Unit-2 Differential Equations

Differential Equations ⇒

↳ D.E. are the eq.s involving an unknown fn and its derivatives.

→ An eq. which involves ordinary derivatives is known as ordinary diff. eq.

→ An eq. which involves partial derivatives is known as partial d.e.

E.g. ⇒ $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5xy = x^2 \rightarrow$ o.d.e

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x^2y \rightarrow$ p.d.e.

Order and degree of a d.e. ⇒

order → Highest order derivative appearing in a d.e.

Degree → The heighest Power of the highest order derivative (free from radicals & fractional Powers)

Eg. ⇒

$\frac{d^2y}{dx^2} + 4y = e^x \Rightarrow O = 2, \quad D = 1$

$\frac{d^2y}{dx^2} = \left[5 + 7\left(\frac{dy}{dx}\right)^2\right]^{3/2} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left[5 + 7\left(\frac{dy}{dx}\right)^2\right]^3$

$\Rightarrow O = 2, \quad D = 2$

(3.) $\dfrac{dy}{dx} + y = \dfrac{1}{\left(\dfrac{dy}{dx}\right)}$

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 + y\left(\dfrac{dy}{dx}\right) = 1$ $\qquad O = 1, \quad D = 2$

(4) $e^{\frac{d^3y}{dx^2}} - x\dfrac{d^2y}{dx^2} + y = 0$ $\qquad O = 3 \quad d =$ Not defined.

(5) $\ln\left(\dfrac{dy}{dx}\right) = ax + by$

$\dfrac{dy}{dx} = e^{ax+by}$ $\qquad O = 1, \quad d = 1$

Solution of D.E. (First order & first degree) $\Rightarrow$

D. E.

(1) Variable separable
↓
Reducible to Var. Sep.

(2) Homogeneous
↓
Reducible to Homogeneous

(3) Linear D.E.
↓
Reducible to Linear (Bernoulli's d.e)

(4)

(4) Exact d.e.
↓
Reducible to exact form

(I) Variable separable form ⇒

General form ⇒ $\frac{dy}{dx} = \frac{P(x)}{Q(y)}$

$$\Rightarrow Q(y)\, dy = P(x)\, dx$$

$$\Rightarrow \int Q(y)\, dy = \int P(x)\, dx + c$$

Q: Solve: $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

Sol:

$$\frac{dy}{y(1-ay)} = \frac{dx}{x+a}$$

$$\left(\frac{1}{y} + \frac{a}{1-ay}\right) dy = \frac{dx}{x+a}$$

Integrating both sides

$$\Rightarrow \log y + a \times \frac{-1}{a} \log(1-ay) = \log(x+a) + \log c$$

$$\Rightarrow \frac{y}{1-ay} = c(x+a) \Rightarrow y = c(x+a)(1-ay)$$ Ans.

(2) Homogeneous Equations $\Rightarrow$

A differential eq. of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{h(x,y)} \quad \text{—(1)}$$

where $f(x,y)$ and $h(x,y)$ $\rightarrow$ are homogeneous fn of same degree.

Method to solve $\Rightarrow$ Put $y = vx$ —(2)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{—(3)}$$

Substitute eq. (2) and (3) in eq. (1) reduces to variable separable form.

Q: Solve: $(xy - x^2)\,dy = y^2\,dx$

Sol:

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} \quad \text{—(1)} \quad \text{(Homog. eq.)}$$

Put $y = vx$ $\Rightarrow$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1)

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2}{v-1}$$

$$\Rightarrow \frac{(v-1)\,dv}{v} = \cancel{\frac{v}{v}}\,\frac{dx}{x}$$

$$\Rightarrow dv - \frac{1}{v}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow v - \log v = \log x + \log c \quad \text{] (integrating)}$$

$\Rightarrow \log v x c = v$

$\Rightarrow v x c = e^{v}$

$\Rightarrow cy = e^{y/x}$ Ay.

