Proper Integration :-

$\int_a^b f(x)dx$ is said to be proper integral if range of integration is finite & integrand $f(x)$ is bounded. (sequence)

Ex. $\int_0^{\pi/2} \sin x\, dx$, $\int_{-5}^{5} x^2 dx$

Improper Integration :-

$\int_a^b f(x)dx$ is said to be improper integral if range of integration is not finite or $f(x)$ is not bounded or both.

Ex. $\int_0^{\infty} \cos x\, dx$

- **Gamma function :-** Known as Euler's Integral of the second kind.

$$\Gamma n = \int_0^{\infty} x^{n-1} \cdot e^{-x} dx$$

Properties of Gamma function

(i) $\Gamma(n+1) = n\Gamma n$

(ii) $\Gamma 1 = 1$

(iii) $\Gamma(n+1) = n!$

Proof : $\Gamma(n+1) = n\Gamma n$

we know that

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx$$

$n \to n+1$

$$\Gamma(n+1) = \int_0^{\infty} \underset{I}{x^n} \, \underset{II}{e^{-x}} dx$$

$$\Gamma(n+1) = \left[x^n(-e^{-x})\right]_0^{\infty} - \int_0^{\infty} nx^{n-1}(-e^{-x})dx$$

$$\Gamma(n+1) = -\frac{x^n}{e^x} - 0 + \int_0^{\infty} nx^{n-1}e^{-x} dx$$

$\Gamma(n+1) = n\int_0^\infty x^{n-1} e^{-x} dx$ $(n>0)$

$\Gamma(n)$

$\boxed{\Gamma(n+1) = n\Gamma(n)}$ H.P.

(ii) Proof

$\Gamma(1) = 1$

$\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

put $n=1$

$\Gamma(1) = \int_0^\infty x^0 e^{-x} dx$

$\Gamma(1) = \int_0^\infty e^{-x} dx$

$\Gamma(1) = -\left[e^{-x}\right]_0^\infty$

$\Gamma(1) = -[e^{-\infty} - e^{-0}]$

$\Gamma(1) = -[0-1]$

$\boxed{\Gamma(1) = 1}$ HP

(iii) Proof

$\Gamma(n+1) = n!$; $(n = 1, 2, \ldots$ $n \in N)$

$\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$\because \Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+1) = n\,\Gamma((n-1)+1)$

$= n(n-1)\,\Gamma(n-1)$

$= n(n-1)\,\Gamma((n-2)+1)$

$\Rightarrow n(n-1)(n-2)\,\Gamma(n-2)$

$\Rightarrow n(n-1)(n-2)(n-3)\,\Gamma(n-3)$

$\ldots\; 3\cdot 2\cdot 1\cdot \Gamma(1)$

$n!$

$\boxed{\Gamma(n+1) = n!}$ HP

* gemma funᵒ is as the generalization of the elementry factorial function.

Transformation of Gamma function :-

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \; (n>0)$

(i) if $x = \log\left(\frac{1}{y}\right)$

Anti log lene pe

$e^x = \frac{1}{y}$

$y = e^{-x}$

$x = 0$	$x = \infty$
$y = e^{-x}$	$y = e^{-\infty}$
$y = e^{-0}$	$y = \frac{1}{e^\infty}$
$y = 1$	$y = 0$

$\frac{dy}{dx} = -e^{-x}$

$dy = -e^{-x} dx$

$\Gamma(n) = -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} dy \Rightarrow$

$\Gamma(n) = -\int_1^0 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy$

$\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy$

(ii) if $x^n = y$

$nx^{n-1} = \frac{dy}{dx}$

$nx^{n-1}dx = dy$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$\because \Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$

$\Gamma n = \int_0^\infty e^{-(y)^{1/n}} \frac{dy}{n}$

$$\boxed{n\Gamma n = \int_0^\infty e^{-(y)^{1/n}} dy}$$

★ $\Gamma 1 = 1$

$\Gamma 0$; $\Gamma -1$, $\Gamma -2$ etc are undefined.

Taking $n = 1/2$

$$\boxed{\frac{1}{2}\Gamma \frac{1}{2} = \int_0^\infty e^{-(y)^2} dy} \Rightarrow \boxed{\Gamma \frac{1}{2} = \sqrt{\pi}}$$

Partial Fraction :-

	Form of fractional function	Form of partial fraction
(i)	$\frac{px+q}{(x-a)(x-b)}$; $a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
(ii)	$\frac{px+q}{(x-a)^2}$:	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
(iii)	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ $a \neq b \neq c$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
(iv)	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
(v)	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$

Q $\int \frac{4x+1}{(x+1)(x-2)} dx$

Let $\frac{4x+1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

$\frac{4x+1}{(x+1)(x-2)} = \frac{Ax-2A+Bx+B}{(x+1)(x-2)}$

comparing the coff.

$4 = A+B$ —(i) $\quad 1 = B-2A$ —(ii)

by eq 2(i) + (ii)

$8 \pm 2A + 2B$

$+1 = B - 2A$

$9 = 3B$

$\boxed{B = 3}$

$\boxed{A = 1}$

now

$\frac{4x+1}{(x+1)(x-2)} = \frac{1}{(x+1)} + \frac{3}{(x-2)}$

$\int \frac{1}{(x+1)} dx + \int \frac{3}{(x-2)} dx$

$\ln(x+1) + 3\ln(x-2) + \log c$

$\ln[(x+1)(x-2)^3 c]$ Ans

Q $\frac{34-12x}{(3x^2-10x-8)}$

$\frac{34-12x}{3x^2-10x-8} = \frac{A}{(x-4)} + \frac{B}{(3x+2)}$

$3x^2-10x-8$

$3x^2-12x+2x-8$

$3x(x-4)+2(x-4)$

$(x-4)(3x+2)$

$\frac{34-12x}{(3x^2-10x-8)} = \frac{A(3x+2)+B(x-4)}{(x-4)(3x+2)}$

$34-12x = 3Ax^2+2Ax+Bx-4B$

Put $x=4$

$-2 = 48A+8A$

$-2 = 56A$

$\boxed{-\frac{1}{13} = A}$

Put $x = -2/3$

$\frac{3}{42} = -\frac{14}{3}B$

$\boxed{B = -9}$

$$\frac{34-12x}{(3x^2-10x-8)} = \frac{A}{(x-4)} + \frac{B}{(3x+2)}$$

$$34-12x = (3x+2)A + (x-4)B$$

put $x=4$ put $x=-2/3$

$-14 = 14A$ $42 = \frac{-14}{3}B$

$\boxed{A=-1}$ $\boxed{B=-9}$

$$f(x) = \int \frac{-1}{(x-4)}dx + \int \frac{-9}{3x+2}dx$$

$$\int \frac{-1}{(x-4)}dx + \int \frac{-3}{x+2/3}dx$$

$$-\ln(x-4) + (-3\ln(x+2/3)] + \ln c$$

$$-\ln\left[\frac{(x-4)\left(\frac{3x+2}{3}\right)^3}{c}\right] \quad \underline{\text{Ans}}$$

① $= \dfrac{3x^2+7x+28}{x(x^2+x+7)}$

$$\frac{3x^2+7x+28}{x(x^2+x+7)} = \cancel{Ax+1}\ \frac{A}{x} + \frac{Bx+C}{(x^2+x+7)}$$

$$3x^2+7x+28 = A(x^2+x+7) + B(x^2)+C(x)$$

Comparing to coff.

$A+B=3$ $A+C=7$ $7A=28$

$\boxed{B=-1}$ $\boxed{C=3}$ $\boxed{A=4}$

$$\int \frac{3x^2+7x+28}{x(x^2+x+7)}dx = \int \frac{4}{x}dx + \int\left(\frac{-x+3}{x^2+x+7}\right)dx \quad \text{—(i)}$$

~~4 ln x~~

$$\int \frac{3-x}{(x^2+x+7)}\,dx = \underline{\quad}$$

$$\hookrightarrow \int \frac{3}{(x^2+x+7)}\,dx - \int \frac{x}{x^2+x+7}\,dx$$

x^2+x+7 पूर्ण वर्ग विधि

let $\cancel{x^2+x+7} = t$

$2x+1 = \frac{dt}{dx}$

$x^2 + x + 7 + \frac{1}{4} - \frac{1}{4}$

$(x+\frac{1}{2})^2 - \frac{27}{4}$

$(x+\frac{1}{2})^2 - (\cancel{\frac{3}{2}})\ \frac{27}{4}$

$$\frac{3-x}{x^2+x+7} = \frac{A}{(x+\frac{1}{2})-\sqrt{27/4}} + \frac{B}{(x+\frac{1}{2})+\sqrt{27/4}}$$

* Very important

$n=0 \Rightarrow \Gamma 0 = \frac{\Gamma 0+1}{0} = \frac{\Gamma 1}{0} = \frac{1}{0} = \infty$

$n=-1 \Rightarrow \Gamma{-1} = \frac{\Gamma{1-1}}{-1} = \frac{\Gamma 0}{-1} = -\infty$

$n=-2 = \Gamma{-2} = \frac{\Gamma{-2+1}}{-2} = \infty$

$n=-3 = \Gamma{-3} = \frac{\Gamma{-3+1}}{-3} = \frac{\Gamma{-2}}{-3} = \frac{\infty}{-3} = -\infty$

if $\Gamma{-n}$: $\begin{cases} n \text{ is even then } +\infty \\ n \text{ is odd then } -\infty \end{cases}$

* $\Gamma{-n} = (-1)^n \times \infty$

(where $n = 1, 2, 3, \ldots$)

Q Show that (i) $\int_0^\infty x^{m-1} e^{-ax} \cos bx\, dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}} \cos m\theta$.

(ii) $\int_0^\infty x^{m-1} e^{-ax} \sin bx\, dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}} \sin m\theta$.

Proof we know -

2nd case by book

3rd Case

$$\frac{\Gamma n}{a^n} = \int_0^\infty x^{n-1} e^{-ax} dx$$

3rd Proof

$$\because \Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

let $x = az$ $\quad x = 0 \to 0$

$dx = a\,dz$ $\quad x = \infty \quad \infty$

$$\Gamma n = \int_0^\infty (az)^{n-1} e^{-az} a\, dz$$

$$\Gamma n = \int_0^\infty z^{n-1} e^{-az} a^n dz$$

$$\boxed{\frac{\Gamma n}{a^n} = \int_0^\infty z^{n-1} e^{-az} dz}$$

Soln (i) $\int_0^\infty x^{m-1} e^{-ax} \cos bx\, dx$

~~replace~~ $\cancel{a} \to x \quad a \to z$

$n \to m$

$\cancel{\int_0^\infty x^{n-1} e^{-} \cos}$

$$\because \frac{\Gamma m}{\cancel{a}^m} = \int_0^\infty x^{m-1} e^{-\cancel{a}x} dx$$

put $z = a - ib$

$$\int_0^\infty x^{m-1} e^{-ax} dx = \frac{\Gamma m}{z^m}$$

$$\int_0^\infty x^{m-1} e^{-(a-ib)x} dx = \frac{\Gamma m}{(a-ib)^m}$$

by doing rationalization

$$\int_0^\infty x^{m-1} e^{-(a-ib)x} dx = \frac{\Gamma m}{(a-ib)^m} \times \frac{(a+ib)^m}{(a+ib)^m}$$

$$\int_0^\infty x^{m-1} e^{-(a-bi)x} dx = \frac{\Gamma m \, (a+ib)^m}{(a^2+b^2)^m} \quad (i^2=-1)$$

let $a = r\cos\theta$
$b = r\sin\theta$ $\quad (b/a = \tan\theta) \Rightarrow \theta = \tan^{-1}(b/a)$

$(a^2+b^2 = r^2)$

$$\int_0^\infty x^{m-1} e^{-(r\cos\theta - ri\sin\theta)x} dx = \frac{\Gamma m \, (r\cos\theta + ri\sin\theta)^m}{(r^2)^m}$$

$$\int_0^\infty x^{m-1} e^{-(a-ib)x} dx = \frac{\Gamma m \, ((\cos\theta + i\sin\theta)^m}{r^m}$$

$\downarrow$

$$\int_0^\infty x^{m-1} e^{-ax} \cdot e^{+bix} dx = \frac{\Gamma m \, (e^{i\theta})^m}{r^m}$$

$$\left(\because \begin{array}{l} \cos\theta + i\sin\theta = e^{i\theta} \\ \cos\theta - i\sin\theta = e^{-i\theta} \end{array} \right)$$

$$\int_0^\infty x^{m-1} e^{-ax} e^{bix} dx = \frac{\Gamma m \, e^{mi\theta}}{r^m}$$

$$\int_0^\infty x^{m-1} [\cos bx + i\sin bx] e^{-ax} dx = \frac{\Gamma m \, [\cos m\theta + i\sin m\theta]}{r^m}$$

$$\int_0^\infty x^{m-1} [\cos bx \cdot e^{-ax} + i e^{-ax}\sin bx] dx = \frac{\Gamma m}{r^m} [\cos m\theta + i\sin m\theta]$$

$$= \frac{\Gamma m}{r^m} [C.$$

Comparing real & imaginary part -

$z_1 = a+ib \qquad z_2 = x+iy$

$z_1 = z_2 \Rightarrow a = x$
$b = y$

$$\int_0^\infty x^{m-1} \cos bx \cdot e^{-ax} dx = \frac{\Gamma m}{r^m} \cos m\theta$$

