Unit-4 Partial Differential Equations

→ An equation is said to be partial differential eq. if it involves two (or more) independent variables x, y (say) and a dependent variable z (say) and its partial derivatives.

Eg. let $z = f(x,y)$

$$p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} \qquad s = \frac{\partial^2 z}{\partial x \partial y} \qquad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Order and Degree ⇒

Eg. ① $$\left[3\left(\frac{\partial^4 u}{\partial x^4}\right)^3 - 4\left(\frac{\partial^5 u}{\partial t^5}\right)^7\right]^2 = 7x^4t^2\left(\frac{\partial u}{\partial t}\right)^4$$

$O = 5 \quad d = 14$

② $$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$$ $O = 3,\ d = 1$

③ $$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{\frac{1}{2}}$$ $O = 2,\ d = 2$

Classify the eq. (RTU-2023)(Marks-2)

④ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,\cancel{xy\left(\frac{\partial u}{\partial x}\right)^3}$$ $O = 2,\ d = 1$

Q: Classify the eq. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (RTU-2024) (Marks-2)

$O = 2,\ d = 1$ ⇒ Linear P.d.e. of order 2.

Formation of Partial Differential Equations ⇒

↳ by ① elemination of arbitary Constants.

② elemination of arbitary function.

① Elemination of arbitary Constant ⇒

Eg. form the p.d.e. by eleminating the arbitary Const.

$$z^2 + ax^2 + by^2 = 1$$

Sol. $z^2 + ax^2 + by^2 = 1$ ——①

$\Rightarrow 2z\frac{\partial z}{\partial x} + 2ax = 0$ (diff w.r.t. x)

$\Rightarrow a = -\frac{z}{x}\frac{\partial z}{\partial x} = -\frac{z}{x}p$ ——② $\left(p = \frac{\partial z}{\partial x}\right)$

Again $\Rightarrow 2z\frac{\partial z}{\partial y} + 2by = 0$ (diff w.r.t. y)

$\Rightarrow b = -\frac{z}{y}\frac{\partial z}{\partial y} = \frac{-z}{y}q$ ——③ $\left(q = \frac{\partial z}{\partial x}\right)$

Substituting values of a & b in eq. ①

$\Rightarrow z(z - xp - yq) = 1$

② Elemination of arbitary function ⇒

Eg. form the p.d.e. by elemination of ϕ from

$$lx + my + nz = \phi(x^2 + y^2 + z^2)$$

Sol: $lx + my + nz = \phi(x^2+y^2+z^2)$ —— ①

$\Rightarrow l + np = \phi'(x^2+y^2+z^2)(2x+2zp)$ —— ② [diff. w.r.t. x]

and $m + nq = \phi'(x^2+y^2+z^2)(2y+2zq)$ —— ③ [diff. w.r.t. y]

on dividing ② by ③,

$$\Rightarrow \frac{l+np}{m+nq} = \frac{2x+2zp}{2y+2zq}$$

$$\Rightarrow (l+np)y + z(lp - mp) = (m+np)x.$$

<u>Linear Partial Differential Equations of first Order (O=1, d=1)</u>

The general form of a quasi-linear partial diff. eq. of the first order is —

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \quad \text{or} \quad \boxed{Pp + Qq = R}$$ —— ①

where, P, Q, R → fn of x, y & z.

The eq. ① is known as Lagrange's Linear Equation.

