higher education & training
Department:
Higher Education and Training
REPUBLIC OF SOUTH AFRICA

MATHEMATICS **N6 SYLLABUS**

COORDINATOR: ENGINEERING STUDIES **NATIONAL CERTIFICATE**

CODE NUMBER

1603164

DATE OF IMPLEMENTATION DATE **January 2023**

Contents

1. SUBJECT AIMS FOR MATHEMATICS N6

1.1 GENERAL SUBJECT AIMS

Mathematics N6 aims to provide learners with the skills to identify, and calculate mathematical problems in N6 and the content form part of engineering calculation problems from industry.

Furthermore, Mathematics N6 will equip students with relevant knowledge to enable them to integrate meaningfully into their trade subjects and also serve as the foundation for mathematics N6 syllabus in order to achieve a National diploma.

Upon completion of this subject, the student should be able to apply

1.1.1 the necessary knowledge of Mathematics to various engineering fields in their respective working environments;

1.1.2 higher cognitive skills pertaining to application, analysis, synthesis and evaluation, logical and critical thought processes;

1.1.3 their understanding in the interpretation of real world problems;

1.1.4 and, promote Mathematics as a tool to be used to trouble shoot in different fields of study.

1.1.5 certain theorems that are not examinable to be calculated.

1.2 SPECIFIC SUBJECT AIMS

1.2.1 The specific aims of Mathematics N6 is to continue and apply Differential and Integral Calculus and serve as a prerequisite for Mathematics N6.

1.2.2 Mathematics N6 strives to assist students to obtain trade-specific calculation knowledge.

1.2.3 Other specific aims of Mathematics N6 also include:

1.2.3.1 Promote correct mathematical terminology;

1.2.3.2 Promote and focus on word problems and the problem solving thereof, in order to prepare the students for their relevant careers

1.2.3.3 Use technology in Mathematics and apply Mathematics to further technology

2. ADMISSION REQUIREMENT

For admission to N6 Mathematics, a student must have passed N5 Mathematics

3. DURATION OF COURSE

The duration of the subject is one trimester on full time, part time or distance learning mode.

4. EVALUATION

Candidates must be evaluated continually as follows:

4.1 **ICASS Trimester Mark**

4.1.1 Assessment marks are valid for a period of one year and are referred to as ICASS Trimester marks.

4.1.2 A minimum of 40% is required for a student to qualify for entry to the final examination.

4.1.3 **Two** formal class tests for full time and part time students (or **Two** assignments for distance learning students only)

4.2 Calculation of trimester mark will be as follows:

4.2.1 Weight of test or assignment 1 = 30% of the syllabus

4.2.2 Weight of test or assignment 2 = 70% of the syllabus

5. EXAMINATION

5.1 A final examination will be conducted in April, August and November of each year. The pass requirement is 40%.

5.2 The final examination shall consist of 100 % of the syllabus

5.3 The duration of the final examination shall be 3 hours

5.4 The final examination will be a closed book examination

5.5 Minimum pass percentage shall be 40%

5.6 Assessments shall be based on the cognitive domain of Bloom's Taxonomy, that is remember, understand, apply, analyse, evaluate, and create The division of these aspects are as follows;

Remember	Understand	Apply	Analyse	Evaluate	Create
20	20	20	10	20	10

6. GENERAL INFORMATION

6.1 Problems should be based on real world scenarios allowing students to relate theory to practice.

6.2 Emphasis of correct mathematical terminology should be encouraged and promoted at all times.

6.3 A systematic approach to problem solving should be adhered to.

6.4 Students should be encouraged to understand rather than memorise the basic formulae applicable to N6 Mathematics.

6.5 Calculators may be used to do mathematical calculations.

6.6 Answers to all calculations must be approximated correctly to three decimal places, unless otherwise stated. Unless otherwise stated, approximations may not be done during calculations. The final answer must be approximated to the stipulated degree of accuracy.

6.7 The weight value of a module gives an indication of the time to be spent on teaching the module as well as the relative percentage of the total marks allocated to the module in the final exam examination (1 mark = 1.8 minutes).

6.8 **LEARNING CONTENT** are given at the end of each module. These guidelines provide relevant **Examples**, appropriate procedures and other pertinent information and may not be deviated from.