&

$$\int_0^\infty x^{m-1} e^{-ax} \sin bx \, dx = \frac{\Gamma m}{r^m} \sin m\theta$$

$$\because r = (a^2+b^2)^{1/2}$$

So
$$\int_0^\infty x^{m-1} \cos bx \, e^{-ax} dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}} \cos m\theta$$

&

$$\int_0^\infty x^{m-1} e^{-ax} \sin bx \, dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}} \sin m\theta$$

Hance Proved

Q Prove that (iii) $\int_0^\infty x^{m-1} \cos bx \cdot dx = \frac{\Gamma m}{b^m} \cos\left(\frac{m\pi}{2}\right)$

(iv) $\int_0^\infty x^{m-1} \sin bx \, dx = \frac{\Gamma m}{b^m} \sin\left(\frac{m\pi}{2}\right)$

we know that -

$$\int_0^\infty x^{m-1} \cos bx \, e^{-ax} dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}} \cos m\theta \quad \text{-(i)}$$

& if we take $a = 0$ in (i)
& $\theta = \pi/2$

$$\int_0^\infty x^{m-1} \cos bx \, dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}} \cos m\theta$$

$$\int_0^\infty x^{m-1} \cos bx \, dx = \frac{\Gamma m}{b^m} \cos \frac{m\pi}{2}$$

Ans

(ii) we know that

$$\int_0^\infty x^{m-1} e^{-ax} \sin bx \, dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}} \sin m\theta$$

if we take

$a=0$ & $\theta = \pi/2$ then

$$\int_0^\infty x^{m-1} \sin bx \, dx = \frac{\Gamma m}{b^m} \sin \frac{m\pi}{2}$$

H.P

$\underline{\underline{10}}$ $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$$

$\underline{\underline{ex}}$ $$\int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta$$

$\cos\theta = t$

$\sin\theta \, d\theta = -dt$

$\int_0^1 t^2 dt$

$\left[\frac{t^3}{3}\right]_0^1$

$1/3$ Ans

$m=1$

$n=2$

$$\Rightarrow \frac{\Gamma 3/2 \cdot 1}{2\Gamma\frac{5}{2}}$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}+1}{2\Gamma 3/2+1}$$ $(\because \Gamma n+1 = n\Gamma n)$

$$\Rightarrow \frac{1/2\,\Gamma 1/2}{3/2 \cdot 2\,\Gamma 3/2}$$ $(\because \Gamma 1/2 = \sqrt{\pi})$

$$\Rightarrow \frac{1/2\,\sqrt{\pi}/2}{3 \cdot 1/2 \cdot \sqrt{\pi}/2}$$

$$\Rightarrow \frac{1}{3} \text{ Ans}$$

Ex $\int_0^{\pi/2} \sin\theta \, d\theta$ $\quad m = 0, \; n = 1$

$$\Rightarrow \frac{\lceil 2/2 \cdot \lceil 1/2}{2 \lceil 3/2} \Rightarrow \frac{\lceil 1 \cdot \cancel{\lceil 1/2}}{2 \cdot 1/2 \cancel{\lceil 1/2}}$$

$$\Rightarrow \left[-\cos\theta\right]_0^{\pi/2}$$

$$\Rightarrow \frac{1}{1}$$

$$\Rightarrow \cos 0 - \cos \pi/2$$

$$\Rightarrow 1 \text{ Ans}$$

$$\Rightarrow 1$$

Q $\int_0^{\pi/2} \sin^5\theta \cos^6\theta \, d\theta$

$m = 5 \quad n = 6$

$$\Rightarrow \frac{\lceil 6/2 \cdot \lceil 7/2}{2 \lceil 13/2}$$

$3 = 1 + 2$

$$\Rightarrow \frac{\lceil 3 \cdot \lceil 1 + 5/2}{2 \lceil \frac{11}{2} + 1} \Rightarrow \frac{2 \lceil 2 \cdot 5/2 \lceil 5/2}{2 \; 11/2 \lceil 11/2}$$

$$\Rightarrow \frac{2 \, 1 \cdot 1 \cdot \frac{5}{2} \cdot 3/2 \lceil 3/2}{2 \cdot \frac{11}{2} \cdot 9/2 \cdot \lceil 9/2}$$

$$\Rightarrow \frac{2 \cdot 5/2 \cdot 3/2 \cdot \lceil 3/2}{2 \cdot \frac{11}{2} \cdot 9/2 \cdot 7/2 \lceil 7/2}$$

$$\Rightarrow \frac{\cancel{2} \cdot \cancel{5/2} \cdot \cancel{3/2} \cancel{\lceil 3/2}}{\cancel{2} \cdot 11/2 \cdot 9/2 \cdot 7/2 \cdot \cancel{5/2} \cancel{\lceil 5/2}}$$

$$\Rightarrow \frac{8}{11 \times 9 \times 7} \Rightarrow \frac{8}{693} \text{ Ans}$$

Q $\int_0^{\pi/6} \cos^4 3\theta \, \sin^2 6\theta \, d\theta$

let $3\theta = t$

$3\,d\theta = dt$

Sol

$\because \sin^2 2t = 2\cos t \sin t$

$\frac{1}{3}\int_0^{\pi/2} \cos^4 t \, \sin^2 2t \, dt$

$\frac{1}{3}\int_0^{\pi/2} \cos^4 t \cdot 4 \cdot \sin^2 t \cos^2 t \, dt$

$\frac{4}{3}\int_0^{\pi/2} \cos^6 t \, \sin^2 t \, dt$

$m = 6 \quad , n = 2$

$$\frac{4}{3}\left(\frac{\Gamma\frac{6+1}{2} \cdot \Gamma\frac{2+1}{2}}{2\,\Gamma\frac{6+2+2}{2}} \right)$$

$$\frac{4}{3\cdot 2}\left(\frac{\Gamma 7/2 \cdot \Gamma 3/2}{\Gamma 5} \right) \Rightarrow \frac{2}{3}\left(\frac{5/2 \cdot 3/2 \cdot 1/2 \, \Gamma 1/2 \cdot 1/2 \, \Gamma 1/2}{5 \times 4 \times 3 \times 2 \times 1} \right)$$

$$\Rightarrow \frac{1}{3} \; \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot 1/2 \cdot \sqrt{\pi} \cdot \sqrt{\pi}}{5 \times 4 \times 3 \times 1}$$

$$\Rightarrow \frac{5\pi}{16 \times 3 \times 4} \Rightarrow \frac{5\pi}{192} \; \underline{\text{Answer}}$$

Q $\int_0^{\pi/2} \sin^6 x \, dx$

$m = 0, \quad n = 6$

$$\Rightarrow \frac{\Gamma\frac{0+1}{2} \cdot \Gamma\frac{6+1}{2}}{2\Gamma\frac{0+6+2}{2}}$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{7}{2}}{2\Gamma 4} \Rightarrow \frac{\sqrt{\pi}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot 3\times 2\times 1}$$

$$\Rightarrow \frac{5\pi}{32} \text{ Ans}$$

Q $\int_0^a x^2 (a^2 - x^2)^{3/2} dx$

let $x = a\sin\theta \qquad a = \frac{x}{\sin\theta}$

$dx = a\cos\theta \, d\theta$

$x = a\sin\theta$

$a = a\sin\theta \quad | \quad 0 = a\sin\theta$

$\theta = \pi/2 \quad | \quad \theta = 0$

$$\int_0^{\pi/2} a^2\sin^2\theta\,(a^2 - a^2\sin^2\theta)^{3/2}\, a\cos\theta \, d\theta$$

$$\int_0^{\pi/2} a^6 \sin^2\theta \cos^4\theta \, d\theta \qquad n = 2,\ m = 4$$

$$a^6 \frac{\Gamma\frac{3}{2}\cdot\Gamma\frac{5}{2}}{2\Gamma\frac{8}{2}} \Rightarrow \frac{a^6\,\Gamma\frac{3}{2}\cdot\Gamma\frac{5}{2}}{2\Gamma 4}$$

$$\Rightarrow \frac{a^6\ \frac{1}{2}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\times 3\times 2\times 1}$$

$$\Rightarrow \frac{\pi a^5}{32} \text{ Ans}$$

$$\cancel{\frac{a^5\,\Gamma 3/2\cdot\Gamma 2}{2\ 5/2\ \Gamma 3/2}}$$

$$\Rightarrow \cancel{\frac{a^5\cdot 1}{2\times 5/2}} \Rightarrow \cancel{\frac{a^5}{5}} \text{ Ans}$$

Q $\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$

$n = 4, \quad m = 2$

$$\Rightarrow \frac{\Gamma\frac{5}{2} \, \Gamma\frac{3}{2}}{2\,\Gamma\frac{8}{2}}$$

$$\Rightarrow \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{1}{2} \sqrt{\pi}}{2 \cdot 3 \cdot 2 \cdot 1}$$

$$\Rightarrow \frac{\pi}{32} \text{ Ans}$$

Q $\int_0^{\pi/2} \cos^6 x \, dx$

$n = 6, \quad m = 0$

$$\frac{\Gamma\frac{1}{2} \, \Gamma\frac{7}{2}}{2\,\Gamma\frac{8}{2}}$$

$$\Rightarrow \frac{\sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{1}{2} \sqrt{\pi}}{2 \cdot 3 \cdot 2 \cdot 1}$$

$$\Rightarrow \frac{5\pi}{32} \text{ Ans}$$

Special Case for Gamma function:-

* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

we know that

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

let $x^n = y \Rightarrow y^{1/n} = x$

$n x^{n-1} dx = dy$

$$\Gamma n = \int_0^\infty \left(e^{-(y)^{1/n}}\right)\frac{dy}{n}$$

$$n\Gamma n = \int_0^\infty e^{-(y)^{1/n}} dy \quad \Big/ \text{ for } n = 1/2$$

$$\frac{1}{2}\Gamma\frac{1}{2} = \int_0^\infty e^{-y^2} dy$$

$$\Gamma\frac{1}{2} = 2\int_0^\infty e^{-y^2} dy \quad \text{--- (i)}$$

Similarly

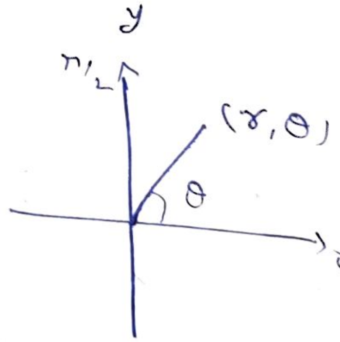
$$\Gamma\frac{1}{2} = 2\int_0^\infty e^{-x^2} dx \quad \text{--- (ii)}$$

multipling by (i) to (ii)

$$\left(\Gamma\frac{1}{2}\right)^2 = 4\int_0^\infty e^{-y^2} dy \cdot \int_0^\infty e^{-x^2} dx$$

$$\left(\Gamma\frac{1}{2}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx \cdot dy$$

for polar form.



let $x = r\cos\theta$ $\quad$ $x^2 + y^2 = r^2$

$y = r\sin\theta$ $\quad$ $\theta = \tan^{-1} y/x$

$dx\,dy = r\,dr\,d\theta$ $\longrightarrow$ (next)

$r \quad 0 \to \infty$

$\theta \quad 0 \to \pi/2$

$$\left(\Gamma\frac{1}{2}\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta$$

$r^2 = t$

$2r\,dr = dt$

$$(\Gamma\tfrac{1}{2})^2 = 4\int_0^{\pi/2} \tfrac{1}{2}\, d\theta$$

$$(\Gamma\tfrac{1}{2})^2 = 2\,[\theta]_0^{\pi/2}$$

$$(\Gamma\tfrac{1}{2})^2 = 2\,[\pi/2]$$

$$\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

$$\frac{dt}{2} = r\,dr$$

$$\frac{1}{2}\int_0^{\infty} e^{-t}\,dt$$

$$\frac{1}{2}[-e^{-t}]_0^{\infty}$$

$$\frac{1}{2}[+1]$$

Exercise 8.1

Q1 (a) $\dfrac{\Gamma 3\ \Gamma 2.5}{\Gamma 5.5}$

$\Rightarrow \dfrac{\Gamma 3 \cdot \Gamma 5/2}{\Gamma 11/2}$

$\Rightarrow \dfrac{2\times 1\times \Gamma 1 \cdot \cancel{\Gamma 5/2}}{9/2 \cdot 7/2 \cdot 5/2 \cdot \cancel{\Gamma 5/2}}$

