• Definition

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$

Example Relation Induced by a Partition

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Theorem

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

• Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

• Definition

Let m and n be integers and let d be a positive integer. We say that ***m* is congruent to *n* modulo *d*** and write

$$\boldsymbol{m \equiv n \ (\mathrm{mod}\ d)}$$

if, and only if, $$d \mid (m - n).$$

Symbolically: $$m \equiv n \ (\mathrm{mod}\ d) \quad \Leftrightarrow \quad d \mid (m - n)$$

Example Evaluating Congruences

Determine which of the following congruences are true and which are false.

a. $12 \equiv 7 \ (\mathrm{mod}\ 5)$ b. $6 \equiv -8 \ (\mathrm{mod}\ 4)$ c. $3 \equiv 3 \ (\mathrm{mod}\ 7)$

Example An Equivalence Relation on a Set of Subsets

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation **R** on X as follows: For all A and B in X,

$$A \mathbf{R} B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

Prove that **R** is an equivalence relation on X.

• Definition

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of *a*,** denoted **[*a*]** and called the **class of *a*** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

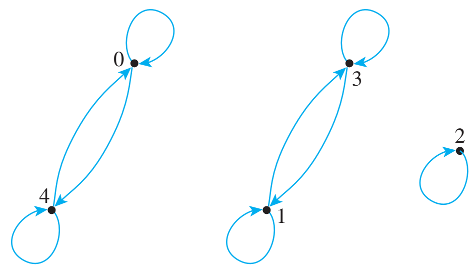
$$[a] = \{x \in A \mid x \; R \; a\}$$

Example Equivalence Classes of a Relation Given as a Set of Ordered Pairs

Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

$$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}.$$

The directed graph for R is as shown below. As can be seen by inspection, R is an equivalence relation on A. Find the distinct equivalence classes of R.



Example Equivalence Classes of the Identity Relation

Let A be any set and define a relation R on A as follows: For all x and y in A,

$$\boxed{x\ R\ y \quad \Leftrightarrow \quad x = y.}$$

Then R is an equivalence relation. *[To prove this, just generalize the argument used in Example 8.2.2.]* Describe the distinct equivalence classes of R.

Lemma

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a \ R \ b$, then $[a] = [b]$.

Lemma

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

Theorem The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.