Unit-4 Multivariable Calculus

A function if two variables

$$\boxed{z = f(x,y)}$$

Ex $f(x,y) = x^2+y^2-5x^2y-3xy^2$

$f(0,1) = 0^2+1^2-5\times0^2\times1-3\times0\times1^2 = 1$

$f(-1,2) = (-1)^2+2^2-5\times(-1)^2\times2-3\times(-1)\times2^2 = 7$

Limit

L_1 when we solve x limit & then solve y limit

L_2 when we solve y limit & then solve x limit

Q Evalute $\lim\limits_{\substack{x\to2\\y\to1}} \dfrac{x^2+2y}{x+y^2}$

Sol^n

$$\lim_{y\to1}\left[\lim_{x\to2}\frac{x^2+2y}{x+y^2}\right]$$

$$\lim_{y\to1}\left[\frac{4+2y}{2+y^2}\right] = \frac{4+2}{2+1} = 6/3 = 2$$

Here $L_1 = L_2 = 2$ Ans

Q Evalute :- $\lim\limits_{\substack{x\to0\\y\to0}} \dfrac{y^2-x^2}{y^2+x^2}.$

Sol^n

$$\lim_{y\to0}\left[\lim_{x\to0}\frac{y^2-x^2}{y^2+x^2}\right]$$

$$\lim_{y\to0}\left[\frac{y^2-0}{y^2+0}\right] = \cancel{\frac{0^2-0}{0+0}} \quad \cancel{= 0 \text{ Answer } (L_1)}$$

$$\lim_{y\to0}(1) = 1 \quad (L_1)$$

Now for L_2

$$\lim_{x \to 0}\left[\lim_{y \to 0} \frac{y^2-x^2}{y^2+x^2}\right]$$

$$\lim_{x \to 0}\left[\frac{-x^2}{x^2}\right]$$

$$\lim_{x \to 0} = -1 \quad L_2$$

Here $L_1 \neq L_2$

limit does not exist.

Q Evalute $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4+y^2}$

Soln $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4+y^2} \Rightarrow \lim\limits_{y \to 0}\left[\lim\limits_{x \to 0} \frac{x^2 y}{x^4+y^2}\right]$

$$= \lim_{y \to 0} \frac{0}{0+y^2}$$

$$= 0 \quad (L_1)$$

for L_2

$$\lim_{x \to 0}\left[\lim_{y \to 0} \frac{x^2 y}{x^4+y^2}\right]$$

$$= \lim_{x \to 0}\left[\lim_{y \to 0} \frac{0}{x^4+\cancel{0}^2}\right]$$

$$= \lim_{x \to 0} [0]$$

$$= 0 \quad (L_2)$$

When $L_1 = L_2 = 0$
then check for
$x = mx$ & $x = mx^2$

$L_1 = L_2 = 0$

Now put $y = mx$.

$$\Rightarrow \lim_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = L_3$$

$$\Rightarrow \lim_{x\to 0} \frac{mx^3}{(x^2+m)\,x^2}$$

$$\Rightarrow \lim_{x\to 0} \frac{mx}{x^2+m}$$

$\Rightarrow$ 0 Answer (L_3)

again now, put $y = mx^2$

$$L_4 = \lim_{x\to 0} \frac{x^2(mx^2)}{x^4+(mx^2)^2}$$

$$L_4 = \lim_{x\to 0} \frac{m}{1+m^2}$$

Hence

$\Rightarrow L_1 = L_2 = L_3 \neq L_4$ Limit does not exist.

Q Evaluate $\lim_{\substack{x\to 0\\ y\to 0}} \frac{2x^3-y^3}{x^2+y^2}$

Soln

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{2x^3-y^3}{x^2+y^2}$$

for L_1

$$\Rightarrow \lim_{y\to 0}\left[\lim_{x\to 0} \frac{2x^3-y^3}{x^2+y^2}\right]$$

$$\Rightarrow \lim_{y\to 0} (-y)$$

$\Rightarrow$ 0 (L_1)

for L_2

$$\lim_{x\to 0}\left[\lim_{y\to 0} \frac{2x^3-y^3}{x^2+y^2}\right]$$

$$\lim_{x\to 0} (2x)$$

$$\lim_{x\to 0} \quad 0 \ (L_2)$$

put $y = mx$

$$\Rightarrow \lim_{x \to 0} \left(\frac{2x^3 - (mx)^3}{x^2 + (mx)^2} \right)$$

$$\Rightarrow \lim_{x \to 0} \frac{(2 - m)x}{1 + m^2}$$

$$\Rightarrow 0 \quad (L_3)$$

put $y = mx^2$

$$\lim_{x \to 0} \left[\frac{2x^3 - (mx^2)^3}{x^2 - (mx^2)^2} \right]$$

$$\lim_{x \to 0} \left(\frac{2 - mx^3}{1 - mx^2} \right) x$$

$$0 \quad (L_4)$$

Hence $L_1 = L_2 = L_3 = L_4$

So Limit exist (Limit $= 0$)

* Continuity :-

Q $f(x, y) = \dfrac{x^2 + 2y}{x + y^2}$ $\quad (x, y) \to (2, 1)$

find that $f(x,y)$ is continous or not ?

Sol^n

$$L_1 = \lim_{x \to 2} \left[\lim_{y \to 1} \frac{x^2 + 2y}{x + y^2} \right]$$

$$L_1 = \lim_{x \to 2} \left(\frac{x^2 + 2}{x + 1} \right)$$

$$L_1 = \frac{4 + 2}{2 + 1}$$

$$\boxed{L_1 = 2}$$

$$L_2 = \lim_{y \to 1} \left[\lim_{x \to 2} \frac{x^2 + 2y}{x + y^2} \right]$$

$$L_2 = \lim_{y \to 1} \left[\frac{4 + 2y}{2 + y^2} \right]$$

$$L_2 = \frac{4 + 2}{2 + 1}$$

$$\boxed{L_2 = 2}$$

So $L_1 = L_2$

then check $f(x, y)$ at $(2, 1)$

$$f(2,1) = \frac{4 + 2}{2 + 1}$$

$$\boxed{f(2,1) = 3}$$

So $L_1 = L_2 = f(x, y) = 2$

Hence $f(x, y)$ is continious.

Q

$$f(x) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & ; \text{ If } (x,y) \neq (0,0) \\ 0 & ; \text{ If } (x,y) \neq (0,0) \end{cases}$$

Continious?

