A study of exponential and Poisson distributions

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Abstract

The exponential distribution and Poisson distribution have traditionally been explained as follows: "The exponential distribution is the distribution of the time interval between the occurrence of rare events, and the Poisson distribution is the distribution of the number of occurrences of rare events". In addition, the Poisson distribution has been derived from the binomial distribution using a method based on the Poisson's law of small numbers. However, from a logical consideration of depression, it was found that it is not possible to say that "the distribution of the time interval between occurrences of rare events is an exponential distribution, and the distribution of the number of occurrences of rare events is a Poisson distribution" unless there is an approximation that the time during which events that rarely occur are occurring is not negligible. Therefore, we considered the exponential distribution and Poisson distribution while taking into account the time during which events are occurring. As a result, it was found that the distribution of the total time, including the time when the event is occurring and the time when it is not occurring, is an exponential distribution, and the distribution of the number of times the event occurs is a Poisson distribution. In addition, the equations that derive the exponential distribution and Poisson distribution were derived, and it was found that the Poisson distribution can be derived without using the assumption of the Poisson's law of small numbers, as in the past, and that the distribution of the number of times any event occurs follows the Poisson distribution. In addition, it was also found that the results obtained in this study encompass the conventional interpretation that "the distribution of the number of occurrences of events that rarely occur is a Poisson distribution, and the distribution of the intervals (i.e., the time when events are not occurring) between occurrences of events that rarely occur is an exponential distribution" if the time during which events are occurring is very short. Finally, we were able to give a stochastic differential equation from which the exponential distribution and Poisson distribution can be derived.

1. Introduction

The exponential distribution and Poisson distribution have traditionally been explained as follows: "The exponential distribution is the distribution of the time interval between the occurrence of rare events, and the Poisson distribution is the distribution of the number of occurrences of rare events" [Rozanov 1977]. In addition, the Poisson distribution has been derived from the binomial distribution using a method based on the Poisson's law of small numbers. However, from a logical consideration of depression, it was found that it is not possible to say that "the distribution of the time interval between occurrences of rare events is an exponential distribution, and the distribution of the number of occurrences of rare events is a Poisson distribution" unless there is an approximation that the time during which events that rarely occur are occurring is not negligible [Yabu 2025].

Therefore, we consider the exponential distribution and Poisson distribution while taking into account the time during which events are occurring

2. About the equation that derives the exponential distribution

2.1. The relationship between the number of times an event occurs and the number of times it does not occur within a certain period of time

Let $N(k, \Delta t)$ be the number of times an event occurs within a certain time period, i.e. let $N(k, \Delta t)$ be the number of times an event occurs k times within a certain time period Δt. In this case, $N(k, \Delta t)$ is given by the following equation.

$$N(k, \Delta t) = \frac{k}{\Delta t} \tag{1}$$

Now, let's consider the relationship between $N(k, \Delta t)$ and the time $T(k)$ when an event has not occurred. In general, it is said that the interval between the occurrence of rare events (i.e. the time when an event has not occurred) and the number of times a rare event occurs are exponential and Poisson distributions, respectively. In that case, the following equation should hold.

$$T(k) = \frac{1}{N(k, \Delta t)} \tag{2}$$

Let H(k) be the time at which an event occurs. If we give the time T(k) at which an event has not occurred as the average value, the following relationship holds. However, we used equation(1).

$$T(k) = \frac{\Delta t - k \cdot H(k)}{N(k, \Delta t) \cdot \Delta t}$$

$$= \frac{1}{N(k, \Delta t)} - \frac{k \cdot H(k)}{N(k, \Delta t) \cdot \Delta t}$$

$$= \frac{1}{N(k, \Delta t)} - \frac{k \cdot H(k)}{\frac{k}{\Delta t} \cdot \Delta t}$$

$$\therefore T(k) = \frac{1}{N(k, \Delta t)} - H(k) \tag{3}$$

In other words, in order for the relationship equation(2) to hold, it is necessary for $H(k) \cong 0$. In other words, the time $H(k)$ when the event occurs must be very short. However, in general, the time when the event occurs is not necessarily short. Therefore, equation(2) may not hold.

In other words, it is important to consider the time H(k) at which the event occurs.

