ENGINEERING MATHEMATICS-II

MATRICES 1

PREVIOUS YEARS QUESTIONS

PART A

Q.1 *State the rank-nullity theorem.* [R.T.U. 2019]

Ans. Let A be an $m \times n$ matrix of rank r_A and nullity N_A then,

$r_A + N_A = n$

i.e. rank of A + nullity of A = number of columns in A

Q.2 *Determine whether the set {(3, 2, 4)}, (1, 0, 2) (1, –1, –1)} of vectors is linearly independent.* [R.T.U. 2019]

Ans.

$$\text{Let } A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 5 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -8 \end{bmatrix}$$

$\Rightarrow P(A) = 3 =$ no. of vectars.

Hence the given vectrs are linearly independed.

Q.3 *What is upper triangular matrix? Give an example.*

Ans. Upper Triangular Matrix : A square matrix in which all the elements below the principal diagonal are zero is called an upper triangular matrix.

Thus $A = (a_{ij})_{n \times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$

$$e.g. \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ is an upper triangular matrix.}$$

Q.4 *Give the properties of matrix multiplication.*

Ans. The properties of matrix multiplication are as follows :

(1) Matrix multiplication is not commutative in general i.e. $AB \neq BA$

(2) Matrix multiplication is associative i.e. $(AB)C = A(BC)$

(3) Matrix multiplication is distributive with respect to matrix addition i.e.

$A.(B + C) = A.B + A.C$

Q.5 *Give any four properties of eigen values and eigen vectors.*

Ans. Properties of Eigen Values and Eigen Vectors

1. Matrices A and A^T have the same eigen values.
2. If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of adj A.
3. The sum of eigen values of a matrix A is equal to the sum of the elements of the principal diagonal of A.
4. If matrix A is singular, then atleast one of its eigen values will be zero.

Q.6 ***State Cayley-Hamilton theorem.***

Ans. Cayley-Hamilton Theorem : Every square matrix A satisfies its own characteristic equation.

i.e. If

$$|A-\lambda I| = (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n] = 0$$

is the characteristic equation of n^{th} order square matrix A, then

$$[A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_nI] = 0$$

Q.7 ***Define diagonalizable matrix.***

Ans. Diagonalizable Matrix : A matrix 'A' is said to be diagonalizable if it is similar to a diagonal matrix.

So a matrix A is diagonalizable if there exist an invertible matrix P such that

$$P^{-1}AP = Q.$$

where Q is a diagonal matrix.

Thus the matrix P is used for diagonalizing A or transforming A to diagonal form.

Q.8 ***What do you understand by orthogonal matrix?***

Ans. Orthogonal Matrix : The matrix A is said to be orthogonal, if

$$AA' = I$$

But, we know that

$$|A| = |A'|$$

$$\Rightarrow \quad |AA'| = |A||A'| = |A|^2$$

$$\Rightarrow \quad |A| = \pm 1 \qquad [\text{Since } |I| = 1]$$

Hence, every matrix which is orthogonal is non-singular.

Q.9 ***Define row matrix and column matrix.***

Ans. Row Matrix : A matrix which has only one row and any number of columns is called a row matrix.

e.g. $[1\ 2\ 3\ 4]_{1\times 4}$

Column Matrix : A matrix having one column and any number of rows is called column matrix.

e.g. $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}_{4\times 1}$

PART B

Q.10 ***Find the rank of the following matrix by reducing it to the normal form:***

$$A = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 5\\ 1 & 5 & 5 & 7\\ 8 & 1 & 14 & 17\end{pmatrix}$$

[R.T.U. 2019]

Ans. $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 8R_1$

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -3 & -2 & -3\\ 0 & 3 & 2 & 3\\ 0 & -15 & -10 & -15\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - 5R_2$

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -3 & -2 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 4C_1$

$$A = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -3 & -2 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$C_2 \to \frac{-C_2}{3}$

$$A = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -2 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$C_3 \to C_3 + 2C_2$

$C_4 \to C_4 + 3C_2$

$$A = \left[\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] = \begin{bmatrix}I_2 & 0\\ 0 & 0\end{bmatrix}$$

$P(A) = 2$

Q.11 (a) ***Defined rank of a matrix.***

(b) ***Find the Eigen values of the matrix :***

$$\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$$

[R.T.U. 2018]

Ans.(a) Rank of a Matrix : A non-negative integer 'r' is said to be the rank of a matrix A, if it has the following properties;-

(1) Atleast one minor of order r is non-zero.

(2) Every minor of order $(r + 1)$, (if any) must be zero.

Ans.(b) Here $A = \begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$

If λ be the eigen value of A, then the characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix}5-\lambda & 4\\ 1 & 2-\lambda\end{bmatrix} = 0$$

$\Rightarrow (5 - \lambda)(2 - \lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 6) = 0$

$\Rightarrow \lambda = 1, 6$

Then the eigen values are 1, 6

Q.12 ***Test the consistency of the following equations and if possible solve it :***

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$3x + 19y - 47z = 32$$

Ans.

$$\begin{bmatrix}2 & -3 & 7\\ 3 & 1 & -3\\ 3 & 19 & -47\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\13\\32\end{bmatrix}$$

Now, the augmented matrix is

$$[A:B] = \begin{bmatrix}2 & -3 & 7\\ 3 & 1 & -3\\ 3 & 19 & -47\end{bmatrix}\begin{bmatrix}5\\13\\32\end{bmatrix}$$

Using $R_2 \to 2R_2 - 3R_1$

$$\begin{bmatrix}2 & -3 & 7\\ 0 & 11 & -27\\ 3 & 19 & -47\end{bmatrix}\begin{bmatrix}5\\11\\32\end{bmatrix}$$

Now, $R_3 \to 2R_3 - 3R_1$

$$\begin{bmatrix}2 & -3 & 7\\ 0 & 11 & -27\\ 0 & 47 & -115\end{bmatrix}\begin{bmatrix}5\\11\\49\end{bmatrix}$$

$\Rightarrow R_3 \to 11R_3 - 47R_2$

$$\begin{bmatrix}2 & -3 & 7\\ 0 & 11 & -27\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}5\\11\\22\end{bmatrix}$$

Which is Echelen form

So $\rho(A) = 3$, $P[A:B]$

Number of variables = 3

Hence the system is consistent and solution.

The above matrix can be written as

$$2x - 3y + 7z = 5$$
$$11y - 27z = 11$$
$$4z = 22$$

On solving

$x = 5$

$y = 29/2$

$z = 11/2$

Q.13 ***Find the rank of the following matrix to the normal form :***

$$\begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 3 & 1 & 0 & 2\\ 1 & 1 & -2 & 0\end{bmatrix}$$

OR

Reduce the matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ *in its normal form and hence find its rank.*

[R.T.U. 2008, 2007]

Ans. Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Interchanging R_1 and R_2

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 \to C_3 + 3C_2 - C_1$

$C_4 \to C_4 + C_2 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Here, 2nd order identity matrix exists.

Hence Rank of matrix A = 2.

Q.14 *Find the eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

[R.T.U. 2015, 2013]

Ans. Here

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

if λ be the eigen value of A, then the characteristic equation is given by $|A - \lambda I| = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$$= (6-\lambda)[(3-\lambda)(3-\lambda)-1] + 2[-2(3-\lambda)+2] + 2[2-2(3-\lambda)] = 0$$

$$= (6-\lambda)[\lambda^2 - 6\lambda + 9 - 1] + 2[-6 + 2\lambda + 2] + 2[2 - 6 + 2\lambda] = 0$$

$$= 6\lambda^2 - 36\lambda + 54 - 6 - \lambda^3 + 6\lambda^2 - 8\lambda - 8 + 4\lambda - 8 + 4\lambda = 0$$

$$= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

if we get the factor of it then $(\lambda - 2)$ is a factor

$$\begin{array}{r} -\lambda^2 + 10\lambda - 16 \\ \lambda - 2 \overline{) -\lambda^3 + 12\lambda^2 - 36\lambda + 32} \\ -\lambda^3 + 2\lambda^2 \\ + \quad - \\ \hline 10\lambda^2 - 36\lambda \\ 10\lambda^2 - 20\lambda \\ - \quad + \\ \hline -16\lambda + 32 \\ -16\lambda + 32 \\ \hline \times \end{array}$$

then

$$= -\lambda^2 + 10\lambda - 16 = 0$$
$$= \lambda^2 - 10\lambda + 16 = 0$$

$$= \lambda^2 - 8\lambda - 2\lambda + 16 = 0$$
$$= \lambda(\lambda - 8) - 2(\lambda - 8) = 0$$
$$= (\lambda - 2)(\lambda - 8) = 0$$

then the eigen value are 2, 2, 8

Now, we find the eigen vector corresponding to eigen value

(i) When $\lambda = 2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 + R_2 \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 \to 2R_2 + R_1 \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$4x_1 - 2x_2 + 2x_3 = 0$$

Let if $x_2 = 0$ then $4x_1 + 2x_3 = 0$

$$x_1 = \frac{-2x_3}{4} = \frac{-x_3}{2}$$

$\frac{x_1}{1} = \frac{x_2}{0} = \frac{-x_3}{2}$ then eigen vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

(ii) when $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 - R_2 \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_1 = \frac{R_1}{2}, R_2 = \frac{R_2}{3} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-x_1 - x_2 + x_3 = 0$ ...(i)

$-x_2 - x_3 = 0$ ...(ii)

from eq. (ii)

$-x_2 = x_3$

Let $x_3 = k$

then, $-x_1 = -2k$

$x_1 = 2k, x_2 = -k, x_3 = k$

for $k = 1$

$x_1 = 2, x_2 = -1, x_3 = 1$

then eigen vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Q.15 *Find the rank of the matrix by reducing it to normal form* $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

[R.T.U. 2015]

Ans.

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$\therefore$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$\therefore$ $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$\therefore \; R_4 \to R_4 - R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$\therefore \; C_3 \to C_3 - 2C_2, \; C_4 \to C_4 + 3C_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$\therefore \; R_2 \to R_2 - R_4, \; R_3 \to -\frac{1}{4} R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$\therefore \; C_3 \to -\frac{1}{3} C_3, \; C_4 \to C_4 - 2C_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore \; R_2 \to R_3, \; R_3 \to R_4$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ is the normal form and hence rank of $A = 3$.

Q.16 *Find the eigen values and eigen vectors of the following matrix A :*

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

[R.T.U. 2014, 10, Raj. Univ. 2003]

Ans. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$\therefore$ Characteristic equation is :

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)[(2-\lambda)^2 - 1] + 1[-2+\lambda+1] + 1[1-2+\lambda] = 0$

$\Rightarrow (2-\lambda)^3 - (2-\lambda) + \lambda - 1 - 1 + \lambda = 0$

$\Rightarrow (2-\lambda)^3 - 4 + 3\lambda = 0 \Rightarrow -\lambda^3 + 8 - 6\lambda(2-\lambda) - 4 + 3\lambda = 0$

$\Rightarrow -\lambda^3 + 4 + 6\lambda^2 - 9\lambda = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\Rightarrow \lambda^2(\lambda - 1) - 5\lambda(\lambda - 1) + 4(\lambda - 1) = 0$

$\Rightarrow (\lambda^2 - 5\lambda + 4)(\lambda - 1) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 4)(\lambda - 1) = 0 \Rightarrow \lambda = 1, 1, 4$

$\therefore$ Eigen values are $\to \lambda_1 = 1; \lambda_2 = 1; \lambda_3 = 4$

Now, matrix equation :

$[A - \lambda I] X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 1$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_2 + R_1; \; R_3 \to R_3 + R_2$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - y + z = 0$

$z = 0, y = 1 \Rightarrow x = 1$

again, let $z = 1, y = 0 \Rightarrow x = -1$

$\therefore$ eigen vectors corresponding to $\lambda_1 = 1 = \lambda_2$ are

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 4$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 \to R_3 + R_1$

$$\Rightarrow \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_2 + R_1$

$$\Rightarrow \begin{bmatrix} -2 & -1 & 1 \\ -3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x - y + z = 0$ ....(i)

and $-3x - 3y = 0 \Rightarrow x = -y$

Let $x = 1 \Rightarrow y = -1$

$\therefore$ (i) $\Rightarrow z = 1$

$\therefore$ eigen vector corresponding to $\lambda_3 = 4$ is :

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Q.17 *Test for consistency of the following system of equations and if possible, Solve them :*

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

[R.T.U. 2014, 13, 07]

Ans. The given system of equations is–

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

These can be written as

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Now the augmented matrix is

$$[A : B] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

Using $\begin{array}{l} R_2 \to 5R_2 - 3R_1 \\ R_3 \to 5R_3 - 7R_1 \end{array} \sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{array}\right]$

Using $R_3 \to 11R_3 + R_2 \sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right]$

which is Echelon form

So $\rho[A] = 2; \rho[A : B] = 2$; No. of variables = 3

Here $\rho[A] = \rho[A : B] = <$ No. of variables

Hence the system is consistent and has an infinite number of solutions.

From above matrix we have.

$5x + 3y + 7z = 4$

$121y - 11z = 33$

Let $z = k$

so $121y - 11k = 33$

$\Rightarrow \quad y = \frac{3+k}{11}$

also $\quad 5x + 3\left(\frac{3+k}{11}\right) + 7k = 4$

$\Rightarrow \quad x = \frac{7-16k}{11}$

Hence $\quad x = \frac{7-16k}{11}; \; y = \frac{3+k}{11}; \; z = k$

Q.18 *Find the characteristic equation of the*

$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Show that the matrix A

it. Hence find A^{-1}.

Ans. $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic equation is given by $|A - \lambda I| =$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)\{(2-\lambda)^2 - 1\} - 1\{-2+\lambda+1\} + 1\{1-(2-\lambda$

$(2-\lambda)^3 - (2-\lambda) - 1(-1+\lambda) + 1(1-2+\lambda) =$

$(2-\lambda)^3 - (2-\lambda) - 1(-1+\lambda) + 1(1-2+\lambda) =$

$(2-\lambda)^3 - (2-\lambda)$

$8 - \lambda^3 - 12\lambda + 6\lambda^2 - 2 + \lambda = 0$

$6 - \lambda^3 - 11\lambda + 6\lambda^2 = 0$

$\lambda^3 + 11\lambda - 6\lambda^2 - 6 = 0$

$A^3 + 11A - 6A^2 - 6I = 0$

or $\quad A^3 - 6A^2 + 11A - 6I = 0$

$A^3 = A^2 \cdot A$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 2(2)+1\cdot(-1)+1(1) & 2(1)+1(2)+1(-1) & 2(1)+1(-1)+1(2) \\ -1(2)+2(-1)+-1(1) & -1(1)+2(2)+-1(-1) & -1(1)+2(-1)+(-1)(2) \\ 1(2)+(-1)(-1)+2(1) & 1(1)+(-1)(2)+2(-1) & 1(1)+(-1)(-1)+2(2) \end{bmatrix}$$

$$=\begin{bmatrix} 4-1+1 & 2+2-1 & 2-1+2 \\ -2-2-1 & -1+4+1 & -1-2-2 \\ 2+1+2 & 1-2-2 & 1+1+4 \end{bmatrix}$$

$$A^2=\begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix}$$

$$A^3=A\cdot A^2$$

$$=\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix}=$$

$$\begin{bmatrix} 2(4)+1(-5)+1(5) & 2(3)+1(4)+1(-3) & 2(3)+1(-5)+1(6) \\ -1(4)+2(-5)+-1(5) & -1(3)+2(4)+-1(-3) & -1(3)+(2)(-5)+(-1)(6) \\ 1(4)+(-1)(-5)+2(5) & +1(3)+(-1)(4)+2(-3) & 1(3)+(-1)(-5)+2(6) \end{bmatrix}$$

$$=\begin{bmatrix} 8-5+5 & 6+4-3 & 6-5+6 \\ -4-10-5 & -3+8+3 & -3-10-6 \\ 4+5+10 & 3-4-6 & 3+5+12 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 7 & 7 \\ -19 & 8 & -19 \\ 19 & -7 & 20 \end{bmatrix}$$

Hence consider the equation (1)

$$A^3-6A^2+11A-6I=0$$

$$\begin{bmatrix} 8 & 7 & 7 \\ -19 & 8 & -19 \\ 19 & -7 & 20 \end{bmatrix}-6\begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix}+11\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$-6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiplying equation (1) by A^{-1}, we get

$$A^2-6A+11I-6A^{-1}=0$$

$$A^{-1}=\frac{1}{6}(A^2-6A+11I)$$

$$6A^{-1}=\begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix}-6\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}+11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}=\frac{1}{6}\begin{bmatrix} 3 & -3 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 5 \end{bmatrix}$$

Q.19 *Find the inverse of the matrix using elementary transformation.*

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

[R.T.U. 2013, 08]

Ans. Let $A=\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

applying $R_1\to\frac{R_1}{3}$ [A] : [I]

$$\sim\begin{bmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}:\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

applying $R_2\to R_2-2R_1$

$$\sim\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix}:\begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

applying $R_2\to R_2(-1)$

$$\sim\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix}:\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

applying $R_3\to R_3+R_2$

$$\sim\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & -1/3 \end{bmatrix}:\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix}$$

applying $R_3\to R_3(-3)$

$$\sim\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{bmatrix}:\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix}$$

applying $R_1\to R_1+R_2$

$$R_2\to R_2+\frac{4}{3}R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}:\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}\sim[I]:[A^{-1}]$$

Therefore

Inverse of matrix $A=A^{-1}=\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

Q.20 *Verify Cayley-Hamilton theorem for the following matrix*

$$A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

[R.T.U. 2013, Raj. Univ. 2006, 02]

Ans. Cayley - Hamilton theorem states that if A be a square matrix, then A satisfies its characteristic equation. The characteristic equation is given by $|A-\lambda I|=0$.

$$\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}-\lambda\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

$$(2-\lambda)\{(2-\lambda)^2-1\}+1\{-2+\lambda+1\}+1\{1-(2-\lambda)\}=0$$

$$(2-\lambda)^3-(2-\lambda)+1+1-(2-\lambda)-(2-\lambda)=0$$

$$(2-\lambda)^3-3(2-\lambda)+2=0$$

$$(2-\lambda-1)\left[(2-\lambda)^2+(2-\lambda)-2\right]=0$$

$$(1-\lambda)(4-\lambda)(1-\lambda)=0$$

$$(1-\lambda)^2(4-\lambda)=0$$

To show $(I-A)^2(4-A)=0$

$$\Rightarrow\ A^3-6A^2+9A-4I=0 \quad ...(1)$$

$$A^2=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$9A=\begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

Now L.H.S.

$$=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}-6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}+\begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}-\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Proved.

Multiplying (1) by A^{-1} we have

$$A^2-6A+9I-4A^{-1}=0$$

$$4A^{-1}=A^2-6A+9I$$

$$=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}-6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}+9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Q.21 *Let A be an m × n matrix. Suppose that the nullspace of A is a plane in R^3 and the range is spanned by a nonzero vector v in R^5. Determine m and n. Also, find the rank and nullity of A.*

Ans. For an $m\times n$ matrix A, the nullspace consists of vectors x such that $Ax=0$. Thus, such an x must be n-dimensional. Since the nullspace is a subspace in R^3, we conclude that $n=3$.

The range of A consists of vector y such that $y=Ax$ for some $x\in R^n$. Hence, y is m-dimensional. As the range is a subspace in R^5, we conclude that $m=5$.

Since a plane is a 2-dimensional subspace, the nullity of A is 2.

The range is spanned by a single vector v. Thus, {v} is basis for the range. Thus, the rank is 1.

Here is another way to see this. By the rank-nullity theorem,

rank of A + nullity of A = n.

Since $n=3$ and the nullity is 2, the rank is 1.

Q.22 *Prove that the vectors $x_1=(1, 2, 4)$, $x_2=(3, 6, 12)$ are linearly dependent.*

Ans. Let $x_1=(1,2,4)$

$x_2=(3,6,12)$

Then $k_1x_1+k_2x_2=0$

or $k_1(1,2,4)+k_2(3,6,12)=(0,0,0)$

Here $k_1=3$

$k_2=-1$

or $3(1,2,4)+(-1)(3,6,12)=(0,0,0)$

Hence x_1 and x_2 are linearly dependent **Proved.**

Q.23 What is meant by homogeneous linear equation and non-homogeneous linear equation?

Ans. Homogeneous linear equation : If in $B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$b_1 = b_2 = \ldots = b_m = 0$ i.e. $B = 0$ then the matrix $AX = B$ reduces to $AX = 0$, which is called homogeneous linear equation.

Non-homogeneous linear equation : If in $B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

atleast one of $b_1, b_2, \ldots b_m$ is non-zero then $B \neq 0$ then the matrix $AX = B$ is known as non-homogeneous linear equation.

Q.24 *Determine bases for $N(A)$ and $N(A^TA)$ when*

$$A=\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, determine the ranks and nullities of the matrix A and A^TA.

Ans. We will first compute

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

and
$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+1 & 1+3 \\ 2+1 & 4+1 & 2+3 \\ 1+3 & 2+3 & 1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 5 \\ 4 & 5 & 10 \end{bmatrix}$$

Next, we will find N(A) by row reducing [A|0]

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the solution to AX = 0 is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

Thus

$$N(A) = \left\{ x \in R^3 \,\middle|\, x = x_3 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \text{for any } x_3 \in R \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Therefore $\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for N(A).

Similarly, we will compute $N(A^TA)$ by row reducing $[A^TA \mid 0]$:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 3 & 5 & 5 & 0 \\ 4 & 5 & 10 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{then} -R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the row reduced matrices for [A|0] and $[A^TA|0]$ are identical, we can immediately conclude that

$$N(A^TA) = \text{Span}\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \right\}$$

and that $\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A^TA)$.

It follows that the nullities of A and A^TA are both 1.

The rank nullity theorem tells

rank of A + nullity of A = 3.

Hence the rank of A is 2. Similarly, the rank of A^TA is 2.

Q.25 *Express the matrix* $B=\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ ***as the sum of a symmetric and a skew symmetric matrix.***

Ans. $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$; $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

$$\frac{1}{2}(B+B') = \frac{1}{2}\left(\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix}$$

$$\frac{1}{2}(B-B') = \frac{1}{2}\left(\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Let
$$P = \frac{1}{2}(B+B') = \begin{bmatrix} 2 & \frac{-3}{2} & \ldots \\ \frac{-3}{2} & 3 & \ldots \\ \frac{-3}{2} & 1 & \ldots \end{bmatrix}$$

$$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Since P' = P

P is a symmetric matrix

Let

$$Q = \frac{1}{2}(B-B') = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{2} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Since Q' = –Q

Q is a skew symmetric matrix.

Q.26 *Let R^2 be the vector space of size-2 co…*
This vector space has an inner produ…
$(v,w) = v^Tw$. A linear transformation T…
called an orthogonal transformation…
$w \in R^2$.
$\langle T(v), T(w) \rangle = \langle v, w \rangle$.
For a fixed angle $\theta \in [0, 2\pi]$, define…

$$[T] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

And the linear transformation $T : R^2$…
$T(v) = [T]\,v$.
Prove that T is an orthogonal transfor…

Ans. Suppose we have vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Then $T(v) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$= \begin{bmatrix} \cos(\theta)v_1 - \sin(\theta)v_2 \\ \sin(\theta)v_1 + \cos(\theta)v_2 \end{bmatrix}$$

and $T(w) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$= \begin{bmatrix} \cos(\theta)w_1 - \sin(\theta)w_2 \\ \sin(\theta)w_1 + \cos(\theta)w_2 \end{bmatrix}$$

Then we find the inner product for these two vectors:

$$\langle T(v), T(w)\rangle = [\cos(\theta)v_1 - \sin(\theta)v_2 \;\; \sin(\theta)v_1 + \cos(\theta)v_2]\begin{bmatrix} \cos(\theta)w_1 - \sin(\theta)w_2 \\ \sin(\theta)w_1 + \cos(\theta)w_2 \end{bmatrix}$$

$$= (\cos(\theta)\, v_1 - \sin(\theta)\, v_2)\,(\cos(\theta)\, w_1 - \sin(\theta)\, w_2) + (\sin(\theta)\, v_1 + \cos(\theta)\, v_2)\,(\sin(\theta)\, w_1 + \cos(\theta)\, w_2)$$

$$= \cos^2(\theta)\,(v_1w_1 + v_2w_2) + \sin(\theta)\cos(\theta)\,(-v_1w_2 - v_2w_1 + v_1w_2 + v_2w_1) + \sin^2(\theta)\,(v_2w_2 + v_1w_1)$$

$$= (\cos^2(\theta) + \sin^2(\theta))\,(v_1w_1 + v_2w_2)$$

$$= v_1w_1 + v_2w_2$$

$$= \langle v, w\rangle.$$

PART C

Q.27 ***Examine whether the following matrix A is diagonalizable. If so, obtain the matrix P such that $P^{-1}AP$ is a diagonal matrix.***

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

[R.T.U. 2019]

Ans. The characteristic equation of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)(-\lambda+\lambda^2-12) - 2(-2\lambda-6) - 3(-4+1-\lambda) = 0$$

$$\Rightarrow -(\lambda^3+\lambda^2-21\lambda-45) = 0$$

$$\Rightarrow \lambda^3+\lambda^2-21\lambda-45 = 0$$

$$\Rightarrow (\lambda+3)^2(\lambda-5) = 0$$

$$\Rightarrow \lambda = -3, -3, 5$$

(i) Eigen vector corresponding to Eigen value $\lambda = -3$, we have matrix equation

$$[A - \lambda I]\, X_1 = 0 \quad \text{where } X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$

Let $x_2 = k$ and $x_3 = k'$

$\Rightarrow x_1 = 3k' - 2k$

where k and k' are arbitraly constant. If we put $k = 1$ and $k' = 0$

$$\text{then } X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Any other vector corresponding to $\lambda = -3$, we take $k = 0$, $k' = 1$, then we get second Eigen vector,

$$X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

(ii) Eigen vector corresponding to Eigen value $\lambda = 5$, we have,

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ 1 & -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 - 5x_3 = 0$

Taking first two we get

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k'' \text{ (Let)}$$

$x_1 = k''$

$x_2 = 2k''$ and and $x_3 = -k''$

Where k'' is any arbitray constant.

Put $k'' = 1$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(Eigen vector corresponding to Eigen value $\lambda = 5$)

Now we have constant modal matrix

$P = [x_1\, x_2\, x_3]$

$$P = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Now we find the inverse of Modal matrix

$|P| = -2[-2] - 3[-1] + 1[1]$

$= 4 + 3 + 1$

$|P| = 8 \neq 0$

Minors of P (M_{ij})

$M_{11} = -2, M_{12} = -1, M_{13} = 1$

$M_{21} = -4, M_{22} = 2, M_{23} = -2$

$M_{31} = 6, M_{32} = -5, M_{33} = -3$

Cofector of P(C_{ij})

$C_{11} = -2, C_{12} = 1, C_{13} = 1$

$C_{21} = 4, C_{22} = 2, C_{23} = 2$

$C_{31} = 6, C_{32} = 5, C_{33} = 3$

$$C_{ij} = \begin{bmatrix} -2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 5 & 3 \end{bmatrix}$$

$\text{adj } P = [C_{ij}]^T$

$$= \begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj } P}{|P|} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

Now diagonal form of given matrix A

$D = P^{-1}AP$

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Q.28 ***(a) Find the rank of the matrix :***

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

after reducing it to the normal form.

(b) Examine the consistency of the system:

$$x + y + z = 6$$
$$2x + y + 3z = 13$$
$$5x + 2y + z = 12$$
$$2x - 3y - 2z = -10$$

and solve them if they are consistent.

[R.T.U. 2018]

Ans.(a) Let $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Applying $C_1 \leftrightarrow C_2 \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1 \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

Applying $R_3 \to 2R_3 - R_2 \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Applying } \begin{matrix} C_3 \to 2C_3 - C_2 \\ C_4 \to C_3 + C_4 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Applying } \begin{matrix} C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - 2C_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Applying } C_2 \to \frac{1}{4}C_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ which is the normal form

and hence rank of A, $\rho(A) = 2$

Ans.(b) $x+y+z=6$

$2x+y+3z=13$

$5x+2y+z=12$

$2x-3y-2z=-10$

Above equation can be reduced to $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 5 & 2 & 1 \\ 2 & -3 & -2 \end{bmatrix};\ X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix};\ B = \begin{bmatrix} 6 \\ 13 \\ 12 \\ -10 \end{bmatrix}$$

Augmented matrix be

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \\ 5 & 2 & 1 & 12 \\ 2 & -3 & -2 & -10 \end{array}\right]$$

Performing $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 2R_1$,

$$\text{we get} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & -3 & -4 & -18 \\ 0 & -5 & -4 & -22 \end{array}\right]$$

...ming $R_3 \to R_3 - 3R_2$

$R_4 \to R_4 - 5R_2$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -7 & -21 \\ 0 & 0 & -9 & -27 \end{array}\right]$$

Performing $R_2 \to -R_2$

$R_3 \to -\frac{1}{7}R_3$

$R_4 \to -\frac{1}{9}R_4$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Performing $R_4 \to R_4 - R_3$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

since the rank of A, $\rho(A)$ = rank of augmented matrix, $\rho[(A:B)] = 3$

$\therefore$ Equations are consistent.

$$\text{Further we have } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x+y+z=6 \\ y-z=-1 \\ z=3 \end{matrix} \quad \Rightarrow \left.\begin{matrix} x=1 \\ y=2 \\ z=3 \end{matrix}\right\}$$

Q.29 ***Find the Eigen values and the corresponding Eigen vectors of the following matrix :***

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

[R.T.U. 201...]

Ans. The characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\{(2-\lambda)(3-\lambda)-2\} - 1\{2-2(2-\lambda)\} = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\Rightarrow \lambda = 1, 2, 3$

These are the eigen values of given matrix A. Now we will find eigen vectors corresponding to each eigen value

(i) When $\lambda = 1$

$[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 1$, we get

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_3 = 0$

$x_1 + x_2 + x_3 = 0$

$2x_1 + 2x_2 + 2x_3 = 0$

so $x_1 = -x_2$ & $x_3 = 0$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0} = k$

$\Rightarrow x_1 = k,\ x_2 = -k,\ x_3 = 0$

$$\text{Thus } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

is eigen vector corresponding to eigen value $\lambda = 1$.

(ii) When $\lambda = 2$

$[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 3 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 2$, we get

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Using } R_2 \to R_2 + R_1 \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 - x_3 = 0$

$2x_1 + 2x_2 + x_3 = 0$

Let $x_3 = k$ (arbitrary)

On solving we get

$x_1 = -k, \quad x_2 = \frac{k}{2}, \quad x_3 = k$

$$\text{Thus } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ \frac{k}{2} \\ k \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \quad (\text{Let } k = 2)$$

is eigen vector corresponding to eigen value $\lambda = 2$.

(iii) When $\lambda = 3$

$[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 3$, we get

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Using } R_2 \to R_2 + R_1 \sim \begin{bmatrix} -2 & 0 & -1 \\ -1 & -1 & 0 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Using } R_3 \to R_3 + 2R_2 \sim \begin{bmatrix} -2 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow -2x_1 - x_3 = 0$

$-x_1 - x_2 = 0$

Let $x_3 = k$

$\Rightarrow x_1 = \frac{-k}{2}$

and $x_2 = \frac{k}{2}$

$$\text{Thus } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-k}{2} \\ \frac{k}{2} \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Hence $X = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is eigen vector correspond... value $\lambda = 3$.

Q.30 ***(a) Find the eigen values and eigen ve... following matrix :***

$$\begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$

(b) State Cayley Hamilton Theorem, ... the matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \text{ and find } A^{-1}.$$

Ans.(a) Given matrix is

$$\begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$

Equation can be solved by (λ) form

$$\begin{bmatrix} -2-\lambda & 1 & 1 \\ -11 & 4-\lambda & 5 \\ -1 & 1 & 0-\lambda \end{bmatrix} = 0 \quad \text{...(i)}$$

$\Rightarrow (-2-\lambda)[(4-\lambda)(-\lambda)-5] - 1[11\lambda+5] + 1[-11+(4-\lambda)] = 0$

$\Rightarrow (-2-\lambda)[-4\lambda+\lambda^2-5] - 11\lambda - 5 - 11 + 4 - \lambda = 0$

$\Rightarrow -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\Rightarrow \lambda = 1, -1, 2$ are the eigen values of the given matrix.

(i) Eigen vector corresponding to $\lambda = 1$ is

$$\begin{bmatrix} -2-1 & 1 & 1 \\ -11 & 4-1 & 5 \\ -1 & 1 & 0-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ -11 & 3 & 5 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying

$R_3 \to R_3 + R_2$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $-3x_1 + x_2 + x_3 = 0$

$-2x_1 + 2x_2 = 0$

$\Rightarrow x_1 = x_3$

Let $x_3 = k$

$\Rightarrow x_1 = k$

and $x_2 = 2k$

Then eigen vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

For $k = 1$, eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(ii) Eigen vector corresponding to $\lambda = -1$

$$\begin{bmatrix} -1 & 1 & 1 \\ -11 & 5 & 5 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying

$R_2 \to R_2 - 5R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ -6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $-y_1 + y_2 + y_3 = 0$

$-6y_1 = 0$

$\Rightarrow y_1 = 0$

$y_2 + y_3 = 0$

$\Rightarrow y_2 = -y_3 = k'$ (Let)

Then eigen vector

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k' \\ -k' \end{bmatrix} = k'\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

For $k' = 1$

Then eigen vector corresponding to $\lambda = -1$ is $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(iii) Eigen vector corresponding to $\lambda = 2$

$$\begin{bmatrix} -4 & 1 & 1 \\ -11 & 2 & 5 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} -4 & 1 & 1 \\ -3 & 0 & 3 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -4 & 1 & 1 \\ -3 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $-4z_1 + z_2 + z_3 = 0$

$-3z_1 + 3z_3 = 0$

$\Rightarrow z_1 = z_3 = k''$ (Let)

$\Rightarrow z_2 = 3k''$

Then eigen vector corresponding to $\lambda = 2$ is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} k'' \\ 3k'' \\ k'' \end{bmatrix} = k''\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Ans.(b) Cayley Hamilton Theorem : Every square matrix satisfies its own characteristic equation *i.e.* for every square matrix of order *n*.

Verification : Characteristic equation of given matrix is :

$|A - \lambda I| = 0$

or $\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda[(2-\lambda)(1-\lambda)-3] - 1[(1-\lambda)-9] + 2(1-3(2-\lambda)) = 0$

$\Rightarrow -\lambda(2-3\lambda+\lambda^2-3) - (1-\lambda-9) + 2(1-6+3\lambda) = 0$

$\Rightarrow -\lambda(\lambda^2-3\lambda-1) - 1(-\lambda-8) + 2(3\lambda-5) = 0$

$\Rightarrow (-\lambda^3+3\lambda^2+\lambda) + (\lambda+8) + (6\lambda-10) = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 8\lambda - 2 = 0 \quad \text{...(1)}$

Equation (1) is the characteristic equation of A.

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix}$$

Now taking $\lambda = A$ on LHS of (1) we get

$-A^3 + 3A^2 + 8A - 2I = 0 \quad \text{...(2)}$

$$\Rightarrow \begin{bmatrix} -19 & -20 & -31 \\ -41 & -38 & -57 \\ -36 & -26 & -36 \end{bmatrix} + \begin{bmatrix} 21 & 12 & 15 \\ 33 & 24 & 33 \\ 12 & 18 & 30 \end{bmatrix} + \begin{bmatrix} 0 & 8 & 16 \\ 8 & 16 & 24 \\ 24 & 8 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Thus, A satisfies its own characteristic equation.

