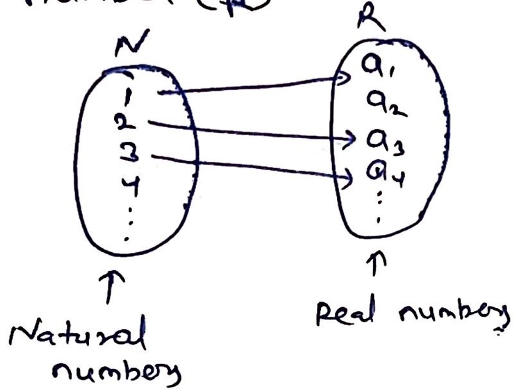
Unit-2 Sequence & series

★ Sequence :- A sequence is a function from the domain set of natural numbers (N) to any set of real number (R)

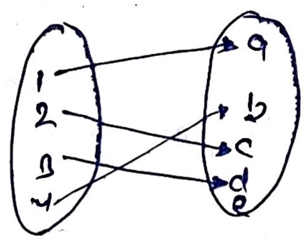


Note

Domain = {1, 2, 3, 4}

Codomain = {a, b, c, d, e}

Range = {a, b, c, d}



★ Real Sequence :-

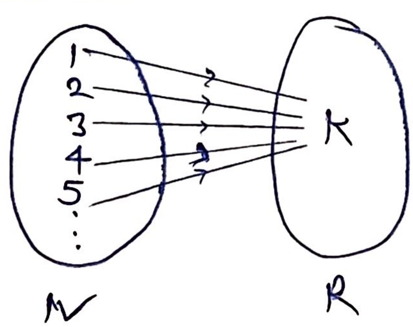
All the element in given sequence → Real.

Ex (1) {1, 2, 3, 4, ...}

(2) {1/2, 1/3, 1/4, ...}

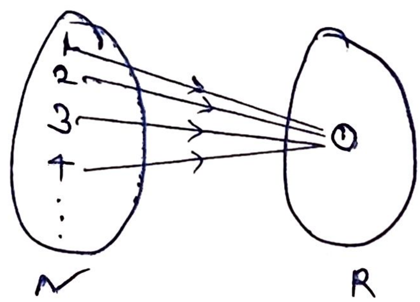
(3) {-1, -2, -3, -4 ...}

★ Constant Sequence :-



{K, K, K, ...}

★ Null Sequence :-



{0, 0, 0, ...}

* Bounded and Unbounded Sequence :-

(i) $\{1, 4, 9, 16 \ldots\}$

L.B. = 1
U.B = Does not exist } Unbounded.

(ii) $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$

Lower value = Does not exist
Upper value = 1 } Unbounded
(0, 1]

(iii) $\{-1, 1, -1, +1, -1, 1 \ldots\}$

Lower value = -1
Upper value = +1 } bounded.

* Monotonic Sequence :-

Ex (i) $\{1, 4, 9, 16, 25 \ldots\}$ → monotonically increasing sequence
$(a_n > a_{n+1})$

$\{1, 1/2, 1/3, 1/4 \ldots\}$ ⇒ monotonically decreasing sequence.
$(a_n < a_{n+1})$

★ Convergence, Divergence & Oscillation of a Sequence

Convergence

Sequence ⇒ $a_1, a_2, a_3, a_4, \ldots a_n, \ldots$

$\lim_{n \to \infty} a_n = L$ (finite value)

- **Divergence** ⇒ $\lim_{n \to \infty} a_n = \pm\infty$

- **Oscillation** $\lim_{n \to \infty} a_n = L$ (finite but not unique)

(a) Finite Oscillation

$\{-1, 1, -1, 1, -1, \ldots\}$ $a_n = (-1)^n \begin{cases} -1 & n: \text{odd} \\ +1 & n: \text{even} \end{cases}$

(b) Infinite Oscillation

$\lim_{n \to \infty} a_n = \pm\infty$ but Oscillation.

$\{-1, +4, -9, 16, -25 \ldots\}$

$a_n = (-1)^n \times n^2$

Q Check Convergence or divergence.

$\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots\}$

$a_n = \frac{n}{n+1}$

$\lim_{n \to \infty} \frac{n}{n+1} \Rightarrow \lim_{n \to \infty} \frac{1}{1} = 1$ Ans = finite

Convergence.