<u>Steps to solve L.P.D.E of first order</u> ⇒

S-① Write down the subsidiary eq. (or Auxiliary Eq.)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

S-② find the solution of A.E. by grouping method, multiplier method or Combined method.

i.e $C_1 =$ Constant and $C_2 =$ Constant

S-③ The general solution of $Pp + Qq = R$ is given by

$$\phi(C_1, C_2) = 0$$

or $C_2 = \phi(C_1)$

① Grouping method ⇒

E.g. ⇒ solve $yzp + zxq = xy$. [RTU-2020] (Marks-5) $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Sol:

$yzp + zxq = xy$ —① is of the form $Pp + Qq = R$

⇒ Lagrange's P.D.E.

$$\Rightarrow \quad P = yz, \quad Q = zx, \quad R = xy.$$

$$A.E \Rightarrow \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \quad \text{—②}$$

Using grouping method, take first two ratios

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x\,dx = y\,dy$$

$$\Rightarrow x^2 = y^2 + C_1 \quad \text{(Integrating)}$$

$$\Rightarrow C_1 = x^2 - y^2$$

taking last two ratios,

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y\,dy = z\,dz$$

$$\Rightarrow y^2 = z^2 + C_2 \quad \text{(Integrating)}$$

$$\Rightarrow C_2 = y^2 - z^2$$

Hence general solution of eq.① ⇒ $\phi(C_1, C_2) = 0$

where $\phi \rightarrow$ arbitrary fn.

② Multiplier method ⇒

Eg. Solve $(mz - ny)p + (nx - lz)q = ly - mx$.

Sol: A.E. ⇒ $$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \lambda \quad \text{—}(A)$$

$$\Rightarrow dx = \lambda(mz-ny) \quad \text{—}(1)$$

$$dy = \lambda(nx-lz) \quad \text{—}(2)$$

$$dz = \lambda(ly-mx) \quad \text{—}(3)$$

$l \times$ eq.(1) $+ m \times$ eq.(2) $+ n \times$ eq.(3)

(Taking) $l, m, n \rightarrow$ multiplier

Here, $l\,dx + m\,dy + n\,dz = 0$

on integrating, $lx + my + nz = C_1$

(Taking) $x, y, z \rightarrow$ multiplier

$x\,dx + y\,dy + z\,dz = 0$

on integrating $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$

∴ Solution of eq.(A) is $\phi(C_1, C_2) = 0$

$\Rightarrow \phi(lx+my+nz,\ x^2+y^2+z^2)$

where, $\phi \rightarrow$ arbitrary Const.

③ Combined method ⇒

Eg. ⇒ Solve: $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Sol: A.E. ⇒ $$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

Q: Solve: $a(p+q) = z$ (RTU-2024) (Marks-2)

Sol: Given $\Rightarrow$ $ap + aq = z$ —— ①

A.E. $\Rightarrow$ $\dfrac{dx}{a} = \dfrac{dy}{a} = \dfrac{dz}{z}$

Considering first two ratios

$$\frac{dx}{a} = \frac{dy}{a} \Rightarrow dx = dy$$

$\Rightarrow x = y + c_1$ (on integrating)

$\Rightarrow c_1 = x - y$

Taking last two ratios,

$$\frac{dy}{a} = \frac{dz}{z} \Rightarrow y = a\log z + c_2$$ (on integrating)

$\Rightarrow c_2 = y - a\log z$

Hence sol. of eq. ① $\Rightarrow \phi(c_1, c_2) = 0$

$\Rightarrow \phi(x-y,\ y - a\log z) = 0$

Q: Solve the following: [RTU-2023] (Marks-5)

$x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol: A.E. $\Rightarrow$ $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)} = \lambda$ —— ①

$dx = x^2(y-z)\lambda$ —— ②

$dy = y^2(z-x)\lambda$ —— ③

$dz = z^2(x-y)\lambda$ —— ④

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \log x + \log y + \log z = \log a$$

$$\Rightarrow xyz = c_1$$

Again taking $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers

$$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_2$$

$\therefore$ Sol. $\Rightarrow$ $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$

where $\phi \rightarrow$ arbitrary Const.