7. SUBJECT MATTER

Mathematics N6 strives to assist students to obtain trade-specific calculation knowledge. Students should be able to acquire in-depth knowledge of the following content:

Module	Topic	Weight Value

1.	Differentiation	(6)
2.	Integration Techniques	(18)
3.	Partial Fractions	(12)
4.	Differential Equations	(12)
5.	Area And Volumes	(15)
6.	Centroids and Centre of Gravity	(10)
7.	Second Moment Of Area, Moment Of Inertia And Centre Of Fluid Pressure	(15)
8.	Combinations of Differentiation and Integration	(12)
	TOTAL:	**[100]**

8. DETAILED SYLLABUS

8.1 MODULE 1: DIFFERENTIATION

LEARNING OUTCOMES

On completion of this module, the student should be able to apply differentiation to:

8.1.1 **first and second order partial** derivatives by

8.1.1.1 Partially differentiating a function consisting of two (or more) variables with respect to one variable only

8.1.1.2 Using successive differentiation to obtain the second derivatives of a function consisting of two variables

8.1.1.3 Calculating specific values of the first and second order partial derivative(s) at specified coordinates

8.1.2 **practical (real – life) problems** by analysing, recreating and applying partial differentiation then interpreting the results

8.1.3 **First and second order parametric equations** by

8.1.1.1 differentiating two functions consisting of the same variable (parameter)

8.1.1.2 Using successive differentiation to obtain the second derivative of two functions consisting of the same variable (parameter)

8.1.1.3 Calculating specific values of the derivative(s) at specified coordinates

LEARNING CONTENT

8.1.1.1 **First order partial derivatives**

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

8.1.1.2 **Second order partial derivatives**

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example

Find the second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

8.1.3 First and second order parametric equations

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane.

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

As we know it is also useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example

Given the following parametric equations $x = t^2$, $y = t^3 - 3t$.

Determine:

a. $\frac{dy}{dx}$

b. $\frac{d^2y}{dx^2}$

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$

b. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$

8.2 MODULE 2: INTEGRATION TECHNIQUES

LEARNING OUTCOMES

On completion of this module, the student should be able to:

8.2.1 **Integrate using Integration by Parts**

Functions of the form $\int f(x)g'(x)\,dx = f(x).g(x) - \int g(x)f'(x)\,dx$, where $f(x)$ and $g(x)$ are not derivatives of each other.

8.2.2 **Integrate Trigonometric functions**

Apply specific integration techniques to the following functions ($m, n \leq 5$ and some constant a)

8.2.2.1 $\sin^m ax$ and $\cos^n ax$

8.2.2.2 $\tan^m ax$ and $\cot^n ax$

8.2.2.3 $\sin^m ax.\cos^n ax$

8.2.3 **Integration by means of completing the square** applied to the following functions

8.2.3.1 $\dfrac{1}{\sqrt{ax^2+bx+c}}$

8.2.3.2 $\dfrac{1}{ax^2+bx+c}$

8.2.3.3 $\dfrac{1}{c+bx-ax^2}$

8.2.3.4 $\dfrac{1}{\sqrt{c+bx-ax^2}}$

LEARNING CONTENT

8.2.1 Integration by Parts Examples

Integrate $\int \sin .xe\, dx^{2x}$

8.2.2 Trigonometric functions

Examples

$\int f'(x)\cdot\sin[f(x)]dx$ [The same for the other trigonometric functions]

$\int \sin^n xdx$, $\int \cos^n xdx$, $\int \tan^n xdx$, and $\int \cot^n xdx$ where $n = 1, 2, 3$

$\int \sin^m x\cdot\cos^n xdx$
where m and n are positive, m and/or n are odd and m and $n < 3$

$\int \sin(ax)dx$, $\int \cos(ax)dx$, $\int \tan(ax)dx$, and $\int \cot(ax)\ dx$

$\int \sin^2(ax)dx$, $\int \cos^2(ax)dx$, $\int \tan^2(ax)dx$, and $\int \cot^2(ax)\ dx$ by using the following

identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$\cot^2 x = \operatorname{cosec}^2 x - 1$ $\tan^2 x$

$= \sec^2 x - 1$

$\int \sin(ax)\cdot\cos(bx)dx$ by transforming:

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$ $\cos$

$A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$ $\sin A$

$\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

8.2.3 **Integration by means of completing the square** If

given $bx = a\sin\theta$, then:

$$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + c$$

If given $bx = a\tan\theta$, then:

$$\int \frac{dx}{b^2x^2 + a^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$$

If given $bx = a\sin^{-1}\theta$, then:

$$\int \sqrt{a^2 - b^2x^2}\,dx = \frac{a^2}{2b}\sin^{-1}\left(\frac{bx}{a}\right) + \frac{x}{2}\sqrt{a^2 - b^2x^2} + c$$