* Standard limits

(1) $\lim_{n\to\infty} \frac{1}{n} = 0$

(4) $\lim_{n\to\infty} (n)^{1/n} = 1$

(2). $\lim_{n\to\infty} \frac{1}{(n)^2} = 0$

(5) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(3) $\lim_{n\to\infty} = \frac{1}{\sqrt{n}} = 0$

(6) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{x} = e^{x}$

(7) $\lim_{n\to\infty} (x)^{1/n} = 1$ for $x>0$

(8) $\lim_{x\to\infty} x^n = 0$ for $|x|<1$

Q Find the limit of the sequence $\langle x_n \rangle$, where

$$x_n = \frac{5n-3}{7n+8} \quad \text{(RTU-2023)}$$

$$a_n = \frac{5n-3}{7n+8} = \frac{5-3/n}{7+8/n}$$

$$\lim_{n\to\infty}\left(\frac{5n-3}{7n+8}\right)$$

$$\frac{5-0}{7+0} = 5/7$$

$= 5/7$ Ans

Convergence

Q $a_n = \frac{n^2-n}{2n^2+n}$

$$\lim_{n\to\infty} \frac{n^2-n}{2n^2+n} = \frac{n^2(\;)}{n^2}$$

$$\lim_{n\to\infty} \frac{n^2(1-1/n)}{n^2(2+1/n)} = \frac{1}{2} \text{ Ans}$$

convergence

Q $a_n = e^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^n = \lim_{n\to\infty} n e^{n-1} \cdot \lim_{n\to\infty} n(n-1) e^{n-2} \ldots$$

$$= \infty \text{ divergence}$$

Q $a_n = e^{-n}$

$$\lim_{n\to\infty} e^{-n} = e^{-\infty} = 0 \text{ Ans}$$

Q. $a_n = \tan h n$ → (Hyperbolic tan)

* $\sinh x = \dfrac{e^x - e^{-x}}{2}$ (hyperbolic Sinx)

* $\cosh x = \dfrac{e^x + e^{-x}}{2}$ (hyperbolic Cos)

$$a_n = \tanh n = \frac{e^n - e^{-n}}{e^n + e^{-n}}$$

$$\lim_{a_n\to\infty} = \lim_{a_n\to\infty} \frac{e^n(1 - e^{-2n})}{e^n(1 + e^{-2n})}$$

$$\lim_{a_n\to\infty} = \lim_{a_n\to\infty} \left(\frac{1 - 0}{1 + 0}\right)$$

$$= 1 \text{ Ans. (Convergence)}$$

Series

if sequence $\Rightarrow a_1, a_2, a_3 \dots a_n \dots$

then series is sum of each term of sequence.

$S_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$

- $S_n = a_1 + a_2 + \dots a_n = \sum_{K=1}^{n} a_K$
- $S_\infty = a_1 + a_2 + a_3 \dots = \sum_{K=1}^{\infty} a_n$

* Convergence, divergence & Oscillation of series :-

(1.) Convergence :-

if $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{K=1}^{n} a_K = L(\text{finite}) = \sum_{n=1}^{\infty} a_n$

then series is convergence.

Ex

(2.) Divergence :-

if $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{K=1}^{n} a_K = \pm\infty$

then series is divergence.

Ex ① $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$ $= \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}}$

$a_n = a\left(\frac{1+r^n}{1-r}\right)$

$= 2$ Ans

series is convergence.

Q 2 $1+2+2^2+2^3+2^4+\ldots$

Sol^n $S_\infty = 1+2+2^2+2^3+2^4+\ldots$

$a_n = (2)^n$

$\lim_{n\to\infty} (2^n) = \infty$ series is divergence.

Q 3 $1^2+2^2+3^2+4^2+\ldots$

$S_\infty = 1+2+2^2+3^2+4^2+\ldots$

$a_n = (n)^2$

$\lim_{n\to\infty} (n^2) = \infty$. series is divergence.

$S_n = \frac{n(n+1)(2n+1)}{6}$ (Sum of n terms).

$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} = \infty$. Ans

(3) Oscillation :-

if $\lim_{n\to\infty} S_n$ = finite (but not unique)

Ex $S = +1-1+1-1+1-1+1\ldots$

$= \begin{cases} 0 & n = \text{even} \\ 1 & n = \text{odd} \end{cases}$ series is finite Oscillation

Ex. $S = 1-2+3-4+5-6+\ldots$

$S = (1+3+5+\ldots) - (2+4+6+\ldots)$ $\begin{cases} -\infty, & n = \text{even} \\ +\infty, & n = \text{odd} \end{cases}$

$S = 1+(n-1)2 - 2(1+(n-1)2)$

$S = -(1+(n-1)2)$

$S = -2n+1$ So series is infinite oscillation.

$$\begin{Bmatrix} -n/2 & n = \text{even} \\ \left(\frac{n+1}{2}\right) & n = \text{odd} \end{Bmatrix} = \begin{Bmatrix} -\infty & n = \text{even} \\ +\infty & n = \text{odd} \end{Bmatrix}$$

Q Check series is convergence or Divergence?