Solⁿ for $L_1 = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{3x^2y}{x^2+y^2}\right]$ for $L_2 = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{3x^2y}{x^2+y^2}\right]$

$L_1 = \lim_{x\to 0}[0]$ $L_2 = \lim_{y\to 0}[0]$

$\boxed{L_1 = 0}$ $\boxed{L_2 = 0}$

Now $f(x,y)$ at $(0,0) = 0$

So $L_1 = L_2 = f(0,0) = 0$

~~[Hence $f(x,y)$ is continious.]~~

Now L_3 L_4

$y = mx$ $y = mx^2$

$L_3 = \lim_{x\to 0}\left[\left(\frac{3x^2 \cdot mx}{x^2+m^2x^2}\right)\right]$ $L_4 = \lim_{x\to 0} \frac{3x^2 \cdot mx^2}{x^2+m^2x^4}$

$L_3 = \lim_{x\to 0} \frac{3mx}{1+m^2}$ $L_4 = \lim_{x\to 0} \frac{3mx^2}{1+m^2x^2}$

$\boxed{L_3 = 0}$ $\boxed{L_4 = 0}$

Finally $L_1 = L_2 = L_3 = L_4 = 0 = f(0,0)$

Limit exists

& $f(x,y)$ is continious.

* Partial Differentiation :

$z = f(x,y)$ (two variable i.e. x, y)

partial derivative of z with respect to $x = \frac{\delta z}{\delta x} = f(x)$

partial derivative of z with respect to $y = \frac{\delta z}{\delta y} = f(y)$

Note

- $f_x = \frac{\delta z}{\delta x}$
- $f_y = \frac{\delta z}{\delta y}$

(first order)

- $\frac{\delta}{\delta x}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta x^2} = f_{xx}$
- $\frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta y\, \delta x} = f_{yx}$

(second order)

$$\frac{\delta}{\delta x}\left(\frac{\delta}{\delta y}\left(\frac{\delta f}{\delta z}\right)\right) = \frac{\delta^3 f}{\delta x\, \delta y\, \delta z}$$

Q Find the first order derivative of $u = \tan^{-1}(y/x)$

Sol^n $U = \tan^{-1}(y/x)$

$$U_x = \frac{\delta U}{\delta x} = \frac{1}{1+(y/x)^2} \cdot \quad y \cdot \ln(x)$$

$$U_x = \frac{x^2 y}{x^2+y^2} \cdot \frac{(-1)}{x^2} = \frac{-y}{x^2+y^2} \text{ Answer}$$

$$U_y = \frac{\delta U}{\delta y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} \cdot (1)$$

$$U_y = \frac{x}{x^2+y^2} \text{ Ans}$$

Q $U = e^{2x}\cos y$ find U_x & U_y

Sol^n $U = e^{2x}\cos y$

$$U_x = \frac{\partial(e^{2x}\cos y)}{\partial x} = \cos y \cdot 2e^{2x}$$

$$U_x = 2e^{2x}\cos y \text{ Ans}$$

$$U_y = \frac{\partial(e^{2x}\cos y)}{\partial y} = e^{2x}(-\sin y)$$

$$U_y = -e^{2x}\sin y \text{ Ans}$$

Q Let $f = y^x$ find $\frac{\partial^2 f}{\partial x \partial y}$ at $x = 2$ & $y = 1$

(RTU 2022)

Sol^n

$$f = y^x$$

$$\frac{\partial f}{\partial y} = \cancel{y^x \log x} \quad \cancel{y^x \log y} \quad (xy^{x-1})$$

$$\frac{\partial^2 f}{\partial x \partial y} = y^{x-1}\log y \cdot x + y^{x-1}$$

$$\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{(2,1)} = y^{2-1}\log(1)\cdot 2 + (1)^{2-1}$$

$$= 1 \text{ Answer}$$

Q. $f = e^{xyz}$ Calculate $\frac{\partial^3 f}{\partial x \, \partial y \, \partial z}$ [RTU 2024]

Sol^n $f = e^{xyz}$

$$\frac{\partial^3 f}{\partial x \, \partial y \, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial z}\right]\right) =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial e^{xyz}}{\partial z}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} xy \cdot e^{xyz}\right)$$

$$= \frac{\partial}{\partial x} x\left[y \cdot xz\, e^{xyz} + 1 \cdot e^{xyz}\right]$$

$$= \frac{\partial}{\partial x}\left[yz \cdot x^2 e^{xyz} + x e^{xyz}\right]$$

$$= yz(2x e^{xyz} + x^2 \cdot yz\, e^{xyz}] + x \cdot yz\, e^{xyz} + e^{xyz}]$$

$$= (2xyz + x^2y^2z^2)\, e^{xyz} + (1 + xyz) e^{xyz}$$

$$= e^{xyz}\left[3xyz + x^2y^2z^2 + 1\right]$$

Q If $U = f(r)$ where, $r^2 = x^2+y^2$

then prove that [PTU 2024]

$$\frac{\delta^2 U}{\delta x^2} + \frac{\delta^2 U}{\delta y^2} = f''(r) + \frac{1}{r} f'(r)$$

Sol^n $r^2 = x^2+y^2$ $U = f(r)$

$2r\frac{dr}{dx} = 2x$ $\frac{dU}{\delta x} = f'(r) \cdot \frac{dr}{dx}$

$\frac{dr}{dx} = x/r$ — (i)

LHS

$$\frac{\delta^2 U}{\delta x^2} + \frac{\delta^2 U}{\delta y^2} = \frac{\delta}{\delta x}\left(\frac{\delta U}{\delta x}\right) + \frac{\delta}{\delta y}\left(\frac{\delta U}{\delta y}\right)$$

$$= \frac{d(f'(r) \cdot dr/dx)}{dx} + \frac{d(f'(r)\, dr/dy)}{dy}$$

$$= f''(r) \cdot \frac{dr}{dx} \cdot \frac{dr}{dx} + f'(r) \cdot \frac{\delta^2 r}{\delta x^2} + f''(r) \frac{\delta r}{\delta y} \cdot \frac{dr}{\delta y}$$

$$+ f'(r) \frac{\delta^2 r}{\delta y^2}$$

$$= f''(r)\left[\left(\frac{dr}{dx}\right)^2 + \left(\frac{dr}{dy}\right)^2\right] + f'(r)\left[\frac{\delta^2 r}{\delta x^2} + \frac{\delta^2 y}{\delta y^2}\right] \quad \text{— (ii)}$$

(i) $\frac{\delta r}{\delta x} = \frac{\delta(\sqrt{x^2+y^2})}{\delta x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{r}$

(ii) $\frac{\delta r}{\delta y} = \frac{\delta \sqrt{x^2+y^2}}{\delta y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{r}$

Now $\frac{\delta^2 r}{\delta x^2} = \frac{\delta (x/r)}{\delta x} = \frac{r \cdot \frac{d(x)}{dx} - x \cdot \frac{dr}{dx}}{r^2}$

$$= \frac{r - x \cdot x/r}{r^2}$$

$\left(\frac{dr}{dx} = x/r \right)$

$$= \frac{r^2 - x^2}{r^3}$$

Similarly for $\frac{\delta^2 r}{\delta y^2} = \frac{\delta (y/r)}{\delta y^2} = \frac{r \cdot 1 - y \cdot \frac{dr}{dy}}{r^2}$