Equation Reducible to Homogeneous form →

A Differential eq. of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}, \quad [a, b, c, A, B, C \rightarrow \text{Const.}$$

and $\boxed{\frac{a}{A} \neq \frac{b}{B}}$

Can be reduced to homog. by substituting

$$x = X+h \quad \text{and} \quad y = Y+k \quad [h, k \rightarrow \text{Constants}$$

Case 1.

Q: Solve $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$

Sol: Put $x = X+h$ and $y = Y+k$ and $\frac{dy}{dx} = \frac{dY}{dX}$

$$\frac{dY}{dX} = \frac{2(X+h)-(Y+k)+1}{(X+h)+2(Y+k)-3}$$

$$\frac{dY}{dX} = \frac{(2X-Y)+(2h-k+1)}{(X+2Y)+(h+2k-3)} \quad \text{—— (1)}$$

Choose h and k in such a way

$$\left.\begin{aligned} 2h-k+1 &= 0 \\ \& \quad h+2k-3 &= 0. \end{aligned}\right\} \Rightarrow h = \frac{1}{5} \text{ and } k = \frac{7}{5}$$

Now eq.(1) becomes $\frac{dY}{dX} = \frac{2X-Y}{X+2Y}$

$$\frac{dY}{dX} = \frac{2X - Y}{X + 2Y} = \frac{2 - \frac{Y}{X}}{1 + 2\frac{Y}{X}} \quad \text{—②}$$

Put $Y = VX$, $\frac{dY}{dX} = V + X\frac{dV}{dX}$ —③

using ② and ③,

$$V + X\frac{dV}{dX} = \frac{2 - V}{1 + 2V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{-2(V^2 + V - 1)}{2V + 1}$$

$$\Rightarrow \frac{2V + 1}{V^2 + V - 1}\, dV = -2\frac{dX}{X}$$

] on integrating

$$\Rightarrow \log(V^2 + V - 1) = -2\log X + \log c$$

$$\Rightarrow V^2 + V - 1 = \frac{c}{X^2}$$

$$\Rightarrow \frac{Y^2}{X^2} + \frac{Y}{X} - 1 = \frac{c}{X^2}$$

$$\Rightarrow Y^2 + XY - X^2 = c$$

But $x = X + h$ $\quad y = Y + k$

$X = x - \frac{1}{5}$ $\quad Y = y - \frac{7}{5}$

$$\Rightarrow \left(y - \frac{7}{5}\right)^2 + \left(x - \frac{1}{5}\right)\left(y - \frac{7}{5}\right) - \left(x - \frac{1}{5}\right)^2 = c$$

$$\Rightarrow xy + y^2 - 3y = x^2 + x + c'$$

③ Linear Differential Equations ⇒

↳ A d.e. is said to be linear if the dependent variable and its derivative occurs in the first degree only.

Case-1 The d.e. $\frac{dy}{dx} + Py = Q$ —① is linear in y.

Where P & Q → f^n of x (or constant) only.

Method to solve ⇒ find I.F. $= e^{\int P dx}$

multiply by I.F. by eq. ①

$$\frac{dy}{dx} e^{\int P dx} + P y e^{\int P dx} = Q e^{\int P dx}$$

$$\frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P dx}$$

on integrating,

$$y e^{\int P dx} = \int Q e^{\int P dx} + C$$

$$\Rightarrow y(\text{I.F.}) = \int Q (\text{I.F}) \, dx + C$$

Case-2 The d.e. $\frac{dx}{dy} + Px = Q$

$$\text{I.F.} = e^{\int P dy}$$

Sol. ⇒ $x(\text{IF}) = \int Q (\text{IF}) \, dy + C$

Q: write the I.F. of the following d.e.