Q: Find the general sol. of the p.d.e. [RTU-2022] (Marks-5)

$$(3-2yz)p + x(2z-1)q = 2x(y-3)$$

Q: Find the general sol. of the p.d.e. [RTU-2019] (Marks-5)

$$xy^2p + y^3q = (zxy^2 - 4x^3)$$

Non-Linear first order Partial Diff. Eq. $\Rightarrow$

↳ PDE $\Rightarrow$ first order and any degree.
(other than 1)

Standard Form – I

The eq. which Contain "p" and "q" terms only, i.e.

$$\boxed{F(p, q) = 0} \quad \text{——} (1)$$

Working Rule $\Rightarrow$

(1) The complete integral of eq. (1) $\Rightarrow$ $z = ax + by + c$ —— (2)

(2) from eq. (2) (differentiating w.r.t. x, y),

$$\left.\begin{aligned} \frac{\partial z}{\partial x} = a \quad &\Rightarrow p = a \\ \text{and } \frac{\partial z}{\partial y} = b \quad &\Rightarrow q = b \end{aligned}\right\} \text{——} (3)$$

(3) Put values of p and q in eq. (1), we get

$$F(a, b) = 0$$

(4) Find b in terms of a i.e. $b = \phi(a)$ —— (4)

(5) Substitute b from eq. (4) into eq. (2),

$$z = ax + \phi(a)\,y + c \quad \Leftarrow \text{Complete integral.}$$

Q: Solve: $p^2 + q^2 = 1$ —— (1)

Sol: Eq. Contain only p and q $\Rightarrow$ std. form – I

$\therefore$ Complete integral $\Rightarrow$ $z = ax + by + c$ —— (2)

where, $\frac{\partial z}{\partial x} = a \Rightarrow p = a$
$\frac{\partial z}{\partial y} = b \Rightarrow q = b$ —— (3)

from (1) & (3), $a^2 + b^2 = 1$

$b = \sqrt{1-a^2}$ —— (4)

$\therefore$ Complete integral $\Rightarrow z = ax + \sqrt{1-a^2}\, y + c$

Q: Solve: $x^2p^2 + y^2q^2 = z^2$

Sol:
$$\frac{x^2}{z^2}p^2 + \frac{y^2}{z^2}q^2 = 1$$

$$\Rightarrow \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{—— (1)}$$

Now, take
$\frac{dx}{x} = dX \Rightarrow \log x = X$
$\frac{dy}{y} = dY \Rightarrow \log y = Y$
$\frac{dz}{z} = dZ \Rightarrow \log z = Z$

Then (1) becomes,

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \quad \text{—— (2)}$$

let $P = \frac{\partial Z}{\partial x}$ and $Q = \frac{\partial Z}{\partial Y}$, then eq. (2) becomes

$P^2 + Q^2 = 1$ $\Rightarrow$ std. Form - I

$\therefore$ Complete integral of (2) is —

$$z = ax + by + c$$

where $a^2 + b^2 = 1 \Rightarrow b = \sqrt{1-a^2}$

$$z = ax + \sqrt{1-a^2}\, y + c$$

$$z = a \log x + \sqrt{1-a^2} \log y + c \quad \text{Ans.}$$

<u>Standard Form - II</u> PDE Containing only z, p and q, not Containing x & y.

$$\boxed{F(z, p, q) = 0} \quad \text{--- (1)}$$

<u>Working Rule</u> $\Rightarrow$

(1) Assume $u = x + ay$ --- (2)

(2) Differentiating eq. (2) $\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = a$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} \times 1 = \frac{dz}{du}$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{\partial z}{\partial u} \times a = a\frac{dz}{du}$$

--- (3)

(3) Put values of p and q in eq. (1), we get

$$F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \quad \text{--- (4)}$$

(4) Solve eq. (4) in z and u by integration. Let eq. (5) recieved.

(5) Put the value of $u = x + ay$ in eq. (5).

