Exercises

Continuous Optimization

Prof. Dr. Peter Ochs

`www.mop.uni-saarland.de/teaching/OPT24`

— Summer Term 2024 —

— Assignment 10 —

Exercise 1. [10 points]

For the linear program in standard form

$$\min_{x\in\mathbb{R}^n} \langle c, x\rangle\,, \quad s.t.\ Ax = b,\ x \geq 0,$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, $m < n$ and suppose that A has full rank. The KKT conditions are given as

$$A^\top \lambda = c, \tag{1}$$

$$Ax = b, \tag{2}$$

$$x \geq 0, \tag{3}$$

$$\mu \geq 0, \tag{4}$$

$$x_i\mu_i = 0,\ i = 1, \dots, n. \tag{5}$$

Derive these Conditions (1-5) using Corollary 15.19 from lecture notes.

Exercise 2. [10 points]

Let $B \in \mathbb{R}^{m\times n}$. Solve the following problem:

$$\min_{X\in\mathbb{R}^{m\times n}} \operatorname{tr}(B^\top X) \quad s.t. \forall j = 1, \dots, n\colon \sum_{i=1}^{m} X_{i,j} \leq 1 \text{ and } \forall i, j\colon\ X_{i,j} \geq 0\,,$$

where $\operatorname{tr}(A) := \sum_{i=1}^{n} A_{i,i}$ is the trace of a matrix $A \in \mathbb{R}^{n\times n}$.

Exercise 3. [5 points]

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, $p \in \mathbb{R}$, $q \in \mathbb{R}$, and the condition $p < q$ holds true. Consider the following optimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 \quad s.t.\ \forall i = 1, \dots, n\colon\ p \leq x_i \leq q\,. \tag{6}$$

Your tasks are the following:

(a) Derive the update step of the Conditional Gradient Method for solving (6).

(b) Derive the update step of the Projected Gradient Method for solving (6).

Exercise 4. [15 points]

We consider an optimization problem that can be used for image deblurring. Let $f \in \mathbb{R}^{n_x \times n_y}$ be a recorded gray-value image that is blurry and noisy. We model the degradation process as follows: There exists an image $h \in \mathbb{R}^{n_x \times n_y}$ (clean image or ground truth image) such that the observed image f is given by

$$f = \mathcal{A}h + n\,,$$

where $\mathcal{A}\colon \mathbb{R}^{n_x \times n_y} \to \mathbb{R}^{n_x \times n_y}$ is a (known) blurr operator and $n \in \mathbb{R}^{n_x \times n_y}$ is a realization of a pixel-wise independent and identically distributed normal distributed random variable (Gaussian noise). In this exercise, the blurr operator implements a convolution with a filter (a small image patch).

In order to reconstruct the clean image as good as possible, we seek for an image $u \in \mathbb{R}^{n_x \times n_y}$ that approximates the inverse of the blurring and looks like an image. Such a reconstruction can be found by minimizing the following optimization problem

$$\min_{u \in [0,1]^{n_x \times n_y}} \frac{1}{2}\|\mathcal{A}u - f\|^2 + \mu \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} \sqrt{((\mathcal{D}u)_{i,j,1})^2 + ((\mathcal{D}u)_{i,j,2})^2 + \epsilon^2}\,, \tag{7}$$

with certain $\mu, \epsilon > 0$. A solution tries to minimize both terms in (7) as much as possible, i.e., it tries to find the best compromise between a small value of the first term, i.e., $\mathcal{A}u \approx f$, and a small value of the second term, i.e., u being smooth, steered by a positive parameter μ. For the definition of the forward difference operator $\mathcal{D}\colon \mathbb{R}^{n_x \times n_y} \to \mathbb{R}^{n_x \times n_y \times 2}$ and the context of image processing applications, we refer to Section 2.3 of the lecture notes, although this knowledge is not required for solving the following problem.

Note that we enforce a box constraint in (7), where the values of the reconstructed image must lie in $[0, 1]$ interval. Box constraints are natural for image reconstruction problems, because any realistic image must have bounded values for pixels, in particular must be non-negative and bounded from above. In our case, we set the upper bound for pixel value as 1, however, this may vary depending on the representation of the image.

We solve the problem in (7) using Projected Gradient Method and Conditional Gradient Method. In the implementation, we consider the vectorized form of the above problem, i.e., we set $N = n_x n_y$ and let $u, f \in \mathbb{R}^N$, $\mathcal{A} \in \mathbb{R}^{N \times N}$, $\mathcal{D} \in \mathbb{R}^{2N \times N}$ and $\mu > 0$. The Lipschitz constant of the gradient of the objective in (7) is $\|\mathcal{A}^T\mathcal{A}\|_2 + \frac{8\mu}{\varepsilon}$, where $\|\mathcal{A}^T\mathcal{A}\|_2$ denotes the induced 2-norm (operator 2-norm) of the matrix $\mathcal{A}^T\mathcal{A}$. The Lipschitz constant of the gradient is stored in the variable `Lip` in `ex09_10_image_deblurring.py`. The exact construction of $\mathcal{D}$ and $\mathcal{A}$ can be found in `make_derivatives2D` and `make_filter2D` function in `myimgtools.py`.

(a) Download the code template from the webpage. Unpack the file using `unzip opt20_ex10.zip`. It consists of files `ex09_10_image_deblurring.py`, `ConditionalGradient.py`. and `ProjectedGradient.py`.

(b) Implement the Projected Gradient Descent by filling in the TODOs in `ProjectedGradient.py`. You may adapt the results from Exercise 1 in this sheet to the problem in (7).

(c) Implement the Conditional Gradient Method by filling in the TODOs in `ConditionalGradient.py`. You may adapt the results from Exercise 1 in this sheet to the problem in (7).

(d) Select the parameter $\mu > 0$ such that the best visual quality of the reconstruction u is achieved. You can use the Peak-Signal-to-noise-ratio (PSNR) as a quality measure, which is defined by

$$\text{PSNR} = 10 \log\left(\frac{MAX^2}{MSE}\right)$$

where $MAX := 255$ is the maximal pixel value and MSE is the mean-squared error defined by

$$MSE := \frac{1}{N}\sum_{i=1}^{N}(u_i - f_i)^2 \,.$$

PSNR is a mathematical construct to measure the quality of an image with respect to another image. Thus, for higher relative image quality, we require higher PSNR. (2 points)

Submission Instructions: This assignment sheet comprises the theoretical and programming parts.

- **Theoretical Part:** Write down your solutions clearly on a paper, scan them and convert them into a file named *theory(Name).pdf*. Take good care of the ordering of the pages in this file. You are also welcome to submit the solutions prepared with LaTeX or some digital handwriting tool. You must write the full names of all the students in your group on the top of first page.
- **Programming part:** Submit your solution for the programming exercise with the filename `ex09_10_image_deblurring(Name).py` where Name is the name of the student who submits the assignment on behalf of the group. You can only use python3.
- **Submission Folder:** Create a folder with the name *MatA_MatB_MatC* where MatA, MatB and MatC are the matriculation number (Matrikelnummer) of all the students in your group; depending on the number of people in the group. For example, if there are three students in a group with matriculation numbers 123456, 789012 and 345678 respectively, then the folder should be named: *123456_789012_345678*.
- **Submission:** Add all the relevant files to your submission folder and compress the folder into *123456_789012_345678.zip* file and upload it on the link provided on Moodle.
- **Deadline:** The submission deadline is 02.07.2024, 2:00 p.m. (always Tuesday 2 p.m.) via Moodle.