$\Rightarrow \dfrac{16}{315}$ Ans

Q1(b) $\Gamma(-5/2)$

let $\Gamma n+1 = n\Gamma n$

$n \to (n-1)$

$(n-1)\Gamma(n-1) = \Gamma n$

$n = 1/2$

$-1/2\ \Gamma -1/2 = \Gamma 1/2$

$\Gamma -1/2 = -2\sqrt{\pi}$

now

$n = -1/2$

$-3/2\ \Gamma -3/2 = \Gamma -1/2$

$\Gamma -3/2 = -\frac{4}{3}\sqrt{\pi}$

now $n = -3/2$

$-5/2\ \Gamma -5/2 = \Gamma -3/2$

$-\frac{5}{2}\Gamma -5/2 = -\frac{4}{3}\sqrt{\pi}$

$\Gamma -5/2 = \frac{8}{15}\sqrt{\pi}$ Ans

Q2. (a) $\int_0^{\pi/2} \sin^6\theta\, d\theta$

$n = 0,\ m = 6$

$\dfrac{\Gamma 1/2\ \Gamma 7/2}{2\,\Gamma 8/2}$

$\Rightarrow \dfrac{\sqrt{\pi}\cdot 5/2 \cdot \cancel{3/2} \cdot \Gamma 1/2 \cdot 1/2}{2 \cdot \cancel{3} \cdot 2 \cdot 1}$

$\Rightarrow \dfrac{5\pi}{32}$ Ans

(b) $\int_0^{\pi/2} \sin^2\theta \cos^5\theta\, d\theta$

$n = 2,\ m = 5$

$\dfrac{\Gamma 3/2 \cdot \Gamma 6/2}{2 \cdot \Gamma 9/2}$

$\Rightarrow \dfrac{\cancel{2}\cdot 1 \cdot \cancel{\Gamma 3/2}}{\cancel{2}\cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot \cancel{\Gamma 3/2}}$

$\Rightarrow \dfrac{8}{105}$ Ans

(c) $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

let $x = \sqrt{\sin\theta}$

$dx = \frac{1}{2}\sin^{-1/2}\theta \, d\theta . \cos\theta$

$\frac{1}{2}\int_0^{\pi/2} \frac{\sin^{-1/2}\theta \cancel{\cos\theta} \, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$

$\frac{1}{2}\int_0^{\pi/2} \sin^{-1/2} d\theta$

$n = -1/2 \ , \ m = 0$

$\frac{1}{2}\left[\frac{\Gamma\frac{1-1/2}{2}\ \Gamma 1/2}{2\Gamma\frac{-1/2+0+2}{2}}\right]$

$\frac{1}{4}\left(\frac{\sqrt{\pi}\ \Gamma 1/4}{\Gamma 3/4}\right)$ Ans

Ex (i) $\Gamma(-1/2)$

$n\Gamma n = \Gamma n+1$

$n \to n-1$

$(n-1)\Gamma(n-1) = \Gamma n$

$n = 1/2$

$-\frac{1}{2}\Gamma -\frac{1}{2} = \Gamma\frac{1}{2}$

$\boxed{\Gamma -1/2 = -2\sqrt{\pi}}$

(ii) $\Gamma -1.5$

$\because \Gamma -1/2 = -2\sqrt{\pi}$

$(n-1)\Gamma(n-1) = \Gamma n$

$n = -1/2$

$-\frac{3}{2}\Gamma -3/2 = \Gamma -1/2$

$\Gamma -3/2 = -2\sqrt{\pi} \times \frac{2}{-3}$

$\boxed{\Gamma -3/2 = +\frac{4\sqrt{\pi}}{3}}$ A

Ex $\int_0^\infty x^2 e^{-x^4} dx \int_0^\infty e^{-x^4} dx$

let $x^4 = t$ $\quad x=0 \ t=0$

$x = t^{1/4}$ $\quad x\to\infty \ t\to\infty$

$dx = \frac{1}{4} t^{-3/4} dt$

$$I = \int_0^\infty t^{1/2} e^{-t} \frac{1}{4} t^{-3/4} dt . \int_0^\infty e^{-t} dt \ \frac{1}{4} t^{-3/4}$$

$$I = \frac{1}{16} \int_0^\infty e^{-t} t^{-1/4} dt . \int_0^\infty e^{-t} t^{-3/4}$$

$$I = \frac{1}{16} \int_0^\infty t^{(3/4-1)} e^{-t} dt \int_0^\infty e^{-t} t^{+1/4-1}$$

$n = 3/4$ $\quad$ $n = 1/4$ $\quad (1+3/4)$

$$I = \frac{1}{16} \Gamma 3/4 \, \Gamma 1/4 = \frac{1}{16} \Gamma 1/4 \, \Gamma 1 - 1/4$$

$$I = \frac{1}{16} \frac{\pi}{\sin \pi/4} \qquad I = \frac{\pi}{\sin n\pi}$$

when $\Gamma n \, \Gamma 1-n$

$$I = \frac{\pi}{8\sqrt{2}} \text{ Ans}$$

Q. Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

let $x^2 = t$

$x = t^{1/2}$

$dx = \frac{1}{2} t^{-1/2} dt$

$\Rightarrow \int_0^\infty e^{-t} \frac{1}{2} t^{-1/2} dt$

$\Rightarrow \frac{1}{2} \int_0^\infty e^{-t} . t^{-1+1/2} dt'$

$\Rightarrow$ if $n = 1/2$

$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$ से,

$\frac{1}{2}\Gamma \frac{1}{2} = \frac{\sqrt{\pi}}{2}$ Hance proved.

Q. $\int_{-\infty}^\infty e^{-a^2x^2} dx$ Soln

let $a^2x^2 = t$

$a^2 2x dx = dt$

$dx = \frac{dt}{a2(t/a^2)^{1/2}}$

$dx = \frac{dt\, t^{-1/2}}{a}$

Let $e^{-a^2x^2} = f(x)$

if $x \to -x$

$\therefore e^{-a^2x^2} = f(-x) = f(x)$

ever function So

$\int_{-a}^{a} f(x) dx = 2\int_0^a f(x) dx$

even & odd function.

in $\int_{-a}^{a} f(x) dx$ $\quad x \to -x$

$f(x) = f(-x)$

if function does not change then even function.

if $f(x)$ $x \to -x$

$f(x) \to -f(x)$

then it is odd function.

in Definite integration.

$\int_{-a}^{a} f(x) dx = 2\int_0^a f(x) dx$

(for even function)

Q $\int_{-a}^{a} f(x) dx = 0$

(for odd function)

$dx = \frac{1}{2a} dt \; t^{-1/2}$

$\Rightarrow 2\int_0^\infty e^{-t} \frac{1}{2a} t^{-1/2} dt$

$\Rightarrow \frac{1}{a}\int_0^\infty t^{-1/2} e^{-t} dt$

$\Rightarrow \frac{1}{a}\sqrt{\tfrac{1}{2}}$

$\Rightarrow \frac{\sqrt{\pi}}{a}$ Answer

Q $\int_0^a x^3(2ax-x^2)^{3/2} dx$

$x = 2a\sin^2\theta$

$x = 0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \pm\pi/4$

$dx = 2a \cdot 2\sin\theta\cos\theta\, d\theta$

put the values in function

$\Rightarrow \int_0^{\pi/4} (2a\sin^2\theta)^3(2a.a\sin^2\theta.g - a^2\sin^4\theta.4)^{3/2}.4a\sin\theta\cos\theta\, d\theta$

$\Rightarrow \int_0^{\pi/4} 2^8 a^7 . \sin^6\theta . \sin^3\theta . \cos^3\theta . \sin\theta\cos\theta\, d\theta$

$\Rightarrow \int_0^{\pi/4} 2^8 a^7 \sin^{10}\theta . \cos^4\theta\, d\theta$

$2^8 a^7 \int_0^{\pi/4} \sin^6\theta\,(\cos\theta\sin\theta)^4 d\theta \times \frac{2^4}{2^4}$

$2\,\frac{\cancel{2^8} a^7}{\cancel{2^4}}\int_0^{\pi/4} \frac{\sin^6\theta \times \cancel{2^3}(\sin 2\theta)^4}{\cancel{2^3}} d\theta$

$(\because 2\sin^2\theta = 1-\cos 2\theta)$

$2a^7 \int_0^{\pi/4} (2\sin^2\theta)^3(\sin 2\theta)^4 d\theta$

let $2\theta = t$

$2d\theta = dt$

$\theta \to 0$, $t \to 0$ | $\theta \to \pi/4$, $t \to \pi/2$

$2a^7 . \int_0^{\pi/4} (1-\cos 2\theta)^3(\sin 2\theta)^4 d\theta$

~~let 2~~

$\frac{\cancel{2}a^7}{\cancel{2}}\int_0^{\pi/2}(1-\cos t)^3(\sin t)^4 dt$

$$\Rightarrow a^7 \int_0^{\pi/2} \sin^4 t\,(1-\cos t)^3\,dt$$

$$\Rightarrow a^7 \int_0^{\pi/2} \sin^4 t\,(1-\cos^3 t - 3\cos t + 3\cos^2 t)\,dt$$

$$\Rightarrow a^7 \int_0^{\pi/2} [\sin^4 t - \sin^4 t\cos^3 t - 3\sin^4 t\cos t + 3\sin^4 t\cos^2 t)\,dt$$

$$\Rightarrow a^7\left[\frac{\sqrt{5/2}\,\sqrt{1/2}}{2\boxed{3}} - \frac{\sqrt{5/2}\,\sqrt{4/2}}{2\sqrt{9/2}} - 3\frac{\sqrt{5/2}\,\sqrt{2/2}}{2\sqrt{7/2}} + 3\frac{\sqrt{5/2}\,\sqrt{3/2}}{2\sqrt{4}}\right]$$

$$\Rightarrow a^7\left[\frac{3/2\cdot 1/2\cdot\pi}{2\cdot 2\cdot 1} - \frac{\cancel{\sqrt{5/2}}\cdot 1}{\cancel{2}\cdot 7/\cancel{2}\cdot 5/2\,\cancel{\sqrt{5/2}}} - \frac{3\,\cancel{\sqrt{5/2}}\cdot 1}{\cancel{2}\cdot 5/\cancel{2}\,\cancel{\sqrt{5/2}}} + \frac{3\cdot 3/2\cdot 1/2\cdot 1/2\cdot\pi}{2\cdot\cancel{3}\cdot 2\cdot 1}\right]$$

$$\Rightarrow a^7\left[\frac{3\pi}{16} - \frac{2}{35} - \frac{3}{5} + \frac{3\pi}{32}\right]$$

$$\Rightarrow a^7\left[\frac{9\pi}{16} - \frac{23}{35}\right]$$ Answer

Q $\int_0^1 x^6(1-x^2)^{1/2}\,dx$ evaluate.

let $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$x = 0 \Rightarrow \sin\theta = 0$

$x = 1 \Rightarrow \sin\theta = \pi/2$

$$\int_0^{\pi/2} \sin^6\theta\,(1-\sin^2\theta)^{1/2}\cdot\cos\theta\,d\theta$$

$$\int_0^{\pi/2} \sin^6\theta\cdot\cos^2\theta\,d\theta$$

$m = 6,\ n = 2$

$$\frac{\sqrt{7/2}\cdot\sqrt{3/2}}{2\sqrt{5}} \Rightarrow \frac{5/2\cdot\cancel{3}/2\cdot 1/2\cdot\sqrt{\pi}\cdot 1/2\cdot\sqrt{\pi}}{2\cdot 4\cdot\cancel{3}\cdot 2\cdot 1} \Rightarrow \frac{5\pi}{28}$$ As

Q $\int_0^{\pi/2} \sin^2\theta \cdot \cos^5\theta \, d\theta$

$n = 2, \quad m = 5$

$$\frac{\Gamma\frac{3}{2} \cdot \Gamma\frac{6}{2}}{2\,\Gamma\frac{9}{2}} \Rightarrow \frac{\cancel{\Gamma\frac{3}{2}} \cdot \cancel{2} \cdot 1 \cdot \Gamma 1}{\cancel{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \cancel{\Gamma\frac{3}{2}}} \Rightarrow \frac{8}{105} \text{ Ans}$$

Prove that

$\Gamma n+1 = n! \quad (n > 0)$

we know that

$\Gamma n+1 = n\,\Gamma n \Rightarrow n \to n-1$

~~$n \to n+1$~~

$\Gamma n = (n-1)\,\Gamma(n-1)$

~~$\Gamma n+2 = (n+1)\,\Gamma n+1$~~

$\because \Gamma n-1 = (n-2)\,\Gamma(n-2)$

$\Gamma n = (n-1)\,\Gamma(n-1)$

$\Gamma n = (n-1)\,\Gamma n-2 \cdot (n-2)$

$\Gamma n = (n-1)(n-2)(n-3)\,\Gamma(n-3)$

$\Gamma n = (n-1)(n-2)(n-3)(n-4) \ldots 3 \cdot 2 \cdot 1$

$\Gamma n = (n-1)!$

$\therefore n \to n+1$

$\Gamma n+1 = n!$ Ans

Beta function Euler's first integral.

Beta function $\beta(m,n)$ defined by

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx \quad \text{--- (i)}}$$

Putting $x = 1-y$ in (i) eq.

