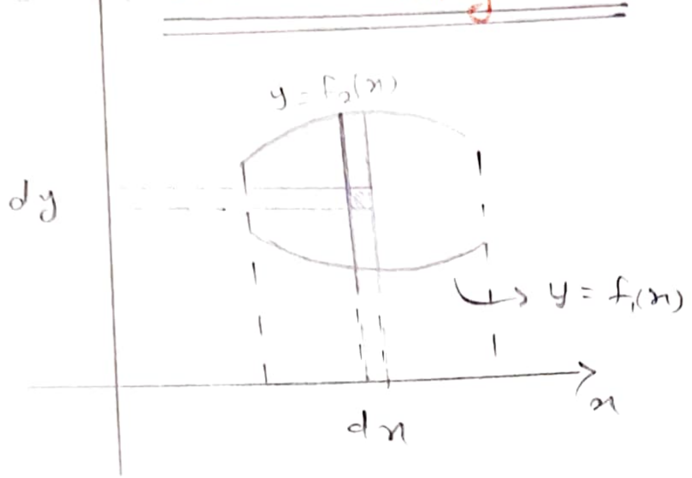
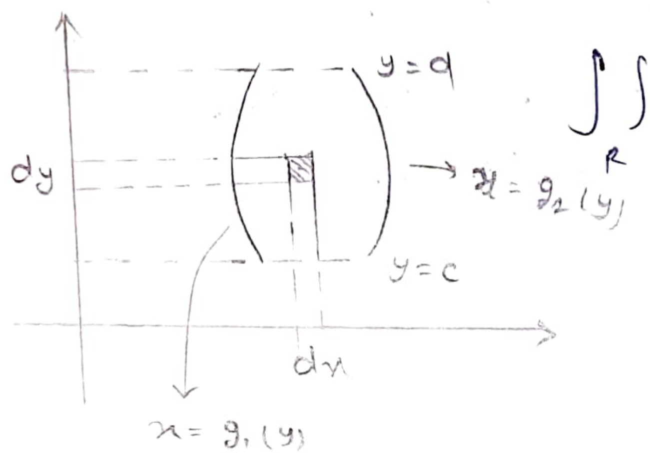
Unit - 5 Multivariable Calculus

Double Integration :-



$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$$



$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dy\,dx$$

Q Evaluate $\displaystyle\int_0^b\int_0^x xy\,dx\,dy$ (RTU - 2024)

Soln

$$= \int_0^b\int_0^x xy\,dx\,dy$$

$$= \int_0^b \left[\frac{x^2}{2}\right]_0^x y\,dy$$

$$= \int_0^b \frac{x^2}{2} y\,dy \Rightarrow \frac{x^2}{2}\left[\frac{y^2}{2}\right]_0^b \Rightarrow \frac{x^2b^2}{4} \text{ Ans}$$

$$\Rightarrow \int_0^b x\left[\frac{y^2}{2}\right]_0^x dx$$

$$\Rightarrow \int_0^b \frac{x^3}{2}\,dx$$

$$\Rightarrow \left[\frac{x^4}{8}\right]_0^b = \frac{b^4}{8} \text{ Ans}$$

Q Evaluate $\int_0^1 \int_0^2 dx\, dy$. (RTU 2019)

Sol^n $\Rightarrow \int_0^1 \int_0^2 dx\, dy$

$\Rightarrow \int_0^1 [x]_0^2\, dy$

$= \left[[xy]_0^2 \right]_0^1$

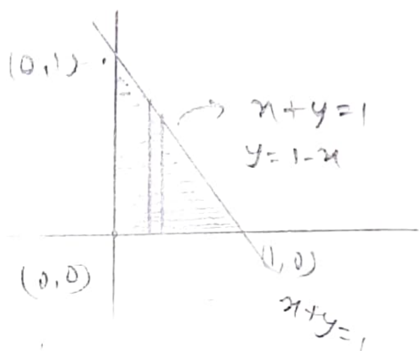
$\Rightarrow 2$

Q Evaluate $\iint xy\, dx\, dy$ where the region of the integration is $x+y<1$ in (+)ive Quadrant

(RTU-2023)

Sol^n

$f(x,y) = \int_0^1 \int_0^{1-x} xy\, dx\, dy$



$f(x,y) = \int_0^1 x \cdot \left[\frac{y^2}{2}\right]_0^{1-x} dx$

$= \int_0^{1-x} x \left[\frac{(1+x)}{2}\right]^2 dx$

$= \left[\frac{x^2}{4}\right]_0^1 dx$

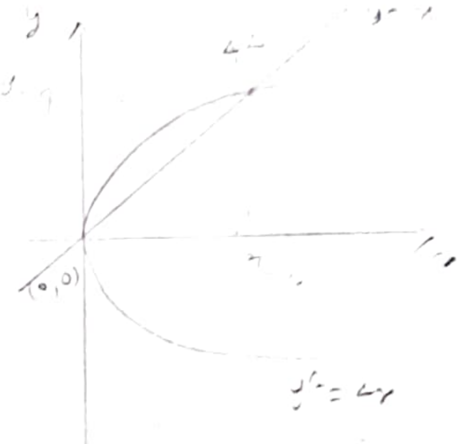
$= \frac{1}{2} \int_0^1 (x^3 + x - 2x^2)\, dx$

$= \frac{1}{2} \left[\frac{x^4}{4} + \frac{x^2}{2} - \frac{2x^3}{3} \right]_0^1$

$= \frac{1}{2} \left[\frac{1}{4} + \frac{1}{2} - \frac{2}{3} \right] = \frac{1}{2 \times 12} = \frac{1}{24}$ Ans

Q Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region bounded by $y = x$ and $y^2 = 4x$

Sol^n



$$y^2 = 4x$$

put ($y = x$)

$$x^2 - 4x = 0$$

$$(x-4)x = 0$$

$$x = 0 \ \& \ x = 4$$

$$A = \int_0^4 \int_x^{\sqrt{4x}} (x^2+y^2)\,dx\,dy$$

$$A = \int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\,dx\,dy$$

$$A = \int_0^4 \left[y + \frac{y^3}{3} \right]_x^{2\sqrt{x}} dx$$

$$A = \int_0^4 \left[2\sqrt{x} + \frac{8x^{3/2}}{3} - x - \frac{x^3}{3} \right] dx$$

$$A = \left[\frac{2x^{3/2}}{3} \times 2 + \frac{8}{3} \times \frac{x^{5/2}}{5} \times 2 - \frac{x^2}{2} - \frac{x^4}{12} \right]_0^4$$

$$A = \left[\frac{4}{3} x^{3/2} + \frac{16}{15} x^{5/2} - \frac{x^2}{2} - \frac{x^4}{12} \right]_0^4$$

$$A = \left(\frac{32}{3} + \frac{16 \times 32}{15} - \frac{16}{2} - \frac{256}{12} \right)$$

$A =$

$$A = \frac{768}{35} \text{ Ans}$$

Q Evaluate the integral $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$ by changing to polar cordinate (RTU 2013)

Sol^n

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

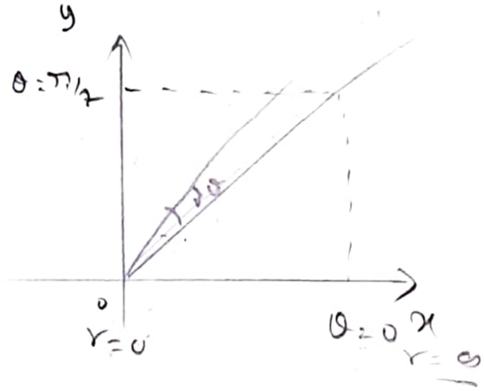
$$dx\,dy \Rightarrow r\,dr\,d\theta$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r \Rightarrow 0 \to \infty$$

$$\theta \Rightarrow 0 \to \pi/2$$



$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta$$

let $r^2 = t$

$$2r\,dr = dt$$

$$r\,dr = dt/2$$

$$I = \frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{-t} dt\,d\theta$$

$$I = -\frac{1}{2} \int_0^{\pi/2} \left[e^{-t}\right]_0^\infty d\theta$$

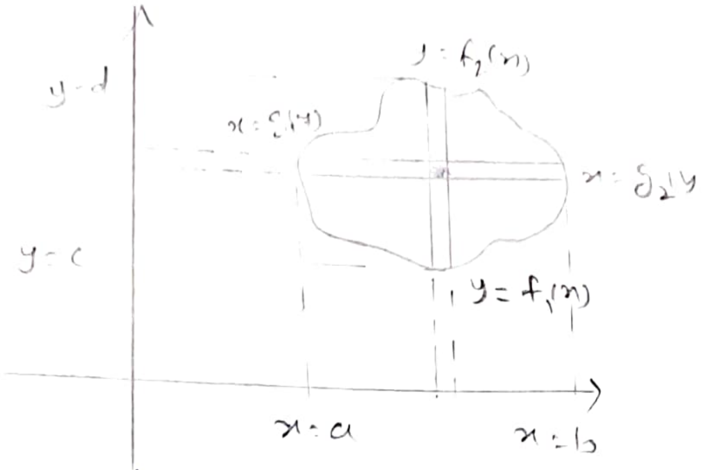
$$I = -\frac{1}{2} \int_0^{\pi/2} (e^{-\infty} - e^{-0})\,d\theta$$