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy \qquad \text{(RTU-2024)}$$

Sol:

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \underbrace{\frac{1}{1+y^2}}_{P}\,x = \underbrace{\frac{\tan^{-1}y}{1+y^2}}_{Q}$$

$$\text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy}$$

$$= e^{\tan^{-1}y} \quad \text{Ans}$$

Q: write the I.F. of the following d.e. (RTU-2023)

$$(x+2y^3)\,dy = y\,dx$$

Sol:

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \underbrace{\frac{1}{y}}_{P}\,x = \underbrace{2y^2}_{Q}$$

$$\text{IF} = e^{\int P\,dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y}$$

$$= \frac{1}{y} \quad \text{Ans}$$

Q: Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ [RTU-2021]

Sol:

$$\frac{dx}{dy} + \frac{1}{(1+y^2)}x = \frac{\tan^{-1}y}{(1+y^2)}$$

$$\frac{dx}{dy} + Px = Q$$

$$I.F. = e^{\int P\,dy} = e^{\int \frac{1}{(1+y^2)}dy} = e^{\tan^{-1}y}$$

$$\therefore \text{Sol.} \Rightarrow x(IF) = \int Q(IF)\,dy + c$$

$$\Rightarrow x\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{(1+y^2)} e^{\tan^{-1}y}\,dy + c$$

$$\Rightarrow x\,e^{\tan^{-1}y} = \int t e^t\,dt + c$$

$$\left\{ \begin{array}{l} \tan^{-1}y = t \\ \frac{1}{1+y^2}dy = dt \end{array} \right.$$

$$\Rightarrow x\,e^{\tan^{-1}y} = (t-1)e^t + c$$

$$\Rightarrow x\,e^{\tan^{-1}y} = (\tan^{-1}y - 1)\,e^{\tan^{-1}y} + c$$

$$\Rightarrow x = \tan^{-1}y - 1 + c\,e^{-\tan^{-1}y}$$

Ans.

Q: Solve: $x\log x \frac{dy}{dx} + y = 2\log x$

Sol:

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$$

$$\frac{dy}{dx} + Py = Q.$$

$$I.F. = e^{\int P\,dx} = e^{\int \frac{1}{x \log x} dx} = \log x \quad (\text{Put } \log x = t)$$

$$\therefore \text{Sol.} \rightarrow y(IF) = \int Q(IF)\,dx + C$$

$$\Rightarrow y \log x = \int \frac{2}{x} \log x \, dx + C$$

$$\Rightarrow y \log x = (\log x)^2 + C \qquad \text{Ans.}$$

Q: Solve $\sin y \frac{dy}{dx} = \cos y (1 - x \cos y)$

Sol:

$$\cancel{\frac{\sin y}{\cos^2 y} \frac{dy}{dx} - \frac{1}{\cos y} = -x}$$

$$\cancel{\frac{dy}{dx} = \frac{\cos y}{\sin y} - x \frac{\cos^2 y}{\sin y}}$$

Equation Reducible to Linear Form ⇒

↳ The eq. of the form

$$\frac{dy}{dx} + Py = Qy^n, \quad n \neq 1$$

where P & Q → fn of x only (or Const.), is called Bernoulli's equation.

It can be made linear by dividing by y^n and then putting $y^{-n+1} = v$.

$$y^{-n}\frac{dy}{dx} + P y^{-n+1} = Q$$

Put $\quad y^{-n+1} = v$

$$(-n+1)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}.$$

$$\therefore \quad \frac{1}{(-n+1)}\frac{dv}{dx} + Pv = Q \quad ; \quad (n \neq 1)$$

$$\Rightarrow \quad \frac{dv}{dx} + (-n+1)Pv = (-n+1)Q ; \quad (n \neq 1)$$

which is linear in v. This can be solved ~~easy~~ easily.

Q: solve $\frac{dy}{dx} = \frac{1}{xy(x^2y^2+1)}$ [Raj Univ. 2001, 2003, RTU- 2007]