The following needs to be reduced to any of the above forms by first completing the square:

$$\frac{1}{\sqrt{ax^2+bx+c}}$$

$$\frac{1}{ax^2+bx+c}$$

$$\frac{1}{c+bx-ax^2}$$

$$\frac{1}{\sqrt{c+bx-ax^2}}$$

Examples

Integrate $\int \frac{1}{\sqrt{2+3x-x^2}}\,dx$

$$2+3x-x^2 = -x^2+3x+2$$

$$=-\left(x^2-3x-2\right)$$

$$=-\left[\left(x^2-3x+\left(\frac{3}{2}\right)^2\right)-2-\frac{9}{4}\right]$$

$$=\frac{17}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\int \frac{1}{\sqrt{2+3x-x^2}}\,dx = \int \frac{1}{\sqrt{\frac{17}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx = \frac{2}{\sqrt{17}}.\arcsin\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

.3 MODULE 3: PARTIAL FRACTIONS

LEARNING OUTCOMES

On completion of this module, the student should be able to use their previous knowledge of partial fractions and apply it to:

8.3.1 **Single Recursive Factor**

Fractions where the denominator has a single recursive factor

8.3.2 **Two Recursive Factors**

Fractions where the denominator has two recursive factors

8.3.3 **Trinomial Factor and Recursive Factors**

Fractions where the denominator has a trinomial factor and recursive factors

8.3.4 **Improper Rational Factors**

Fractions where the denominator has a higher degree polynomial and has to be reduced using long division to:

a. Fractions where the denominator has two recursive factors
b. Fractions where the denominator has a trinomial factor and recursive factors

LEARNING CONTENT

8.3.1 **Single Recursive Factor**

$\int \frac{f(x)\,dx}{(ax \pm b)^n}$ where $n \leq 3$; $a\ b$, are integers

Examples

$$\int \frac{2\,dx}{(x+3)^3} = \int \frac{A\,dx}{(x+3)^3} + \int \frac{B\,dx}{(x+3)^2} + \int \frac{C\,dx}{x+3}$$

8

8.3.2 Two Recursive Factors

$$\int \frac{f(x)\,dx}{(ax\pm b)^m (cx\pm d)^n} \text{ where } m, n \le 3;\ a, b, c, d \text{ are integers}$$

Example

$$\int \frac{5x\,dx}{(x-1)^2(2x-5)^2} = \int \frac{A\,dx}{(x-1)^2} + \int \frac{B\,dx}{x-1} + \int \frac{C\,dx}{(2x-5)^2} + \int \frac{D\,dx}{2x-5}$$

8.3.3 Trinomial Factor and Recursive Factors

$$\int \frac{f(x)\,dx}{(ax^2 \pm bx + c)(dx\pm e)^n} \text{ where } n \le 3;\ a, b, c, d, e \text{ are integers}$$

Example

$$\int \frac{f(x)\,dx}{(x^2+1)(2x-3)^2} = \int \frac{Ax+B}{x^2+1}\,dx + \int \frac{C}{(2x-3)^2}\,dx + \int \frac{D}{2x-3}\,dx$$

8.3.4 Improper Rational Factors

Example

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

.4 MODULE 4: DIFFERENTIAL EQUATIONS

LEARNING OUTCOMES

On completion of this topic the student should be able to between determine

8.4.1 **First Order Linear Differential Equations** 8.4.1.1

By first writing it in standard form

$\frac{dy}{dx} + P(x) = Q(x)$ where P and Q are continuous functions.

8.4.1.2 And then calculating the integrating factor $I(x) = \int e^{P(x)} dx$

8.4.1.3 to solve the equation $y^{\int e^{P(x)dx}} = \int Q^{\int e^{P(x)dx}} dx$

8.4.2 **Second Order Differential Equations**

8.4.2.1 By first writing it in standard form

$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R(x)$ where a and b are real numbers.

8.4.2.2 determine the complimentary function

$m^2 + am + b = 0$

8.4.2.3 determine the particular function

$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R(x)$ where

A, for some constant A

$mx + c$, (linear)

8

$R(x) = ax^2 + bx + c$, (parabola)

Ae^{ax} or Axe^{ax} when $m = a$, (exponential)

LEARNING CONTENT

8.4.1 **First Order Linear Differential Equations**

Example

Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$.