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots \infty$$

$$a_n = \frac{1}{n(n+1)} = \frac{(n+1) - n}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

$$S_n = \sum_{n=\infty} \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{n} - \frac{1}{(n+1)}\right]$$

$$S_n = \left\{ \begin{array}{l} 1 - \cancel{1/2} \\ \cancel{1/2} - \cancel{1/3} \\ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \\ \cancel{1}\cdot \quad -- \\ \cdot\ \cdot \quad -- \\ \cancel{1/n} - \cancel{1/n+1} \\ \cancel{1}\cdot \quad - \\ \cdot\ \cdot \quad - \\ \frac{1}{0} - \frac{1}{0} \end{array} \right. = 1 - 0 = 1$$

$S_n = 1$ (finite)

series is convergence.

Test for convergence or divergence

(1.) **GP test**

Series $= 1 + x + x^2 + x^3 + \dots$

(i) if $|x| < 1$ or $-1 < x < 1$, then series is convergence

(ii) if $x \geq 1$, then series is divergence

(iii) if $x = -1$, then series is oscillation (finite)

(iv) if $x < -1$, then series is oscillation (Infinite)

Q Series $= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

Soln by GP test

$x = 1/2$ i.e. $|x| < 1$

series is convergence,

Q Series $= 1 + 3 + 3^2 + 3^3 + \dots$

Soln by G.P. test

$x = 3$

i.e. $|x| \geq 1$

series is divergence.

Q Series $= 1 + (-2) + (-2)^2 + (-2)^3 + (-2)^4 + \dots$

Soln by G.P. test,

$x = -2$

i.e. $\cancel{|x| < 1}$ $x < -1$

Series is oscillation (Infinite)

(2) Auxillary Series Test / Harmonic series test ((P Series)

series $= \sum_{n=1}^{\infty} \frac{1}{(n)^p}$ $\left[a_n = \frac{1}{(n)^p}\right]$

$= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

(i) if $p > 1$ then series is convergence

(ii) if $p \leq 1$ then series is divergence

Q Series $U_n = \frac{1}{\sqrt{2n}}$ test for C/D?

$U_n = \frac{1}{\sqrt{2n}}$

Series $= \frac{1}{\sqrt{2}}\left[\frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots\right]$

$= \frac{d}{\sqrt{\cdot}}$ $p = 1/2$

$p \leq 1$

So series is divergence

Q Series $1^{-2} + 2^{-2} + 3^{-2} + 4^{-2} \cdots$

Series $= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

$p = 2$

$\Rightarrow p > 1$

So series is convergence.

* Limit of comparision test -

$\sum U_n$ = Given

Take $\sum V_n$ such that

$$\lim_{n \to \infty} \frac{U_n}{V_n} = l \text{ (finite)}$$

If V_n is convergence then U_n is convergence

& If V_n is divergence then U_n is divergence

Q Test of Convergence?

$$\frac{1.2}{3.4.5} + \frac{2.3}{4.5.6} + \frac{3.4}{5.6.7} + \ldots$$

Sol^n $U_n = \dfrac{n(n+1)}{(n+2)(n+3)(n+4)}$

So let take $V_n = \dfrac{1}{n}$

$$\lim \frac{U_n}{V_n} \Rightarrow \lim_{n \to \infty} \frac{n(n+1)}{(n+2)(n+3)(n+4)} \times \frac{n}{1}$$

$$\Rightarrow \lim_{n \to \infty} \frac{\cancel{n} \times (1+1/n) \times \cancel{n}}{\cancel{n} \times \cancel{n} \times \cancel{n}(1+2/n)(1+3/n)(1+4/n)} \times \cancel{n}$$

$$\Rightarrow \lim_{n \to \infty} \frac{(1+1/n)}{(1+2/n)(1+3/n)(1+4/n)}$$

$\Rightarrow$ 1 (finite) So Now check for $V_n = 1/n$

by P series test (P is power = 1)

$$P = \sum_{n=1}^{\infty} 1/n = \frac{1}{1} + 1/2 + 1/3 + \ldots \Rightarrow \boxed{\cancel{P \leq 1}}$$ ~~for~~ $P \leq 1$

V_n is ~~Convergence~~ Divergence, Hence U_n is also divergence.