Sol:

$$\frac{dx}{dy} = x^3y^3 + xy$$

$$\frac{dx}{dy} - xy = x^3y^3$$

$$\frac{1}{x^3}\frac{dx}{dy} - \frac{y}{x^2} = y^3 \quad \text{--- (1)} \quad (\text{dividing by } x^3)$$

let $\frac{-1}{x^2} = u \Rightarrow \frac{2}{x^3}\frac{dx}{dy} = \frac{du}{dy}$

Hence eqn (1) $\Rightarrow \frac{1}{2}\frac{du}{dy} + uy = y^3$

$$\Rightarrow \frac{du}{dy} + 2uy = 2y^3 \quad (\text{Linear in } u)$$

$$IF = e^{\int 2y\,dy} = e^{y^2}$$

$\therefore$ sol. $\rightarrow$ $u.e^{y^2} = \int 2y^3 e^{y^2} dy + C$

let $y^2 = t \Rightarrow 2y\,dy = dt$

$$\Rightarrow ue^{y^2} = \int te^t\,dt + C$$

$$ue^{y^2} = (t-1)e^t + C$$

$$u = (y^2-1) + Ce^{-y^2}$$

$$\Rightarrow \frac{-1}{x^2} = (y^2-1) + Ce^{-y^2} \quad \text{Ans.}$$

Q: Solve: $\frac{dy}{dx} = e^{x-y}(e^{x} - e^{y})$ (Raj Univ. 2007)

Sol:

$$\frac{dy}{dx} = e^{2x} \cdot e^{-y} - e^{x}$$

$$\frac{dy}{dx} + e^{x} = e^{2x} \cdot e^{-y}$$

$$e^{y}\frac{dy}{dx} + e^{x} \cdot e^{y} = e^{2x} \quad \text{——} (1)$$

$$e^{y} = v$$

$$e^{y}\frac{dy}{dx} = \frac{dv}{dx} \quad \text{——} (2)$$

Using (1) and (2) $\frac{dv}{dx} + e^{x} v = e^{2x}$ (L. diff eq. in v)

$$\text{I.F.} = e^{\int e^{x} dx} = e^{e^{x}}$$

Therefore Sol. $\Rightarrow$ $v e^{e^{x}} = \int e^{2x} e^{e^{x}} dx + c$

$$= \int e^{x} \cdot e^{x} e^{e^{x}} + c$$

$e^{x} = t$ (let)

$$= \int t e^{t} dt + c$$

$$\Rightarrow v e^{e^{x}} = (t-1) e^{t} + c$$

$$\Rightarrow e^{y} e^{e^{x}} = e^{e^{x}}(e^{x} - 1) + c$$

$$\Rightarrow e^{y} = e^{x} - 1 + c e^{-e^{x}}$$ Ans.

(4) Exact differential equations →

↳ A d.e. is called 'exact' if it can be obtained from its primitive by direct differentiation (without any subsequent simplification involving elimination or cancellation of common factors etc.)

An eq. of first order and first degree given by –

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{—— (1)}$$

is said to be exact diff if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{—— (2)}$$

Condition of exactness
(RTU-2023)

Method to solve:

(1) Check the eq. of the form (1) is exact or not with the help of the condition (2)

(2). If the eq. → exact, then compute

$$U(x,y) = \int M\,dx \quad (\text{keeping } y \to \text{Const.})$$

(3) Compute $\frac{\partial U}{\partial y}$

(4) Compute $V(y) = \int \left(N - \frac{\partial U}{\partial y}\right) dy$

(5) The complete sol. is $U(x,y) + V(y) = c$

Q: Solve: $(1+4xy+2y^2)\,dx + (1+4xy+2x^2)\,dy = 0$

Sol: $M = (1+4xy+2y^2)$ $\quad\quad N = 1+4xy+2x^2$

$\frac{\partial M}{\partial y} = 4x+4y$ $\quad\quad \frac{\partial N}{\partial x} = 4y+4x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ given eq. is exact

now, $U = \int M\,dx = \int (1+4xy+2y^2)\,dx$

$= x + 2x^2y + 2xy^2$ $\quad (y \rightarrow \text{const})$

$\frac{\partial U}{\partial y} = 2x^2 + 4xy$

$V(y) = \int \left(N - \frac{\partial U}{\partial y}\right) dy$

$= \int (1)\,dy = y$

$\therefore$ Sol. is $U + V = C$

$\Rightarrow x + 2x^2y + 2xy^2 + y = C$ Ans.