Therefore, we will further transform equation(3).

$$T(k) + H(k) = \frac{1}{N(k, \Delta t)} \tag{4}$$

Equation(4) holds without approximation. In other words, the number of times an event occurs within a certain period of time, $N(k, \Delta t)$, and the sum of the time when an event is not occurring and the time when an event is occurring, $T(k) + H(k)$, are inversely proportional. In other words, the distribution of time consisting of a pair of times when an event is not occurring and a time when an event is occurring, and the number of times an event occurs, can be thought of as being related to the exponential distribution and Poisson distribution.

2.2. Equations for deriving the exponential distribution

We derive the exponential distribution from an equation. This is to avoid assuming, without logical explanation, that a pair of times, one when an event has not occurred and one when an event has occurred, can be expressed as an exponential distribution, or that a pair of times, one when an event has not occurred and one when an event has occurred, can be considered as an exponential distribution.

First, consider a pair of times: time when an event is not occurring and time when an event is occurring. The time that passes from the occurrence of the kth event to the occurrence of the next event is $T(k) + H(k)$. However, $T(0) = 0$ and $H(0) = 0$. Therefore, when the time that an event begins is t_k, the following equation holds.

$$T(k) + H(k) = \int_{t_k}^{t_k+t} ds \tag{5}$$

Differentiate both sides of equation(5) with respect to time t.

$$\frac{d}{dt}(T + H) = 1 \tag{6}$$

The probability density function $p(t)$ is multiplied on both sides, and a time integral is performed from time 0 to ∞.

$$\int_0^\infty \frac{d(T + H)}{dt} pdt = \int_0^\infty pdt \tag{7}$$

Partial integrate the left side. Here, $p(\infty) = 0$. Also, from equation(5), $T(k) + H(k) = 0$ when $t = 0$.

$$[(T(k) + H(k)) \cdot p(t)]_0^\infty - \int_0^\infty (T + H) \frac{dp}{dt} dt = \int_0^\infty p dt$$

$$\therefore -\int_0^\infty (T + H) \frac{dp}{dt} dt = \int_0^\infty p(t) dt \tag{8}$$

From Formula(8), the following equation holds.

$$-(T + H) \frac{dp}{dt} = p$$

$$\therefore \frac{dp}{dt} = -\frac{1}{(T + H)} p \tag{9}$$

Solving equation(9) gives the following equation. However, C is a real number that is an integration constant.

$$p(t) = Ce^{-\frac{1}{(T+H)}t} \tag{10}$$

Now, let's find C. $\frac{1}{T+H}$ means how much time is passing without an event occurring and how much time is passing with an event occurring per second when $T(k) + H(k)$ is a pair of time when an event is not occurring and time when an event is occurring. Therefore, $\frac{1}{T+H} t$ means the amount of time that has passed since the occurrence of an event at a certain time t. Therefore, p is the probability density function of a continuous probability distribution. Therefore, the following equation holds.

$$\int_0^\infty p(t) dt = \int_0^\infty Ce^{-\frac{1}{(T+H)}t} dt = 1$$

$$\therefore C \left[-(T(k) + H(k)) e^{-\frac{1}{(T(k)+H(k))}t} \right]_0^\infty = C(T(k) + H(k))(-e^{-\infty} + e^0) = C \cdot (T(k) + H(k)) = 1$$

$$\therefore C = \frac{1}{(T + H)} \tag{11}$$

Therefore, Equation(10) becomes as follows.

$$p(t) = \frac{1}{(T + H)} e^{-\frac{1}{(T+H)}t} \tag{12}$$

Therefore, from Equation(12), we can see that the distribution is an exponential distribution for pairs of times when an event has not occurred and when an event has occurred.

2.3. About the Poisson distribution (Poisson process)

The number of times an event occurs within a certain period of time is represented by

$N(k, \Delta t)$. However, $N(0, \Delta t) = 0$. At this time, the following relationship is established from equation(4) in section 2.1.

$$N(k, \Delta t) = \frac{1}{(T(k) + H(k))} \tag{13}$$

In other words, the reciprocal of the number of pairs of times when an event has not occurred and times when an event has occurred can be interpreted as the number of times an event occurs per a certain period of time. Therefore, equation(10) can also be interpreted as an expression for a discrete probability mass function. Therefore, equation(10) can be written in the following form using $N(k, \Delta t)$. However, $p(n, t) = p_n(t)$ can be considered to be a probability mass function that represents the distribution of the number of times an event occurs n times during a certain period of time t when the number of times a person becomes depressed per certain period of time is $N(k, \Delta t)$. In addition, C_n is a real number integral constant.