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}\cdot y^{n-1}\,(-dy)$$

$(dx = -dy)$

$$\beta(m,n) = \int_0^1 (1-y)^{m-1}\, y^{n-1}\,dy \quad \text{--- (ii)}$$

$x \to y$

$$\beta(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

★ In terms of Improper integral

(i) $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \dfrac{y}{1+y}$ $\qquad x = 0 \Rightarrow y = 0$

$x + xy = y$ $\qquad x = 1 \Rightarrow y = \infty$

$dx + x\,dy + y\,dx = dy$

$(1+y)\,dx = (1-x)\,dy$

$$\frac{dx}{(\cancel{1-x})} = \frac{dy}{(1+y)}(1-x), \quad x = \frac{y}{1+y}$$

$$dx = \frac{dy}{(1+y)^2}$$

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(1-\frac{y}{y+1}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}\cdot(y+1)^{n-1}}\frac{dy}{(y+1)^2}$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}\,dy}{(1+y)^{m+n}}$$

$y \to x$

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}$$

(ii)

* $$\beta(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$ $\quad x=0 \Rightarrow \theta = 0$

$$\beta(m,n) = \int_0^{\pi/2}(\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\sin 2\theta\,d\theta$$ $\quad x = 1$, $\theta = \pi/2$

$$\beta(m,n) = 2\int_0^{\pi/2}(\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\sin\theta\cos\theta\,d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2}\sin^{2n-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

* Beta function :-

For any positive integer m, n such that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Note 1. If $x = 1-y$ $\qquad dx = -dy$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1}(-dy)$$

$$\beta(m,n) = \int_0^1 (1-y)^{m-1}(y)^{n-1}\,dy = \beta(n,m)$$

$$\boxed{\beta(m,n) = \beta(n,m)}$$

2. If $x = \frac{y}{1+y} \Rightarrow x + xy = y \Rightarrow y = \frac{x}{1-x}$

$$dx + y\,dx + x\,dy = dy$$

$$(1+y)\,dx = dy(1-x)$$

$$dx = \frac{dy}{(1+y)^2}$$

$$\beta(m,n) = \int_0^1 (x)^{m-1}(1-x)^{n-1}\,dx$$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = \infty$

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(1-\frac{y}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

m-1-

$x \to y$

$$\boxed{\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx}$$

3. If $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$ $\qquad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$ $\qquad \beta(m,n) = \int_0^{\pi/2} (\sin\theta)^{2m-2}(\cos\theta)^{2m-2}\,d\theta\,(2\sin\theta\cos\theta$

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$$

** Relation between Beta & Gamma function:-

(RTU-2023)

we know that

$$\Gamma n = \int_0^\infty x^{n-1}e^{-x}\,dx$$

Let $x = zy$ $\qquad z =$ Constant

$dx = z\,dy$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\Gamma n = \int_0^\infty (zy)^{n-1}e^{-zy}\,dy \cdot z$$

$$\Gamma n = \int_0^\infty (z)^n y^{n-1}e^{-zy}\,dy$$

$$\frac{\Gamma n}{z^n} = \int_0^\infty y^{n-1}e^{-zy}\,dy$$

$y \to x$

$$\frac{\Gamma n}{z^n} = \int_0^\infty x^{n-1}e^{-zx}\,dx \longrightarrow (i)$$

eq. (i) $\times z^{m-1} e^{-z}$

$$\frac{\Gamma n\,(z^{m-1})\, e^{-z}}{z^n} = \int_0^\infty e^{-zx}(x)^{n-1}\,dx \times z^{m-1} e^{-z}$$

$$\Gamma n \times z^{m-1} e^{-z} = \int_0^\infty e^{-z(1+x)}(x)^{n-1} z^{n+m-1}\,dx$$

$$(\Gamma n)\, z^{m-1} e^{-z} = \int_0^\infty z^{m+n-1} e^{-z(1+x)} x^{n-1}\,dx$$

integration with respect to z taking limit 0 to ∞

$$\int_0^\infty (\Gamma n) z^{m-1} e^{-z}\,dz = \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-z(1+x)} x^{n-1}\,dx\right] dz$$

$$\Gamma n \int_0^\infty z^{m-1} e^{-z}\,dz = \int_0^\infty x^{n-1}\left[\int_0^\infty z^{m+n-1} e^{-z(1+x)}\,dz\right] dx$$

let $z(1+x) = y \rightarrow z = \frac{y}{(1+x)}$

$dz\,(1+x) = dy$

$z = 0 \Rightarrow y = 0$

$z = \infty \Rightarrow y = \infty$

$$\Gamma n \int_0^\infty z^{m-1} e^{-z}\,dz = \int_0^\infty x^{n-1}\left[\int_0^\infty \left(\frac{y}{1+x}\right)^{m+n-1} e^{-y} \frac{dy}{(1+x)}\right] dx$$

$$\Gamma n \cdot \Gamma m = \int_0^\infty x^{n-1}\left[\int_0^\infty \frac{y^{m+n-1} e^{-y}\,dy}{(1+x)^{m+n}}\right] dx$$

$$\Gamma n \cdot \Gamma m = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left(\int_0^\infty y^{m+n-1} e^{-y}\,dy\right) dx$$

$$\Gamma n \cdot \Gamma m = \beta(m,n)\,\Gamma m+n \Rightarrow \boxed{\beta(m,n) = \frac{\Gamma n \cdot \Gamma m}{\Gamma m+n}}$$

Q Find the value of integration

$$\int_0^1 x^6 (1-x)^7 \, dx$$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

compare

$m-1 = 6 \qquad n-1 = 7$

$m = 7 \qquad n = 8$

$$\beta(7,8) = \frac{\Gamma 7 \cdot \Gamma 8}{\Gamma 7+8}$$

$$\Rightarrow \frac{6! \times 7!}{\Gamma 15 = 14!} = \frac{6! \times 7!}{14!}$$

$$\Rightarrow \frac{6 \times 5 \times 4 \times 3 \times 2 \times \cancel{7!}}{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times \cancel{7!}}$$

$$\Rightarrow \frac{1}{14 \times 13 \times 11 \times 4 \times 3} = \frac{1}{1848 \times 13} \Rightarrow \frac{1}{24024}$$ Ans

Q $\int_0^{\infty} \sqrt{x}\, e^{-3\sqrt{x}}\, dx$

Let $3\sqrt{x} = t$

$\frac{3\,dx}{2\sqrt{x}} = dt \Rightarrow 3\,dx = 2dt \times \frac{t}{3}$

$\Rightarrow \cancel{\int_0^{\infty} \frac{t^2}{9} \times e^{-t}\, \frac{2}{3} dt \times \frac{t^2}{9}}$

$\Rightarrow \cancel{\frac{2}{3^7} \int_0^{\infty} t^4 e^{-t}\, dt}$

$\Rightarrow \cancel{\frac{2}{3^7} \Gamma 4}$

$\Rightarrow \cancel{\frac{2}{3^7} \times 3 \times 2 \times 1}$

$\Rightarrow \cancel{\frac{4}{3^6}}$ Ans

Let $3\sqrt{x} = t$

$\frac{3}{2\sqrt{x}} dx = dt$

$dx = \frac{2\sqrt{x}}{3} dt$

$dx = \frac{2t}{9} dt$

$\int_0^{\infty} \frac{t}{3} \times e^{-t} \times \frac{2t}{9}\, dt$

$\frac{2}{27} \int_0^{\infty} t^2 e^{-t}\, dt$ $\quad n-1 = 2$, $n = 3$

$\frac{2}{27} \Gamma 3 = \frac{4}{27}$ Ans

Q $\int_0^{\infty} x^2 e^{-x^4} dx \times \int_0^{\infty} e^{-x^4} dx$

$x^4 = t$

$4x^3 dx = dt$

$dx = \frac{dt}{4x^3}$

$\int_0^{\infty} t^{1/2} e^{-t} \frac{dt}{4(t)^{3/4}} \times \int_0^{\infty} e^{-t} \times \frac{dt}{4(t)^{3/4}}$

$\frac{1}{4}\int_0^{\infty} t^{-1/4} e^{-t}\, dt \times \frac{1}{4}\int_0^{\infty} t^{-3/4} e^{-t}\, dt$

$n-1 = -1/4$, $n = 3/4$ $\quad n-1 = -3/4$, $n = 1/4$

$\frac{1}{16} \times \Gamma 3/4 \times \Gamma 1/4$

$\frac{1}{16} \times \Gamma 3/4 \times \Gamma (1-3/4)$

$\frac{1}{16} \times \frac{\pi}{\sin \frac{3}{4}\pi}$

$\frac{1}{16} \times \frac{\pi\sqrt{2}}{1} = \frac{\sqrt{2}\pi}{16}$ Ans

$\boxed{\star\ \Gamma n \times \Gamma (1-n) = \frac{\pi}{\sin n\pi}}$

$$B(P,q) = 2\int_0^{\pi/2} \sin^{2P-1}\theta \cos^{2q-1}\theta\, d\theta$$

$$\frac{B(P,q)}{2} = \int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta \qquad m = 2p-1, \quad n = 2q-1$$

$$\frac{1}{2}\left[B\left\{\frac{m+1}{2}, \frac{n+1}{2}\right\}\right] = \int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta$$

$$\frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}} = \int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta.$$

H.P

Q Evaluate $\int_0^1 x^2(1-x)^3 dx$

$\int_0^1 x^{m-1}(1-x)^{n-1} dx$ = fins

Compare

$m-1 = 2 \qquad n-1 = 3$

$m = 3 \qquad n = 4$

$$f(x) = \frac{\Gamma 3 \cdot \Gamma 4}{\Gamma 7} = \frac{2\times 1 \times 3\times 2\times 1}{6\times 5\times 4\times 3\times 2\times 1}$$

$$\Rightarrow \frac{1}{60} \text{ Ans}$$

Q $B(n,3) = \frac{1}{3} = \frac{\Gamma n \times \Gamma 3}{\Gamma n+3}$

$n = (+)$ int $\quad n = ?$

$$\frac{1}{3} = \frac{\cancel{(n-1)}!\,(n-1)! \times 2\times 1}{\Gamma n+3}$$

$$\frac{1}{6} = \frac{\cancel{(n-1)}!}{(n+2)(n+1)\,n \times \cancel{(n-1)}!}$$

$$(n+2)(n+1)n = 6$$

$$\boxed{n = 1}$$

Q $\displaystyle\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta$ (RTU-22)

$n = 6, \quad m = 7$

$$\frac{\Gamma\frac{6+1}{2}\,\Gamma\frac{7+1}{2}}{2\Gamma\frac{6+7+2}{2}} \Rightarrow \frac{\Gamma\frac{7}{2}\,\Gamma 4}{2\,\Gamma\frac{15}{2}}$$

$$\Rightarrow \frac{\cancel{\Gamma\frac{7}{2}} \cdot 3\times2\times1}{2\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\cancel{\Gamma\frac{7}{2}}\times\frac{13}{2}}$$

$$\Rightarrow \frac{8}{11\times7\times13\times3} = \frac{16}{77\times13\times3} \text{ Ans} = \frac{16}{3003}$$

Q $\displaystyle\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta$ (RTU-13, 08)

$$2^2\int_0^{\pi/6} \cos^4 3\theta \sin^2 3\theta \cdot \cos^2 3\theta \, d\theta$$

let $3\theta = t$ $\qquad \theta = \pi/6 \Rightarrow t = \pi/2$

$d\theta = dt/3$ $\qquad \theta = 0 \Rightarrow t = 0$

$$\frac{2^2}{3}\int_0^{\pi/2} \cos^6 t \sin^2 t \, dt$$

$n = 6, \quad m = 2$

$$\frac{4}{3} \times \frac{\Gamma\frac{7}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{10}{2}} \Rightarrow \frac{\cancel{2}}{3} \times \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1}$$

$$\Rightarrow \frac{5\pi}{3\times2^6} = \frac{5\pi}{192} \text{ Ans}$$

64×3

192

Q Evaluate $\int_0^1 x^4(1-x^2)^{5/2}\cdot dx$

let $x^2 = t$ $\qquad x=0 \Rightarrow t=0$

$2x\,dx = dt$ $\qquad x=1 \Rightarrow t=1$

$dx = \dfrac{dt}{2t^{1/2}}$

$$\int_0^1 t^2(1-t)^{5/2}\frac{dt}{2t^{1/2}}$$

$$\frac{1}{2}\int_0^1 t^{3/2}(1-t)^{5/2}\,dt$$

$m-1 = 3/2 \qquad n-1 = 5/2$

$m = 5/2 \qquad n = 7/2$

by Beta function.

$$= \frac{1}{2}\cdot\frac{\Gamma 5/2\cdot\Gamma 7/2}{\Gamma 12/2} = \frac{3/2\cdot 1/2\cdot\sqrt{\pi}\cdot 5/2\cdot 3/2\cdot 1/2\sqrt{\pi}}{2\,\Gamma 6}$$

$$\Rightarrow \frac{3\cdot\cancel{5}\cdot\cancel{3}\cdot\pi}{2^6\cdot\cancel{5}\cdot 4\cdot\cancel{3}\cdot 2\cdot 1}$$

$$\Rightarrow \frac{3\pi}{2^9} \Rightarrow \frac{3\pi}{512}$$

Q $\int_0^\infty \frac{x^2\,dx}{(1+x^4)^3}$

$x^2 = \tan\theta$ $\quad$ $x=0 \Rightarrow \theta = 0$

$2x\,dx = \sec^2\theta\,d\theta$ $\quad$ $x=\infty \Rightarrow \theta = \pi/2$

$$\int_0^{\pi/2} \frac{\tan\theta \cdot \sec^2\theta\,d\theta}{(\sec^2\theta)^3\, 2\times(\tan\theta)^{1/2}}$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}} \times \frac{\cos^{\cancel{6}\,4\,3}\theta}{\cancel{\cos^2\theta}}\,d\theta \cdot \frac{\cos\theta^{1/2}}{(\sin\theta)^{1/2}}$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\, \cos^{3+1/2}\theta\, d\theta$$