Q Check the convergence ? (RTU - 2022)

$$\frac{1.2}{3^2 \cdot 4^2} + \frac{3.4}{5^2 \cdot 6^2} + \frac{5.6}{7^2 \cdot 8^2} + \ldots ,$$

Series is

$$a_n = \frac{(2n+1)(2n+2)}{(2n+3)^2 (2n+4)^2} \quad \text{for } n = 0 \text{ to } \infty$$

$$U_n = a_n = \frac{(2n-1)(2n)}{(2n+1)^2 (2n+2)^2} \quad \text{for } n = 1 \text{ to } \infty.$$

let $V_n = 1/n^2$

appling limit

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{(2n-1)(2n)}{(2n+1)^2 (2n+2)^2} \times \frac{n^2}{1}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{(2-1/n)(2)\cancel{n}\cdot\cancel{n}}{(2+1/n)^2 \cdot \cancel{n^2} (2+2/n)^2 \cdot n^2} \times n^2$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{(2-1/n)\cdot 2}{(2+1/n)^2 (2+2/n)^2}$$

$$= \frac{\cancel{2}\cdot 2}{\cancel{2^2}\cdot 2^2}$$

$$= \frac{4}{} \; 1/4 \text{ (finite)}$$

So now check for $V_n = 1/n^2$

$\cdot$ P (Power = 2)

So $P > 1$ series is (V_n) convergence

Hence U_n series is convergence.

Q Check for convergence?

$$S_n = \sum_{n=1}^{\infty} \left[\frac{(2^n+3)}{(3^n+1)}\right]^{1/2}$$

Sol^n Given series

$$U_n = \left(\frac{2^n+3}{3^n+1}\right)^{1/2}$$

let $V_n = \left(\frac{2}{3}\right)^{n/2}$

applying limit on $\frac{U_n}{V_n}$.

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \left(\frac{2^n+3}{3^n+1}\right)^{1/2} \times \left(\frac{3}{2}\right)^{n/2}$$

$$= \lim_{n\to\infty} \frac{2^{n/2}\left(1+\frac{3}{2^n}\right)^{1/2}}{3^{n/2}\left(1+\frac{1}{3^n}\right)^{1/2}} \times \left(\frac{3}{2}\right)^{n/2}$$

$$= \lim_{n\to\infty} \cancel{\left(\frac{2}{3}\right)^{n/2}} \left[\frac{1+3/2^n}{1+1/3^n}\right]^{1/2} \times \cancel{\left(\frac{3}{2}\right)^{n/2}}$$

$$= \lim_{n\to\infty} \left[\frac{1+3/2^n}{1+1/3^n}\right]^{1/2}$$

$= 1$ Answer (finite) $V_n = \frac{2^{n/2}}{3^{n/2}} = \frac{1}{(3/2)^{n/2}}$

~~So V_n is~~

by GP test $r = \sqrt{2/3} \in (-1, 1)$ ~~$p = n/2 = 1/2$~~

So V_n is ~~divergence~~ convergence.

Hence U_n is also convergence.

***Note** Necessary condition for tests. (Convergence)

(1) If Series is convergence then

$$\lim_{n\to\infty} U_n = 0$$

but reverse is not true.

If ~~series~~ limit of U_n :

$$\lim_{n\to\infty} U_n = 0$$

then series may be convergence or divergence.

(2) If $\lim_{n\to\infty} U_n \neq 0$ (finite)

Series is divergence.

(4) Cauchy's n^{th} roof test / n^{th} root test / Root test :-

$\sum U_n \to$ series

↳ all terms $\Rightarrow$ (+)ive

$$\lim_{n\to\infty} (U_n)^{1/n} = l \text{ (finite)}$$

(i) $l > 1 \Rightarrow$ Series is divergence

(ii) $l < 1 \Rightarrow$ Series is convergence.

(iii) $l = 1 \Rightarrow$ test fail.

Ex $\cancel{\left[\frac{(1+1/n)}{e^n}\right]^{2n}} = S_n = \sum_{n=1}^{\infty} \frac{(1+1/n)^{2n}}{e^n}$ Series is convergence or divergence

Sol^n $\sum_{n=1}^{\infty} \frac{(1+1/n)^{2n}}{e^n}$

let limit of

$$\lim_{n\to\infty} (U_n)^{1/n} \Rightarrow \lim_{n\to\infty} \left[\frac{(1+1/n)^{2n}}{e^n}\right]^{1/n}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1+1/n}{e}\right)$$

$\Rightarrow$ $1/e < 1$ So series is convergence.

Q $\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n}$ Check series is convergence or divergence

Soln $U_n = \left(\frac{x^2}{2}\right)^n$

applying n^{th} root test

$$\Rightarrow \lim_{n\to\infty}\left[\left(\frac{x^2}{2}\right)^n\right]^{1/n}$$ if $\frac{x^2}{2} < 1$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{x^2}{2}\right)$$ then $x < \sqrt{2}$

then Series is convergence.

& if $\frac{x^2}{2} > 1$

then $x > \sqrt{2}$

series will be divergence.

$\frac{x^2}{2} = 1$

$x = \sqrt{2}$

Test fail.

let put $x = \sqrt{2}$ in U_n

$U_n = 1$

$$\lim_{n\to\infty}(U_n) = \lim_{n\to\infty}(1) = 1$$

From necessary condition of convergence.

at $x = \sqrt{2}$

Series is divergence.