Q: Solve: $9(p^2 z + q^2) = 4$ [RTU-2024]

(Marks-5)

Sol: $\quad$ —①

let $u = x + ay$ —②

$p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ —③

from eq.①,

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4$$

$$\Rightarrow 9\left(\frac{dz}{du}\right)^2 (z^2 + a^2) = 4$$

$$\Rightarrow 3\sqrt{z^2 + a^2}\, dz = 2du$$

$$\Rightarrow (z^2 + a^2)^{3/2} = u + c$$ on integrating

$$\Rightarrow (z^2 + a^2)^{3/2} = x + ay + c$$ from eqⁿ ②

$$\Rightarrow z^2 + a^2 = (x + ay + c)^2$$

Q: Solve, $z^2(p^2 x^2 + q^2) = 1$

Sol: It is a non-linear first order PDE but it do not belongs to any std. form.

But it can be reduced to std form-II

so, $z^2\left[(xp)^2 + (q)^2\right] = 1$ —①

$$\Rightarrow z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = 1$$ —②

$\Rightarrow$ Assume $\Rightarrow$ $\frac{\partial x}{x} = \partial X$, $\partial y = \partial Y$, $\partial z = \partial Z$

Now integrating both sides, we get

$\log x = X$, $y = Y$, $z = Z$ —— (3)

Now,

$$Z^2\left[\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2\right] = 1$$

$$Z^2(P^2 + Q^2) = 1 \quad \text{—— (4)}$$

which is std. form – II.

Put $u = \cancel{x+ay}$. $X + aY$

$P = \frac{dz}{du}$ $\quad Q = a\frac{dz}{du}$

∴ from eq. (4),

$$\Rightarrow Z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1$$

$$\Rightarrow z^2(1+a^2)\left(\frac{dz}{du}\right)^2 = 1$$

$$\Rightarrow z\,dz = \frac{1}{\sqrt{1+a^2}}\,du$$

$$\Rightarrow \frac{z^2}{2} = \frac{1}{\sqrt{1+a^2}}u + c'$$

$$\Rightarrow \cancel{z^2} \sqrt{1+a^2}\,z^2 = 2u + c$$

$$\Rightarrow \sqrt{1+a^2}\,z^2 = 2(X+aY) + c$$ from eq. (3)

after solving $\Rightarrow \sqrt{1+a^2}\,z^2 = \log x^2 + 2ay + c$ Ans.

Standard form - III ⇒

$$\boxed{f_1(x,p) = f_2(y,q)} \quad \text{---(1)}$$

Working Rule ⇒

(1) let $f_1(x,p) = f_2(y,q) = a$ ---(2)

(2) From eq.(2), $p = \phi_1(x,a)$ and $q = \phi_2(y,a)$ ---(3)

(3) Consider $\cancel{\frac{dz}{dx} \neq \frac{\partial z}{\partial x}}$ $dz = p\,dx + q\,dy$ ---(4)

(4) $\Rightarrow dz = \phi_1(x,a)dx + \phi_2(y,a)dy$ ---(5)

Integrate eq.(5),

$$z = \int \phi_1(x,a)\,dx + \int \phi_2(y,a)\,dy + C.$$

Q: Solve: $p^2 + q^2 = x + y$ [RTU - 2021] (Marks - 5)

Sol. The Given P.D.E ⇒ $p^2 - x = y - q^2$ ⇒ std form - III

let $p^2 - x = y - q^2 = a$

$\Rightarrow p = \sqrt{x+a}$ and $q = \sqrt{y-a}$

Now,
$$dz = p\,dx + q\,dy$$
$$dz = \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$$
integrating.
$$z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + C$$ Ans.

Q: Solve: $z^2(p^2+q^2) = x^2+y^2$ (RTU 2019) (5-Marks)

Sol:

Given $\Rightarrow$ $\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2$

let $z\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P$ and $z\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q$

So, $P^2 + Q^2 = x^2+y^2$

$\Rightarrow P^2 - x^2 = y^2 - Q^2 = a$ (let)

Now,

$dZ = P\,dx + Q\,dy$

$dZ = (a^2+x^2)^{\frac{1}{2}}dx + (y^2-a^2)^{\frac{1}{2}}dx$

on integrating, $Z = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a}{2}\log\{x+\sqrt{x^2+a^2}\}$

$+ \frac{y}{2}\sqrt{y^2-a^2} + \frac{a}{2}\log\{y+\sqrt{y^2-a^2}\} + C$

or $z^2 = x\sqrt{x^2+a^2} + a\log\{x+\sqrt{x^2+a^2}\} + y\sqrt{y^2-a^2} + a\log\{y+\sqrt{y^2-a^2}\} + C$

A.