$$I = \frac{1}{2} \int_0^{\pi/2} 1\,d\theta$$

$$I = \frac{1}{2}\left[\frac{\pi}{2}\right]$$

$$I = \pi/4$$

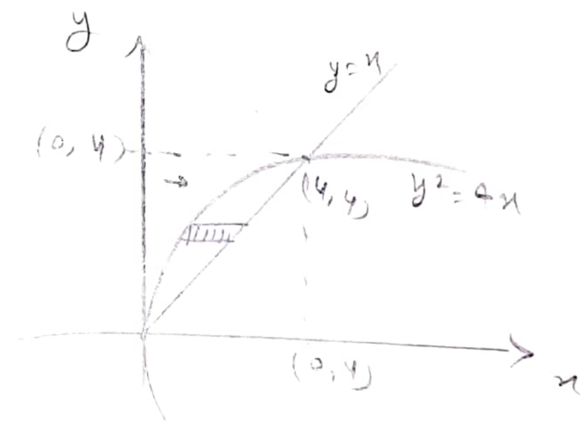
Change of order of integration:-



$$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

Q Change the order of integration of following double integration. (RTU-2023)

$$\int_{0}^{4}\int_{y=x}^{2\sqrt{x}} f(x,y)\,dx\,dy$$



Sol^n

$y=x$ to $y=2\sqrt{x}$

$y^2=4x$

put

$x=0$ to $x=4$

$y \to 0$ to 4

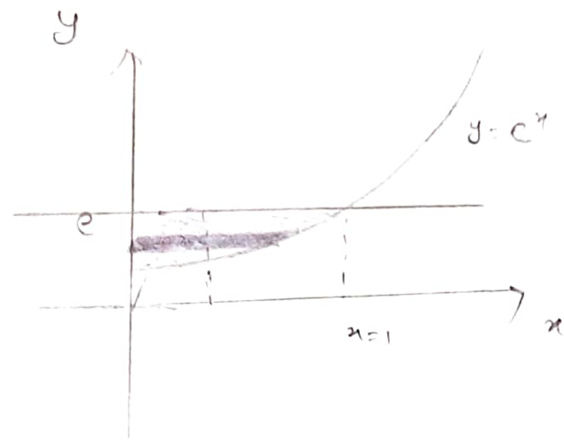
then now order

$$\int_{y=0}^{4}\int_{y^2/4}^{y} f(x,y)\,dy\,dx.$$

Q Change the ~~odrd.~~ order of integration & evaluate :-

$$\int_{0}^{1}\int_{y=e^{x}}^{e} \frac{1}{\log(y)}\,dx\,dy$$

(RTU 2019 - 2024)



Sol^n

$$\int_{y=1}^{e}\int_{x=0}^{\log(y)} \frac{1}{\log(y)}\,dy\,dx$$

$$\Rightarrow \int_{y=1}^{e}\int_{x=0}^{\log y} \frac{1}{\log(y)}\,dx\,dy$$

$$\Rightarrow \int_{y=1}^{e}\left[\frac{1}{\log(y)}\cdot x\right]_{x=0}^{\log(y)} dy$$

$$\Rightarrow \int_{y=1}^{e}[1]\,dy$$

$$\Rightarrow [y]_{1}^{e}$$

$$\Rightarrow e-1$$ Answer

Area & Volume as double integration:

Area: $\iint_R dx\,dy$ or $\iint_R r\,dr\,d\theta$

Volume: $\iint_R f(x,y)\,dx\,dy$ or $\iint_R f(r,\theta)\,r\,dr\,d\theta$.

Q. Find the area between $y^2=4x$ & $x^2=4y$

(RTU-2022)

Soln

$x^2=4y$ & $y^2=4x$

$y=2\sqrt{x}$

$x^2=4\times2\sqrt{x}$

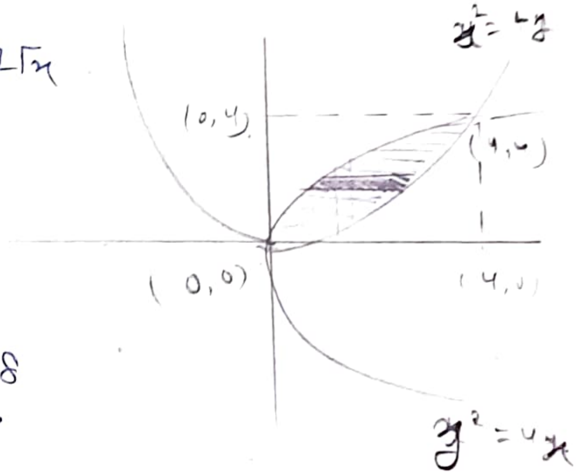
$x^2=8\sqrt{x}$

$\sqrt{x}(x^{3/2}-8)=0$

$x=0 \;|\; x^{3/2}=8$

$\boxed{x=4}$

& $y=0$ & $y=4$



$$A=\iint_R dx\,dy$$

$$A=\int_0^4\int_{\sqrt{4y}}^{y^2/4} dx\,dy$$

$$A=\int_0^4 [x]_{2\sqrt{y}}^{y^2/4}\,dy$$

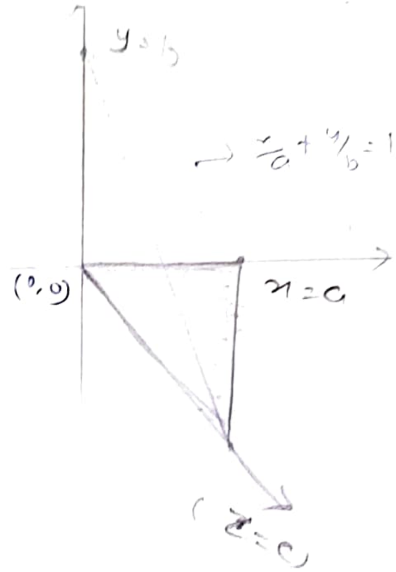
$$A=\int_0^4\left(\frac{y^2}{4}-2\sqrt{y}\right)dy \Rightarrow \left[\frac{y^3}{12}-4\cdot\frac{y^{3/2}}{3}\right]_0^4$$

$$A=\left[\frac{16}{3}-\frac{4\times8}{3}\right]=\frac{1}{3}\times16(1-2)=\frac{16}{3}\text{ Ans}$$

Q Find the volume of the tetrahedron bounded by the co-ordinate planes & the plane $\frac{x}{a}+\frac{y}{b}+z/c=1$

(RTU-2023)

Sol^n



$$\text{Volume} = \iint_R f(x,y)\,dx\,dy$$

$$\text{volume} = \iint_R \left(\frac{x}{a}+y/b+z/c\right) dx\,dy$$

in terms of z

let $\frac{x}{a}+y/b+z/c=1$

$$z/c = (1 - x/a + y/b)$$

$$z = c[1 - x/a - y/b]$$

when $x \Rightarrow 0$ to a

they

$y = (1-\frac{x}{a})b$

to

$y=0$

$$\text{Volume} = \int_{y=0}^{(1-\frac{x}{a})b}\int_0^a c[1 - x/a - y/b]\,dx\,dy$$

$$\text{Volume} = \int_{x=0}^{a}\int_{y=0}^{(1-x/a)b}\left[c - \frac{cx}{a} - \frac{cy}{b}\right]dx\,dy$$

$$\text{volume} = \int_{x=0}^{a}\left[cy - \frac{cxy}{a^2} - \frac{cy^2}{2b}\right]_{y=0}^{(1-x/a)b} dx$$

$$\text{volume} = \int_{x=0}^{a}\left[cb\left(1-\frac{x}{a}\right) - \frac{c}{a^2}\left(1-\frac{x}{a}\right)bx - \frac{c}{2b}\left[\left(1-\frac{x}{a}\right)b\right]^2\right]dx$$

$$\text{volume} = \int_{x=0}^{a}\left[\frac{cb}{a}(a-x) - \frac{cb}{a^3}[ax-x^2] - \frac{bc}{2a^2}[1+x^2-2x]\right]dx$$

$$\cancel{\text{volume}} = \left[\cancel{cbx} - \cancel{\frac{cbx^2}{2}} - \cancel{\frac{cb}{a^2}\frac{x^2}{2}} - \cancel{\frac{cbx^3}{a^3\,3}} - \cancel{\frac{bc}{2a^2}\left[x+\frac{x^3}{3}-\frac{2x^2}{2}\right]}\right]_0^a$$

$$\text{Volume} = \left[abc - \frac{a^2bc}{2} - \frac{cb}{2} + \frac{cb}{3} - \frac{bc}{2a}\left[1 + \frac{a^2}{2} - a\right]\right]$$

$$\text{Volume} = \left[abc - \frac{a^2bc}{2} - \frac{5cb}{6} - \frac{bc}{2a} + \frac{abc}{4} - \frac{abc}{2}\right]$$

$\times$ $+$

$$\int_0^a \left[\frac{b}{a}(a-x)c - \frac{x}{a}\left(\frac{b}{a}(a-x)\right) - \frac{b}{a^2}\left(\frac{(a-x)^2}{2}\right)dx\right]$$

$$\int \frac{bc}{a}\left(ax - \frac{x^2}{2}\right) - \frac{bc}{a^2}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right) - \frac{bc}{2a^2}\left(a^2x + \frac{x^3}{3} - \frac{2ax^2}{2}\right)\Bigg|_0^a$$

$$\Rightarrow \frac{bc}{a}\left(a^2 - \frac{a^2}{2}\right) - \frac{bc}{a^2}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) - \frac{bc}{2a^2}\left(a^3 + \frac{a^3}{3} - \frac{2a^3}{2}\right)$$

$$\Rightarrow \frac{bc}{a}\left(\frac{a^2}{2}\right) - \frac{bc}{a^2}\left(\frac{a^3}{6}\right) - \frac{bc}{2a^2}\left(\frac{a^3}{3}\right)$$

$$\Rightarrow \frac{abc}{2} - \frac{abc}{6} - \frac{abc}{6}$$

$$\Rightarrow -\frac{abc}{3} + \frac{abc}{2}$$

$$\Rightarrow \frac{abc}{6} \quad \text{Ans}$$

- ## Tripple Integrations:

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$$

Q Evaluate $\displaystyle\int_{x_1=0}^{1}\int_{y_1=1}^{2}\int_{z_1=2}^{3}(x+y+z)\,dx\,dy\,dz$.