Finally

$x < \sqrt{2} \Rightarrow$ Convergence

$x \geq \sqrt{2} \Rightarrow$ Divergence

(5) D'Alembert's Ratio Test / Ratio test :-

$\sum U_n \Rightarrow$ series with positive terms.

then $\lim \dfrac{U_n}{U_{n+1}} = l$ or $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n} = l$

(i) $l > 1 \Rightarrow$ Convergence
(ii) $l < 1 \Rightarrow$ Divergence
(iii) $l = 1 \Rightarrow$ Test fail

(i) $l > 1 \Rightarrow$ Divergence
(ii) $l \nless 1 \Rightarrow$ Convergence
(iii) $l = 1 \Rightarrow$ Test Fail.

Q $\dfrac{2}{1} + \dfrac{2\cdot5}{1\cdot5} + \dfrac{2\cdot5\cdot8}{1\cdot5\cdot9} + \ldots,$ Check for C/D

Soln

$$U_n = \frac{2\cdot5\cdot8\cdots\cdots(3n+2)(3n-1)}{1\cdot5\cdot9\cdots\cdots(4n-3)(4n+1)}$$

Applying ; test (Ratio test)

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(4n+1)}{(3n+2)}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{4+1/n}{3+2/n}\right)$$

$\Rightarrow \; 4/3 > 1$.

then series is Convergence.

Q. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$ or $\sum_{n=1}^{\infty} \left(\frac{x^{2n} \cdot n}{n^2+1}\right)^{1/2}$

Sol^n $U_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$ $U_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n^2+2n+2)}} x^{n+1}$

Now applying Ratio test -

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{\sqrt{n}\, x^n}{\sqrt{n^2+1}} \times \frac{\sqrt{n^2+2n+2}}{\sqrt{n+1}\; x^{n+1}}$$

$$\Rightarrow \frac{1}{x} \lim_{n\to\infty} \cdot \frac{\cancel{\sqrt{n}}}{\cancel{n}\sqrt{1+1/n^2}} \cdot \frac{\cancel{n}\sqrt{1+2/n+2/n^2}}{\cancel{\sqrt{n}}\sqrt{1+1/n}}$$

$$\Rightarrow \frac{1}{x} \lim_{n\to\infty} \left[\frac{(1+2/n+2/n^2)}{(1+1/n^2)(1+1/n)}\right]^{1/2}$$

$\Rightarrow \frac{1}{x}$ (i) if $1/x > 1 \Rightarrow x < 1 \Rightarrow$ Convergence

(ii) if $1/x < 1 \Rightarrow x > 1 \Rightarrow$ divergence

(iii) if $1/x = 1 \Rightarrow x = 1 \Rightarrow$ Test fail.

$U_n = \frac{\sqrt{n}}{\sqrt{n^2+1}}$ $\left(V_n = 1/\sqrt{n}\right)$

$$\lim_{n\to\infty}\left(\frac{U_n}{V_n}\right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} \times \frac{\sqrt{n}}{1}$$

(finite value)

let $V_n = 1/\sqrt{n}$

by P Series

$P = 1/2 < 1 \Rightarrow$ Convergence

at $x=1$, series is Convergence

(6) Raabe's test :-

$\Sigma U_n \rightarrow$ Series with +ive terms

such that

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = l \text{ (finite)}$$

(i) $l > 1 \Rightarrow \Sigma U_n \Rightarrow$ Convergence (iii) $l = 1 \Rightarrow$ test fails

(ii) $l < 1 \Rightarrow \Sigma U_n \Rightarrow$ divergence.

Note It is applicable after failure of ratio test.

Q Test for convergence?

$$\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \dots$$

Sol^n

$$U_n = \frac{1.3.5.7\dots(2n-1)}{2.4.6.8\dots(2n)}$$

$$U_{n+1} = \frac{1.3.5.7\dots(2n-1)(2n+1)}{2.4.6.8\dots 2n(2n+2)}$$

applying Ratio test

$$\lim_{n\to\infty}\left(\frac{U_n}{U_{n+1}}\right) = \lim_{n\to\infty}\frac{(2n+2)}{(2n+1)}$$

$$\Rightarrow \lim_{n\to\infty}(1)$$

$\Rightarrow 1$ Answer

test fail

applying raabe's test,

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right)$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right)$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{1}{2n+1}\right)$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{2}\right)$$

$\Rightarrow \frac{1}{2} < 1$ Divergence

Q Test for convergence? [RTU-2019]

$$\sum_{n=1}^{\infty} \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2}$$

Soln

$$U_n = \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2}$$

$$U_{n+1} = \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots (4n-3)^2 (4n+1)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2 (4n+4)^2}$$

$$\lim_{n\to\infty} \left(\frac{U_n}{U_{n+1}}\right) \Rightarrow \lim_{n\to\infty} \frac{(4n+4)^2}{(4n+1)^2}$$

$\Rightarrow 1$ Test fail.

applying raabe's test

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right)$$

$$\lim_{n\to\infty} n\left(\frac{(4n+4)^2}{(4n+1)^2} - \right)$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{(4n+4)^2-(4n+1)^2}{(4n+1)^2}\right)$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{15+24n}{(4n+1)^2}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{24}{16}\right)$$

$\Rightarrow \frac{3}{2} > 1$ Convergence.

