Exercises

Continuous Optimization

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www.mop.uni-saarland.de/teaching/OPT24

— Summer Term 2024 —

— Assignment 3 —

Exercise 1. [5 points]

Correct the 10 errors in the proof of the following statement.

Descent Lemma:
Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with L-Lipschitz continuous gradient, $L > 0$, i.e., L satisfies

(1)
$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|\,, \quad \forall x, y \in \mathbb{R}^n\,.$$

Then, for any point $\bar{x} \in \mathbb{R}^n$, the following holds

(2)
$$f(x) \leq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x}\rangle + \frac{L}{2}\|x - \bar{x}\|^2\,, \quad \forall x \in \mathbb{R}^n\,.$$

Proof. Using the fundamental theorem of linear algebra, the following holds

$$f(x) - f(\bar{x}) = \int_0^1 \langle \nabla f(x + t(x - \bar{x})), x - \bar{x}\rangle \ \mathrm{d}t\,.$$

The right-hand side can be rewritten as

$$\int_0^1 \langle \nabla f(\bar{x} + t(x - \bar{x})), x - \bar{x}\rangle \ \mathrm{d}t \leq \int_0^1 \langle \nabla f(\bar{x}), x - \bar{x}\rangle \ \mathrm{d}t + \Big|\int_0^1 \langle \nabla f(\bar{x} + t(x - \bar{x})) - \nabla f(\bar{x}), x - \bar{x}\rangle \ \mathrm{d}t\Big|\,.$$

Now, using $|\int_0^1 \varphi(t)\,\mathrm{d}t| \geq \int_0^1 |\varphi(t)|\,\mathrm{d}t$ and Fenchel inequality, we obtain

$$\Big|\int_0^1 \langle \nabla f(\bar{x} + t(x - \bar{x})) - \nabla f(\bar{x}), x - \bar{x}\rangle \ \mathrm{d}t\Big| \leq \int_0^1 \|x - \bar{x}\|\,\|\nabla f(\bar{x} + t(x - \bar{x})) - \nabla f(\bar{x})\|\,\mathrm{d}t\,.$$

Note that this inequality also requires the Bolzano–Weierstrass theorem. Putting everything together, and incorporating the Lipschitz continuity of f, (2) implies:

$$f(x) - f(\bar{x}) \leq \langle \nabla f(\bar{x}), x - \bar{x}\rangle + \|x - \bar{x}\| \int_0^1 \frac{Lt}{2}\|x - \bar{x}\|\,\mathrm{d}t = \langle \nabla f(\bar{x}), x - \bar{x}\rangle + \frac{L}{2}\|x - \bar{x}\|^2\,.$$

Finally, the statement follows by compactness of bounded sets, which induces a result on convergence of subsequences, which is required to proof the statement. □

Exercise 2. [5 + 3 = 8 points]

Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix.

(a) Then, the following holds:

$$\lambda_{\min}(Q)\|x\|^2 \leq \langle x, Qx\rangle \leq \lambda_{\max}(Q)\|x\|^2 .$$

(b) For any $\tau \in \mathbb{R}$, the eigenvalues of $(I - \tau Q)^2$ are equal to $(1 - \tau\lambda_i)^2$ where $\lambda_i, i = 1, \dots, n$, are the eigenvalues of Q.

Exercise 3. [7 points]

Consider the space $\mathbb{R}^n$ with the inner product given by $\langle x\,,\,y\rangle = x^\top Q y$, where Q is a symmetric positive definite matrix. Next, consider a subspace of $\mathbb{R}^n$ spanned by vectors $\{u_1,\,u_2,\,\dots\,,u_m\}$ with $1 \leq m \leq n$ and let $A = [u_1,\,u_2,\,\dots,\,u_m]$ be the matrix formed by stacking these vectors as columns. For a vector $v \in \mathbb{R}^n$, perform the following tasks,

(a) Pose an optimization problem to find the projection of v onto the subspace spanned by vectors $\{u_1,\,u_2,\,\dots\,,u_m\}$.

(b) Solve the problem to find an expression for projection of vector v onto the given subspace with $m = 1$.

(c) Solve the problem to find a general formula for projection of vector v onto the given subspace with m vectors.

Exercise 4. [20 points]

We consider an optimization problem that can be used for image deblurring. Let $f \in \mathbb{R}^{n_x\times n_y}$ be a recorded gray-value image that is blurry and noisy. We model the degradation process as follows: There exists an image $h \in \mathbb{R}^{n_x\times n_y}$ (clean image or ground truth image) such that the observed image f is given by

$$f = \mathcal{A}h + n\,,$$

where $\mathcal{A}\colon \mathbb{R}^{n_x\times n_y} \to \mathbb{R}^{n_x\times n_y}$ is a (known) blurr operator and $n \in \mathbb{R}^{n_x\times n_y}$ is a realization of a pixel-wise independent and identically distributed normal distributed random variable (Gaussian noise). In this exercise, the blurr operator implements a convolution with a filter (a small image patch).

