* Unit 3 Fourier Series *

*** Periodic Function** A function $f(x)$ is said to be periodic if its Time period 'T'

if $f(x+T) = f(x)$

Ex Sin x is periodic function, $T = 2\pi$ as well as Cosec x
Cos x is periodic function, $T = 2\pi$ as well as Sec x
tan x is periodic function, $T = \pi$ as well as Cot x

$\sin 2x \Rightarrow T.P. = 2\pi/2 \Rightarrow T/2$

$\sin nx \Rightarrow T.P. = 2\pi/n$

*** Even & Odd function**

if $f(-x) = f(x) \Rightarrow$ even function

if $f(-x) = -f(x) \Rightarrow$ odd function

otherwise none (neither odd nor even) function.

Ex $x, x^3, x^5 \ldots x^{2n+1}$ odd function
$x^2, x^4, x^6 \ldots x^{2n}$ even function.

$\sin x, \tan x \Rightarrow$ odd function
$\cos x \Rightarrow$ even function

$e^x, x+x^2 \Rightarrow$ none

*** Property**

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & f(x) \text{ is odd;} \\ 2\int_0^a f(x)\,dx & f(x) \text{ is even;} \end{cases}$$

→ Odd functions are always symmatric about origin.
→ even functions are symmatric about Y-axis.

Fourier Series:-

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{--- (i)}$$

$\star$ $c < x < c + 2\pi$ [Generally $c = 0$ or $-\pi$]

where

$$a_0 = \frac{1}{2\pi} \int_c^{c+2\pi} f(x)\, dx \; ; \quad \text{--- (ii)} \quad (n \in I)$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx \; ; \quad \text{--- (iii)}$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx \; ; \quad \text{--- (iv)}$$

Formule (2), (3) & (4) known as Euler's formula [RTU - 2022]

put $c = -\pi$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Note for limit $-\pi$ to $+\pi$

(i) if $f(x)$ is even function then $b_n = 0$;

(ii) if $f(x)$ is odd function then $a_0 = 0$, $a_n = 0$;

Q Find Fourier series of $f(x) = |x|$, $-\pi < x \leq \pi$

Hence show that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Soln

$f(x) = |x|$ (even function)

$(b_n = 0)$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin nx\, dx$$

$b_n = 0$

So $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx$

$a_0 = \frac{2}{2\pi}\int_{0}^{\pi} (x)\,dx$

$a_0 = \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{x^2}{2}\right]_0^{\pi}$

$a_0 = \frac{1}{2\pi}\left[\pi^2 - 0\right]$

$a_0 = \frac{\pi}{2}$ (RTU 2024)

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx.$

$a_n = \frac{2}{\pi}\int_{0}^{\pi} \underset{I}{x}\,\underset{II}{\cos nx}\,dx$

$a_n = \frac{2}{\pi}\left[x\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\frac{d(x)}{dx}\cdot\int_0^{\pi}\cos nx\,dx\right]$

$-\sin nx$

$a_n = \frac{2}{\pi}\left[\left[\frac{x\sin nx}{n}\right]_0^{\pi} + \left[\frac{\cos nx}{n^2}\right]_0^{\pi}\right]$

$a_n = \frac{2}{\pi}\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi - \cos 0}{n^2}\right]$

$a_n = \frac{2}{\pi}\left[0\ \frac{\cos n\pi - 1}{n^2} - \frac{\cancel{\cos n\pi}}{n^2}\right]$

~~$a_n = \frac{-4}{\pi n^2}$~~ $a_n = \frac{2}{n^2\pi}(\cos n\pi - 1)$

$a_n = \begin{cases} \frac{-4}{\pi n^2}; & n \to \text{odd} \\ 0; & n \to \text{even} \end{cases}$

Now the series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$f(x) =$ for n even

$$|x| = \pi/2 + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[\cos n\pi - 1] \cos nx + 0$$

$$|x| = \pi/2 + \left(-\frac{4}{\pi 1^2}\cos x + 0 - \frac{4}{\pi 3^2}\cos 3x - \frac{\cos 5x}{\pi 5^2} - \right.$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$ Answer

Now put $x = 0$

$$0 = \pi/2 - 4/\pi\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Hence Proved

Q. Find the fourier Series of the function
$f(x) = x^2, \quad -\pi < x < \pi$
Hence Show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots$$ (RTU-2023-24)