Q: Solve: $\left(3x^2y + \frac{y}{x}\right) dx + (x^3 + \log x)\,dy = 0$ (RTU-2021)

Sol: $M = 3x^2y + \frac{y}{x}$ $\quad\quad N = x^3 + \log x$

$\frac{\partial M}{\partial y} = 3x^2 + \frac{1}{x}$ $\quad\quad \frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ given eq. is exact.

Now, $\quad U = \int M\,dx = \int \left(3x^2y + \frac{y}{x}\right) dx \qquad y \rightarrow \text{Const}$

$$= x^3y + y\log x$$

$$\frac{\partial U}{\partial y} = x^3 + \log x$$

$$V = \int \left(N - \frac{\partial U}{\partial y}\right) dy$$

$$= \int (x^3 + \log x - x^3 - \log x)\,dy$$

$$= \cancel{x}\ 0 \quad \text{or (any Const.)}$$

$\therefore$ Sol. $\Rightarrow \quad U + V = C$

$$\Rightarrow x^3y + y\log x + 0 = C$$

$$\Rightarrow x^3y + y\log x = C \qquad \text{A.}$$

Q:

Equations Reducible to Exact Form ⇒

↳ D.E. which is not exact can be made exact by multiplying it with a suitable f^n of x & y, known as integrating factor (I.F.)

Method for finding I.F. ⇒

① By insepection

② If d.e. $M\,dx + N\,dy = 0$ is of the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$$

then it's I.F. $= \dfrac{1}{Mx - Ny}$ where $Mx - Ny \neq 0$

③ If $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] = f(x)$

then I.F. $= e^{\int f(x)\,dx}$

④ If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) = \phi(y)$

then I.F. $= e^{\int \phi(y)\,dy}$

⑤ If a d.e. of the form

$$x^a y^b (my\,dx + nx\,dy) + x^c d^c (py\,dx + qx\,dy) = 0$$

then I.F. $= x^h y^k$

Where h and k determined by $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

⑥ If eq. is homogeneous then,

$$I.F. = \frac{1}{Mx + Ny}, \quad Mx + Ny \neq 0$$

Q: Solve: $(x^4y^4 + x^2y^2 + xy)y\,dx + (x^4y^4 - x^2y^2 + xy)x\,dy = 0$

(RTU - 2019, 2023, 2024)

Sol. Form $\Rightarrow f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

$$\therefore I.F. = \frac{1}{Mx - Ny}, \text{ where } Mx + Ny = 0$$

Now, $M = (x^4y^4 + x^2y^2 + xy)y$

$N = (x^4y^4 - x^2y^2 + xy)x$

$$\because Mx - Ny = 2x^3y^3 \quad \Rightarrow I.F. = \frac{1}{2x^3y^3}$$

Multiplying the above eq. by $\frac{1}{2x^3y^3}$, we get —

$$\Rightarrow \frac{1}{2x^3y^3}\left[(x^4y^4 + x^2y^2 + xy)y\,dx\right] + \frac{1}{2x^3y^3}\left[(x^4y^4 - x^2y^2 - xy)x\,dy\right] = 0$$

$$\Rightarrow \underbrace{\left[xy^2 + \frac{1}{x} + \frac{1}{x^2y}\right]}_{M_1}dx + \underbrace{\left[x^2y - \frac{1}{y} + \frac{1}{xy^2}\right]}_{N_1}dy = 0$$

$$\left[\text{Here } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}\right]$$

$$U = \int M_1\,dx = \int\left(xy^2 + \frac{1}{x} + \frac{1}{x^2y}\right)dx$$

$$U = \frac{x^2y^2}{2} + \log x - \frac{1}{xy}$$

$$\Rightarrow \frac{\partial U}{\partial y} = x^2 y + \frac{1}{xy^2},$$

$$\therefore V = \int \left(N_1 - \frac{\partial U}{\partial y}\right) dy$$

$$= \int -\frac{1}{y}\, dy = -\log y$$

Hence sol. is $U + V = C$

$$\Rightarrow \frac{x^2y^2}{2} + \log\left(\frac{x}{y}\right) - \frac{1}{xy} = C \quad \text{A.}$$