The given equation is linear since it has the form
$P(x) = 3x^2$ and $Q(x) = 6x^2$. An integrating factor is

$$I(x) = e^{\int 3x^2 dx} = e^{x^3}$$

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

or

$$\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3}$$

Integrating both sides, we have

$$e^{x^3}y = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

8.4.2 **Second Order Differential Equations Example**

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10x^2$$

.5 MODULE 5: AREA AND VOLUMES

LEARNING OUTCOMES

On completion of this module, the student should be able to:

i. Sketch a function on a given interval

ii. Calculate the Areas and Volumes of a given function using a definite integral. iii. Calculate the points of intersection of Areas and Volumes under two functions. iv. Sketch the points of intersection of Areas and Volumes under two functions

v. Calculate the Areas and Volumes of two given functions using a definite integral.

with respect to:

8.5.1 **Areas**

8.5.1.1 Calculate the area between a curve and one of the reference axis using

$$A_x = \int_a^b y\,dx \qquad\qquad A_y = \int_c^d x\,dy$$

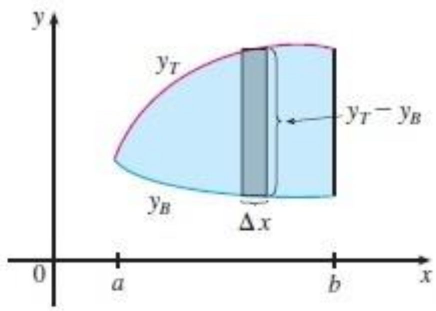
Where $a \leq x \leq b$ — where $c \leq y \leq d$

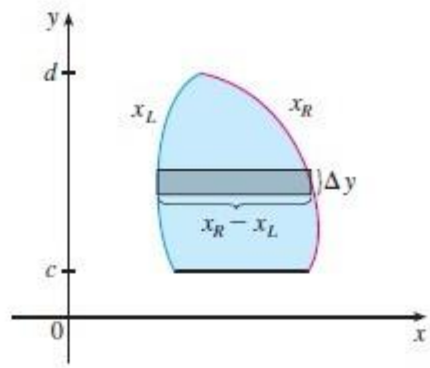
8.5.1.2 Calculate the area between two curves using

$$A_x = \int_a^b \left(y_T - y_B\right)dx \qquad\qquad A_y = \int_c^d \left(x_R - x_L\right)dy$$

Where a and b are the x coordinates of the intersections between the two curves

Where c and d are the y coordinates of the intersections between the two curves





8.5.2 Volumes

8.5.2.1 Disk Method

a. Calculate the volume between a curve and one of the reference axis.

$$V_x = \pi \int_a^b y^2\,dx \qquad V_y = \pi \int_c^d x^2\,dy$$

Where $a \leq x \leq b$ Where $c \leq y \leq d$

b. Calculate the volume between two curves

$$V_x = \pi \int_a^b \left(y_B^2 - y_T^2\right) dx \qquad V_y = \pi \int_c^d \left(x_R^2 - x_L^2\right) dy$$

Where a and b are the x coordinates of the intersections between the two curves

Where c and d are the y coordinates of the intersections between the two curves

8.5.2.2 Shell Method

c. Calculate the volume between a curve and one of the reference axis.

$$V_x = 2\pi \int_a^b xy\,dx \qquad V_y = 2\pi \int_c^d yx\,dy$$

Where $a \le x \le b$ | Where $c \le y \le d$

d. Calculate the volume between two curves

$$V_x = 2\pi\int_a^b x\,(y_T - y_B)\,dx \qquad V_y = 2\pi\int_c^d y\,(x_R - x_L)\,dy$$

Where a and b are the x coordinates of the intersections between the two curves

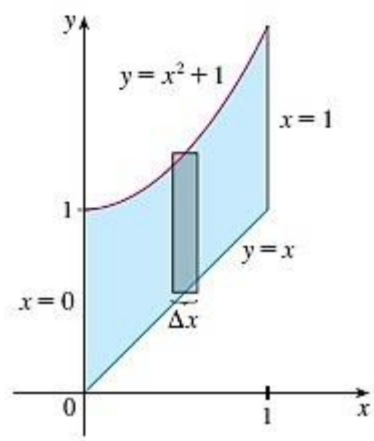
Where c and d are the y coordinates of the intersections between the two curves