Soln Given $f(x) = x^2$ (even function)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$b_n = \int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0$$

Now for

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\mathrm{I}}{x^2} \underset{\mathrm{II}}{\cos nx}\, dx$$

by by-parts

$$a_n = \frac{2}{\pi}\left[x^2 \int_0^{\pi} \cos nx\, dx - \int_0^{\pi} \frac{d(x^2)}{dx} \int_0^{\pi} \cos nx\, dx \right]$$

$$a_n = \frac{2}{\pi}\left[\left[x^2\left(\frac{\sin nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} \frac{2x \sin nx}{n}\, dx\right]$$

$$a_n = \frac{2}{\pi}\left[\left[\frac{x^2 \sin nx}{n}\right]_0^{\pi} - 2\left[-\left(\frac{x \cdot \cos nx}{n^2}\right)_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n^2}\, dx\right]\right]$$

$$a_n = \frac{2}{\pi}\left[0 - 2\left[-\left(\frac{x \cos nx}{n^2}\right)_0^{\pi} + \left(\frac{\sin nx}{n^3}\right)_0^{\pi}\right]\right]$$

$$a_n = \frac{2}{\pi}\left[-2\left[-\frac{\pi \cos n\pi}{n^2} - 0\right] + 0\right]$$

$$a_n = \frac{2}{\pi}\left[+\frac{2\pi \cos n\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}\cos n\pi$$

★ [Note $\sin n\pi = 0$
$\cos n\pi = (-1)^n$] ★

Now the fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi \cdot \cos nx$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=0}^{\infty} \frac{4}{n^2} \cos n\pi \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 0 - 4\cos x + \cos 2x - \frac{4}{3^2}\cos 3x$$

$$+ \frac{4}{4^2}\cos 4x \; ----$$

$$x^2 = \frac{\pi^2}{3} - \frac{4\cos x}{1^2} + \frac{4\cos 2x}{2^2} - \frac{4}{3^2}\cos 3x + \frac{4}{4^2}\cos 4x \ldots$$

Now Put $x = 0$

$$\Rightarrow \quad 0 = \frac{\pi^2}{3} - \frac{4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \frac{4}{4^2} - \frac{4}{5^2} + \frac{4}{6^2} + \ldots$$

$$\Rightarrow \quad \frac{\pi^2}{3} \times \frac{-1}{4} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \; ----$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \; ----$$

Hence Proved

Q Find the Fourier Series of the function

$$f(x) = \begin{cases} \pi + x \; ; & -\pi < x < 0 \\ \pi - x \; ; & 0 < x < \pi \end{cases}$$

for the interval $-\pi < x < \pi$

Soln for limit $(-\pi, 0)$ $\quad f(x) = \pi + x$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} (\pi + x)\,dx$$

$$a_0 = \frac{1}{2\pi} \left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} \Rightarrow \frac{1}{2\pi}\left[\pi^2 + \frac{\pi^2}{2}\right] \Rightarrow \frac{3\pi}{4}$$

for limit $(0,\pi)$ $f(x) = \pi - x$

$$a_0 = -\left(\pi^2 - \frac{\pi^2}{2}\right)\frac{1}{2\pi} \qquad (0,\pi)$$

$$a_0 = \frac{-\pi}{4}$$

Now Complete.

$$a_0 = \frac{3\pi}{4} + \frac{\pi}{4}$$

$$a_0 = \pi/2$$

Now for

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\frac{\pi \sin nx}{n} + \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\pi \sin nx}{n} - \frac{x\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{1}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{2}{n^2} - \frac{2\cos n\pi}{n^2}\right]$$

$$a_n = \frac{2}{n^2\pi}\left[1 - \cos n\pi\right] \quad A$$

$$a_n = \begin{cases} \frac{4}{n^2\pi} & n \to \text{odd} \\ 0 & n \to \text{even} \end{cases}$$

for $b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\sin nx\,dx$

$b_n = \frac{1}{\pi}\left[\frac{-\pi \cos nx}{n} + \right.$

(odd)

$b_n = 0$

So fourier series

$f(x) = \frac{\pi}{2} + \frac{4\cos x}{1^2\pi} + 0 \frac{4}{3^2\pi}\cos 3x + \frac{4\cos 5x}{5^2\pi} \dots$

$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \dots\right]$

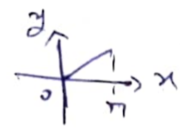
* Half range Series:-

When the fourier Series is required in the half interval that is $0 < x < \pi$, not in full interval, We can assume $f(x)$ to be even function or odd function in interval $-\pi < x < \pi$.

