Continuous Optimization: Assignment 7

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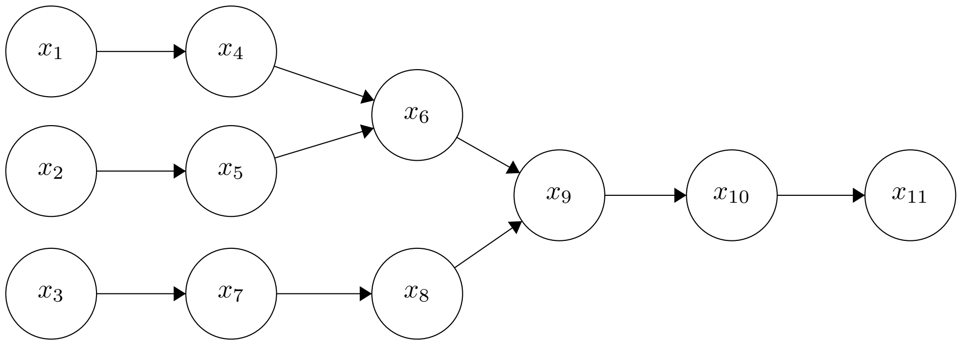
Exercise 1

The computational graph for function f is expressed as such



where

$$
\begin{aligned}
x_4 &= x_1^2 \\
x_5 &= x_2^3 \\
x_7 &= x_3^4 \\
x_6 &= x_4 \cdot x_5 \\
x_8 &= \sin(x_7) \\
x_9 &= x_6 + x_8 \\
x_{10} &= \exp(x_9) \\
x_{11} &= x_{10}^2
\end{aligned}
$$

and we have the following derivatives

$$
\begin{aligned}
\frac{\partial x_4}{\partial x_1} &= 2x_1 \\
\frac{\partial x_5}{\partial x_2} &= 3x_2^2 \\
\frac{\partial x_7}{\partial x_3} &= 4x_3^3 \\
\frac{\partial x_6}{\partial x_4} &= x_5 \\
\frac{\partial x_6}{\partial x_5} &= x_4 \\
\frac{\partial x_8}{\partial x_7} &= \cos(x_7) \\
\frac{\partial x_9}{\partial x_6} &= 1 \\
\frac{\partial x_9}{\partial x_8} &= 1 \\
\frac{\partial x_{10}}{\partial x_9} &= \exp(x_9) \\
\frac{\partial x_{11}}{\partial x_{10}} &= 2x_{10}
\end{aligned}
$$

Exercise 1 continued on next page...

Forward Mode

We propogate the tangents through the computational graph to compute the derivative of f with respect to x_1, x_2 and x_3. Normally, we would have to propogate all the bases i.e. $\hat{i}$, $\hat{j}$ and $\hat{k}$ one after another in order to all the partial derivatives but here each operation of the computational graph is a scalar operation and the starting node x_1, x_2 and x_3 are also scalas so we can simply set $\dot{x}_1 = 1$ when calculating $\frac{\partial f}{\partial x_1}$ and so on. To calculate the partial derivatives at point $(x_1, x_2, x_3) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, we have

(i) $\frac{\partial f}{x_1}$ Set $\dot{x}_1 = 1, \dot{x}_2 = 0, \dot{x}_3 = 0$

$$\begin{aligned}
x_1 &= \tilde{x}_1 & \dot{x}_1 &= 1 \\
x_2 &= \tilde{x}_2 & \dot{x}_2 &= 0 \\
x_3 &= \tilde{x}_3 & \dot{x}_3 &= 0 \\
x_4 &= x_1^2 = \tilde{x}_1^2 & \dot{x}_4 &= 2x_1\dot{x}_1 = 2\tilde{x}_1\dot{x}_1 = 2\tilde{x}_1 \\
x_5 &= x_2^3 = \tilde{x}_2^3 & \dot{x}_5 &= 3x_2^2\dot{x}_2 = 0 \\
x_6 &= x_4 \cdot x_5 = \tilde{x}_1^2 \cdot \tilde{x}_2^3 & \dot{x}_6 &= x_5\dot{x}_4 + x_4\dot{x}_5 = 2\tilde{x}_1\tilde{x}_2^3 \\
x_7 &= x_3^4 = \tilde{x}_3^4 & \dot{x}_7 &= 4x_3^3\dot{x}_3 = 0 \\
x_8 &= \sin(x_7) = \sin(\tilde{x}_3^4) & \dot{x}_8 &= \cos(x_7)\dot{x}_7 = 0 \\
x_9 &= x_6 + x_8 = \tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4) & \dot{x}_9 &= \dot{x}_6 + \dot{x}_8 = 2\tilde{x}_1\tilde{x}_2^3 \\
x_{10} &= \exp(x_9) = \exp(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)) & \dot{x}_{10} &= \exp(x_9)\dot{x}_9 = 2\exp(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4))\tilde{x}_1\tilde{x}_2^3 \\
x_{11} &= x_{10}^2 = \exp(2(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4))) & \dot{x}_{11} &= 2x_{10}\dot{x}_{10} = 4\exp(2(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)))\tilde{x}_1\tilde{x}_2^3
\end{aligned}$$