$\left(\frac{dy}{dx} = y/r \right)$ $\qquad \frac{r^2 - y^2}{y^3}$

So by eq (ii)

$\Rightarrow f''(r) \left[\frac{x^2}{r^2} + \frac{y^2}{r^2} \right] + f'(r) \left[\frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} \right]$

$\Rightarrow f''(r) \left[\frac{x^2 + y^2}{r^2} \right] + f'(r) \left[\frac{2r^2 - (x^2 + y^2)}{r^3} \right]$

$\because x^2 + y^2 = r^2$

$\Rightarrow f''(r)(1) + f'(r) \times \frac{1}{r}$

$\Rightarrow f''(r) + \frac{1}{r} f'(r)$ Ans

Homogeneous function :

(RTU-2023)

$$f(x,y) = a_0 x^n y^0 + a_1 x^{n-1} y^1 + a_2 x^{n-2} y^2 + \dots + a_n x^0 y^n$$

Summition of power of x & y should be constant in every term.

$$= x^n \left[a_0 \left(\frac{y}{x}\right)^0 + a_1 \left(\frac{y}{x}\right)^{n-1} + \dots + a_n \left(\frac{y}{x}\right)^n \right]$$

$$\boxed{f(x,y) = x^n \phi(y/x)}$$

(Here $n \to$ Real number +ve / -ve / 0)

(Homogeneous function)

$\Rightarrow$ n is degree of function

Q. $f(x,y) = \dfrac{\sqrt{x} + \sqrt{y}}{x^2 + y^2}$ Homo f^n ?

Solⁿ

$$f(x,y) = \frac{\sqrt{x}}{x^2} \left[\frac{1 + \sqrt{y/x}}{1 + (y/x)^2} \right]$$

$$f(x,y) = \frac{1}{x^{3/2}} \phi(y/x)$$

It is a homogeneous function.
Degree $= -3/2$

(Euler's theorem)

** Theorem :-

If $u = f(x,y)$ is a Homo. function

then $\boxed{x \dfrac{\delta f}{\delta x} + y \dfrac{\delta f}{\delta y} = nf}$

Proof

Let $f(x,y) = f = x^n \phi(y/x)$

Let LHS

$$\frac{\delta f}{\delta x} = \frac{\delta(x^n \phi(y/x))}{\delta x} = n(x^{n-1})\,\phi(y/x) + \frac{-y}{x^2}\, x^n \phi'(y/x)$$

$$\cancel{\frac{\delta f}{\delta x} = \phi(y/x)\left[n x^{n-1} - y x^{n-2}\right]}$$

$$\frac{\delta f}{\delta x} = x^{n-2}\left[n x \phi(y/x) - y\,\phi'(y/x)\right]$$

Now

$$\frac{\delta f}{\delta y} = x^n \phi'(y/x) \cdot \frac{1}{x} = x^{n-1} \phi'(y/x)$$

Hence

$$\Rightarrow x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y}$$

$$\Rightarrow x\left[x^{n-2}\left(n x \phi(y/x) - y\,\phi'(y/x)\right)\right] + y.\left[x^{n-1}\phi'(y/x)\right]$$

$$\Rightarrow n x^n \phi(y/x) - x^{n-1}\cancel{y\phi'(y/x)} + \cancel{y x^{n-1}\phi'(y/x)}$$

$$\Rightarrow n x^n \phi(y/x)$$

$$\Rightarrow nf$$

RHS

Hence Proved

Imp Results

(1.) $x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} + z\frac{\delta f}{\delta z} = nf$

(2.) $x^2\frac{\delta^2 f}{\delta x^2} + 2xy\frac{\delta^2 f}{\delta x \delta y} + y^2\frac{\delta^2 f}{\delta y^2} = n(n-1)f$

Q If $U = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$ (RTU -2015)

Prove that $x\frac{\delta U}{\delta x} + y\frac{\delta U}{\delta y} = \frac{1}{20}\tan U$

by theorem $x\frac{\delta U}{\delta x} + y\frac{\delta U}{\delta y} = nf$

let f^n $U = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$

$\sin U = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$ → Homogenous fn

$\Rightarrow V = \sin U = x^{1/4-1/5}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$

$\Rightarrow V = \sin U = x^{1/20}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$

$\Rightarrow V = \sin U = x^{1/20}\phi(y/x)$

$n = 1/20$

$\frac{\delta \sin U}{\delta x} =$

$\frac{\delta \sin U}{\delta y} =$

by theorem

$x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} = \frac{1}{20}\times v$

$x\frac{\delta \sin U}{\delta x} + y\frac{\delta \sin U}{\delta y} = \frac{1}{20}\sin U$

$$\Rightarrow x\cos u \frac{\partial u}{\partial x} + y\cos u \frac{\partial u}{\partial y} = \frac{1}{2a}\sin u$$

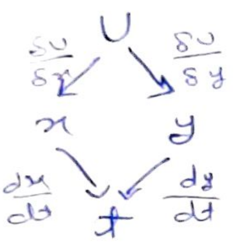
$$\Rightarrow \cos u \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right] = \frac{1}{2a}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2a}\tan u$$

Hence Proved

* Total Derivative :-

Case 1 (a)



$u = f(x, y)$

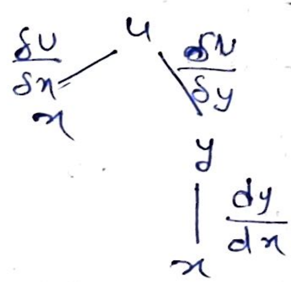
$x = g(t)$

$y = h(t)$

Total Deri $\left(\frac{du}{dt}\right) = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$

(b) $U = f(x, y)$

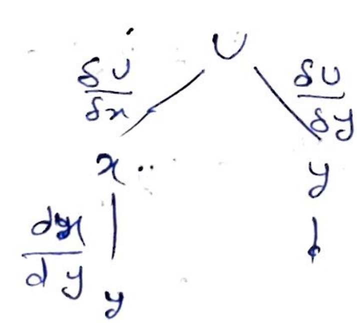
$y = g(x)$



Total Deri. $= \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \times \frac{dy}{dx}$

(c) $U = f(x, y)$

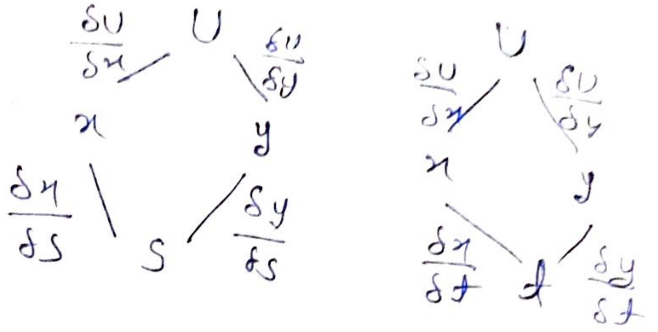
$x = g(y)$



Total Deri. $= \frac{du}{dy} = \frac{\partial u}{\partial x} \times \frac{dx}{dy} + \frac{\partial u}{\partial y}$