$n = 1/2, \quad m = 7/2$

$$\Rightarrow \cancel{\frac{\Gamma\frac{1+1}{2}\cdot\Gamma 4/2}{2\,\Gamma 6/2}} \Rightarrow \cancel{\frac{\Gamma\cdot\sqrt{2}}{2\cdot\sqrt{3}}} \Rightarrow \cancel{\frac{1\cdot 2}{2\cdot 2\cdot 1}}$$

$$\Rightarrow \cancel{\frac{1}{4}} \text{ Ans}$$

$$\Rightarrow \frac{1}{2}\cdot\frac{\Gamma\frac{1/2+1}{2}\cdot\Gamma\frac{7/2+1}{2}}{2\,\Gamma\frac{8/2+2}{2}}$$

$$\Rightarrow \frac{1}{4}\cdot\frac{\Gamma 3/4\cdot\Gamma 9/4}{\Gamma 3} \Rightarrow \frac{1}{4}\,\frac{\Gamma 3/4\cdot\Gamma 9/4}{\times 2\cdot 1}$$

$$\Rightarrow \frac{1}{8}\,\Gamma 3/4\cdot\Gamma 9/4 \text{ A}$$

$$\Rightarrow \frac{1}{8}\times\frac{5}{4}\cdot\frac{1}{4}\cdot\Gamma 3/4\cdot\Gamma 1/4 \Rightarrow \frac{5\sqrt{2}\,\pi}{128} \text{ A}$$

method 2

$$\int_0^\infty \frac{x^2\,dx}{(1+x^4)^3}$$

$$\beta(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx$$

let $x^4 = t$

$4x^3\,dx = dt$

$dx = \frac{dt}{4t^{3/4}}$

$\frac{1}{2}-\frac{3}{4}$

$$= \int_0^\infty \frac{t^{1/2}}{(1+t)^3}\frac{dt}{4t^{3/4}}$$

$$= \frac{1}{4}\int_0^\infty \frac{t^{-1/4}}{(1+t)^3}\,dt$$

$$\boxed{\Gamma\tfrac{3}{4}\cdot\Gamma\tfrac{1}{4} = \sqrt{2}\pi}$$

by comparing

$p-1 = -1/4$

$p = 3/4$

$p+q = 3$

$3/4 + q = 3$

$-q = \frac{9}{4}$

$$\Rightarrow \frac{\frac{1}{4}\Gamma 3/4 \cdot \Gamma 9/4}{\Gamma \frac{12}{4}}$$

$$\Rightarrow \frac{\frac{1}{4}\Gamma 3/4 \cdot \Gamma 9/4}{2!}$$

$$\Rightarrow \frac{1}{8}\Gamma 3/4 \cdot \Gamma 9/4 \quad \text{Ans}$$

$$\Rightarrow \frac{1}{8}\Gamma 3/4 \cdot \frac{5}{4}\cdot\frac{1}{4}\Gamma 1/4$$

$2^3 \cdot 2^4$

$$\Rightarrow \frac{1}{8}\times\frac{5}{16}\times\sqrt{2}\pi$$

$$\Rightarrow \frac{5\sqrt{2}\pi}{2^7} \Rightarrow \frac{5\sqrt{2}\pi}{128} \quad \text{Ans}$$

Q $\displaystyle\int_0^\infty \frac{1}{1+x^4}\,dx$ (RTU - 2019)

method 1

let $x^4 = t$

$4x^3\,dx = dt$

$dx = \dfrac{dt}{4(t)^{3/4}}$

$\dfrac{1}{4}\displaystyle\int_0^\infty \frac{dt}{(1+t)^1}\, t^{-3/4}$

$p-1 = -3/4$ $\qquad p+q = 1$

$p = 1/4$ $\qquad q = 3/4$

$\Rightarrow \dfrac{1}{4}\,\dfrac{\Gamma\frac{1}{4}\cdot\Gamma\frac{3}{4}}{\Gamma\frac{4}{4}}$

$\Rightarrow \dfrac{1}{4}\cdot\sqrt{2}\,\pi$

$\Rightarrow \dfrac{\sqrt{2}\,\pi}{4}$ Ans

method 2

let $x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\,d\theta$

$dx = \dfrac{\sec^2\theta\,d\theta}{2\tan^{1/2}\theta}$

$\dfrac{1}{2}\displaystyle\int_0^{\pi/2} \frac{1}{\cancel{1+\tan^2\theta}}\cdot\frac{\cancel{\sec^2\theta}}{\tan^{1/2}\theta}\,d\theta$

$\dfrac{1}{2}\displaystyle\int_0^{\pi/2} \frac{(\cos\theta)^{1/2}}{(\sin\theta)^{1/2}}\,d\theta$

$\dfrac{1}{2}\displaystyle\int_0^{\pi/2} (\cos\theta)^{1/2}\cdot(\sin\theta)^{-1/2}\,d\theta$

$\dfrac{1}{2}\,\dfrac{\Gamma\frac{1+1/2}{2}\cdot\Gamma\frac{1-1/2}{2}}{2\cdot\Gamma\frac{1/2-1/2+2}{2}}$

$\Rightarrow \dfrac{1}{4}\,\dfrac{\Gamma\frac{3}{4}\cdot\Gamma\frac{1}{4}}{\Gamma 1}$

$\Rightarrow \dfrac{\sqrt{2}\,\pi}{4}$ Ans

Q $\displaystyle\int_0^\infty \frac{x^a}{a^x}\,dx$

let $a^x = e^t$

$\log_e$ लेने पर

$\log_e a^x = \log_e e^t$

$x\log_e a = t$

$x = \dfrac{t}{\log_e a}$

$dx = \dfrac{dt}{\log_e a}$

limit

$x = 0 \to 0$

$x \to \infty \to \infty$

then function is

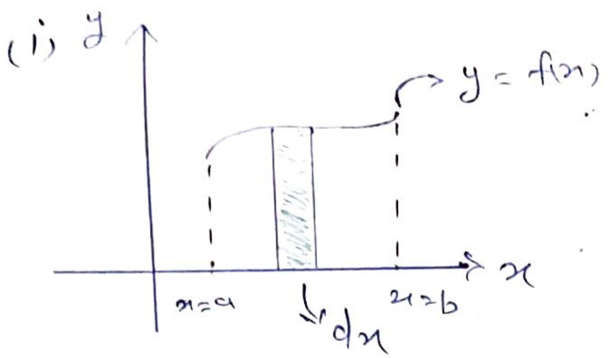
$= \displaystyle\int_0^\infty \left(\frac{t}{\log_e a}\right)^a\cdot\frac{1}{e^t}\,\frac{dt}{\log_e a}$

$= \dfrac{1}{(\log_e a)^{a+1}}\displaystyle\int_0^\infty t^a e^{-t}\,dt$

$= \dfrac{\Gamma(a+1)}{(\log_e a)^{a+1}}$ Ans

* Definite integral & it's application:-

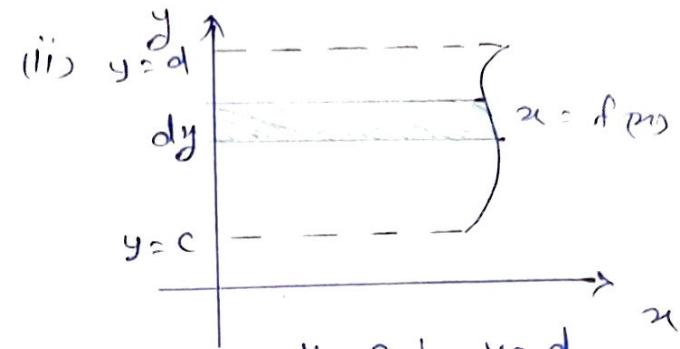
(i)



$x = a$ to $x = b$

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

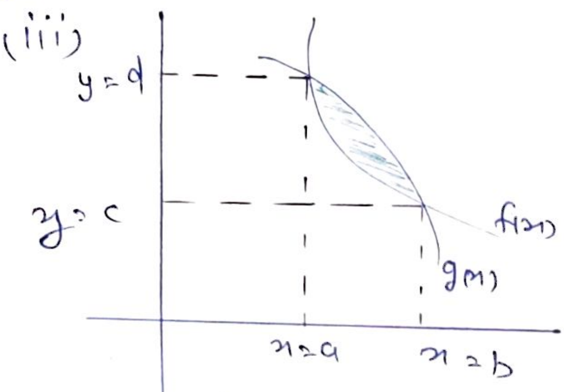
(ii)



$y = c$ to $y = d$

$$A = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

(iii)



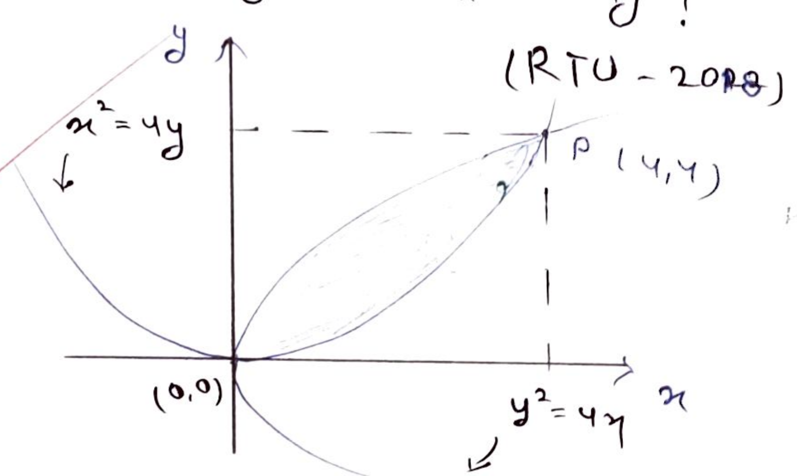
$$A = \int_a^b [f(x) - g(x)]\,dx$$

Q Find area of the curve $y^2 = 4x$ & $x^2 = 4y$?

(RTU - 2020)

$y^2 = 4x$ — (i)

$x^2 = 4y$ — (ii)



from eq (i)

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$x^4 = 64x$$

$$x^4 - 64x = 0$$

$$x(x^3 - 64) = 0$$

$x = 0$ | $x^3 = 64$, $x = 4$

by eq (i)

at $x = 0$, $y = 0$

at $x = 4$, $y = 4$

Now Area,

let $y^2 = 4x = f(x)$

$x^2 = 4y = g(x)$

$$A = \int_0^4 [f(x) - g(x)]\,dx$$

$$A = \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$A = \left[\frac{2 \cdot x^{3/2} \times 2}{3} - \frac{x^3}{3 \cdot 4}\right]_0^4$$

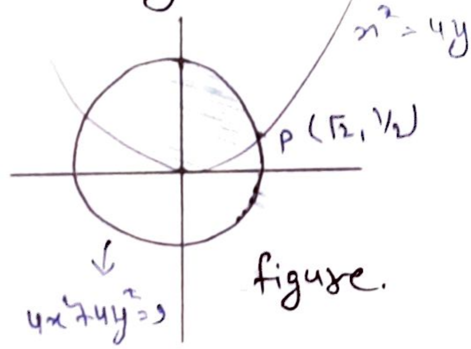
$$A = \frac{1}{3}\left[4x^{3/2} - \frac{x^3}{4}\right]_0^4$$

$$A = \frac{1}{3}\left[4 \cdot 4 \cdot 2 - \frac{4^3}{4} - 0\right]$$

$$A = \frac{1}{3} \cdot 4^2 [2 - 1]$$

$$A = \frac{16}{3} \text{ Answer}$$

Q Find the area of the shaded region.

