals to 0, we need  $\sum_j^m A_{1ij} = \sum_j^m A_{2ij}$  for each row  $i$ .

### Correct Answer

No need to expand any term. Just treat  $\|f(x+h) - f(x) - Ah\|$  like  $f(x+h) - f(x) - Ah$  without the norm.

(ii)

If  $A = Df(x)$  is admitted as the derivative of  $f$  at  $x$  then we can write

$$\begin{aligned}
 f(x+h) &= f(x) + Ah \\
 &\Rightarrow \lim_{h \rightarrow 0} f(x+h) = \lim_{h \rightarrow 0} (f(x) + Ah) \\
 &\Rightarrow \lim_{h \rightarrow 0} f(x+h) = f(x) \\
 &\Rightarrow \lim_{h \rightarrow 0} f(x+h) = f(x)
 \end{aligned}$$

which implies continuity of  $f$  at  $x$ .

### Correct Answer

Let  $h = x - a$ .

$$\begin{aligned}
 f(x+h) &= f(x) + Df(x)h + o(\|h\|) \Rightarrow f(x) = f(a) + Df(a) \cdot (x-a) + o(\|x-a\|) \\
 &\Rightarrow \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} f(a) + \lim_{x \rightarrow a} (Df(a) \cdot (x-a)) + \lim_{x \rightarrow a} o(\|x-a\|) \\
 &\Rightarrow \lim_{x \rightarrow a} f(x) = f(a) + Df(a) \cdot \lim_{x \rightarrow a} (x-a) + \lim_{x \rightarrow a} o(\|x-a\|) \\
 &\Rightarrow \lim_{x \rightarrow a} f(x) = f(a) + Df(a) \cdot 0 + \lim_{x \rightarrow a} o(\|x-a\|) \\
 &\Rightarrow \lim_{x \rightarrow a} f(x) = f(a) + \lim_{x \rightarrow a} o(\|x-a\|) \\
 &\Rightarrow \lim_{x \rightarrow a} f(x) = f(a) + \lim_{x \rightarrow a} \left( \frac{\|x-a\| o(\|x-a\|)}{\|x-a\|} \right) \\
 &\Rightarrow \lim_{x \rightarrow a} f(x) = f(a)
 \end{aligned}$$

(iii)

We can continue to derive from part (i):

$$\begin{aligned}
& \lim_{t \rightarrow 0} \left( \frac{1}{nt^2} \sum_i^n (f'_i - f_i)^2 \right) - \lim_{t \rightarrow 0} \left( \frac{2}{nt} \sum_i^n (f'_i - f_i) \sum_j^m A_{ij} \right) + \frac{1}{n} \sum_i^n \left( \sum_j^m A_{ij} \right)^2 = 0 \\
& \Rightarrow \frac{1}{n} \left( \sum_i^n \lim_{t \rightarrow 0} \frac{(f'_i - f_i)^2}{t^2} - 2 \sum_i^n \sum_j^m A_{ij} \lim_{t \rightarrow 0} \frac{f'_i - f_i}{t} + \sum_i^n \left( \sum_j^m A_{ij} \right)^2 \right) = 0 \\
& \Rightarrow \frac{1}{n} \sum_i^n \left( \lim_{t \rightarrow 0} \frac{(f'_i - f_i)^2}{t^2} - 2 \sum_j^m A_{ij} \lim_{t \rightarrow 0} \frac{f'_i - f_i}{t} + \left( \sum_j^m A_{ij} \right)^2 \right) = 0 \\
& \Rightarrow \frac{1}{n} \sum_i^n \left( \lim_{t \rightarrow 0} \frac{f'_i - f_i}{t} - \sum_j^m A_{ij} \right)^2 = 0 \\
& \Rightarrow \lim_{t \rightarrow 0} \frac{f'_i - f_i}{t} = \sum_j^m A_{ij}
\end{aligned}$$

which implies that the sum of each row  $i$  of  $A$  is how sensitive  $f_i$  is to a small change of  $t$ .  $A$  is the Jacobian matrix of  $f$  and is express as:

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

### Correct Answer

Let  $h = te_j$  where  $e_j$  is the  $j$ th canonical basis vector.