If function → even ⇒ $b_n = 0$ ⇒ Cosine series
function → odd ⇒ $a_n = a_0 = 0$ ⇒ Sine series

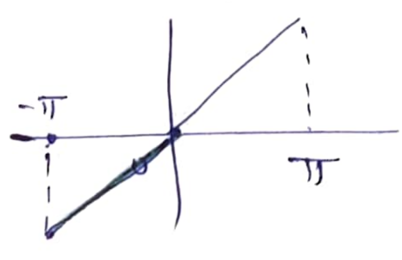
f^n → even	f^n → odd
$b_n = 0$	$a_0 = a_n = 0$
$f(x) = \sum_{n=1}^{\infty} a_n \cos nx + a_0$	$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$
Cosine Series	Sine Series

Q. find the fourier Sine & Cosine series of $f(x) = x$ in interval $0 < x < \pi$. →



Sol^n $f(x) = x$ ~~(odd function)~~

① Sine Series ⇒



(odd function)

$f(x) = x$; $-\pi < x < \pi$

$a_0 = 0$, $a_n = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{I}}{x}\, \underset{\text{II}}{\sin nx}\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[x\int \sin nx\, dx - \int \frac{d(x)}{dx}\int \sin nx\, dx\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \int 1 \cdot \frac{\cos nx}{n}\, dx\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{(-\pi)\cos n\pi}{n}\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n}\right]$$

$$\Rightarrow b_n = -\frac{2}{n}\cos n\pi$$

So fourier Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}\cos n\pi \sin nx$$

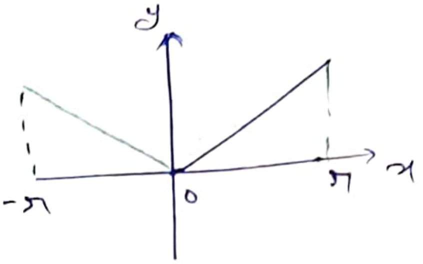
$$f(x) = 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \frac{2}{5}\sin 5x - \ldots$$

$$f(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{\sin 4x}{2} + \frac{2}{5}\sin 5x - \ldots$$

(2) Now for Cosine Series

function should be even

$f(x) = |x| \qquad -\pi < x < \pi$



$b_n = 0$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, d(x)$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\, dx$

$a_0 = \frac{1\times 2}{2\pi}\int_{0}^{\pi} x\, dx$

$a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$

$\boxed{a_0 = \pi/2}$

for $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx$

$a_n = \frac{2}{\pi}\int_{0}^{\pi} \underset{\text{I}}{x} \underset{\text{II}}{\cos nx}\, dx$

$a_n = \frac{2}{\pi}\left[x\int \cos nx\, dx - \int \frac{d(x)}{dx}\int \cos nx\, dx\right]_0^{\pi}$

$a_n = \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$

$a_n = \frac{2}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2} - 0\right]$

$a_n = \frac{2(\cos n\pi - 1)}{n^2\pi}$

So series is

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$f(x) = \frac{\pi}{2} + \frac{(-4)}{\pi}\cos x + 0 - \frac{4}{3^2\pi}\cos 3x + 0 - \frac{4}{\pi 5^2}\cos 5x \ldots$

$f(x) = \pi/2 - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$

* Change of Interval :-

$$(-l < x < l)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right]$$

("General formula for fourier series")

where,
$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Q Find half range cosine series of function $f(x) = (2x-1)$ in $0 < x < 1$ [RTU 2023]

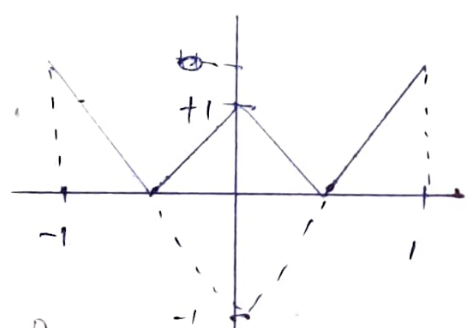
Sol^n Given function

$$f(x) = (2x-1) \qquad 0 < x < 1$$

for cosine series, we assume even function.