(ii) $\frac{\partial f}{x_2}$ Set $\dot{x}_1 = 0, \dot{x}_2 = 1, \dot{x}_3 = 0$. Repeat the same process as above we get

$$\frac{\partial f}{\partial x_2} = 6\tilde{x}_1^2\tilde{x}_2^2\exp(2(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)))$$

(iii) $\frac{\partial f}{x_3}$ Set $\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = 1$. Repeat the same process as above we get

$$\frac{\partial f}{\partial x_3} = 8\tilde{x}_3^3\cos(\tilde{x}_3^4)\exp(2(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)))$$

Backward Mode

We propogate normal vector of the hyperplane defined by $\langle \bar{y}, \nabla F(x) \rangle = c$ backward through the computational graph where $\bar{y}$ is the desired variation of the function F. For each partial derivatives, we set the normal vector to be the corresponding basis vector, i.e. $\bar{y} = \hat{i}$ for $\frac{\partial f}{\partial x_1}$ and so on.

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(i) $\frac{\partial f}{x_1}$

$$\begin{aligned}
x_1 &= \tilde{x}_1 \\
x_2 &= \tilde{x}_2 \\
x_3 &= \tilde{x}_3 \\
x_4 &= \tilde{x}_1^2 \\
x_5 &= \tilde{x}_2^3 \\
x_6 &= \tilde{x}_1^2 \cdot \tilde{x}_2^3 \\
x_7 &= \tilde{x}_3^4 \\
x_8 &= \sin(\tilde{x}_3^4) \\
x_9 &= \tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4) \\
x_{10} &= \exp(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)) \\
x_{11} &= \exp(2(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)))
\end{aligned}$$

$$\begin{aligned}
\bar{x}_{11} &= 1 \\
\bar{x}_{10} &= 2\exp(\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)) \\
\bar{x}_9 &= 2\exp(2\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)) \\
\bar{x}_6 &= 2\exp(2\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)) \\
\bar{x}_4 &= 2\tilde{x}_2^3\exp(2\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4)) \\
\bar{x}_1 &= 4\tilde{x}_1\tilde{x}_2^3\exp(2\tilde{x}_1^2 \cdot \tilde{x}_2^3 + \sin(\tilde{x}_3^4))
\end{aligned}$$

Similarly we can calculate the other partial derivatives. The result is the same as the forward mode.

Exercise 2

(a)

We construct a matrix $\tilde{P}_j$ for each column j of P such that $\tilde{P}_j \in \mathbb{R}^{2\times 6}$

$$\tilde{P}_j := \begin{pmatrix} P_{1j} & 0 & P_{2j} & 0 & 1 & 0 \\ 0 & P_{1j} & 0 & P_{2j} & 0 & 1 \end{pmatrix}$$

and we have $\tilde{P} \in \mathbb{R}^{2n\times 6}$ as following

$$\tilde{P} := \begin{pmatrix} \tilde{P}_1 \\ \vdots \\ \tilde{P}_n \end{pmatrix}$$

At last, we construct $\tilde{Q} \in \mathbb{R}^{2n\times 1}$:

$$\tilde{Q} := \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix}$$

where Q_j denotes the j-th column of Q.

(b)

The linear least squares problem can be formulated as

$$\min_{x\in\mathbb{R}^3} \frac{1}{2} \|F(P,Q)\|^2$$

where

$$x := \begin{pmatrix} t_1 \\ t_2 \\ \theta \end{pmatrix} \qquad F(P,Q) := \tilde{P}A(x) - \tilde{Q}$$

with $\tilde{P}$ and $\tilde{Q}$ defined in (a).
The linear system of equations that need to be solved is

$$\nabla F(x^{(k)})\nabla F(x^{(k)})^\top d^{(k)} = -\nabla F(x^{(k)})F(x^{(k)})$$

in order to calculate the descent direction $d^{(k)}$. $\nabla F(x^{(k)})$ can be calculated analytically. It can also be approximated by finite difference.