Sol^n

$$\Rightarrow \int_{x_1=0}^{1}\int_{y_1=1}^{2}\int_{z_1=2}^{3}(x+y+z)\,dx\,dy\,dz$$

$$\Rightarrow \int_{0}^{1}\int_{1}^{2}\left[xz+yz+\frac{z^2}{2}\right]_2^3 dx\,dy$$

$$\Rightarrow \int_{0}^{1}\int_{1}^{2}\left[3x+3y+\frac{9}{2}-2x-2y-2\right] dx\,dy$$

$$\Rightarrow \int_{0}^{1}\int_{1}^{2}\left[x+y+\frac{5}{2}\right] dx\,dy$$

$$\Rightarrow \int_{0}^{1}\left[xy+\frac{y^2}{2}+\frac{5}{2}y\right]_1^2 dx$$

$$\Rightarrow \int_{0}^{1}\left[2x+2+5-x-\frac{1}{2}-\frac{5}{2}\right] dx$$

$$\Rightarrow \int_{0}^{1}[x+4]\,dx \Rightarrow \left[\frac{x^2}{2}+4x\right]_0^1$$

$$\Rightarrow \frac{1}{2}+4 = \frac{9}{2} \text{ Ans}$$

Q. Evaluate $\int_{z=1}^{2}\int_{y=1}^{z}\int_{0}^{yz}(xyz)\,dx\,dy\,dz$

Soln

$$\int_{z=1}^{2}\int_{y=1}^{z}\int_{x=0}^{yz}(xyz)\,dx\,dy\,dz$$

$$\Rightarrow \int_{1}^{2}\int_{1}^{z}\left[yz\frac{x^2}{2}\right]_0^{yz} dz\,dy$$

$$\Rightarrow \int_{1}^{2}\int_{y=1}^{2}\left(yz\cdot\frac{y^2z^2}{2}-0\right)dz\,dy$$

$$\Rightarrow \cancel{\int_{1}^{2}\int_{y=1}^{2}\left(\frac{y^3}{z^3}\right)dz\,dy}$$

$$\Rightarrow \int_{1}^{2}\int_{y=1}^{2}(y^3z^3)\,dz\,dy$$

$$\Rightarrow \int_{1}^{2}\left[\frac{z^3y^4}{4}\right]_1^2 dz$$

$$= \int_{1}^{2}\left(\frac{z^7}{4}-\frac{z^3}{4}\right)dz$$

$$\Rightarrow \left[\frac{z^8}{4\times 8}-\frac{z^4}{4\times 4}\right]_1^2$$

$$\Rightarrow \left[4^3-4-\frac{1}{32}-\frac{1}{16}\right]$$

$$\Rightarrow \left(60-\frac{3}{32}\right)\Rightarrow \frac{1917}{32}$$ Answer

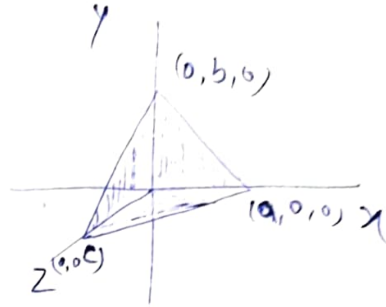
Q) Volume by tripple Integration.

$$\text{Vol} = \iiint_S dx\,dy\,dz$$

Q Find the volume bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Sol^n

$$V = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}(a-x)} \int_{z=0}^{\frac{c}{ab}(ab-bx-ay)} dx\,dy\,dz$$



$$V = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}(a-x)} \frac{c}{ab}(ab - bx - ay)\, dy\, dx$$

$$V = \int_{x=0}^{a} \left[\frac{c}{ab}\left[aby - bxy - \frac{ay^2}{2} \right]_0^{\frac{b}{a}(a-x)} \right] dx$$

$$V = \int_{x=0}^{a} \left[\frac{c}{ab}\left[ab \cdot \frac{b}{\cancel{a}}(a-x) - bx \cdot \frac{b}{\cancel{a}}(a-x) - \frac{\cancel{a}}{2} \cdot \frac{b^2}{a^{\cancel{2}}}(a-x)^2 \right] \right] dx$$

$$V = \int_{x=0}^{a} \frac{c}{ab}\left[b^2(a-x) - \frac{xb^2}{a}(a-x) - \frac{b^2}{2a}(a^2+x^2-2ax) \right] dx$$

$$V = \int_{x=0}^{a} \cancel{\frac{cb}{a}(a} \left[bc - \frac{cb}{a}x - \frac{cb\,\cancel{a}x}{a^2} + \frac{cb}{a^2}x^2 - \frac{cb}{2} - \frac{cb}{2a^2}x^2 + \frac{bc}{a}x \right] dx$$

for y limit

$z = 0$

$1 - \frac{x}{a} = y/b$

$y = \frac{b}{a}(a-x)$

for z limit

$z = c\left(1 - \frac{x}{a} - y/b\right)$

$z = \frac{c}{ab}(ab - bx - ay)$

for x limit

$z = 0\,,\; y = 0$

$\boxed{x = a}$

$$V = \left[bcx - \frac{cbx^2}{2a} - \frac{cbx^2}{2a} + \frac{cbx^3}{3a^2} - \frac{cbx}{2} - \frac{cb}{2a^2}\frac{x^3}{3} + \frac{bcx^2}{2a} \right]_0^a$$

$$V = \left(b\cancel{c}a - \frac{b\cancel{c}a}{\cancel{2}} - \frac{bc\cancel{a}}{\cancel{2}} + \frac{bca}{3} - \frac{b\cancel{c}a}{\cancel{2}} - \frac{bca}{6} + \frac{b\cancel{c}a}{\cancel{2}} \right]$$

$$\boxed{V = \frac{bca}{6}}$$ Answer

Line integral:-

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Note

If $\vec{F}$ = force

$\vec{r}$ = displacement

then word done

$\vec{w} = \int_C \vec{F}\cdot d\vec{r}$

line integral of $\vec{F}$ along curve "c"

$$= \int_C \vec{f}\cdot d\vec{r} \quad , \text{ where } d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Q If $\phi = 2xyz^2$ and c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to 1, then evaluate the line integral $\int_C \phi\, dr$.

Sol^n

$$\phi = 2xyz^2$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

$$d\vec{r} = (2t\,\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$$

$$\int \vec{\phi}\cdot d\vec{r} = \int_0^1 (2t\,\hat{i} + 2\hat{j} + 3t^2\hat{k}) \cdot [2(t^2)(2t)(t^3)^2]\,dt$$

$$= \int_0^1 (2t\,\hat{i} + 2\hat{j} + 3t^2\hat{k})[4t^9]\,dt$$

$$= \int_0^1 [8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k}]\,dt$$

$$\Rightarrow \left[\frac{8t^{11}}{11}\hat{i} + \frac{8t^{10}}{10} + \frac{12t^{12}}{12}\right]_0^1 \Rightarrow \boxed{\frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}}$$

Q If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a particle in the x-y plane from (0,0) to (1,4) along the curve $y = 4x^2$, find the work done.

Sol^n $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

in x-y Plane $\quad y = 4x^2$

$z = 0 \quad dy = 8x$

$\vec{r} = x\hat{i} + 4x^2\hat{j}$

$d\vec{r} = (\hat{i} + 8x\hat{j})dx$

$$\int_0^1 \vec{F}\cdot d\vec{r} = \int_0^1 [(2x^2)(4x^2)\,\hat{i} + 3x\,4(x^2)\hat{j}]\cdot(\hat{i} + 8x\hat{j})\,dx$$

$$\vec{F}\cdot d\vec{r} = \int_0^1 8x^4dx + \int_0^1 96x^4dx$$

$$\vec{F}\cdot d\vec{r} = \left[\frac{8x^5}{5}\right]_0^1 + \left[\frac{96x^5}{5}\right]_0^1$$

$$\vec{F}\cdot d\vec{r} = 8/5 + 96/5$$

$$\vec{F}\cdot d\vec{r} = 104/5 \text{ Answer}$$

Q. Find the work done in moving a particle in the force field.

$$\vec{f} = 3x^2\hat{i} + (2xz-y)\hat{j} + 2\hat{k} \text{ along}$$

(a) Straight line from $A(0,0,0)$ to $B(2,1,3)$

(b) Curve C: $x = 2t^2$; $y = t$; $z = 4t^2 - t$ from ($t=0$ to $t=1$)

Sol^n $\vec{f} = 3x^2\hat{i} + (2xz-y)\hat{j} + 2\hat{k}$

(a) point $A(0,0,0)$ to $B(2,1,3)$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$$\left[\because \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = k\right]$$