Value of A^{-1} : Now Multiplying both side of (2) by A^{-1} we have,

$A^2 - 3A - 8I + 2A^{-1} = 0$

$\Rightarrow 2A^{-1} = 8I + 3A - A^2$

$$= 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$$

$$2A^{-1} = \begin{bmatrix} 8+0-7 & 0+3-4 & 0+6-5 \\ 0+3-11 & 8+6-8 & 0+9-11 \\ 0+9-4 & 0+3-6 & 8+3-10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Q.31 *(a) Show that the three equations $-2x + y + z = a$, $x - 2y + z = b$ and $x + y - 2z = c$ have no solution unless $a + b + c = 0$, in which case they have infinitely many solutions. Find these solutions when $a = 1, b = 1, c = -2$.*

(b) Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$ *Hence find the value of* $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

[R.T.U. 2015]

Ans. (a) Augmented matrix,

$$[A : B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right] \qquad [\rho(A) = 2]$$

Operating R_{13}

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & c \\ 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \end{array}\right]$$

Operating $R_{21}(-1)$, $R_{31}(2)$

$$\sim\begin{bmatrix}1 & 1 & -2 & \vdots & c\\ 0 & -3 & 3 & \vdots & b-c\\ 0 & 3 & -3 & \vdots & a+2c\end{bmatrix}$$

Operating $R_{32}(1)$

$$\sim\begin{bmatrix}1 & 1 & -2 & \vdots & c\\ 0 & -3 & 3 & \vdots & b-c\\ 0 & 0 & 0 & \vdots & a+b+c\end{bmatrix}$$

Case I : If $a+b+c \neq 0$

$\rho[A : B] = 3 \neq \rho(A)$

where A is the coefficient matrix.

Hence the system,being inconsistent,have no solution.

Case II : If $a+b+c = 0$

$\rho[A : B] = 2 = \rho(A)$ $\quad [\because \rho(A) < 3]$

Hence the system has infinite number of solution.

Equivalent system of equations is

$x+y-2z = -2$ $\quad$ [Putting $b = 1, c = -2$]

$-3y+3z = 3$

Let $z = k$, k being an arbitrary constant.

$y = k-1$

$x = k-1$

Hence the solutions are $x = k-1, y = k-1, z = k$.

Ans. (b) The characteristic equation of A is given by,

$|A - \lambda I_3| = 0$

$$\Rightarrow \begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 2-\lambda\end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)[(1-\lambda)(2-\lambda)] - 1[0] + 1[-(1-\lambda)] = 0$

$\Rightarrow (2-\lambda)(2-\lambda-2\lambda+\lambda^2)+\lambda-1 = 0$

$\Rightarrow 4-6\lambda+2\lambda^2-2\lambda+3\lambda^2-\lambda^3+\lambda-1 = 0$

$\Rightarrow -\lambda^3+5\lambda^2-7\lambda+3 = 0$

$\Rightarrow \lambda^3-5\lambda^2+7\lambda-3 = 0$ $\quad$ ....(i)

This is characteristic equation.

Now λ is replaced by A, then

$A^3-5A^2+7A-3I_3 = 0$ $\quad$ ....(ii)

Now the given matrix,

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+Z$

We know that

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+ZI_3$

$\Rightarrow A^5[A^3-5A^2+7A-3I_3]+A[A^3-5A^2+7A-3I_3]$

$+A^2+A+ZI_3$ $\quad$ ...(iii)

From eq. (iii)

A^2+A+ZI_3

$$= \begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix} + \begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix} + Z\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix} + \begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix} + \begin{bmatrix}Z & 0 & 0\\ 0 & Z & 0\\ 0 & 0 & Z\end{bmatrix}$$

Thus matrix is

$$= \begin{bmatrix}7+Z & 5 & 5\\ 0 & 2+Z & 0\\ 5 & 5 & 7+Z\end{bmatrix}$$

Q.32 (a) *Find out for what values of λ the equations*

$$x+y+z = 1,\ x+2y+4z = \lambda$$

and $x+4y+10z = \lambda^2$

have a solution and solve them completely in each case.

(b) If $A = \begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}$.

Find the characteristics equation of A. Prove that A satisfies this equation and hence find A^{-1}. **[R.T.U. 2012]**

Ans. (a) Writing the given equation in the matrix nominations, we have

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 4 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ \lambda\\ \lambda^2\end{bmatrix}$$

$AX = B$

We apply rank method

$$A = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 4 & 10\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, B = \begin{bmatrix}1\\ \lambda\\ \lambda^2\end{bmatrix}$$

$$\text{Consider } [A : B] = \begin{bmatrix}1 & 1 & 1 & \vdots & 1\\ 1 & 2 & 4 & \vdots & \lambda\\ 1 & 4 & 10 & \vdots & \lambda^2\end{bmatrix}$$

Operating, $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix}1 & 1 & 1 & \vdots & 1\\ -1 & 0 & 2 & \vdots & \lambda-2\\ -3 & 0 & 6 & \vdots & \lambda^2-4\end{bmatrix}$$

Operating, $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix}1 & 1 & 1 & \vdots & 1\\ -1 & 0 & 2 & \vdots & \lambda-2\\ 0 & 0 & 0 & \vdots & \lambda^2-3\lambda+2\end{bmatrix} \quad ...(i)$$

The system has a solution if $r[A : B] = r(A)$

which is possible if $\lambda^2-3\lambda+2 = 0$

thus, $\lambda = 1, 2$

Case I

When $\lambda = 1$, from Eq.(i)

$$[A : B] = \begin{bmatrix}1 & 1 & 1 & : & 1\\ -1 & 0 & 2 & : & -1\\ 0 & 0 & 0 & : & 0\end{bmatrix}$$

Here $\quad r[A : B] = r(A) = 2 < 3$. Therefore, the system has infinite solution.

Take $\quad z = k_1$

Expanding by R_2,

$-x+2z = -1$

$-x = -1-2k_1$

$x = 1+2k_1$

Expanding by R_1

$x+y+z = 1$

$y = 1-x-z$

$= 1-(1+2k_1)-k_1$

$= -3k_1$

Case II

when $\quad \lambda = 2$, from Eq.(i)

$$[A : B] = \begin{bmatrix}1 & 1 & 1 & : & 1\\ -1 & 0 & 2 & : & 0\\ 0 & 0 & 0 & : & 0\end{bmatrix}$$

Here $\quad r[A : B] = r(A) = 2 < 3$. (= No. of unknowns)

Therefore, the system has infinite solution.

Take $\quad z = k_2$

Expanding by R_2, $-x+2z = 0$

$x = 2k_2$

Expanding by R_1, $x+y+z = 1$

$y = 1-x-z$

$= 1-2k_2-k_2$

$= 1-3k_2$

Hence, $x = 2k_2$, $y = 1-3k_2$ and $z = k_2$ is the required solution.

Ans. (b) The characteristic equation is given by $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{bmatrix}1-\lambda & 0 & -1\\ 3 & 4-\lambda & 5\\ 0 & -6 & -7-\lambda\end{bmatrix}$$

$(1-\lambda)\{(4-\lambda)(-7-\lambda)+30\} + (-1)\{(-18)-(0)$

$(4-\lambda)\} = 0$

$(1-\lambda)\{(4-\lambda)(-7-\lambda)+30\} - 1(-18) = 0$

$\lambda^3+2\lambda^2-\lambda-20 = 0$

This is the characteristics equation of A

By the Cayley-Hamilton theorem

$A^3+2A^2-A-20I = 0$

Verification

$$\text{We have, } A^2 = \begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}\times\begin{bmatrix}1 & 0 & \\ 3 & 4 & \\ 0 & -6 & \end{bmatrix}$$

$$= \begin{bmatrix}1 & 6 & 6\\ 15 & -14 & -18\\ -18 & 18 & 19\end{bmatrix}$$

Also, $A^3 = AA^2$

$$= \begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}\times\begin{bmatrix}1 & 6 & 6\\ 15 & -14 & -18\\ -18 & 18 & 19\end{bmatrix}$$

$$= \begin{bmatrix}19 & -12 & -13\\ -27 & 52 & 41\\ 36 & -42 & -25\end{bmatrix}$$

Now $A^3+2A^2-A-20I$

$$= \begin{bmatrix}19 & -12 & -13\\ -27 & 52 & 41\\ 36 & -42 & -25\end{bmatrix} + 2\begin{bmatrix}1 & 6 & 6\\ 15 & -14 & -18\\ -18 & 18 & 19\end{bmatrix} - \begin{bmatrix}1 & 0 & \\ 3 & 4 & \\ 0 & -6 & \end{bmatrix}$$

$$-20\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}19 & -12 & -13\\ -27 & 52 & 41\\ 36 & -42 & -25\end{bmatrix} + \begin{bmatrix}2 & 12 & 12\\ 30 & -28 & -36\\ -36 & 36 & 38\end{bmatrix} - \begin{bmatrix}1 & 0 & \\ 3 & 4 & \\ 0 & -6 & \end{bmatrix}$$

$$-\begin{bmatrix}20 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 20\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 0$$

Hence, Cayley-Hamilton theorem is ve

Now, we shall compute A^{-1}.

Multiplying Eq. (i) by A^{-1}, we get

$A^2+2A-I-20A^{-1} = 0$

$$\therefore A^{-1} = \frac{1}{20}(A^2+2A-I)$$

$$20A^{-1} = \begin{bmatrix}1 & 6 & 6\\ 15 & -14 & -18\\ -18 & 18 & 19\end{bmatrix} + 2\begin{bmatrix}1 & 0 & \\ 3 & 4 & \\ 0 & -6 & \end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \frac{1}{20}\begin{bmatrix}2&6&4\\21&-7&-8\\-18&6&4\end{bmatrix}$$

Q.33 (a) *Examine whether the following equations are consistent and solve them if they are consistent.*

$x + 2y - z = 3$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

$x - y + z = -1$

(b) *Find the eigen values and the corresponding eigen vectors of the following matrix :*

$$A = \begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$$

[R.T.U. 2012]

Ans. (a) Coefficient matrix,

$$A = \begin{bmatrix}1&2&-1\\3&-1&2\\2&-2&3\\1&-1&1\end{bmatrix}$$

$\rho(a) = 3$ since minor $\begin{vmatrix}1&2&-1\\3&-1&2\\2&-2&3\end{vmatrix} \neq 0$

Augmented matrix,

$$[A:B] = \left[\begin{array}{ccc:c}1&2&-1&3\\3&-1&2&1\\2&-2&3&2\\1&-1&1&-1\end{array}\right]$$

Operating $R_{21}(-3)$, $R_{31}(-2)$, $R_{41}(-1)$, we have

$$\left[\begin{array}{ccc:c}1&2&-1&3\\0&-7&5&-8\\0&-6&5&-4\\0&-3&2&-4\end{array}\right]$$

Operating $R_{32}(-1)$, we have

$$\left[\begin{array}{ccc:c}1&2&-1&3\\0&-7&5&-8\\0&1&0&4\\0&-3&2&-4\end{array}\right]$$

Operating R_{23}, we have

$$\left[\begin{array}{ccc:c}1&2&-1&3\\0&1&0&4\\0&-7&5&-8\\0&-3&2&-4\end{array}\right]$$

Operating $R_{32}(7)$, $R_{42}(3)$, we have

$$\left[\begin{array}{ccc:c}1&2&-1&3\\0&1&0&4\\0&0&5&20\\0&0&2&8\end{array}\right]$$

Operating $R_{43}(-2/5)$, we have

$$\left[\begin{array}{ccc:c}1&2&-1&3\\0&1&0&4\\0&0&5&20\\0&0&0&0\end{array}\right]$$

$\rho = (A:B) = 3$

Since $\rho(A:B) = \rho(A) = 3$ (no. of variables).

Hence the given system of equations is consistent and has a unique solution.

Equivalent system of equations is

$x + 2y - z = 3$

$y = 4$

$5z = 20$

On solving, we get

$x = -1$

$y = 4$

$z = 4$

Ans.(b) The characteristics equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix}8-\lambda&-6&2\\-6&7-\lambda&-4\\2&-4&3-\lambda\end{vmatrix} = 0$$

$(8-\lambda)\{(7-\lambda)(3-\lambda)-16\} + 6\{(-6)(3-\lambda)+8\} + 2\{24 - 2(7-\lambda)\} = 0$

$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda(\lambda-3)(\lambda-15) = 0$

$\lambda = 0, 3, 15$

Hence, the eigen values are 0, 3, 15.

The eigen vector corresponding to $\lambda_1 = 0$ is given by

$(A - \lambda_1 I)X_1 = 0$

where, $X_1 = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$$\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$ ...(i)

$-6x_1 + 7x_2 - 4x_3 = 0$ ...(ii)

$2x_1 - 4x_2 + 3x_3 = 0$ ...(iii)

Solving Eqs. (i) and (ii) by cross multiplication,

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k_1 \text{ (say)}$$

where, $k_1 \neq 0$

$x_1 = k_1$

$x_2 = 2k_1$

$x_3 = 2k_1$

which satisfy Eq. (iii) also.

$\therefore$ The required eigen vector X_1 corresponding to eigen value $\lambda_1 = 0$ is given by

$$X_1 = \begin{bmatrix}k_1\\2k_1\\2k_1\end{bmatrix} = k_1\begin{bmatrix}1\\2\\2\end{bmatrix}$$

The eigen vector X_2 corresponding to $\lambda_2 = 3$ is given by

$(A - \lambda_2 I)X_2 = 0$

where, $X_2 = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$$\begin{bmatrix}5&-6&2\\-6&4&-4\\2&-4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$ ...(iv)

$-6x_1 + 4x_2 - 4x_3 = 0$ ...(v)

$2x_1 - 4x_2 = 0$ ...(vi)

Solving Eqs. (iv) and (v) by cross-multiplication, we get

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k_2 \text{ (say)}$$

where, $k_2 \neq 0$

$\therefore x_1 = 2k_2$

$x_2 = k_2$

$x_3 = -2k_2$

which satisfy Eq. (vi) also.

$\therefore$ The required eigen vector X_2 is

$$X_2 = \begin{bmatrix}2k_2\\k_2\\-2k_2\end{bmatrix} = k_2\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

The eigen vector corresponding to $\lambda_3 = 15$ is given by

$(A - \lambda_3 I)X_3 = 0$

where, $X_3 = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$$\begin{bmatrix}-7&-6&2\\-6&-8&-4\\2&-4&-12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$ ...(vii)

$-6x_1 - 8x_2 - 4x_3 = 0$ ...(viii)

$2x_1 - 4x_2 - 12x_3 = 0$ ...(ix)

Solving Eqs. (vii) and (viii) by cross-multiplication, we get

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k_3 \text{ (say)}$$

where, $k_3 \neq 0$

$\therefore x_1 = 2k_3$

$x_2 = -2k_3$

$x_3 = k_3$

which satisfy Eq. (ix) also.

$\therefore$ The required eigen vector X_3 is

$$X_3 = \begin{bmatrix}2k_3\\-2k_3\\k_3\end{bmatrix} = k_3\begin{bmatrix}2\\-2\\1\end{bmatrix}$$

□□□

First Order Ordinary Differential Equations 2

Previous Years Questions

Part A

Q.1 ***Write the Bernoulli's equation.*** [R.T.U. 2019]

Ans. The equation of the form

$$\frac{dy}{dx}+Py=Qy^n;\ n\neq 1$$

where P and Q are some functions of x only (or constant) is called Bernoulli's equation.

Q.2 ***Write the Clairut's equation.*** [R.T.U. 2019]

Ans. A equation of the form

$y=Px+f(p)$

is called clairaut's equation.

Q.3 ***Define order and degree of a differential equation.***

Ans. Order of a differential equation : The order of the highest order derivative involved in a differential equation is called the order of the differential equation.

Degree of a differential equation : The power of the highest order differential coefficient after removing the redical sign and fraction in a equation is called its degree.

Q.4 $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$.

Ans. Separating the variables, we have $\frac{dy}{dx}=\frac{e^x+x^2}{e^y}$

Integrating, we get

$$\int e^y dy=\int (e^x+x^2)dx$$

or $$e^y=e^x+\frac{x^3}{3}+C$$ **Ans.**

Q.5 $\frac{dy}{dx}=\frac{xy+y}{xy+x}$.

Ans. The given equation can be written as $\frac{dy}{dx}=\frac{y(1+x)}{x(1+y)}$

or $$\left(\frac{y+1}{y}\right)dy=\left(\frac{1+x}{x}\right)dx$$

or $$\left(1+\frac{1}{y}\right)dy=\left(\frac{1}{x}+1\right)dx$$

Integrating, we get $(y+\log y)=(x+\log x)+\log C$ **Ans.**

Q.6 ***Define homogeneous differential equation.***

Ans. Homogeneous Differential Equations : A differential equation of the form $\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$ is called a homogeneous equation, if each $f(x,y)$ and $\phi(x,y)$ are homogeneous *i.e.*, of same degree.

Q.7 $x\frac{dy}{dx}+y\log y=xye^x$.

Ans. $x\frac{dy}{dx}+y\log y=xye^x$

dividing the equation by xy, we get

$$\frac{1}{y}\frac{dy}{dx}+\frac{\log y}{x}=e^x$$

put $\log y=z\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ the given equation becomes $\frac{dz}{dx}+\frac{z}{x}=e^x$

$$\text{I.F.}=e^{\int\frac{dx}{x}}=e^{\log x}=x$$

Hence the complete solution is

$$zx=\int xe^x dx+C=(x-1)e^x+C$$

or $$x\log y=(x-1)e^x+C$$ **Ans.**

Q.8 $(x^2+y^2)dx+2xy\,dy=0$.

Ans. Here, $M=x^2+y^2,\quad N=2xy\quad \frac{\partial M}{\partial y}=2y=\frac{\partial N}{\partial x}$

So, the given equation is exact, so

$$\int M dx=\frac{x^3}{3}+xy^2$$

and $$\frac{\partial}{\partial y}\int M dx=2xy$$

Thus the solution is

$$\int M dx+\int\left\{N-\frac{\partial}{\partial y}\int M dx\right\}dy=C$$

$$\frac{x^3}{3}+xy^2+\int(2xy-2xy)dy=C$$

or $$\frac{x^3}{3}+xy^2=C$$ **Ans.**

Part B

Q.9 ***Solve the differential equation-***

$$(3x^2y^3e^y+y^3+y^2)dx+(x^3y^3e^y-xy)dy=0$$

[R.T.U. 2019]

Ans. $(3x^2y^3e^y+y^3+y^2)dx+(x^3y^3e^y-xy)dy=0$

The given differential equation of the form,

$Mdx+Ndy=0$

Let, $M=3x^2y^3e^y+y^3+y^2$

$N=x^3y^3e^y-xy$

$$\frac{\partial M}{\partial y}=3x^2(3y^2e^y+y^3e^y)+3y^2+2y$$

$$\frac{\partial N}{\partial x}=3x^2y^3e^y-y$$

$$\because \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-3y-3y^2-9x^2y^2e^y}{3x^2y^3e^y+y^3+y^2}$$

$$=\frac{-3y(1+y+3x^2ye^y)}{y^2(1+y+3x^2ye^y)}$$

$$=\frac{-3}{y}=f(y)$$

So, $\text{I.F.}=e^{\int\frac{-3}{y}dy}=e^{-3\log y}=\frac{1}{y^3}$

Multiply equation (1) by I.F., we get

$$\left(3x^2e^y+1+\frac{1}{y}\right)dx+\left(x^3e^y-\frac{x}{y^2}\right)dy=0$$

Now $\frac{\partial M}{\partial y}=3x^2e^y-\frac{1}{y^2},\ \frac{\partial N}{\partial x}=3x^2e^y-\frac{1}{y^2}$

$$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

So equation (2) is an exact diff. equation.

Now we find the solution of equation (2) by sim

(i) $U(x,y)=\int Mdx$ · keeping y as a constat

$$=\int\left(3x^2e^y+1+\frac{1}{y}\right)dx$$

$$=x^3e^y+x+\frac{x}{y}$$

(ii) $\frac{\partial U}{\partial y}=x^3e^y-\frac{x}{y^2}$

(iii) $V(y)=\int\left(N-\frac{\partial U}{\partial y}\right)dy$

$=\int 0dy$

$=C$

(iv) Hence complete solution

$u(x, y) + v(y) = C_1$ (constant)

$$\Rightarrow x^2e^y + x + \frac{x}{y} = C'$$

where $C_1 - C = C'$

Q.10 Solve: $p^2 + 2py \cot x = y^2$, where $p = \frac{dy}{dx}$. [R.T.U. 2019]

Ans. The given differential equation

$$p^2 + 2py \cot x - y^2 = 0 \quad ...(i)$$

which is a quadratic in p, therefore

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= -y \cot x \pm y \operatorname{cosec} x$$

on taking +ve symbol

$$p = -y \cot x + y \operatorname{cosec} x$$

$$\frac{dy}{dx} = -y \cot x + y \operatorname{cosec} x$$

$$\frac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$$

$$\frac{dy}{y} = \frac{1-\cos x}{\sin x} dx$$

$$\frac{dy}{y} = \frac{1-\cos^2 x}{(1+\cos x)\sin x} dx$$

$$\frac{dy}{y} = \frac{\sin x}{1+\cos x} dx$$

on integration, we get

$$\log y = -\log(1+\cos x) + \log c$$

$$\log y = \log\left(\frac{c}{1+\cos x}\right)$$

$$y = \frac{c}{1+\cos x} \quad ...(2)$$

on taking –ve symbol

$$p = -y \cot x - y \operatorname{cosec} x$$

$$\frac{dy}{dx} = -y(\operatorname{cosec} x + \cot x)$$

$$\frac{dy}{y} = -\left(\frac{1+\cos x}{\sin x}\right) dx$$

$$\frac{dy}{y} = -\left(\frac{\sin x}{1-\cos x}\right) dx$$

on integration, we get

$$\log y = -\log(1-\cos x) + \log c$$

$$\log y = \log\left(\frac{c}{1-\cos x}\right)$$

$$y = \frac{c}{1-\cos x} \quad ...(3)$$

By taking (2) and (3)

$$\left(y - \frac{c}{1+\cos x}\right)\left(y - \frac{c}{1-\cos x}\right) = 0$$

which is the general solution of given differential equation (1).

Q.11 *Define linear differential equation by giving an example.* [R.T.U. 2018]

Ans. Linear Differential Equation : The differential equation in which the dependent variable and its derivatives present only in the first degree and are not multiplied together is called linear differential equation.

Example : $x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$

The given equation can be written as

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right)y = \frac{\sec x}{x}$$

$$\text{I.F.} = e^{\int\left(\tan x + \frac{1}{x}\right)dx} = e^{\log\sec x + \log x} = x\sec x$$

Hence the required solution is $yx\sec x = \int \sec^2 x \cdot dx$

or $yx\sec x = \tan x + C$ **Ans.**

Q.12 *Solve the following differential equation :*

$$\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(1+y^2)\,x\,dy = 0$$

[R.T.U. 2018]

Ans. $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(1+y^2)x\,dy = 0$

Here $M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2, N = \frac{1}{4}(x + xy^2)$

$$\frac{\partial M}{\partial y} = 1 + y^2, \frac{\partial N}{\partial x} = \frac{1}{4}(1+y^2)$$

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so the equation is not exact.

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\frac{1}{4}(x+xy^2)}\left[\frac{3}{4}(1+y^2)\right]$$

$$= \frac{3}{x} \text{ which is a function of x alone}$$

$$\therefore \quad \text{I.F.} = e^{\int\frac{3}{x}dx} = e^{3\log x} = x^3$$

Multiplying given equation by x^3, we get

$$\left(x^3y + \frac{1}{3}x^3y^3 + \frac{1}{2}x^5\right)dx + \frac{1}{4}(1+y^2)x^4dy = 0$$

$$\int M dx = \frac{x^4y}{4} + \frac{1}{12}x^4y^3 + \frac{x^6}{12}$$

$$\frac{\partial}{\partial y}\int M dx = \frac{x^4}{4} + \frac{1}{4}x^4y^2$$

Thus, the solution is $\int M dx + \int\left\{N - \frac{\partial}{\partial y}\int M dx\right\}dy = C$

$$\Rightarrow \frac{x^4y}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} + \int 0\, dy = C$$

$$\Rightarrow \frac{x^4y}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = C$$

$$\Rightarrow 3x^4y + x^4y^3 + x^6 = 12C = C'$$

Q.13 *Solve the following differential equation:*

$$(x-y)^2\frac{dy}{dx} = a^2$$

[R.T.U. 2015]

Ans. Let $x - y = u$

$$1 - \frac{dy}{dx} = \frac{du}{dx}$$

or $\frac{dy}{dx} = \left(1 - \frac{du}{dx}\right)$

Substitute above values in the given equation, we have

$$u^2\left(1 - \frac{du}{dx}\right) = a^2$$

$$u^2 - a^2 = u^2\frac{du}{dx}$$

$$\frac{u^2du}{u^2-a^2} = dx \text{ (By separation of variables)}$$

$$du + \left(\frac{a^2}{u^2-a^2}\right)du = dx$$

On integration

$$u + a^2\frac{1}{2a}\log\left(\frac{u-a}{u+a}\right) = x + C$$

$$-y + \frac{a}{2}\log\left(\frac{x-y-a}{x-y+a}\right) = C$$

or $y - \frac{a}{2}\log\left(\frac{x-y-a}{x-y+a}\right) = C$

Q.14 *Solve the following differential equation:*

$$(1+y^2)dx = (\tan^{-1}y - x)dy$$

[R.T.U. 2015, 2010; MREC Auto 2001]

Ans. The differential equation is linear in x, since it can be written as

$$(1+y^2)\frac{dx}{dy} + x = \tan^{-1}y$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

$$\therefore \quad \text{I.F.} = e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Hence, the solution is

$$xe^{\tan^{-1}y} = \int e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}dy + K \quad (\text{Putting } \tan^{-1}y = t)$$

$$= \int te^t dt + K$$

$$= te^t - e^t + K = e^{\tan^{-1}y}(\tan^{-1}y - 1) + K$$

$$\Rightarrow \quad x = \tan^{-1}y - 1 + Ke^{-\tan^{-1}y}.$$

Q.15 *Solve the following differential equation:*

$$\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$$

[R.T.U. 2015, Raj. Univ. 2005]

Ans. The given equation may be written as

$$e^{x/y}\left(1-\frac{x}{y}\right)+\left(1+e^{x/y}\right)\frac{dx}{dy}=0$$

Putting $x = vy$ or $\frac{dx}{dy}=v+y\frac{dv}{dy}$

$$\Rightarrow e^{v}(1-v)+(1+e^{v})\left(v+y\frac{dv}{dy}\right)=0$$

$$\Rightarrow e^{v}-ve^{v}+v+ve^{v}+(1+e^{v})\,y\frac{dv}{dy}=0$$

$$\Rightarrow (v+e^{v})+(1+e^{v})\,y\frac{dv}{dy}=0$$

$$\Rightarrow \frac{(1-e^{v})dv}{(v+e^{v})}+\frac{dy}{y}=0$$

Integrating,

$$\log(v+e^{v})+\log y=\log k$$

$$\Rightarrow \log[(v+e^{v})\,y]=\log k$$

$$\Rightarrow (v+e^{v})\,y=k$$

Putting $v=x/y$

$$\Rightarrow \left[\left(\frac{x}{y}\right)+e^{x/y}\right]y=k$$

$$\Rightarrow (x+ye^{x/y})=k$$ **Ans.**

Q.16 *Form the differential equation by eliminating arbitrary constants.*

$$y^2=Ax^2+Bx+C$$

Ans. $y^2=Ax^2+Bx+C$

On differentiating $2y\frac{dy}{dx}=2Ax+B$

Again differentiating $2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=2A$

Again differentiating, we have

$$y\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=0$$

or $$y\frac{d^3y}{dx^3}+3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=0$$ **Ans.**

Q.17 $(x-y-2)dx-(2x-2y-3)dy=0$.

Ans. The equation can be written as

$$\frac{dy}{dx}=\frac{x-y-2}{2(x-y)-3}\quad \text{Let } x-y-2=z$$

$$1-\frac{dy}{dx}=\frac{dz}{dx}\quad\text{so}\quad 1-\frac{dz}{dx}=\frac{dy}{dx}$$

putting the value of $\frac{dy}{dx}$ and $x-y-2$, we have

$$1-\frac{dz}{dx}=\frac{z}{2(z+2)-3}$$

or $$\frac{dz}{dx}=1-\frac{z}{2z+1}=\frac{z+1}{2z+1}$$

or $$\int\frac{2z+1}{z+1}dz=\int dx$$

or $$\int 2dz-\int\frac{dz}{1+z}=\int dx$$

or $$2z-x=\log(1+z)+C$$

or $$2x-2y-4-x=\log(x-y-1)+C$$

or $$\log(x-y-1)=x-2y+C$$ **Ans.**

Q.18 *Solve the differential equation:*

$$x+yp^2+(1+xy)p=0$$

Ans. Given differential equation,

$$yp^2+(1+xy)p+x=0 \qquad \ldots(i)$$

which is quadratic equation in p, hence,

$$p=\frac{-(1+xy)\pm\sqrt{(1+xy)^2-4xy}}{2y}$$

$$=\frac{-(1+xy)\pm\sqrt{1+x^2y^2+2xy-4xy}}{2y}$$

$$=\frac{-(1+xy)\pm(1-xy)}{2y}$$

On taking + symbol,

$$p=\frac{-(1+xy)+(1-xy)}{2y}$$

$$=\frac{-1-xy+1-xy}{2y}=-x$$

$$p=-x \qquad \ldots(ii)$$

On taking –ve symbol,

$$p=\frac{-(1+xy)-(1-xy)}{2y}$$

$$=\frac{-1-xy-1+xy}{2y}$$

$$=-\frac{1}{y} \qquad \ldots(iii)$$

if $p=-x$ then,

$$p=-x$$

$$\Rightarrow \frac{dy}{dx}=-x$$

$$\Rightarrow dy=-xdx$$

On integrating both sides,

$$\int dy=-\int xdx+c$$

$$\Rightarrow y=-\frac{x^2}{2}+c \qquad \ldots(iii)$$

if $p=-\frac{1}{y}$ then,

$$\Rightarrow p=-\frac{1}{y}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{y}$$

$$\Rightarrow ydy+dx=0$$

On integration, both sides,

$$\int ydy+\int dx=c$$

$$\frac{1}{2}y^2+x=c \qquad \ldots(iv)$$

By equations, (iii) and (iv),

$$\left(\frac{1}{2}y^2+x-c\right)\left(y+\frac{x^2}{2}-c\right)=0$$

which is the required general solution of the given differential equation (i).

Q.19 *Solve the differential equation:*

$$xy^2(p^2+2)=2py^3+x^3$$

Ans. The given differential equation,

$$xy^2(p^2+2)=2py^3+x^3$$

$$\Rightarrow xy^2p^2-2y^3p+x(2y^2-x^2)=0$$

equation (ii) is a quadratic equation in p,

$$\Rightarrow p=\frac{2y^3\pm\sqrt{4y^6-4xy^2x(2y^2-x^2)}}{2xy^2}$$

$$=\frac{2y^3\pm\sqrt{4y^6-8x^2y^4+4y^2x^4}}{2xy^2}$$

$$\Rightarrow p=\frac{2y^3\pm 2y\sqrt{y^4-2x^2y^2+x^4}}{2xy^2}$$

On taking +ve symbol,

$$p=\frac{y^2+(x^2-y^2)}{xy}=\frac{x^2}{xy}$$

$$\Rightarrow p=\frac{x}{y}$$

On taking –ve symbol,

$$p=\frac{y^2-(x^2-y^2)}{xy}=\frac{2y^2-x^2}{xy}$$

$$\Rightarrow p=\frac{2y^2-x^2}{xy}$$

By equation (iii)

$$p=\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{y}$$

$$\Rightarrow ydy=xdx$$

On integrating both sides,

$$y^2=x^2+c$$

By equation (iv)

$$p=\frac{2y^2-x^2}{xy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y^2 - x^2}{xy}$$

which is homogeneous equation,

Then $\quad y = ux$

$$\text{or} \quad \frac{dy}{dx} = u + x.\frac{du}{dx}$$

$$u + x.\frac{du}{dx} = \frac{2u^2x^2 - x^2}{x.ux}$$

$$= \frac{2u^2 - 1}{u}$$

$$\text{or} \quad x.\frac{du}{dx} = \frac{2u^2 - 1}{u} - u$$

$$= \frac{2u^2 - 1 - u^2}{u} = \frac{u^2 - 1}{u}$$

$$\Rightarrow \quad \frac{2u}{u^2 - 1}du = \frac{2dx}{x}$$

On integration,

$$\log(u^2 - 1) = 2\log x + \log c$$

$$\Rightarrow \log(u^2 - 1) = \log c\,x^2$$

$$\Rightarrow \quad u^2 - 1 = cx^2$$

on putting $y = ux$

$$\frac{y^2}{x^2} - 1 = cx^2$$

$$\text{or} \quad y^2 = x^2 + cx^4 \qquad \ldots(vi)$$

By equation (v) and (vi),

$$(y^2 - x^2 - c)(y^2 - x^2 - cx^4) = 0$$

which is the general solution of the given differential equation (i).

Q.20 ***Solve the differential equation:*** $x^2 + xp^2 = yp$

Ans. The given differential equation,

$$yp = x^2 + xp^2 \qquad \ldots(i)$$

$$\text{or} \quad y = \frac{x^2}{p} + px \qquad \ldots(ii)$$

on differentiating with respect to x,

$$\frac{dy}{dx} = \frac{2x}{p} - \frac{x^2}{p^2}\frac{dp}{dx} + p + x.\frac{dp}{dx}$$

$$\text{or} \quad p = \frac{2x}{p} - \frac{x^2}{p^2}\frac{dp}{dx} + p + x.\frac{dp}{dx}$$

$$\text{or} \quad \frac{2x}{p} + \left(x - \frac{x^2}{p^2}\right)\frac{dp}{dx} = 0$$

$$\text{or} \quad (p^2 - x)\frac{dp}{dx} + 2p = 0$$

$$\text{or} \quad \frac{dx}{dp} - \frac{x}{2p} = -\frac{1}{2}p \qquad \ldots(iii)$$

equation (iii) is a linear equation, hence solution,

$$x.\frac{1}{\sqrt{p}} = \int -\frac{1}{2}p.\frac{1}{\sqrt{p}}dp + c$$

$$= -\frac{1}{2}\int p^{1/2}dp + c$$

$$\frac{x}{\sqrt{p}} = -\frac{1}{3}p^{3/2} + c$$

$$x = c\sqrt{p} - \frac{1}{3}p^2 \qquad \ldots(iv)$$

On putting the value of x is equation (ii), we have

$$y = \frac{1}{p}\left(c\sqrt{p} - \frac{1}{3}p^2\right)^2 + p\left(c\sqrt{p} - \frac{1}{3}p^2\right) \qquad \ldots(v)$$

equation (iv) and (v) combinely gives general solution to given differential equation (i).

Q.21 ***Solve the differential equation,***

$$y = -px + p^2x^4$$

Ans. The given differential equation,

$$y = -px + p^2x^4 \qquad \ldots(i)$$

on differentiating with respect to x, equation (i),

$$\frac{dy}{dx} = 2px^4.\frac{dp}{dx} + 4p^2x^3 - p - x.\frac{dp}{dx}$$

$$\Rightarrow \quad \left(2p + x.\frac{dp}{dx}\right) = 2px^3\left(2p + x.\frac{dp}{dx}\right)$$

$$\Rightarrow \quad (1 - 2px^3)\left(2p + x.\frac{dp}{dx}\right) = 0 \qquad \ldots(ii)$$

equation (ii) have two factors. First factor provide singular solution, while second factor provides general solution, hence,

$$\left(2p + x.\frac{dp}{dx}\right) = 0$$

$$\Rightarrow \quad 2\frac{dx}{x} + \frac{dp}{p} = 0 \qquad \ldots(iii)$$

On integrating equation (iii),

$$2\log x + \log p = \log c$$

$$\text{or} \quad px^2 = c$$

$$\Rightarrow \quad p = c/x^2$$

On eliminating p from equation (i), we have,

$$y = -\frac{c}{x^2}x + \frac{c^2}{x^4}x^4$$

$$xy = -c + c^2x$$

which is the general solution for the given differential equation (i).