Standard form - IV ⇒

$$\boxed{z = px + qy + f(p, q)} \quad \text{—(1)}$$

Working Procedure ⇒

① The complete integral of eq. ① ⇒ $z = ax + by + f(a, b)$ —②

b/c
② from eq. ② (differentiating w.r.t. x)

$$p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = b \quad \text{—(3)}$$

③ Put values of p and q in eq. ①,

$$z = ax + by + f(a, b) \quad \text{—(4)}$$

④ Find b in terms of a i.e. $b = \phi(a)$ —⑤

⑤ Substitute b from eq. ⑤ into eq. ④

$$\therefore \; z = ax + \phi(a) y + \phi(a, \phi(a)) \quad \text{—(6)}$$

⑥ Eleminate "a" by solving $z = 0$ and $\frac{\partial z}{\partial a} = 0$

Q: Find the p.d.e. from $z = px + qy + p^2 + q^2$

Sol: Complete Integral $\Rightarrow$ $z = ax + by + a^2 + b^2$ —— ①

Now diff. eq.① w.r.t. a and b,

$$x + 2a = 0 \Rightarrow a = \frac{-x}{2}$$

& $$y + 2b = 0 \Rightarrow b = \frac{-y}{2}$$

Now put these value in eq.①

$$z = \frac{-x}{2}(x) + \left(\frac{-y}{2}\right)y + \left(\frac{-x}{2}\right)^2 + \left(\frac{-y}{2}\right)^2$$

$$z = \frac{-(x^2+y^2)}{2} \Rightarrow 4z = -(x^2+y^2) \text{ Ans.}$$

Q: Solve. $z = px + qy + 2\sqrt{pq}$

Sol: Complete integral $\Rightarrow$ $z = ax + by - 2\sqrt{ab}$ —— ①

Now diff. eq.① w.r.t. a and b.

$$\frac{\partial z}{\partial a} = x - 2 \times \frac{1}{2}(ab)^{-1/2} \times b = 0 \Rightarrow x = \sqrt{\frac{b}{a}} \quad \text{—— ②}$$

$$\frac{\partial z}{\partial b} = y - \frac{a}{\sqrt{ab}} = 0 \Rightarrow y = \sqrt{\frac{a}{b}} \quad \text{—— ③}$$

Put these value in eq.①

$$z = a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} - 2\sqrt{ab}$$

$$z = \sqrt{ab} + \sqrt{ab} - 2\sqrt{ab} = 0$$

or from eq.② & ③

$$\boxed{xy = 1} \text{ A}$$

Q: find the p.d.e. from $z = ax + by + ab$ (RTU. 2023) (5. Marks)

Sol: Given $z = ax + by + ab$ —— (1)

now differentiating eq. (1) w.r.t. a & b,

$\frac{\partial z}{\partial a} = 0 \Rightarrow x + b = 0 \Rightarrow b = -x$

$\frac{\partial z}{\partial b} = 0 \Rightarrow y + a = 0 \Rightarrow a = -y$

now substitute the above values of a & b in eq. (1),

$z = -xy - xy + xy$

$z = -xy$ Ans.