$x = 2t$; $y = t$; $z = 3t$

$$\vec{r} = 2t\hat{i} + t\hat{j} + 3t\hat{k}$$

$$d\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k})\,dt$$

$$W = \vec{f}\cdot d\vec{r} = \int (2\hat{i}+\hat{j}+3\hat{k}).[3x^2\hat{i} + (2x\cancel{z}-y)\hat{j} + 2\hat{k}]\,dt$$

$$W = \vec{f}\cdot d\vec{r} = \int (2\hat{i}+\hat{j}+3\hat{k}).[12t^2\hat{i} + (12t^2-t)\hat{j} + 3t\hat{k}]\,dt$$

$$W = \int_0^1 [24t^2\hat{i} + (12t^2 - t)\hat{j} + 9t]\,dt$$

$$W = \left[\frac{24t^3}{3} + \frac{12t^3}{3} - \frac{t^2}{2} + \frac{9t^2}{2}\right]_0^1$$

$$W = \left[8 + 4 + 8/2\right] = 16 \text{ Answer}$$

(b) Curve c $x = 2t^2$; $y = t$; $z = 4t^2 - t$

$(t = 0 \text{ to } t = 1)$

$$\int_c \vec{F} \cdot d\vec{r} = [3x^2 \hat{i} + (2xz - y)\hat{j} + 2\hat{k}]\, d(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\vec{r} = 2t^2 \hat{i} + t\hat{j} + (4t^2 - t)\hat{k}$$

$$d\vec{r} = \left(\frac{2t^3}{3}\hat{i} + \frac{t^2}{2}\hat{j} + \frac{4t^3}{3}\hat{k} - \frac{t^2}{2}\hat{k}\right) dt$$

$$W = \int_0^1 [3(2t^2)^2 \hat{i} + [2(2t^2)(4t^2 - t) - t]\hat{j} + (4t^2 - t)\hat{k}] \cdot \left(\frac{2t^3}{3}\hat{i} + \frac{4t^2}{2}\hat{j} + \frac{4t^3}{2}\hat{k} - \frac{t^3}{2}\hat{k}\right) dt$$

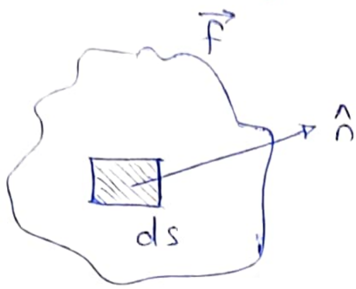
$$W = \int_0^1 8t^7 \hat{i} + \left(8t^6 - 2t^3 - \frac{t^3}{2}\right)\hat{j} + \left(\frac{16}{3}t^5 - \frac{4t^4}{3} - 2t^4 + \frac{t^3}{3}\right)\hat{k}\, dt$$

$$W = \left[\frac{8t^8}{8}\right]_0^1 \hat{i} + \left(\frac{8t^7}{7} - \frac{2t^4}{4} - \frac{t^4}{8}\right)\hat{j} + \left[\frac{16}{3 \times 6}t^6 - \frac{4t^5}{5} - \frac{2t^5}{5} + \frac{t^4}{12}\right]\hat{k}$$

$$W = \hat{i} + \left(\frac{64 - 35}{56}\right) + \left(\frac{8}{9} - \frac{6}{5} + \frac{1}{12}\right)$$

$W \approx 14.2$ Joule.

★ Surface Integral :-



$$\iint \vec{f}\cdot d\vec{s} = \iint_R f\cdot\hat{n}\, ds$$

where

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

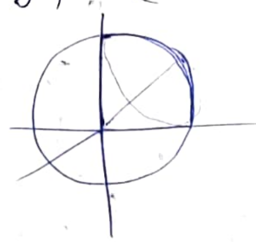
Surface $\Rightarrow \phi$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

Q. Evaluate double integration $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\,ds$, where S is surface of the sphere : $x^2+y^2+z^2=1$ in the first octant.

↳ xy plane



$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$f = yz + zx + xy$$

$$ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\phi = x^2+y^2+z^2-1$$

$$\nabla\phi = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\nabla\phi| = 2\sqrt{x^2+y^2+z^2} = 2(1)$$

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$\hat{n}\cdot\hat{k} = \frac{\cancel{2}\hat{k}\cdot z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{1}$$

So $$ds = \frac{dx\,dy\,(\sqrt{x^2+y^2+z^2})}{z} = \frac{dx\,dy}{z}$$

Now Surface integral.

$$\Rightarrow \iint_S \hat{f}\cdot\hat{n}\, ds$$

$$\Rightarrow \iint_S \sqrt{x^2+y^2+z^2}\,\frac{dx\,dy}{z}\left[yz\hat{i}+zx\hat{j}+xy\hat{k}\right]\cdot\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \iint_S \frac{dx\,dy}{z}\left[yz\hat{i}+zx\hat{j}+xy\hat{k}\right]\cdot\left[x\hat{i}+y\hat{j}+z\hat{k}\right]$$

$$\Rightarrow \iint_S (xyz+yzx+xyz)\frac{dx\,dy}{z}$$

limit

$x^2+y^2+z^2=1$

$z=0$ for I^{st} Octan

$x^2+y^2=1$

$x \to 0$ to 1

& $y=\sqrt{1-x^2}$

$$\Rightarrow \iint_S 3xy\cdot dx\,dy$$

$$\Rightarrow \cancel{\int_0 3x\left(\frac{y^2}{2}\right)_0^1 dx}$$

$$\Rightarrow \cancel{\frac{3}{4}\ \text{Ans}}$$

$$\Rightarrow 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dx\,dy$$

$$\Rightarrow 3\int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$\Rightarrow 3\int_0^1 \frac{x(1-x^2)}{2}\,dx$$

$$\Rightarrow \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$\Rightarrow \frac{3}{2}\times\frac{1}{4} = \frac{3}{8}\ \text{Ans}$$

Q Evaluate $\iint \vec{f}\cdot\hat{n}\, ds$ (RTU 2015)

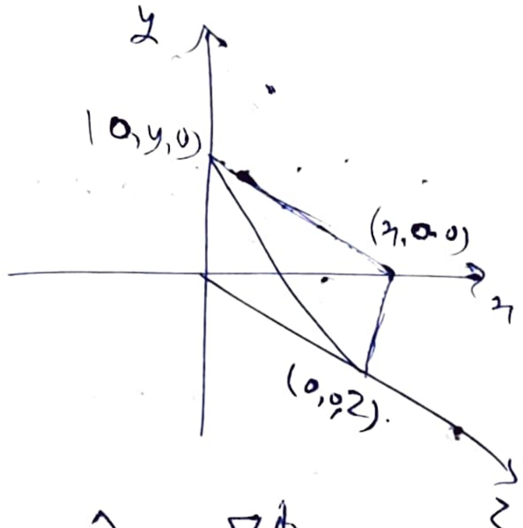
where $\vec{f} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

S → part of plane $x+y+z=1$ located in 1st octant.

Soln

$\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

$\vec{S} = x+y+z=1 = \phi$



~~$\nabla\phi = x\hat{i} + y\hat{j} + z\hat{k}$~~

~~$\nabla\phi =$~~ $\phi = x+y+z-1$

$\nabla\phi = \hat{i} + \hat{j} + \hat{k}$

$\nabla\phi = \sqrt{(1)^2+(1)^2+(1)^2} = \sqrt{3}$

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Now

$\vec{F}\cdot\hat{n} = \dfrac{1}{\sqrt{3}}(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})$

$\vec{F}\cdot\hat{n} = \dfrac{x^2+y^2+z^2}{\sqrt{3}}$

$\hat{n}\cdot\hat{k} = \left(\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)\cdot\hat{k} = 1/\sqrt{3}$

$ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \sqrt{3}\,dx\,dy$

Now surface integral

$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_S \left(\dfrac{x^2+y^2+z^2}{\sqrt{3}}\right)\cdot\sqrt{3}\,dx\,dy$

$$= \iint_S (x^2+y^2+z^2)\,dx\,dy$$

put $z = 1-x-y$

$$= \iint_S [x^2+y^2+(1-x-y)^2]\,dx\,dy$$

$$= \iint_S [x^2+y^2+1+x^2+y^2+2xy-2y-2x]\,dx\,dy$$

$$\Rightarrow \iint_S [1+2x^2+2y^2+2xy-2y-2x)\,dx\,dy$$

$z+x+y+1=0$

first Octant

$z=0$

$x \to 0$ to 1

& $y = 1-x$

$$\Rightarrow \int_0^1 \int_0^{1-x} [1+2x^2+2y^2+2xy-2y-2x]\,dx\,dy$$

$$\Rightarrow \int_0^1 \left[y+2x^2y+\frac{2y^3}{3}+\frac{2xy^2}{2}-\frac{2y^2}{2}-2xy\right]_0^{1-x} dx$$

$$\Rightarrow \int_0^1 \Big[1-x+\cancel{2}x^2-2x^3+\frac{2(1-x)^3}{3}+\frac{\cancel{2}x(1-x)^2}{\cancel{2}}+\frac{\cancel{2}(1-x)^2}{\cancel{2}} \;-2x+2x^2$$

$$\Rightarrow \int_0^1 \Big[\cancel{1}-3x+4x^2-2x^3+\tfrac{2}{3}[1-x^3-3x+3x^2]+x^3+\cancel{x}-2x^2 \;\cancel{-1}-x^2+2\cancel{x}\Big]\,dx$$

$$\Rightarrow \int_0^1 \left[1 + 3x^2 - x^3 + \frac{2}{3} - \frac{2x^3}{3} - 2x \right] dx$$

$$\Rightarrow \int_0^1 \left(\frac{2}{3} - 2x - \frac{5}{3}x^3 + 3x^2 \right) dx$$

$$\Rightarrow \left[\frac{2}{3}x - \frac{2x^2}{2} - \frac{5x^4}{12} + \frac{3x^3}{3} \right]_0^1$$

$$\Rightarrow \left(\frac{2}{3} - \frac{5}{12} - 1 + 1 \right)$$

$$\Rightarrow \frac{8-5}{12} = \frac{3}{12} = \frac{1}{4} \text{ Ans}$$

02/12/24

Green's theorem :-

$$\oint_c \vec{f} \cdot d\vec{r} = \oint_c M\,dx + N\,dy = \iint_S \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right) dx\,dy$$

Q. A vector field $\vec{F}$ is given by $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the line integral $\int_c \vec{f} \cdot d\vec{r}$, where c is the circular path given by $x^2 + y^2 = a^2$.