LEARNING CONTENT

8.5.1 **Areas Example**

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$



The region is shown . The upper boundary curve is $y = x^2 + 1$ and the lower boundary curve is $y = x$. So we use the area formula (2) with $f(x) = x^2 + 1$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 [(x^2 + 1) - x]\,dx = \int_0^1 (x^2 - x + 1)\,dx$$

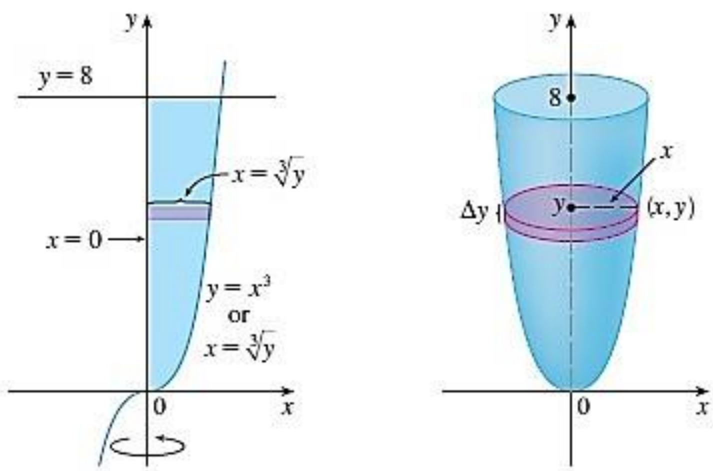
$$= \frac{x^3}{3} - \frac{x^2}{2} + x\Bigg]_0^1 = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$$

8.5.2 **Volumes**

8.5.2.1 **Disk Method**

Example

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



(a) (b)

The region is shown and the resulting solid is shown .

Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis (obtaining circular cross-sections) and therefore to integrate with respect to y. If we slice at height y, we get a circular disk with radius x, where $x = \sqrt[3]{y}$. So the area of a cross-section through y is

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

and the volume of the approximating cylinder pictured

$$A(y)\,\Delta y = \pi y^{2/3}\,\Delta y$$

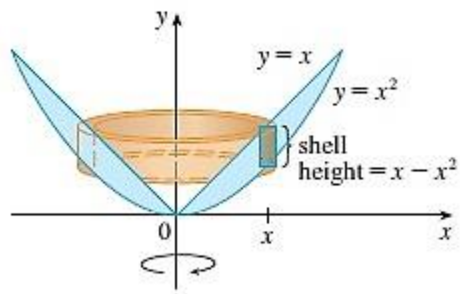
Since the solid lies between $y = 0$ and $y = 8$, its volume is

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\left[\tfrac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

8.5.2.2 **Shell Method**

Example

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



The region and a typical shell are shown . We see that the shell has radius x, circumference $2\pi x$, and height $x - x^2$. So the volume is

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

8.6 MODULE 6: CENTROIDS AND CENTRE OF GRAVITY

LEARNING OUTCOMES

On completion of this module, the student should be able to:

Sketch a function on a given interval

Calculate the Areas and Volumes of a given function using a definite integral.

Calculate the points of intersection of Areas and Volumes under two functions.

Sketch the points of intersection of Areas and Volumes under two functions Calculate the Areas and Volumes of two given functions using a definite integral.

8.6.1 **Centroids**

Calculate the distance from any of the reference axes to the centroid of the area between a given curve and an axis, or between two given curves.

8.6.2 **Centre of Gravity**

Calculate the distance from a reference axis to the centre of gravity of a solid of revolution generated when the area between two given curves or a given curve and an axis is rotated about a reference axis.