$$\begin{aligned}
& f(x + te_j) - f(x) = Df(x)te_j + o(\|te_j\|) \\
& \Rightarrow \frac{f(x + te_j) - f(x)}{t} = \frac{Df(x)te_j}{t} + \frac{o(\|te_j\|)}{t} \\
& \Rightarrow \frac{f(x + te_j) - f(x)}{t} = Df(x)e_j + \frac{o(\|te_j\|)}{t} \\
& \Rightarrow \lim_{t \rightarrow 0} \frac{f(x + te_j) - f(x)}{t} = \lim_{t \rightarrow 0} Df(x)e_j + \lim_{t \rightarrow 0} \frac{o(\|te_j\|)}{t} \\
& \Rightarrow \lim_{t \rightarrow 0} \frac{f(x + te_j) - f(x)}{t} = Df(x)e_j + \lim_{t \rightarrow 0} \frac{o(\|te_j\|)}{t} \\
& \Rightarrow \lim_{t \rightarrow 0} \frac{f(x + te_j) - f(x)}{t} = Df(x)e_j + \lim_{t \rightarrow 0} \frac{|t|o(\|e_j\|)}{t} \\
& \Rightarrow \lim_{t \rightarrow 0} \frac{f(x + te_j) - f(x)}{t} = Df(x)e_j
\end{aligned}$$

The left hand side denotes the directional derivative of  $f$  at  $x$  in the direction of  $e_j$  while the right hand side is the definition of the  $j$ th column of the matrix  $Df(x)$ . Thus each column of  $Df(x)$  is the directional derivative of  $f$  at  $x$  along the direction of the canonical basis vectors i.e. the partial derivatives of  $f$  at  $x$ .

## Exercise 4

(a)

The function can be rewritten as

$$f(u) = \frac{1}{2} \|u - c\|^2 + \frac{\mu}{2} \|Au\|^2$$

where  $u \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{N \times N}$  and each element at row  $i < N$  and column  $j$  is defined as

$$A_{ij} := \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

and its last row is defined as  $A_{Nj} := 0$

(b)

For each entry of  $\nabla f(u)$  we have

$$\frac{\partial f}{\partial u_i} = \begin{cases} u_i - c_i + \mu(u_i - u_{i+1}) & \text{if } i = 1 \\ u_i - c_i + \mu(-u_{i-1} + 2u_i - u_{i+1}) & \text{if } 1 < i < n \\ u_i - c_i + \mu(-u_{i-1} + u_i) & \text{if } i = n \end{cases}$$

(c)

$$\nabla f(u) = u - c + \mu A^\top A u$$

which can be verified since each element of  $A^\top A$  can be expressed as

$$\begin{aligned} (A^\top A)_{1j} &= \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases} \\ (A^\top A)_{ij} &= \begin{cases} -1 & \text{if } j = i - 1 \\ 2 & \text{if } j = i \\ -1 & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}, 2 \leq i \leq N - 2 \\ (A^\top A)_{Nj} &= \begin{cases} -1 & \text{if } j = N - 1 \\ 1 & \text{if } j = N \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(d)

$$\begin{aligned} \nabla f(u) = 0 &\Rightarrow u - c + \mu A^\top A u = 0 \\ &\Rightarrow (\mu A^\top A + I)u = c \end{aligned}$$

$\mu A^\top A + I$  is symmetric. if  $\det(\mu A^\top A + I) \neq 0$  then  $\mu A^\top A + I$  is invertible and we can write

$$u = (\mu A^\top A + I)^{-1} c$$



(e)

We need to show that the matrix  $\mu A^\top A + I$  is invertible, *i.e.*,  $\mu A^\top A + I$  is positive definite. Let  $\lambda$  be an eigenvalue of  $\mu A^\top A + I$  and  $v$  be the corresponding eigenvector. Then we have

$$(\mu A^\top A + I)v = \lambda v.$$

By multiplying both sides by  $v^\top$ , we get

$$v^\top (\mu A^\top A + I)v = \lambda v^\top v.$$

Since  $\mu A^\top A + I$  is symmetric, we have  $v^\top (\mu A^\top A + I)v = \mu \|Av\|^2 + \|v\|^2$ .

Since  $\mu > 0$ , we have  $\mu \|Av\|^2 + \|v\|^2 > 0$  and thus, we get  $\lambda > 0$ .

This shows that all eigenvalues of  $\mu A^\top A + I$  are positive, and hence it is positive definite and invertible.