Soln

$$\vec{f} \cdot d\vec{r} = [\sin y\,\hat{i} + x(1+\cos y)\hat{j}] \cdot [dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]$$

$$\vec{f} \cdot d\vec{r} = \sin y\,dx + x(1+\cos y)\,dy$$

here $M = \sin y$

$N = x(1+\cos y)$

by Green theorem

$$\oint_c \vec{f} \cdot d\vec{r} = \iint_S \left[-\frac{\delta(\sin y)}{\delta y} + \frac{\delta(x(1+\cos y))}{\delta x} \right] dx\,dy$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left[\frac{-\delta \sin y}{\delta y} + \frac{\delta x(1+\cos y)}{dx}\right] dx\,dy$$

$$\oint_C \vec{f}\cdot d\vec{r} \quad \iint_C [1 + \cancel{\cos} y - \cancel{\cos} y]\, dx\,dy$$

$$\oint_C \vec{f}\cdot d\vec{r} = \iint_R dx\,dy$$

for limit

$x^2 + y^2 = a^2$

$x \Rightarrow 0$ to a

& $y = \sqrt{a^2 - x^2}$.

(radius $= r = a$)

area of circle

$A \Rightarrow \pi a^2$ Answer

Q Using Green's theorem, evaluate

$$\int_C (xy + y^2)\,dx + x^2 dy,$$

where c is the closed curve of the region bounded by $y = x$ & $y = x^2$.

Soln $f(x) = \int_C (xy + y^2)\,dx + x^2 dy.$

by Green theorem

$$\oint (M dx + N dy) = \iint_C \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right)$$

$M = xy + y^2$

& $N = x^2$

So

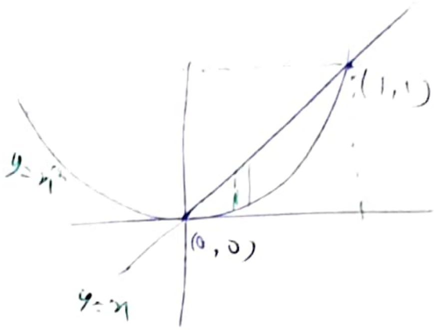
$$f(x) = \iint_C \left[\frac{\delta x^2}{\delta x} - \frac{\delta (xy + y^2)}{\delta y}\right] dx\,dy$$

$$f(x) = \iint [2x - x \cancel{+} 2y]\, dx\,dy$$

$$f(x) = \iint_C (x - 2y)\, dx\,dy$$

$$f(x) = \iint_c (x-2y)\,dx\,dy$$

$$f(x) = \int_0^1 \int_{x^2}^{x} [x - 2y]\,dx$$



$$f(x) = \int_0^1 [xy - y^2]_{x^2}^{x}\,dx$$

$$f(x) = \int_0^1 \left([x^2 - x^2] - [x^3 - x^4]\right) dx$$

$$f(x) = \int_0^1 [x^4 - x^3]\,dx$$

$$f(x) = \left[\frac{x^5}{5}\right]_0^1 - \left[\frac{x^4}{4}\right]$$

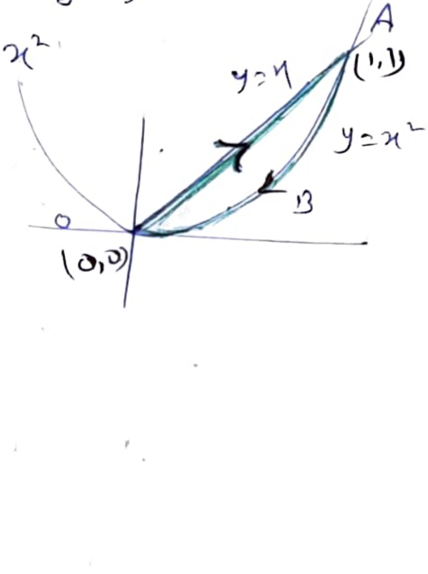
$$f(x) = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20} \quad \text{Ans}$$

by line integral

$$\oint_c (M\,dx + N\,dy)$$

$M = xy + y^2$

$N = x^2$



for Path OA ($y = x$)

$$\cancel{\vec{f} = xy\,\hat{i} + y^2\,\hat{j}}$$

$$\vec{f} = (xy + y^2)\,\hat{i} + x^2\,\hat{j}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = x\hat{i} + x\hat{j}$$

$$\int_0^1 \vec{f}\cdot d\vec{r} = \int_0^1 [(x(x) + x^2)\,\hat{i} + x^2\,\hat{j}] \cdot [dx\,\hat{i} + dx\,\hat{j}]$$

$$\int_0^1 \vec{f}\cdot d\vec{r} = \int_0^1 (2x^2 + x^2)\,dx$$

$$\Rightarrow \int_0^1 \vec{F}\cdot d\vec{r} = \int_0^1 (3x^2)\, dx$$

$$\Rightarrow \int_0^1 \vec{f}\cdot d\vec{r} = \frac{3x^{3}}{3} = 1 \; . \; -1)$$

Now for path ABO $\quad (y = x^2)$

$dy = 2x\,dx$

$$\int \vec{f}\cdot d\vec{r} = \int_0^1 [(x^2y + y^2)\hat{i} + x^2\hat{j}]\cdot[dx\,\hat{i} + 2x\,dx\,\hat{j}]$$

$$= \int_0^1 [[x^3 + x^4]\hat{i} + x^2\hat{j}]\cdot[dx\,\hat{i} + 2x\,dx\,\hat{j}]$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\, dx$$

$$= \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \left[3/4 + 1/5\right]$$

$$\Rightarrow \frac{15+4}{20} = 19/20$$

So (path AB $-$ path ABO)

$$= -1 + 19/20$$

$$= -1/20 \text{ Answer}$$

Q Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region ~~deg~~ defined by $x=0$; $y=0$ & $x+y=1$.

Soln

$$\int_C (3x^2 + 8y^2)\,dx - (4y - 6xy)\,dy$$

by Green's theorem.

$$\oint_C = \oint (M\,dx + N\,dy) = \iint \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right)$$

$$M = 3x^2 - 8y^2$$

$$N = -(4y - 6xy)$$

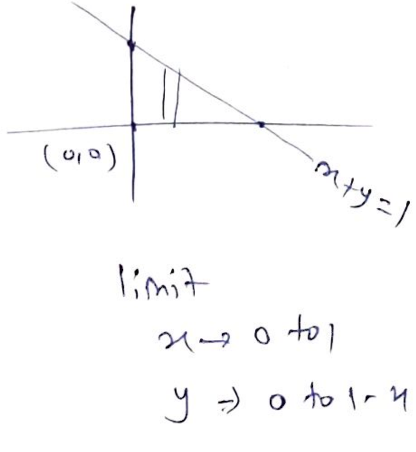
$$\oint_C = \iint_C \left[\frac{-\delta(4y - 6xy)}{\delta x} - \frac{\delta(3x^2 - 8y^2)}{\delta y} \right] dx\,dy$$

$$\oint_C = \iint_C \left[6y + \cancel{\frac{8y^3}{3}}\ 16y \right] dx\,dy$$

$x + y = 1$

$y = 1 - x$

$$\oint_C = \iint_C (y)\,dx\,dy$$



$$\oint_C = \int_0^1 \int_0^{1-x} [y]\,dx\,dy$$

limit

$x \to 0$ to 1

$y \to 0$ to $1-x$

$$\oint_C = \int_0^1 \left(\left[\frac{y^2}{2} \right]_0^{1-x} \right) dx$$

$$\oint_C = \int_0^1 \frac{(1-x)^2}{2}\,dx = \int_0^1 (1 + x^2 - 2x)\,dx$$

$$= \left(x + \frac{x^3}{3} - x^2 \right)_0^1 = \frac{1}{3} \text{ Ans}$$

* Stoke's Theorem :

$$\int_C \vec{f}\cdot d\vec{r} = \iint_C \text{curl}\,\vec{f}\cdot\hat{n}\,ds = \iint_C (\nabla\times\vec{f})\cdot\hat{n}\,ds$$

where, $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

& $ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$ or $\dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$ or $\dfrac{dx\,dy}{|\hat{n}\cdot\hat{j}|}$

Q Using Stock theorem $\int_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$

where $x^2+y^2=1$ is the circle with radius 1, corresponding to the surface of unit radius.