Q.22 ***Solve the differential equation,***

$$p^3 - 4xyp + 8y^2 = 0$$

Ans. The given differential equation,

$$p^3 - 4xyp + 8y^2 = 0 \qquad \ldots(i)$$

$$\text{or} \quad 4xyp = 8y^2 + p^3$$

$$\text{or} \quad x = \frac{2y}{p} + \frac{p^2}{4y} \qquad \ldots(ii)$$

differentiating equation (ii) with respect to y,

$$\frac{1}{p} = \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy} - \frac{p^2}{4y^2} + \frac{p}{2y}\frac{dp}{dy}$$

$$\text{or} \quad \left(\frac{1}{p} - \frac{2y}{p^2}\frac{dp}{dy}\right) + \left(\frac{p}{2y}\frac{dp}{dy} - \frac{p^2}{4y^2}\right) = 0$$

$$\Rightarrow \quad -\frac{p^2}{4y^2}\left(1 - \frac{2y}{p}\frac{dp}{dy}\right) + \frac{1}{p}\left(1 - \frac{2y}{p}\frac{dp}{dy}\right) = 0$$

$$\Rightarrow \quad \left(1 - \frac{2y}{p}\frac{dp}{dy}\right)\left(\frac{1}{p} - \frac{p^2}{4y^2}\right) = 0 \qquad \ldots(iii)$$

Equation (iii) have two factors, second factor provides singular solutions while first factor provides general solutions, hence,

$$1 - \frac{2y}{p}\frac{dp}{dy} = 0$$

$$\Rightarrow \quad \frac{dy}{y} - 2\frac{dp}{p} = 0$$

On integration,

$$\log y - 2\log p + \log c = 0$$

$$\log y + \log c = 2\log p$$

$$\Rightarrow \quad p^2 = cy$$

On substitution in equation (ii), we get,

$$x = \frac{2y}{\sqrt{cy}} + \frac{cy}{4y}$$

$$x - \frac{c}{4} = \frac{2\sqrt{y}}{\sqrt{c}}$$

$$\frac{4x - c}{4} = 2\sqrt{\frac{y}{c}}$$

squaring both sides,

$$\frac{(4x - c)^2}{16} = 4\frac{y}{c}$$

$$c(4x - c)^2 = 64y$$

which is the general solution of the given differential equation.

Q.23 ***Solve the differential equation,***

$$y^2 \log y = xyp + p^2$$

Ans. The given differential equation

$$xyp + p^2 = y^2 \log y \qquad \ldots(i)$$

$$\text{or} \quad x = \frac{1}{p}y\log y - \frac{p}{y} \qquad \ldots(ii)$$

differentiating equation (ii) with respect to y,

$$\frac{dx}{dy} = \left(\log y + y.\frac{1}{y}\right)\frac{1}{p} - y\log y\frac{1}{p^2}\frac{dp}{dy} - \left[-\frac{1}{y^2}p + \frac{1}{y}\frac{dp}{dy}\right]$$

$$\Rightarrow \quad \frac{1}{p} = \frac{\log y}{p} + \frac{1}{p} + \frac{p}{y^2} - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$$

or $\left(\frac{p}{y}-\frac{dp}{dy}\right)\left(\frac{1}{y}+\frac{y\log y}{p^2}\right)=0$ ...(iii)

In equation (iii), second factor provides singular solution while first factor provides general solution, hence,

$$\frac{p}{y}-\frac{dp}{dy}=0$$

$$\Rightarrow \quad \frac{dp}{p}=\frac{dy}{y}$$

On integration,

$$\log p=\log y+\log c$$

$$\Rightarrow \quad p=cy$$

by putting the value of p in equation (ii), we have

$$cx=\log y-c^2$$

$$\Rightarrow \quad \log y=cx+c^2$$

which is the general solution of the given differential equation (i).

PART C

Q.24 *Solve :*

(i) $(1+y^2)+\left(x-e^{-\tan^{-1}y}\right)\frac{dy}{dx}=0$

(ii) $x\frac{dy}{dx}+y=y^2\log x$ *[R.T.U. 2016]*

Ans.(i) The given equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

Equation (i) is a linear differential equation in x with

$$P_1=\frac{1}{(1+y^2)},$$

$$Q_1=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

$$\therefore \quad \text{I.F.}=e^{\int P_1 dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Hence the solution is

$$x(\text{I.F.})=\int Q(\text{I.F.})dy+C$$

$$\Rightarrow \quad xe^{\tan^{-1}y}=\int\frac{e^{-\tan^{-1}y}}{1+y^2}\cdot e^{\tan^{-1}y}dy+C$$

$$\Rightarrow \quad xe^{\tan^{-1}y}=\tan^{-1}y+C$$

Ans.(ii) The given equation can be written as

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{x}\frac{1}{y}=\frac{1}{x}\log x \quad \text{...(i)}$$

Putting $-\frac{1}{y}=v$ so that $\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

We get from equation (i)

$$\frac{dv}{dx}-\frac{1}{x}v=\frac{1}{x}\log x \quad \text{...(ii)}$$

Equation (ii) is a linear differential equation. So,

$$\text{I.F.}=e^{\int-\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$$

Now, solution of equation (ii) is

$$v\,(\text{I.F.})=\int Q(\text{I.F.})dx+C$$

$$v\frac{1}{x}=\int\frac{1}{x}\log x.\frac{1}{x}dx+C$$

$$\Rightarrow \quad \frac{v}{x}=\int te^{-t}dt+C$$

$$\left[\begin{array}{l}\text{Putting } \log x=t\Rightarrow\frac{1}{x}dx=dt\\ x=e^t\end{array}\right]$$

$$\frac{v}{x}=-(t+1)e^{-t}+C$$

$$\frac{-1}{xy}=C-(1+\log x).\frac{1}{x} \quad \left[\because v=\frac{-1}{y}\right]$$

Thus, $(1+\log x)y-Cxy=1$

Q.25 *Solve :*

(i) $(3x^2y+y/x)\,dx+(x^3+\log x)\,dy=0$ *[R.T.U. 2016]*

(ii) $(xy\sin xy+\cos xy)\,ydx+(xy\sin xy-\cos xy)\,xdy=0.$ *[R.T.U. 2016, 2007, Raj. Univ. 2006]*

Ans. (i) Here $M=3x^2y+\frac{y}{x}, N=x^3+\log x$

Therefore $\frac{\partial M}{\partial y}=3x^2+\frac{1}{x}=\frac{\partial N}{\partial x}$ and given equation is an exact differential equation.

Now,

(a) $P=\int Mdx=\int\left(3x^2y+\frac{y}{x}\right)dx=x^3y+y\log x$

(b) Therefore $\frac{\partial P}{\partial y}=x^3+\log x$

(c) Now $N-\frac{\partial P}{\partial y}=(x^3+\log x)-(x^3+\log x)=0$

(d) $f(y)=\int\left(N-\frac{\partial P}{\partial y}\right)dy=0$

The solution is $P+f(y)=C$

$$x^3y+y\log x=C$$

(ii) Integrating factor

$$\frac{1}{Mx-Ny}=\frac{1}{(xy\sin xy+\cos xy)xy-(xy\sin xy-\cos xy)xy}$$

$$=\frac{1}{2xy\cos xy}$$

Multiplying both sides of the given equation by this integrating factor, we get

$$\frac{1}{2}\left(\tan xy+\frac{1}{xy}\right)ydx+\frac{1}{2}\left(\tan xy-\frac{1}{xy}\right)xdy=0$$

$$(\tan xy)(ydx+xdy)+\left(\frac{1}{x}\right)dx-\left(\frac{1}{y}\right)dy=0$$

$$\tan z\,dz+\frac{1}{x}dx-\frac{1}{y}dy=0$$

where $\quad z=xy$

Integrating term by term we get

$$\log(\sec z)+\log x-\log y=\log c$$

$$\log\left\{\frac{x\sec z}{y}\right\}=\log c\,;\ \frac{x}{y}(\sec z)=c$$

$$x\sec(xy)=cy \quad \textbf{Ans.}$$

Q.26 *Solve the following differential equations:*

(i) $\left(x+2y^3\right)\frac{dy}{dx}=y$

(ii) $2\frac{dy}{dx}=\frac{y}{x}+\frac{y^2}{x^2}$

(iii) $\left(xy^2+2x^2y^3\right)dx+\left(x^2y-x^3y^2\right)dy=0$ *[R.T.U. 2014]*

Ans. (i) We have

$$(x+2y^3)\frac{dy}{dx}=y$$

$$\Rightarrow \quad y\frac{dx}{dy}=x+2y^3$$

$$\Rightarrow \quad \frac{dx}{dy}-\frac{1}{y}x=2y^2$$

Which is linear differential equation

Here $\quad P=\frac{-1}{y}, Q=2y^2$

$$\text{I.F.}=e^{-\int\frac{dy}{y}}=e^{-\log y}=\frac{1}{y}$$

Hence solution is given as

$$x(I.F.)=\int Q.(I.F.)dy+c$$

$$\Rightarrow \quad x.\frac{1}{y}=\int\left(2y^2.\frac{1}{y}\right)dy+c$$

$$\Rightarrow \quad \frac{x}{y}=2\int ydy+c\Rightarrow\frac{x}{y}=2.\frac{y^2}{2}+c$$

$\Rightarrow \quad x=y^3+cy$ which is required solutio

Ans.(ii) We have,

$$2\frac{dy}{dx}-\frac{y}{x}=\frac{y^2}{x^2}$$

The given equation is in Bernoulli's form, m it by y^2, we have

$$\frac{2}{y^2}\frac{dy}{dx}-\frac{1}{xy}=\frac{1}{x^2}$$

Let $\quad \frac{1}{y}=v\Rightarrow\frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$. Thus the equation reduces to-

$$-2\frac{dv}{dx}-\frac{v}{x}=\frac{1}{x^2}\Rightarrow\frac{dv}{dx}+\frac{v}{2x}=\frac{-1}{2x^2} \text{ which is li}$$

So, $\quad \text{I.F.}=e^{\int\frac{dx}{2x}}=e^{\log x^{1/2}}=\sqrt{x}$

Thus, the solution will be

$$v.\sqrt{x}=\int-\frac{1}{2x^2}.\sqrt{x}dx+c\Rightarrow v\sqrt{x}=-\int\frac{x^{-3/2}}{2}dx$$

$$\Rightarrow \quad v.\sqrt{x}=-\frac{x^{-1/2}}{2\left(-\frac{1}{2}\right)}+c\Rightarrow\sqrt{x}=\left(\frac{1}{\sqrt{x}}\right)y$$

$$\Rightarrow \quad \boxed{x=y+cy\sqrt{x}}$$

Ans.(iii) $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$

Rewriting we get,

$$(xy^2dx+x^2ydy)+(2x^2y^3dx-x^3y^2dy)=0$$

$$\Rightarrow \quad xyd(xy)+y^3x^2\frac{d(x^2)}{x}-x^3y^2dy=0$$

$$\Rightarrow \quad xyd(xy)+xy^2\left[yd(x^2)-x^2d(y)\right]=0$$

$$\Rightarrow \quad d(xy)+y.y^2d\left(\frac{x^2}{y}\right)=0$$

Let $xy=u,\ \frac{x^2}{y}=v$ so $\frac{u^2}{v}=y^3$

$$\Rightarrow \quad du+\frac{u^2}{v}dv=0$$

$$\Rightarrow \quad \int\frac{du}{u^2}+\int\frac{dv}{v}=0\Rightarrow\frac{-1}{u}+\log v=c$$

$$\Rightarrow \quad \boxed{\frac{-1}{xy}+\log\left(\frac{x^2}{y}\right)=C}$$ **Ans.**

Q.27 (a) *Solve* $\sin y\frac{dy}{dx}=\cos y(1-x\cos y)$

[R.T.U. 2013, Raj. Univ. 2004, MREC 2000]

(b) *Solve* $(e^y+1)\cos x dx+e^y\sin x dy=0$

[R.T.U. 2013]

Ans. (a) We have, $\sin y\frac{dy}{dx}=\cos y(1-x\cos y)$

$$\sin y\frac{dy}{dx}-\cos y=-x\cos^2y$$

Dividing by $\cos^2y$, we get

$$\tan y\cdot\sec y\cdot\frac{dy}{dx}-\sec y=-x$$

Let $\sec y=v$

Then, $\sec y\cdot\tan y\cdot\frac{dy}{dx}=\frac{dv}{dx}$

So, we get

$$\frac{dv}{dx}-v=-x$$

Which is linear differential equation with

$$P=-1,\ Q=-x$$

$\therefore$ I.F. $=e^{\int P\,dx}$

$=e^{\int -1dx}$

$=e^{-x}$

The solution is given by

$$Ve^{-x}=\int -xe^{-x}dx$$

$=xe^{-x}+e^{-x}+c$

$=e^{-x}(x+1)+C$

$V=(1+x)+Ce^x$

$\sec y=(1+x)+Ce^x$ **Ans.**

Ans. (b) The given equation is of the form $Mdx+Ndy=0$

$M=(e^y+1)\cos x$

$N=e^y\sin x$

$\therefore \frac{\partial M}{\partial y}=e^y\cos x$

$\frac{\partial N}{\partial x}=e^y\cos x$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

therefore, the given equation is exact.

Hence, its solution is

$$\int(e^y+1)\cos x dx=c$$

[Since N has no term of y only]

or $(e^y+1)\sin x=c$

Q.28 (a) *Solve* $y(2xy+e^x)dx-e^xdy=0$

[R.T.U. 2013, Raj. Univ. 200]

(b) *Solve* $(y^2+2x^2y)dx+(2x^3-xy)dy=0$

[R.T.U. 2013, Raj. Univ. 200]

Ans. (a) The given equation can be written as

$$e^x\frac{dy}{dx}=e^xy+2xy^2$$

$$\frac{dy}{dx}-y=2xe^{-x}y^2$$

Dividing both sides by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}=2xe^{-x} \quad ...(i)$$

Now put $-1/y=v$ so that $(1/y^2).(dy/dx)=dv/dx$ with these substitutions, the equation (i) becomes

$$\frac{dv}{dx}+V=2xe^{-x}$$

which is linear with V as the dependent variable.

Here P = 1 and Q = $2xe^{-x}$

$\therefore$ I.F. $=e^{\int Pdx}$

$=e^{\int dx}$

$=e^x$

$\therefore$ The solution is $Ve^x=\int 2xe^{-x}\cdot e^xdx+C$

$=x^2+C$

$(-1/y).e^x=x^2+C$ $\quad$ [$\because v=-1/y$]

$e^x+y(C+x^2)=0$ **Ans.**

Ans. (b) The given differential equation is

$$(y^2+2x^2y)dx+(2x^3-xy)dy=0$$

This can be re-written as

$$x^2(2y\,dx+2x\,dy)+y(y\,dx-xdy)=0$$

which is of the form

$$x^ay^b(mydx+ndx)+x^cy^d(pydx+qx\,dy)=0$$

a = 2, b = 0, m = 2, n = 2, c = 0, d = 1, p = 1,

q = –1

which are all constants.

If for the equation of this form is $x^h\cdot y^k$, where h and k are so chosen that after multiplication by $x^h\cdot y^k$, the equation becomes exact.

The value of h and k can be determined from the relations.

$$\frac{a+h+1}{m}=\frac{b+k+1}{n}$$

and $\frac{c+h+1}{p}=\frac{d+k+1}{q}$

This implies that

$$\frac{3+h}{2}=\frac{k+1}{2}$$

and $\frac{h+1}{1}=\frac{k+2}{-1}$

$h-k=-1$

$h+k=-3$

$h=-5/2$

$k=-1/2$

$\therefore$ If is $x^{-5/2}y^{-1/2}$

Now multiplying the given differential equation by $x^{-5/2}y^{-1/2}$, we get

$$(x^{-5/2}.y^{3/2}+2x^{-1/2}y^{1/2})dx+(2x^{1/2}y^{-1/2}-x^{-3/2}y^{1/2})dy=0$$

which is an exact equation

Its solution is given by $\mu=C$, where

$$\mu=\int_{y\ constant}(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2})dx$$

$$=y^{3/2}\left(-\frac{2}{3}\cdot x^{-3/2}\right)+y^{1/2}(4x^{1/2})$$

$$=4\sqrt{xy}-\frac{2}{3}\left(\frac{y}{x}\right)^{3/2}$$

Here F(y) = 0 since N does not contain any term having y only.

Hence the solution is given by

$$4\sqrt{xy}-\frac{2}{3}\left(\frac{y}{x}\right)^{3/2}=C$$

Q.29 (a) *Solve :*

$$\frac{dy}{dx}=\left(\frac{2x-y+1}{x+2y-3}\right)$$ *[R.T.U. 2012, Raj. Univ. 1998]*

(b) $\frac{dy}{dx}=\frac{x^2+y^2+1}{2xy}$ *[R.T.U. 2012]*

Ans. (a) Substituting x = X + h and y = Y + k into given equation where h and k are to be determined, the given differential equation becomes

$$\frac{dY}{dX}=\frac{2(X+h)-(Y+k)+1}{(X+h)+2(Y+k)-3}$$

$$\Rightarrow \quad \frac{dY}{dX}=\frac{(2X-Y)+(2h-k+1)}{(X+2Y)+(h+2k-3)} \quad ...(1)$$

Choose h and k in such a way that

$2h-k+1=0$

and $h+2k-3=0$

On solving these two equation, we get

$$h=\frac{1}{5}, k=\frac{7}{5}$$

Now, equation (1) becomes

$$\frac{dY}{dX}=\frac{(2X-Y)}{(X+2Y)}$$

$$\Rightarrow \quad \frac{dY}{dX}=\left(\frac{2-\frac{Y}{X}}{1+2\frac{Y}{X}}\right) \quad \text{...(2)}$$

Put $\quad Y = VX \quad$ ...(3)

$$\Rightarrow \quad \frac{dY}{dX}=\left(V+X\frac{dV}{dX}\right) \quad \text{...(4)}$$

Using equation (3) and (4) into equation (2), we get

$$V+X\frac{dV}{dX}=\left(\frac{2-V}{1+2V}\right)$$

$$\Rightarrow \quad X\frac{dV}{dX}=\left(\frac{2-V}{1+2V}\right)-V$$

$$\Rightarrow \quad X\frac{dV}{dX}=\left(\frac{2-V-V-2V^2}{(1+2V)}\right)$$

$$\Rightarrow \quad X\frac{dV}{dX}=-\frac{2(V^2+V-1)}{(2V+1)}$$

$$\Rightarrow \quad \left(\frac{2V+1}{V^2+V-1}\right)=-2\frac{dX}{X}$$

On integration

$$\Rightarrow \quad \log(V^2+V-1)=-2\log X+\log c$$

$$\Rightarrow \quad V^2+V-1=\frac{c}{X^2}$$

$$\Rightarrow \quad \frac{Y^2}{X^2}+\frac{Y}{X}-1=\frac{c}{X^2}$$

$$\Rightarrow \quad Y^2+XY-X^2=c \quad \text{...(5)}$$

But $\quad x = X + h$ and $y = Y + k$

so, $\quad x = X+\frac{1}{5}$ and $y = Y+\frac{7}{5}$

$$\Rightarrow \quad X=\left(x-\frac{1}{5}\right) \text{ and } Y=\left(y-\frac{7}{5}\right)$$

Putting these values of X and Y into equation (5), we get

$$\left(y-\frac{7}{5}\right)^2+\left(x-\frac{1}{5}\right)\left(y-\frac{7}{5}\right)-\left(x-\frac{1}{5}\right)^2=c$$

$$\Rightarrow \quad (5y-7)^2+(5x-1)(5y-7)-(5x-1)^2=25c=c_1 \text{ (let)}$$

$$\Rightarrow \quad 25y^2-25x^2+25xy-75y-25x+55=c_1$$

$$\Rightarrow \quad y^2-x^2+xy-3y-x=c_2 \text{ where } c_2=\left(\frac{c_1-55}{25}\right)$$

$$\Rightarrow xy+y^2-3y=x^2+x+c_2 \qquad \textbf{Ans.}$$

Ans. (b) $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}$

$$\Rightarrow (x^2+y^2+1)dx-2xy\,dy=0$$

Here $\quad M = x^2+y^2+1$

$\quad N = -2xy$

Then, $\dfrac{\partial M}{\partial y}=2y; \dfrac{\partial N}{\partial x}=-2y$

Now $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{1}{-2xy}[2y-(-2y)]=\dfrac{-2}{x}$

$$\text{I.F.}=e^{\int -\frac{2}{x}dx}=\frac{1}{x^2}$$

Multiplying the given equation by I.F. we get

$$\left(1+\frac{1+y^2}{x^2}\right)dx-\frac{2y}{x}dy=0$$

$$dx+\left(\frac{1+y^2}{x^2}\right)dx-\frac{2y}{x}dy=0$$

$$dx-\frac{x(2y)dy-(1+y^2)dx}{x^2}=0$$

$$dx-d\left(\frac{1+y^2}{x}\right)=0$$

On integrating $\quad x-\dfrac{1+y^2}{x}=C$

$$\Rightarrow \quad x^2-y^2-1=Cx \qquad \textbf{Ans.}$$

Q.30. (a) $\dfrac{dy}{dx}=\dfrac{1}{xy(x^2y^2+1)}$

[R.T.U. 2012; 2007; Raj. Univ. 2003, 2001, 1998]

(b) $(xy^3+y)\,dx+2(x^2y^2+x+y^4)\,dy=0$

[R.T.U. 2012]

Ans.(a) The given equation can be written as

$$\frac{dx}{dy}=x^3y^3+xy \quad \text{...(i)}$$

$$\frac{dx}{dy}-xy=x^3y^3$$

$$\frac{1}{x^3}\frac{dx}{dy}-\frac{1}{x^2}y=y^3 \quad \text{...(ii)}$$

Let $\quad \dfrac{1}{x^2}=t$ on differentiating,

$$\Rightarrow \quad \frac{-2}{x^3}\frac{dx}{dy}=\frac{dt}{dy}$$

$$\Rightarrow \quad \frac{1}{x^3}\frac{dx}{dy}=\frac{-1}{2}\frac{dt}{dy} \quad \text{...(iii)}$$

Using eq.(iii) in equation (ii), we get

$$\frac{-1}{2}\frac{dt}{dy}-ty=y^3$$

$\dfrac{dt}{dy}+2ty=-2y^3$ is a linear differential equation

$$\therefore \quad \text{I.F.}=e^{\int 2y\,dy}=e^{y^2}$$

Then solution is

$$t\,(\text{I.F.})=\int(-2y^3)e^{y^2}dy+c \qquad (\because y^2=u)$$

$$te^{y^2}=-\int ue^{u}du+c \qquad (\text{and } 2y\,dy=du)$$

$$\Rightarrow \quad te^{y^2}=-e^{u}(u-1)+c$$

$$\Rightarrow \quad \frac{e^{y^2}}{x^2}=-e^{y^2}(y^2-1)+c \Rightarrow \frac{1}{x^2}=(1-y^2)+ce^{-y^2} \qquad \textbf{Ans.}$$

Ans.(b) $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$

Here $\quad M = xy^3+y$

$\quad N = 2(x^2y^2+x+y^4)$

Now $\dfrac{\partial M}{\partial y}=3xy^2+1$

$$\frac{\partial N}{\partial x}=2(2xy^2+1)=4xy^2+2$$

Using formula

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{xy^3+y}(4xy^2+2-3xy^2-1)$$

$$\frac{1}{y(xy^2+1)}(xy^2+1)=\frac{1}{y}$$

$$\text{I.F.}=e^{\int\frac{1}{y}dy}=y$$

Multiplying both sides by I.F,

$$y(xy^3+y)dx+2y(x^2y^2+x+y^4)dy=0$$

$$(xy^4+y^2)dx+2(x^2y^3+xy+y^5)dy=0$$

$$(xy^4\,dx+2x^2y^3dy)+(y^2dx+2xy\,dy)+2y^5dy=0$$

$$\frac{1}{2}d(x^2y^4)+d(y^2x)+d\left(\frac{1}{3}y^6\right)=0$$

on integrating, we get

$$\frac{x^2y^4}{2}+y^2x+\frac{y^6}{3}=C$$

$$3x^2y^4+6y^2x+2y^6=6C=C'$$

How to use change of ind. vari?

ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDERS 3

PREVIOUS YEARS QUESTIONS

PART A

Q.1 *If the roots of A.E. are $100 \pm \sqrt{500}$ then C.F. is ___* **[R.T.U. 2019]**

Ans. C.F.

$$y = e^{100x}\left[c_1 \cosh\sqrt{500}x + c_2 \sinh\sqrt{500}x\right]$$

Q.2 *What is the order and degree of the ODE $\frac{d^4y}{dx^4} = \cos\left(\frac{d^3y}{dx^3}\right)$?* **[R.T.U. 2019]**

Ans. The order of given diff. equation is 4 and degree is 1

Q.3 *Write the Euler-Cauchy equation.* **[R.T.U. 2019]**

OR

Define Cauchy-Euler homogeneous linear differential equation.

Ans. An equation,

$$x^n\frac{d^ny}{dx^n} + C_n x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + C_{n-1}x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots + C_1 y = P(x)$$

Where $C_1, C_2, \dots, C_n$ are constants, is called Cauchy Euler homogeneous linear equation.

Q.4 *Find CF of following differential equation :*

$$\left(D^3 - 13D + 12\right)y = 0, D = \frac{d}{dx}$$ **[R.T.U. 2018]**

Ans. $(D^3 - 13D + 12)y = 0$

Now, auxiliary equation is

$\Rightarrow m^3 - 13m + 12 = 0$

$\Rightarrow m^2(m-1) + m(m-1) - 12(m-1) = 0$

$\Rightarrow (m-1)(m^2 + m - 12) = 0$

$\Rightarrow (m-1)(m-3)(m+4) = 0$

$\Rightarrow M = 1, 3, -4$

$\therefore$ C.F. $= C_1 e^x + c_2 e^{3x} + c_3 e^{-4x}$

Q.5 $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$ **[R.T.U. 2007; R.U. 2002]**

Ans. The given differential equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$$

Therefore, the auxiliary equation corresponding to the given equation is

$m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0$

$\Rightarrow \quad m = 1, 2$

$\Rightarrow \quad$ C.F. $= C_1 e^x + C_2 e^{2x}$

and $\quad$ P.I. $= \frac{1}{D^2 - 3D + 2}e^{5x} = \frac{1}{25 - 15 + 2}e^{5x} = \frac{e^{5x}}{12}$

$\therefore \quad$ The complete solution is given by

$$y = C_1 e^x + C_2 e^{2x} + \frac{e^{5x}}{12}$$ **Ans.**

Q.6 *Explain symbols $D, \frac{1}{D}, \frac{1}{D^2}$ in solving linear differential equation.*

Ans. Symbol D stands for the operation of differential i.e.

$$Dy = \frac{dy}{dx}, D^2y = \frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration, $\frac{1}{D^2}$ stands for the operation of integration twice.

Q.7 *Explain method for finding complementary function when auxiliary equations have imaginary roots.*

Ans. If the roots are $\alpha \pm i\beta$ then the solution will be

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left[C_1 e^{i\beta x} + C_2 e^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x\right]$$

$$= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$

Q.8 $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 17\frac{dy}{dx} - 13y = 0$

Ans. The given differential equation is

$$\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 17\frac{dy}{dx} - 13y = 0$$

Converting it in operator form

We have $(D^3 - 5D^2 + 17D - 13) = 0$, $D = \frac{d}{dx}$

Therefore, the auxiliary equation is

$m^3 - 5m^2 + 17m - 13 = 0$

or $\quad (m-1)\{(m-2)^2 + 3^2\} = 0$

or $\quad m = 1, 2 \pm i3$

Hence, the general solution is

$$y = C_1 e^x + e^{2x}\{C_2\cos 3x + C_3\sin 3x\}$$ **Ans.**

Q.9 $(D^4 - 2D^2 + 1)y = 0$

Ans. The given differential equation is $(D^4 - 2D^2 + 1)y = 0$

Therefore, the auxiliary equation is

$m^4 - 2m^2 + 1 = 0$

or $\quad (m-1)(m^3 + m^2 - m - 1) = 0$

or $\quad (m-1)^2(m+1)^2 = 0$

or $\quad m = 1, 1, -1, -1$

Hence, the general solution is

$$y = (C_1 + C_2 x)e^{-x} + (C_3 + C_4 x)e^x$$ **Ans.**

Q.10 *Explain working rule for finding P.I. when x is of the form $e^{ax}V$, where V is any function of x.*

Ans. $\frac{1}{f(D)}e^{ax}V = \frac{e^{ax}}{f(D+a)}V$

where $\frac{1}{f(D+a)}V$ is calculated with the help of previous methods. To get $f(D+a)$, replace each D by D + a. This method will be used, if V is cos ax or sin ax or x^m or a polynomial of degree m.

Q.11 *Define simultaneous differential equations.*

Ans. Simultaneous differential equations are generally defined on three variables and can be given as

$Pdx + Qdy + Rdz = 0$

$Adx + Bdy + Cdz = 0$

Hence A, B, C, P, Q and R are the functions of x, y, z.

Q.12 *Explain the condition for a second order linear differential equation to be exact.*

Ans. A second order differential equation is said to be exact if it can be obtained from a first order differential equation by direct differentiation.

Q.13 *What is meant by integrating factor for a linear differential equation of second order?*

Ans. Integrating Factor : Sometimes an equation which is not exact can be made, by multiplying the equation by some suitable function of x is known as integrating factor.

Q.14 *Explain the method of undetermined coefficients in brief.*

Ans. The particular integral of n^{th} order linear differential equation with constant coefficients

$$(D^n + c_1 D^{n-1} + \dots + c_n)y = P(x)$$

can be calculated by the method of undetermined coefficients.

Method : Consider $m_1, m_2, m_3, \ldots, m_n$ be the roots of auxiliary equation $\phi(m) = 0$ of the given differential equation (1), and $y = \{y_1, y_2, \ldots, y_n\}$ be the set of n linearly independent solution of the differential equation $f(D)\,y = 0$ having n roots $m_1, m_2, \ldots, m_n$. Then the complimentary function can be given by

$$y = a_1 y_1 + a_2 y_2 + \ldots + a_n y_n.$$

Q.15 ***Write down the condition for which the equation***

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \text{ has } u = e^{ax} \text{ as a part of C.F.}$$

Ans. If $1 + \frac{P}{q} + \frac{Q}{a^2} = 0$ then $u = e^{ax}$ will be a part of C.F.

Q.16 ***Show that for equation***

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

$y = x$ is a part of C.F.

Ans. Writing the equation in the standard form we have

$$\frac{d^2y}{dx^2} - 2\left(\frac{1}{x}+1\right)\frac{dy}{dx} + 2\left(\frac{1}{x^2}+\frac{1}{x}\right)y = x$$

Here $P = -2\left(\frac{1}{x}+1\right),\ Q = 2\left(\frac{1}{x^2}+\frac{1}{x}\right)$

By inspection $P + Qx = 0$

Hence $y = x$ is a part of C.F.

PART B

Q.17 ***Solve:*** $(D^3 + 1)\,y = (e^x + 1)^2$, ***where*** $D = \frac{d}{dx}$. **[R.T.U. 2019]**

Ans. The auxiliary equation is

$$m^3 + 1 = 0$$
$$(m+1)(m^2 - m + 1) = 0$$
$$m = -1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

then C.F.

$$y = c_1e^{-x} + e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^3+1}(e^x+1)^2$$
$$= \frac{1}{D^3+1}(e^{2x} + 2e^x + 1)$$
$$= \frac{1}{D^3+1}e^{2x} + 2\frac{1}{D^3+1}e^x + \frac{1}{D^3+1}e^{0x}$$
$$= \frac{1}{9}e^{2x} + 2\frac{1}{2}e^x + 1$$
$$\text{P.I.} = \frac{1}{9}e^{2x} + e^x + 1$$

Hence the complet solutaion is

$$y = \text{C.F.} + \text{P.I.}$$

$$\therefore\ y = c_1e^{-x} + e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{9}e^{2x} + e^x + 1$$

Q.18 ***Solve:*** $D^2x + m^2y = 0;\ D^2y - m^2x = 0$, ***where*** $D = \frac{d}{dt}$. **[R.T.U. 2019]**

Ans. $D^2x + m^2y = 0$ ... (1)

$D^2y - m^2x = 0$ ... (2)

Differentiate twice times of (1) w.r.t. to t, we get

$D^4x + m^2D^2y = 0$

or $D^4x + m^2.m^2x = 0$ [Using equation (2)]

or $(D^4 + m^4)\,x = 0$ ... (3)

A.E.

$$k^4 + m^4 = 0$$
$$(k^2 + m^2) = 2k^2m^2$$
$$k^2 + m^2 = \pm\sqrt{2}km$$
$$\Rightarrow k^2 - \sqrt{2}km + m^2 = 0$$

or $k^2 + \sqrt{2}km + m^2 = 0$

on solving, we get

$$k = \frac{m}{\sqrt{2}}(1 \pm i) \text{ or } k = \frac{-m}{\sqrt{2}}(1 \pm i)$$

C.F.

$$x(t) = e^{\frac{m}{\sqrt{2}}t}\left[c_1\cos\frac{m}{\sqrt{2}}t + c_2\sin\frac{m}{\sqrt{2}}t\right] + e^{-\frac{m}{\sqrt{2}}t}\left[c_3\cos\frac{m}{\sqrt{2}}t + c_4\sin\frac{m}{\sqrt{2}}t\right] \quad \ldots (4)$$

Puting the value of $D^2x(t)$ in (1), we get

$$2\frac{m}{\sqrt{2}}e^{\frac{m}{\sqrt{2}}t}\left[\frac{-m}{\sqrt{2}}c_1\sin\frac{m}{\sqrt{2}}t + \frac{m}{\sqrt{2}}c_2\cos\frac{m}{\sqrt{2}}t\right] - 2\frac{m}{\sqrt{2}}e^{\frac{-m}{\sqrt{2}}t}$$
$$\times\left[\frac{-m}{\sqrt{2}}c_3\sin\frac{m}{\sqrt{2}}t + \frac{m}{\sqrt{2}}c_4\cos\frac{m}{\sqrt{2}}t\right] + m^2y = 0$$

or $$m^2e^{\frac{m}{\sqrt{2}}t}\left[-c_1\sin\frac{m}{\sqrt{2}}t + c_2\cos\frac{m}{\sqrt{2}}t\right] + m^2e^{\frac{-m}{\sqrt{2}}t}$$
$$\left[c_3\sin\frac{m}{\sqrt{2}}t - c_4\cos\frac{m}{\sqrt{2}}t\right] + m^2y = 0$$

or $$y = -e^{\frac{m}{\sqrt{2}}t}\left[-c_1\sin\frac{m}{\sqrt{2}}t + c_2\cos\frac{m}{\sqrt{2}}t\right] - e^{\frac{-m}{\sqrt{2}}t}$$
$$\left[c_3\sin\frac{m}{\sqrt{2}}t - c_4\cos\frac{m}{\sqrt{2}}t\right]$$

Q.19 ***Solve:*** $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$ **[R.T.U. 2014]**

OR

Find PI of following differential equation :

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2 \quad \textbf{[R.T.U. 2018]}$$

Ans. (Given–homogeneous differential equation)

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$$

Putting $z = \log x$, the differential equation becomes

$$[D(D-1) - 3D + 4]\,y = 2e^{2z} \text{ where, } D \equiv \frac{d}{dz}$$
$$\Rightarrow (D^2 - 4D + 4)\,y = 2e^{2z}$$

A.E. is $m^2 - 4m + 4 = 0 \Rightarrow m = 2,2$

$$\therefore \text{C.F.} = (C_1 + C_2z)e^{2z}$$
$$\text{P.I.} = \frac{1(2e^{2z})}{(D-2)^2} = 2e^{2z}\frac{1}{D^2}.1 = e^{2z}z^2$$