Q: solve: $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$

Sol: let I.F. $= x^h y^k$

multiplying the given eq. by $x^h y^k$, we get

$$\left(2x^h y^{k+1} + 6x^{h+1}y^{k+2}\right) dx + \left(3x^{h+1}y^k + 8x^{h+2}y^{k+1}\right) dy = 0 \quad \text{—(1)}$$

$\because$ the eq. (1) is exact.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow 2(k+1)x^h y^k + 6(k+2)x^{h+1}y^{k+1} = 3(h+1)x^h y^k + 8(h+2)x^{h+1}y^{k+1}.$$

$$\left.\begin{aligned} &\Rightarrow 2(k+1) = 3(h+1) \\ &\& \quad 6(k+2) = 8(h+2) \end{aligned}\right\} \Rightarrow h = 1,\ k = 2$$

$$\therefore \text{I.F.} = xy^2$$

from eq^n(1) $\Rightarrow (2xy^3 + 6x^2y^4)\,dx + (3x^2y^2 + 8x^3y^3)\,dy = 0$

$\therefore \quad U = \int M dx = \int (2xy^3 + 6x^2y^4)\, dx$

$$= x^2y^3 + 2x^3y^4$$

$$\frac{\partial U}{\partial y} = 3x^2y^2 + 8x^3y^3$$

$$V = \int \left(N - \frac{\partial U}{\partial y}\right) dy = \int 0\, dy = C_1$$

$\therefore$ Sol. $\Rightarrow \quad U + V = C_2$

$\Rightarrow x^2y^3 + 2x^3y^4 = C$ Ans. $[C = C_2 - C_1]$

A d.e. of first order but not of first degree $\Rightarrow$

form $\Rightarrow$ $P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p + P_n = 0$

$(n \geq 2, \; n \in N)$

or $f(x, y, p) = 0$ ——— ①

where $p = \dfrac{dy}{dx}$

and $P_0, P_1 \dots P_n$ are fn of x & y.

While solving eq. ①, the following 4- cases may arise.

Case-1 Equation solvable for p $\Rightarrow$

Q: Solve the following : $p^2 + p(x+y) + xy = 0$

Sol: $p^2 + px + py + xy = 0$

$\Rightarrow (p+x)(p+y) = 0$

$\Rightarrow p + x = 0$ or $p + y = 0$

$\dfrac{dy}{dx} + x = 0$ $\qquad \Rightarrow \dfrac{dy}{dx} + y = 0$

$dy + x\,dx = 0$ $\qquad \Rightarrow \dfrac{dy}{y} + dx = 0$

$y + \dfrac{x^2}{2} = c_1$ $\qquad \Rightarrow \log y + x = c$

$2y + x^2 = c$

$\therefore$ Complete sol. $\Rightarrow$ $(2y + x^2 - c)(\log y + x - c) = 0$ Ans.

Q: Solve: $p^2 + 2px - 3x^2 = 0$

Sol:

$$(p+3x)(p-x) = 0$$

$p = -3x$ | $p = x$

$\frac{dy}{dx} + 3x = 0$ | $\frac{dy}{dx} - x = 0$

$dy + 3x\,dx = 0$ | $dy - x\,dx = 0$

$y + 3\frac{x^2}{2} = c_1$ | $y - \frac{x^2}{2} = c_2$

$2y + 3x^2 = c$ | $2y - x^2 = c$

The complete sol. $\Rightarrow$ $(2y + 3x^2 - c)(2y - x^2 - c) = 0$ Ans

Case-2 $\Rightarrow$ Equations solvable for y $\Rightarrow$

Q: Solve : $y = 2px - p^2$ [RTU-2023]

Sol.