7.) De Morgans test / Bertrand's test :-

$\Sigma U_n \rightarrow$ series with +ve terms

such that,

$$\lim_{n\to\infty}\left[n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right]\log n = l \text{ (finite)}$$

(i) $l > 1 \rightarrow$ Series is convergence

(ii) $l < 1 \rightarrow$ Series is divergence

(iii) $l = 1 \rightarrow$ test fail.

Q Check for convergence ?

$$\frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \cdots$$

Sol^n $U_n = \dfrac{1+2+3+\dots+n}{1^2+2^2+3^2+\dots+n^2} = \dfrac{n\left(\frac{n+1}{2}\right)}{\frac{n(n+1)(2n+1)}{6}}$

$U_n = \dfrac{3}{(2n+1)}$ now $U_{n+1} = \dfrac{1+2+3+\dots n+1}{1^2+2^2+3^2+\dots(n+1)^2}$

$$U_{n+1} = \frac{3}{(2n+3)}$$

applying Ratio test

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{\frac{3}{2n+1}}{\frac{3}{2n+3}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2n+3}{2n+1}$$

$\Rightarrow 1$ test fail

applying Raabe's test

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right)$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{2n+3}{2n+1} - 1\right)$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{1}{2n+1}\right)$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{2}{2+1/n}\right)$$

$\Rightarrow 1$ test fail

Now

Applying De-morgan's test

$$\lim_{n\to\infty} \left[n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right] \log n$$

$$\lim_{n\to\infty} \left[\frac{2}{2+1/n} - 1\right] \log n$$

$$\lim_{n\to\infty} \frac{-1/n}{2+1/n} \times \log n$$

$$\lim_{n\to\infty} \frac{-\log n}{2n+1} \quad (L)$$

$$\lim_{n\to\infty} \frac{\left(-\frac{1}{n}\right)}{2}$$

$$\lim_{n\to\infty} \frac{-1}{2n} = 0 < 1$$

So series is divergency

8.) Logarithmic test :-

$\sum U_n \rightarrow$ series with +ve terms

such that

$$\lim_{n\to\infty} n \log \frac{U_n}{U_{n+1}} = l \text{ (finite)}$$

(i) $l > 1 \rightarrow$ Series is convergence

(ii) $l < 1 \rightarrow$ series is divergence

(iii) $l = 1 \rightarrow$ Series test is failed.

Q Test for convergence?

$$1 + \frac{2^1 x^1}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \cdots$$

Soln

$$U_n = \frac{(n)^{n-1} x^{n-1}}{(n)!} = \frac{(nx)^{n-1}}{n!}$$

$$U_n = \frac{(n+1)^n x^n}{(n+1)!}$$

Applying ratio that

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{n^{n-1} x^{n-1}}{n!} \times \frac{(n+1)!}{(n+1)^n x^n}$$

$$\frac{1}{x} \lim_{n\to\infty} \frac{(n+1)\, n^{n-1}}{(n+1)^n}$$

$$\frac{1}{x} \lim_{n\to\infty} \frac{n^{n-1}}{(n+1)^{n-1}} \qquad \text{let} \left(\frac{n}{n+1}\right)^{n-1} = \frac{1}{e}$$

when $\frac{1}{xe} > 1 \Rightarrow x < \frac{1}{e} \Rightarrow$ Convergence;

$\frac{1}{xe} < 1 \Rightarrow x > \frac{1}{e} \Rightarrow$ divergence

$\frac{1}{e} = x \Rightarrow$ test fail.

Applying log test

$$\lim_{n\to\infty} n \log \frac{U_n}{U_{n+1}}$$

$$\lim_{n\to\infty} n \log \left(\frac{n}{n+1}\right)^{n-1} \times e$$

$$\lim_{n\to\infty} n \left[\log \left(\frac{n}{n+1}\right)^{n-1} \cancel{+ \log} e\right]$$

$$\lim_{n\to\infty} n \left\{(n-1)\left[\log\left(\frac{1}{1+1/n}\right)\right) + \log e\right]\right\}$$

$$\lim_{n\to\infty} n\left((1-n)\log\left(1+1/n\right) + 1\right)$$

$$\lim_{n\to\infty} (n-n^2)\log\left(1+1/n\right) + n$$

$$\lim_{n\to\infty} (n-n^2)\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots + n\right]$$

$$\lim_{n\to\infty} \left(1 - \frac{1}{2n} + \frac{1}{3n^2} + \cdots - 1 + 1/2 - \cdots + n\right]$$

$1 + 1/2 = 3/2 > 1$

at $x > 1/e \Rightarrow$ series is con

final result.