Hence, the general solution is

$$\boxed{y = (C_1 + C_2\log x)x^2 + x^2.(\log x)^2}$$

Q.20 ***Solve*** $(D^2+1)^2y = 24x\cos x$ ***where,*** $D = d/dx$ **[R.T.U.**

OR

$(D^2+1)^2y = 24x\cos x;\ D = d/dx$, ***Solve it.*** **[R.T.U.**

Ans. The auxiliary equation is

$$(m^2+1)^2 = 0 \Rightarrow m = -i, -i, +i, +i$$
$$\therefore \text{C.F.} = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin$$
$$\text{P.I.} = \frac{1}{(D^2+1)^2}(24x\cos x)$$
$$= \text{Real part of}\left[24.\frac{1}{(D^2+1)^2}xe^{ix}\right]$$
$$= 24 \text{ Real part of}\left[e^{ix}\frac{1}{[(D+i)^2+1]^2}x\right]$$
$$= 24 \text{ Real part of}\left[e^{ix}\frac{1}{(D^2+2iD)^2}x\right]$$
$$= 24 \text{ Real part of}\left[e^{ix}\frac{1}{D^2(D+2i)^2}x\right]$$
$$= 24 \text{ Real part of}\left[e^{ix}\frac{1}{D^2}\left(\frac{-1}{4}\right)\left(1+\right.\right.$$
$$= 24 \text{ Real part of}\left[e^{ix}\frac{1}{D^2}\left(\frac{-1}{4}\right)\left(1-\right.\right.$$
$$= 24 \text{ Real part of}\left[\frac{-1}{4}e^{ix}\frac{1}{D^2}(x+i\right.$$
$$= (-6) \text{ Real part of}\left[e^{ix}\left(\frac{x^3}{6}+i\frac{x^2}{2}\right.\right.$$
$$= (-6) \text{ Real part of}$$
$$\left[\frac{x^3}{6}(\cos x + i\sin x) + \frac{1}{2}ix^2(\cos x + i\right.$$
$$= -x^3\cos x + 3x^2\sin x.$$

$\therefore$ The general solution is

$$y = (C_1 + C_2x)\cos x + (C_3 + C_4x)$$
$$+ (3x^2\sin x - x^3\cos x)$$

Q.21 Solve :

$$\frac{d^2x}{dt^2}+2n\cos\alpha\frac{dx}{dt}+n^2x=a\cos nt,$$

such that x = 0 and dx/dt = 0 at t = 0

[R.T.U. 2016, 2010, 2007]

Ans. The auxiliary equation is

$$m^2+2mn\cos\alpha+n^2=0$$

or $(m+n\cos\alpha)^2=i^2n^2\sin^2\alpha$

so $m=-n\cos\alpha\pm i\,n\sin\alpha$

$$\therefore \text{C.F.}=e^{-nt\cos\alpha}\left[C_1\cos(nt\sin\alpha)+C_2\sin(nt\sin\alpha)\right]$$

$$\text{P.I.}=\frac{1}{D^2+n^2+2n\cos\alpha D}a\cos nt$$

$$=\frac{1}{-n^2+n^2+2n\cos\alpha D}a\cos nt$$

$$=\frac{a}{2n\cos\alpha}\int\cos nt\,dt=\frac{a}{2n^2\cos\alpha}\sin nt$$

So, the general solution is

$$=e^{-nt\cos\alpha}\left[C_1\cos(nt\sin\alpha)+C_2\sin(nt\sin\alpha)\right]+\frac{a}{2n^2\cos\alpha}\sin nt$$

Q.22 Solve :

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\{\log(1+x)\}$$

[R.T.U. 2016, 2012]

Ans. Here, $(1+x)=e^z$

$\Rightarrow$ $\log(1+x)=z$

$\therefore$ $(1+x)^2\frac{d^2y}{dx^2}=D(D-1)y$

and $(1+x)\frac{dy}{dx}=Dy$

On substituting above value in the given equation, we get

$$[D(D-1)+D+1]y=4\cos z$$

$$(D^2+1)y=4\cos z$$

Auxiliary equation is

$$m^2+1=0\,;\ m=\pm i$$

$\therefore$ $\text{C.F.}=C_1\cos z+C_2\sin z$

and $\text{P.I.}=\frac{1}{D^2+1}4\cos z=\text{R.P. of }\frac{4e^{iz}}{D^2+1}$

$$=\text{R.P. of }4e^{iz}\frac{1}{[(D+i)^2+1]}e^{0z}$$

$$=\text{R.P. of }4e^{iz}.\frac{1}{D^2+2Di}e^{0z}$$

$$=\text{R.P. of }4e^{iz}.\frac{1}{D(D+2i)}e^{0z}$$

$$=\text{R.P. of }4e^{iz}.\frac{1}{2iD}e^{0z}$$

$$=\text{R.P. of }\frac{2(\cos z+i\sin z)z}{i}$$

$$=\text{R.P. of }(-2z\cos z+2z\sin z)$$

$\therefore$ The general solution is

$$y=\text{C.F.}+\text{P.I.}$$

$$=C_1\cos z+C_2\sin z+2z\sin z$$

$$=C_1\cos\{\log(1+x)\}+C_2\sin\{\log(1+x)\}+2\log(1+x)\sin\{\log(1+x)\}$$ **Ans.**

Q.23 Solve :

$$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+(2\sin^2x)y=\sin^2x\,e^{-\cos x}$$

[R.T.U. 2016, 2009; Raj.Univ. 2005]

Ans. Changing the independent variable from x to z

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \qquad ...(1)$$

$$\text{Where, }P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2},Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2};R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Choose z such that

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{2\sin^2x}{\left(\frac{dz}{dx}\right)^2}=\text{constant}=2(say)$$

$$\left(\frac{dz}{dx}\right)^2=\sin^2x$$

$$z=-\cos x$$

$$\left(\frac{dz}{dx}\right)=\sin x\Rightarrow\frac{d^2z}{dx^2}=\cos x$$

$$\text{So,}\quad P_1=\frac{\cos x+(3\sin x-\cot x)(\sin x)}{\sin^2x}$$

$$=\frac{\cos x+3\sin^2x-\cos x}{\sin^2x}=3$$

$$R_1=\frac{e^{-\cos x}\sin^2x}{\sin^2x}=e^{-\cos x}=e^z$$

So the equation (1) becomes

$$\frac{d^2y}{dz^2}+3\frac{dy}{dz}+2=e^z$$

Auxiliary equation

$$m^2+3m+2=0$$

$\Rightarrow$ $(m+2)(m+1)=0$

$\Rightarrow$ $m=-1,-2$

$$\text{C.F}=c_1e^{-z}+c_2e^{-2z}=C_1e^{\cos x}+c_2e^{2\cos x}$$

$$\text{P.I.}=\frac{1}{D^2+3D+2}e^z=e^z\frac{1}{1+3+2}$$

$$=\frac{e^z}{6}=\frac{1}{6}e^{-\cos x}$$

$\therefore$ Complete solution is $y=\text{C.F.}+\text{P.I.}$

$$y=c_1e^{\cos x}+c_2e^{2\cos x}+\frac{1}{6}e^{-\cos x}$$ **Ans.**

Q.24 Solve:

$$(2x^2+3x)\frac{d^2y}{dx^2}+(6x+3)\frac{dy}{dx}+2y=(x+1)e^x$$

[R.T.U. 2016, 2015, 2010; Raj. Univ. 2000]

Ans. Here $P_0=2x^2+3x$

$$P_1=6x+3,P_2=2,Q=(x+1)e^x,$$

$$P_2-P'_1+P''_0=2-6+4=0$$

Here $P_2-P'_1+P''_0=0$ so, given equation is exact. Its first integral is

$$P_0\frac{dy}{dx}+(P_1-P'_0)y=\int Qdx+C_1$$

$$(2x^2+3x)\frac{dy}{dx}+(6x+3-4x-3)y=xe^x+C_1$$

$$(2x^2+3x)\frac{dy}{dx}+(2x)y=xe^x+C_1$$

$$\frac{dy}{dx}+\frac{2x}{x(2x+3)}y=\frac{xe^x}{x(2x+3)}+\frac{C_1}{x(2x+3)}$$

The equation is linear, therefore

$$\text{I.F.}=e^{2\int\frac{1}{2x+3}dx}=e^{\log(2x+3)}=2x+3$$

Solution is $y(2x+3)=\int\left(e^x+\frac{C_1}{x}\right)dx+C_2$

or $y(2x+3)=e^x+C_1\log x+C_2$ **Ans.**

Q.25 *Solve by the method of variation of parameters -*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x.$$

[R.T.U. 2016, 2010, Raj. Univ. 2003]

Ans. For finding C.F., we have the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \qquad ...(1)$$

Changing the independent variable from x to z by putting $x=e^z$ or $z=\log x$

$$x\frac{d}{dx}=\frac{d}{dz}=D\text{ and }x^2\frac{d^2}{dx^2}=D(D-1)$$

The equation (1) becomes

$$[D(D-1)+D-1]y=0$$

$$(D^2-1)y=0$$

$$\therefore \text{C.F.}=C_1e^z+C_2e^{-z}=C_1x+\frac{C_2}{x}$$

Thus $u=x,v=\frac{1}{x}$ as parts of C.F.

Now let $y=A.x+\frac{B}{x}$ where A and B are functions of x.

We further let $A_1x+\frac{1}{x}B_1=0$

Now the values of A_1 and B_1 are given by

$$R=e^x+A_1u_1+B_1v_1$$

$$A_1-\frac{B_1}{x^2}-e^x=0$$

$$xA_1+\frac{B_1}{x}+0=0$$

$$\frac{A_1}{\frac{e^x}{x}}=\frac{B_1}{-xe^x}=\frac{1}{\frac{1}{x}+\frac{x}{x^2}}=\frac{1}{2/x}$$

So, $A_1=\frac{1}{2}\int e^xdx=\frac{1}{2}e^x$

$$B_1=\frac{1}{2}\int x^2e^xdx=\left(-\frac{x^2}{2}+x-1\right)e^x$$

Hence the complete solution is

$$y=C_1x+\frac{C_2}{x}+\frac{1}{2}e^x.x+\frac{1}{x}\left(-\frac{x^2}{2}+x-1\right)e^x$$

$$=C_1x+\frac{C_2}{x}+e^x-x^{-1}e^x$$ **Ans.**

Q.26 Solve:

$$x^3\frac{d^2y}{dx^2}+3x^2\frac{dy}{dx}+xy=\sin(\log x)$$ *[R.T.U. 2015]*

Ans. $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{\sin(\log x)}{x}$ ...(1)

Put $x=e^z$ or $z=\log x$

So $x\frac{dy}{dx}=Dy$ and $x^2\frac{d^2y}{dx^2}=D(D-1)y$

where $D=\frac{d}{dz}$

Hence from eq. (1)

$$D(D-1)y+3Dy+y=\frac{\sin z}{e^z}$$

or $(D^2+2D+1)y=e^{-z}\sin z$ ...(2)

A.E. is $m^2+2m+1=0$

$m=-1,-1$

C.F. $=(C_1+C_2z)e^{-z}$

and P.I. $=\frac{1}{D^2+2D+1}e^{-z}\sin z$

$$=\frac{1}{(D+1)^2}e^{-z}\sin z$$

$$e^{-z}\frac{1}{(D-1+1)^2}\sin z$$

$$e^{-z}\frac{1}{D^2}\sin z$$

$$=-e^{-z}\sin z$$

Solution of (2)

$$y=(C_1+C_2z)e^{-z}-e^{-z}\sin z$$

Put $z=\log x$ and $e^z=x$

we get Solution of eq. (1)

$$y=(C_1+C_2\log x).\frac{1}{x}-\frac{\sin(\log x)}{x}$$

Q.27 *Solve the following differential equation:*

$(D^2-4D+4)y=8x^2e^{2x}\sin 2x$ *[R.T.U. 2015, 2011]*

Ans. Auxiliary equation is $m^2-4m+4=0$.

⇒ $(m-2)^2=0$

⇒ $m=2,2$

C.F. = $(c_1+c_2x)e^{2x}$

P.I. $=\frac{1}{(D-2)^2}8x^2e^{2x}\sin 2x$

$$=8e^{2x}\cdot\frac{1}{(D+2-2)^2}x^2\sin 2x$$

$$=8e^{2x}\frac{1}{D^2}(x^2\sin 2x)$$

$$=8e^{2x}\text{ I.P. of}\left[\frac{1}{D^2}x^2e^{2ix}\right]$$

$$=8e^{2x}\text{ I.P. of}\left[e^{2ix}\cdot\frac{1}{(D+2i)^2}x^2\right]$$

$$=8e^{2x}\text{ I.P. of}\left[e^{2xi}\cdot\frac{1}{(2i)^2}\left(1+\frac{D}{2i}\right)^{-2}x^2\right]$$

$$=8e^{2x}\text{ I.P. of}\left[-\frac{1}{4}e^{2xi}\left(1-\frac{D}{i}+\frac{3}{4i^2}D^2\right)x^2\right]$$

$$=8e^{2x}\text{ I.P. of}\left[-\frac{1}{4}e^{2xi}\left(x^2-\frac{2x}{i}-\frac{3}{2}\right)\right]$$

$$=8e^{2x}\text{ I.P. of}\left[-\frac{1}{4}(\cos 2x+i\sin 2x)\left(x^2+2ix-\frac{3}{2}\right)\right]$$

$$=8e^{2x}\left[-\frac{2x}{4}\cos 2x-\frac{1}{4}\sin 2x\left(x^2-\frac{3}{2}\right)\right]$$

$$=-2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

∴ Complete solution is

$$y=(c_1+c_2x)e^{2x}-2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$ **Ans.**

Q.28 *Solve:* $x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=x^5$ *[R.T.U. 2014]*

Ans. We have,

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=x^5$$

⇒ $\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-4x^2y=x^4$ ...(i)

Let x be changed to z so that (i) becomes

$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$ ...(ii)

Taking $Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{-4x^2}{\left(\frac{dz}{dx}\right)^2}=-1$, we have

⇒ $\frac{dz}{dx}=2x$ ⇒ $\boxed{z=x^2}$

Now, $P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{2+\left(-\frac{1}{x}\right)2x}{(2x)^2}=0$

and, $R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{x^4}{(2x)^2}=\frac{1}{4}x^2=\frac{z}{4}$

Hence, eq.(ii) becomes

$$\frac{d^2y}{dz^2}-y=\frac{z}{4}$$

⇒ $(D^2-1)y=\frac{z}{4}$, where $D=\frac{d}{dz}$

A.E. is $m^2-1=0 \Rightarrow m=\pm1$

C.F. $=C_1e^z+C_2e^{-z}=C_1e^{x^2}+C_2e^{-x^2}$

Now, P.I. $=\frac{1}{(D^2-1)}\left(\frac{z}{4}\right)$

$$=-\frac{1}{4}(1-D^2)^{-1}(z)=\frac{-1}{4}(1+D^2+D^4)(z)$$

$$=-\frac{1}{4}(z)=\frac{-z}{4}=\frac{-x^2}{4}$$

Hence, the complete solution is given by-

$$\boxed{y=C.F.+P.I.=C_1e^{x^2}+C_2e^{-x^2}-\frac{x^2}{4}}$$

Ans.

Q.29 *Solve the differential equation:*

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\log x$$

by using the method of variation of parameters.

[R.T.U. 2014, 09; Raj. Univ. 2004]

Ans. We have,

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\log x$$

The two integrals of the complementary function are e^x and xe^x.

Let the complete solution be

$$y=Ae^x+Bxe^x$$

where A and B are functions of x. Choosing A & B such that

$$e^x\frac{dA}{dx}+xe^x\frac{dB}{dx}=0$$

⇒ $\frac{dA}{dx}+x\frac{dB}{dx}=0$ ...(i)

and, $\frac{dA}{dx}e^x+\frac{dB}{dx}e^x(1+x)=e^x\log x$

⇒ $\frac{dA}{dx}+(1+x)\frac{dB}{dx}=\log x$ ...(ii)

From (i) ⇒ $\frac{dA}{dx}=-x\frac{dB}{dx}$

Using this in eq. (ii) gives

⇒ $\frac{dB}{dx}[-x+1+x]=\log x$

⇒ $\boxed{B=x\log x-x+C_1}$

Also, $\frac{dA}{dx}=-x\log x$

⇒ $\boxed{A=\frac{x^2}{4}-\frac{x^2}{2}\log x+C_2}$

Hence, the complete solution of D.E. is

$$y=\left(\frac{x^2}{4}-\frac{x^2}{2}\log x+C_2\right)e^x+(x\log x-x+C_1)x e^x$$

$$=e^x\log x\left(x^2-\frac{x^2}{2}\right)+e^xx^2\left(\frac{1}{4}-1\right)+e^x(C_2+C_1x)$$

$$=\frac{x^2e^x\log x}{2}-\frac{3}{4}x^2e^x+e^x(C_2+C_1x)$$

⇒ $\boxed{y=e^x\left[\frac{x^2\log x}{2}-\frac{3}{4}x^2+C_2+C_1x\right]}$

Q.30 *Solve* $(D^2-2D-4)y=e^x\cos x-\sin^2 x$

Ans. A.E: $m^2-2m-4=0$

$$m=\frac{2\pm\sqrt{4+16}}{2}$$

$m=1\pm\sqrt{5}$ (real roots)

CF: $y_c=C_1e^{m_1x}+C_2e^{m_2x}$

Where, $m_1=1+\sqrt{5}, m_2=1-\sqrt{5}$

Now P.I: $y_p=\frac{1}{D^2-2D-4}[e^x\cos x-\sin^2 x$

$$=\frac{1}{D^2-2D-4}(e^x\cos x)-\frac{1}{D^2-2D-4}(\sin^2 x$$

$$=\frac{1}{D^2-2D-4}(e^x\cos x)-\frac{1}{D^2-2D-4}\left\{\frac{1}{2}(1-\right.$$

$$=\frac{1}{D^2-2D-4}(e^x\cos x)-\frac{1}{2}\frac{1}{D^2-2D-4}$$

$= I_1 - \frac{1}{2}(I_2 - I_3)$

$I_1 = \frac{1}{D^2 - 2D - 4}(e^x \cos x)$

$= \frac{e^x}{(D+1)^2 - 2(D+1) - 4}.\cos x$

$= \frac{e^x}{D^2 - 5}.\cos x$

Replacing D^2 by -1^2, we have

$\frac{e^x}{-1^2 - 5}.\cos x$

$I_1 = -\frac{e^x.\cos x}{6}$ ---(ii)

$I_2 = \frac{1}{D^2 - 2D - 4}.1$

$I_2 = \frac{1}{D^2 - 2D - 4}e^{0x}$

Replacing D by 0, we have

$I_2 = \frac{1}{0 - 0 - 4} = -\frac{1}{4}$ ---(iii)

$I_3 = \frac{1}{D^2 - 2D - 4}(\cos 2x)$

Replacing D^2 by -2^2, we get

$I_3 = \frac{1}{-2^2 - 2D - 4}\cos 2x$

$= -\frac{1}{(2D+8)}.\cos 2x$

Rewrite this to get D^2 terms as

$I_3 = -\frac{(2D-8)}{(2D+8)(2D-8)}.\cos 2x$

$= -\frac{(2D-8)}{(4D^2 - 8^2)}.\cos 2x$

Replace D^2 by -2^2, we get

$I_3 = -\frac{(2D-8)}{\{4(-2^2) - 8^2\}}.\cos 2x$

$= -\frac{(2D-8)}{(-16-64)}\cos 2x$

$= \frac{1}{80}(2D-8)(\cos 2x)$

$= \frac{1}{80}(-4\sin 2x - 8\cos 2x)$

$= \frac{-4}{80}(\sin 2x + 2\cos 2x)$

$I_3 = \frac{-1}{20}(\sin 2x + 2\cos 2x)$ ...(iv)

Thus, $y_p = I_1 - \frac{1}{2}(I_2 - I_3)$

$= \frac{-e^x.\cos x}{6} - \frac{1}{2}\left[-\frac{1}{4} + \frac{1}{20}(\sin 2x + 2\cos 2x)\right]$

$\therefore y = y_c + y_p$

$= C_1 e^{(1+\sqrt{5})x} + C_2 e^{(1-\sqrt{5})x} - \frac{e^x.\cos x}{6}$

$-\frac{1}{2}\left[-\frac{1}{4} + \frac{1}{20}(\sin 2x + 2\cos 2x)\right]$

Q.31 *Solve* $(D^2 + 3D + 2)y = x^2 \cos x$ [R.T.U. 2013]

Ans. A.E : $m^2 + 3m + 2 = 0$

$m = -1, -2$ (real roots)

C.F : $y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ ...(i)

where $m_1 = -1, m_2 = -2$

Now P.I : $y_p = \frac{1}{D^2 + 3D + 2}x^2 \cos x$

$= \frac{1}{D^2 + 3D + 2}x.(x\cos x)$

Applying

$$\frac{1}{f(D)}\{xV(x)\} = x\frac{1}{f(D)}V(x) - \frac{f'(D)}{\{f(D)\}^2}V(x) \quad ...(ii)$$

We have,

$y_p = \frac{x}{D^2 + 3D + 2}(x\cos x) - \frac{2D+3}{(D^2 + 3D + 2)^2}.(x\cos x)$

$= I_1 - I_2$

$I_1 = \frac{x}{D^2 + 3D + 2}(x\cos x)$

Applying eq. (ii), we have

$I_1 = x\left[x\frac{1}{D^2 + 3D + 2}\cos x - \frac{2D+3}{(D^2 + 3D + 2)^2}\cos x\right]$

Replacing D^2 by -1^2, we have

$I_1 = x\left[\frac{x}{-1^2 + 3D + 2}\cos x - \frac{2D+3}{(-1^2 + 3D + 2)^2}\cos x\right]$

$= x\left[\frac{x}{3D+1}\cos x - (2D+3)\frac{1}{(3D+1)^2}.\cos x\right]$

$= x\left[\frac{x}{10}(3\sin x + \cos x) - \frac{1}{100}(34\sin x - 12\cos x)\right]$

$I_2 = \frac{2D+3}{(D^2 + 3D + 2)^2}.x\cos x$

Applying eq. (ii), we have

$I_2 = (2D+3)$

$\left[x\frac{1}{(D^2 + 3D + 2)^2}\cos x - \frac{2(D^2 + 3D + 2)(2D+3)}{(D^2 + 3D + 2)^4}\right]\cos x$

Replacing D^2 by -1^2, we have

$I_2 = (2D+3)\left[x.\frac{1}{(-1^2 + 3D + 2)}.\cos x - \frac{2(2D+3).\cos x}{(-1^2 + 3D + 2)^3}\right]$

$= (2D+3)\left[x.\frac{1}{(3D+1)^2}.\cos x - \frac{2(2D+3).\cos x}{(3D+1)^3}\right]$

$= (2D+3)\left[x.\frac{(6\sin x - 8\cos x)}{100} + 2(42\sin x + 54\cos x)\right]$

$= \frac{1}{100}[3612\sin x + 49184\cos x + 34x\sin x - 12x\cos x]$

$y_p = I_1 - I_2$

$= x\left[\frac{x}{10}(3\sin x + \cos x) - \frac{1}{100}(34\sin x - 12\cos x)\right]$

$-\frac{1}{400}[3612\sin x + 49184\cos x + 34x\sin x - 12x\cos x]$

$y = y_c + y_p$

$= C_1 e^{-x} + C_2 e^{-2x} +$

$x\left[\frac{x}{10}(3\sin x + \cos x) - \frac{1}{100}(34x\sin x - 12\cos x)\right]$

$-\frac{1}{100}[3612\sin x + 49184\cos x + 34x\sin x - 12x\cos x]$

Q.32 *Solve* $(D^2 + 4)y = 4\tan 2x$, *by variation of parameters method.* [R.T.U. 2013, 08, Raj. Univ. 2001]

Ans. The given equation is a second order linear differential equation with constant coefficient. Therefore A.E. is

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i.$

C.F $= c_1 \cos 2x + c_2 \sin 2x$

Let $\quad y = A\cos 2x + B\sin 2x$ ...(1)

be the complete solution of the given equation

Choose the values of A and B such that $A_1 u + B_1 v = 0$

$A_1 \cos 2x + B_1 \sin 2x = 0$ ...(2)

$A_1 u_1 + B_1 v_1 = R$

$-2A_1 \sin 2x + 2B_1 \cos 2x = 4\tan 2x$ ...(3)

Solving equation (2) and (3) we get

$A_1 = \frac{dA}{dx} = -2\frac{(1 - \cos^2 2x)}{\cos 2x}$

$B_1 = \frac{dB}{dx} = 2\sin 2x$

Integrating A_1 and B_1 w.r.t x, we have

$A = \sin 2x - \log(\sec 2x + \tan 2x) + c_1$

$B = -\cos 2x + c_2$

Hence complete solution is

$y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

Ans.

Q.33 $(D-1)^2 (D^2 + 1)^2 y = \sin^2 \frac{x}{2} e^x$ [R.T.U. 2012]

Ans. $(D-1)^2 (D^2 + 1)^2 y = \sin^2 \frac{x}{2} e^x$

Auxiliary equation is

$(m-1)^2 (m^2 + 1)^2 = 0$

$\Rightarrow \quad m = 1, 1, \pm i, \pm i$

C.F $= (C_1 + C_2 x)e^x + (C_3 + C_4 x)\sin x + (C_5 + C_6 x)\cos x.$

$$\text{P.I.} = \frac{1}{(D-1)^2(D^2+1)^2}\sin^2\frac{x}{2}e^x$$

$$= e^x\frac{1}{(D+1-1)^2\{(D+1)^2+1\}^2}\left(\frac{1-\cos x}{2}\right)$$

$$= \frac{e^x}{2}\frac{1}{D^2(D^2+2D+2)^2}(1-\cos x)$$

$$= \frac{e^x}{2}\frac{1}{(D^2+2D+2)^2}\left\{\frac{x^2}{2}+\sin x\right\}$$

$$= \frac{e^x}{2}\left[\frac{1}{4}\left(1+\frac{D^2+2D}{2}\right)^{-2}\frac{x^2}{2}+\frac{1}{(D^2+2D+2)^2}\sin x\right]$$

$$= \frac{e^x}{2}\left[\frac{1}{8}\left\{1-2\frac{D^2+2D}{2}+3\left(\frac{D^2+2D}{2}\right)^2+\ldots\right\}x^2+\frac{1}{(2D+1)^2}\sin x\right]$$

$$= \frac{e^x}{2}\left[\frac{1}{8}\left\{1-2D-D^2+3D^2+3D^3+\frac{3}{4}D^4+\ldots\right\}x^2+\frac{1}{4D^2+4D+1}\sin x\right]$$

$$= \frac{e^x}{2}\left[\frac{1}{8}\{x^2-4x-2+6\}+\frac{1}{4D-3}\sin x\right]$$

$$= \frac{e^x}{2}\left[\frac{1}{8}(x^2-4x+4)+\frac{4D+3}{16D^2-9}\sin x\right]$$

$$= \frac{e^x}{2}\left[\frac{(x-2)^2}{8}-\frac{1}{25}\{4\cos x+3\sin x\}\right]$$

$$= \frac{e^x}{2}\left[\frac{(x-2)^2}{8}-\frac{4\cos x+3\sin x}{25}\right]$$

Ans.

Q.34 $(D^2-3D+2)\,y = \sin 3x + x^2 + x + e^{4x}$ **[R.T.U. 2012]**

Ans. Auxiliary equation

$m^2-3m+2=0$

$(m-1)(m-2)=0 \Rightarrow m=1,2$

$\text{C.F.} = C_1e^x + C_2e^{2x}$

Now, $\text{P.I.} = \dfrac{\sin 3x}{D^2-3D+2}+\dfrac{x^2}{D^2-3D+2}+\dfrac{x}{D^2-3D+2}+\dfrac{e^{4x}}{D^2-3D+2}$

$$\frac{e^{4x}}{D^2-3D+2} = \frac{e^{4x}}{4^2-3(4)+2} = \frac{e^{4x}}{6}$$

$$\frac{x}{D^2-3D+2} = \frac{1}{2}\frac{x}{\left[1+\frac{D^2-3D}{2}\right]} = \frac{1}{2}\left[1+\frac{D^2-3D}{2}\right]^{-1}x$$

$$= \frac{1}{2}\left[1-\frac{D^2+3D}{2}\right]x$$

$$= \frac{1}{2}\left(x-\frac{3}{2}\right)$$

$$\frac{x^2}{D^2-3D+2} = \frac{1}{2}\left[\frac{x^2}{1+\frac{D^2-3D}{2}}\right] = \frac{1}{2}\left[1+\frac{D^2-3D}{2}\right]^{-1}x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2-3D}{2}\right)+\frac{9D^2}{4}\right]x^2$$

$$= \frac{1}{2}\left[x^2-1+3x+\frac{9}{2}\right]$$

$$= \frac{1}{2}\left[x^2+3x+\frac{7}{2}\right]$$

$$\frac{\sin 3x}{D^2-3D+2} = \frac{\sin 3x}{-9-3D+2} = \frac{\sin 3x}{-3D-7}$$

$$\frac{3D-7}{-(9D^2-49)}\sin 3x = \frac{3D\sin 3x - 7\sin 3x}{(-130)}$$

$$\frac{9\cos 3x - 7\sin 3x}{98} = \frac{7\sin 3x - 9\cos 3x}{130}$$

$$\text{P.I.} = \frac{e^{4x}}{6}+\frac{1}{2}\left(x-\frac{3}{2}\right)+\frac{1}{2}\left(x^2+3x+\frac{7}{2}\right)+\frac{7\sin 3x}{130}-\frac{9}{130}\cos 3x$$

Complete solution is

$y = \text{C.F.} + \text{P.I.}$

$$= C_1e^x + C_2e^{2x} + \frac{e^{4x}}{6}+\frac{1}{2}\left(x-\frac{3}{2}\right)+\frac{1}{2}\left(x^2+3x+\frac{7}{2}\right)+\frac{7\sin 3x}{130}-\frac{9}{130}\cos 3x$$

Ans.

Q.35 *Solve* $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}=1+x^2$ **[R.T.U. 2011]**

Ans. $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}=1+x^2$

Auxiliary equation is given by

$m^3-m^2-6m=0$

$m(m^2-m-6)=0$

$m=0,-2,3$

$\therefore\ \text{C.F.} = C_1+C_2e^{-2x}+C_3e^{3x}$

$$\text{P.I.} = \frac{1}{(D^3-D^2-6D)}(1+x^2)$$

$$= \frac{1}{(1+D^3-D^2-6D-1)}(1+x^2)$$

$$= \{1+(D^3-D^2-6D-1)\}^{-1}(1+x^2)$$

$$= [1-(D^3-D^2-6D-1)+(D^3-D^2-6D-1)^2+\ldots](1+x^2)$$

$$= 1+x^2+1+x^2+6(2x)+2$$

$$= 4+2x^2+12x$$

$\therefore$ General solution of given differential equation is

$$y = C_1+C_2e^{-2x}+C_3e^{3x}+(2x^2+12x+4)$$

Q.36 *Solve* $\dfrac{d^2y}{dx^2}+a^2y=\sec ax.$

[R.T.U. 2011, 2010, 2009; MREC 2000; Raj. Univ. 1996, 1995]

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OR

Solve the following differential equation

$\dfrac{d^2y}{dx^2}+4y=\sec 2x.$ *[Raj. Univ. 2003]*

Ans. A. E. $\Rightarrow m^2+a^2=0 \Rightarrow m=\pm\, ia$

C. F. $= k_1\cos ax + k_2\sin ax$

$$\text{P. I.} = \frac{1}{D^2+a^2}.\sec ax = \frac{1}{(D+ai)(D-ai)}\sec ax$$

Method I : $\dfrac{1}{2ai}\left[\dfrac{1}{D-ai}-\dfrac{1}{D+ai}\right]\sec ax$

$$= \frac{1}{2ai}\left[e^{iax}\int e^{-iax}\sec ax\,dx - e^{-iax}\int e^{iax}\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\int(\cos ax - i\sin ax)\sec ax\,dx - e^{-iax}\int(\cos ax + i\sin ax)\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\left(x+\frac{i\log\cos ax}{a}\right)-e^{-iax}\left(x-\frac{i\log\cos ax}{a}\right)\right]$$

$$= \frac{1}{2ai}x(e^{iax}-e^{-iax})+\frac{1}{a}\frac{\log\cos ax}{2a}[e^{iax}+e^{-iax}]$$

$$= x\left[\frac{e^{iax}-e^{-iax}}{2ai}\right]+\frac{1}{a^2}\log\cos ax\left[\frac{e^{iax}+e^{-iax}}{2}\right]$$

$$= \frac{1}{a}x.\sin ax + \frac{1}{a^2}\log\cos ax.\cos ax$$

Solution is

$$y = k_1\cos ax + k_2\sin ax + \frac{x}{a}.\sin ax + \frac{1}{a^2}\log\cos ax.\cos ax$$

Ans.

Method II : Auxiliary equation is $m^2+a^2=0$

$\Rightarrow\ m=\pm ai$

$\therefore$ C.F. $= c_1\cos ax + c_2\sin ax$

$$\text{P.I.} = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]\sec ax$$

Let, $\dfrac{1}{D-ai}\sec ax = v_1$.

$\dfrac{dv_1}{dx}-aiv_1=\sec ax$

Which is linear differential equation whose integ factor is

$e^{\int -ai\,dx}=e^{-aix}$

$\therefore$ Solution is $v_1.e^{-aix}=\int e^{-aix}\sec ax\,dx$

$$v_1 = e^{aix}\int e^{-aix}\sec ax\,dx$$

$$= e^{aix}\int(\cos ax - i\sin ax)\sec ax\,dx$$

$$= e^{aix}\int(1-i\tan ax)\,dx$$

$$= e^{aix}\left[x-\frac{i}{a}\log\sec ax\right]$$

Similarly if, $\dfrac{1}{D+ai}\sec ax = v_2$

$$v_2 = e^{-aix}\left[x+\frac{i}{a}\log\sec ax\right]$$

$$\therefore\ \text{P.I.} = \frac{1}{2ai}\left[e^{aix}\left(x-\frac{i}{a}\log\sec ax\right)-e^{-iax}\left(x+\frac{i}{a}\log\right.\right.$$

$$= \frac{1}{2ai}\left[x(e^{iax}-e^{-iax})+\frac{\log\sec xa}{a}i(-e^{-iax}-e^{iax}\right.$$

$$= \frac{1}{2ai}\left[x(2i\sin ax)+\frac{\log\sec xa}{a}\cdot i(-2\cos ax)\right]$$

$$= \frac{1}{a}\left[x\sin ax - \frac{1}{a}\cos ax\log\sec ax\right]$$

Hence general solution is

$$y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax\log$$

Q.37 *Solve* $(D^3-3D^2+3D-1)y = xe^x+e^x$

Ans. Auxiliary equation is $m^3-3m^2+3m-1=0$

$\Rightarrow\ (m-1)^3=0$

$\Rightarrow\ m=1,1,1$

$$C.F. = (c_1+c_2x+c_3x^2)e^x$$

$$P.I. = \frac{1}{(D-1)^3}\left[xe^x+e^x\right] = \frac{1}{(D-1)^3}e^x(x+1)$$

$$= e^x\frac{1}{(D+1-1)^3}(x+1) = e^x\frac{1}{D^3}(x+1)$$

$$= e^x\frac{1}{D^2}\left(\frac{x^2}{2}+x\right) = e^x\frac{1}{D}\left(\frac{x^3}{6}+\frac{x^2}{2}\right)$$

$$= e^x\left(\frac{x^4}{24}+\frac{x^3}{6}\right)$$

General solution is

$$y = C.F. + P.I.$$

$$y=(c_1+c_2x+c_3x^2)e^x+e^x\left(\frac{x^4}{24}+\frac{x^3}{6}\right) \text{ Ans.}$$

Q.38 Solve $x^2\dfrac{d^2y}{dx^2}-(x^2+2x)\dfrac{dy}{dx}+(x+2)y=x^3e^x$

[R.T.U. 2011]

Ans. $x^2\dfrac{d^2y}{dx^2}-(x^2+2x)\dfrac{dy}{dx}+(x+2)=x^3e^x$

$$\frac{d^2y}{dx^2}-\left(1+\frac{2}{x}\right)\frac{dy}{dx}+\left(\frac{1}{x}+\frac{2}{x^2}\right)y=xe^x \quad ...(1)$$

Hence $P=-\left(1+\dfrac{2}{x}\right)$

$$Q=\left(\frac{1}{x}+\frac{2}{x^2}\right)$$

$$R=xe^x$$

Here $P+Qx=0$

Hence $y=x$ is a part of C.F.