LEARNING CONTENT

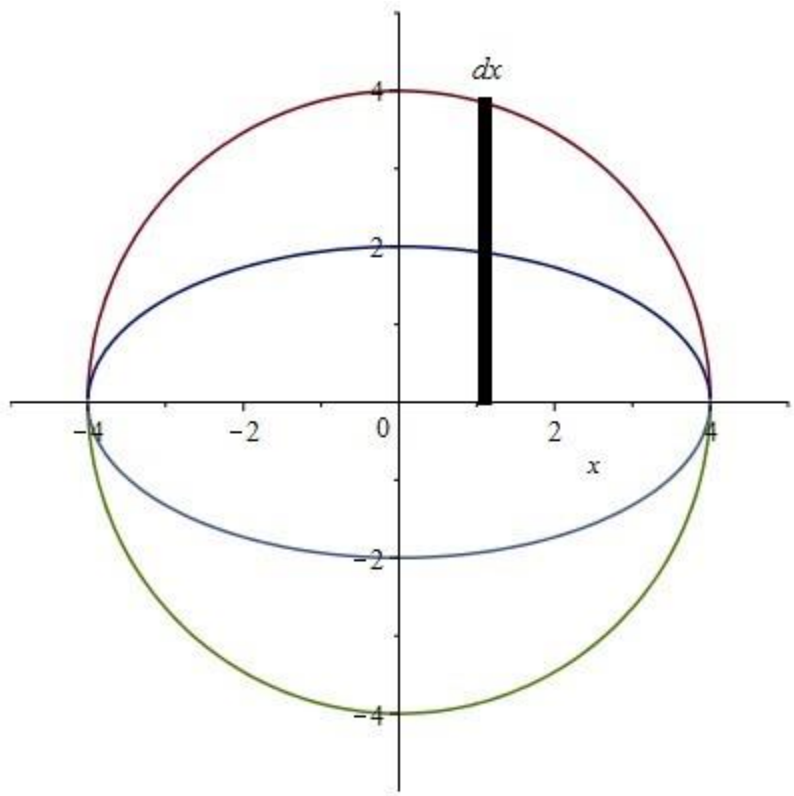
8.6.1 Centroids

Example

Sketch the graphs of $x^2 + 4y^2 = 16$ and $x^2 + y^2 = 16$.
Show the area bounded by the graphs in the first quadrant.
Show a representative strip perpendicular to the x-axis.

Calculate the area

Calculate the distance of the centroid from the y-axis of the area



b

$Area = \int \; y_1 - y \; dx_2$

a

$$= \int_0^4 \sqrt{16-x^2} - \frac{1}{2}\sqrt{16-x^2}\,dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{16-x^2}\,dx$$

$$= \frac{1}{2}\left[\frac{16}{2}\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{16-x^2}\right]_0^4$$

$$= \frac{1}{2}\left[\left(8\sin^{-1}\frac{4}{4} + \frac{4}{2}\sqrt{16-4^2}\right) - \left(8\sin^{-1}\frac{0}{4} + \frac{0}{2}\sqrt{16-0^2}\right)\right]$$

$$= \frac{1}{2}\left[8\sin^{-1}1\right] = 6{,}283 \text{ or } 2\pi \text{ units}^2$$

$$\bar{x} = \frac{A m_y}{A}$$

$$A m_y = \int_a^b r\,dA$$

$$= \int_0^4 x\left[\sqrt{16 - x^2} - \frac{1}{2}\sqrt{16 - x^2}\right]dx$$

$$= \frac{1}{2}\int_0^4 x\sqrt{16 - x^2}\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right)\int_0^4 -2x\,(16 - x^2)^{\frac{1}{2}}\,dx$$

$$= -\frac{1}{4}\left[\frac{(16 - x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= -\frac{1}{6}\left[(16 - x^2)^{\frac{3}{2}}\right]_0^4$$

$$= -\frac{1}{6}\left[(16 - 4^2)^{\frac{3}{2}} - (16 - 0^2)^{\frac{3}{2}}\right]$$