$$p_n(t) = C_n e^{-Nt} \tag{14}$$

Now, let's find C_n. N means the number of events that occur in a certain amount of time. Therefore, Nt means the number of events that occur in a certain amount of time t. Therefore, p_n is the probability mass function of the discrete probability distribution. Therefore, the following equation holds.

$$\Sigma_{n=0}^{\infty} p_n(t) = \Sigma_{n=0}^{\infty} C_n e^{-Nt} = 1$$

$$\therefore \Sigma_{n=0}^{\infty} C_n = e^{Nt} = \Sigma_{n=0}^{\infty} \frac{(Nt)^n}{n!} \tag{15}$$

Therefore, Equation(14) becomes as follows.

$$p_n(t) = \frac{(Nt)^n}{n!} e^{-Nt} \tag{16}$$

Therefore, from Equation(16), we can see that the number of times an event occurs within a certain period of time follows a Poisson process (or Poisson distribution if we consider Nt to be the number of times). In other words, from Equation(16) we can see that the distribution of the number of times an event occurs follows a Poisson distribution, not just for rare events.

3. The relationship between the exponential distribution and the Poisson distribution

From Chapter 2, we found that the distribution of pairs of times when events have not occurred and times when events have occurred is an exponential distribution, and that the number of times when events have occurred is a Poisson distribution. We were also able to derive the Poisson distribution by deriving the exponential distribution from an equation and changing the interpretation of the results of the same equation. In other words, we found that the Poisson distribution can be derived without assuming the Poisson's law of a small number and deriving it from the binomial distribution.

Of course, if the time $H(k)$ in which an event occurs is very short, then $T(k) + H(k) \cong T(k)$, so the traditional interpretation that "The exponential distribution is the distribution of the time interval between the occurrence of rare events (i.e. the time in which no events occur), and the Poisson distribution is the distribution of the number of occurrences of rare events" also holds true in this paper. In other words, it also encompasses the traditional way of thinking.

4. Derivation of the exponential distribution and Poisson distribution using stochastic differential equations

4.1. About the random variable k and its infinitesimal increment

In conventional explanations, the Poisson process is considered to be a stochastic process that jumps by $+1$. In the case of this paper, The random variable k related to time is shown below. However, $H(t)$ is the Heaviside step function given by Equation(18). In addition, τ_i is the time at which an event occurred.

$$k = \Sigma_{i=0}^{k} H(t - \tau_i) \ \ (0 \leq t, \tau_i \leq \Delta t) \tag{17}$$

$$H(t) = \begin{cases} 1 \ \ (t > 0) \\ 0 \ \ (t < 0) \end{cases} \tag{18}$$

Consider the infinitesimal increment dk on the left-hand side of Equation(17). dk is given by the following equation. Here, $\delta(t)$ is the Dirac delta function. We also used the fact that the derivative of the Heaviside step function is the Dirac delta function.

$$dk = \begin{cases} H(t - \tau_i) = \dfrac{d}{dt} H(t - \tau_i) dt = \left(\lim\limits_{\varepsilon \to 0} \displaystyle\int_{\tau_i - \varepsilon}^{\tau_i + \varepsilon} \delta(s - \tau_i) ds \right) dt = dt \ \ (t = \tau_i) \\ H(t - \tau_i) = \dfrac{d}{dt} H(t - \tau_i) dt = \left(\lim\limits_{\varepsilon \to o} \displaystyle\int_{t - \varepsilon}^{t + \varepsilon} \delta(s - \tau_i) ds \right) dt = 0 \ \ (t \neq \tau_i) \end{cases} \tag{19}$$

Also, considering the square of dk, the following equation holds true from equation(19).

$$(dk)^2 = \begin{cases} (dt)^2 \ \ (t = \tau_i) \\ \ \ 0 \ \ (t \neq \tau_i) \end{cases} \tag{20}$$

Therefore, dk and $(dk)^2$ can be considered as random variables related to time.

Therefore, we consider the expected value of dk. As shown in Equation(17), there are k values of τ_i. Therefore, from Equation(19), the sum of these values is k. Therefore, since the mean and expected value are the same, the expected value of dk is $\frac{k}{k} dt = dt$.