So let $y=vx$ is the complete solution.

$$\frac{dy}{dx}=x\frac{dv}{dx}+v$$

$$\frac{d^2y}{dx^2}=x\frac{d^2v}{dx^2}+2\frac{dv}{dx}$$

Putting the values in (1), given equation reduces to

$$\left(x\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right)-\left(1+\frac{2}{x}\right)\left(x\frac{dv}{dx}+v\right)+\left(\frac{1}{x}+\frac{2}{x^2}\right)vx = xe^x$$

$$x\frac{d^2v}{dx^2}+\frac{2dv}{dx}-x\frac{dv}{dx}-v-\frac{2dv}{dx}-\frac{2v}{x}+v+\frac{2v}{x}=xe^x$$

$$\frac{xd^2v}{dx^2}-\frac{xdv}{dx}=xe^x$$

Putting $\dfrac{dv}{dx}=z$

$$\frac{d^2v}{dx^2}=\frac{dz}{dx}$$

The above equation reduces to

$$\frac{dz}{dx}-z=e^x$$

$$I.F=e^{\int -dx}=e^{-x}$$

$$ze^{-x}=\int e^{-x}.e^x+C_1$$

$$z.e^{-x}=x+C_1$$

$$\frac{dv}{dx}.e^{-x}=x+C_1$$

$$\frac{dv}{dx}=(x+C_1).e^x$$

$$\int dv=\int (x+C_1)e^x dx$$

$$v=xe^x+C_1e^x-e^x+C_2$$

$$\frac{y}{x}=xe^x+C_1e^x-e^x+C_2$$

$$\boxed{y=xe^x(x+C_1-1+C_2e^{-x})} \quad \text{Ans.}$$

Q.39 *Use method of variation of parameters to solve*

$$(1+x)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y(1-x)^2$$

[R.T.U. 2011, Raj. Univ. 2003, 2000]

Ans. For the C.F. we consider the equation

$$\frac{d^2y}{dx^2}+\frac{x}{1-x}\frac{dy}{dx}-\frac{1}{1-x}y=0 \quad\text{or}\quad \frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$$

Here $\quad P=\dfrac{x}{1-x}, Q=-\dfrac{1}{1-x},\quad$ so $\quad P+Qx=0$

Hence $\quad u=x$ is a part of C.F. Also $1+P+Q=0$

So, other part v of C.F. is $v=e^x$

let $\quad y=Ax+Be^x$ be a solution

Then we let $A_1u+B_1v=0$

$A_1u_1+B_1v_1-R_1=0$ where $R_1=1-x, u_1=x, v=e^x$

So, $\quad A_1+B_1e^x-(1-x)=0$

$$A_1x+B_1e^x-0=0$$

So, $\dfrac{A_1}{e^x(1-x)}=\dfrac{B_1}{x(1-x)}=\dfrac{1}{e^x(1-x)}$

$$A_1=+1, B_1=xe^{-x}$$

$\therefore \quad A=\int +1dx=x, B=-xe^{-x}+e^{-x}$

Hence the complete solution is

$$y=C_1x+C_2e^x+x(x)+e^x\left(xe^{-x}+e^{-x}\right)$$

or $\quad y=C_1x+C_2e^x+x^2+x+1$

is the required solution.

Q.40 *Solve*

$$\frac{d^2y}{dx^2}+(\tan x-3\cos x)\frac{dy}{dx}+2y\cos^2 x=\cos^4 x$$

[R.T.U. 2011, 2010, Raj. Univ. 2003, MREC Auto 2001]

Ans. Changing the independent variables from x to z, the equation becomes

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$$

where, $Q_1=\dfrac{2\cos^2 x}{\left(\dfrac{dz}{dx}\right)^2}$.

(Let us take $Q_1=2$)

So, $\quad \dfrac{dz}{dx}=\cos x$ or $z=\sin x$

Now $\quad P_1=\dfrac{1}{\left(\dfrac{dz}{dx}\right)^2}\left[\dfrac{d^2z}{dx^2}+P\dfrac{dz}{dx}\right]$

$$=\frac{\left[-\sin x+(\tan x-3\cos x).\cos x\right]}{\cos^2 x}=-3$$

$$R_1=\frac{\cos^4 x}{\cos^2 x}=\cos^2 x$$

$\therefore$ The reduced equation will be

$$\frac{d^2y}{dz^2}-3\frac{dy}{dz}+2y=\cos^2 z=(1-z^2)$$

A.E. is $m^2-3m+2=0 \Rightarrow m=1, 2$

$\therefore \quad C.F.=c_1e^z+c_2e^{2z}$

$$P.I.=\frac{1}{2-3D+D^2}(1-z^2)$$

$$=\frac{1}{2}\left\{1-\frac{3D}{2}+\frac{D^2}{2}\right\}^{-1}(1-z^2)$$

$$=\frac{1}{2}\left[1+\frac{3D}{2}+\left(\frac{3D^2}{2}\right)^2-\frac{D^2}{2}....\right](1-z^2)$$

$$=\frac{1}{2}\left[(1-z^2)-3z-\frac{7}{2}\right]=-2-\frac{3}{2}z-\frac{1}{2}z^2$$

Hence the complete solution is

$$y=\left[c_1e^{\sin x}+c_2e^{2\sin x}+\left(-\frac{5}{4}-\frac{3}{2}\sin x-\frac{1}{2}\sin^2 x\right)\right]$$

Q.41 *Solve the differential equation*

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \text{ in series.}$$

[R.T.U. 2008]

Ans. Here $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0 \quad ...(i)$

Since $x=0$ is an ordinary point of eq. (i).

So, put $m=0$ in $y=\sum_{r=0}^{\infty} a_r x^{m+r}$

So its series solution be $y=\sum_{r=0}^{\infty} a_r x^r \quad ...(ii)$

Then $\dfrac{dy}{dx}=\sum_{r=0}^{\infty} ra_rx^{r-1}$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} r(r-1)a_rx^{r-2}$$

Substituting the values of $y, \dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in eq. (i), we get

$$(1+x^2)\sum_{r=0}^{\infty} r(r-1)a_rx^{r-2}+x\sum_{r=0}^{\infty} ra_rx^{r-1}-\sum_{r=0}^{\infty} a_rx^r=0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty} r(r-1)a_rx^{r-2}+\sum_{r=0}^{\infty} (r^2-1)a_rx^r=0$$

$$\Rightarrow \quad \sum_{r=2}^{\infty} r(r-1)a_rx^{r-2}+\sum_{r=0}^{\infty} (r^2-1)a_rx^r=0$$

$$\Rightarrow \quad (2a_2-a_0)+(3.2a_3)x+(3a_4+3a_2)x^2+...$$

$$\{(n+2)(n+1)a_{n+2}+(n^2-1)a_n\}x^n+...=0$$

Equating to zero the co-efficient of various power of x, we get

$$a_2=\frac{a_0}{2}, a_3=0 \quad ...(iii)$$

In general

$$a_{n+2}=\frac{-(n^2-1)a_n}{(n+1)(n+2)}=\frac{-(n-1)}{n+2}a_n \rightarrow \text{Recurrence relation} \quad ...(iv)$$

Putting n = 2, 3, 4, 5 in eq. (iv) we get

$$a_4=-a_0/8$$

$$a_5=0$$

$a_6 = a_0/16$

$a_7 = 0$

$a_8 = -\frac{5}{8\times16}a_0$

Putting the values of a^r in eq. (ii), we get

$$y = a_0 + a_1x + \frac{a_0}{2}x^2 - \frac{a_0}{8}x^4 + \frac{a_0}{16}x^6 \dots$$

$$y = a_0\left(1+\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16}\dots\right) + a_1 x$$

where a_0 and a_1 are arbitrary constants.

Q.42 *Solve Legendre's equation*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

[R.U. 2006]

Ans. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx}+n(n+1)y=0$ ...(i)

Since $x = 0$ is an ordinary point as $P_0(x) = (1-x^2) \neq 0$ at $x=0$

so, let $y=\sum_{r=0}^{\infty} a_r x^r$ ...(ii)

$$\Rightarrow \frac{dy}{dx} = \sum_{r=0}^{\infty} r a_r x^{r-1} \text{ and } \frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} r(r-1)a_r x^{r-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq. (i), we have

$$(1-x^2)\sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - 2x\sum_{r=0}^{\infty} r a_r x^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} [n(n+1) - 2r - r(r-1)]a_r x^r + \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} = 0$$

equating the coefficient of x^r by putting $r = r+2$ in the second part to get the recurrence relation (step 3)

$$\sum_{r=0}^{\infty} [n(n+1) - r - r^2]a_r x^r - \sum_{r=0}^{\infty} (r+2)(r+1)a_{r+2}x^r = 0$$

$$\Rightarrow a_{r+2} = -\frac{n(n+1)-r^2-r}{(r+1)(r+2)}a_r \rightarrow \text{Recurrence relation} \quad \text{...(iii)}$$

Putting $r = 0, 1, 2, 3, \dots$ in eq. (iii), we have

$$a_2 = \frac{-n(n+1)}{2}a_0,$$

$$a_3 = \frac{-n(n+1)-2}{3\cdot2}a_1 = \frac{-(n-1)(n+2)}{3\cdot2}a_1$$

$$a_4 = \frac{(2-n)(2+n+1)}{3\cdot4}a_2 = \frac{(n-2)n(n+1)(n+3)}{1\cdot2\cdot3\cdot4}a_0$$

substituting these values of $a_2, a_3, a_4, \dots$ in eq. (ii) we have

$$y = a_0 + a_1 x - \frac{n(n+1)}{2}a_0x^2 - \frac{(n-1)(n+2)}{3\cdot2}a_1x^3 + \frac{(n-2)n(n+1)(n+3)}{4\cdot3\cdot2\cdot1}a_0x^4 + \dots$$

$$a_0\left[1 - \frac{n(n+1)}{2}x^2 + \frac{(n-2)(n+1)n(n+3)}{1\cdot2\cdot3\cdot4}x^4 + \dots\right] + a_1x\left[1 - \frac{(n-1)(n+2)}{2\cdot3}x^2+\dots\right]$$

Q.43 *Solve* $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0$

[R.U. 2005, 2003, 2001]

Ans. Given

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y=0 \quad \text{...(i)}$$

Here $P_0(x)=0$, let $y=\sum_{r=0}^\infty a_r x^{m+r}$ be the solution of the given differential equation.

$$\therefore \frac{dy}{dx} = \sum_{r=0}^{\infty} (m+r)a_r x^{m+r-1}, \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (m+r)(m+r-1)a_r x^{m+r-2}$$

Now putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$x^2\sum_{r=0}^{\infty} (m+r)(m+r-1)a_r x^{m+r-2} + x\sum_{r=0}^{\infty} (m+r)a_r x^{m+r-1} + (x^2-1)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} [(m+r)(m+r-1) + (m+r) - 1]a_r x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

Equating the coefficient of lowest power of x to zero by putting $r = 0$, to get the indicial equation, we have

$$(m)(m-1) + (m-1) = 0 \Rightarrow (m-1)(m+1)=0 \Rightarrow m = -1, 1$$

Again equating the coefficient of x^{m+r+2} to zero by putting $r = r+2$ in first expression, we have

$$\sum_{r=0}^{\infty}[(m+r+2)(m+r+1)+(m+r+1)]a_{r+2} + \sum_{r=0}^{\infty} a_r = 0$$

$$\Rightarrow a_{r+2} = \frac{-a_r}{(m+r+1)(m+r+3)} \rightarrow \text{Recurrence relation}$$

Putting $r = 0, 1, 2, 3, \dots\dots$ we have

$$a_2 = \frac{-a_0}{(m+1)(m+3)}, \quad a_3 = \frac{-a_1}{(m+2)(m+4)}$$

$$a_4 = \frac{a_0}{(m+1)(m+3)(m+3)(m+5)},$$

$$a_5 = \frac{a_1}{(m+4)(m+6)(m+2)(m+4)}$$

$\therefore$ The general solution is

$$y = x^m\left[a_0 - \frac{a_0x^2}{(m+1)(m+3)} + \frac{a_0}{(m+1)(m+3)(m+3)(m+5)}x^4 - \dots\right] \quad \text{...(ii)}$$

$\because$ $a_1 = 0$, so $a_3 = a_5 = 0$

Putting $m = 1$, in the general solution, we have first solution as

$$y_I = xa_0\left[1 - \frac{x^2}{2\cdot4} + \frac{x^4}{2\cdot4\cdot4\cdot6}\dots\dots\right]$$

For $m = -1$, the solution becomes infinite, therefore to get the second solution, we multiply both sides of eq. (ii) by $(m+1)$ and differentiate w.r.t. m partially.

$$\frac{\partial}{\partial m}(m+1)y = a_0x^m\left[1 + \frac{x^2}{(m+3)^2} - \dots\dots\right] + a_0x^m\log x\left[1 - \frac{x^2}{(m+3)} + \dots\dots\right]$$

Putting $m = -1$, we have second solution.

$$y_{II} = \frac{a_0}{x}\log x\left[-\frac{1}{2}x^2 + \frac{1}{2^2}x^4 - \dots\dots\right] + \frac{a_0}{x}\left[1 + \frac{x^2}{2^2} - \frac{5}{2^2\cdot4^2}x^4 + \dots\dots\right]$$

$\therefore$ The complete solution is

$$y = A(y)_{at\ m=1} + B\left\{\frac{\partial}{\partial m}(m+1)y\right\}_{at\ m=-1}$$

$$\Rightarrow y = Ay_I + By_{II}$$

Ans.

Q.44 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x x^2$.

Ans. The given differential equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x x^2$

Therefore, auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, \left[\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$$

$\therefore$ C.F. $= (C_1 + C_2x)e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2}e^x x^2 = \frac{e^x}{(D+1-1)^2}x^2 = \frac{e^x}{D^2}x^2 = \frac{e^x}{D}\frac{x^3}{3} = \frac{x^4}{12}e^x$$

$\because$ $y = \text{C.F.} + \text{P.I.} = (C_1 + C_2x)e^x + \frac{x^4}{12}e^x$

which is the required general solution of the differential equation.

Q.45 *What is the method of multipliers to simultaneous differential equation?*

Ans. Method of multipliers :

For a given system of equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

we will choose multipliers a, b, c which necessarily constants such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{adx+bdy+cdz}{aP+bQ+cR}$$

such that $aP + bQ + cR = 0$

and $adx + bdy + cdz = 0$ and can be solve

$f_1(x, y, z) = C$

Let we can choose another set of multiplie

Such that $a_1P + b_1Q + c_1R = 0$

Therefore, $a_1dx + b_1dy + c_1dz = 0$

which on solving give us

$f_2(x, y, z) = C_2$

Set of function f_1 and f_2 together form the solution.

Q.46 *Solve*

$$x^2D^2 - (2m-1)xD + (m^2+n^2)y = n^2x$$

Ans. The given differential equation is

$$x^2D^2-(2m-1)xD+(m^2+n^2))y=n^2x^m\log x$$

Since this equation is homogeneous, therefore, on substitution $x=e^z \Rightarrow z=\log x$, the given equation reduces to

$$\left[D(D-1)-2(m-1)D+(m^2+n^2)\right]y=n^2ze^{mz}$$

$$(D^2-2mD+m^2+n^2)=n^2ze^{mz}$$

$\therefore$ C.F. $=e^{mz}(C_1\cos nz+C_2\sin nz)$

and

$$\text{P.I.}=\frac{1}{(D-m)^2+n^2}n^2ze^{mz}=n^2e^{mz}\frac{1}{(D+m-m)^2+n^2}\cdot z$$

$$=n^2e^{mz}\cdot\frac{1}{D^2+n^2}\cdot z=\frac{n^2e^{mz}}{n^2}\left[1+\frac{D^2}{n^2}\right]^{-1}$$

$$=e^{mz}\left(1-\frac{D^2}{n^2}\right)z=ze^{mz}$$

$y=$ C.F. + P.I.

$$y=x^m\left[C_1\cos(n\log x)+C_2\sin(n\log x)\right]+x^m\log x$$

which is the required solution of the given differential equation. **Ans.**

Q.47 Solve $\dfrac{d^2y}{dx^2}-\dfrac{1}{x}\dfrac{dy}{dx}+4x^2y=x^4$

Ans. Here $P=-\dfrac{1}{x}, Q=4x^2, R=x^4$

In transformed equation choose z in such a way that $Q_1=1$

$$\therefore\quad 1=\frac{4x^2}{\left(\frac{dz}{dx}\right)^2}\Rightarrow\frac{dz}{dx}=2x \text{ and } z=x^2,\frac{d^2z}{dx^2}=2$$

The reduced equation is

$$\frac{d^2y}{dz^2}+\frac{2+\left(-\frac{1}{x}\right)}{\left(\frac{dz}{dx}\right)^2}\cdot\frac{dy}{dz}+y=\frac{x^4}{(2x)^2}$$

or $$\frac{d^2y}{dz^2}+y=\frac{x^2}{4}=\frac{z}{4}$$

A.E. is $m^2+1=0\Rightarrow m=\pm i$, C.F. $=C_1\cos(z+C_2)$

and $$\text{P.I.}=\frac{1}{D^2+1}\left(\frac{z}{4}\right)=\left\{(1+D^2)^{-1}\right\}\frac{z}{4}=\frac{z}{4}$$

Hence the complete solution is given by

$$y=C_1\cos(x^2+C_2)+\frac{x^2}{4}$$

Ans.

Q.48 *Find the C.F. for the following equation*

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\log x\sin(\log x)+1}{x}$$

Ans. The given differential equation is

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\log x\sin(\log x)+1}{x}$$

The equation is in homogeneous form

$x=e^z \Rightarrow z=\log x$

Therefore, on substitution the equation will reduce to

$$[D(D-1)-3D+1]y=\frac{z\sin z+1}{e^z}=e^{-z}[1+z\sin z]$$

$$\Rightarrow\quad [D^2-4D+1]y=e^{-z}[1+z\sin z]$$

$$\therefore\quad \text{C.F.}=e^{2z}\left[C_1\cosh(\sqrt{3}z)+C_2\sinh(\sqrt{3}z)\right]$$

Q.49 *Find the C.F. for the following equation*

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}$$

Ans. The given differential equation is

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}$$

Therefore, on substitution $x=e^z$ or $z=\log x$, we have

$$x\frac{dy}{dx}=Dy,\quad x^2\frac{d^2y}{dx^2}=D(D-1)y$$

So, the given differential equation will reduce to

$$[D(D-1)+3D+1]y=\frac{1}{(1-e^z)^2}$$

$$\Rightarrow\quad (D^2+2D+1)y=\frac{1}{(1-e^z)^2}$$

$$\text{C.F.}=(C_1+C_2z)e^{-z}=(C_1+C_2\log x)\frac{1}{x}$$

PART C

Q.50 *Solve by the method of variation of parameters:*

$$x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$$

[R.T.U. 2019]

OR

Solve :

$$x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$$

[R.T.U. 2014, 09; Raj. Univ. 2004]

OR

Solve the following differential equation :

$$x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$$

[R.T.U. 2018]

Ans. Given equation can be written as

$$\frac{d^2y}{dx^2}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(1+x)}{x^2}y=x \quad ...(1)$$

Here $P+Qx=0$

$\therefore\quad y=x$ is a part of C.F.

Put $\quad y=vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Putting these values in equation (1)

$$x\frac{d^2v}{dx^2}+2\frac{dv}{dx}-\frac{2(1+x)}{x}\left[v+x\frac{dv}{dx}\right]+\frac{2(1+x)}{x^2}vx=0$$

$$x\frac{d^2v}{dx^2}+2\frac{dv}{dx}-\frac{2v}{x}-2v-2\frac{dv}{dx}-2x\frac{dv}{dx}+\frac{2v}{x}+2v=0$$

$$\frac{d^2v}{dx^2}-2\frac{dv}{dx}=0$$

Auxiliary equation is $m^2-2m=0 \Rightarrow m=0, m=2$

C.F. $=c_1e^{0x}+c_2e^{2x}=c_1+c_2e^{2x}$

$y=vx=c_1x+c_2xe^{2x}$

Now suppose that the solution of equation (1) is $y=Ax+Bxe^{2x}$, where A and B are function of x and they are choosen in such a way that

$$x\cdot\frac{dA}{dx}+xe^{2x}\frac{dB}{dx}=0 \quad ...(2)$$

$$y=Ax+Bxe^{2x}$$

$$\frac{dy}{dx}=A+Be^{2x}+2Bxe^{2x}+x\frac{dA}{dx}+xe^{2x}\frac{dB}{dx}$$

$$=A+Be^{2x}+2Bxe^{2x} \text{ (By equation 2)}$$

$$\frac{d^2y}{dx^2}=\frac{dA}{dx}+2Be^{2x}+e^{2x}\frac{dB}{dx}$$
$$+2Be^{2x}+4Bxe^{2x}+2xe^{2x}\frac{dB}{dx}$$

Putting these value in eq.(1)

$$\frac{dA}{dx}+2Be^{2x}+e^{2x}\frac{dB}{dx}+2Be^{2x}+4Bxe^{2x}$$
$$+2xe^{2x}\frac{dB}{dx}-\frac{2(1+x)}{x}(A+Be^{2x}+2Bxe^{2x})$$
$$+2\frac{(1+x)}{x^2}(Ax+Bxe^{2x})=x$$

$$\Rightarrow\quad \frac{dA}{dx}+2Be^{2x}+\frac{dB}{dx}e^{2x}+2Be^{2x}$$
$$+4Bxe^{2x}+2xe^{2x}\frac{dB}{dx}-\frac{2A}{x}-\frac{2B}{x}e^{2x}$$
$$-4Be^{2x}-2A-2Be^{2x}-4Bxe^{2x}$$
$$+\frac{2A}{x}+\frac{2Be^{2x}}{x}+2A+2Be^{2x}=x$$

$$\Rightarrow\quad \frac{dA}{dx}+\frac{dB}{dx}(1+2x)e^{2x}-x=0 \quad ...(3)$$

$$x\frac{dA}{dx}+\frac{dB}{dx}xe^{2x}=0 \quad ...(4)$$

Solving eq. (3) and (4)

$$\frac{\frac{dA}{dx}}{x^2e^{2x}}=\frac{\frac{dB}{dx}}{-x^2}=\frac{1}{xe^{2x}-x(1+2x)e^{2x}}$$

$$\Rightarrow\quad \frac{dA}{dx}=\frac{x^2e^{2x}}{-2x^2e^{2x}}=-\frac{1}{2} \text{ and}$$

$$\frac{dB}{dx}=\frac{-x^2}{-2x^2e^{2x}}=-\frac{1}{2}e^{-2x}$$

$$dA=-\frac{1}{2}dx \qquad B=-\frac{1}{2}\frac{e^{-2x}}{-2}+c_2$$

$$A=-\frac{x}{2}+c_1 \qquad B=\frac{e^{-2x}}{4}+c_2$$

Thus the complete solution is

$$y=\left(-\frac{1}{2}x+c_1\right)x+\left(\frac{e^{-2x}}{4}+c_2\right)xe^{2x}$$ **Ans.**

Q.51 Solve in series :

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$$ **[R.T.U. 2019]**

Ans. Let the solution of given differential equation be,

$$y=\sum_{r=0}^{\infty}A_r x^{m+r}\ ;\ A_0\neq 0 \quad \dots(1)$$

Putting $y=x^m$ in the L.H.S. of the given differential equation, we have,

$$x.m(m-1)x^{m-2}+mx^{m-1}+x.x^m=0$$

$$s=(m+1)-(m-1)=2$$

Putting the value of S = 2 in (1), we get

$$y=\sum_{r=0}^{\infty}A_r x^{m+2r}\ ;\ A_0\neq 0 \quad \dots(2)$$

Finding $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then substituting in the given differential equation, we have

$$\sum_{r=0}^{\infty}A_r(m+2r)^2x^{m+2r-1}+\sum_{r=0}^{\infty}A_r x^{m+2r+1}=0$$

or $$A_0m^2x^{m-1}+\sum_{r=1}^{\infty}A_r(m+2r)^2x^{m+2r-1}+\sum_{r=1}^{\infty}A_{r-1}x^{m+2r-1}=0$$

or $$A_0m^2x^{m-1}+\sum_{r=1}^{\infty}\left[A_r(m+2r)^2+A_{r-1}\right]x^{m+2r-1}=0 \quad \dots(3)$$

The indicial equation is

$A_0m^2=0$

$m=0,0 \quad \because A_0\neq 0$

Recurrence relation is,

$$\frac{A_r}{A_{r-1}}=-\frac{1}{(m+2r)^2} \quad \dots(4)$$

Putting r = 1, 2, 3, ... and we get,

$$\frac{A_1}{A_0}=\frac{-1}{(m+2)^2},\ \frac{A_2}{A_0}=\frac{1}{(m+4)^2(m+2)^2},\ \text{etc.}$$

equation (2) now becomes

$$y=\sum_{r=0}^{\infty}A_r x^{m+2r}$$

$$=A_0x^m\left[1+\frac{A_1}{A_0}x^2+\frac{A_2}{A_0}x^4+\cdots\right]$$

$$y=A_0x^m\left[1-\frac{1}{(m+2)^2}x^2+\frac{1}{(m+2)^2(m+4)^2}x^4+\cdots\right] \quad \dots(5)$$

Which will give only one solution, if we put m = 0 in equation (5).

The first soultion is now obtained by puting m = 0 in equation (5) and is,

$$y=1-\frac{1}{2^2}x^2+\frac{1}{2^24^2}x^4-\dots=y_1\ \text{(say)} \quad \dots(6)$$

Now Differentiating equation (5) partially w.r.t. m, we have

$$\frac{\partial y}{\partial m}=A_0x^m\left[1-\frac{1}{(m+2)^2}x^2+\frac{1}{(m+2)^2(m+4)^2}x^4\cdots\right]\log x$$

$$+A_0x^m\left[\frac{2}{(m+2)^3}x^2-\frac{2}{(m+2)^2(m+4)^2}\left\{\frac{1}{(m+2)}+\frac{1}{(m+4)}\right\}x^4+\cdots\right] \quad \dots(7)$$

Putting m = 0 in equation (7), we get second solution and is given by,

$$\left(\frac{\partial y}{\partial m}\right)_{m=0}=A_0\left[1-\frac{1}{2^2}x^2+\frac{1}{2^24^2}x^4\cdots\right]\log x+$$

$$A_0\left\{\frac{1}{2^2}x^2-\frac{1}{2^24^2}\left(1+\frac{1}{2}\right)x^4+\cdots\right\}$$

$$=y_1\log x+A_0\left(\frac{1}{2^2}x^2-\frac{1}{2^24^2}\cdot\frac{3}{2}x^4+\cdots\right)=y_2\ \text{(say)}$$

The complete solution, therfore is given by

$$y=Ay_1+By_2$$

Q.52 (a) Solve :

$$(D^2+2D+1)y=e^x+x^2-\sin x$$

(b) Solve :

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}-3y=x^2\log x$$ **[R.T.U. 2018]**

Ans.(a) A.E. is $(m^2+2m+1)=0$

$\Rightarrow (m+1)^2=0$

$\Rightarrow M=-1,-1$

So, C.F. $=(C_1+C_2x)e^{-x}$ ...(i)

$$\text{P.I.}=\frac{1}{(D+1)^2}e^x+\frac{1}{(D+1)^2}x^2-\frac{\sin x}{(D+1)^2}$$

$$=\frac{1}{(D^2+2D+1)}e^x+(D+1)^{-2}x^2$$

$$-\frac{1}{(D^2+2D+1)}\sin x$$

$$=\frac{1}{1+2+1}e^x+(1-2D+3D^2-4D^3.....)x^2$$

$$-\left(\frac{1}{-1^2+2D+1}\right)\sin x$$

$$=\frac{1}{4}e^x+(x^2-4x+6)-\frac{1}{2D}\sin x$$

$$=\frac{1}{4}e^x+x^2-4x+6-\frac{1}{2}\int\sin x\,dx \quad \left\{\because \frac{1}{D}=\int\right\}$$

$$\text{P.I.}=\frac{1}{4}e^x+x^2-4x+6+\frac{1}{2}\cos x \quad \dots(ii)$$

∴ The complete solution is y = C.F. + P.I.

$$y=(C_1+C_2x)e^{-x}+\frac{1}{4}e^x+x^2-4x+6+\frac{1}{2}\cos x$$

Ans.(b) Putting $x=e^z\Rightarrow z=\log x$, then the given differential equation transformed into

$\Rightarrow \{D(D-1)-D-3\}y=e^{2z}.z$

$\Rightarrow (D^2-2D-3)y=ze^{2z}$

Now, the auxiliary equation (A.E.) is

$m^2-2m-3=0$

$\Rightarrow (m+1)(m-3)=0$

$\Rightarrow m=-1,3$

$\therefore$ C.F. $=C_1e^{-z}+C_2e^{3z}=C_1x^{-1}+C_2x^3$

$=\frac{C_1}{x}+C_2x^3$

$$\text{P.I.}=\frac{1}{(D-3)(D+1)}e^{2z}.z$$

$$=e^{2z}\frac{1}{(D+2-3)(D+2+1)}z$$

$$=e^{2z}\frac{1}{(D-1)(D+3)}z$$

$$=e^{2z}\frac{1}{4}\left[\frac{1}{D-1}-\frac{1}{D+3}\right]z$$

$$=\frac{-1}{4}e^{2z}(1-D)_z^{-1}-\frac{1}{12}e^{2z}\left(1+\frac{D}{3}\right)_z^{-1}$$

$$=\frac{-1}{4}e^{2z}(1+D+D^2....)z-\frac{1}{12}e^{2z}\left(1-\frac{D}{3}+\cdots\right)$$

$$=-\frac{1}{4}e^{2z}(z+1)-\frac{1}{12}e^{2z}\left(z-\frac{1}{3}\right)$$

$$=-\frac{1}{3}ze^{2z}-\frac{2}{9}e^{2z}$$

$$=-\frac{1}{3}x^2\log x-\frac{2}{9}x^2$$

Thus, the general solution is

y = C.F. + P.I.

$$y=\frac{c_1}{x}+c_2x^3-\frac{1}{3}x^2\log x-\frac{2}{9}x^2$$

Q.53 Solve: $\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}-2y\cos^3x=2\cos^5x$

[R.T.U. 2015, 2013, Raj. Univ. 2005]

OR

Solve: $\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}-(2\cos^3x)y=2\cos^5x$

[R.T.U. 2011, Raj. Univ. 2003]

Ans. Dividing by cos x, given equation in standard form

$$y''+\tan x\,y'-(2\cos^2x)y=2\cos^4x$$

comparing (1) with $y''+Py'+Qy=R$, we have

$P=\tan x$, $Q=-2\cos^2x$ and $R=2\cos^4x$

Choose z such that $\left(\frac{dz}{dx}\right)^2=2\cos^2x$ or

$$\left(\frac{dz}{dx}\right)=\sqrt{2}\cos x$$

∴ $dz=\sqrt{2}\cos x\,dx$ so that $z=\sqrt{2}\sin x$ ...(3)

with this z eq.(1) transform to

$$\left(\frac{d^2y}{dz^2}\right)+P_1\left(\frac{dy}{dz}\right)+Q_1y=R_1 \qquad ...(4)$$

where

$$P_1=\frac{\left(\frac{d^2z}{dx^2}\right)+P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}=\frac{-\sqrt{2}\sin x+\tan x.\sqrt{2}\cos x}{2\cos^2 x}=0$$

$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=-1$ and

$$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{2\cos^4 x}{2\cos^2 x}$$

$=\cos^2 x=1-\sin^2 x=1-\frac{z^2}{2}$ by, (2) and (3)

∴ Eq.(4) gives $(D_1^2-1)y=1-\frac{z^2}{2}$ ...(5)

where $D_1=d/dz$

Auxiliary equation of (5) is $D_1^2-1=0$ giving $D_1=\pm1$

∴ C.F of $(5)=c_1e^z+c_2e^{-z}$. where c_1 and c_2 are arbitrary constants.

and P.I $=\frac{1}{D_1^2-1}\left[1-\frac{1}{2}z^2\right]$

$$=\frac{1}{D_1^2-1}e^{0z}+\frac{1}{2}\frac{1}{(1-D_1^2)}z^2$$

$$=\frac{1}{0^2-1}e^{0z}+\frac{1}{2}(1-D_1^2)^{-1}z^2$$

$$=-1+\left(\frac{1}{2}\right)\times(1+D_1^2+...)z^2$$

$$=-1+\left(\frac{1}{2}\right)\times(z^2+2)=\frac{z^2}{2}$$

Hence the required solution is $y=C.F+P.I...$, i.e..

$$y=c_1e^z+c_2e^{-z}+\frac{z^2}{2}$$

or $y=c_1e^{\sqrt{2}\sin x}+c_2e^{-\sqrt{2}\sin x}+\sin^2 x$, as $z=\sqrt{2}\sin x$.

Q.54 *Solve the following differential equation :*

$$(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x$$

[Raj. Univ. 2004, 2000]

OR

Solve by the method of variation of parameters:

$$(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x$$

[R.T.U. 2015]

Ans. Given equation can be written as-

$$\frac{d^2y}{dx^2}-\frac{2x+5}{x+2}\cdot\frac{dy}{dx}+\frac{2y}{x+2}=\frac{(x+1)e^x}{(x+2)} \qquad ...(1)$$

Let us compare with

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad ...(2)$$

So $P=\frac{-2x+5}{x+2}, Q=\frac{2}{x+2}$ and $R=\frac{x+1}{x+2}e^x$

To get C.F. of (1) let us re-write the equation (1) as-

$$\frac{d^2y}{dx^2}-\frac{2x+5}{x+2}\cdot\frac{dy}{dx}+\frac{2y}{x+2}=0 \qquad ...(3)$$

Now the solution of this equation will be the C.F. of equation (1). So to solve this let us compare equation (3) with equation (2), then

$$P=-\frac{2x+5}{x+2}, Q=\frac{2}{x+2} \text{ and } R=0$$

As $4+2p+Q=4-\frac{2(2x+5)}{x+2}+\frac{2x}{x+2}$

$$=\frac{4x+8-4x-10+2}{(x+2)}=0$$

⇒ $y=u=e^{2x}$ is a part of C.F.

Let $y=u_1.v_1$ ...(4) is complete solution of (3)

where v_1 can be obtained from-

$$\frac{d^2v_1}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv_1}{dx}=\frac{R}{u}$$

$$\Rightarrow \frac{d^2v_1}{dx^2}+\left[-\frac{2x+5}{x+2}+\frac{2}{e^{2x}}.2e^{2x}\right]\frac{dv}{dx}=0$$

Let $\frac{dv_1}{dx}=s\Rightarrow\frac{d^2v_1}{dx^2}=\frac{ds}{dx}$

so $\frac{ds}{dx}+\left[4-\frac{2x+5}{x+2}\right]s=0$

$$\Rightarrow \frac{ds}{dx}-\frac{2x+3}{x+2}s=0 \qquad ...(5)$$

If $=e^{-\int\frac{2x+3}{x+2}dx}=e^{-\int\frac{2(x+2)-1}{x+2}dx}=\frac{e^{2x}}{x+2}$

So solution of (5) is given by-

$$s.\frac{e^{2x}}{x+2}=c_1$$

$$\Rightarrow s=\frac{dv_1}{dx}=c_1(x+2).e^{-2x}$$

$$\Rightarrow v_1=c_1\int(x+2)e^{-2x}dx+c_2$$

$$=c_1\left[(x+2)\frac{e^{-2x}}{2}-\frac{e^{-2x}}{(-2)^2}\right]+c_2$$

$$=c_1\left[-\frac{1}{2}(x+2)e^{-2x}-\frac{1}{4}e^{-2x}\right]+c_2$$

$$\Rightarrow v_1=-\frac{c_1}{4}(2x+5)e^{-2x}+c_2$$

So solution of (3) is given by-

$$y=u_1.v_1=e^{2x}\left[-\frac{c_1}{4}(2x+5)e^{-2x}+c_2\right]$$

$$=-\frac{c_1}{4}(2x+5)+c_2e^{2x}$$

or $y=c_1'(2x+5)+c_2e^{2x}$ ...(6)

which is C.F. of (1) and can be written as

or $y=c_1'u+c_2v$ ...(7)

Where $u=2x+5$ and $v=e^{2x}$.