Case 2

$$U = f(x, y)$$
$$x = g(s, t)$$
$$y = h(s, t)$$



$$\frac{dU}{ds} = \frac{\delta U}{\delta x} \times \frac{\delta x}{\delta s} + \frac{\delta U}{\delta y} \times \frac{\delta U}{\delta s} \quad \Bigg| \quad \frac{dU}{dt} = \frac{\delta U}{\delta x} \times \frac{\delta x}{\delta t} + \frac{\delta U}{\delta y} \times \frac{\delta y}{\delta t}$$

Case 3

$f(x,y) \rightarrow$ implicit function.

$$f(x,y) = 0$$

$$\frac{dy}{dx} = -\frac{\delta f / \delta x}{\delta f / \delta y}$$

Ex $x^3 + y^3 - 3xy = 0$ (Implicit f^n)

$$\frac{\delta f}{\delta x} = 3x^2 \cancel{+} 3y \quad \Bigg| \quad \frac{\delta f}{\delta y} = 3y^2 - 3x$$

$$\frac{dy}{dx} = -\frac{\delta f/\delta x}{\delta f/\delta y} = -\frac{(3x^2 - 3y)}{3y^2 - 3x}$$

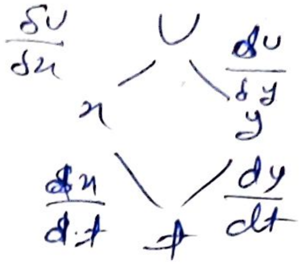
$$\frac{dy}{dx} = \cancel{3y - 3x} \quad 3y$$

$$\boxed{\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}}$$ Ans

Q $U = x^2y - y^2$ Total Derivative at $t = 0$?

$x = \sin t$

$y = e^t$



So $\frac{\delta U}{\delta x} = \frac{\delta(x^2y - y^2)}{\delta x} = 2xy$

$$\frac{\delta U}{\delta y} = x^2 - 2y$$

$$\frac{dx}{dt} = \frac{d(\sin t)}{dt} = \cos t \qquad \frac{dy}{dt} = \frac{d(e^t)}{dt} = e^t$$

Formula is

$$\frac{dU}{dt} = \frac{\delta U}{\delta x} \times \frac{dx}{dt} + \frac{\delta U}{\delta y} \times \frac{dy}{dt}$$

$$\frac{dy}{dt} = 2xy \times \cos t + (x^2 - 2y)\, e^t$$

$$\frac{dU}{dt} = 2xy \cos t + (x^2 - 2y)e^t$$

at $t = 0$

at $t = 0$

$x = \sin t = 0$

$y = 1$

$$\frac{dU}{dt} = 2xy + (x^2 - 2y)(1)$$

$$\frac{dU}{dt} = -2 \text{ Ans}$$

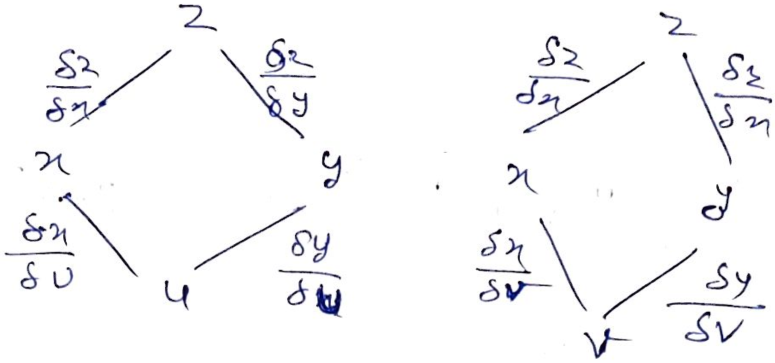
Q $z = f(x, y)$, where

$x = e^{u}\cos v$,

$y = e^{u}\sin v$

Show that $y\dfrac{\delta z}{\delta u} + x\dfrac{\delta z}{\delta v} = e^{2u}\dfrac{\delta z}{\delta y}$

Sol^n



$\dfrac{\delta z}{\delta x}$, $\dfrac{\delta z}{\delta y}$ (Unknown)

So

$$\frac{\delta x}{\delta u} = \frac{\delta(e^{u}\cos v)}{\delta u} = e^{u}\cos v$$

$$\frac{\delta y}{\delta u} = \frac{\delta(e^{u}\sin v)}{\delta u} = e^{u}\sin v$$

and. $\dfrac{\delta x}{\delta v} = -\sin v(e^{u})$ $\qquad \dfrac{\delta y}{\delta v} = \cos v\,(e^{u})$

Formula

$$\frac{\delta z}{\delta u} = \frac{\delta z}{\delta x} \times e^{u}\cos v + \frac{\delta z}{\delta y} \cdot e^{u}\sin v$$

$$\frac{\delta z}{\delta v} = \frac{\delta z}{\delta x}\cos v(e^{u}) + -\frac{\delta z}{\delta y}\sin v\, e^{u}$$

taking LHS

$= y\frac{\delta z}{\delta u} + x\frac{\delta z}{\delta v}$

$= y.e^{u}\cos v\frac{\delta z}{\delta x} + ye^{u}\sin v\frac{\delta z}{\delta y} + x\cos v.e^{u}\frac{\delta z}{\delta x} -$

$x\sin v e^{u}\frac{\delta z}{\delta y}$

$= e^{u}\sin v.\frac{\delta z}{\delta y}[y-x] + e^{u}\cos v\frac{\delta z}{\delta x}[y+x]$

$x = e^{u}\cos v \qquad y = e^{u}\sin v$

$x^2+y^2 = e^{2u}$

$= e^{u}\sin v\frac{\delta z}{\delta y}.e^{u}\sin v - \cancel{e^{u}\sin v\frac{\delta z}{\delta x}[e^{u}\cos v]}$

$+ e^{u}\cos v\frac{\delta z}{\delta y}[e^{u}\cos v] + \cancel{e^{u}\cos v\frac{\delta z}{\delta x}e^{u}\sin v}$

