Unit - 5. P.D.E. - Higher Order

Classification of P.D.E. of second order ⇒

Gen. form ⇒ $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$ —(1)

where A, B, C are function of x & t.

Now eq. (1) can be classified in three categories:

(i) Elliptic if $B^2 - 4AC < 0$

(ii) Parabolic if $B^2 - 4AC = 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Example ⇒

(1) One dimensional wave eq. ⇒

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$ (RTU-2024)

is hyperbolic as $B^2 - 4AC = 0 - 4(c^2)(-1) = 4c^2 > 0$

(2) One dimensional Heat eq. ⇒

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$ (RTU-2023, 2021)

is parabolic as $B^2 - 4AC = 0 - 4(c^2)(0) = 0$

(3) Two dimensional Laplace's eq. ⇒

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is elliptic as $B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$

Method of separation of variables ⇒ (RTU-2023)

let u be a fn of two independent variables x & t.

$$u(x,t) = X(x)\,T(t) \quad \text{——— (1)}$$

$X(x) \Rightarrow$ fn of x alone $\Rightarrow \frac{\partial X}{\partial t} = 0$

$T(t) \Rightarrow$ fn of t alone $\Rightarrow \frac{\partial T}{\partial x} = 0$

Now, $\frac{\partial u}{\partial x} = TX'$ and $\frac{\partial u}{\partial t} = XT'$ ——— (2)

Substitute the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ in given p.d.e,

which reduces

$$f(X, X', X'', \dots) = g(T, T', T'', \dots) = k \quad \text{(let)}$$

Now, $f(X, X', X'', \dots) = k$ ——— (3)

and $g(X, X', X'', \dots) = k$ ——— (4)

Now eq. (3) and eq. (4) are o.d.e can be solved by known methods. we get X and T.

Now substitute values of X and T in eq. (1).

we get desired solution.

Q: Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ by method of separation of variables where $u(x,0) = 6e^{-3x}$. (RTU-2024) (5-M)

Sol: Given that $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ —— (1)

let the sol. of eq. (1) $\Rightarrow u(x,t) = X(x).T(t) \Rightarrow u = XT$

Now $\frac{\partial u}{\partial x} = TX'$ and $\frac{\partial u}{\partial t} = XT'$ —— (2)

Put the values of eq.(2) in eq.(1),

$$X'T = 2XT' + XT$$

$$\Rightarrow X'T - XT = 2XT'$$

$$\Rightarrow T(X' - X) = 2XT'$$

$$\Rightarrow \frac{X' - X}{X} = \frac{2T'}{T} = K$$

Now, $\frac{X' - X}{X} = K$ and $\frac{\cancel{2}T'}{T} = \frac{K}{2}$

$\frac{X'}{X} - 1 = K$ $\Rightarrow \log T = \frac{K}{2}t + \log c_1$

$\log X - x = Kx$ $\Rightarrow T = c_1 e^{\frac{Kt}{2}}$

$\log X = x(K+1)$

$X = e^{x(K+1)}$

so, $u(x,t) = c_1 e^{(k+1)x} \cdot c_2 e^{kt/2}$

$$u(x,t) = c\, e^{(k+1)x + \frac{kt}{2}} \quad \text{--- (3)}$$

Put $t = 0$

$$u(x,0) = 6e^{-3x} = c\, e^{(k+1)x}$$

$\Rightarrow c = 6$ and $k+1 = -3$

$k = -4$

so, from eq. (3)

$$u(x,t) = 6e^{-3x-2t}$$

Ans.

$$\boxed{u(x,t) = 6e^{-(3x+2t)}}$$

Solution of Laplace Equations in two dimension ⇒ (RTU-2024, 23, 2021) (10-M)

The sol. of Laplace eq. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —— ①

Can be obtained by method of separation of variables.