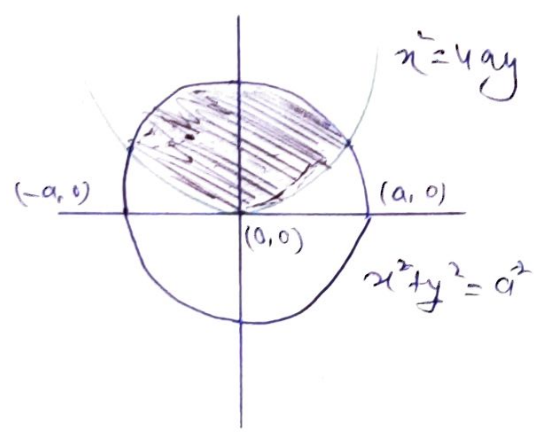


Sol^n

let $f(x)$ is circle of radius a and $g(x)$ is perabola of focal length of a

then $f(x) \Rightarrow y^2 + x^2 = b^2$ — (i)

& $g(x) \Rightarrow x^2 = 4ay$ — (ii)



by eq (i)

$$y^2 + 4ay = b^2$$

in question $a = 1$, $b = 9/2$

$$y^2 + 4y = 9/4 \Rightarrow$$

$\Rightarrow 16y + 4y^2 = 9$

$\Rightarrow 4y^2 + 16y - 9 = 0$

$\Rightarrow 4y^2 + 18y - 2y - 9 = 0$

$\Rightarrow 2y(2y+9) - 1(2y-9) = 0$

$\Rightarrow 2y = 1 \quad 2y = 9$

$y = 1/2$ & $y = 9/2$

$y = 1/2$ पर $x = \sqrt{2}$

$y = 9/2$ पर $x = -\sqrt{2}$

then Area will be $x = -\sqrt{2}$ to $x = \sqrt{2}$

$$A = 2\int_0^{\sqrt{2}} \left[\sqrt{\frac{9}{4} - x^2} - x^2/4\right] dx$$

$$A = 2\int_0^{\sqrt{2}} \left[\sqrt{(3/2)^2 - x^2} - x^2/4\right] dx$$

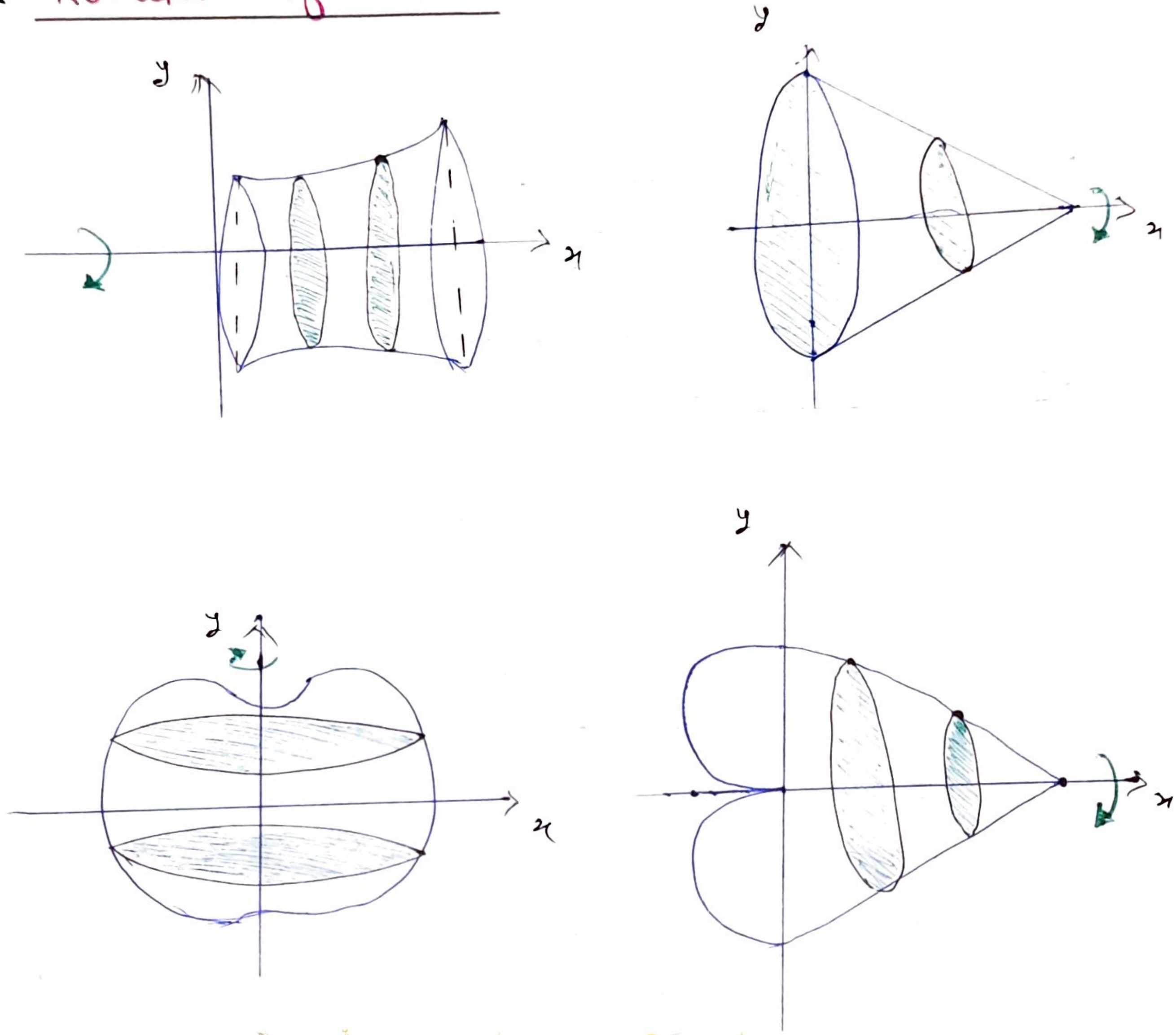
$$A = 2\left[\frac{x}{2}\sqrt{9/4 - x^2} + \frac{9/4}{2}\sin^{-1}\frac{x}{3/2}\right]_0^{\sqrt{2}} + 2\left[\frac{x^3}{3\cdot 4}\right]_0^{\sqrt{2}}$$

$$A = \cancel{2}\left[\frac{\sqrt{2}}{\cancel{2}}\sqrt{9/4 - 2} + \frac{9}{4\cdot\cancel{2}}\sin^{-1}\frac{2\sqrt{2}}{3}\right] - 0 - \frac{\cancel{2}}{3\cdot\cancel{4}_2}\left[2\sqrt{2} - 0\right]$$

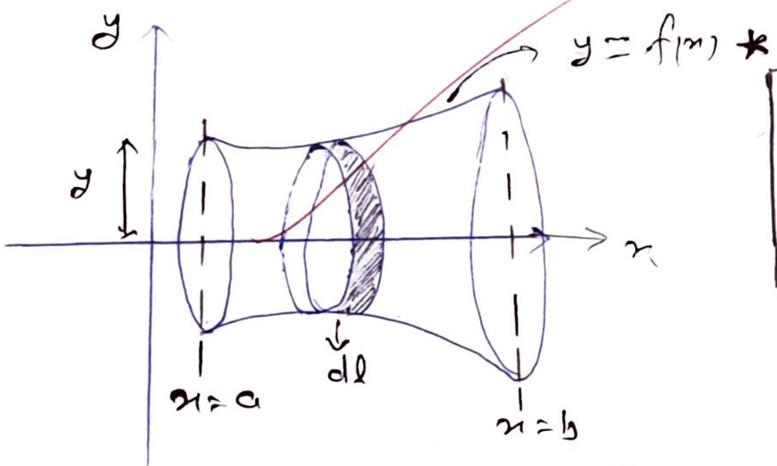
$$A = \frac{\sqrt{2}}{2} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{3}$$

$$A = \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$$ Ans

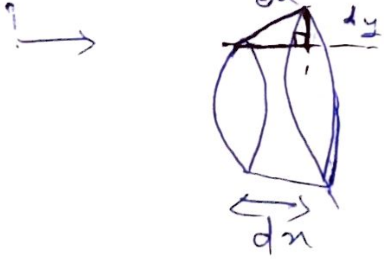
* Rotation of Solids :-

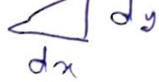


Area of Surface of a Solid of revolution :-



$$\boxed{A = \int_a^b 2\pi y \cdot dl} \quad \text{(i)}$$





$$(dl)^2 = (dx)^2 + (dy)^2$$

$$\left(\frac{dl}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{dl}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

by eq (i)

$$A = \int_a^b 2\pi y \frac{dl}{dx} \cdot dx$$

$$\boxed{A = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx}$$

1. Carteion form :-

(a) Rotate about x-axis then

$$A = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad (y=0)$$

(b) Rotate about y-axis then

$$A = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad (x=0)$$

(c) Rotate about $y = c$

$$A = \int_a^b 2\pi (y-c) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

(d) Rotate about $x = c$

$$A = \int_c^{b} 2\pi (x-c) \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

2. Polar form :-

$r = f(\theta)$

(a) Rotation about x - Axis

$y = r\sin\theta$

$x = r\cos\theta$

$$A = \int_{\theta_1}^{\theta_2} 2\pi (r\sin\theta) \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

(b) Rotation about y - Axis

$$A = \int_{\theta_1}^{\theta_2} 2\pi r\cos\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

3. Parametric form

by formula

$$S = \int_a^b 2\pi y \frac{dl}{dt} \cdot dt$$

by

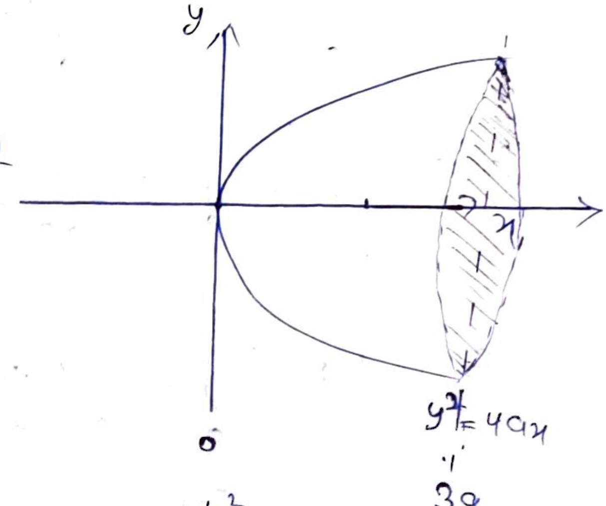
$$(dl)^2 = (dx)^2 + (dy)^2$$

$$\left(\frac{dl}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\star \quad S = \int_a^b 2\pi\, y(t) \left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right) dt$$

Q Find the area of surface of a solid generated by revolving the parabola $y^2 = 4ax$, $0 \le x \le 3a$ about x-axis

Soln



$$S = \int_a^b 2\pi\, y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$y^2 = 4ax \Rightarrow \text{let } x = at^2$$
$$y = 2at$$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$x = at^2$

at $x = 0$, $t = 0$ | $x = 3a$, $t = \sqrt{3}$

$$S = \int_0^{\sqrt{3}} 2\pi (2at) \left(\sqrt{(2at)^2 + (2a)^2}\right) dt$$

$$S = 8\pi \int_0^{\sqrt{3}} a^2 \left(t\sqrt{t^2+1}\right) dt$$

let $t^2 + 1 = m$

$2t\,dt = dm$

$t\,dt = dm/2$

$t = 0,\ m = 1$

$t = \sqrt{3},\ m = 4$

$$S = 8\pi a^2 \int_1^4 \sqrt{m} \cdot \frac{dm}{2}$$

$$S = \frac{8a^2\pi}{2} \left[\frac{2m^{3/2}}{3}\right]_1^4$$

$$S = \frac{8}{3} a^2 \pi \left[4 \cdot 2 - 1\right] \Rightarrow S = \frac{56}{3} \pi a^2 \text{ Ans}$$

Method 2

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2\sqrt{4ax}}$$

$$A = \int_0^{3a} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$A = \int_0^{3a} 2\pi\sqrt{4ax}\sqrt{1+\left(\frac{4a}{2\sqrt{4ax}}\right)^2}\, dx$$

$$A = 4\pi\int_0^{3a}\sqrt{ax}\sqrt{1+\frac{a^2}{ax}}\, dx$$

$$A = 4\pi a^{1/2}\int_0^{3a}\sqrt{x}\sqrt{\frac{x+a}{x}}\, dx$$

$$A = 4\pi a^{1/2}\int_0^{3a}\sqrt{x+a}\, dx$$

let $x+a = t$ | $x=0,\ t=a$

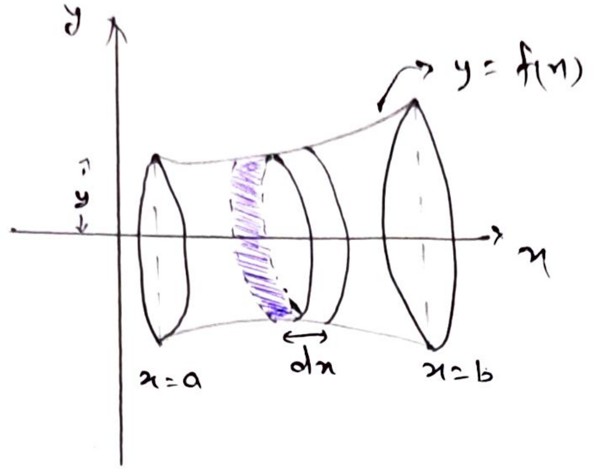
$dx = dt$ | $x=3a,\ t=4a$

$$A = 4\pi a^{1/2}\int_a^{4a}\sqrt{t}\, dt$$

$$A = 4\pi a^{1/2}\left[\frac{2(t)^{3/2}}{3}\right]_a^{4a}$$

$$A = \frac{8\pi}{3}a^{1/2}\left[8a\sqrt{a} - a\sqrt{a}\right] \Rightarrow \frac{56}{3}\pi a^2 \text{ Ans}$$