If we consider the expected value of $(dk)^2$ in the same way, it becomes $(dt)^2$. Therefore, if we use $E[\ \ ]$ as the symbol for the expected value, the following is true.

$$E[dk] = dt \tag{21}$$

$$E[(dk)^2] = (dt)^2 \tag{22}$$

Therefore, the variance of dk is given by the following formula. However, $V[\ \ ]$ is used as the variance symbol.

$$V[dk] = E[(dk)^2] - (E[dk])^2 = (dt)^2 - (dt)^2 = 0 \tag{23}$$

Therefore, if we are following the law of large numbers, the probability variable dk will have a probability of 1 and the following equation will be true.

$$dk = dt \tag{24}$$

In other words, the infinitesimal increment of k, dk, is equal to the infinitesimal increment of time, dt. Note that equation(24) is not a stochastic differential equation.

4.2. Derivation of the exponential distribution

We will consider the derivation of the exponential distribution and Poisson distribution using stochastic differential equations.

We assume that the following stochastic differential equation holds.

$$d\big(T(k) + H(k)\big) = -dt \tag{25}$$

Here, we will calculate the expected value of the random variable $T(k) + H(k)$.

$$E[T + H] = \Sigma_{k=0}^{\infty}\big(T(k) + H(k)\big)p \tag{26}$$

We can transform equation(26) as follows. However, $o(dt)$ is the term where the multiplier is greater than dt. We also used equations(24).

$$E[T + H] = T(0) + H(0) + E\left[\frac{d(T + H)}{dk}dk + o(dk)\right]$$

$$= T(0) + H(0) + E\left[\frac{d(T + H)}{dt}dt + o(dt)\right] \tag{27}$$

We will now perform a more specific equation transformation. Here, since the 0-th time does not exist according to the definition, $T(0) + H(0) = 0$. In addition, $o(dt)$ is ignored as a higher-order infinitesimal term.

$$E[T + H] = \Sigma_{k=0}^{\infty}\left(\frac{d(T + H)}{dt}dt \cdot p + o(dt) \cdot p\right)$$

$$= \Sigma_{k=0}^{\infty}\frac{d(T + H)}{dt}pdt$$

$$\therefore \frac{d}{dt}E[T + H] = \Sigma_{k=0}^{\infty}\frac{d(T + H)}{dt}p \tag{28}$$

Also, from equation(26), the following equation holds as well.

$$\frac{d}{dt}E[T + H] = \Sigma_{k=0}^{\infty}(T + H)\frac{dp}{dt} \tag{29}$$

From Equations(28) and (29), the following equation holds.

$$(T+H)\frac{dp}{dt}=\frac{d(T+H)}{dt}p \tag{30}$$

From here, from Equation(25), $\frac{d(T+H)}{dt}=-1$, the following equation holds.

$$\frac{dp}{dt}=-\frac{1}{(T+H)}\,p \tag{31}$$

Equation(31) is the same as Equation(9). Therefore, the exponential distribution and Poisson process (Poisson distribution) can be obtained in the same way as in Sections 2.2 and 2.3.

Therefore, we have been able to give equation(25) as a stochastic differential equation from which the exponential distribution and Poisson distribution can be derived.

4.3. Derivation of Poisson distribution

We will consider deriving the Poisson distribution directly using a stochastic differential equation.

We assume that the following stochastic differential equation holds.

$$d\big(kT(k)+kH(k)\big)=-kdt \tag{32}$$

Here, we will calculate the expected value of the random variable $kT(k)+kH(k)$.

$$E[kT+kH]=\Sigma_{k=0}^{\infty}\big(kT(k)+kH(k)\big)p \tag{33}$$

We can transform equation(33) as follows. However, $o(dt)$ is the term where the multiplier is greater than dt. We also used equations(24).

$$\begin{aligned} E[kT+kH] &= 0\cdot T(0)+0\cdot H(0)+E\left[\frac{d(kT+kH)}{dk}dk+o(dk)\right] \\ &= E\left[\frac{d(kT+kH)}{dt}dt+o(dt)\right] \end{aligned} \tag{34}$$

We will now perform a more specific transformation of the formula. Here, we will ignore $o(dt)$ as a higher-order infinitesimal term. We will also assume that $p(k,t) = p_k(t)$.

$$\begin{aligned} E[kT+kH] &= \Sigma_{k=0}^{\infty}\left(\frac{d(kT+kH)}{dt}dt\cdot p_k+o(dt)\cdot p_k\right) \\ &= \Sigma_{k=0}^{\infty}\frac{d(kT+kH)}{dt}p_k dt \end{aligned}$$

$$\therefore \frac{d}{dt}E[kT+kH]=\Sigma_{k=0}^{\infty}\frac{d(kT+kH)}{dt}p_k \tag{35}$$