$$y = 2px - p^2 \quad \text{——— (1)}$$

$$\underset{\hookrightarrow p}{\frac{dy}{dx}} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow (2x - 2p)\frac{dp}{dx} + p = 0$$

$$\Rightarrow p\frac{dx}{dp} + 2x - 2p = 0$$

$$\Rightarrow \frac{dx}{dp} + \frac{2}{p}x = 2 \quad \text{——— (2)}$$

(Linear in x)

$$\text{I.F.} = e^{\int \frac{2}{p}dx} = e^{2\log p} = e^{\log p^2} = p^2$$

$\therefore$ Sol. of (2) is

$$x(IF) = \int Q(IF)\,dy + c$$

$$\Rightarrow xp^2 = \int 2p^2\,dp + c$$

$$\Rightarrow xp^2 = \frac{2}{3}p^3 + c \Rightarrow x = \frac{2}{3}p + c p^{-2} \quad \text{——— (3)}$$

Putting value of x in eq. (1)

$$\Rightarrow y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2 =$$

$$y = \frac{1}{3}p^2 + 2cp^{-1} \quad \text{Ans}$$

Q: Solve: $y + px = x^4 p^2$ —— (1)

Sol: diff. w.r.t. x,

$$p + p + x\frac{dp}{dx} = 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

$$\Rightarrow \left(2p + x\frac{dp}{dx}\right) - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\Rightarrow (1-2px^3)\left(2p + x\frac{dp}{dx}\right) = 0$$

doesn't involve $\frac{dp}{dx}$

$\therefore$ $\cancel{\frac{dp}{dx}}$. $2p + x\frac{dp}{dx} = 0$

$$\Rightarrow \frac{dp}{p} + 2\frac{dx}{x} = 0$$

$$\Rightarrow \log p + 2\log x = \log c$$

$$\Rightarrow \log px^2 = \log c$$

$$\Rightarrow p = \frac{c}{x^2} \quad —— (2)$$

eleminating p b/w (1) & (2),

$$y = -\left(\frac{c}{x^2}\right)x + x^4\left(\frac{c^2}{x^4}\right)$$

$$y = -\frac{c}{x} + c^2 \qquad A$$

Case-3. Equations solvable for x $\Rightarrow$

Q: Solve the following: $y = 3px + 6p^2y^2$ —— (1)

Sol: Solving for x,

$$x = \frac{y}{3p} - 2py^2$$

differentiating w.r.t. x

$$\Rightarrow \frac{dx}{dy} = \frac{1}{p} = \frac{1}{3p} - \frac{y}{3p^2}\frac{dp}{dy} - 4py - 2y^2\frac{dp}{dy}$$

$$\Rightarrow 3p = p - y\frac{dp}{dy} - 12p^3y - 6p^2y^2\frac{dp}{dy}$$

$$\Rightarrow 2p(1+6p^2y) + y\frac{dp}{dy}(1+6p^2y) = 0$$

$$\Rightarrow (1+6p^2y)\left(2p + y\frac{dp}{dy}\right) = 0$$

Neglecting the first factor (does not involve $\frac{dp}{dy}$).

$$\Rightarrow 2p + y\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{dp}{p} + 2\frac{dy}{y} = 0$$

$$\Rightarrow \log p + 2\log y = \log c \qquad \text{(integrating)}$$

$$\Rightarrow py^2 = c \quad \text{—— (2)}$$

Eleminating "p" from (1) & (2)

$$y^3 = 3cx + 6c^2$$ Ans.

Q: Solve: $y = 2px + p^2y$ — (1) (RTU-2024, 2022)

Sol.

$$x = \frac{y}{2p} - \frac{py}{2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2p} + \frac{y}{2}\left(\frac{-1}{p^2}\right)\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$\frac{1}{p}$

$$\Rightarrow \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \frac{-1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{-1}{2p}\left(1 + \frac{y}{p}\frac{dp}{dy}\right) - \frac{p}{2}\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

$$\Rightarrow \left(1 + \frac{y}{p}\frac{dp}{dy}\right)\left(\frac{-1}{2p} - \frac{p}{2}\right) = 0$$

It does not have derivative term

$$\Rightarrow 1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{y}{p}\frac{dp}{dy} = -1$$

$$\Rightarrow \frac{dp}{p} = -\frac{dy}{y}$$

$$\Rightarrow \log p = -\log y + \log c$$

$$\Rightarrow py = c \Rightarrow p = \frac{c}{y} \text{ — (2)}$$

Hence, $y = 2\left(\frac{c}{y}\right)x + \left(\frac{c}{y}\right)^2 y$

Case-4. Clairaut's Equation ⇒ (RTU-2024)

↳ An equation of the form $y = px + f(p)$ —①
is called Clairaut's equation.