Sol^n

$$\int_C \left((2x-y)\,dx - yz^2\,dy - y^2z\,dz\right) = \int \vec{f}\cdot d\vec{r}$$

here

$$\vec{f} = (2x-y)\,\hat{i} - (yz^2)\,\hat{j} - y^2z\,\hat{k}$$

$\phi = x^2+y^2+z^2-1$

$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$|\nabla\phi| = 2\sqrt{x^2+y^2+z^2}$

$|\nabla\phi| = 2$

$\hat{n} = \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2}$

$\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$

for curl $\vec{f} = \vec{\nabla}\times\vec{f}$

$$\vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ (2x-y) & -yz^2 & -zy^2 \end{vmatrix}$$

$\vec{\nabla}\times\vec{f} = (-2zy + 2zy)\,\hat{i} - (0-0) + (0+1)\,\hat{k}$

$\vec{\nabla}\times\vec{f} = \hat{k}$

Now

LHS

$$\Rightarrow \iint_C (\text{curl}\,\vec{f} \cdot \hat{n})\, ds \qquad ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$ds = \frac{dx\,dy}{z}$$

$$\Rightarrow \iint_C [\hat{k} \cdot (x\hat{i} + y\hat{j} + z\hat{k})]\, \frac{dx\,dy}{z}$$

$$\Rightarrow \iint_C z \times \frac{dx\,dy}{z}$$

limit

$x \to 0$ to 1

$$\Rightarrow \iint_C dx\,dy$$

then $y = 0$ to $\sqrt{1-x^2}$

$$\Rightarrow \int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx = \pi \text{ Answer}$$

$$\Rightarrow \int_0^1 [y]_0^{\sqrt{1-x^2}}\, dx$$

$$\Rightarrow \left[\sqrt{1-x^2}\right]_0^1$$

$$\Rightarrow -1$$

Now for RHS

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\iint_C (\vec{\nabla} \times \vec{f}) \cdot \hat{r}\, ds \qquad \hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \iint_C \hat{k} \cdot (x\hat{i} + y\hat{j} + z\hat{k})\, \frac{dx\,dy}{z}$$

$$\Rightarrow \iint_C dx\,dy = \pi(1)^2 = \pi$$

Hence Proved

Q Apply by using the stock's theorem for $\int_C y\,dx + z\,dy + x\,dz$ where c is the intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$ (RTU-2022)

Soln

$$\int_C (y\,dx + z\,dy + x\,dz) = \int_C (y\hat{i}+z\hat{j}+x\hat{k})\cdot(dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k})$$

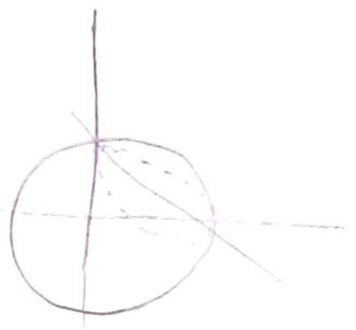
$$\vec{f} = y\hat{i}+z\hat{j}+x\hat{k} \qquad ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$d\vec{r} = dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}$$

$$\phi = x+z-a$$

$$\nabla\phi = \hat{i}+\hat{k}$$

$$|\nabla\phi| = \sqrt{2}$$



$$\text{curl}\,\vec{f} = \vec{\nabla}\times\vec{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y & z & x \end{vmatrix}$$

$$= (-1)\hat{i} - (1)\hat{j} + -(1)\hat{k}$$

$$= -(\hat{i}+\hat{j}+\hat{k})$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}+\hat{k}}{\sqrt{2}}$$

$$\Rightarrow \hat{n}\cdot\hat{k}$$

$$\Rightarrow \left(\frac{\hat{i}+\hat{k}}{\sqrt{2}}\right)\cdot(\hat{k})$$

$$\Rightarrow 1/\sqrt{2}$$

So $ds = \sqrt{2}\,dx\,dy$

So

$$\int \vec{f}\cdot d\vec{r} = \iint (\vec{\nabla}\times\vec{f})\cdot\hat{n}\,ds$$

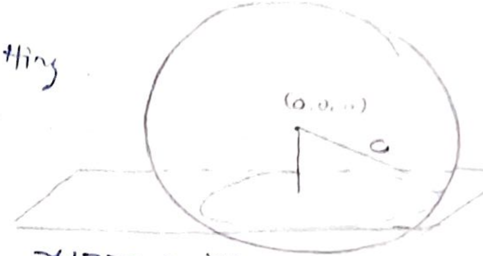
$$= \iint_C -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{i}+\hat{k}}{\sqrt{2}}\right)ds$$

$$\int \vec{f}\cdot d\vec{r} = \iint_C (-\sqrt{2})\cdot\sqrt{2}\,dx\,dy$$

$\int \vec{f}\cdot d\vec{r} = -2\iint_C dx\,dy$

↳ region of cutting circle

$\cancel{\int \vec{f}\cdot d\vec{r} = -2(\pi)a^2}$

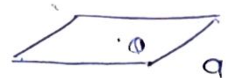


$x+z=a \Rightarrow \frac{x}{a}+\frac{z}{a}=1$

Shortest distance

$P(\alpha,\beta,\gamma)$

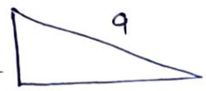
$$d = \left| \frac{0+0+0-1}{\sqrt{(1/a)^2+(1/a)^2}} \right|$$



$ax+by+cz=1$

$$d = \left| \frac{a\alpha+b\beta+c\gamma-1}{\sqrt{a^2+b^2+c^2}} \right|$$

$$d = \frac{a^2}{\sqrt{2}}$$



$$\sqrt{a^2 - \frac{a^2}{2}} = r$$

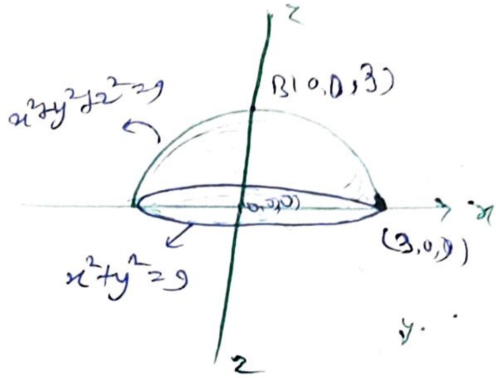
$$r = \frac{a}{\sqrt{2}}$$

then area of circle $= \frac{\pi a^2}{\sqrt{2}}$

So $-2\iint_C \frac{\pi a^2}{(\sqrt{2})^2} = \pi a^2$ Answer

Q Verify the stocks theorem for the hemisphere $x^2+y^2+z^2=9$, $z\geq 0$ its boundry circle $x^2+y^2=9$, $z=0$ and the field $\vec{f}=y\hat{i}-x\hat{j}$ (2018)

Soln $\vec{f}=y\hat{i}-x\hat{j}$



$\phi = x^2+y^2+z^2-9$

$\nabla\phi = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2}}$

$\hat{n} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$

$ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$= \dfrac{dx\,dy}{\frac{(z\hat{k}+x\hat{i}+y\hat{j})\cdot\hat{k}}{3}}$

$ds = \dfrac{3\,dx\,dy}{z}$

$\text{Curl}\,\vec{f} = \vec{\nabla}\times\vec{f}$

$\text{Curl}\,\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y & -x & 0 \end{vmatrix}$

$\text{Curl}\,\vec{f} = 0-0+(-1-1)\hat{k}$

$\text{Curl}\,\vec{f} = -2\hat{k}$

$\hat{n}\cdot\hat{k} = \left(\dfrac{x\hat{i}+y\hat{j}}{3}\right)\cdot\hat{k} = 0$

$\hat{n}\cdot\hat{j} = \left(\dfrac{x\hat{i}+y\hat{j}}{3}\right)\hat{j} = y/3$

$ds =$

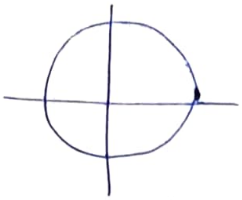
$\Rightarrow \int \vec{f}\cdot d\vec{r} =$

$\Rightarrow \iint (\vec{\nabla}\times\vec{f})\cdot\hat{n}\,ds$

$= \iint (-2\hat{k})\cdot\left(\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\cancel{3}}\right)\cdot\dfrac{\cancel{3}\,dx\,dy}{z}$

$\Rightarrow -2\iint dx\,dy = -2\pi(3)^2 = -18\pi$ Answer

Proof



$\theta = 0$ to 2π

$x = 3\cos\theta$

$y = 3\sin\theta$

$dx = -3\sin\theta\, d\theta$

$dy = 3\cos\theta\, d\theta$

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$

$\Rightarrow \int (y\hat{i} - x\hat{j}) [dx\hat{i} + dy\hat{j} + dz\hat{k}]$

$\Rightarrow \int_C (y\,dx - x\,dy)$

$\Rightarrow \int_0^{2\pi} [[3\sin\theta(-3\sin\theta\, d\theta) - 3\cos\theta\,[3\cos\theta\, d\theta]]]$

$\Rightarrow -9\int_0^{2\pi} [\sin^2\theta + \cos^2\theta]\, d\theta$

$\Rightarrow -9\int_0^{2\pi} d\theta \Rightarrow -9[\theta]_0^{2\pi} = -18\pi$ Ans

Hence Proved

- Gauss Divergence Theorem :-

$$\iint \vec{f}\cdot\hat{n}\,ds = \iiint_S \operatorname{div}\vec{f}\,dv$$

(here $dv = dx\,dy\,dz$)

Q Find $\iint_S \vec{f}\cdot\hat{n}\,ds$ where $\vec{f} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$

where S is surface of the sphere having centre (3, -1, 2) and radius 3.