let sol of eq. ① ⇒ $u(x,y) = X(x)\, Y(y)$

then $\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2}$

$= Y \frac{d^2 X}{dx^2}$ $\qquad = X \frac{d^2 Y}{dy^2}$

So, from eq. ①

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2}$$

$$\Rightarrow Y \frac{d^2 X}{dx^2} = -X \frac{d^2 Y}{dy^2}$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2} = K \quad \text{—— ②}$$

↑ Constant

Then $\frac{d^2 X}{dx^2} - KX = 0$ —— ③

and $\frac{d^2 Y}{dy^2} + KY = 0$ —— ④

If $K > 0$ ⇒ sol. of eq. ③ ⇒ $X = A e^{\sqrt{K}x} + B e^{-\sqrt{K}x}$

$X = A \cosh \sqrt{K}x + B \sinh \sqrt{K}x$ (or)

and sol. of eq. ④ ~~and~~ $\quad Y = A \cos \sqrt{K}y + B \sin \sqrt{K}y$

$\therefore$ solution of eq. (1) is $\Rightarrow$ $u = XY$

$$u = \left(Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}\right)\left(A\cos\sqrt{k}y + B\sin\sqrt{k}y\right)$$

or $$u = \left(A\cosh\sqrt{k}x + B\sinh\sqrt{k}x\right)\left(A\cos\sqrt{k}y + B\sin\sqrt{k}y\right) \quad \text{--- (5)}$$

If $k < 0$ (let $k = -\lambda^2$)

eq. (3) $\Rightarrow$ $$\frac{d^2X}{dy^2} + \lambda^2 X = 0$$

Sol. $\Rightarrow$ $X = A\cos\lambda x + B\sin\lambda x$

eq. (4) $\Rightarrow$ $$\frac{d^2Y}{dx^2} - \lambda^2 Y = 0$$

Sol. $\Rightarrow$ $Y = Ce^{\lambda y} + De^{-\lambda y}$

$\therefore$ solution of eq. (1) is $\Rightarrow$ $u = XY$

$$u = \left(A\cos\lambda x + B\sin\lambda x\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right) \quad \text{--- (6)}$$

The nature of solution will depends on the boundary Conditions given in the problem.

One Dimensional wave Eq. →

Q: A string is streched between the fixed points $(0,0)$ and $(l,0)$ and released from rest from position $y = A\sin\frac{\pi x}{l}$. find $y(x,t)$.

Sol: The d.e of wave eq. ⇒ $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— (1)

let the sol. of (1) ⇒ $y = X(x)\,T(t)$.

$$y = XT \quad \text{—— (2)}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = X\frac{d^2 T}{dt^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = T\frac{d^2 X}{dx^2}$$

∴ Eq.(1) gives $X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2}$

$$\Rightarrow \frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2$$

then $\frac{1}{c^2T}\frac{d^2T}{dt^2} = -k^2$ and $\frac{1}{X}\frac{d^2X}{dx^2} = -k^2$

$$\Rightarrow \frac{d^2T}{dt^2} + c^2k^2T = 0 \qquad \frac{d^2X}{dx^2} + k^2X = 0$$

whose solutions are

$$T = A_1\cos ckt + A_2\sin ckt$$

and $X = A_3\cos kx + A_4\sin kx$.

eq.(2) gives ⇒ $y = (A_1\cos ckt + A_2\sin ckt)(A_3\cos kx + A_4\sin kx)$ —— (3)

Given boundary conditions $\Rightarrow$

$x = 0,\ y = 0$ — (a) and $x = l,\ y = 0$ — (b)

Using (a) in eq.(3),

$$0 = (A_1 \cos ckt + A_2 \sin ckt) A_3 \Rightarrow A_3 = 0 \quad \text{— (4)}$$

Using (b) and (4) in eq.(3)

$$0 = (A_1 \cos ckt + A_2 \sin ckt) A_4 \sin kl$$

$$\Rightarrow \sin kl = 0$$

($\because A_4 \neq 0$ as $A_3 = 0 = A_4$ gives $x = 0$ which is not possible).

$$\Rightarrow kl = n\pi \quad \Rightarrow k = \frac{n\pi}{l}, \quad n \in I$$

$\therefore$ eq(3) becomes $\Rightarrow$ $$y = \left(A_n \cos \frac{n\pi c}{l} t + B_n \sin \frac{n\pi c}{l} t\right) \sin \frac{n\pi x}{l} \quad \text{— (5)}$$

where $A_n = A_1 A_4$ and $B_n = A_2 A_4$.