$$= \frac{32}{3}\ units^2 \text{ or } 10{,}667\,units^2$$

$$\bar{x} = \frac{32/3}{6{,}283} = 1{,}698\ units$$

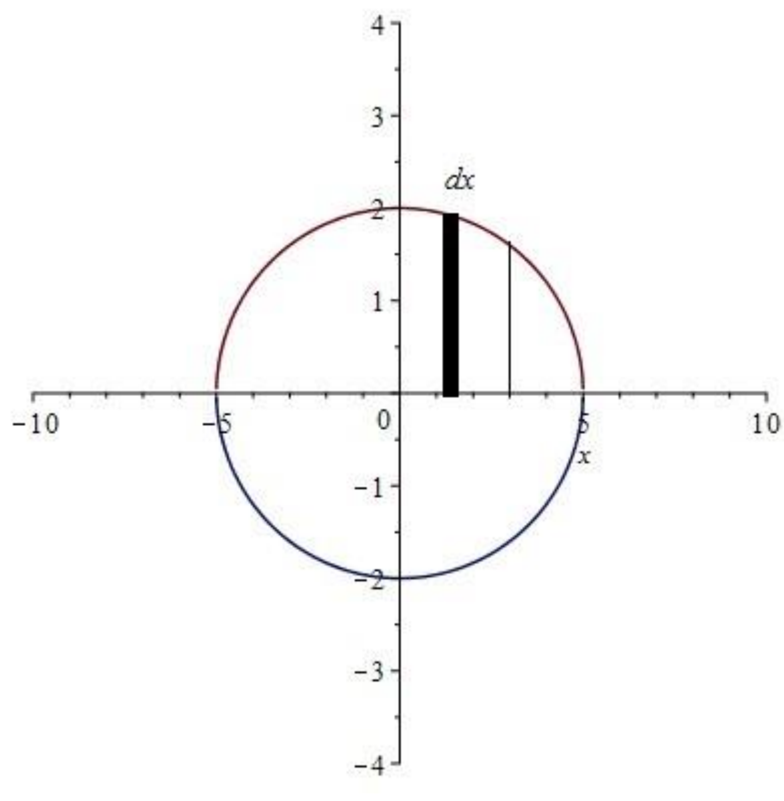
Centre of Gravity

Example

Sketch the graph of $\frac{x^2}{25}+\frac{y^2}{4}=1$.

Show the area in the first quadrant, bounded by the graph, and the lines $x=0, x=3$ and $y=0$. Show the representative strip/element that you will use to calculate the volume generated when the area rotates about the x-axis.

Calculate the x-coordinate of the centre of gravity of the solid obtained.



$$\bar{x} = \frac{V_{m-y}}{V_x}$$

$$V_{m-y} = \int_a^b r dV \surd = \int_a^b x \pi y^2 dx$$

$$= \pi \int_0^3 x \frac{4}{25}(25 - x^2) dx$$

$$= \frac{4}{25} \pi \int_0^3 (25x - x^3) dx$$

$$= \frac{4}{25} \pi \left[25\frac{x^2}{2} - \frac{x^4}{4} \right]_0^3$$

$$= \frac{4}{25} \pi \left[25\frac{3^2}{2} - \frac{3^4}{4} - 0 \right]_0^3 = 14{,}76\pi \quad or\, 46{,}370 units^4$$

$$\bar{x} = \frac{14{,}76\pi}{10{,}56\pi} \quad or \quad \frac{46{,}370}{33{,}175} = 1{,}398 units$$

8.7 MODULE 7: SECOND MOMENT OF AREA, MOMENT OF INERTIA AND CENTRE OF FLUID PRESSURE

LEARNING OUTCOMES

On completion of this module, the student should be able to:

Sketch a function on a given interval

Calculate the Areas and Volumes of a given function using a definite integral.

Calculate the points of intersection of Areas and Volumes under two functions.

Sketch the points of intersection of Areas and Volumes under two functions Calculate the Areas and Volumes of two given functions using a definite integral.

8.7.1 **Second Moment Of Area**

Calculate the second moment of area of an area enclosed between two given curves, or a given curve and an axis, with respect to a reference axis.

8.7.2 **Moments Of Inertia**

Calculate the moment of inertia of a solid of revolution generated when the area between two given curves or a given curve and an axis is rotated about an axis. If the mass of the solid is not given, the answer must be given in terms of the mass *m*.

8.7.3 **Centre of Fluid Pressure**

Calculate the depth of the centre of fluid pressure on a vertical plane submerged *in* the fluid with respect to the surface of the fluid.

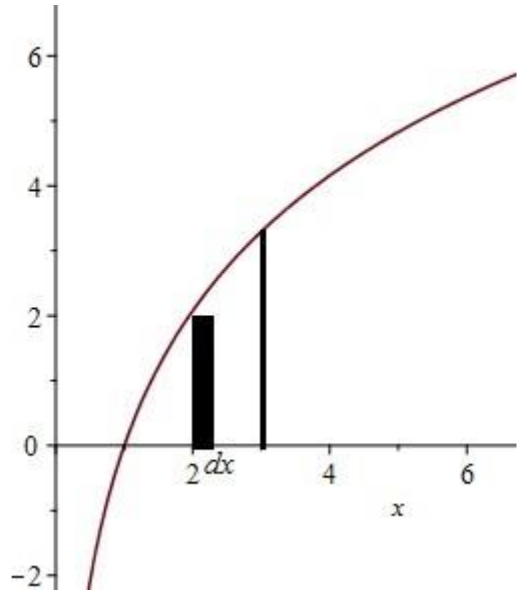
LEARNING CONTENT

8.7.1 **Second Moment Of Area**

Example

Sketch the graph of $y = 3\ln x$. Show the area bounded by the graph, the x-axis and the line $x = 3$. Use a representative strip/element perpendicular to the x-axis.