$= e^{2u}\frac{\delta z}{\delta y}[\sin^2 v + \cos^2 v]$

$= e^{2u}\frac{\delta z}{\delta y}$ Ans

RHS

Q $x^5 + y^5 = 5a^3x^2$ Find $\frac{dy}{dx} = ?$

Solⁿ $5x^4\,dx + 5y^4\,dy = 10a^3x\,dx$

$\cancel{5}y^4\,dy = \cancel{5}x[2a^3 - x^3]\,dx$

$$\frac{dy}{dx} = \frac{(2a^3 - x^3)x}{y^4}$$

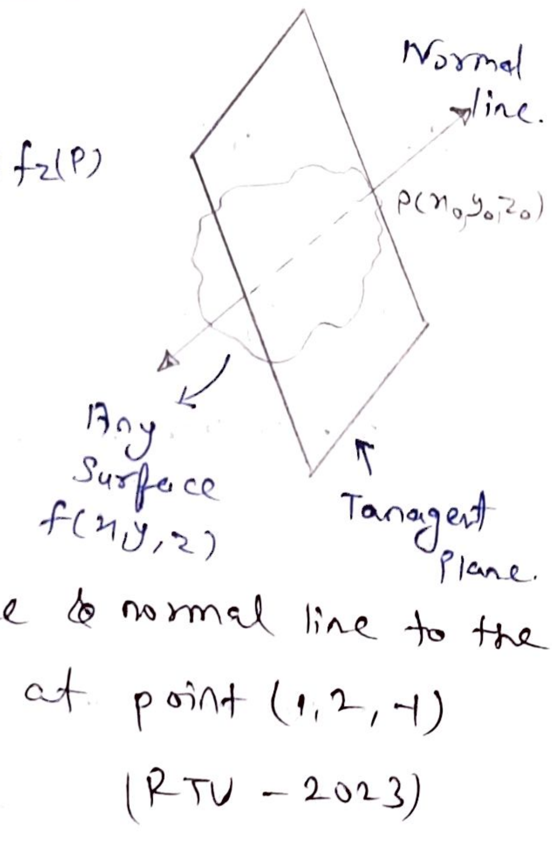
* Tangent Plane & Normal line :-

Equation of tangent Plane

$(x - x_0) f_x(P) + (y - y_0) f_y(P) + (z - z_0) f_z(P)$

Equation of Normal line

$$\frac{x - x_0}{f_x(P)} = \frac{y - y_0}{f_y(P)} = \frac{z - z_0}{f_z(P)}$$



Q Find the eq. of tangent plane & normal line to the surface $x^3 + y^3 + 3xyz = 3$ at point $(1, 2, -1)$

(RTU - 2023)

$f(x,y,z) = x^3 + y^3 + 3xyz - 3$

$(x_0, y_0, z_0) \to (1, 2, -1)$

$f_{(x)} = 3x^2 + 3yz$

$f_{(x)P} = 3 - 6 = -3$

$f_{(y)} = 3y^2 + 3xz$

$f_{(y)P} = 3\times4 - 3\times1 = 9$

$f_{(z)} = 3xy$

$f_{(z)P} = 3\times1\times2 = 6$

So tangent Plane

~~$(x-1)[3x^2+3yz]$~~

$= (x-1)\times -3 + (y-2)9 + (z+1)6$

$\Rightarrow 6z + 9y - 3x - 9 = 0$

$\Rightarrow \boxed{x - 3y - 2z + 3 = 0}$

& eq of Normal line

$$\boxed{\frac{x-1}{-3} = \frac{y-2}{9} = \frac{z+1}{6}}$$

$-3 - 18 + 6$

Q Find the equation of tangent plane and normal to the surface $e^{xy^2} + zy^4 - 61 + \frac{z^2}{x+1}$

at point $(0,-2,6)$

Sol^n Surface $e^{xy^2} + zy^4 - 61 + \frac{z^2}{x+1}$

$$f(x) = e^{xy^2} \cdot y^2 + \frac{z^2}{(x+1)^2} \Rightarrow f(x)_p = 4 + \frac{36}{1} = 40$$

$$f(y) = e^{xy^2} \cdot 2y + 4zy^3 \Rightarrow f(y)_p = -4 - 8\times4\times6 = -196$$

$$f(z) = y^4 - \frac{2z}{x+1} \Rightarrow f(z)_p = 16 - 12 = 4$$

So eq of tangent plane

$$\Rightarrow (x-0)40 + (y+2)\times -196 + (z-6)4 = 0$$

$\Rightarrow$

eq to Normal line

$$\boxed{\frac{x-0}{40} = \frac{y+2}{-196} = \frac{z-6}{4}}$$ Ans

* Gradient " Scaler Product "

liner differential operator ⇒ "del"

denoted by " ∇ "

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}}$$

(Represent direction)

Scaler quantity ⇒ ϕ = have magnitude.
(function)

$\text{Grad}(\phi) = \nabla\phi$

$$\boxed{\text{Grad}(\phi) = \nabla\phi = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta z}{\delta x}\hat{k}}$$

Q Find the gradient of function $f(x,y,z)$
$f(x,y,z) = x^2y^2 + xy^2 - z^2$ at $(3,1,1)$ (RTU - 2024)

Soln
$$\frac{\delta\phi}{\delta x} = 2xy^2 + y^2 \text{ at } (3,1,1) \Rightarrow 6 + 1 = 7 = \frac{\delta\phi}{\delta x}(P)$$

$$\frac{\delta\phi}{\delta y} = +2x^2y + 2xy \Rightarrow 18 + 6 = 24 = \frac{\delta\phi}{\delta y}(P)$$

$$\frac{\delta\phi}{\delta z} = -2z \Rightarrow -2 = \frac{\delta\phi}{\delta z}(P)$$

$$\boxed{\text{Grad}(\phi) = 7\hat{i} + 24\hat{j} - 2\hat{k}}$$ Answer

Q Find the gradient (grad(f)) for

$$f(x,y,z) = (x^2+y^2+z^2)^{-5/2} = \frac{1}{r^5}$$

Sol^n $\frac{\delta f}{\delta x} = -\frac{5}{2}(x^2+y^2+z^2)^{-7/2} \cdot 2x = -5x(x^2+y^2+z^2)^{-7/2}$

$\frac{\delta f}{\delta y} = -\frac{5}{2}(x^2+y^2+z^2)^{-7/2} \cdot 2y = -5y(x^2+y^2+z^2)^{-7/2}$

$\frac{\delta f}{\delta z} = -\frac{5}{2}(x^2+y^2+z^2)^{-7/2} \cdot 2z = -5z(x^2+y^2+z^2)^{-7/2}$

$\text{grad}(f) = -5[x\hat{i} + y\hat{j} + z\hat{k}](x^2+y^2+z^2)^{-7/2}$ Answer

or

$\text{grad}(f) = -5\vec{r}\left(\frac{1}{r^7}\right)$ Ans $= -5\hat{r} \cdot \frac{1}{r^6}$

* **Directional Derivative:-**

Directional Derivative of ϕ at any point in the direction of $\vec{a}$ is $\nabla\phi \cdot \hat{a}$

Q find the directional derivative of the scaler function $f(x,y,z) = x^2 + 2y^2 + z$ at the point $(1,1,2)$ in the direction of the vactor $(3\hat{i} - 4\hat{j})$ (RTU-2022)

Sol^n $f(x,y,z) = x^2 + 2y^2 + z$

$\frac{\delta f}{\delta x} = 2x$ $DD\vec{a} = \nabla\phi \cdot \hat{a}$

$DD\vec{a} = (2x\hat{i} + 4y\hat{j} + 1\hat{k}) \cdot \frac{3\hat{i} - 4\hat{j}}{5}$

$\frac{\delta f}{\delta y} = 4y$ $DD\vec{a} = \frac{6 - 16}{5} \Rightarrow -2$ Answer

$\frac{\delta f}{\delta z} = 1$

Divergence & Curl :-

Divergence => Dot Product

$$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

$$\text{div}\,\vec{f} = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\cdot(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}) = \vec{\nabla}\cdot\vec{f}$$

$$\text{div}\,\vec{f} = \frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y} + \frac{\delta f_3}{\delta z}$$

Note

$\vec{\nabla}\cdot\vec{f} = 0 \Rightarrow$ field $\Rightarrow$ Solenoidal.