Let $y=Au+Bv$ ...(8)

is complete solution of equation (1), where-

$$A=\int\frac{-Rv}{u.\frac{dv}{dx}-v.\frac{du}{dx}}dx+c_3$$

$$=-\int\frac{\frac{x+1}{x+2}.e^xe^{2x}}{(2x+5).2e^{2x}-e^{2x}.2}dx+c_3$$

$$=-\frac{1}{4}\int\frac{\frac{x+1}{x+2}e^x}{(x+2)}dx+c_3$$

$$=-\frac{1}{4}\int\frac{(x+2)-1}{(x+2)^2}e^xdx+c_3$$

$$=-\frac{1}{4}\int\left[\frac{1}{x+2}-\frac{1}{(x+2)^2}\right]e^xdx+c_3$$

$$A=-\frac{1}{4}\frac{e^x}{x+2}+c_3$$

and $B=\int\frac{Ru}{u.\frac{dv}{dx}-v.\frac{du}{dx}}dx+c_4$

$$\Rightarrow B=\int\frac{\frac{x+1}{x+2}.e^x(2x+5)}{4(x+2)e^{2x}}dx+c_4$$

$$=\frac{1}{4}\int\frac{(x+1)(2x+5)e^{-x}}{(x+2)^2}dx+c_4$$

$$=\frac{1}{4}\int\frac{2x^2+7x+5}{x^2+4x+4}e^{-x}dx+c_4$$

$$=\frac{1}{4}\int\left\{2-\frac{x+3}{(x+2)^2}\right\}e^{-x}dx+c_4$$

$$=-\frac{1}{2}e^{-x}-\frac{1}{4}\int\frac{(x+2)+1}{(x+2)^2}e^{-x}dx+c_4$$

$$=-\frac{1}{2}e^{-x}-\frac{1}{4}\int\left\{\frac{1}{x+2}+\frac{1}{(x+2)^2}\right\}e^{-x}dx+c_4$$

$$=-\frac{1}{2}e^{-x}-\frac{1}{4}\left[\frac{-e^{-x}}{x+2}-\int\frac{1}{(x+2)^2}(-e^{-x})dx+\int\frac{e^{-x}}{(x+2)^2}dx\right]+c_4$$

$$=-\frac{1}{2}e^{-x}+\frac{1}{4}\frac{-e^{-x}}{x+2}-\frac{1}{4}\left[-\int\frac{1}{(x+2)^2}(-e^{-x})dx+\int\frac{e^{-x}}{(x+2)^2}dx\right]+c_4$$

$$B=-\frac{1}{2}e^{-x}+\frac{1}{4}\frac{e^{-x}}{x+2}+c_4$$

So from (8), complete solution of given equation is given by-

$$y=\left[-\frac{1}{4}\frac{e^x}{x+2}+c_3\right](2x+5)+\left[-\frac{1}{2}e^{-x}+\frac{1}{4}\frac{e^{-x}}{x+2}+c_4\right]e^{2x}$$

$$\Rightarrow\quad y=c_3(2x+5)+c_4e^{2x}-\frac{2x+5}{4(x+2)}e^x-\frac{1}{2}e^x+\frac{1}{4}\frac{e^x}{x+2}$$

$$\Rightarrow\quad y=c_3(2x+5)+c_4e^{2x}+\frac{e^x}{4(x+2)}[-2x-5-2(x+2)+1]$$

$$\Rightarrow\quad y=c_3(2x+5)+c_4e^{2x}+\frac{e^x}{4(x+2)}[-4x-8]$$

$$\Rightarrow\quad y=c_3(2x+5)+c_4e^{2x}-e^x.$$

Q.55 *Solve the following differential equation:*

(a) $(D^4+2D^3-3D^2)y=3e^{2x}+4\sin x$

(b) $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+\frac{dy}{dx}=e^{2x}+x^2+x$

[R.T.U. 2015]

Ans. (a) A.E. is $m^4+2m^3-3m^2=0$

$m^2(m^2+2m-3)=0$

$m=0,0$; and

$m^2+2m-3=0$

$m^2+3m-m-3=0$

$m(m+3)-1(m+3)=0$

$(m-1)(m+3)=0$

So $m=1,-3$

Thus $m=0,0,1,-3$

So $C.F.=(c_1+c_2x)e^{0x}+c_3e^x+c_4e^{-3x}$

$=c_1+c_2x+c_3e^x+c_4e^{-3x}$

$$P.I.=\frac{1}{D^4+2D^3-3D^2}(3e^{2x}+4\sin x)$$

$$=3\frac{1}{D^4+2D^3-3D^2}e^{2x}+4\frac{1}{D^4+2D^3-3D^2}\sin x$$

$$=3\frac{1}{16+16-12}e^{2x}+4\frac{1}{1-2D+3}\sin x$$

$$\left\{\begin{array}{l}\text{Using }\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}\\ \frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax\end{array}\right\}$$

$$=\frac{3}{20}e^{2x}+2\frac{1}{2-D}\frac{(2+D)}{(2+D)}\sin x$$

$$=\frac{3}{20}e^{2x}+2\frac{(2+D)}{(4-D^2)}\sin x$$

$$=\frac{3}{20}e^{2x}+\frac{2}{5}(2\sin x+\cos x)$$

Solution is $y=C.F+P.I.$

$$=c_1+c_2x+c_3e^x+c_4e^{-3x}+\frac{3}{20}e^{2x}+\frac{2}{5}(2\sin x+\cos x)$$

Ans.(b) $(D^3+2D^2+D)y=e^{2x}+x^2+x$

A.E. $m^3+2m^2+m=0$

$m(m^2+2m+1)=0$

$m(m+1)^2=0$

$m=0,-1,-1$

$C.F.=c_1+(c_2+c_3x)e^{-x}$

$$P.I.=\frac{1}{D^3+2D^2+D}(e^{2x}+x^2+x)$$

$$=\frac{1}{D^3+2D^2+D}e^{2x}+\frac{1}{D(D^2+2D+1)}(x^2+x)$$

$$=\frac{1}{8+8+2}e^{2x}+\frac{1}{D(D+1)^2}(x^2+x)$$

$$=\frac{1}{18}e^{2x}+\frac{1}{D}(1-2D+3D^2+.....)(x^2+x)$$

$$=\frac{1}{18}e^{2x}+\frac{1}{D}(x^2+x-2(2x+1)+3(2))$$

$$=\frac{1}{18}e^{2x}+\frac{1}{D}(x^2-3x+4)$$

$$\frac{1}{18}e^{2x}+\frac{x^3}{3}-\frac{3x^2}{2}+4x$$

Solution is $y=C.F+P.I.$

$$=c_1+(c_2+c_3x)e^{-x}+\frac{1}{18}e^{2x}+\frac{x^3}{3}-\frac{3x^2}{2}+4x$$

Q.56 *Solve the following differential equations:*

(i) $(D^3-2D^2+4D-8)y=0$

(ii) $(D^2-4D+4)y=e^{2x}+\sin 2x$

(iii) $(D^3-D^2-6D)y=1+x^2$ *[R.T.U. 2014]*

Ans. (i) We have,

$(D^3-2D^2+4D-8)y=0$

⇒ $(D-2)(D^2+4)y=0$

A.E. is $(m-2)(m^2+4)=0$

⇒ $m=2,\pm 2i$

Hence solution is

$y=C_1e^{2x}+e^{0x}(C_2\cos 2x+C_3\sin 2x)$

⇒ $\boxed{y=C_1e^{2x}+C_2\cos 2x+C_3\sin 2x}$ **Ans.**

Ans.(ii) The auxiliary equation corresponding to equation

$m^2-4m+4=0\Rightarrow(m-2)^2=0\Rightarrow m=2,2$

∴ C.F. = $(C_1+C_2x)e^{2x}$

$$P.I.=\frac{1}{(D-2)^2}(e^{2x}+\sin 2x)$$

$$=\frac{1}{(D-2)^2}e^{2x}+\frac{1}{(D-2)^2}\sin 2x$$

$$=e^{2x}\frac{1}{D^2}\cdot 1+\frac{1}{D^2-4D+4}(\sin 2x)$$

$$=e^{2x}\frac{x^2}{2}-\frac{1}{4D}(\sin 2x)=\frac{e^{2x}\cdot x^2}{2}+\frac{1}{8}\cos 2x$$

∴ The complete solution

$$y=(C_1+C_2x)e^{2x}+\frac{x^2e^{2x}}{2}+\frac{\cos 2x}{8}$$ **Ans.**

Ans.(iii) We have,

$(D^3-D^2-6D)y=1+x^2$

The A.E. is

$m^3-m^2-6m=0\Rightarrow m(m^2-m-6)=0$

⇒ $m(m-3)(m+2)=0\Rightarrow m=0,-2,3$

Hence $C.F.=C_1+C_2e^{-2x}+C_3e^{3x}$

$$P.I.=\frac{1}{D^3-D^2-6D}(1+x^2)$$

$$=\frac{1}{D(D^2-D-6)}(1+x^2)=\frac{1}{-6D\left(1+\frac{D-D^2}{6}\right)}(1+x^2)$$

$$=\frac{-1}{6D}\left(1+\frac{D-D^2}{6}\right)^{-1}(1+x^2)$$

$$=\frac{-1}{6D}\left[1-\frac{D-D^2}{6}+\frac{(D-D^2)^2}{36}+...\right](1+x^2)$$

$$=\frac{-1}{6}\left(\frac{1}{D}-\frac{1}{6}+\frac{D}{6}+\frac{D}{36}-\frac{D^2}{18}+...\right)(1+x^2)$$

$$=\frac{-1}{6}\left(x+\frac{x^3}{3}-\frac{1}{6}-\frac{x^2}{6}+\frac{7x}{18}-\frac{1}{9}\right)$$

$$=-\frac{25}{108}x+\frac{x^2}{36}-\frac{x^3}{18}+\text{constant}$$

Thus, the general solution is

y = C.F. + P.I.

$$\Rightarrow\quad \boxed{y=C_1+C_2e^{-2x}+C_3e^{3x}-\frac{25}{108}x+\frac{1}{36}x^2-\frac{1}{18}x^3}$$

Q.57 (a) *Solve* $(x^2D^2-3xD+1)y=\frac{\log x.\sin\log x}{x}$

$D\equiv\frac{\partial}{\partial x}$

(b) *Solve* $\sqrt{x}\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+3y=x$

[R.T.U. 2013, Raj. Univ. 200]

Ans. (a) Put $x=e^t$ then, $xD=D, x^2D^2=D(D-1)$, the Euler-Cauchy DE reduces to

$$(D(D-1)-3D+1)y=\frac{t\sin t+1}{e^t}$$

A E: $m^2-4m+1=0$

$$m=\frac{4\pm\sqrt{16-4}}{2}$$

$m=2\pm\sqrt{3}$ (real roots)

$$CF: y_c = C_1e^{m_1t} + C_2e^{m_2t} \quad \text{...(i)}$$

where, $m_1 = 2+\sqrt{3}, m_2 = 2-\sqrt{3}$

Now P I: $y_p = \dfrac{1}{D^2-4D+1}\left[te^{-t}.\sin t + e^{-t}\right]$

$= y_1 + y_2$ ...(ii)

where, $y_1 = \dfrac{1}{D^2-4D+1}\left(te^{-t}.\sin t\right)$

$y_2 = \dfrac{1}{D^2-4D+1}e^{-t}$

Then $y_2 = \dfrac{1}{D^2-4D+1}e^{-t} = \dfrac{1}{1+4+1}e^{-t} = \dfrac{1}{6x}$ ... (iii)

Use exponential shift, then replace D by D – 1

$y_1 = \dfrac{1}{D^2-4D+1}\left(te^{-t}.\sin t\right)$

$= \dfrac{e^{-t}}{(D-1)^2-4(D-1)+1}(t\sin t)$

$= e^{-t}\dfrac{1}{D^2-6D+6}(t\sin t)$

Use $\dfrac{1}{f(D)}\{xV(x)\} = x\dfrac{1}{f(D)}V(x) - \dfrac{f'(D)}{\{f(D)\}^2}V(x)$

$y_1 = e^{-t}\left[t.\dfrac{1}{D^2-6D+6}\sin t - \dfrac{2D-6}{(D^2-6D+6)^2}\sin t\right]$ ...(iv)

$= I_1 + I_2$

where,

$I_1 = \dfrac{1}{D^2-6D+6}\sin t = \dfrac{1}{-1-6D+6}\sin t$

$= \dfrac{1}{5-6D}\sin t = \dfrac{5+6D}{25-36D^2}\sin t$

$= \dfrac{5+6D}{25-36(-1)}\sin t = \dfrac{1}{61}[5+6D]\sin t$

$I_1 = \dfrac{1}{61}[5\sin t + 6\cos t]$ ...(v)

Here

$I_2 = \dfrac{2D-6}{(D^2-6D+6)^2}\sin t$

$= \dfrac{2(D-3)}{(-1-6D+6)^2}\sin t = \dfrac{2(D-3)}{(5-6D)^2}\sin t$

$= \dfrac{2(D-3)}{25+36D^2-60D}\sin t = \dfrac{2(D-3)}{-11-60D}\sin t$

$= -\dfrac{2(D-3)(11-60D)\sin t}{121-3600D^2}$

$= -\dfrac{2}{3721}\left[60D^2-191D+33\right]\sin t$

$I_2 = -\dfrac{2}{3721}\left[-60\sin t - 191\cos t + 33\sin t\right]$ ...(vi)

Substituting (v) and (vi) in (iv), we get

$y_1 = e^{-t}\left[t.\dfrac{1}{61}(5\sin t + 6\cos t)\right]$

$-e^{-t}\left[\dfrac{2}{3721}\{60\sin t + 191\cos t - 33\sin t\}\right]$

Then P.I : $y_p = y_1 + y_2$ where y_1 is from (vii) and y_2 is from (iii). The G.S. is $y = y_c + y_p$ where y_c is from (i).

Replace t by log x. Thus the G.S. is

$y = C_1e^{(2+\sqrt{3})}\log x + C_2e^{(2-\sqrt{3})}\log x$

$+\dfrac{1}{x}\left[\log x\dfrac{1}{61}(5\sin\log x + 6\cos\log x)\right] - \dfrac{2}{3721}$

$\times\dfrac{1}{x}\{60\sin\log x + 191\cos\log x - 33\sin\log x\} + \dfrac{1}{6x}$

Ans. (b) We see that the given equation is not exact (verify yourself as usual). Let its integrating factor be x^m. Multiplying the given equation by x^m, we get

$x^{m+1/2}(d^2y/dx^2) + 2x^{m+1}(dy/dx) + 3x^my = x^{m+1}$ ...(1)

which must be exact, comparing (1) with $P_0y'' + P_1y' + P_2y = \phi(x)$, here

$P_0 = x^{m+1/2}, P_1 = 2x^{m+1}, P_2 = 3x^m, \phi(x) = x^{m+1}$ ...(2)

Now eq.(1) is exact if $P_2 - P'_1 + P_0'' = 0$

or $3x^m - 2(m+1)x^m + (m+1/2)(m-1/2)x^{m-3/2} = 0$

or $(1-2m)x^m + (1/4)(2m-1)(2m-1)x^{m-3/2} = 0$

or $(1-2m)[x^m - (1/4)(2m+1)x^{m-3/2}] = 0$

or $1-2m = 0$ for all x. Hence $m = 1/2$.

Putting this value of m in (1) we get

$$x\frac{d^2y}{dx^2} + 2x^{3/2}\frac{dy}{dx} + 3x^{1/2}y = x^{3/2}$$

which must be exact. For this equation [using (2)], we have

$P_0 = x, P_1 = 2x^{3/2}, P_2 = 3x^{1/2}, \phi(x) = x^{3/2}$

and hence its integral is

$$P_0\frac{dy}{dx} + (P_1 - P'_0)y = \int\phi(x)dx + c_1$$

or $x\dfrac{dy}{dx} + (2x^{3/2}-1)y = \dfrac{2}{5}x^{5/2} + \dfrac{c_1}{x}$

which is not exact. Dividing by x, we get

$\dfrac{dy}{dx} + \left(2x^{1/2} - \dfrac{1}{x}\right) = \dfrac{2}{5}x^{3/2} + \dfrac{c_1}{x}$, which is linear ...(3)

its I.F. $= e^{\int(2x^{1/2}-1/x)} = e^{(4/3)x^{3/2}-\log x} = (1/x)e^{(4/3)x^{3/2}}$

$\therefore$ The solution of (3) is

$\dfrac{y}{x}e^{(4/3)x^{3/2}} = \dfrac{2}{5}\int e^{(4/3)x^{3/2}}.x^{1/2}dx + c_1\int\dfrac{1}{x^2}e^{(4/3)x^{3/2}}dx + c_2$ ...(4)

Now,

$\int e^{(4/3)x^{3/2}}.x^{1/2}dx = \dfrac{2}{3}\int e^{4t/3}dt = \dfrac{2}{3}.\dfrac{3}{4}e^{4t/3} = \dfrac{1}{3}e^{(4/3)x^{3/2}}$

[Putting $x^{3/2} = t$ so that $x^{1/2}dx = (2/3)dt$]

Putting this in (3), the required solution is

$\dfrac{y}{x}e^{(4/3)x^{3/2}} = \dfrac{1}{5}e^{(4/3)x^{3/2}} + c_1\int\dfrac{1}{x^2}e^{(4/3)x^{3/2}}dx + c_2$ **Ans.**

Q.58 *Solve*

(a) $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^3-3)y = e^{x^2}$,

(b) $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$ *[R.T.U. 2012]*

Ans. (a) Here, $P = -4x, Q = 4x^2-3, R = e^{x^2}$

Let, $y = u.v$ is the complete solution

and $u = e^{-\frac{1}{2}\int Pdx} = e^{-\frac{1}{2}\int -4xdx}$

$= e^{x^2}$

Then, the given equation reduces to

$$\frac{d^2v}{dx^2} + Iv = s$$

where, $I = Q - \dfrac{1}{2}\dfrac{dP}{dx} - \dfrac{P^2}{4}$

$= (4x^2-3) - \dfrac{1}{2}(-4) - \dfrac{16x^2}{4}$

$= 4x^2 - 3 + 2 - 4x^2$

$= -1$

and $s = \dfrac{R}{u} = \dfrac{e^{x^2}}{e^{x^2}} = 1$

$\therefore$ Equation will be

$$\frac{d^2v}{dx^2} - v = 1$$

which is a second order differentiation equation with constant coefficient.

$\therefore$ Auxiliary equation is

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

C.F. $= C_1e^x + C_2e^{-x}$

Now, P.I. $= \dfrac{1}{D^2-1} = -1$

$\therefore$ The solution is

$v = $ C.F. + P.I.

$= C_1e^x + C_2e^{-x} - 1$

Here, the complete solution is

$y = uv = (C_1e^x + C_2e^{-x} - 1)e^{x^2}$ **Ans.**

Ans. (b) Given equation $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$ ...(1)

Here $P_0 = \sin 2x$, $P_1 = 0$, $P_2 = 0$ therefore condition of exactness is

$P_2 - P'_1 + P''_0 = 2\cos 2x \neq 0$

Hence the given equation is not exact therefore, we have to use an integration factor here to change the given equation into exact form. Now multiply both the sides of given equation by 'cotx', we get

$\cot x\sin^2 x\dfrac{d^2y}{dx^2} - 2y\cot x = 0$

$\cot x\dfrac{d^2y}{dx^2} - \dfrac{2y}{\sin^2 x}\cot x = 0$

$\cot x\dfrac{d^2y}{dx^2} - 2\cot x\,\text{cosec}^2 xy = 0$ ...(2)

$\therefore$ For the above equation

$P_0 = \cot x$, $P_1 = 0$, $P_2 = -2\cot x\,\text{cosec}^2 x$

Now again check the condition of exactness,

$P_2 - P'_1 + P''_0 = 0$

which is satisfied here

$\therefore$ The equation is now become exact. So the primitive of the equation (2) is

$\cot x\dfrac{dy}{dx} + (\text{cosec}^2 x)y = C$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{\text{cosec}^2 x}{\cot x}y = C\tan x$ ...(3)

Which is a linear differential equation in y.

$\therefore \quad I.F. = e^{\int \frac{\sec^2 x}{\tan x}dx} = e^{\log \tan x} = \tan x$

Multiply the equation (3) with I.F., we get

$\therefore \quad (y \tan x) = \int C \tan^2 x \, dx + C_1$

$y \tan x = C(\tan x - x) + C_1$ **Ans.**

Q.59 (a) $(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$

(b) *Use the method of variation of parameters to solve.*

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$ *[R.T.U. 2012, Raj. Univ. 2004, 2003]*

Ans. (a) $(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}y = 0$$

Let $\quad Q_1 = 4$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \frac{\frac{4}{(1+x^2)^2}}{\left(\frac{dz}{dx}\right)^2} = 4$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = \frac{1}{(1+x^2)^2}$$

$$\int \frac{dz}{dx} = \int \frac{1}{1+x^2}$$

$$z = \tan^{-1}x$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{-2x}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2}}{\left(\frac{1}{1+x^2}\right)^2} = 0$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

Therefore, given equation reduces to $\frac{d^2y}{dz^2} + 4y = 0$

$(D^2+4)y = 0$

Auxiliary equation $m^2 + 4 = 0$

$m = \pm 2i$

$\therefore$ Solution is $y = C_1 \cos 2z + C_2 \sin 2z$

$y = C_1 \cos 2(\tan^{-1}x) + C_2 \sin 2(\tan^{-1}x)$ **Ans.**

Ans. (b) The given equation is a second order linear differential equation with constant coefficient

- Therefore the C.F. of the given equation is

$$y = C_1e^x + C_2e^{-x}$$

Let us assume

$$y = Ae^x + Be^{-x} \quad ...(1)$$

be the complete solution of the given equation, where A and B are functions of x.

$$A_1u + B_1v \Rightarrow A_1e^x + B_1e^{-x} = 0 \quad ...(2)$$

$$A_1u_1 + B_1v_1 = R \Rightarrow A_1e^x - B_1e^x = \frac{2}{1+e^x} \quad ...(3)$$

where $\quad A_1 = \frac{dA}{dx}, B_1 = \frac{dB}{dx}$

Solving equation (2) and (3), we get

$$A_1 = \frac{dA}{dx} = \frac{e^{-x}}{1+e^x} \Rightarrow A = \log\left(\frac{1+e^x}{e^x}\right) - e^{-x}$$

$$B = \frac{dB}{dx} = \frac{-e^x}{1+e^x} \Rightarrow B = -\log(1+e^x)$$

Putting the values of A and B in (1), we get the complete solution.

$$y = C_1e^x + C_2e^{-x} + e^x \log\left(\frac{1+e^x}{e^x}\right) - 1 - e^{-x}\log(1+e^x)$$ **Ans.**

□□□

PARTIAL DIFFERENTIAL EQUATIONS - FIRST ORDER 4

PREVIOUS YEARS QUESTIONS

PART A

Q.1 *When is a nonlinear PDE semilinear?* *[R.T.U. 2019]*

Ans. A differential equation of the form,

$$P(x, y)\,p + Q(x, y)\,q = R(x, y, z)$$

is said to be nonlinear PDE of first order if p and/or q occur in more than one degree but this PDE is said to be semilinear if it is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

Where P(x, y) and Q(x, y) are function of x and y only and do not cantain any term of z and R(x, y, z) cantains same terms that all not of first degree in z. p and q occurs in first degree only.

Q.2 *What is the homogeneous PDE?* *[R.T.U. 2019]*

Ans. If each term of a linear PDE contains either the dependent variable or one of its partial derivatives then the equation is said to be homogeneous PDE.

Q.3 *Define partial differential equation.*

Ans. Partial Differential Equation : An equation involving partial differential coefficients of function of two or more variables is known as partial differential equation.

Q.4 *What is meant by "Solution of a differential equation"?*

Ans. Solution or Integral : A solution of a differ… equation is a relation between the variables from which… differential equation can be obtained. It is free from deriv…

Q.5 *What do you understand by complete integral?*

Ans. Complete Integral : A relation between the varia… containing as many arbitrary constants as there are indep… variables and which satisfies the given differential equ…

As $\phi(x, y, z, a, b) = 0$ is the complete integ… $F(x, y, z, p, q) = 0$.

Q.6 *Define singular integral.*

Ans. Singular Integral : The equation of the env… of the surfaces represented by the complete integ… the given partial differential equation is called sin… integral. Thus singular integral is obtained by elimin… a and b from $\frac{\partial F}{\partial a} = 0$, $\frac{\partial F}{\partial b} = 0$ and $\phi(x, y, z, a, b)$, w… $\phi(x, y, z, a, b)$ is complete integral.

Q.7 *Give the working rule while using Charpit's me…*

Ans. Working Rule While Using Charpit's Meth…

Step-1 : Let $f(x, y, z, p, q) = 0$ be the given p… differential equation.

Step-2 : Put the values in the Charpit's auxiliary equ…

Step-3 : Find the values of p and q using charpit's au… equation of the given equation.

Step-4 : Put the values of p and q in

$$dz = p\,dx + q\,dy$$

Step-5 : Integrating the above equation gives co… integral.

Q.8 Write down auxiliary equation of Charpit method.

Ans. $$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(fx + pf_z)} = \frac{dq}{-(fy + qf_z)}$$

Q.9 Discuss the procedure for solving non-linear partial differential equation of standard form-III.

Ans. Procedure for solving non-linear partial differential equation of standard form - III :

Step-1 : Let $f(p,q,z) = 0$ be the given equation. ...(1)

Step-2 : Put $u = x + ay$. ...(2)

Step-3 : Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in eq. (1).

Step-4 : Solve the ordinary differential equation.

Step-5 : Put $u = x + ay$ to get the complete integral.

Step-6 : Calculate the general and singular integral by usual methods.

PART B

Q.10 Solve $z^2(p^2 + q^2) = x^2 + y^2$ **[R.T.U. 2019, 12, Raj. Univ. 2003, 2000]**

Ans. The given equation can be written as

$$z^2(p^2 + q^2) - x^2 - y^2 = 0$$

$$\Rightarrow \quad z^2p^2 - x^2 = y^2 - z^2q^2 = a^2 \text{ (say)}$$

Which is of the form III

On separating we have

$$p = \frac{\sqrt{x^2 + a^2}}{z} \ \& \ q = \frac{\sqrt{y^2 - a^2}}{z}$$

Substituting the values of p and q in $dz = p\,dx + q\,dy$, we get

$$z\,dz = \sqrt{x^2 + a^2}\,dx + \sqrt{y^2 - a^2}\,dy$$

Integrating, we get

$$\frac{z^2}{2} = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right)$$
$$+ \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\log\left(y + \sqrt{y^2 - a^2}\right) + b$$

Thus, the complete integral is given by

$$z^2 = x\sqrt{x^2 + a^2} + y\sqrt{y^2 - a^2} + a^2\log\left(\frac{x + \sqrt{x^2 + a^2}}{y + \sqrt{y^2 - a^2}}\right) + b$$

Q.11 Find the general solution of the partial differential equation- $xy^2p + y^3q = (zxy^2 - 4x^3)$ **[R.T.U. 2019]**

Ans. The given differential equation of the form,

$$Pp + Qq = R$$

Here, $P = xy^2, Q = y^3, R = zxy^2 - 4x^3$

Auxiliary equation,

$$\frac{dx}{xy^2} = \frac{dy}{y^3} = \frac{dz}{zxy^2 - 4x^3} \quad \text{... (1)}$$

Taking first two pairs, we get

$$\frac{dy}{xy^2} = \frac{dy}{d_1^3}$$

$$\text{or} \quad \frac{dx}{x} = \frac{dy}{y}$$

on integration

$$\log x = \log y + \log c_1$$

$$x = c_1 y \quad \text{... (2)}$$

Also taking 1^{st} and last term of equation (1)

$$\frac{dx}{xy^2} = \frac{dz}{zxy^2 - 4x^3}$$

$$\frac{dx}{xy^2} = \frac{dz}{x\left(zy^2 - 4x^2\right)}$$

$$\frac{dx}{xy^2} = \frac{dz}{x\left(zy^2 - 4c_1^2y^2\right)} \quad \text{[using equation (2)]}$$

$$\text{or} \quad \frac{dx}{y^2} = \frac{dz}{y^2\left(z - 4c_1^2\right)}$$

$$\text{or } dx = \frac{dz}{z - 4c_1^2}$$

on integration, we get

$$x = \log\left(z - 4c_1^2\right) + \log c_2$$

$$\text{or } c_2 = \frac{e^x}{z - 4c_1^2}$$

Hence the complete solution of given differential equation.

$$\phi(c_1, c_2) = 0$$

$$\text{or} \quad \phi\left(\frac{x}{y}, \frac{e^x}{z - 4\frac{x^2}{y^2}}\right) = 0$$

$$\text{or} \quad \phi\left(\frac{x}{y}, \frac{e^x y^2}{zy^2 - 4x^2}\right) = 0$$

Q.12 Solve : $(y + z)p + (z + x)q = x + y$ **[R.T.U. 2018, 2012]**

Ans. Given :

$$(y + z)p + (z + x)q = x + y \quad \text{...(i)}$$

Comparing Eq. (i) with $Pp + Qq = R$, we get

$P = y + z$

$Q = z + x$

$R = x + y$

Consider Lagrange's auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \quad \frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}$$

$$\Rightarrow \quad \frac{dx - dy}{y + z - z - x} = \frac{dy - dz}{z + x - x - y}$$

$$= \frac{dz - dx}{x + y - y - z}$$

$$= \frac{dx + dy + dz}{2(x + y + z)}$$

$$\Rightarrow \quad \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

$$= \frac{dx + dy + dz}{2(x + y + z)}$$

From first and second member, we have

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

On integration, we get

$$\log(x - y) = \log(y - z), \log C_1$$

$$\therefore \quad \frac{x - y}{y - z} = C_1 \quad \text{...(ii)}$$

From first and fourth member, we have

$$\frac{dx - dy}{y - z} = \frac{dx + dy + dz}{2(x + y + z)}$$

$$\frac{2(dx - dy)}{x - y} = \frac{dx + dy + dz}{x + y + z}$$

Integrating, we get

$$-2\log(x - y) = \log(x + y + z) + \log C_2'$$

$$\Rightarrow -2\log(x - y) - \log(x + y + z) = \log C_2'$$

$$\Rightarrow \log(x - y)^2(x + y + z) = \log C_2$$

where, $\log C_2 = -\log C_2'$

$$(x - y)^2(x + y + z) = C_2 \quad \text{...(iii)}$$

From Eq. (ii) and (iii), the general solution of Eq. (i) is given by

$$\phi\left(\frac{x - y}{y - z}, (x - y)^2(x + y + z)\right) = 0$$

Q.13 Solve $9\left(p^2z + q^2\right) = 4$ **[R.T.U. 2014]**

Ans. The given equation is of the form $f(p,q,z) = 0$

Let $u = x + ay$, where a is an arbitrary constant.

$$\therefore \quad p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

So replacing p and q in the given equation, we have

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4$$

$$\text{or} \quad \left(\frac{dz}{du}\right)^2 = \frac{4}{9\left(z + a^2\right)} \text{ or } du = \pm\left(\frac{3}{2}\right)\left(z + a^2\right)^{1/2} dz$$

Integrating, we have

$$u + b = \pm\left(\frac{3}{2}\right)\left[\frac{2}{3}\left(z + a^2\right)^{3/2}\right]$$

$$\text{or} \quad u + b = \pm\left(z + a^2\right)^{3/2} \text{ or } (u + b)^2 = \pm\left(z + a^2\right)^3$$

$$\text{or} \quad (x + ay + b)^2 = \left(z + a^2\right)^3$$

Which is the required complete integral.

Q.14 Solve in series :

$$x(1 - x)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0 \quad \text{[R.T.U. 2014, July 2011]}$$

Ans. As $x = 0$ is regular singular point for the given equation

so let $y = \sum_{r=0}^{\infty} a_r x^{m+r}; a_0 \neq 0$ is solution of the given equation.

Putting (1) in the given equation

$$x(x-1)\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-2}$$

$$-3x\sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1} - \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r} - \sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-1}$$

$$-3\sum_{r=0}^{\infty} a_r(m+r)x^{m+r} - \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\sum_{r=0}^{\infty}[(m+r)(m+r-1)-3(m+r)-1]a_r x^{m+r}$$

$$-\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-1} = 0$$

$$\sum_{r=0}^{\infty}[(m+r)^2-4(m+r)-1]a_r x^{m+r}$$

$$-\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-1} = 0$$

Equating to zero, coefficient of various powers of x, starting with lowest power of x i.e.

Coefficient of x^{m-1}

$a^0 m(m-1) = 0$

$m = 0, 1 \quad (\because a_0 \neq 0)$

Coefficient of x^{m+n-1}

$[(m+n-1)^2 - 4(m+n-1) - 1]a_{n-1} - (m+n)(m+n-1)a_n = 0$

$$a_n = \frac{(m+n-1)^2 - 4(m+n-1) - 1}{(m+n)(m+n-1)} a_{n-1}$$

where $n = 1, 2, \ldots$

Putting values of $n = 1, 2, 3, \ldots$

$$a_1 = \frac{m^2 - 4m - 1}{(m+1)m} a_0$$

$$a_2 = \frac{(m+1)^2 - 4(m+1) - 1}{(m+2)(m+1)}$$

$$a_2 = \frac{((m+1)^2 - 4(m+1) - 1)(m^2 - 4m - 1)}{(m+2)(m+1)^2 m} a_0$$

$$a_3 = \frac{(m+2)^2 - 4(m+2) - 1}{(m+3)(m+2)} a_2$$

$$= \frac{[(m+2)^2 - 4(m+2) - 1][(m+1)^2 - 4(m+1) - 1][m^2 - 4m - 1]}{(m+3)(m+2)^2(m-1)^2 m}$$

and so on

$$y = a_0 x^m \left[1 + \frac{m^2 - 4m - 1}{(m+1)m} x + \frac{((m+1)^2 - 4(m+1) - 1)(m^2 - 4m - 1)}{(m+2)(m+1)^2 m} x^2 + \ldots\right]$$

at $m = 0$, coefficient of $x, x^2, \ldots$ becomes infinite so that

$a_0 = b_0(m-1) = b_0 m$ so

$$y = b_0 x^m \left[m + \frac{m^2 - 4m - 1}{m+1} x + \frac{(m+1)^2 - 4(m+1) - 1(m^2 - 4m - 1)}{(m+2)(m+1)^2} x^2 + \ldots\right]$$

at $m = 0$

$(y)_{m=0} = y_1 = b_0[-x + 2x^2 = \ldots]$

Differentiate (3) w.r. to m

$$\frac{\partial y}{\partial m} = b_0 x^m \log x \left[m + \frac{m^2 - 4m - 1}{(m+1)} x + \ldots\right]$$

$$+ b_0 x^m \left[1 + \left(\frac{2m - 4}{m+1} - \frac{m^2 - 4m - 1}{(m+1)^2}\right) x + \ldots\right]$$

Put $m = 0$

$$\left(\frac{\partial y}{\partial m}\right)_{m=0} = y_2 = b_0 \log x(-x + 2x^2 + \ldots) + b_0(1 - 3x + \ldots)$$

Complete solution is

$y = c_1 y_1 + c_2 y_2$

So $y = c_1 b_0(-x + 2x^2 + \ldots) + c_2 b_0 \log x(-x + 2x^2 + \ldots)$

$+ c_2 b_0 (1 - 3x + \ldots)$

$y = (A + B \log x)(-x + 2x^2 + \ldots) + B(1 - 3x + \ldots)$

where $A = C_1 b_0$ and $B = C_2 b_0$

Q.15 *Solve in series* $(1 - x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + y = 0$ **[R.T.U. 2013]**

Ans. Since $x = 0$ is an ordinary point of given equation let its series solution be

$$y = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n + \ldots = \sum_{k=0}^{\infty} a_k x^k \quad \ldots (i)$$

then

$$\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$(1 - x^2)\sum_{k=2}^{\infty} k(k-a)a_k x^{k-2} + 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$(1 - x^2)\begin{bmatrix} 2.1a_2 + 3.2a_3 x + \ldots + n(n-1)a_n x^{n-2} + \\ (n+1)na_{n+1}x^{n-1} + (n+2)(n+1)a_{n+2}x^n + \ldots \end{bmatrix}$$

$$+ 2x[a_1 + 2a_2 x + \ldots + na_n x^{n-1} + \ldots]$$

$$+ [a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots] = 0$$

Equating to zero the coefficients of various powers of x

$$a_2 = -\frac{a_0}{2}$$

$$a_3 = -\frac{a_1}{2} \text{ and so on}$$

Equating to zero the coefficients of x^n, we have

$$(n+2)(n+1)a_{n+2} + (n^2 + n + 1)a_n = 0$$

$$a_{n+2} = -\frac{(n^2 + n + 1)}{(n+2)(n+1)} a_n$$

Setting $n = 2, 3, 4, 5, \ldots$ in above Eq., we get

$$a_4 = -\frac{7}{12} a_2 = \frac{7}{24} a_0$$

$$a_5 = -\frac{13}{20} \quad a_3 = \frac{13}{40} a_1$$

$$a_6 = -\frac{21}{30} \quad a_4 = -\frac{49}{240} a_0$$

$$a_7 = -\frac{31}{42} \quad a_5 = -\frac{403}{1680} a_1$$

$$a_8 = -\frac{43}{56} \quad a_6 = -\frac{2107}{13440} a_0$$

Substituting the values of a_i's & in Eq. (i), we get

$$y = a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{2} x^3 + \frac{7}{24} a_0 x^4 + \frac{13a_1}{40} x^5$$

$$- \frac{49a_0}{240} x^6 - \frac{403a_1}{1680} x^7 + \frac{2107a_0}{13440} x^8 + \ldots$$

$$y = \left(1 - \frac{x^2}{2} + \frac{7x^4}{24} - \frac{49x^6}{240} + \frac{2107x^8}{13440}\right) a_0 + \left(x - \frac{x^3}{2} + \frac{13x^5}{40} - \frac{403x^7}{1680}\right) a_1$$

this is the required general solution containing two arbitrary constants a_0 and a_1

Q.16 *Solve* $2xz + qp = px^2 + 2qxy$ **[R.T.U. 2013]**

Ans. Here $f = 2xz - px^2 - 2qxy + pq$

$$\therefore \quad \frac{\partial f}{\partial x} = 2z - 2px - 2qy$$

$$\frac{\partial f}{\partial y} = -2qx, \frac{\partial f}{\partial z} = 2x$$

$$\frac{\partial f}{\partial p} = -x^2 + q$$

$$\text{and} \quad \frac{\partial f}{\partial q} = -2xy + p$$

The auxiliary equation is

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{df}{0}$$

or

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 + 2xyq} = \frac{df}{0}$$

from above, $dq = 0$ or $q = a$

Putting $q = a$ in the given equation, we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

$\therefore$ Substituting values of p and q in $dz = p\,dx + q\,dy$ we get

$$dz = \frac{2x(z - ay)}{x^2 - a} dx + a\,dy \quad \text{or} \quad \frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}$$

Integrating,

$$\log(z - ay) = \log(x^2 - a) + \log b = \log b(x^2 - a)$$

$$\therefore \quad z = ay + b(x^2 - a)$$

This is the complete solution.