$x \le 1/e \Rightarrow$ Conv.

$x > 1/e \Rightarrow$ div.

9. Cauchy's Integral test :-

$f(x) \Rightarrow$ decreasing function with +ve terms in $(1, \infty)$

then, $\int_1^\infty f(x)\,dx$ = finite ~~the~~ (+)'ve number.

then series is convergence

otherwise divergence.

Q Check for Convergence.

$$\sin \pi + \frac{1}{2^2} \sin \frac{\pi}{2} + \frac{1}{3^2} \sin \frac{\pi}{3} + \dots$$

Soln

$$f(x) = \frac{1}{x^2} \sin \frac{\pi}{x}$$

applying cauchy's into test

$$\Rightarrow \int_1^\infty f(x)\,dx$$

$$\Rightarrow \int_1^\infty \frac{1}{x^2} \sin \frac{\pi}{x}\,dx$$

let $\frac{\pi}{x} = t$

$$-\frac{\pi}{x^2}\,dx = dt$$

limit

$x = 1$, $t = \pi$

$x = \infty$, $t = 0$

$$\int_\pi^0 \frac{-1}{\pi} \sin t\,dt$$

$$\Rightarrow \frac{1}{\pi} [-\cos t]_0^\pi$$

$$\Rightarrow \frac{2}{\pi} \text{ (finite)}$$

So series is convergence

Q Test for convergence -

$$\frac{1}{2} + \frac{4}{9} + \frac{9}{28} + \cdots$$

Soln

$$\frac{1}{2} + \frac{4}{9} + \frac{9}{28} + \cdots$$

$$\Rightarrow U_n = \frac{n^2}{n^3+1}$$

So $f(x) = \dfrac{x^2}{x^3+1}$

applying Cauchy's integral test (CIT)

$$\int_1^{\infty} f(x)\,dx \Rightarrow \int_1^{\infty} \frac{x^2}{x^3+1}\,dx$$

let $1+x^3 = t$

d.w.r. to x

$3x^2\,dx = dt$

$x^2\,dx = dt/3$

limit

$x = 1 \Rightarrow t = 2$

$x \to \infty \Rightarrow t \to \infty$

$$\Rightarrow \int_2^{\infty} \frac{dt}{3t}$$

$$\Rightarrow \frac{1}{3}\left[\log t\right]_2^{\infty}$$

$\Rightarrow$ $\because$ $\log(\infty)$. Not define value

So Series is divergence.

*Alternating Series

An infinite series whose terms are positive and negative alternatively is called an alternating series i.e. $u_1 - u_2 + u_3 - u_4 + \dots = \sum (-1)^{n-1} u_n$

1° Leibnitz's Test :-

An all series $\sum (-1)^{n-1} U_n$, $(U_n > 0)$ is convergent if

(i) $U_1 \geq U_2 \geq U_3 \geq U_4 \geq \dots$

(ii) $\lim_{n \to \infty} U_n = 0$

Q Check for convergence ?

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \dots$?

Sol^n

$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} \dots$

means for U_n, $S_n = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{2^{n-1}}$

$\left(U_n = \frac{1}{2^{n-1}} \right)$

$U_{n+1} = \frac{1}{2^n}$

$\because 2^n > 2^{n-1}$

$\therefore \frac{1}{2^{n-1}} > \frac{1}{2^n}$

$\boxed{U_n > U_{n+1}}$

① first condition setisfied ✓

$U_1 \geq U_2 \geq U_3 \dots$ ✓

②

$\lim_{n \to \infty} \left(\cancel{(-1)}^{n-1} \frac{1}{2^{n-1}} \right)$

$= \frac{1}{2^\infty}$

$= 0$

So Both Conditions are setisfied by Leibnitz test Hence, Series is convergence.

$\underline{Q}$ $\frac{1}{6} + \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \ldots$

Check for convergence?

$\underline{\text{Sol}^n}$ Series is

$$\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \ldots$$

$$\cancel{b_n} = A_n = (-1)^{n-1} \frac{n}{(1+5n)}$$

$$U_n \text{ is } = \frac{n}{(1+5n)}$$

Here, applying leibnitz's test

(i) $\lim_{n \to \infty} (U_n) = \lim_{n \to \infty} \frac{n}{(1+5n)}$

$$= \lim_{n \to \infty} \frac{1}{(1/n + 5)}$$

= Finite value. (not equals to zero)

(ii) $1/6 \geq \frac{2}{11} \geq 3/16 \geq 4/21 \geq \ldots$

So (i)st condition is not satisfied by leibnitz test... So The series is divergence.