~~As~~ As different solutions are possible for varying values of n, the most general solution is obtained by adding all these solutions.

$$y(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi c t}{l} + B_n \sin \frac{n\pi c t}{l}\right] \sin \frac{n\pi x}{l} \quad \text{— (6)}$$

Applying initial conditions

Cond. (I) $\Rightarrow$ $y = A \sin \frac{\pi x}{l}$ for $t = 0$ in eq(6) we get

$$A \sin \frac{\pi x}{l} = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$$A \sin \frac{\pi x}{l} = A_1 \sin \frac{\pi x}{l} + A_2 \sin \frac{2\pi x}{l} + \cdots$$

on Comparing

$$A = A_1 \quad \text{and} \quad A_2 = A_3 = \cdots = 0 \quad \text{—— (7)}$$

Cond. (2) $\quad \frac{\partial y}{\partial t} = 0 \quad$ for $t = 0$

eq. (6) become $\Rightarrow$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left[-\frac{n\pi c}{l} A_n \sin \frac{n\pi c t}{l} + \frac{n\pi c}{l} B_n \cos \frac{n\pi c t}{l} \right] \sin \frac{n\pi}{l}.$$

using Cond (2) $\Rightarrow$

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow B_1 = B_2 = \cdots = 0 \quad \Rightarrow \quad B_n = 0 \quad \text{—— (8)}$$

Using eq. (7) & (8) in eq. (6),

we get $\quad y(x,t) = A \cos \frac{\pi c t}{l} \sin \frac{\pi x}{l}$. Ans.

One Dimensional Heat Eq. →

Heat eq. ⇒ $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ —— ①

let solution of eq. ① ⇒ $u(x,t) = X(x)\,T(t)$ —— ②

$u = XT$

$$\Rightarrow \frac{\partial u}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$$

∴ eq. ① reduces ⇒

$$X\frac{dT}{dt} = c^2 T \frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{dT}{dt} = K$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = K \qquad\qquad \frac{1}{c^2T}\frac{dT}{dt} = K$$

$$\frac{d^2X}{dx^2} - KX = 0 \qquad\qquad \frac{dT}{dt} - Kc^2T = 0$$

Case-1 $\boxed{K = 0}$

$$\frac{d^2X}{dx^2} = 0 \qquad\qquad \frac{dT}{dt} = 0$$

$$\Rightarrow X = Ax + B \qquad\qquad T = C$$

$$\Rightarrow u(x,t) = (Ax+B)(C)$$

Case-2 $\quad k = \lambda^2 > 0$

$$\frac{d^2X}{dx^2} - \lambda^2 x = 0 \qquad \frac{1}{T}\frac{dT}{dt} = \lambda^2 c^2 \cancel{T}$$

$$X = Ae^{\lambda x} + Be^{-\lambda x} \qquad \log_e T = \lambda^2 c^2 \cancel{\tfrac{t^2}{2}}\, t \cancel{+D} + \log D$$

$$T = De^{\lambda^2 c^2 t}$$

$$u(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(\cancel{\lambda^2 c^2 \tfrac{t^2}{2} + D}\right)$$

$$\left(De^{\lambda^2 c^2 t}\right)$$

Case-3. $\quad k = -\lambda^2 < 0$

$$\frac{d^2X}{dx^2} + \lambda^2 x = 0 \qquad \frac{dT}{dt} = -\lambda^2 c^2 T$$

$$T = e^{-\lambda^2 c^2 t} + \log D$$

$$X = A\cos\lambda x + B\sin\lambda x \qquad T = De^{-\lambda^2 c^2 t}.$$

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\left(De^{-\lambda^2 c^2 t}\right)$$