Calculate the second moment of area about the y-axis



$$I_y = \int_a^b r^2\,dA$$

$$= \int_a^b x^2(y_1 - y_2)\,dx \quad or \quad \int_a^b x^2 y\,dx$$

$$= \int_1^3 x^2 3\ln x\,dx$$

$$= 3\left[\ln x\frac{x^3}{3} - \int\frac{1}{x}\frac{x^3}{3}dx\right]_1^3$$

$$= 3\left[\ln x\frac{x^3}{3} - \frac{1}{3}\int x^2 dx\right]_1^3$$

$$= 3\left[\ln x\frac{x^3}{3} - \frac{1}{3}\frac{x^3}{3}\right]_1^3$$

$$= 3\left[\ln 3\frac{3^3}{3} - \frac{1}{3}\frac{3^3}{3} - \left(\ln 1\frac{1^3}{3} - \frac{1}{3}\frac{1^3}{3}\right)\right]$$

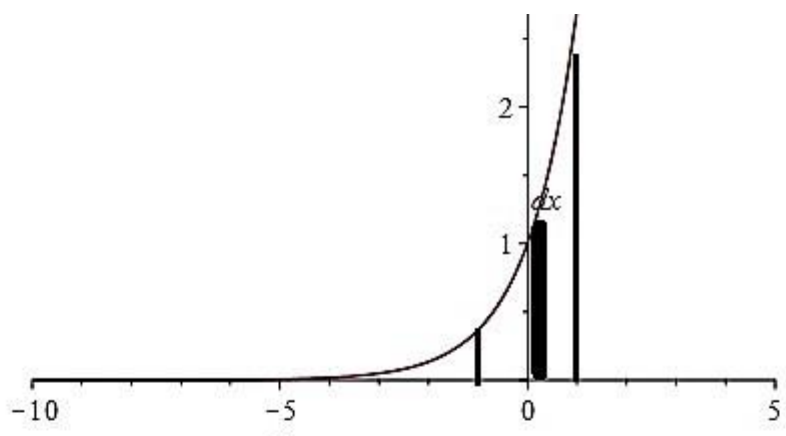
$$= 20{,}996 units^4$$

8.7.2 **Moments Of Inertia**

Example

Make a neat sketch of the curve $y = e^x$ and show the area bounded by the curve, the x-axis and the lines $x = -1$ and $x = 1$.
Show the representative strip that you will use to calculate the volume when this area rotates about the x-axis.

Calculate the moment of inertia about the x-axis of the solid obtained when the area described rotates about the x-axis.
Express the answer in terms of mass.



$$
\begin{aligned}
I_x &= \frac{1}{2}\pi\rho\int_a^b y^4\,dx \\
&= \frac{1}{2}\pi\rho\int_{-1}^{1}(e^x)^4\,dx \\
&= \frac{1}{2}\pi\rho\left[\frac{e^{4x}}{4}\right]_{-1}^{1} \\
&= \frac{1}{2}\pi\rho\left[\frac{e^4}{4}-\frac{e^{-4}}{4}\right] \quad or \; \frac{1}{8}\pi\rho\left(e^4-e^{-4}\right) \\
&= 6{,}823\pi\rho \quad or \; 21{,}434\rho \\
&= \frac{6{,}823\pi m}{3{,}627\pi} = \frac{21{,}434m}{11{,}394} \\
&= 1{,}881m
\end{aligned}
$$

8.8 MODULE 8: COMBINATIONS OF DIFFERENTIATION AND INTEGRATION

LEARNING OUTCOMES

On completion of this module, the student should be able to calculate:

8.8.1 **Lengths of Curves**

using the arc length of a given curve between two given points followed by applying differentiation and integration formulae as indicated below.

a. $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ when $y = f(x)$

b. $S = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,dx$ for paramedic equations

8.8.2 **Surfaces of Revolution**

of the surface area generated when the arc of a curve, between two points, revolves through a full revolution about an *axis,* using:

c. $$A=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \text{ when } y=f(x)$$

d. $$A=\int_a^b 2\pi y\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,dx$$ for paramedic equations

LEARNING CONTENT

8.8.1 **Lengths of Curves**

Example

Calculate the length of the curve given by:

$x=e^{\theta}\sin\theta$ *and* $y=e^{\theta}\cos\theta$ from $\theta=0$ to $\theta=\frac{\pi}{3}$

8.8.2 **Surfaces of Revolution**

Example

Calculate the surface area generated when the curve $y=\frac{3}{2}x$ between the points (2;3) and (4;6) rotates about the x-axis.