Q: find the p.d.e. from – $az + b = a^2x + y$ (RTU-2021) (Marks - 2)

Sol: $z = ax + \frac{1}{a}y - \frac{b}{a}$

Complete integral is $z = ax + \frac{y}{a} - \frac{b}{a}$ —— (1)

now diff. eq. (1) w.r.t. a and b

$\frac{\partial z}{\partial a} = x + \left(-\frac{1}{a^2}\right)y - \left(-\frac{1}{a^2}\right)b = 0$

$\frac{\partial z}{\partial b} = -\frac{1}{a} = 0$

Charpit's Method for Solving General P.D.E. ⇒

Let, the equation to be solved be denoted by,

$$f(x,y,z,p,q) = 0$$

Charpit's equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Q: Find Complete integral of $px + qy = pq$ [RTU-2024, RTU-2023] (10-M.)

Sol: $f(x,y,z,p,q) = 0 \Rightarrow px + qy - pq = 0$ —— ①

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = +x - q, \quad \frac{\partial f}{\partial q} = y - p$$

Charpit's equations are,

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{df}{0}$$

Taking first two, we get

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p - \log q = a \Rightarrow \frac{p}{q} = a \Rightarrow p = aq$$ ②

Putting, p in the given eq., we get

$$aqx + qy = aq(q)$$

$$\Rightarrow q = \frac{y + ax}{a}$$

∴ from ② $p = y + ax$.

now, $dz = p\,dx + q\,dy$

$$dz = (y+ax)\left\{dx + \left(\frac{y+ax}{a}\right)dy\right.$$

$$dz = y\,dx + x\,dy + ax\,dx + \frac{y}{a}\,dy$$

$$dz = xy + a\frac{x^2}{2} + \frac{1}{a}\frac{y^2}{2} + c' \quad \Big] \text{ integrating.}$$

$$z = \frac{a^2x^2 + 2axy + y^2}{2} + c$$

$$z = \frac{(ax+y)^2}{2} + c \quad \text{Ans.}$$

Q: Solve: $2zx - px^2 - 2qxy + pq = 0$ by Charpit's eq.

[RTU - 2020 - 2008] (10-m)

[MNIT - 2006]

Sol:

$$f(x, y, z, p, q) = 0$$

$$\Rightarrow 2zx - px^2 - 2qxy + pq = 0$$

$$\frac{\partial f}{\partial p} = -x^2 + q, \quad \frac{\partial f}{\partial q} = -2xy \quad \frac{\partial f}{\partial z} = 2x$$

$$\frac{\partial f}{\partial x} = 2z - 2px - qy \qquad \frac{\partial f}{\partial y} = -2qx$$

Charpit's eq.,

$$\frac{dp}{2z - 2px - 2qy + 2px} = \frac{dq}{-2qx + 2qx}$$

$$\Rightarrow \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\Rightarrow dq = 0$

$\Rightarrow q = a$ (constant)

$\therefore$ from eq. (1), $p = \dfrac{2xz - 2axy}{x^2 - a}$

Now, $dz = p\,dx + q\,dy$

$$dz = \frac{2xz - 2axy}{x^2 - a}\,dx + a\,dy$$

$$\Rightarrow dz - a\,dy = 2x\frac{(z - ay)}{(x^2 - a)}\,dx$$

$$\Rightarrow \frac{dz - a\,dy}{z - ay} = \frac{2x}{x^2 - a}\,dx$$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

$\Rightarrow z - ay = b(x^2 - a)$

$\Rightarrow z = ay + b(x^2 - a)$ A.

Q: Solve: $px + qy + z = xq^2$ [RTU 2019] (10-m)

Sol:

$$f(x,y,z,p,q) = 0$$

$$\Rightarrow px + qy + z - xq^2 = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial p} = x \quad \frac{\partial f}{\partial q} = y - 2qx, \quad \frac{\partial f}{\partial x} = p - q^2, \quad \frac{\partial f}{\partial y} = q \quad \frac{\partial f}{\partial z} = 1$$