Sol^n

$$\iint_S \vec{f}\cdot\hat{n}\,ds = \iiint \operatorname{div}\vec{f}\,dx\,dy\,dz$$

$$\vec{f} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$$

$$\operatorname{div}\vec{f} = \vec{\nabla}\cdot\vec{f}$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\cdot\left((2x+3z)\hat{i} + (xz+y)\hat{j} + (y^2+2z)\hat{k}\right)$$

$$\operatorname{div}\vec{f} = 2 - 1 + 2$$

$$\operatorname{div}\vec{f} = 3$$

Sphere is

$$(x-3)^2 + (y+1)^2 + (z-2)^2 = 9$$

$$x^2 + \cancel{9} + y^2 + 2y + z^2 + 4 - 6x + 1 - 4z = \cancel{9}$$

$$x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$$

* for limit

when

$z = 0$

$$x^2 + y^2 - 6x + 2y + 5 = 0$$

$$(x-3)^2 + (y+1)^2 \cancel{+}5 = \cancel{0}$$

$$(y+1) = \sqrt{5 - (x-3)^2}$$

$\cancel{(1+y) = x^2}$

$\Rightarrow \iiint_S 3\,dv$

$\Rightarrow \iiint_S 3\,dx\,dy\,dz$ [Volume of sphere]

$\Rightarrow 3\left(\frac{4}{3}\pi(3)^3\right)$

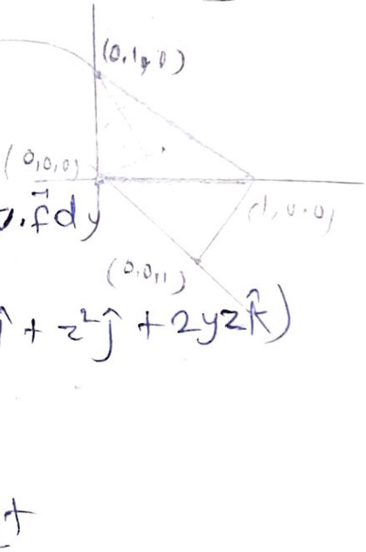
$\Rightarrow 27\times4\times\pi$

$\Rightarrow 108\pi$ Ans

Q. Verify Gauss Divergence theorem $\vec{f} = xy\hat{i} + z^2\hat{j} + 2yz\hat{k}$ on the ~~tetra~~hedron $x=y=z=0$ & $x+y+z=1$. (RTU 2023)

Sol^n Given

$\vec{f} = xy\hat{i} + z^2\hat{j} + 2yz\hat{k}$



$\iint_S \vec{f}\cdot\hat{n}\,ds = \iiint_S \nabla\cdot\vec{f}\,dv$

Divergence (RHS)

$\operatorname{div}\vec{f} = \nabla\cdot\vec{f} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\cdot\left(xy\hat{i} + z^2\hat{j} + 2yz\hat{k}\right)$

$= y + 0 + 2y$

$\operatorname{div}\vec{f} = 3y$

limit

$x+y+z=1$

when $z=0$

$x=0$ to 1

$y = 1-x$

means $y=0$ to $1-x$

& $z = 0$ to $1-x-y$.

$\Rightarrow \iiint_S \nabla\cdot\vec{f}\,dv$

$\Rightarrow \iiint_S 3y.\,dx\,dy\,dz$

$\Rightarrow 3.\int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dx\,dy\,dz$ Or.

$$\Rightarrow 3\int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx$$

$$\Rightarrow 3\int_0^1\int_0^{1-x}\left[yz\right]_0^{1-x-y} dy\,dx$$

$$\Rightarrow 3\int_0^1\int_0^{1-x}\left[y-xy-y^2\right] dy\,dx$$

$$\Rightarrow 3\int_0^1\left[\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right]_0^{1-x} dx$$

$$\Rightarrow 3\int_0^1\left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right] dx$$

$$\Rightarrow 3\int_0^1\left(\frac{x^2+1-2x}{2}-\frac{x^3+x-2x^2}{2}-\left(\frac{1-x^3-3x(1-x)}{3}\right)\right) dx$$

$$\Rightarrow 3\int_0^1\left[\frac{3x^2+3-\cancel{6x}-3x^3+3x+\cancel{6x^2}-2+2x^3+\cancel{6x}-\cancel{6x^2}}{6}\right] dx$$

$$\Rightarrow 3\int_0^1\left[\frac{3x^2-x^3-3x+1}{6}\right] dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{\cancel{3}x^3}{\cancel{3}}-\frac{x^4}{4}-\frac{3x^2}{2}+x\right]_0^1$$

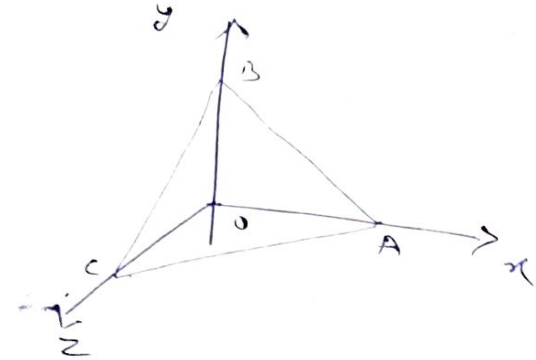
$$\Rightarrow \frac{1}{2}\left[1-\frac{1}{4}-\frac{3}{2}+1\right]$$

$$\Rightarrow \frac{1}{2}\left[2-\frac{7}{4}\right]$$

$$\Rightarrow \frac{1}{2}\times\frac{1}{4}=\frac{1}{8}\ \text{Answer}$$

|LHS|

$\Rightarrow \iint \vec{f}\cdot\hat{n}\, ds$



$\Rightarrow \vec{f} = xy\hat{i} + z^2\hat{j} + 2yz\hat{k}$

$$\iint \vec{f}\cdot\hat{n}\, ds = \iint_{OAB} \vec{f}\cdot\hat{n}\, ds + \iint_{OBC} \vec{f}\cdot\hat{n}\, ds + \iint_{OAC} \vec{f}\cdot\hat{n}\, ds + \iint_{ABC} \vec{f}\cdot\hat{n}\, ds$$

$$\iint \vec{f}\cdot\hat{n}\, ds = I_1 + I_2 + I_3 + I_4$$

For $I_1 = \iint_{OAB} \vec{f}\cdot\hat{n}\, ds$

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$, ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$\phi = x + y = 1$

$\nabla\phi = \hat{i} + \hat{j}$

$\Rightarrow ds = dx\,dy$

$(z=0)$

$|\nabla\phi| = \sqrt{1^2+1^2} = \sqrt{2}$

$(\hat{n} = -\hat{k})$

$$I_1 = \iint_R (xy\hat{i} + z^2\hat{j} + 2yz\hat{k})\cdot(-\hat{k})\, dx\,dy$$

$\vec{f} = xy\hat{i} + z^2\hat{j} + 2yz\hat{k}$

for $z=0$ (x-y plane)

$\vec{f} = xy\hat{i}$

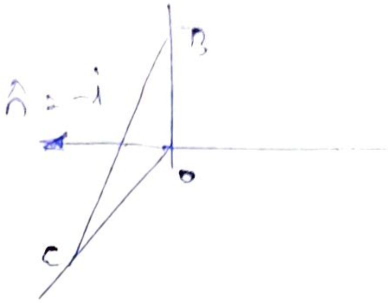
$$I_1 = -\iint_R 2yz\, dx\,dy \quad \because z=0$$

$\boxed{I_1 = 0}$ $\rightarrow$ (i)

For

$$I_2 = \iint_{OBC} \vec{f}\cdot\hat{n}\, ds$$

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$



for yz plane

$x = 0$

So

$\vec{f} = xy\,\hat{i} + z^2\hat{j} + 2yz\hat{k}$

$\vec{f} = 2yz\hat{k} + z^2\hat{j}$

$\vec{f}\cdot\hat{n} = (2yz\hat{k} + z^2\hat{j})\cdot(-\hat{i})$

$\vec{f}\cdot\hat{n} = 0$

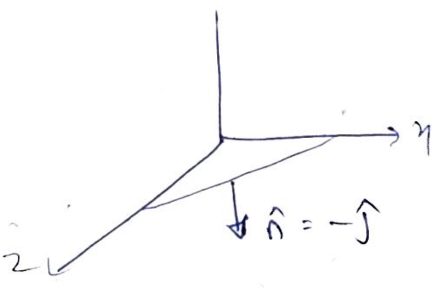
$$I_2 = \iint_R \vec{f}\cdot\hat{n}\, ds$$

$$\boxed{I_2 = 0} \quad \text{—(ii)}$$

for

$$I_3 = \iint_{OAC} \vec{f}\cdot\hat{n}\, ds$$

$$ds = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$



Plane

$x + z = 1$

for xz plane

$y = 0$

So

$\vec{f} = z^2\hat{j}$

$ds = dx\,dz$

So $I_2 = \iint_R (z^2\hat{j})\cdot(-\hat{j})\, dx\,dz$

$$I_2 = -\int_0^1\int_0^{1-x} z^2\, dx\,dz$$

$$I_2 = -\int_0^1 \left[\frac{(1-x)^3}{3}\right] dx$$

$$I_3 = +\left[\frac{(1-x)^4}{12}\right]_0^1$$

$$I_3 = -1/12 \quad \text{—(iii)}$$

Now for ABC

$\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Plane

$x+y+z=1$

$\phi = x+y+z-1$

$\nabla\phi = \hat{i}+\hat{j}+\hat{k}$

$|\nabla\phi| = \sqrt{3}$

So $\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

$ds = \dfrac{dx\,dy}{(\hat{n}\cdot\hat{k})}$

$\vec{F}\cdot\hat{n} = (xy\hat{i} + z^2\hat{j} + 2yz\hat{k})\dfrac{(\hat{i}+\hat{k}+\hat{j})}{\sqrt{3}}$