In order to reconstruct the clean image as good as possible, we seek for an image $u \in \mathbb{R}^{n_x\times n_y}$ that approximates the inverse of the blurring and looks like an image. Such a reconstruction can be found by minimizing the following optimization problem

$$\min_u \frac{1}{2}\|\mathcal{A}u - f\|^2 + \frac{\mu}{2}\|\mathcal{D}u\|^2 . \tag{3}$$

A solution tries to minimize both terms as much as possible, i.e., it tries to find the best compromise between a small value of the first term, i.e., $\mathcal{A}u \approx f$, and a small value of the second term, i.e., u being smooth, steered by a positive parameter μ. For the definition of the forward difference operator $\mathcal{D}\colon \mathbb{R}^{n_x\times n_y} \to \mathbb{R}^{n_x\times n_y\times 2}$ and the context of image processing applications, we refer to Section 2.3 of the lecture notes, although this knowledge is not required for solving the following problem.

In the implementation, we consider the vectorized form of the above problem, i.e., we set $N = n_x n_y$ and let $u, f \in \mathbb{R}^N$, $\mathcal{A} \in \mathbb{R}^{N\times N}$, $\mathcal{D} \in \mathbb{R}^{2N\times N}$ and $\mu > 0$. The exact construction of $\mathcal{D}$ and $\mathcal{A}$ can be found in `make_derivatives2D` and `make_filter2D` function in `myimgtools.py`.

(a) Convert the optimization problem in (3) into the form $\frac{1}{2}\|\tilde{\mathcal{A}}u - \tilde{f}\|^2$ and clearly specify $\tilde{\mathcal{A}}$ and $\tilde{f}$. (3 points)

(b) Now, rewrite the obtained optimization problem into the standard form $\frac{1}{2}\langle u, Qu\rangle + \langle b, u\rangle + c$. Clearly specify Q, b, c in terms of the variables of (3). State the gradient in terms of the variables in (3) and step size for exact line search in terms of the gradient. (3 points)

(c) Download the code template from moodle. It consists of files `ex03_04_image_deblurring.py`, `GradientDescent_ELS.py`.

(d) Implement the Gradient Descent Method with Exact Line Search by filling in the TODOs in `GradientDescent_ELS.py`. (6 points)

(e) Implement the Gradient Descent Method with constant step size. Comment on the best choice of step size.(6 points)

(f) Select the parameter $\mu > 0$ such that the best visual quality of the reconstruction u is achieved. You can use the Peak-Signal-to-noise-ratio (PSNR) as a quality measure, which is defined by

$$\mathrm{PSNR} = 10 \log\left(\frac{MAX^2}{MSE}\right)$$

where $MAX := 255$ is the maximal pixel value and MSE is the mean-squared error defined by

$$MSE := \frac{1}{N}\sum_{i=1}^{N}(u_i - f_i)^2 \,.$$

PSNR is a mathematical construct to measure the quality of an image with respect to another image. Thus, for higher relative image quality, we require higher PSNR. (2 points)

Submission Instructions: This assignment sheet comprises the theoretical and programming parts.

- **Theoretical Part:** Write down your solutions clearly on a paper, scan them and convert them into a file named *theory(Name).pdf* where Name denotes the name of the student submitting on behalf of the group. Take good care of the ordering of the pages in this file. You are also welcome to submit the solutions prepared with LaTeX or some digital handwriting tool. You must write the full names of all the students in your group on the top of first page.
- **Programming part:** Submit your solution for the programming exercise with the filename `ex03_04_image_deblurring_solution_Name.py` where Name is the name of the student who submits the assignment on behalf of the group. You can only use python3.
- **Submission Folder:** Create a folder with the name *MatA_MatB_MatC* where MatA, MatB and MatC are the matriculation number (Matrikelnummer) of all the students in your group; depending on the number of people in the group. For example, if there are three students in a group with matriculation numbers 123456, 789012 and 345678 respectively, then the folder should be named: *123456_789012_345678*.
- **Submission:** Add all the relevant files to your submission folder and compress the folder into *123456_789012_345678.zip* file and upload it on the link provided on Moodle.
- **Deadline:** The submission deadline is 14.05.2023, 2:00 p.m. (always Tuesday 2 p.m.) via Moodle.