Q.17 Solve in series $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0$

[R.T.U. 2013, July 11, 08]

Ans. Since x = 0 is a regular singular point of the given equation, let its series solution be

$$y=x^m\left(a_0+a_1x+a_2x^2+...\right)=\sum_{k=0}^{\infty}a_kx^{m+k} \quad ...(i)$$

Then

$$\frac{dy}{dx}=\sum_{k=0}^{\infty}(m+k)\,a_kx^{m+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}$$

Substituting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we get

$$=x\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}$$

$$+\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}+\sum_{k=0}^{\infty}a_kx^{m+k}=0$$

$$=x\left[m(m-1)a_0x^{m-2}+(m+1)(m)a_1x^{m-1}\right.$$

$$\left.+(m+2)(m+1)a_2x^m+(m+3)(m+2)a_3x^{m+1}+...\right]+$$

$$\left[ma_0x^{m-1}+(m+1)a_1x^m+(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+...\right]+$$

$$\left[a_0x^m+a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+...\right]=0$$

The lowest power of x is x^{m-1}. Equating the coefficient of x^{m-1} to zero, we get

$$m(m-1)a_0+ma_0=0$$

$$m^2a_0=0$$

since $a_0\neq 0$

$$m^2=0$$

which is the indicial equation. Its roots are m = 0, 0 which are equal, Equating the coefficient of x^m to zero, we get

$$m(m+1)a_1+(m+1)a_1+a_0=0$$

$$(m+1)^2a_1+a_0=0$$

$$a_1=-\frac{a_0}{(m+1)^2}$$

Equating the coefficient of x^{m+1} to zero, we get

$$(m+1)(m+2)a_2+(m+2)a_2+a_1=0$$

$$(m+2)^2a_2+a_1=0$$

$$a_2=-\frac{a_1}{(m+2)^2}$$

$$a_2=-\frac{a_0}{(m+1)^2(m+2)^2}$$

so the solution is (putting values of a_i's & in Eq. (i))

$$y=x^m\left[a_0-\frac{a_0}{(m+1)^2}x+\frac{a_0}{(m+1)^2(m+2)^2}x^2...\right] \quad ...(ii)$$

As m = 0, the first solution is

$$y_1=a_0\left[1-x+\frac{x^2}{4}+...\right]$$

The second solution is $\mu=\left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

Differentiating Eq. (ii) w.r.t. m, partially, we get

$$\frac{\partial y}{\partial m}=a_0x^m\log x\left[1-\frac{x}{(m+1)^2}+\frac{x^2}{(m+1)^2(m+2)^2}\right]$$

$$+a_0x^m\left[\frac{2}{(m+1)^3}x-\frac{2(2m+3)}{\{(m+1)(m+3)\}^3}x^2...\right]$$

The second solution is

$$y_2=\left(\frac{\partial y}{\partial m}\right)_{m=0}=a_0\log xy_1+a_0\left(2x-\frac{6x^2}{8}...\right)$$

$$=a_0\log xy_1+a_0\left(2x-\frac{3x^2}{4}...\right)$$

The complete solution is $y=Ay_1+By_2$

Q.18 Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$

[R.T.U. 2012 RAJ. UNIV. 2005]

Ans. Here the Lagrange's auxiliary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

$$\Rightarrow \quad \frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$

$$=\frac{dz-dx}{(z-x)(x+y+z)}$$

$$\Rightarrow \quad \frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$$

Taking first two fractions of the above, we have

$$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

Integrating both sides, we have

$$\frac{x-y}{y-z}=C_1$$

Again taking last two fraction, we have

$$\frac{dz-dx}{z-x}=\frac{dy-dz}{y-z} \quad \Rightarrow \quad \frac{y-z}{z-x}=C_2$$

Hence the general solution is given by

$$\phi\left(\frac{x-y}{y-z},\frac{y-z}{z-x}\right)=0$$

ϕ being arbitrary function.

Q.19 Solve $(x^2+y^2)(p^2+q^2)=1$

[R.T.U. 2012, Raj. Univ. 2005, 2003]

Ans. Putting $x=r\cos\theta$ and $y=r\sin\theta$ ...(i)

Then, $r^2=x^2+y^2$ and $\theta=\tan^{-1}(y/x)$ ...(ii)

Differentiating eq. (ii) partially with respect to x and y, we get

$$2r(\partial r/\partial x)=2x \text{ and } 2r(\partial r/\partial y)=2y$$

$$\Rightarrow \quad \frac{\partial r}{\partial x}=\frac{r\cos\theta}{r}=\cos\theta \text{ and } \frac{\partial r}{\partial y}=\frac{r\sin\theta}{r}=\sin\theta \quad ...(iii)$$

and

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$$

$$=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r} \quad ...(iv)$$

$$\frac{\partial\theta}{\partial y}=\frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

$$=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r} \quad ...(v)$$

Given equation is

$$\left(x^2+y^2\right)\left(p^2+q^2\right)=1 \quad ...(vi)$$

Now, $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial x}$,

$$=\cos\theta\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}$$

by eq. (iii) and eq. (iv) and

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial y},$$

$$=\sin\theta\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\frac{\partial z}{\partial\theta} \text{ by eq. (iii) and eq. (v)}$$

Hence $p^2+q^2=(\partial z/\partial r)^2+(1/r^2)(\partial z/\partial\theta)^2$ ...(vii)

$\therefore$ Eq. (vi) becomes $r^2\left[(\partial z/\partial r)^2+(1/r^2)(\partial z/\partial\theta)^2\right]=1$,

using eq. (ii) and eq. (vii)

or $\left(r\dfrac{\partial z}{\partial r}\right)^2+\left(\dfrac{\partial z}{\partial\theta}\right)^2=1$ ...(viii)

Let R be a new variable such that $(1/r)dr=dR$ so that

$\log r=R$ ...(ix)

Then eq. (viii) becomes $(\partial z/\partial R)^2+(\partial z/\partial\theta)^2=1$ or

$P^2+Q^2=1$, ...(x)

Where $P=\partial z/\partial R$ and $Q=\partial z/\partial\theta$. Eq. (x) is of the form $f(P,Q)=0$.

$\therefore$ Solution of eq. (iv) is $z=aR+b\theta+c$, ...(xi)

Where $a^2+b^2=1$ or $b=\sqrt{1-a^2}$, putting a for P and b for Q in eq. (x),

$\therefore$ from eq. (xi), the required complete integral is

$$z=aR+\theta\sqrt{1-a^2}+c$$

or $$z=a\log r+\theta\sqrt{1-a^2}+c$$

or $$z=a\log\left(x^2+y^2\right)^{1/2}$$

$$+\tan^{-1}(y/x)\cdot\sqrt{1-a^2}+c, \text{ by eq. (ii)}$$

or $$z=(a/2)\log\left(x^2+y^2\right)$$

$$+\sqrt{1-a^2}\tan^{-1}(y/x)+c.$$

Q.20 Apply Charpit's method to find complete integral of $(p+q)(px+qy)-1=0$

[R.T.U. 2012, Raj. Univ. 2003, 2001]

Ans. $f=(p+q)(px+qy)-1=0$ ...(i)

Charpit's first two A.E. are

$$\Rightarrow \quad \frac{dp}{p^2+pq}=\frac{dq}{pq+q^2} \Rightarrow \frac{dp}{p}=\frac{dq}{q}$$

On integration $p=aq$

putting this value of p in equation (i)

$$(aq+q)(aqx+qy)=1$$

or $$q=\frac{1}{[(1+a)(ax+y)]^{1/2}}$$

which gives $p=\frac{a}{\{(1+a)+(ax+y)\}^{1/2}}$

Putting the values of p and q in $dz=p\,dx+q\,dy$

we get $dz=\frac{1}{\sqrt{(1+a)(ax+y)}}[a\,dx+dy]$

Integrating we have $z=\frac{2}{\sqrt{(1+a)}}\sqrt{(ax+y)}+c$

Q.21 *Apply Charpit's method to find the complete integral of $(x^2-y^2)pq-xy(p^2-q^2)=1$*

[R.T.U. July-2011, Raj. Univ. 2005]

OR

Use Charpit's method to solve

$(x^2-y^2)pq-xy(p^2-q^2)=1$

Where $p=\frac{\partial z}{\partial x}$ *and* $q=\frac{\partial z}{\partial y}$ [R.T.U. 2009]

Ans. Here

$$f(x,y,z,p,q)=(x^2-y^2)pq-xy(p^2-q^2)-1=0$$

Then Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}} \quad ...(1)$$

$$\frac{\partial f}{\partial x}=2pqx-y(p^2-q^2);\ \frac{\partial f}{\partial y}=-2ypq-x(p^2-q^2);\ \frac{\partial f}{\partial z}=0$$

$$\frac{\partial f}{\partial p}=(x^2-y^2)q-2pxy;\ \frac{\partial f}{\partial q}=(x^2-y^2)p+2qxy$$

Putting these values in eq.(1) we have

$$\frac{dp}{2pqx-y(p^2-q^2)}=\frac{dq}{-2pqy-x(p^2-q^2)}=\frac{dx}{-(x^2-y^2)q+2pxy}$$

$$=\frac{dy}{-(x^2-y^2)p-2qxy}$$

using x, y, p, q as multipliers each fraction is

$$=\frac{x\,dp+y\,dq+p\,dx+q\,dy}{0}=\frac{d(xp)+d(yq)}{0}$$

$\Rightarrow d(xp+yq)=0 \Rightarrow xp+yq=a \Rightarrow p=\frac{a-qy}{x}$ ...(2)

Putting the value of p in eq.(1)

$$(x^2-y^2)\left(\frac{a-qx}{x}\right)q-xy\left[\left(\frac{a-qy}{x}\right)^2-q^2\right]-1=0$$

or $\frac{a-qy}{x}\{(x^2-y^2)q-(a-qy)y\}+xyq^2-1=0$

or $\left(\frac{a-qy}{x}\right)(x^2q-ay)+xyq^2-1=0$

or $(a-qy)(x^2q-ay)+x^2yq^2-x=0$

or $aq(x^2+y^2)=a^2y+x$

$\therefore$ $q=\frac{a^2y+x}{a(x^2+y^2)}$

and $p=\frac{1}{x}\left[a-\frac{(a^2y+x)y}{a(x^2+y^2)}\right]=\frac{a^2x-y}{a(x^2+y^2)}$

Substituting these values in $dz=p\,dx+q\,dy$, we have

$$dz=\frac{(a^2x-y)dx+(a^2y+x)dy}{a(x^2+y^2)}$$

$$dz=a\frac{x\,dx+y\,dy}{x^2+y^2}+\frac{x\,dy-y\,dx}{a(x^2+y^2)}$$

Integrating $z=\frac{a}{2}\log(x^2+y^2)+\frac{1}{a}\tan^{-1}\frac{y}{x}+b$

Q.22 *Discuss working rule for solving Pp + Qq = R by Lagrange's method.*

Ans. Working Rule for Solving $Pp+Qq=R$ by Lagrange's Method

Step-1 : Put the given linear partial differential equation of the first order in the standard form

$Pp+Qq=R$ ...(1)

Step-2 : Write down Lagrange's auxiliary equations for eq. (1) namely,

$dx/P=dy/Q=dz/R$ ...(2)

Step-3 : Solve eq. (2) by using the well known methods. Let $u(x,y,z)=C_1$ and $v(x,y,z)=C_2$ be two independent solutions of eq. (2).

Step-4 : The general solution (or integral) of eq. (1) is then written in one of the following three equivalent forms :

$\phi(u,v)=0,\ u=\phi(v)$ or $v=\phi(u)$.

Q.23 *Explain briefly to solve partial differential equation for, (I) Type 1 (II) Type 2*

Ans. (i) **Type 1 : For solving** $dx/P=dy/Q=dz/R$ ...(1)

Suppose that one of the variables is either absent or cancels out from any two fractions of given eq. (1). Then an integral can be obtained by the usual method. The same procedure can be repeated with another set of two fractions of given eq. (1).

(ii) **Type 2 : For solving** $dx/P=dy/Q=dz/R$ ...(2)

Suppose that one integral of eq. (2) is known by using rule I explained above and suppose also that another integral cannot be obtained from above. Then one integral known to us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of first integral should be removed later on.

Q.24 *Find the complete and singular integral of $z=px+qy+\log(pq)$*

Ans. The complete integral is $z=ax+by+\log(ab)$ ...(i)

or $z=ax+by+\log a+\log b$

To Find Singular Integral

Differentiating eq. (i) partially *w.r.t.* a and b, we have

$0=x+\frac{1}{a}$ and $0=y+\frac{1}{b}$

$\Rightarrow$ $a=-\frac{1}{x}$ and $b=-\frac{1}{y}$

$\therefore$ $z=-1-1+\log\left(\frac{1}{xy}\right) \Rightarrow z=-2-\log xy$

which is the required singular integral. **Ans.**

Q.25 *Find the complete integral of $p^2+q^2=x+y$*

Ans. The given equation can be written as

$p^2-x=y-q^2=a$ (say)

$\therefore$ $p=(x+a)^{1/2},\ q=(y-a)^{1/2}$

$\therefore$ Putting the value of p and q in $dz=p\,dx+q\,dy$, we get

$dz=(x+a)^{1/2}dx+(y-a)^{1/2}dy$

Integrating, we have

$z+b=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y-a)^{3/2}$

Which is the required complete integral. **Ans.**

Q.26 *Solve $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$*

Ans. Lagrange's subsidiary equations are

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}$$

Using the multiplier $x, y, -1$, we get

$x\,dx+y\,dy-dz=0 \Rightarrow x^2+y^2-2z=C_1$

Again using $1/x$, $1/y$, $1/z$ as multipliers, we obtain

$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

$\Rightarrow$ $\log x+\log y+\log z=\log C_2$ or $xyz=C_2$

$\Rightarrow$ Hence the required general solution

$\phi(x^2+y^2-2z, xyz)=0$, where ϕ is an arb

constants.

PART C

Q.27 *Find the complete integral of the partial diffe equation :*

$px+qy+z=xq^2$ [R.T.U.

Ans. Hence $f=px+qy+z-xq^2=0$

$\therefore$ $\frac{\partial f}{\partial x}=p-q^2, \frac{\partial f}{\partial y}=q, \frac{\partial f}{\partial z}=1$

$\frac{\partial f}{\partial p}=x, \frac{\partial f}{\partial q}=y-2xq$

The charpit auxiliary equation is.

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$$

$$=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}$$

or $=\frac{dp}{p-q^2+p}=\frac{dq}{q+q}$

$$= \frac{dx}{-x} = \frac{dy}{-(y-2xq)} = \frac{dz}{-px-qy+2xq^2}$$

or $$\frac{dp}{2p-q^2} = \frac{dq}{2q} = \frac{dx}{-x} = \frac{dy}{2xq-y} = \frac{dz}{-Px-qy+2xq^2}$$

on taking $\frac{dq}{2q} = \frac{dx}{-x}$

on integration, we get

$$\frac{1}{2}\log q + \log x = \log c$$

$$\sqrt{q}x = c$$

or $\sqrt{q} = \frac{c}{x}$

or $q = \frac{c^2}{x^2}$ ... (2)

Putting $q = \frac{c^2}{x^2}$ in eq. (1), we get

$$px + \frac{c^2}{x^2}y + z - x\frac{c^4}{x^4} = 0$$

or $$p = \frac{\frac{c^4}{x^3} - z - \frac{c^2}{x^2}y}{x}$$

or $$p = \frac{c^4 - zx^3 - c^2xy}{x^4}$$

or $$p = \frac{c^4}{x^4} - \frac{z}{x} - \frac{c^2y}{x^3}$$

using $dz = pdx + qdy$

$$dz = \left(\frac{c^4}{x^4} - \frac{z}{x} - \frac{c^2y}{x^3}\right)dx + \frac{c^2}{x^2}dy$$

or $$dz + \frac{z}{x}dx = \frac{c^4}{x^4}dx - \frac{c^2}{x^3}ydx + \frac{c^2}{x^2}dy$$

or $$\frac{xdz + zdx}{x} = \frac{c^4}{x^4}dx - \frac{c^2y}{x^3}dx + \frac{c^2}{x^2}dy$$

or $$xdz + zdx = \frac{c^4}{x^3}dx - \frac{c^2}{x^2}ydx + \frac{c^2}{x}dy$$

or $$d(xz) = \frac{c^4}{x^3}dx + \frac{c^2}{x^2}(xdy - ydx)$$

or $$d(xz) = \frac{c^4}{x^3}dx + c^2 d\left(\frac{y}{x}\right)$$

on integration, we get

$$xz = \frac{-c^4}{2x^2} + c^2\frac{y}{x}$$

or $$z = \frac{-c^4}{2x^3} + \frac{c^2y}{x^2}$$

Q.28 (a) ***Form the partial differential by using the elimination of arbitrary function :***

$$z = f\left(\frac{y}{x}\right)$$

(b) Solve the following differential equation by using Charpit's method :

$$pxy + pq + qy = yz$$

[R.T.U. 2018]

Ans.(a) $z = f\left(\frac{y}{x}\right)$

Partially Differentiating w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = p = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad ...(i)$$

and $$\frac{\partial z}{\partial y} = q = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad ...(ii)$$

Dividing equation (i) by equation (ii)

We get $\frac{p}{q} = -\frac{y}{x}$

$$px + qy = 0$$

Ans.(b) The given PDE can be written as

$$pxy + pq + qy - yz = 0 \quad ...(i)$$

and $f = pxy + pq + qy - yz$

By Charpit's method, the auxiliary equations are

$$\frac{dp}{(\partial f/\partial x) + p(\partial f/\partial z)} = \frac{dq}{(\partial f/\partial y) + q(\partial f/\partial z)} = \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)}$$

$$= \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q}$$

$$\frac{dp}{py - py} = \frac{dq}{(px+q) - qy} = ...$$

$\therefore \; dp = 0$ or $p = C_1$

Putting this value of p in eq. (i), we get

$$C_1xy + C_1q + qy - yz = 0 \text{ or } q = \frac{yz - C_1xy}{C_1 + y}$$

Putting the value of p and q in $dz = p\,dx + q\,dy$

$$dz = C_1dx + \frac{yz - C_1xy}{C_1 + y}dy$$

or $$(dz - C_1dx) = y\frac{(z - C_1x)}{C_1 + y}dy$$

or $$\frac{dz - C_1dx}{z - C_1x} = \frac{y}{C_1 + y}dy = \left(1 - \frac{C_1}{C_1 + y}\right)dy$$

Integrating, $\log(z - C_1x) + C_2 = y - C_1\log(C_1 + y)$

or $$\log\left\{(z - C_1x).(C_1 + y)^{C_1}\right\} = y - C_2$$

or $$(z - C_1x)(C_1 + y)^{C_1} = e^{y - C_2} = be^y$$

Which is the required solution of the given PDE. **Ans.**

Q.29 (a) ***Solve the following differential equation in series.***

$$\frac{d^2y}{dx^2} + x^2y = 0$$

(b) Solve :

$$x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$$

[R.T.U. 2016]

Ans.(a) Here $P_0(x) = 1$, $P_1(x) = 0$ and $P_2(x) = x^2$

Since $P_0(x) = 1 \neq 0$ at $x = 0$, so $x = 0$ is an ordinary point.

So, let $y = a_0 + a_1x + a_2x^2 + ...$ be the solution of the given differential equation.

$$\Rightarrow \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} ra_rx^{r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} r(r-1)a_rx^{r-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$\sum_{r=0}^{\infty} r(r-1)a_rx^{r-2} + x^2\sum_{r=0}^{\infty} a_rx^r = 0 \quad ...(i)$$

Equating the coefficient of x^{r+2}, by putting $r = r + 4$ in first term (here we here x^{r+2} as the highest degree term), we have

$$\sum_{r=0}^{\infty} [(r+4)(r+3)a_{r+4} + a_r]x^{r+2} = 0$$

$$\Rightarrow a_{r+4} = \frac{-a_r}{(r+3)(r+4)} \rightarrow \text{Recurrence relation}$$

Putting $r = 0, 1, 2, 3, 4, 5, 6, ...$, we have

$$a_4 = -\frac{a_0}{3.4}, a_5 = -\frac{a_1}{4.5}, a_6 = -\frac{a_2}{5.6}$$

$$a_7 = -\frac{a_3}{6.7}, a_8 = -\frac{a_4}{7.8} = \frac{a_0}{3.4.7.8}$$

$$a_9 = -\frac{a_5}{8.9} = -\frac{a_1}{4.5.8.9} \text{ and so on.}$$

here we have four constants a_0, a_1, a_2 and a_3 but in the solution of second order differential equation, we can have only two constants so considering the first two constants a_0 and a_1 and neglecting the other two by putting $r = 2$ and 3 in the lower degree term (equation i), we have

$$2 \cdot a_2 = 0 \text{ and } 3 \cdot 2 \cdot a_3 = 0$$

$$\Rightarrow a_2 = 0 = a_3$$

$\therefore$ the general solution is given by

$$y = a_0\left[1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - ...\right] + a_1x\left[1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - ...\right]$$

Ans.(b) $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$

The auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} \quad ...(i)$$

Taking x, y, z as multipliers, each fraction in (i)

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e. $x\,dx + y\,dy + z\,dz = 0$

So that $x^2 + y^2 + z^2 = C_1$

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we have

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating $xyz = C_2$

Hence the general solution of the given equation is

$f[x^2 + y^2 + z^2, xyz] = 0$

Q.30 *(a)* ***Solve :***

$$x^2p^2 + y^2q^2 = z^2$$

(b) ***Find a complete integral of***

$$q = (z + px)^2$$

by using charpit's method. *[R.T.U. 2016]*

Ans.(a) Given eq. $= x^2p^2 + y^2q^2 = z^2$

$$x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 = z^2$$

or $$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$$

Put $x = \log x, y = \log y, z = \log z$

$$dx = \frac{1}{x}dx$$

$$dy = \frac{1}{y}dy$$

$$dz = \frac{1}{z}dz$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$$

or $P^2 + Q^2 = 1$

where $P = \frac{\partial z}{\partial x}, Q = \frac{\partial z}{\partial y}$,

Therefore its complete integral is

$$Z = ax + by + C$$

where $a^2 + b^2 = 1$ or $b = \sqrt{1 - a^2}$

or $\log z = a \log x + b \log y + \log C$

or $\log z = \cos\alpha \log x + \sin\alpha \log y + \log C$

where

$a = \cos\alpha$

$b = \sqrt{1 - a^2} = \sqrt{1 - \cos^2\alpha} = \sin\alpha$

or $\log z = \log x^{\cos\alpha} + \log y^{\sin\alpha} + \log C$

or $\boxed{z = x^{\cos\alpha}.y^{\sin\alpha}.C}$

Ans.(b) Given $\quad q = (px + z)^2$

Here $\quad f = -q + (px + z)^2 = 0$

Charpit's auxiliary equation are

$$\frac{dp}{4p(px+z)} = \frac{dq}{2q(px+z)} = \frac{dz}{(-q)(-1) - 2p(px+z)x}$$

$$= \frac{dx}{-2x(px+z)} = \frac{dy}{1} = \frac{df}{0}$$

Taking the first and fourth term we have

$$\frac{dq}{q} + \frac{dx}{x} = 0$$

$\therefore \quad \log q + \log x = \log a$

or $\quad q = \frac{a}{x}$

$\therefore$ Putting $q = ax$ in the given equation,

$$\frac{a}{x} = (px + z)^2$$

or $$\sqrt{\frac{a}{x}} = px + z$$

$$p = \frac{\sqrt{a}}{x^{3/2}} - \frac{z}{x}$$

Now substituting these values of p and q in $dz = pdx + qdy$, we have

$$dz = \frac{a}{x}dy + \left(\frac{\sqrt{a}}{x^{3/2}} - \frac{z}{x}\right)dx$$

$$x\,dz + z\,dx = a\,dy + \frac{\sqrt{a}}{\sqrt{x}}dx$$

Integrating, $xz = ay + 2\sqrt{ax} + b$ which is the complete integral.

Q.31 ***Solve :***

(i) $(x + 2z)p + (4zx - y)q = 2x^2 + y$

(ii) $pq = x^m y^n z^{2l}$

(iii) ***Solve in series :***

$$x^2\frac{d^2y}{dx^2} + (x + x^2)\frac{dy}{dx} + (x - 9)y = 0$$

[R.T.U. 2015]

Ans. (i) Here $P = x + 2z, Q = 4zx - y, R = 2x^2 + y$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{x+2z} = \frac{dy}{4zx - y} = \frac{dz}{2x^2 + y} \qquad ...(1)$$

Now choosing $2x, -1, -1$ as multipliers, we have each fraction

$$= \frac{2xdx - dy - dz}{2x(x+2z) - (4zx - y) - (2x^2 + y)}$$

$$= \frac{2xdx - dy - dz}{0}$$

$\therefore \quad 2xdx - dy - dz = 0$ integrating, we get

$x^2 - y - z = a \qquad ...(2)$

Again, choosing $y, x, -2z$ as multipliers, we have each fraction

$$= \frac{ydx + xdy - 2zdz}{y(x+2z) + x(4zx - y) - 2z(2x^2 + y)}$$

$$= \frac{ydx + xdy - 2zdz}{0}$$

$\therefore \quad y\,dx + x\,dy - 2z\,dz = 0$

$\Rightarrow d(xy) - 2z\,dz = 0$

Integrating, we get $xy - z^2 = b \qquad ...(3)$

From eq. (2) and (3) the required general solution is $\phi(x^2-y-z, xy-z^2) = 0$, where ϕ is an arbitrary function.

Ans. (ii) We have $pq = x^m y^n z^{2l} \qquad ...(1)$

$$\Rightarrow \frac{z^{-2l}}{x^m y^n}pq = 1$$

$$\Rightarrow \left(\frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}\right) = 1 \qquad ...(2)$$

Let X, Y, Z be new variables such that

$dX = x^m dx, dY = y^n dy, dZ = z^{-l}dz$

$\therefore$ By using integration, we have

$$X = \frac{x^{m+1}}{m+1}, Y = \frac{y^{n+1}}{n+1}, Z = \frac{z^{1-l}}{1-l}$$

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = z^{-l}\cdot\frac{\partial z}{\partial x}\cdot\frac{1}{x^m}$$

$$= \frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}$$

and $$Q = \frac{\partial Z}{\partial Y} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial y}\cdot\frac{dy}{dY} = z^{-l}\cdot\frac{\partial z}{\partial y}\cdot\frac{1}{y^n}$$

$$= \frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}$$

$\therefore \quad$ By eq. (2) $PQ = 1 \qquad ...(3)$

This equation is of the form $g(P,Q) = 0$

Let $z = aX + \phi(a)Y + c \qquad ...(4)$

be the complete solution of (3), where $\phi(a)$ is so[me] function of a.

By eq. (4) $\Rightarrow P = \frac{\partial Z}{\partial X} = a$ and $Q = \frac{\partial Z}{\partial Y} = \phi(a)$

$\therefore \quad$ By eq. (3) $\Rightarrow a\,\phi(a) = 1$ or $\phi(a) = 1/a$

$\therefore \quad$ The complete solution of eq. (3) is $Z = aX + \frac{1}{a}Y + c$

$\therefore \quad$ The complete solution of eq. (1) is

$$\frac{z^{1-l}}{1-l} = \frac{a}{m+1}x^{m+1} + \frac{1}{(n+1)a}y^{n+1} + c,$$ where a and c arbitrary constants and $a \neq 0$.

Ans. (iii) Given $x^2\frac{d^2y}{dx^2} + (x + x^2)\frac{dy}{dx} + (x - 9)y = 0 \quad ...(1)$

Here $x = 0$ is a regular singular point. So

Let $y = \sum_{r=0}^{\infty} a_r x^{m+r}$ be the solution of the given differen[tial] equation.

$$\therefore \quad \frac{dy}{dx} = \sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-2}$$

Putting the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq. (1), we

$$x^2\sum_{r=0}^{\infty}a_r(m+r)(m+r+1)x^{m+r-2}$$

$$+ (x + x^2)\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1} + (x-9)\sum_{r=0}^{\infty}a_r x^{m+r}$$

or $\sum_{r=0}^{\infty} a_r\{(m+r)(m+r-1)(m+r)-9\}x^{m+r}$

$$+\sum_{r=0}^{\infty} a_r\{(m+r)+1\}x^{m+r+1}=0$$

or $\sum_{r=0}^{\infty} a_r\{(m+r-3)(m+r+3)x^{m+r}\}$

$$+\sum_{r=0}^{\infty} a_r(m+r+1)x^{m+r+1}=0$$

Equating to zero the coefficient of lowest power of x i.e. x^{m+r}, we get

$$a_r(m+r-3)(m+r+3)+a_{r-1}(m+r)=0$$

or $a_r=-\frac{(m+r)}{(m+r-3)(m+r+3)}a_{r-1}$ ...(2)

Taking $m=-3$

$$a_r=-\frac{(r-3)}{(r-6)r}a_{r-1} \quad ...(3)$$

Therefore for $r=1, 2, 3, 4, 5, 6...$ from eq. (3), we have

$$a_1=\frac{2}{5}a_0,\ a_2=-\frac{1}{8}a_1=\frac{2}{5}\cdot\frac{1}{8}a_0,$$

$a_3=0,\ a_4=0,\ a_5=0$, but $a_6=\frac{0}{0}$ (because $a_5=0$)

[indeterminate form]

Hence $a_7=-\frac{4}{7}a_6,\ a_8=-\frac{5}{16}a_7=+\frac{5}{16}\cdot\frac{4}{7}a_6$ and so on.

$\therefore$ The general solution of eq. (1) is

$$y=x^m[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+...]$$

$$y=x^{-3}\left[a_0-\frac{2}{5}a_0x+\frac{2}{5}\cdot\frac{1}{8}a_0x^2+0x^3+0x^4+0x^5\right]$$

$$+x^{-3}\left[a_6x^6+\left(-\frac{4}{7}a_6\right)x^7+\frac{5}{16}\cdot\frac{4}{7}a_6x^8+...\right]$$

$$\Rightarrow\ y=a_0x^{-3}\left(1-\frac{2}{5}x+\frac{2}{5}\cdot\frac{1}{8}x^2\right)$$

$$+a_6x^{-3}\left(x^6-\frac{4}{7}x^7+\frac{4.5}{7.16}x^8...\right)$$

or $y=a_0x^{-3}\left(1-\frac{2}{5}x+\frac{2}{5}\cdot\frac{1}{8}x^2\right)$

$$+a_6x^3\left(1-\frac{4}{7}x+\frac{4.5}{7.16}x^2...\right)$$

This contains two arbitrary constants a_0 and a_6, hence may be taken as complete solution.

Note : If we substitute $r=3$ in eq.(3), we get

$$y=a_0x^3\left(1-\frac{4}{7}x+\frac{4.5}{7.16}x^2...\right)$$

which gives no new independent solution.

Q.32 *(a) Solve in series :*

$$4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$$

[R.T.U. 2015]

(b) Solve :

(i) $(y^2+z^2-x^2)p-2xyq=-2xz$ [R.T.U. 2015, 14]

(ii) $z=p^2x+q^2y$ [R.T.U. 2015]

Ans. (a) Given

$$4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0 \quad ....(i)$$

Here $P_0(x)=0$ at $x=0$, so let $y=\sum_{r=0}^{\infty}a_rx^{m+r}$ is the solution of the given differential equation.

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}(m+r)a_rx^{m+r-1},\ \frac{d^2y}{dx^2}=[(m+r)(m+r-1)a_rx^{m+r-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$4x\sum_{r=0}^{\infty}(m+r)(m+r-1)a_rx^{m+r-2}$$

$$+2\sum_{r=0}^{\infty}(m+r)a_rx^{m+r-1}+\sum_{r=0}^{\infty}a_rx^{m+r}=0$$

equating the coefficient of lowest power of x to zero, by putting $r=0$, we have

$$[4(m)(m-1)+2m]a_0=0 \quad \because a_0\neq 0$$

$$\Rightarrow\quad 4m(m-1)+2m=0$$

$$\Rightarrow\quad 4m^2-2m=0\Rightarrow m=0, \frac{1}{2}$$

Again equating the coefficient of x^{m+r} by putting $r=r+1$ in the first expression, we have

$$\sum_{r=0}^{\infty}[4(m+r+1)(m+r)+2(m+r+1)]a_{r+1}+\sum_{r=0}^{\infty}a_r=0$$

$$\Rightarrow a_{r+1}=\frac{-a_r}{4(m+r+1)(m+r)+2(m+r+1)}$$

$$=\frac{-a_r}{2(m+r+1)(2m+2r+1)}\rightarrow \text{Recurrence relation}$$

Putting $r=0, 1, 2, 3, ...$ we have

$$a_1=\frac{-a_0}{2(m+1)(2m+1)}$$

$$a_2=\frac{-a_1}{2(m+2)(2m+3)}=\frac{a_0}{4(m+1)(m+2)(2m+1)(2m+3)}$$

$$a_3=\frac{-a_2}{2(m+3)(2m+5)}$$

$$=-\frac{-a_0}{8(m+1)(m+2)(m+3)(2m+1)(2m+3)(2m+5)}$$

$\therefore$ the general solution is

$$y=x^m\left[a_0-\frac{a_0x}{2(m+1)(2m+1)}+\frac{a_0x^2}{4(m+1)(m+2)(2m+1)(2m+3)}\right.$$

$$\left.-\frac{a_0x^3}{8(m+1)(m+2)(m+3)(2m+1)(2m+3)(2m+5)}+...\right]$$

Putting $m=0$, we will get the first solution as

$$y_I=a_0\left[1-\frac{x}{2}+\frac{x^2}{24}-\frac{x^3}{720}+....\right]$$

Putting $m=\frac{1}{2}$, we have the second solution.

$$y_{II}=x^{1/2}a_0\left[1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+...\right]$$

Therefore, the complete solution is

$$y=A\left[1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+....\right]+Bx^{1/2}\left[1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+...\right]$$

Ans. (b) (i) Here the Lagrange's auxiliary equations of the given equation are

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz} \quad ...(i)$$

Taking the last two fractions of eq. (i), we have

$$(1/y)dy=(1/z)dz \text{ so that } (1/y)dy-(1/z)dz=0$$

Integrating, $\log y-\log z=\log c_1$ or $y/z=c_1$ ...(ii)

Choosing x, y, z as multipliers, each fraction of eq. (i) will reduce to

$$=\frac{x\,dx+y\,dy+z\,dz}{xy^2+xz^2-x^3-2xy^2-2xz^2}=\frac{x\,dx+y\,dy+z\,dz}{-x(x^2+y^2+z^2)} \quad ...(iii)$$

Combining the third fraction of eq. (i) with fraction eq. (iii), we have

$$\frac{x\,dx+y\,dy+z\,dz}{-x(x^2+y^2+z^2)}=\frac{dz}{-2xz}$$

or $\frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}-\frac{dz}{z}=0$

Integrating, $\log(x^2+y^2+z^2)-\log z=\log c_2$

or $(x^2+y^2+z^2)/z=c_2$ ...(iv)

From eq. (ii) and eq. (iv), the required general solution is given by

$\phi\left(y/z,(x^2+y^2+z^2)/z\right)=0$, ϕ being an arbitrary function.