$\vec{F}\cdot\hat{n} = \dfrac{xy}{\sqrt{3}} + \dfrac{z^2}{\sqrt{3}} + \dfrac{2yz}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}(xy+z^2+2yz)$

$(\hat{n}\cdot\hat{k}) = 1/\sqrt{3}$

So $ds = \sqrt{3}\,dx\,dy$

Hence $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \frac{1}{\cancel{\sqrt{3}}}(xy+z^2+2yz)\,dx\,dy\,(\cancel{\sqrt{3}})$

$\displaystyle = \iint_S (xy+z^2+2yz)\,dx\,dy$

$\displaystyle = \int_0^1\int_0^{1-x} (xy+z^2+2yz)\,dx\,dy$

$(\because x+y+z=1$

$z = 1-x-y)$

$\displaystyle = \int_0^1\int_0^{1-x} \left[xy + (1-x-y)^2 + 2y(1-x-y)\right] dx\,dy$

$\displaystyle = \int_0^1\int_0^{1-x} \left[xy + 1 + x^2 + y^2 - 2x - \cancel{2y} + 2xy + \cancel{2y} - 2xy - 2y^2\right] dx\,dy$

$$= \int_0^1 \int_0^{1-x} [1 + x^2 + y^2 + xy - 2x - 2y^2)\,dx\,dy$$

$$= \int_0^1 \left[y + x^2 y + \frac{y^3}{3} + \frac{xy^2}{2} - 2xy - \frac{2y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[1 - x + x^2(1-x) + \frac{(1-x)^3}{3} - \frac{x(1-x)^2}{2} - 2x(1-x) - \frac{2(1-x)^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[1 - x + x^2 - x^3 + \frac{(1-x)^3}{3} - \frac{x}{2}(1 + x^2 - 2x) - 2x + 2x^2 \right] dx$$

$$= \int_0^1 \left[1 - 3x + 3x^2 - x^3 - \left(\frac{1 - x^3 - 3x(1-x)}{3} \right) - \frac{x}{2} - \frac{x^3}{2} + x^2 \right) dx$$

$$= \int_0^1 \left[1 - \frac{7x}{2} + 4x^2 - \frac{3x^3}{2} - \frac{1}{3} + \frac{x^3}{3} + \frac{3x}{3} + 3x^2 \right] dx$$

$$= \int_0^1 \left[\frac{2}{3} - \frac{5}{2}x + 5x^2 - \frac{7x^3}{6} \right] dx$$

$$= \left[\frac{2}{3}x - \frac{5}{2}\frac{x^2}{2} + \frac{5x^3}{3} - \frac{7x^4}{24} \right]_0^1$$

$$= \frac{2}{3} - \frac{5}{4} + \frac{5}{3} - \frac{7}{24}$$

$$= \frac{16 - 30 + 40 - 7}{24}$$

$$= \frac{5}{24}$$

So $\iint \vec{f} \cdot \hat{n}\, ds = I_1 + I_2 + I_3 + I_4$

$= 0 + 0 + 0 - \frac{1}{12} + \frac{5}{24}$

$= \frac{1}{8}$ Ans

Centre of Gravity & Centre of Mass :-

for a plane of area A, if the density at a point (x, y) be $\rho = f(x, y)$

then its

total mass $= m = \iint_A \rho \, dx dy$

Also if $(\bar{x}, \bar{y})$ is required centre of gravity,

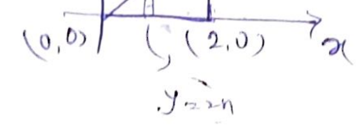
then $\bar{x} = \dfrac{\iint \rho x \, dx dy}{\iint \rho \, dx dy}$ and $\bar{y} = \dfrac{\iint \rho y \, dx dy}{\iint \rho \, dx dy}$

Q A triangular thin plate with vertices $(0,0)$, $(2,0)$ & $(2,4)$ has density $\rho = 1 + x + y$, then find -

(a) the mass of the plate.

(b) the position of its centre of gravity.

Sol^n $\rho = 1 + x + y$



(a) $M = \int_{x=0}^{2} \int_{y=0}^{2x} (1 + x + y) \, dx dy$

$M = \int_{x=0}^{2} \left[y + xy + \frac{y^2}{2} \right]_0^{2x} dx$

$M = \int_0^2 \left[2x + 2x^2 + 2x^2 \right] dx$

$M = \left[\frac{2x^2}{2} + \frac{4x^3}{3} \right]_0^2$

$M = 4 + \frac{32}{3} = \frac{44}{3}$ Ans

(b) the position of its centre of gravity.

Centre of mass for plate.

$$x = \frac{0+2+2}{3} = 4/3$$

$$y = \frac{0+0+4}{3} = 4/3$$

So
$$\bar{x} = \frac{\iint \rho x \, dx \, dy}{\iint \rho \, dx \, dy}$$

$$\bar{x} = \frac{3 \int_0^2 \int_0^{2x} (1+x+y)\, x \, dx \, dy}{44}$$

$$\bar{x} = \frac{3}{44} \int_0^2 \int_0^{2x} [x + x^2 + xy] \, dx \, dy$$

$$\bar{x} = \frac{3}{44} \int_0^2 \left[xy + x^2 y + \frac{xy^2}{2} \right]_0^{2x} dx$$

$$\bar{x} = \frac{3}{44} \int_0^2 [2x^2 + 2x^3 + 2x^3] \, dx$$

$$\bar{x} = \frac{3}{44} \int_0^2 [2x^2 + 4x^3] \, dx$$

$$\bar{x} = \frac{3}{44} \left[\frac{2x^3}{3} + \frac{4x^4}{4} \right]_0^2$$

$$\bar{x} = \frac{3}{44} \left[\frac{2 \times 8 + 3 \times 16}{12} \right] = \frac{3}{44} \times \frac{\cancel{64}^{16}(\cancel{4})}{\cancel{12}_{4}}$$

$$= 16/11 \text{ Ans}$$

for $\bar{y} = \frac{3}{44}\int_0^2\int_0^{2x}(1+x+y)\,y\,dx\,dy$

$$\bar{y} = \frac{3}{44}\int_0^2\int_0^{2x}[y+xy+y^2)\,dx\,dy$$

$$\bar{y} = \frac{3}{44}\int_0^2\left[\frac{y^2}{2}+\frac{xy^2}{2}+\frac{y^3}{3}\right]_0^{2x}dx$$

$$\bar{y} = \frac{3}{44}\int_0^2\left[2x^2+2x^3+\frac{8x^3}{\cancel{3}}\right]dx$$

$$\bar{y} = \frac{3}{44}\left[\frac{2x^3}{3}+\frac{2x^4}{\cancel{4}_2}+\frac{\cancel{8}^2x^4}{\cancel{4}\times3}\right]_0^2$$

$$\bar{y} = \frac{3}{44}\left[\frac{2\cdot8}{3}+\frac{16}{2}+\frac{2\times16}{3}\right]$$

$$\bar{y} = \frac{3}{\cancel{44}_{11}}\left[\frac{1}{3}+\frac{1}{2}+\frac{2}{3}\right]\times\cancel{16}^4$$

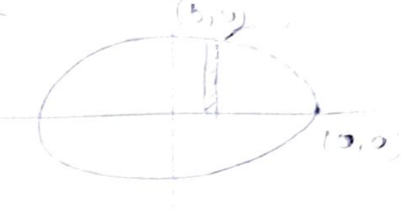
$$\bar{y} = \frac{3}{11}\left[\frac{2+3+4}{\cancel{6}_2}\right]\times\cancel{4}^2$$

$$\bar{y} = \frac{9\times2}{11} = \frac{18}{11} \text{ Ans}$$

Q Find the mass of an ellipatic plate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if the density at any point (x, y) on it is μxy.

Sol^n Curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\qquad M = \iint_R \rho \, dx dy$

$$M = 4 \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \mu xy \, dx dy$$



$$M = 4\mu \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} xy \, dx dy$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y = \frac{b}{a}\sqrt{a^2 - x^2}$

$$M = 4\mu \int_0^a \left[x \, y^2/2 \right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$M = 4\mu \int_0^a \left[\frac{x}{2} \cdot \frac{b^2}{a^2} (a^2 - x^2) \right] dx$$

$$M = \frac{2\mu b^2}{a^2} \int_0^a (a^2 x - x^3) \, dx$$

$$M = \frac{2\mu b^2}{a^2} \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a$$

$$M = \frac{2\mu b^2}{a^2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right]$$

$$M = 2\mu b^2 a^2 \times 1/4$$

$$M = \frac{\mu b^2 a^2}{2} \text{ Answer}$$