$$f(x) = |2x-1|$$

by half range series -



$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$$

$$a_0 = \frac{1}{2(1)} \int_{-1}^{1} |2x-1|\, dx \quad \frac{1}{2}\left[-\int_{-1}^{0} (2x-1) + \int_{0}^{1} (2x-1)\, dx \right]$$

$$a_0 = \frac{1}{2} \int (2x-1)\, dx \quad \frac{1}{2}\left[[-x^2+x]_{-1}^{0} + [x^2-x]_0^1 \right]$$

$$\frac{1}{2}\left(-0+0-(-1-1)\right) \Rightarrow$$

$$a_0 = \frac{2}{2}\left[\frac{2x^2}{2} - x\right]_0^1$$

$$a_0 = \frac{2}{2}\left[1 - 1 - (0) + (0)\right].$$

$$\boxed{a_0 = 0} \qquad b_n = 0$$

$\int_{-1}^{-1/2}(\ldots)\,dx + \int_{-1/2}^{1/2} + \int_0^{1/2}$

$-[x^2-x]_{-1}^{-1/2} + [x^2-x]_{-1/2}^{0}$

$-(x^2-x)_0^{1/2} + (x^2-x)$

$\Rightarrow -\left(\frac{1}{4}+\frac{1}{2}\right)+(1+1)+-\frac{1}{4}+\frac{1}{2}$

$-\frac{1}{4}+\frac{1}{2}+1+1+\frac{1}{4}$

$-\frac{1}{2}\times 2 = (-1)$

Now for $a_n = \frac{1}{l}\int_{-1}^{1} f(x)\,\frac{\cos n\pi x}{l}\,dx$

$$a_0 = \frac{1}{1}\int_{-1}^{1}|2x-1|\,\frac{\cos n\pi x}{1}\,dx$$

$$a_0 = 2\int_0^1 \underset{\mathrm{I}}{(2x-1)}\,\underset{\mathrm{II}}{\cos n\pi x}\,dx$$

$$a_0 = 2\left[\left[(2x-1)\frac{\sin n\pi x}{n\pi}\right]_0^1 - \int_0^1 2\,\frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$a_0 = 2\left[1\cdot\frac{\sin n\pi}{n\pi} - 0 + \left(\frac{2\cos n\pi x}{n^2\pi^2}\right)_0^1\right]$$

$$a_0 = 4\left[\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 = \frac{4}{n^2\pi^2}\left[\cos n\pi - \cos(0)\right]$$

$$a_0 = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \quad \begin{cases} -\frac{8}{n^2\pi^2} & ; \; n \in \text{odd} \\ 0 & ; \; n \in \text{even} \end{cases}$$

So the series is

$$f(x) = -1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos n\pi x$$

$$f(x) = -1 - \frac{8}{\pi^2}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right] \text{ Ans}$$

05/11/24

* Parseval's Theorem :-

let $f(x) \rightarrow$ Periodic function

(Period $\Rightarrow 2l$)

then

$$\boxed{\frac{1}{2l}\int_{c}^{c+2l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

where,

$$a_0 = \frac{1}{2l}\int_{c}^{c+2l} f(x)\,dx \qquad b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Q Find the fourier series for $f(x) = x^2$ in $(-\pi, \pi)$ Hence using parseval's the : prove that -

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

(RTU - 2019 & 24)

$(\because c \rightarrow -l)$

Sol^n from the previous question -

$$a_0 = \frac{\pi^2}{3} \qquad a_n = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

$$b_n = 0$$

So fourier series

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi \cos nx$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right]$$

Now for Parseval's

$$\frac{1}{2l}\int_{c}^{c+2l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$

$(l=\pi)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2)^2\,dx = \left(\frac{\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(\frac{4}{n^2}(-1)^n\right)^2+0^2\right]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}x^4\,dx = \frac{\pi^4}{9} + \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{16}{n^4}\right]$$

$$\frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{\pi^4}{9} + 8\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right]$$

$$\frac{1}{2\pi}\left[\frac{\pi^5}{5}+\frac{\pi^5}{5}\right] = \frac{\pi^4}{9} + 8\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)$$

$$\pi^4\left(\frac{1}{5}-\frac{1}{9}\right) = 8\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)$$

$$\pi^4\left(\frac{\cancel{4}}{45}\right) = \overset{2}{\cancel{8}}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)$$

$$\boxed{\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}}$$

Hence Proved