Charpit's method,

$$\frac{dp}{p-q^2+p} = \frac{dq}{q+q} = \frac{dz}{-px-q(y-2qx)} = \frac{dx}{-x} = \frac{dy}{y-2qx} = \frac{df}{1}$$

taking, $\dfrac{dq}{2q} = \dfrac{dx}{-x}$

$$\frac{dq}{q} + 2\frac{dx}{x} = 0$$

on integrating, $\log q + 2\log x = \log a$

$$\Rightarrow qx^2 = a \quad \Rightarrow q = \frac{a}{x^2}$$

from eq.(1),

$$px + \frac{a}{x^2}y + z - x \times \frac{a^2}{x^4} = 0$$

$$\Rightarrow px = \frac{a^2}{x^3} - \frac{ay}{x^2} - z$$

$$\Rightarrow p = \frac{a^2}{x^4} - \frac{ay}{x^3} - \frac{z}{x}$$

Putting values of p and q in

$$dz = p\,dx + q\,dy$$

$$dz = \left[\frac{a^2}{x^4} - \frac{ay}{x^3} - \frac{z}{x}\right]dx + \frac{a}{x^2}dy$$

$$\Rightarrow x\,dz + z\,dx = \frac{a^2}{x^3}dx - \frac{ay}{x^2}dx + \frac{a}{x}dy$$

$$\Rightarrow x\,dz + z\,dx = a\left[\frac{1}{x}dy - \frac{y}{x^2}dx\right] + \frac{a^2}{x^3}dx$$

$$\Rightarrow x\,dz + z\,dx = a\left[\frac{x\,dy - y\,dx}{x^2}\right] + \frac{a^2}{x^3}dx$$

on integrating,

$$\Rightarrow xz = \frac{ay}{x} - \frac{2a^2}{x^2} + b$$

$$\Rightarrow z = \frac{ay}{x^2} - \frac{2a^2}{x^3} + \frac{b}{x} \quad \text{Ans.}$$

Q: Find the Complete integral of [RTU-2012]

$$(p+q)(px+qy) = 1$$

Sol:

$$f = (p+q)(px+qy) - 1 = 0 \quad \text{—— (1)}$$

Charpit's first two A.E. are

$$\frac{dp}{p^2+pq} = \frac{dq}{pq+q^2}$$

$$\Rightarrow \frac{dp}{q} = \frac{dq}{p}$$

on integrating, $p = aq$ —— (2)

Putting value of p in eq. (1), we get

$(aq+q)(aqx+qy) = 1 \Rightarrow q = \dfrac{1}{\sqrt{1+a}\sqrt{ax+y}}$

Putting the values of p and q in

$$dz = p\,dx + q\,dy,$$

$$dz = \frac{1}{\sqrt{1+a}\sqrt{ax+y}}[a\,dx + dy]$$

on integrating,

$$z = \frac{2}{\sqrt{1+a}}\sqrt{ax+y} + c \qquad \text{A.}$$

Q: Find the complete integral of $z^2 = pqxy$ (PYQ)

Sol: $f: pqxy - z^2 = 0$ —— (1)

Charpit's A.E. are

$$\frac{dp}{pqy - 2pz} = \frac{dq}{pqx - 2qz} = \frac{dz}{-2pqxy} = \frac{dx}{-qxy} = \frac{dy}{-pxy}$$

$\therefore$ we have $\dfrac{x\,dp + p\,dx}{-2pxz} = \dfrac{y\,dq + q\,dy}{-2pyz}$

or $\dfrac{d(px)}{px} = \dfrac{d(qy)}{qy}$

on integrating,

$$\log px = \log qy + \log a^2$$

$$px = a^2 qy \quad \text{—— (2)}$$

Solving ① and ②,

$$p = \frac{az}{x} \qquad q = \frac{z}{ay}$$

$$\therefore \quad dz = p\,dx + q\,dy$$

$$dz = \frac{az}{x}dx + \frac{z}{ay}dy$$

$$\frac{dz}{z} = \frac{a}{x}dx + \frac{1}{ay}dy$$

integrating,

$$\log z = a\log x + \frac{1}{a}\log y + \log b$$

$$z = b x^{a} y^{1/a}$$

Ans.