On differentiating eq.① w.r.t. x,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad p = p + [x + f'(p)]\frac{dp}{dx}$$

$$\Rightarrow \quad [x + f'(p)]\frac{dp}{dx} = 0$$

$$\Rightarrow \quad \frac{dp}{dx} = 0 \quad \text{—(i)}$$

and $x + f'(p)$ —(ii)

integrating (i), we get $p = c$ (Constant)

Substituting $p = c$ in ①, we get the required sol.
$$y = cx + f(c)$$

Thus the solution of Clairaut's eq. can be obtained by putting c for p in the eq.

Q: Solve the following : $(y-px)(p-1) = p$

Sol:

$$y - px = \frac{p}{p-1}$$

$$y = px + \frac{p}{p-1}$$

It is of clairaut's form, hence its sol. is –

$$y = cx + \frac{c}{c-1}$$

Q: solve: $\sin px \cos y = \cos px \sin y + p$

Sol:

$$\sin px \cos y - \cos px \sin y = p$$

$$\Rightarrow \sin(px - y) = p \qquad \cancel{or} \qquad \cancel{px}$$

$$\Rightarrow px - y = \sin^{-1} p$$

$$\Rightarrow y = px - \sin^{-1} p$$

It is ~~Claur claire~~ Calirant's form and hence its sol. is

$$y = cx - \sin^{-1} c.$$

Q: Solve: $x^2(y - px) = yp^2$ —— ①

Sol:

Let $x^2 = X$ and $y^2 = Y$

$2x\,dx = dX$ $\qquad 2y\,dy = dY$

$$\frac{dY}{dX} = \frac{y}{x}\frac{dy}{dx} \Rightarrow P = \frac{y}{x}p \text{ —— ②} \Rightarrow p = \frac{x}{y}P$$

Eq.(1) becomes,

$$\Rightarrow x^2\left(y-\frac{x^2}{y}p\right) = y\frac{x^2}{y^2}p^2$$

$$\Rightarrow (y^2-x^2p) = p^2$$

$$\Rightarrow Y = PX + P^2 \qquad \text{(Which is Clairaut's form)}$$

$\therefore$ The sol is $\Rightarrow$ $Y = cX + c^2$

$$y^2 = cx^2 + c^2 \qquad \underline{A}.$$

Q: solve: $(px-y)(py+x) = a^2p$ —— (1)

Sol:

$$\cancel{\left(\frac{x^2}{y}p - y\right)(xp+x) = a^2\frac{x}{y}p} \qquad \cancel{(\text{multiply by } \frac{x}{y})}$$

let $x^2 = X$ and $y^2 = Y$

$2x\,dx = dX$ $\qquad$ $2y\,dy = dY$

$$\Rightarrow \frac{dY}{dX} = \frac{y}{x}\frac{dy}{dx}$$

$$\Rightarrow P = \frac{y}{x}p$$

$$\Rightarrow p = \frac{xP}{y}$$

Putting value of p in eq. (1),

$$\Rightarrow \left(\frac{x^2}{y}P - y\right)(xP + x) = a^2\frac{x}{y}P$$

$$\Rightarrow (x^2P - y^2).\,x(P+1) = a^2xP$$

$$\Rightarrow (XP - Y)(P+1) = a^2P$$

$$\Rightarrow xp - y = \frac{a^2 p}{p+1}$$

$$\Rightarrow y = px - \frac{a^2 p}{p+1} \qquad \text{(Clairaut's form)}$$

$$\therefore \text{Sol.} \rightarrow \quad y = cx - \frac{a^2 c}{c+1}$$

$$\Rightarrow y^2 = cx^2 - \frac{ca^2}{c+1} \qquad \text{A.}$$