Curl => "Cross product"

$$\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

$$\text{curl}\,\vec{f} = \vec{\nabla}\times\vec{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Note $\vec{\nabla}\times\vec{f} = 0 \Rightarrow$ field = irrotational.

Q If $\vec{f} = x^2y\,\hat{i} - 2xy^2z\,\hat{k} + 3x^2z\,\hat{k}$

find div $\vec{f}$ ~~and~~ ~~curl f~~ at point $(3,-1,-2)$

Sol^n $\vec{f} = x^2y\,\hat{i} - 2xy^2z\,\hat{k} + 3x^2z\,\hat{k}$ (RTU 2023)

$f_1 = x^2y$ $\quad f_3 = 3x^2z$

$f_2 = -2xy^2z$

$$\text{div}\,\vec{f} = 2xy - 4xyz + 3x^2$$

at point $(3,-1,-2)$

$$\text{div}\,\vec{f} = -6 - 24 + 27 = -3 \text{ Ans}$$

Q Curl of vector $\vec{v} = 2x^2\hat{i} + 3z^2\hat{j} + y^3\hat{k}$ at $x = y = z = 1$ is ______ (PTU. 2022)

Sol^n $\text{Curl}(\vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$f_1 = 2x^2$

$f_2 = 3z^2$

$f_3 = y^3$

$= \hat{i}\left(\frac{\delta f_3}{\delta y} - \frac{\delta f_2}{\delta z}\right) - \hat{j}\left(\frac{\delta f_3}{\delta x} - \frac{\delta f_1}{\delta z}\right)$

$+ \hat{z}\left(\frac{\delta f_2}{\delta x} - \frac{\delta f_1}{\delta y}\right)$

$= \hat{i}(3y^2 - 6z) - \hat{j}(0 - 0) + \hat{z}(0 - 0)$

$\Rightarrow \hat{i}(3y^2 - 6z)$ at $x = y = z = 1$

$\Rightarrow (3 - 6)\hat{i}$

$\Rightarrow -3\hat{i}$ Ans

Q Prove that $\text{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{Curl}\,\vec{a} - \vec{a} \cdot \text{Curl}\,\vec{b}$

Sol^n LHS

$\text{div}(\vec{a} \times \vec{b})$

$\Rightarrow \vec{\nabla} \cdot (\vec{a} \times \vec{b})$

$\Rightarrow \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \cdot (\vec{a} \times \vec{b})$

$\Rightarrow \vec{\nabla} \cdot (\vec{a} \times \vec{b}) = [\vec{\nabla}\, \vec{a}\, \vec{b}]$

$\Rightarrow$

$$\Rightarrow \hat{i}\,\frac{\delta}{\delta x}(\vec{a}\times\vec{b}) + \hat{j}\,\frac{\delta}{\delta y}(\vec{a}\times\vec{b}) + \hat{k}\,\frac{\delta(\vec{a}\times\vec{b})}{\delta z}$$

$$\Rightarrow \sum i\,\frac{\delta(\vec{a}\times\vec{b})}{\delta x}$$

$$\Rightarrow \sum i\cdot\left(\vec{a}\times\frac{d\vec{b}}{dx} + \vec{b}\times\frac{d\vec{a}}{dx}\right)$$

$$\Rightarrow \sum i\cdot\left(\vec{a}\times\frac{d\vec{b}}{dx}\right) + \sum i\cdot\left(\vec{b}\times\frac{d\vec{a}}{dx}\right)$$

$\because \vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot c$

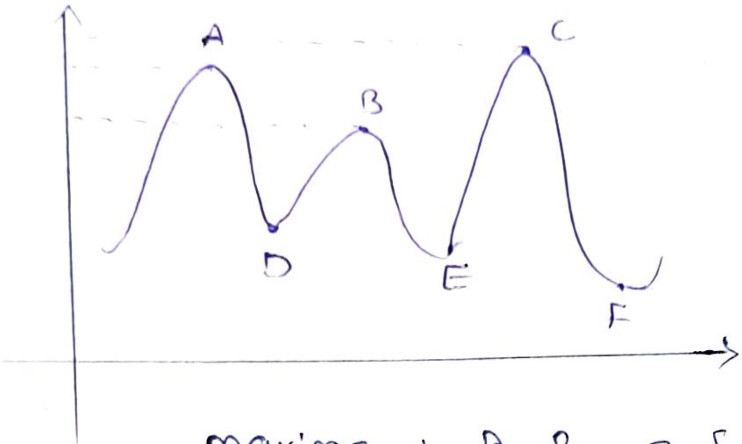
$$\Rightarrow \sum\left(i\times\frac{d\vec{a}}{dx}\right)\cdot\vec{b} + \left(-\sum i\cdot\left(\frac{d\vec{b}}{dx}\times\vec{a}\right)\right)$$

$$\Rightarrow \sum\underbrace{\left(i\times\frac{d\vec{a}}{dx}\right)}_{\text{curl }\vec{a}}\cdot\vec{b} - \sum\underbrace{\left(i\times\frac{d\vec{b}}{dx}\right)}_{\text{curl } b}\cdot\vec{a}$$

$$\Rightarrow \vec{b}\cdot\text{curl }\vec{a} - \vec{a}\cdot\text{curl }\vec{b}$$

Hence Proved.

★ Maxima & Minima :-



maxima => A, B, C [local maxima]

minima => D, E, F (Local minima]

Here C is Global max.

& F is Global min.

