# Continuous Optimization: Assignment 7

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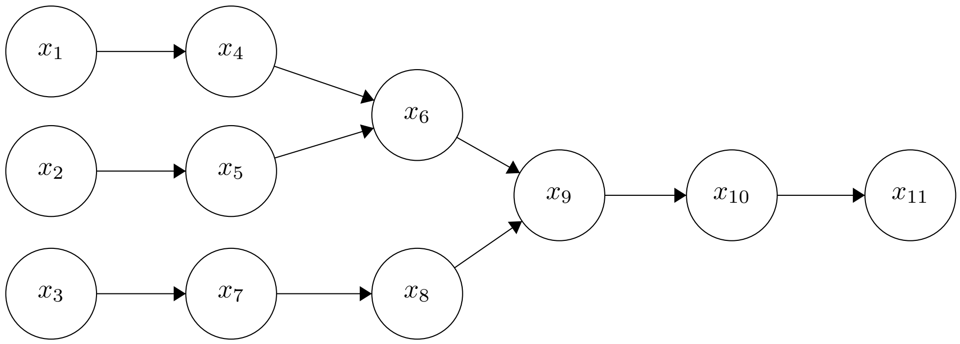
# Exercise 1

The computational graph for function $f$ is expressed as such




where

$$
\begin{aligned}
x_4 &= x_1^2 \\
x_5 &= x_2^3 \\
x_7 &= x_3^4 \\
x_6 &= x_4 \cdot x_5 \\
x_8 &= \sin(x_7) \\
x_9 &= x_6 + x_8 \\
x_{10} &= \exp(x_9) \\
x_{11} &= x_{10}^2
\end{aligned}
$$

and we have the following derivatives

$$
\begin{aligned}
\frac{\partial x_4}{\partial x_1} &= 2x_1 \\
\frac{\partial x_5}{\partial x_2} &= 3x_2^2 \\
\frac{\partial x_7}{\partial x_3} &= 4x_3^3 \\
\frac{\partial x_6}{\partial x_4} &= x_5 \\
\frac{\partial x_6}{\partial x_5} &= x_4 \\
\frac{\partial x_8}{\partial x_7} &= \cos(x_7) \\
\frac{\partial x_9}{\partial x_6} &= 1 \\
\frac{\partial x_9}{\partial x_8} &= 1 \\
\frac{\partial x_{10}}{\partial x_9} &= \exp(x_9) \\
\frac{\partial x_{11}}{\partial x_{10}} &= 2x_{10}
\end{aligned}
$$

Exercise 1 continued on next page...

## Forward Mode

We propogate the tangents through the computational graph to compute the derivative of $f$ with respect to $x_1$, $x_2$ and $x_3$. Normally, we would have to propogate all the bases i.e. $\hat{i}$, $\hat{j}$ and $\hat{k}$ one after another in order to all the partial derivatives but here each operation of the computational graph is a scalar operation and the starting node $x_1$, $x_2$ and $x_3$ are also scalas so we can simply set $\dot{x}_1 = 1$ when calculating $\frac{\partial f}{\partial x_1}$ and so on. To calculate the partial derivatives at point $(x_1, x_2, x_3) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, we have

(i) $\frac{\partial f}{x_1}$ Set $\dot{x}_1 = 1, \dot{x}_2 = 0, \dot{x}_3 = 0$

$$
\begin{aligned}
x_1 &= \bar{x}_1 & \dot{x}_1 &= 1 \\
x_2 &= \bar{x}_2 & \dot{x}_2 &= 0 \\
x_3 &= \bar{x}_3 & \dot{x}_3 &= 0 \\
x_4 &= x_1^2 = \bar{x}_1^2 & \dot{x}_4 &= 2x_1\dot{x}_1 = 2\bar{x}_1\dot{x}_1 = 2\bar{x}_1 \\
x_5 &= x_2^3 = \bar{x}_2^3 & \dot{x}_5 &= 3x_2^2\dot{x}_2 = 0 \\
x_6 &= x_4 \cdot x_5 = \bar{x}_1^2 \cdot \bar{x}_2^3 & \dot{x}_6 &= x_5\dot{x}_4 + x_4\dot{x}_5 = 2\bar{x}_1\bar{x}_2^3 \\
x_7 &= x_3^4 = \bar{x}_3^4 & \dot{x}_7 &= 4x_3^3\dot{x}_3 = 0 \\
x_8 &= \sin(x_7) = \sin(\bar{x}_3^4) & \dot{x}_8 &= \cos(x_7)\dot{x}_7 = 0 \\
x_9 &= x_6 + x_8 = \bar{x}_1^2 \cdot \bar{x}_2^3 + \sin(\bar{x}_3^4) & \dot{x}_9 &= \dot{x}_6 + \dot{x}_8 = 2\bar{x}_1\bar{x}_2^3 \\
x_{10} &= \exp(x_9) = \exp(\bar{x}_1^2 \cdot \bar{x}_2^3 + \sin(\bar{x}_3^4)) & \dot{x}_{10} &= \exp(x_9)\dot{x}_9 = 2\exp(\bar{x}_1^2 \cdot \bar{x}_2^3 + \sin(\bar{x}_3^4))\bar{x}_1\bar{x}_2^3 \\
x_{11} &= x_{10}^2 = \exp(2(\bar{x}_1^2 \cdot \bar{x}_2^3 + \sin(\bar{x}_3^4))) & \dot{x}_{11} &= 2x_{10}\dot{x}_{10} = 4\exp(2(\bar{x}_1^2 \cdot \bar{x}_2^3 + \sin(\bar{x}_3^4)))\bar{x}_1\bar{x}_2^3
\end{aligned}
$$

(ii) $\frac{\partial f}{x_2}$ Set $\dot{x}_1 = 0, \dot{x}_2 = 1, \dot{x}_3 = 0$. Repeat the same process as above we get

$$\frac{\partial f}{\partial x_2} = 6\bar{x}_1^2\bar{x}_2^2 \exp(2(\bar{x}_1^2 \cdot \bar{x}_2^3 + \sin(\bar{x}_3^4)))$$

(iii) $\frac{\partial f}{x_3}$ Set $\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = 1$. Repeat the same process as above we get

$$\frac{\partial f}{\partial x_3} = 8\bar{x}_3^3 \cos(\bar{x}_3^4) \exp(2(\bar{x}_1^2 \cdot \bar{x}_2^3 + \sin(\bar{x}_3^4)))$$

## Backward Mode

Backward mode has the same equation as forward mode but the order of the operations is reversed. We can only start backpropagating after evaluating all intermediate functions (because we need the value for the last node) as such

$$\frac{\partial f}{\partial x_1} = \left(\frac{\partial f}{\partial x_{11}} \frac{\partial x_{11}}{\partial x_{10}}\right) \frac{\partial x_{10}}{\partial x_9} \frac{\partial x_9}{\partial x_6} \frac{\partial x_6}{\partial x_4} \frac{\partial x_4}{\partial x_1}$$

the partial derivatives are all evaluated with the previous intermediate nodes values. This will give us the same result as the forward mode.