* Volume of Solids of revolutions :-



$$\boxed{\text{Volume} = \int_a^b \pi y^2\,dx}$$

1. Cartesion form

(a) Rotation about x-axis

$$V = \int_a^b \pi y^2\,dx$$

(b) Rotation about y-axis

$$V = \int_a^b \pi x^2\,dy$$

(c) Rotation about y = c

$$V = \int_a^b \pi (y-c)^2\,dx$$

(d) Rotation about x = c

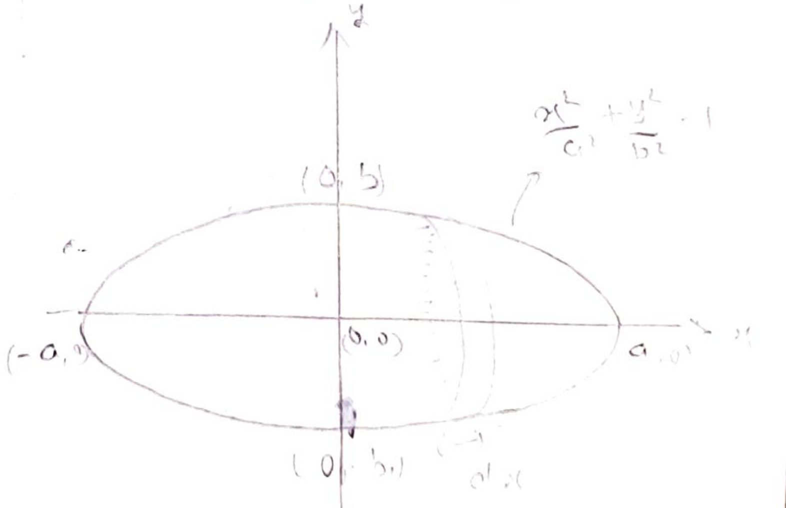
$$V = \int_a^b \pi (x-c)^2\,dy$$

Q Find the surface area and the volumé of the Solid formed by revolving the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ about the major axis (x axis)

(RTU - 1994, 2015, 16, 19, 19)

Sol^n $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



for area

$$A = \int_{-a}^{a} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$\longrightarrow$ (i)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} = 0$$

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

Squaring both side

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{b^2}{a^2}\right)^2 \times (x^2)\,\frac{1}{y^2}$$

putting $y^2 = b^2\left(\frac{a^2 - x^2}{a^2}\right)$

$$\left[\because e = \sqrt{1 - \frac{b^2}{a^2}} \quad (e<1)\right]$$

$$ae = \sqrt{a^2 - b^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{\cancel{b^4}\,b^2}{\cancel{a^4}\,a^2}\,x^2 \times \frac{\cancel{a^2}}{\cancel{b^2}} \times \frac{1}{(a^2 - x^2)}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{b^2}{a^2}\frac{x^2}{(a^2 - x^2)}$$

by eq (i)

$$A = \int_{-a}^{a} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$A = \int_{-a}^{a} 2\pi\, \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{b^2x^2}{(a^2-x^2)\,a^2}}\, dx$$

$$A = 2\pi \int_{-a}^{a} \frac{b}{a}\,\cancel{\sqrt{a^2 - x^2}}\,\sqrt{\frac{a^2 - x^2 + b^2x^2}{a\,\cancel{\sqrt{a^2 - x^2}}}}\, dx$$

$$A = \frac{2\pi b}{a^2}\int_{-a}^{a} \sqrt{a^4 - a^2x^2 + b^2x^2}\, dx$$

$$A = \frac{4\pi b}{a^2}\int_{0}^{a} \sqrt{a^4 - (a^2 - b^2)x^2}\, dx$$

$$A = \frac{4\pi b\left(\sqrt{a^2 - b^2}\right)}{a^2}\int_{0}^{a} \sqrt{\frac{a^4}{a^2 - b^2} - x^2}\, dx$$

by formula

$$\int \sqrt{A^2-x^2}\,dx = \frac{x}{2}\sqrt{A^2-x^2} + \frac{A^2}{2}\sin^{-1}\frac{x}{A} + c$$

$$A = \frac{4\pi b}{a^2}\sqrt{a^2-b^2}\int_0^a \sqrt{\frac{a^4}{a^2-b^2}-x^2}\,dx$$ here $\left(A = \frac{a^4}{a^2-b^2}\right)$

$$A = \frac{4\pi b}{a^2}\sqrt{a^2-b^2}\left[\frac{x}{2}\sqrt{\frac{a^4}{a^2-b^2}-x^2} + \frac{a^4}{2(a^2-b^2)}\sin^{-1}\frac{x\sqrt{(a^2-b^2)}}{a^2}\right]_0^a$$

$$A = \frac{4\pi b}{a^2}\sqrt{a^2-b^2}\left[\frac{a}{2}\sqrt{\frac{a^4}{a^2-b^2}-a^2} + \frac{a^4}{2(a^2-b^2)}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a} - 0\right]$$

$$A = \frac{2\pi b\sqrt{a^2-b^2}}{a^2}\left[a^2\sqrt{\frac{a^2}{a^2-b^2}-1} + \frac{a^4}{a^2-b^2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]$$

$$A = \frac{2\pi b}{\cancel{a^2}}\cancel{\sqrt{a^2-b^2}}\left[1\sqrt{\frac{b^2}{\cancel{a^2-b^2}}} + \frac{a^2}{\cancel{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]\cancel{a^2}$$

$$A = 2\pi b\left[\frac{b}{1} + \frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]$$ Answer

$$\left[A = 2\pi b\left[\frac{b}{1} + \frac{a}{e}(\sin^{-1}e)\right]\right]$$ (in terms of e)

for Volume

$$V = \int_{-a}^{a} \pi y^2\,dx \qquad \because y^2 = b^2\left(1-\frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2-x^2)$$

$$V = 2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$V = \frac{2\pi b^2}{a^2}\int_0^a (a^2-x^2)\,dx = \frac{2\pi b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$V = \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$\boxed{V = \frac{4\pi b^2 a}{3}}$$ Answer

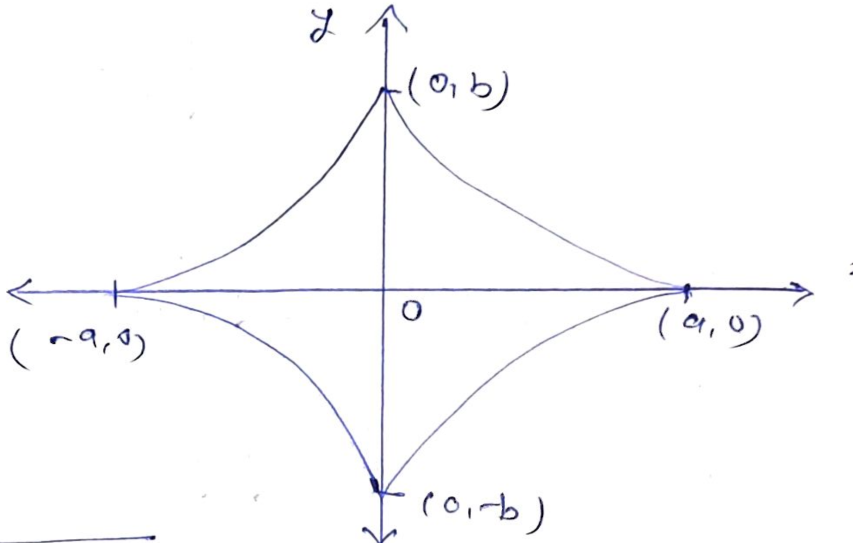
Q Find the surface area & volume of the solid formed by revolving astroid.

$$x^{2/3} + y^{2/3} = a^{2/3}$$

about the x axis.

[RTU - 2023, 2011, 2008, 2004, 2001]

Soln



for area

$$A = \int_{-a}^{a} 2\pi y\, dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$A = 2\pi \int_{-a}^{a} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$A = 4\pi \int_{0}^{a} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{--- i,}$$

$x^{2/3} + y^{2/3} = a^{2/3}$

Let $x = (a \cos^3\theta)$

$y = (a \sin^3\theta)$

$\frac{x}{y} = \tan^3\theta$

$$\frac{dx}{d\theta} = -3a \cos^2\theta \sin\theta$$

limit $y = a \sin^3\theta$

$y = 0$.

$(0, \pi/2)$

$$\frac{dy}{d\theta} = 3a \sin^2\theta \cos\theta$$

$$A = 4\pi \int_{0}^{a} a \sin^3\theta . \sqrt{(3a \sin\theta \cos^2\theta)^2 + (3a \sin^2\theta \cos\theta)^2}\, d\theta$$

$$A = a.4\pi \int_{0}^{\pi/2} \sin^3\theta . 3a \sin\theta \cos\theta \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$$

$$A = 4\pi a \int_0^{\pi/2} \sin^4\theta \cdot \cos\theta \cdot 3a \, d\theta$$

$$A = 12\pi a^2 \int_0^{\pi/2} \sin^4\theta \cos\theta \, d\theta$$

$$A = 12\pi a^2 \frac{\sqrt{\frac{4+1}{2}} \cdot \sqrt{\frac{1+1}{2}}}{2\sqrt{\frac{4+1+2}{2}}}$$

$$A = \frac{12\pi a^2}{2} \frac{\sqrt{5/2} \cdot 1}{\sqrt{7/2}} = \frac{6\pi a^2}{5/2}$$

$$A = \frac{12\pi a^2}{5}$$

Mid term 1

pattern

Sec A (1 marks)
$10 \times 1 = 10$

Sec B (3 marks)
$5 \times 3 = 15$

Sec C (5 mark)
$3 \times 5 = 15$

Total = 40 mark

for volume

$$v = \int_{-a}^{a} \pi y^2 \, dx$$

$$v = 2\int_0^a \pi y^2 \, dx$$

$y = a\sin^3\theta, \quad x = a\cos^3\theta \Rightarrow$

$$v = -2\pi \int_0^{\pi/2} (a\sin^3\theta)^2 (-3a\cos^2\theta)(\sin\theta) \, d\theta$$

$$v = 6\pi a^3 \int_0^{\pi/2} \sin^7\theta \cdot \cos^2\theta \, d\theta$$

$$v = 6\pi a^3 \cdot \frac{\sqrt{8/2} \cdot \sqrt{3/2}}{2 \cdot \sqrt{11/2}} \Rightarrow \frac{3\,\cancel{6}\pi a^3}{\cancel{2}} \frac{\sqrt{4} \cdot \cancel{\sqrt{3/2}}}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \cancel{\sqrt{3/2}}}$$

$$\Rightarrow \frac{\cancel{3} \times 2 \times 1 \times \pi a^3}{3\,\frac{\cancel{9}}{2} \cdot \frac{7}{2} \cdot \frac{5}{2}} \Rightarrow \frac{32\pi a^3}{105}$$

Q. Find S.A. & volume of the solid generated by revolving ~~th~~ by the tractrix.

$$x = a\cos t + \frac{a}{2}\log(\tan^2 t/2) \quad (\text{RTU} - 2003, 2015)$$

$$y = a\sin t.$$

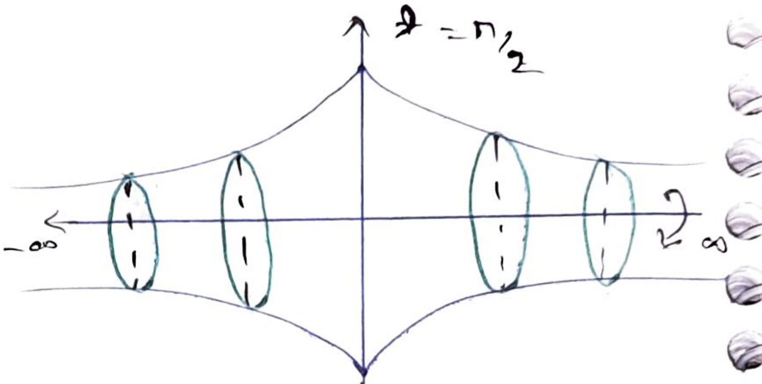
(जिस line पर curve ∞ पर match करता हुआ हो।)

about x axis or its asymptotes.