Absolutely & Conditionally Convergence :-

Infinite alterenating Series $\Rightarrow \sum(-1)^{n-1} U_n$

i.e. $U_1 - U_2 + U_3 - U_4 + \cdots$

$\sum |U_n| \Rightarrow$ Series $\Rightarrow$ Convergence.

Then given series is Absolutely convergence.

otherwise series is Conditionaly convergence

Q Test for Absolutely / Conditionally convergence?

S $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \cdots$

Soln by Leibnitz's test

Given series is convergence

Now, $\sum |U_n|$

$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$r = \frac{1}{2}$ (Common ratio)

by G.P. test

$-1 \leq r \leq 1$

So Series(U_n) is convergence

Hence Complete series is Absolutely convergence

* Power Series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

* Taylor's Series

(i) $f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$

(ii) $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

where $f'(a), f''(a) \dots$ derivative of f^n at point a.

In (i) eq. $(x \to 0)$

$$* \left[f(h) = f(0) + \frac{h}{1!} f'(0) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(0) \right]^*$$

" Maclaurin's Series "

* Exponential Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

* Trigonometric Series

$\sin x = f(x)$

$f(0) = 0 \quad = \quad f''(0) = -0$

$f'(0) = 1 \quad f'''(0) = 1$

So expension is

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

for $\cos x$

$f(0) = 1$ $\qquad$ $f''(0) = -1$

$f'(0) = 0$ $\qquad$ $f'''(0) = 0$

So expension is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

for $\tan x$

$f(0) = \tan(0) = 0$ $\qquad$ $f''(0) = 2\sec(0)\sec(0)\tan(0) = 0$

$f'(0) = \sec^2(0) = 1$ $\qquad$ $f'''(0) = 4\sec^2(0)\tan(0) + 2\sec^4(0)$
$= 2$

So expension is

$$\tan x = \frac{x}{1!} + \frac{2x^3}{3!} + \frac{2}{15}x^5$$ Ans

* <u>logarithm Series</u> :- (RTU 2023)

for $\log(1+x) \Rightarrow$ $f(0) = 0$, $f'(0) = 1$

$f''(0) = -1$ $\qquad$ $f'''(0) = 2$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots \ldots$$

$x \in (-1, 1]$

for $\log(1-x) \Rightarrow$ $f(0) = 0$ $\qquad$ $f'(0) = -1$

$f''(0) = -1$ $\qquad$ $f'''(0) = -2$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

$x \in [-1, 1)$

Test for convergence of Power Series :-

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\left(\frac{a_{n+1} x^{n+1}}{a_n x^n}\right)$$

$$= x \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \qquad \text{Here } \frac{a_{n+1}}{a_n} = L$$

$$= xL$$

(i) If $|xL| < 1 \Rightarrow$ Series is convergence

(ii) If $|xL| > 1 \Rightarrow$ series is divergence

Q In a Taylor's series expansion of e^x about $x = 2$, find the coeffecient of $(x-2)^4$ (RTU.2023)

we know that

expansion of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

by Taylor's series -

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \frac{e^2}{4!}(x-2)^4 + \dots$$

Coff. of $(x-2)^4$ is $\frac{e^2}{24}$ Ans

Q. Find Taylor's series expension of $\cos 5x^2$ about the point $x = \pi$ (2018)

Soln We know that Taylor's series -

$f(x) = \cos 5x^2$

$a = \pi$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

$$f(a) = \cos 5\pi^2$$

$$f'(a) = -2\times 5 \sin 5\pi^2 = \pi\times -10 \sin 5\pi^2 = -10\pi \sin 5\pi^2$$

$$f''(a) = -100\pi^2 \cos 5\pi^2 - 10 \sin 5x^2$$

$$f'''(a) = +1000\pi^3 \sin 5\pi^2$$

$$f(x) = \cos 5\pi^2 - 10\pi \sin 5\pi^2 (x-\pi) - \frac{100\pi^2 \cos 5\pi^2 (x-\pi)^2}{2} - \frac{10 \sin 5\pi^2 (x-\pi)^2}{2} + \frac{1000\pi^3 \sin 5\pi^2 (x-\pi)^3}{3!} + \ldots$$

$$f(x) = \cos 5\pi^2 - (10\pi)^1 \sin 5\pi^2 (x-\pi)^1 - \frac{(10\pi)^2 (x-\pi)^2 \cos 5\pi^2}{2} - \frac{10^2 \sin 5\pi^2 (x-\pi)^2}{2} + (10\pi)^3 (x-\pi)^3 \sin 5\pi^2 + \ldots$$