Also, from equation(33), the following equation holds as well.

$$\frac{d}{dt}E[kT + kH] = \Sigma_{k=0}^{\infty}(kT + kH)\frac{dp_k}{dt} \tag{36}$$

From Equations(35) and (36), the following equation holds.

$$(kT + kH)\frac{dp_k}{dt} = \frac{d(kT + kH)}{dt}p_k \tag{37}$$

Here, since $\Delta t = kT(k) + kH(k)$ and $\frac{d(kT+kH)}{dt} = -1$ from equation(32), the following equation holds.

$$\frac{dp_k}{dt} = -\frac{k}{\Delta t}\, p_k \tag{38}$$

Using equation(1) here, the following equation holds.

$$\frac{dp_k}{dt} = -N(k, \Delta t)\, p_k \tag{39}$$

From Equation(39), $p_k(t)$ can be obtained as follows. However, C_k is an integral constant that is a real number.

$$p_k = C_k e^{-Nt} \tag{40}$$

Since k is a symbol, it can be replaced with n. In other words, the following equation is obtained.

$$p_n = C_n e^{-Nt} \tag{41}$$

If we transform the equation as shown in section 2.3, starting from equation(14), we can obtain the Poisson process (Poisson distribution).

Therefore, we have been able to give equation(32) as a stochastic differential equation that can be used to derive the Poisson distribution.

4.4. The relationship between the exponential distribution and the Poisson distribution

We will discuss the relationship between the exponential distribution and the Poisson distribution.

Equation(25), which is a stochastic differential equation that can be used to derive the exponential distribution, is multiplied by k to become equation(32), which is a stochastic differential equation that can be used to derive the Poisson distribution.

In other words, the distribution of the sum of the time when an event occurs and the time when it does not occur is the exponential distribution, and the distribution of the number of pairs of times when an event occurs and the time when it does not occur is the Poisson distribution. This basic relationship can also be seen from the relationship between the stochastic differential equations(25) and (32).

We have been discussing the relationship between the exponential distribution and the Poisson distribution in terms of time, but the relationship between the exponential distribution and the Poisson distribution also holds true for other values rather than time. For example, if we consider $T(k)$, $H(k)$ and Δt in equation(4) to be monetary values, and if there are k transactions in a certain period, the total transaction amount is Δm, the sales amount is $T(k)$ and the payment amount is $H(k)$, the following equation, which is similar to Equation(4), holds true.

$$T(k) + H(k) = \frac{1}{N(k, \Delta m)} \tag{42}$$

Therefore, based on the relationship between the exponential distribution and the Poisson distribution, it is possible to predict $T(k)$ and $H(k)$, and to decide whether to sell or buy now.

We can also think about this in terms of the stock market. Let k be the number of transactions, and Δm be the total amount of money spent on purchasing $(H(k))$ and selling $(T(k))$ shares. Assuming that few people would specify 1 yen or 1 cent as the amount spent on a single transaction, we can assume that $k < \Delta m$. Therefore, the following equation holds.

$$T + H = \frac{1}{N(k, \Delta m)} = \frac{\Delta m}{k} > 1 \tag{43}$$

In other words, if the number of transactions in the stock market is high, the amount of money spent on each purchase and sale is likely to increase. Therefore, we can expect the average stock price to increase as well. Therefore, if we assume that the number of transactions in the stock market is high and the amount of money spent on each transaction is not a small amount, such as 1 yen or 1 cent, the average stock price will increase, and we can say that index investment is expected to produce a profit.

5. Conclusion

We considered the exponential distribution and Poisson distribution, taking into account the time at which events occur. As a result, we found that the distribution of the time when events occur plus the time when they do not occur is an exponential distribution, and the distribution of the number of times events occur is a Poisson distribution. We also derived the equations that lead to the exponential and Poisson distributions, and found that we can derive the Poisson distribution without using the assumption of Poisson's law of small numbers, as in the past, and that the distribution of the number of times any event occurs follows the Poisson distribution. In addition, it was also found that the results obtained in this study encompass the conventional interpretation that “the distribution of the number of occurrences of events that rarely occur is a Poisson distribution, and the distribution of the intervals (i.e., the time when events are not occurring) between occurrences of events that rarely occur is an exponential distribution” if the time during which events are occurring is very short. Finally, we were able to give a stochastic differential equation from which the exponential distribution and Poisson distribution can be derived.

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