~~So that~~ Show that S.A. = (S.A. of sphere of radius a)

& volume = $\frac{1}{2}$(volume of sphere of radius a)

Soln



$$x = a\cos t + \frac{a}{2}\log(\tan^2 t/2)$$

$$y = a\sin t$$

$$\frac{dx}{dt} = -a\sin t + \frac{a}{\cancel{2}}\frac{1}{\tan^2 t/2}\cdot\cancel{2}\tan t/2\cdot\sec^2 t/2\left(\frac{1}{2}\right)$$

$$\frac{dx}{dt} = -a\sin t + \frac{a}{\cancel{2}}\frac{\cos t/2}{\sin t/2\cdot\cos^2 t/2}$$

$$\Rightarrow \frac{dx}{dt} = -a\sin t + a\times\frac{1}{\sin(t)} = \frac{-a\sin^2 t + a}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\cos t$$

$$S.A. = \int_a^b 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$S.A. = 2\cdot 2\pi\int_0^{\pi/2} a\sin t\sqrt{\left(-a\sin t + \frac{a}{\sin t}\right)^2 + (a\cos t)^2}\,dt$$

$$S.A. = 4\pi\int_0^{\pi/2} a\sin t\sqrt{a^2\sin^2 t + a^2\cos^2 t + \frac{a^2}{(\sin^2 t)} - 2a^2}$$

$$SA = 4\pi \int_0^{\pi/2} a \sin t \sqrt{a^2 + \frac{a^2}{\sin^2 t} - 2a^2}\, dt$$

$$SA = 4\pi a^2 \int_0^{\pi/2} \sin t \sqrt{\frac{1 - \sin^2 t}{\sin t}}\, dt$$

$$SA = 4\pi a^2 \int_0^{\pi/2} \sqrt{1 - \sin^2 t}\, dt$$

$$SA = 4\pi a^2 \int_0^{\pi/2} \cos t\, dt$$

$$SA = 4\pi a^2 [\sin t]_0^{\pi/2}$$

$$SA = 4\pi a^2 [1 - 0]$$

$$\boxed{SA = 4\pi a^2 = \text{S.A. of Sphere of radius } r}$$

For volume

$$V = 2\int_0^{\pi/2} \pi y^2\, dx$$

$$V = 2\pi \int_0^{\pi/2} (a \sin t)^2 \left(-a \sin t + \frac{a}{\sin t}\right) dt$$

$$V = 2\pi a^3 \int_0^{\pi/2} [(-\sin^3 t) + \sin t]\, dt$$

$$V = 2\pi a^3 \left[-\frac{\Gamma\frac{3+1}{2} \cdot \Gamma\frac{1}{2}}{2 \cdot \Gamma\frac{5}{2}} + \frac{\Gamma\frac{2}{2} \cdot \Gamma\frac{1}{2}}{2\Gamma\frac{3}{2}} \right]$$

$$V = 2\pi a^3 \left[-\frac{\Gamma 2 \cdot \cancel{\Gamma\frac{1}{2}}}{\cancel{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cancel{\Gamma\frac{1}{2}}} + \frac{1 \cdot \cancel{\Gamma\frac{1}{2}}}{2 \cdot \frac{1}{2} \cdot \cancel{\Gamma\frac{1}{2}}} \right]$$

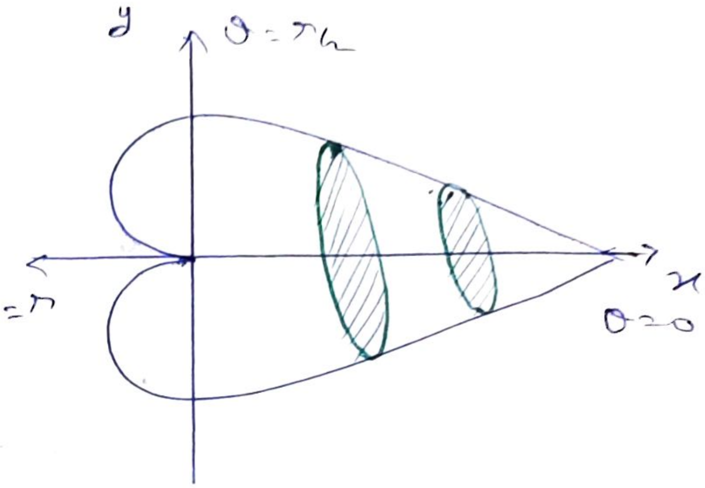
$$V = 2\pi a^3 \left[-\frac{2}{3} + 1 \right]$$

$$\boxed{V = \frac{2}{3}\pi a^3 = \frac{1}{2}(\text{Volume of Sphere of radius } r)}$$

Q Find out surface area and volume of the solid generated by the revolving a Cardioid.

$r = a(1+\cos\theta)$ about the initial line.

Solⁿ $r = a(1+\cos\theta)$



polar form (r, θ)

Cartiection form (x, y)

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan^{-1}(y/x) = \theta$

$\frac{dr}{d\theta} = a(-\sin\theta)$

So

$x = a(1+\cos\theta)\cos\theta$

$x = a\cos\theta + a\cos^2\theta$

$dx = a(-\sin\theta + 2\cos\theta(-\sin\theta))d\theta$

$dx = -a\sin\theta(1 + 2\cos\theta)d\theta$

$y = a(1+\cos\theta)\sin\theta$

$y = a\sin\theta + a\cos\theta\sin\theta$

$\frac{dy}{d\theta} = a\cos\theta + a(-\sin^2\theta) + a\cos^2\theta$

$dy = a[\cos\theta + \cos^2\theta - \sin^2\theta]d\theta$

$\because \cos 2\theta = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta$

$\because SA = \int_0^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

in Polar form

$SA = \int_{\theta_1}^{\theta_2} 2\pi\, y(\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$SA = 2\pi\int_0^{\pi}(r\sin\theta)\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$

$$SA = 2\pi \int_0^{\pi} a(1+\cos\theta)\sin\theta \sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}\, d\theta$$

$$SA = 2\pi a^2 \int_0^{\pi} (1+\cos\theta)\sin\theta \sqrt{2+2\cos\theta}\, d\theta$$

$$SA = 2\sqrt{2}\,\pi a^2 \int_0^{\pi} (1+\cos\theta)\sin\theta \sqrt{1+\cos\theta}\, d\theta$$

let $(1+\cos\theta) = t$ $\qquad \theta = 0 \Rightarrow t = 2$

$-\sin\theta\, d\theta = dt$ $\qquad \theta = \pi \Rightarrow t = 0$

M1

$$SA = 2\sqrt{2}\,\pi a^2 \int_2^0 (t)\, (t)^{1/2}$$

$$SA = 2\sqrt{2}\,\pi a^2 \int_2^0 t \cdot t^{1/2}(-dt)$$

$$SA = 2\sqrt{2}\,\pi a^2 \int_0^2 t^{3/2}\, dt$$

$$SA = 2\sqrt{2}\,\pi a^2 \left[\frac{2t^{5/2}}{5}\right]_0^2$$

$$SA = \frac{4\sqrt{2}}{5}\pi a^2 \left[t^{5/2}\right]_0^2$$

$$SA = \frac{4\sqrt{2}}{5}\pi a^2 (2^{3/2}\cdot 2^1)$$

$$\boxed{SA = \frac{32}{5}\pi a^2}$$

Let M2

$1+\cos\theta = 2\cos^2\theta/2$

$\sin\theta = 2\sin\theta/2 \cos\theta/2$

$$= 2\pi a^2 \int_0^{\pi} 2\cos^2\theta/2 \sin\theta/2 \cos\theta/2 \sqrt{2(2\cos^2\theta/2)} \times 2\, d\theta$$

$$= 8\times 2\pi a^2 \int_0^{\pi} \cos^4\theta/2 \sin\theta/2\, d\theta$$

let $\theta/2 = t$ $\quad | \quad \pi \to \pi/2$

$2dt = d\theta$ $\quad | \quad 0 \to 0$

$$= 16\pi a^2 \int_0^{\pi/2} \cos^4 t \sin t\, dt \times 2$$

$$= 32\pi a^2 \int_0^{\pi/2} \cos^4 t \sin t\, dt$$

$$SA = 32\pi a^2 \left(\frac{\Gamma(5/2)\cdot\Gamma(2/2)}{2\,\Gamma(7/2)}\right)$$

$$\boxed{SA = \frac{32}{5}\pi a^2}$$ Answer

for volume

$$V = \int_a^b \pi y^2 \, dx$$

$x = r\cos\theta$

$x = a(1+\cos\theta)\cos\theta =$

$dx = -a\sin\theta(1+2\cos\theta)\,d\theta$

$y = r\sin\theta$

$y = a(1+\cos\theta)\sin\theta$

$$V = \pi\int_0^{\pi} a^2(1+\cos\theta)^2\sin^2\theta \cdot (-a\sin\theta(1+2\cos\theta)\,d\theta)$$

$$V = 2\pi a^3 \int_0^{\pi/2} (1+\cos\theta)^2(1+2\cos\theta)\sin^3\theta \, d\theta$$

let $\cos\theta = t \Rightarrow -\sin\theta \, d\theta = dt$

$\theta = 0 \Rightarrow t = 1$

$\theta = \pi/2; \; t = 0$

$$V = 2\pi a^3 \int_1^0 (1+t)^2(1+2t)(1-t^2)(dt).$$

$$V = -2\pi a^3 \int_{-1}^{1} (1+t^2+2t)(1-2t^3-t^2+2t)\,dt$$

$$V = -2\pi a^3 \int_{-1}^{1} (1+t^2+2t)(1+2t-2t^3-t^2)\,dt$$

$$V = -2\pi a^3 \int_{-1}^{1} (1+4t+4t^2-2t^3-5t^4-2t^5)\,dt$$

$$V = -2\pi a^3 \left[t + \frac{4t^2}{2} + \frac{4t^3}{3} - \frac{2t^4}{4} - \frac{5t^5}{5} - \frac{2t^6}{6}\right]_{-1}^{1}$$

$$V = -2\pi a^3 \left[1 + 2 + \frac{4}{3} - \frac{1}{2} - 1 - \frac{1}{3} + 1 - 2 + 1 \times \frac{4}{3} + \frac{1}{2} - 1 + \frac{1}{3}\right]$$

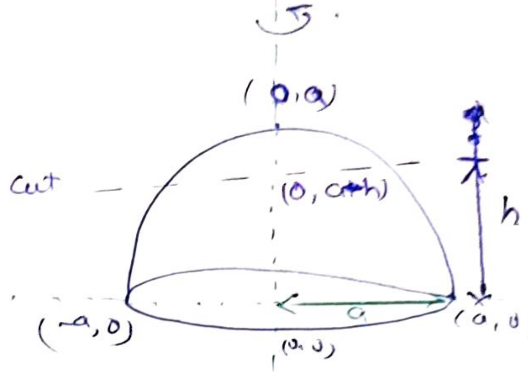
$$V = -2\pi a^3 \left[3 - \tfrac{1}{2}\right\} - 2 + \tfrac{5}{3} + \tfrac{1}{2}\Big]$$

$$\boxed{V = -\frac{8\pi a^3}{3} \quad \text{Ans}}$$

Q Show that the volume of a spherical cap of height h cut off from a sphere of radius a is $\pi h^2\left(a - \frac{h}{2}\right)$

A =

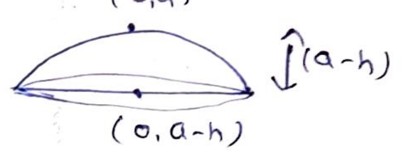
$$V = \int_a^b \pi y^2\,dx$$



$$V = \int_{a-h}^{h} \pi x^2\,dy \quad (\text{for } y \text{ axis})$$

eq $= x^2 + y^2 = a^2$

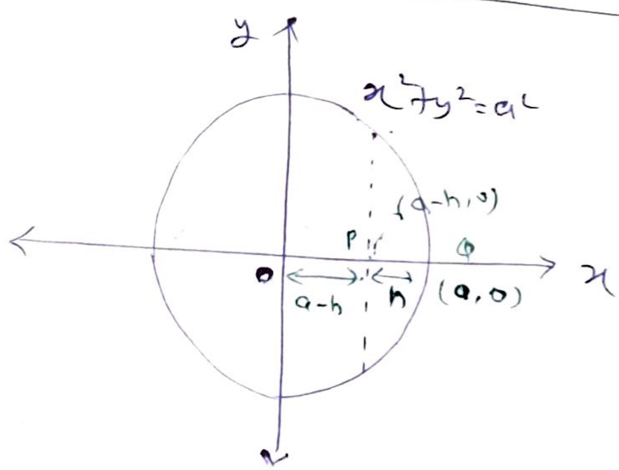
⇓



[for y axis]

$$V = \int_{a-h}^{a} \pi y^2\,dx \quad (\text{for } x \text{ axis}$$

$$y^2 = a^2 - x^2$$



$$V = \int_{a-h}^{a} \pi(a^2 - x^2)\,dx$$

$$V = \pi \int_{a-h}^{a} (a^2 - x^2)\,dx$$

$$V = \pi\left[a^2x - \frac{x^3}{3}\right]_{a-h}^{a}$$

$$\boxed{(a-b)^3 = a^3 - b^3 - 3ab(a-b)}$$

$$V = \pi\left[a^2 a - a^2(a-h) - \frac{a^3}{3} + \frac{(a-h)^3}{3}\right]$$

$$V = \pi\left[\cancel{a^3} - \cancel{a^3} + a^2h - \frac{a^3}{3} + \frac{a^3 - h^3 - 3ah(a-h)}{3}\right]$$

$$V = \pi\left[\frac{3a^2h - a^3 + a^3 - h^3 - 3a^2h + 3ah^2}{3}\right]$$

$$\cancel{V = \pi\left[\frac{3a^2h - 2a^3 - 2h^3 + 3ah^2}{3}\right]}$$

$$v = \frac{\pi a h}{3}\left[3a - 2a^2 - 2h^2 + 3h - \right]$$

$$v = \frac{\pi a h}{3}\left[3(a+h) - 2(a^2+h^2)\right]$$

$$v = \frac{\pi}{\cdot}\left[\frac{3ah^2 - h^3}{3}\right]$$

$$\boxed{v = \pi h^2\left[a - \frac{h}{3}\right]}$$

Hance proved