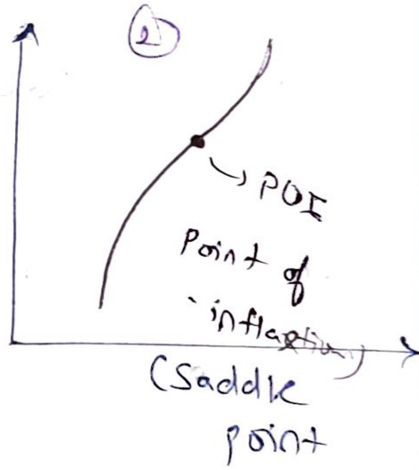
Note ①

$f'(\alpha) = f''(\alpha) = f'''(\alpha) = \dots = f^{n-1}(\alpha) = 0$

and $f^n(\alpha) \neq 0$

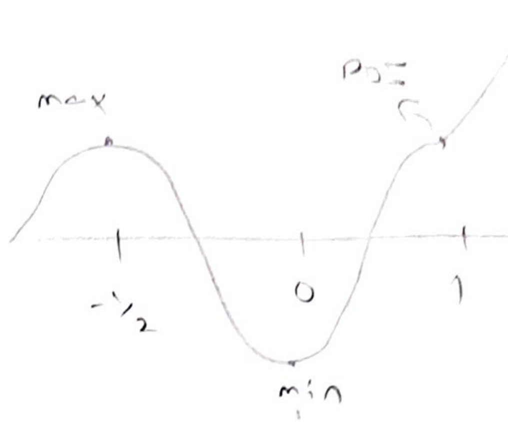
n - even	$f^n(\alpha) > 0$	local max
n - even	$f^n(\alpha) < 0$	local min
n - odd	$f^n(\alpha) > 0$	P.O.I & inc
n - odd	$f^n(\alpha) < 0$	P.O.I & Dec

②



Q Find the greatest and the least value of the function $f(x) = 3x^5 - 15x^4 + 10x^2$

Sol^n $f(x) = 8x^5 - 15x^4 + 10x^2$



$f'(x) = 40x^4 - 60x^3 + 20x$

Now $f'(x) = 0$

$40x^4 - 60x^3 + 20x = 0$

$20x(2x^3 - 3x^2 + 1) = 0$

$x = 0$ | $(x-1)[2x^2 - x - 1] = 0$

$(x-1)[2x^2 - 2x + x - 1] = 0$

$2x(x-1) + 1(x-1) = 0$

$(x-1)(x-1)(2x+1) = 0$

$x = 0$; $x = 1$; $x = -1/2$ (3 points)

Now second derivative

$f''(x) = 160x^3 - 180x^2 + 20$

at $x = 0$

$160 \times 0 - 180 \times 0 + 20 = f''(0)$

$f''(0) = 20$

$f''(0) > 0$

[minima]

at $x = 1$

$f''(1) = 160 - 180 + 20$

$f''(1) = 0$

at $x = 1$

at $x = -1/2$

$f''(-1/2) = \overset{-20}{160 \times -\frac{1}{8}} - \overset{45}{180 \times \frac{1}{4}} + 20$

$f''(-1/2) = -45$

$f''(-1/2) < 0$

(maxima)

Third derivative

$f'''(x) = 480x^2 - 360x$

$f'''(1) = 480 - 360$

$\boxed{f'''(1) > 0}$

Point of inflaxtion and increasing.

Maxima & minima :-

(For function of two variable)

$f(x, y) \rightarrow f^n$ of two variable

$$\left.\frac{\delta f}{\delta x}\right|_{P(a,b)} = 0 \qquad \left.\frac{\delta f}{\delta y}\right|_{P(a,b)} = 0$$

$$\frac{\delta^2 f}{\delta x^2} = r \cdot \quad \frac{\delta^2 f}{\delta y^2} = t, \quad \frac{\delta^2 f}{\delta x \delta y} = s.$$

$$A = \begin{bmatrix} r & s \\ s & t \end{bmatrix}$$

Hessian Matrix

(i) $A_1 = |r|$

(ii) $A_2 = \begin{vmatrix} r & s \\ s & t \end{vmatrix}$

Case 1

$\left.\begin{matrix} A_1 \rightarrow (+) \\ A_2 \rightarrow (+) \end{matrix}\right\}$ matrix is (+)ive definite at point $P(a,b) \rightarrow$ minima

Case 2

$\left.\begin{matrix} A_1 \rightarrow (-) \\ A_2 \rightarrow (+) \end{matrix}\right\}$ matrix is (-)ive definite at point $P(a,b) \rightarrow$ maxima

Case 3 other then (1) & (2) at point $(a,b) \rightarrow$ Saddle point

$A_1 \rightarrow 0$ or, $A_2 \rightarrow 0$

$A_1 \rightarrow +$
$A_2 \rightarrow +$
$A_3 \rightarrow +$
positive definite
(minima)

$A_1 \rightarrow -$
$A_2 \rightarrow +$
$A_3 \rightarrow -$
Negative definite
maxima.

Q $f(x,y) = 20x + 26y + 4xy - 4x^2 - 3y^2$

find (x,y) to maximize

Solⁿ

$$f(x,y) = 20x + 26y + 4xy - 4x^2 - 3y^2$$

$$\frac{\delta f}{\delta x} = 20 + 4y - 8x = 0$$

$$y - 2x + 5 = 0 \quad \text{— (i)}$$

$$\frac{\delta f}{\delta y} = 26 + 4x - 6y = 0$$

$$2x - 3y + 13 = 0 \quad \text{— (ii)}$$

add (i) and (ii)

$$\begin{array}{r} y - 2x + 5 = 0 \\ + \; 2x - 3y + 13 = 0 \\ \hline -2y + 18 = 0 \end{array}$$

$$y = 18/2 = 9$$

$$\boxed{y = 9}$$

& $y - 2x + 5 = 0$

$$y = 9$$

$$9 - 2x + 5 = 0$$

$$14 = 2x$$

$$\boxed{x = 7}$$

point $(7, 9) \to (a, b)$

Now

$$\frac{\delta^2 f}{\delta x^2} = \frac{\delta(20 + 4y - 8x)}{\delta x^2}$$

$$\frac{\delta^2 f}{\delta x^2} = -8 = r \;;\; \frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = 4 = s$$

&

$$\frac{\delta^2 f}{\delta y^2} = -6 = t$$

Now Matrix $A = \begin{bmatrix} -8 & 4 \\ 4 & -6 \end{bmatrix}$

$A_1 = |-8| = -8$

$A_2 = \begin{vmatrix} -8 & 4 \\ 4 & -6 \end{vmatrix} = (48 - 16) = 32$ (+)ive

$\left.\begin{array}{l} A_1 \rightarrow (-)\text{ive} \\ A_2 \rightarrow (+)\text{ive} \end{array}\right\}$ Maxima at $(7, 9)$
matrix is negative definite

Q2 Show that $u = x^3y^2(1-x-y)$ is max at $x = 1/2$ and $y = 1/3$

Soln $f(x,y) = x^3y^2(1-x-y)$

$$\frac{\delta f}{\delta x} = 3x^2y^2(1-x-y) + (-1)x^3y^2$$

$$\frac{\delta f}{\delta x} = 3x^2y^2(1-x-y) - x^3y^2 = 0$$

$$\frac{\delta f}{\delta x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \quad \text{— (i)}$$