(ii) Let $f(x, y, z, p, q)=p^2x+q^2y-z=0$

Now $\frac{\partial f}{\partial x}=p^2;\ \frac{\partial f}{\partial p}=2px$

$$\frac{\partial f}{\partial y}=q^2;\ \frac{\partial f}{\partial q}=2qy$$

$$\frac{\partial f}{\partial z}=-1$$

Consider Charpit's auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-\left(p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}\right)}$$

$$=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}$$

$$\Rightarrow\frac{dp}{p^2+(-p)}=\frac{dq}{q^2-q}=\frac{dz}{-(2p^2x+2q^2y)}$$

$$=\frac{dx}{-2px}=\frac{dy}{-2qy}=\frac{dF}{0}$$

Consider

$$\frac{p^2dx+2px\,dp}{-2p^3x+2p^3x-2p^2x}=\frac{q^2dy+2qy\,dy}{-2q^3y+2q^3y-2q^2y}$$

$$\Rightarrow\frac{p^2dx+2px\,dp}{p^2x}=\frac{q^2dy+2qy\,dy}{q^2y}$$

Integrating $\int\frac{p^2dx+2px\,dp}{p^2x}=\int\frac{q^2dy+2qy\,dy}{q^2y}$

$$\Rightarrow\log(p^2x)=\log(q^2y)+\log c=\log q^2yc$$

$$\left[\int\frac{f'(x)}{f(x)}dx=\log f(x)\right]$$

$$p^2x = cq^2y \quad ...(1)$$

$$p^2x + q^2y = z$$

$$cq^2y + q^2y = z$$

$$(c+1)q^2y = z$$

$$\Rightarrow q^2 = \frac{z}{y(c+1)}$$

$$q = \sqrt{\frac{z}{y(c+1)}}$$

Hence from eq. (1),

$$p^2x = cy\frac{z}{y(c+1)} = \frac{c}{c+1}z$$

$$\Rightarrow p = \sqrt{\frac{c}{c+1}\cdot\frac{z}{x}}$$

Now consider $dz = pdx + qdy$

$$dz = \sqrt{\frac{c}{c+1}\cdot\frac{z}{x}}dx + \sqrt{\frac{1}{c+1}\cdot\frac{z}{y}}dy$$

$$z^{-1/2}dz = \sqrt{\frac{c}{c+1}}x^{-1/2}dx + \sqrt{\frac{c}{c+1}}y^{-1/2}dy$$

Integrating

$$2\sqrt{z} = \sqrt{\frac{c}{c+1}}2\sqrt{x} + \frac{1}{\sqrt{c+1}}2\sqrt{y} + b$$

$$\Rightarrow \sqrt{c+1}\sqrt{z} = \sqrt{cx} + \sqrt{y} + \frac{b}{2\sqrt{c+1}} \text{ (say)}$$

$$\left[\text{Take } a = \frac{b}{2\sqrt{c+1}}\right]$$

$\Rightarrow \sqrt{c+1}\sqrt{z} = \sqrt{cx} + \sqrt{y} + a$, is the required complete solution.

Q.33 (a) *Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$*

[R.T.U. 2014, 2013, July 2011, 2008]

(b) *Solve $z = px + qy + C\sqrt{(1+p^2+q^2)}$*

[R.T.U. 2014, July 2011, MREC 2000]

(c) *Find the complete integral of $2(z+xp+yq) = yp^2$*

[R.T.U. 2014]

OR

Solve the Charpit's method $2(z + px + qy) = p^2y$

[RTU June/July - 2011]

Ans.(a) Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad ...(i)$$

Choosing $1/x, 1/y, 1/z$ as multipliers, each fraction of eq. (i)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - (x^2+z) + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow \quad (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that

$$\log x + \log y + \log z = \log c_1$$

$$\Rightarrow \quad \log(xyz) = \log c_1 \quad \Rightarrow \quad xyz = c_1 \quad ...(ii)$$

Choosing $x, y, -1$ as multipliers, each fraction of eq. (i)

$$= \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{x\,dx + y\,dy - dz}{0}$$

$\Rightarrow x\,dx + y\,dy - dz = 0$ so that $x^2 + y^2 - 2z = c_2$ ...(iii)

From eq. (ii) and eq. (iii), the required general solution is

$\phi(x^2 + y^2 - 2z, xyz) = 0$, ϕ is being an arbitrary function.

Ans.(b) The complete integral (C.I.) is

$$z = ax + by + C\sqrt{(1+a^2+b^2)} \quad ...(i)$$

To Find Singular Integral :

Differentiating eq. (i) partially *w.r.t.* 'a' and 'b', we get

$$0 = x + \frac{Ca}{\sqrt{(1+a^2+b^2)}} \quad ...(ii)$$

$$0 = y + \frac{bC}{\sqrt{(1+a^2+b^2)}} \quad ...(iii)$$

$$\therefore \quad x^2 + y^2 = \frac{C^2(a^2+b^2)}{(1+a^2+b^2)}$$

or $$C^2 - (x^2+y^2) = C^2 - \frac{C^2(a^2+b^2)}{(1+a^2+b^2)} = \frac{C^2}{1+a^2+b^2}$$

or $$1 + a^2 + b^2 = \frac{C^2}{C^2 - (x^2+y^2)}$$

or $$\frac{1}{\sqrt{(1+a^2+b^2)}} = \frac{\sqrt{(C^2-x^2-y^2)}}{C}$$

$$\therefore \quad x = -\frac{aC}{\sqrt{(1+a^2+b^2)}} = -aC\cdot\frac{\sqrt{(C^2-x^2-y^2)}}{C}$$

$$= -a\sqrt{(C^2-x^2-y^2)}$$

$$\Rightarrow \quad a = -\frac{x}{\sqrt{(C^2-x^2-y^2)}}. \text{ Similarly}$$

$$b = -\frac{y}{\sqrt{(C^2-x^2-y^2)}}$$

Now putting these values of a and b in eq. (i), we get

$$z = -\frac{x^2}{\sqrt{(C^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(C^2-x^2-y^2)}} + \frac{C^2}{\sqrt{(C^2-x^2-y^2)}}$$

or $$z = \sqrt{(C^2-x^2-y^2)}$$

or $$x^2 + y^2 + z^2 = C^2.$$

Ans.(c) We have

$$f \equiv 2(z + xp + yq) - yp^2 = 0 \quad .... (i)$$

Charpit's auxiliary equations are

$$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p(2x-2yp)-2py}$$

$$= \frac{dx}{-2x+2yp} = \frac{dy}{-2y}$$

Taking $\dfrac{dp}{4p} = \dfrac{dy}{-2y}$

or $\dfrac{dp}{p} + 2\dfrac{dy}{y} = 0$, we have

$py^2 = a$. (on integration)

Putting $p = \dfrac{a}{y^2}$ in eq.(i), we have

$$z + \frac{ax}{y^2} + yq = \frac{a^2}{2y^3}$$

$$\therefore \quad q = -\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}$$

Putting the values of p and q in $1dz = pdx + qdy$, we get

$$dz = \frac{a}{y^2}dx - \left(\frac{z}{y} + \frac{ax}{y^3} - \frac{a^2}{2y^4}\right)dy,$$

or $$ydz + zdy = a\left(\frac{ydx - xdy}{y^2}\right) + \frac{a^2}{2y^3}dy$$

Integrating we get

$$yz = \frac{ax}{y} - \frac{a^2}{4y^2} + b \quad \text{or} \quad z = \frac{ax}{y^2} - \frac{a^2}{4y^3} + \frac{b}{y}$$

which is the complete integral.

Q.34 (a) *Solve in series : $(2x+x^3)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0$*

[RTU June/July - 2

(b) *Solve the partial differential equation*

$$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$$

[RTU June/July - 2011. Raj. Univ. 2

Ans. (a) Since $x = 0$ is a regular singular point, We assume the solution in the form

$$y = \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty}(m+r)a_r x^{m+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-2}$$

Putting these values in the given equation, we get

$$(2x+x^3)\left[\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r-2}\right]$$

$$-\left[\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}\right] - 6x\sum_{r=0}^{\infty}a_r x^{m+r} = 0$$

or $$\sum_{r=0}^{\infty}a_r[(m+r)(2m+2r-3)]x^{m+r-1} + \sum_{r=0}^{\infty}a_r(m+r-3)$$

$$(m+r+2)x^{m+r+1} = 0 \quad ...(i)$$

Which is an identity.

Now, equating to zero, the coefficient of the smallest power of x, *i.e.* x^{m-1} (put $r = 0$), then the equation gives indicial equation or graduate equation in m.

$$a_0 m(2m-3) = 0$$

$$m = 0 \text{ or } 3/2 \text{ as } a_0 \neq 0$$

So, the roots of the indicial equation are unequal and differ by an integer. Putting $r = 1$ in the first summation gives the next lowest degree terms. Its coefficient is $(m+1)(2m-1)a_1 = 0$ *i.e.*, $a_1 = 0$.

Again equating the coefficient of x^{m+r-1} to zero. We have

$$a_r(m+r)(2m+2r-3) + a_{r-2}(m+r-5)(m+r)$$

Which implies that

$$a_r = \frac{-(m+r-5)}{(2m+2r-3)}a_{r-2} \quad ...(ii)$$

Since $a_1 = 0$, this gives $a_3 = 0$ and consequently $a_5 = a_9 = a_{11} = ... = 0$.

From equation (ii), we find

$$a_2 = \frac{-(m-3)}{(2m+1)}a_0$$

$$a_4 = \frac{-(m-1)}{(2m+5)}a_2 = \frac{(m-1)(m-3)}{(2m+1)(2m+5)}a_0$$

$$a_6 = \frac{-(m-1)}{(2m+9)}a_4 = \frac{(m-1)(m-3)(m+1)}{(2m+1)(2m+5)(2m+9)}a_0$$

For $m=0$

$$a_2 = 3a_0$$

$$a_4 = \frac{3}{5}a_0$$

$$a_6 = \frac{3}{45}a_0$$

So, the first solution is

$$y_1 = a_0\left(1+3x^2+\frac{3}{5}x^4-\frac{3.1}{5.9}x^6+\frac{3.1.3}{5.9.13}x^8 \ldots\right)$$

For $m = 3/2$,

$$a_2 = \frac{3}{8}a_0$$

$$a_4 = -3\cdot\frac{1}{8}\cdot\frac{1}{16}a_0$$

Therefore the second solution is

$$y_2 = a_0 x^{3/2}\left(1+\frac{3}{8}x^2-\frac{3.1}{8.16}x^4+\frac{3.1.5}{8.16.24}x^6 \ldots\right)$$

Hence, the general solution is $A\,y_1 + B\,y_2$.

Ans. (b) Let us compare the given equation with

$$Pp + Qq = R \quad \ldots (i)$$

$$P = z^2 - 2yz - y^2,\ Q = xy + zx,\ R = xy - zx$$

$\therefore$ The subsidiary equation

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx}$$

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx} = \frac{xdx+ydy+zdz}{0}$$

By taking last two factors

$$\frac{xdx+ydy+zdz}{0} = \frac{dz}{xy-zx}$$

$$xdx + ydy + zdz = 0 \quad \ldots (ii)$$

$$x^2 + y^2 + z^2 = C_1$$

By

$$\frac{dy}{xy+zx} = \frac{dz}{xy-zx}$$

$$\frac{dy}{dz} = \frac{y+z}{y-z} \quad \ldots (iii)$$

Let $y + z = u$ & $y - z = v$

$$y = \frac{u+v}{2} \ \&\ z = \frac{u-v}{2}$$

$$\therefore\ dy = \frac{1}{2}(du+dv) \ \&\ dz = \frac{1}{2}(du-dv)$$

so from eq.(iii), we have

$$\frac{du+dv}{du-dv} = \frac{u}{v}$$

or $vdu + vdv = udu - udv$

$udu - vdv = udv + vdu$

$udu - vdv = d(uv)$

$$\frac{u^2}{2} - \frac{v^2}{2} = uv - \frac{C_2}{2}$$

$$u^2 - v^2 - 2uv = C_2$$

$$(y+z)^2 - (y-z)^2 - 2(y+z)(y-z) = C_2$$

$$y^2 + z^2 = 2yz - y^2 - z^2 + 2yz - 2y^2 + 2z^2 = C_2$$

$$z^2 - y^2 + 4yz = C_2$$

Hence the solution is

$$\phi(C_1, C_2) = 0$$

$$\Rightarrow \phi\,[x^2 + y^2 + z^2,\ z^2 - y^2 + 4yz] = 0.$$

Q.35 (a) Solve $q^2y^2 = z(z - px)$ **[RTU June/July - 2011]**

(b) Solve $x^2y^3p^2q = z^3$

[RTU June/July - 2011, July 2011, Raj. Univ. 2003]

Ans. (a) The given equation is

$$q^2y^2 + zpx = z^2$$

$$\text{or}\quad zx\left(\frac{\partial z}{\partial x}\right) + y^2\left(\frac{\partial z}{\partial y}\right)^2 = z^2$$

$$\text{or}\quad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right) + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$$

This is now of the form $P + Q^2 = 1$

Put $X = \log x$, $Y = \log y$, $Z = \log z$

$$dX = \frac{1}{x}dx,\ dY = \frac{1}{y}dy,\ dZ = \frac{1}{z}dz$$

$$\left(\frac{dZ}{dX}\right) + \left(\frac{dZ}{dY}\right)^2 = 1$$

$$P + Q^2 = 1$$

where $P = \dfrac{dZ}{dX}$ and $Q = \dfrac{dZ}{dY}$

Therefore its complete integral is

$$Z = aX + bY + C$$

where $a + b^2 = 1$

$$b = \sqrt{1-a}$$

$$\log z = a\log x + b\log y + \log c$$

If we put $a = \cos^2\alpha$

$$b = \sqrt{1-a}$$

$$= \sqrt{1-\cos^2\alpha} = \sin\alpha$$

$$\log z = \cos^2\alpha\log x + \sin\alpha\log y + \log c$$

$$\log z = \log x^{\cos^2\alpha} + \log y^{\sin\alpha} + \log c$$

$$\boxed{Z = x^{\cos^2\alpha}\cdot y^{\sin\alpha}\cdot c}$$

Ans. (b) Given equation is $x^2y^3p^2q = z^3$

Putting $u = \log z$, $\dfrac{\partial u}{\partial x} = \dfrac{1}{z}\dfrac{\partial z}{\partial x} = p$, $\dfrac{\partial u}{\partial y} = \dfrac{1}{z}\dfrac{\partial z}{\partial y} = q$

$$\therefore\ x^2y^3\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2\left(\frac{1}{z}\frac{\partial z}{\partial y}\right) = 1$$

$$\text{or}\quad x^2y^3\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial u}{\partial y}\right) = 1$$

$$\text{or}\quad x^2p^2 = \frac{1}{y^3q} = a \text{ (say)}$$

$$\text{or}\quad p = \frac{\sqrt{a}}{x};\ q = \frac{1}{ay^3} \text{ (say)}$$

Now $$du = pdx + qdy = \frac{\sqrt{a}}{x}dx + \frac{1}{ay^3}dy$$

$$u = \sqrt{a}\log x - \frac{1}{2ay^2} + b$$

$$\log z = \lambda\log x - \frac{1}{2\lambda^2y^2} + b$$

□□□

Partial Differential Equations - Higher Order

5

Previous Years Questions

Part A

Q.1 ***Classify the following PDE as to type in the second quadrant of the xy - plane***

$$\sqrt{x^2+y^2}\,u_{xx}+2(x-y)u_{xy}+\sqrt{x^2+y^2}\,u_{yy}$$

[R.T.U. 2019]

Ans. Rewritting the given equation, we get

$$\sqrt{x^2+y^2}\,r+2(x-y)s+\sqrt{x^2+y^2}\,t \qquad \text{... (1)}$$

Comparing equation (1) with the following expression,

$Rr + Ss + Tt + f(x, y, u, p, q) = 0$

We get, $R = \sqrt{x^2+y^2}$; $S = 2(x-y)$; $T = \sqrt{x^2+y^2}$

So,

$$S^2-4RT=\left[2(x-y)\right]^2-4\left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+y^2}\right)$$

$= 4(x-y)^2 - 4(x^2+y^2)$

$= 4[x^2-2xy+y^2] - 4x^2 - 4y^2$

$= -8xy$

The equation is elliptic if, $S^2 - 4RT < 0$ i.e. $xy > 0$

The equation is parabolic if, $S^2 - 4RT = 0$ i.e. $xy = 0$

The equation is hyperbolic if, $S^2 - 4RT > 0$ i.e. $xy < 0$

Q.2 ***Write Laplace equation in two dimensions?***

Ans. Laplace equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Q.3 ***Write one dimensional wave equation?***

Ans. One dimensional wave equation :

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

Q.4 ***Write one dimensional heat flow equation?***

Ans. One dimensional heat flow equation

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

Q.5 ***Let $u(x, t)$ be a solution of the wave equation***
$u_{tt} - c^2 u_{xx} = 0$
which is defined in the whole plane. Assume that u is constant on the line $x = 2 + ct$.
Prove that $u_t + cu_x = 0$.

Ans. The solution $u(x, t)$ has the form $u(x, t) = F(x + ct) + G(x - ct)$. Since $u(2 + ct, t)$ = constant, it follows that

$$F(2 + 2ct) + G(2) = \text{constant.}$$

Set $s = 2 + 2ct$, we have $F(s)$ = constant. Consequently $u(x, t) = G(x - ct)$.

Computing now the expression $u_t + cu_x$, we obtain

$$u_t + cu_x = -cG'(x - ct) + cG'(x - ct) = 0.$$

Part B

Q.6 ***Describe the classification of second order partial differential equations?***

Ans. Classification of a second order PDE

A second order linear partial differential equation in two independent variables x and y in its general form is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad \text{...(i)}$$

where A, B, C, D, E, F and G are constants or functions of the variables x and y. A second order partial differential equation (i) is usually classified into three basic classes of equations namely:

1. Parabolic : Parabolic equation is an equation that satisfies the property

$$B^2 - 4AC = 0 \qquad \text{...(ii)}$$

Examples of parabolic equations are heat flow and diffusion processes equations. The heat transfer equation

$$u_t = ku_{xx} \qquad \text{...(iii)}$$

2. Hyperbolic : Hyperbolic equation is an equation that satisfies the property

$$B^2 - 4AC > 0 \qquad \text{...(iv)}$$

Examples of hyperbolic equations are wave propagation equations. The wave equation is

$$u_{tt} = c^2 u_{xx} \qquad \text{...(v)}$$

3. Elliptic : Elliptic equation is an equation that satisfies the property

$$B^2 - 4AC < 0 \qquad \text{...(vi)}$$

Examples of elliptic equation are Laplace's equation and Schrodinger equation. The Laplace equation in a two dimensional space is

$$u_{xx} + u_{yy} = 0 \qquad \text{...(vii)}$$

The Laplace's equation is often called the potential equation because $u(x, y)$ defines the potential function.

Q.7 ***Classify the following second order partial differential equations as parabolic, hyperbolic or elliptic:***

(a) $u_t = 4u_{xx}$

(b) $u_{tt} = 4u_{xx}$

(c) $u_{xx} + u_{yy} = 0$

Ans.(a) $u_t = 4u_{xx}$ ...(i)

$A = 4, B = C = 0$

This means that

$$B^2 - 4AC = 0.$$

Hence, the equation (i) is parabolic.

(b) $u_{tt} = 4u_{xx}$ ...(ii)

$A = 4, B = 0, C = -1$

This means that

$$B^2 - 4AC = 16 > 0.$$

Hence, the equation (ii) is hyperbolic.

(c) $u_{xx} + u_{yy} = 0$ ...(iii)

$A = 1, B = 0, C = 1$

This means that

$$B^2 - 4AC = -4 < 0.$$

Hence, the equation (iii) is elliptic.

Q.8 ***A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $u(x, 0) = u_0 \sin^3(\pi x/l)$. If it is released from reset from this position, find the displacement $u(x, t)$.***

Ans. The displacement function $u(x, t)$ is given by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}, 0<x<l, t>0, \qquad \text{...(i)}$$

With boundary conditions

$$u(0, t) = 0,\ u(l, t) = 0,\ t \geq 0, \qquad \text{...(ii)}$$

And initial conditions

$$u(x,0) = u_0\sin^3(\pi x/l);\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\ 0 \leq x \leq l. \qquad \text{...(iii)}$$

The solution of the one-dimensional wave equation (i) subject to the boundary conditions (ii) and the initial condition

$\left.\frac{\partial u}{\partial t}\right|_{t=0}=0$, is given by

$$u(x,t)=\sum_{n=1}^{\infty} C_n \cos\frac{n\pi ct}{l}\sin\left(\frac{n\pi x}{l}\right), \quad ...(iv)$$

Where C_n are constants to be determined using the second initial condition $u(x, 0) = u_0 \sin^3 (\pi x/l)$ given in (iii). Using this eq. (iv) becomes

$$u_0\sin^3(\pi x/l)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}$$

Or, $$\frac{u_0}{4}\left[3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right]=\sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l}. \quad ...(v)$$

Comparing the coefficients of the like terms on both sides of (v) gives

$$C_1=\frac{3u_0}{4}, C_2=0, C_3=-\frac{u_0}{4}, C_4=C_5=...=0$$

And hence, (iv) gives

$$u(x,t)=\frac{u_0}{4}\left[3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l}-\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right],$$

the desired solution.

Q.9 ***Solve by the method of separation of variables :***

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$$

Ans. Let us consider the solution is

$$u = X(x)\, Y(y) \quad ...(i)$$

Substituting this value of u in the given equation, we have,

$$X''y - 2X'Y + Xy' = 0$$

Separation of variables,

$$\frac{x''-2x'}{x}=-\frac{y'}{y} \quad ...(ii)$$

Since x and y are independent variables, therefore, (ii) be true if each side of (ii) is equal to some constant value, say, a

$$\Rightarrow \quad \frac{x''-2x'}{x}=a$$

and $$-\frac{y'}{y}=a$$

$$\Rightarrow \quad x''-2x'-ax=0 \quad ...(iii)$$

$$y'+ay=0 \quad ...(iv)$$

For equation (iii) we have auxiliary equation

$$m^2-2m-a=0$$

$$m=1\pm\sqrt{1+a}$$

⇒ Solution of the equation (iii) is

$$X=c_1e^{x(1+\sqrt{1+a})}+c_2e^{x(1-\sqrt{1+a})}$$

and solution of (iv)

$$\frac{y'}{y}+a=0$$

$$\log y + a = \log c_3$$

$$y = C^3 e^{-ay}$$

Hence

$$u=\left[c_1e^{x(1+\sqrt{1+a})}+c_2e^{x(1-\sqrt{1+a})}\right]c_3e^{-ay}$$

or $$u=\left[n_1e^{x(1+\sqrt{1+a})}+n_2e^{x(1-\sqrt{1+a})}\right]e^{-ay}$$

Which is the required complete solution of the given partial differential equation.

Q.10 ***Using the method of separation of variables :***

Solve $\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u,\ u(x,0)=6e^{-3x}$

Ans. The given differential equation is

$$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u \quad ...(i)$$

Let us consider the solution is

$$u = X(x)\, T(t)$$

Substituting in the given differential equation.

$$x'T = 2xT' + xT$$

or $(x'-x)T = 2xT'$

or $$\frac{x'-x}{2x}=\frac{T'}{T}=a \text{ (Say)}$$

$$\Rightarrow \quad X'-X=2a \text{ and } \frac{T'}{T}$$

or $$\frac{X'}{X}=1+2a$$

$$\log x = (1+2a)x+\log c$$

or $$x = ce^{(1+2a)x}$$

$$\frac{T'}{T}=a$$

$$\log T = at + \log c_1$$

or $$T = c_1 e^{at}$$

$$\Rightarrow \quad u = XT = cc_1e^{(1+2a)x}e^{at}$$

but at $t = 0$

$$6e^{-3x}=cC_1e^{(1+2a)x}$$

$$\Rightarrow \quad cc_1 = 6$$

$$= 2a+1=-3,\ a=-2$$

Combining all value, we have

$$u = XT = 6e^{-3x-2t}$$

$$= 6e^{-(3x+2t)}$$

Which is required solution of the given differential equation.

Q.11 ***Classify the following second order partial differential equations as parabolic, hyperbolic or elliptic:***

(a) $u_{tt} = u_{xx} - u_t$

(b) $u_t = u_{xx} - u_x$

(c) $u_{xx} - xu_{yy} = 0$

Ans.(a) $A = 1, B = 0, C = -1$

This means that $B^2 - 4AC = 4 > 0$.

Hence, the equation in (a) is hyperbolic.

(b) $A = 1, B = C = 0$

This means that

$$B^2 - 4AC = 0$$

Hence, the equation in (b) is parabolic.

(c) $A = 1, B = 0, C = x$

This means that

$$B^2 - 4AC = -4x$$

Hence, the equation in (a) is parabolic if $x = 0$, hyperbolic if $x < 0$, and elliptic if $x > 0$.

PART C

Q.12 ***Use the method of separation of variables to solve the following PDE:*** $\frac{\partial^2 z}{\partial x^2}=\frac{1}{k}\frac{\partial z}{\partial t}$, ***where*** $z = z(x, t)$ ***with the conditions*** $z(0, t) = z(l, t) = 0$ ***for all t.*** [R.T.U. 2019]

Ans. Let the solution of given differential equation is

$$Z = X(x)\, T(t) \quad ...(1)$$

$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=X'T\\ \frac{\partial^2 z}{\partial x^2}&=X''T\\ \frac{\partial z}{\partial t}&=XT'\end{aligned}\right\} \quad ...(2)$$

substituting equation (2) in given differential equation,

$$X''T=\frac{1}{k}XT'\ \lambda \text{ (say)}.$$

$$\frac{X''}{X}=\frac{1}{k}\frac{T'}{T}=\lambda$$

on taking $\frac{X''}{X}=\lambda$

or $$\frac{d^2X}{dx^2}-\lambda X=0$$

or $(D^2-\lambda)x=0$

A.E.

$$m^2-\lambda=0$$

$$m=\pm\sqrt{\lambda}$$

C.F.

$$X(x) = c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x} \quad \ldots (3)$$

on taking $\frac{1}{k}\frac{T'}{T} = \lambda$

or $\frac{dT}{dt} - k\lambda T = 0$

or $\frac{dT}{T} - k\lambda dt = 0$

on integration

$\log T - k\lambda t = \log c$

$\log\frac{T}{c} = k\lambda t$

$$T = ce^{k\lambda t} \quad \ldots (4)$$

Putting equation (3) and equation (4) in equation (1), we get

$$Z = \left(c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x}\right)\cdot ce^{k\lambda t}$$

$$Z = Ae^{\sqrt{\lambda}x+k\lambda t} + Be^{-\sqrt{\lambda}x+k\lambda t} \quad \ldots (5)$$

Where, $A = cc_1$; $B = cc_2$

Using conditions,

$Z(0, t) = 0$ and $z(1, t) = 0$ in equation (5), we get

$$0 = Ae^{k\lambda t} + Be^{k\lambda t} \quad \ldots (6)$$

$$\text{and } 0 = Ae^{\sqrt{\lambda}+k\lambda t} + Be^{-\sqrt{\lambda}+k\lambda t} \quad \ldots (7)$$

on solving equation (6) and equation (7), we get

$$B = e^{2\sqrt{\lambda}} \text{ and } A = -e^{-2\sqrt{\lambda}} \quad \ldots (8)$$

Putting the value of A and B in equation (5), we get

$$Z(x, t) = -e^{-2\sqrt{\lambda}}\cdot e^{\sqrt{\lambda}x+k\lambda t} + e^{-2\sqrt{\lambda}}\cdot e^{-\sqrt{\lambda}x+k\lambda t}$$

$$= e^{-2\sqrt{\lambda}+k\lambda t}\left[e^{-\sqrt{\lambda}x} - e^{\sqrt{\lambda}x}\right]$$

$$z(x, t) = -e^{-2\sqrt{\lambda}+k\lambda t}\left(2\sinh\sqrt{\lambda}x\right) \quad \text{Ans.}$$

Q.13 ***Show that the equation*** $\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2}$

Possesses solution of the form

$$\Sigma C_r e^{-kt}\cos(\alpha_r x + \epsilon_r)\cos(w_r t + \delta_r)$$

Where $C_r, \alpha_r, \epsilon_r, \omega_r, \delta_r$ ***are constants***

and $\omega_r = \delta_r^2 c^2 - k^2$

Ans. Let $\frac{\partial y}{\partial t} = D$; $\frac{\partial y}{\partial x} = D'$;

Therefore $(D^2 + 2kD)y = c^2D'^2y$

Or $$(D^2 + 2kD - c^2D'^2)y = 0 \quad \ldots (i)$$

It is irreducible, therefore let $y = e^{at+bx}$

Similarly we get

$$Dy = ae^{at+bx} \text{ and } D^2y = a^2e^{at+bx}$$

Therefore form (i) we have

$$a^2e^{at+bx} + 2kae^{at+bx} + c^2b^2e^{at+bx} = 0$$

$$\Rightarrow (a^2 + 2ka + c^2b^2)e^{at+bx} = 0$$

$$\Rightarrow a^2 + 2ka + c^2b^2 = 0$$

$$\Rightarrow a = \frac{-2k \pm \sqrt{4k^2 + 4b^2c^2}}{2}$$

$$\Rightarrow a = -k \pm \sqrt{k^2 + b^2c^2}$$

In general $a_r = -k \pm \sqrt{k^2 + b_r^2c^2}$

If $b_r^2 = -\alpha_r^2$

Then $a_r = -k \pm \sqrt{k^2 - \alpha_r^2c^2}$

$= -k \pm i\omega_r$

Where $\omega_r^2 = \alpha_r^2c^2 - k^2$

Therefore $y = e^{b_r x}e^{a_r t}$

$= e^{-kt}e^{-i\omega_r t}e^{-i\alpha_r x}$

$$y = \sum_{r=0}^{\infty} c_r e^{-kt}\cos(\alpha_r x + \epsilon_r)\cos(\omega_r t + \delta_r)$$

Where $C_r, \alpha_r, \epsilon_r, \omega_r, \delta_r$ are constants and $\omega_r = \delta_r^2c^2 - k^2$

Q.14 ***A tightly stretched string with fixed end points*** $x = 0$ ***and*** $x = l$ ***is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity*** $v_0 x(l-x)$, ***find the displacement of the string at any distance x from one end at any time t.***

Ans. The displacement function u(x,t) is given by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, 0 < x < l, t > 0 \quad \ldots (i)$$

With boundary conditions

$$u(0, t) = 0, u(l, t) = 0, t > 0. \quad \ldots (ii)$$

and initial conditions

$$u(x, 0) = 0,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_0x(l-x), 0 \le x \le l. \quad \ldots (iii)$$

The solution of the wave equation (i) subject to the boundary conditions (i) and the initial condition $u(x,0) = 0$, is given by

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right), \quad \ldots (iv)$$

Where the constants D_n, n = 1,2 ..... are to be determined using the second initial condition

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_0x(l-x).$$

Differentiating (iv) w.r.t t and substituting t = 0 and using $\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_0x(l-x)$, it becomes

$$u_0x(l-x) = \sum_{n=1}^{\infty} D_n\frac{n\pi c}{l}\sin\frac{n\pi x}{l}. \quad \ldots (v)$$

Next, expanding $u_0x(l-x)$ in a half range sine series in $[0, l]$, we have

$$u_0x(l-x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}. \quad \ldots (vi)$$

$$\text{Where } b_n = \frac{2}{l}\int_0^l u_0x(l-x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2u_0}{l}\left[-\frac{l}{n\pi}x(l-x)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l-2x)\sin\frac{n\pi x}{l}\right.$$

$$\left. -2\frac{l^2}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4u_0l^2}{n^3\pi^3}[1 - \cos n\pi] = \frac{4u_0l^2}{\pi^3n^3}\left[1 - (-1)^n\right].$$

and hence

$$b_n = \begin{cases} 0, & \text{When n is even} \\ \dfrac{8u_0l^2}{\pi^3n^3} & \text{when n is odd.} \end{cases} \quad \ldots (vii)$$

From (v), (vi) and (vii), we have

$$D_n = \begin{cases} 0, & \text{When n is even} \\ \dfrac{8u_0l^3}{\pi^4n^4c} & \text{when n is odd.} \end{cases}$$

Substituting for D_n in (iii) the required solution is given by

$$u(x,t) = \sum_{n=1,3,5}^{\infty}\frac{8u_0l^3}{\pi^4n^4c}\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{8u_0l^3}{\pi^4c}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi ct}{l}\sin\frac{(2n-1)\pi x}{l}.$$

Q.15 ***A homogeneous laterally insulated bar of length 100 cm has its ends kept at zero temperature. Find the temperature distribution u(x,t), if the initial temperature is***

$$f(x) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$$

Ans. The temperature distribution u(x, t) is modelled as the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, 0 < x < 100, t > 0 \quad \ldots (i)$$

Subject to the boundary conditions

$$u(0, t) = 0, u(100, t) = 0, t \ge 0, \quad \ldots (ii)$$

And the initial condition

$$f(x) = u(x, 0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases} \quad \ldots (iii)$$

The solution of the heat equation (i) subject to the boundary conditions (ii),

$$\text{Now } u(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{100}e^{-c^2n^2\pi^2t/(100)^2},$$

$$0 \le x \le 100, t \ge 0 \quad \ldots (iv)$$

Using the initial condition (iii), (iv) gives

$$f(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{100},\ 0\le x\le 100$$

Which is Fourier half-range sine series expansion of f(x) in the interval [0,100]. Hence, using (iii) the coefficients B_n are given by

$$B_n=\frac{2}{100}\int_0^{100}f(x)\sin\frac{n\pi x}{100}dx$$

$$=\frac{1}{50}\left[\int_0^{50}x\sin\frac{n\pi x}{100}dx+\int_{50}^{100}(100-x)\sin\frac{n\pi x}{100}dx\right].$$

Integrating and simplifying, we obtain

$$B_n=\begin{cases}0, & \text{if } n \text{ is even,}\\ (-1)^{\frac{n-1}{2}}\dfrac{400}{n^2\pi^2} & \text{if } n \text{ is odd.}\end{cases}$$

Substituting for B_n in (iv), we get

$$u(x,t)=\frac{400}{\pi^2}\left[\sin\frac{\pi x}{100}e^{-\frac{c^2\pi^2}{(100)^2}t}-\frac{1}{9}\sin\frac{3\pi x}{100}e^{\frac{-9c^2\pi^2 t}{(100)^2}}\right.$$

$$\left.+\frac{1}{25}\sin\frac{5\pi x}{100}e^{\frac{-25c^2\pi^2 t}{(100)^2}}-\dots\right]$$

as the desired solution.

□□□