&

$$\frac{\delta f}{\delta y} = 2x^3y(1-x-y) + (-1)x^3y^2$$

$$\frac{\delta f}{\delta y} = 2x^3y - 2x^4y - 3x^3y^2 = 0 \quad \text{— (ii)}$$

by eq (i) - (ii)

$$3x^2y^2 - 3x^3y^2 - 3x^2y^3$$
$$2x^3y - 2x^4y - 2x^3y^2$$
$$+ \quad + \quad +$$

$$3x^2y^2 - 2x^3y - x^3y^2 + 2x^4y - 3x^2y^3 = 0$$

$$\frac{\delta f}{\delta x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2[3 - 4x - 3y] = 0 \quad \text{--- (ii)}$$

& $$\frac{\delta f}{\delta y} = (2x^3y - 2x^4y - 3x^3y^2) = 0$$

$$= xy[2x^2 - 2x^3 - 3x^2y] = 0$$

$$x^3y[2 - 2x - 3y] = 0 \quad \text{--- (iii)}$$

by eq (ii) & (iii)

$(x=0$ & $y=0)$ ✗

$$4x + 3y - 3 = 0$$

$$\underline{\begin{array}{r} 3y + 2x - 2 = 0 \\ - \quad - \quad + \end{array}}$$

$$2x = 1$$

$$\boxed{x = 1/2}$$

& $$4(1/2) + 3y - 3 = 0$$

$$3y - 1 = 0$$

$$\boxed{y = 1/3}$$

point $(1/2, 1/3)$

$$\left.\frac{\delta^2 f}{\delta x^2}\right)_{(1/2, 1/3)} = 6xy^2 - 12x^2y - 6xy^3 = -1/9 = r$$

$$\left.\frac{\delta^2 f}{\delta y^2}\right)_{(1/2, 1/3)} = 2x^3 - 2x^4 - 6x^3y = -1/8 = t$$

$$\left.\frac{\delta^2 f}{\delta x \delta y}\right|_{(1/2, 1/3)} = 6x^2y - 8x^3y - 9x^2y^2 = -1/12 = s$$

Matrix $\begin{bmatrix} -1/9 & -1/12 \\ -1/12 & -1/8 \end{bmatrix}$

$A_1 = -1/9 < 0$

$A_2 = \begin{vmatrix} -1/9 & -1/12 \\ -1/12 & -1/8 \end{vmatrix} = \frac{1}{9} \times \frac{1}{8} - \frac{1}{144}$

$= \frac{1}{72} - \frac{1}{144} > 0$

$A_2 > 0$

So at $(1/2, 1/3)$, we will find maxima.

* **Langrang's multiple method:-**

$z = f(x,y)$ -> function with two variables.

max / min => $z = f(x,y)$ => objective function.

Subject to $g(x,y) = 0$ → constrain (equality)

$x, y \geq 0$

Q Find the largest product of the no. x, y, z when $x^2 + y^2 + z^2 = 9$

Sol^n Max $= P = x \cdot y \cdot z$

obj. => $x^2 + y^2 + z^2 = 9$

$L(x,y,z,\lambda) = xyz + \lambda(x^2 + y^2 + z^2 - 9)$. — (i)

λ = langrang's multipler

$\frac{\delta L}{\delta x} = 0 \Rightarrow yz + 2\lambda x = 0$ — (ii)

$\frac{\delta L}{\delta y} = xz + 2\lambda y = 0$ — (iii) $\frac{\delta L}{\delta z} = xy + 2\lambda z = 0$ — (iv)

$$\frac{\delta L}{\delta \lambda} \quad (x^2+y^2+z^2-9) = 0 \quad \text{---(v)}$$

eq. (ii) $\times x$ − eq. (iii) $\times y$

$\Rightarrow (xyz + 2\lambda x^2) - (xyz + 2\lambda y^2) = 0$

$\Rightarrow 2\lambda(x^2-y^2) = 0$

$x^2 = y^2$ ---(vi)

Similarly from eq (iii) & (iv) $y^2 = z^2$ ---(vii)

Similarly from eq (vi) & (vii) $x^2 = z^2$ ---(viii)

So $x^2 = y^2 = z^2$

$x = y = z$

So $x^2+y^2+z^2 = 9$

$3x^2 = 9$

$x = y = z = \sqrt{3}$

Hence $\max f = xyz$

$\max(f) = \sqrt{3}\cdot\sqrt{3}\cdot\sqrt{3}$

$\max(f) = 3\sqrt{3}$

Q Find the max & min distance of the point $(3,4,12)$ from $x^2+y^2+z^2=1$ (RTU-2024)

Sol^n $x^2+y^2+z^2=1$

Let a point on the sphere is (x,y,z)

distance $= \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$ ---(i)

max / min $f = (x-3)^2+(y-4)^2+(z-12)^2$

subject to $x^2+y^2+z^2=1$

$$L(x, y, z, \lambda) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-1)$$

$$\frac{\delta L}{\delta x} = 0 = 2(x-3) + 2\lambda x = 0$$

$$x = \frac{3}{1+\lambda} \quad \text{—(iii)}$$

$$\frac{\delta L}{\delta y} = 2(y-4) + 2\lambda y = 0$$

$$y = \frac{4}{1+\lambda} \quad \text{—(iv)}$$

$$\frac{\delta L}{\delta z} = 2(z-12) + 2\lambda z = 0$$

$$z = \frac{12}{1+\lambda} \quad \text{—(v)}$$

&

$$\frac{\delta L}{\delta \lambda} = x^2 + y^2 + z^2 - 1 = 0$$

$$x^2 + y^2 + z^2 = 1 \quad \text{—(vi)}$$

from eq (iii), (iv), (v) & (vi)

$$\left(\frac{3}{1+\lambda}\right)^2 + \left(\frac{4}{1+\lambda}\right)^2 + \left(\frac{12}{1+\lambda}\right)^2 = 1$$

$$9 + 16 + 144 = (1+\lambda)^2$$

$$\pm\sqrt{169} = 1+\lambda$$

$$1+\lambda = \pm 13$$

(+)

$$\boxed{\lambda = 12}$$

$x = 3/13$

$y = 4/13$

$z = 12/13$

(−)

$$\boxed{\lambda = -14}$$

$x = 3/-13$

$y = 4/-13$

$z = 12/-13$

min distance

$$\min = \sqrt{\left(\frac{3}{13}-3\right)^2+\left(\frac{4}{13}-4\right)^2+\left(\frac{12}{13}-12\right)^2}$$

$$\min = \sqrt{(3^2+4^2+12^2)\left(\frac{12}{13}\right)^2}$$

$$\min = \sqrt{169}\times\frac{12}{13}$$

$$\boxed{\min = 12}$$

max distance

$$\max = \sqrt{\left(\frac{3}{-13}-3\right)^2+\left(\frac{4}{-13}-4\right)^2+\left(\frac{12}{-13}-12\right)^2}$$

$$\max = \sqrt{169}\times\frac{14}{13}$$

$$\boxed{\max = 14}$$