

# Introduction to Electromagnetism

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- Divergence of vector field
- Curl of vector field

## ➤ Maxwell Equation

- Gauss law of electrostatic
- Gauss law of Magnetostatic
- Faraday's Law of Electromagnetic Induction
- Ampere's circuital law
- Concept of Displacement Current

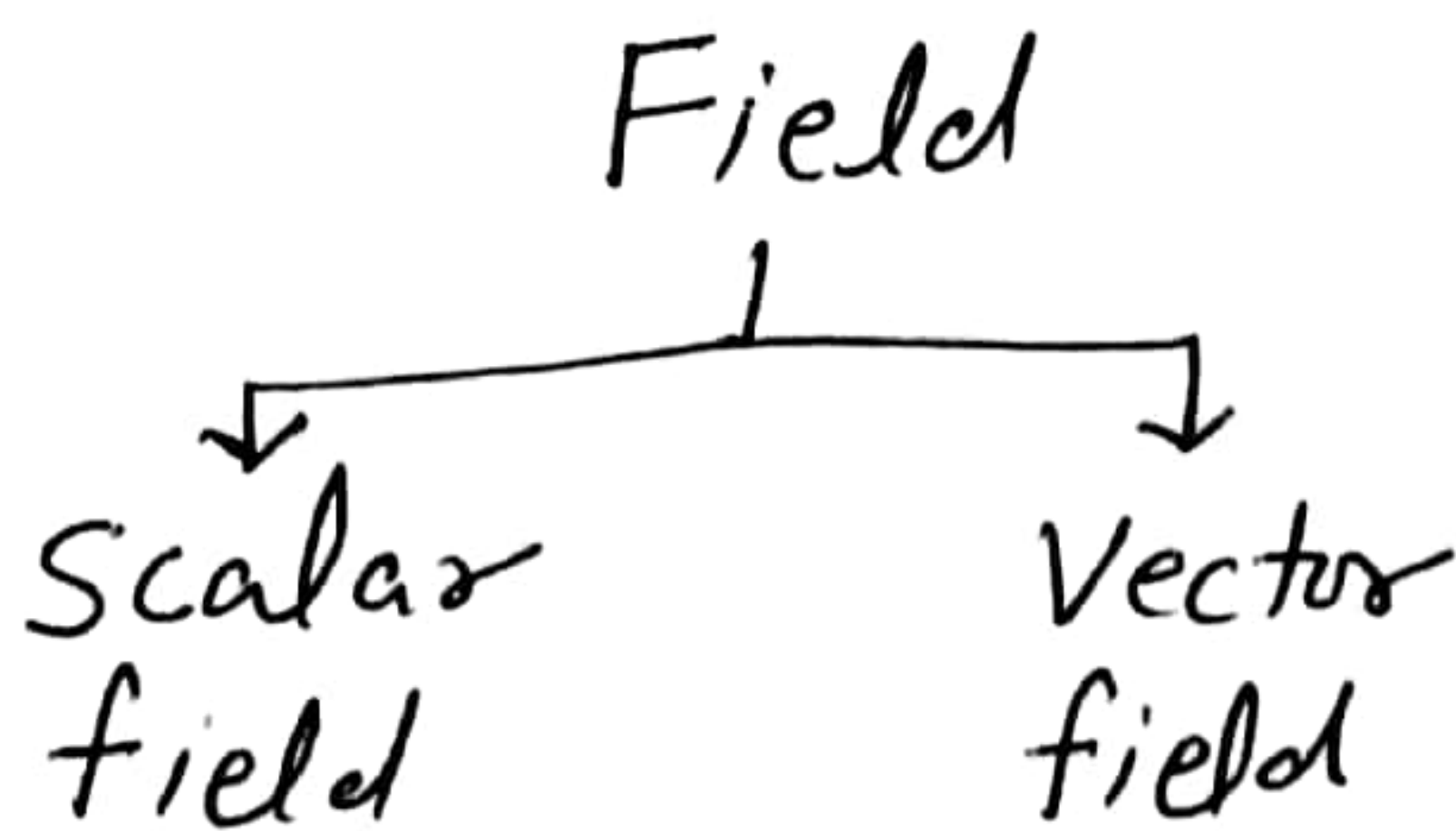
## ➤ Poisson's and Laplace equation

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# Introduction to Electromagnetism

Fields: In physics two types of quantities are studied which are known as scalar and vector quantities. To express the effect of these quantities it is necessary to express these as a function of space and time coordinates. The space in which a physical quantity is expressed and the effect of this quantity can be studied on coordinates of each point of space is called field.



Scalar field: Space in which value of scalar quantity expressed as a function of coordinates of each point of space is called scalar field.  
If coordinates of a point in space is  $(x, y, z)$  and

Quantity expressed in space be  $\phi$ , then

$$\phi = \phi(x, y, z)$$

Example: Gravitational potential, Electric potential, Temperature, density etc. are scalar fields.

★ Vector field: If vector quantity depends on the coordinates of observable points, then that field is called vector field

Example: Gravitational field, electric field and magnetic field etc.

If vector quantity  $\vec{A}$  is defined in vector field and  $(x, y, z)$  be the coordinates of observable point, then vector field can be expressed as

$$\vec{A} = \vec{A}(x, y, z)$$

★ Del operator ( $\vec{\nabla}$ )

$$\vec{\nabla} \equiv \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

$\vec{\nabla}$  Perform two vector product

$$(I) \text{ Divergence } \vec{A} = \vec{\nabla} \cdot \vec{A} \quad (II) \text{ Curl } \vec{A} = \vec{\nabla} \times \vec{A}$$

\* Area: (i) Area is a vector quantity

\* open surface: A close path always bound an open surface.

Example: Circle, Square, Triangle

Direction of area vector is always perpendicular to the surface

\* Close surface: A close surface always bound volume

example: Sphere, Cone, Cylinder

In this case direction of area vector is ~~at~~ always normal outwards.

Divergence of a vector field

Total out going flux per unit volume for closed surface enclosing small volume element  $dV$  of any vector field  $\vec{A}(x, y, z)$  is called divergence of vector  $\vec{A}$ .

This can be written as

$$\text{div } \vec{A} = \lim_{dV \rightarrow 0} \frac{1}{dV} \left[ \oint_S \vec{A} \cdot d\vec{S} \right] = \vec{\nabla} \cdot \vec{A}$$

Here symbol  $\oint_S$  represents integration over close surface

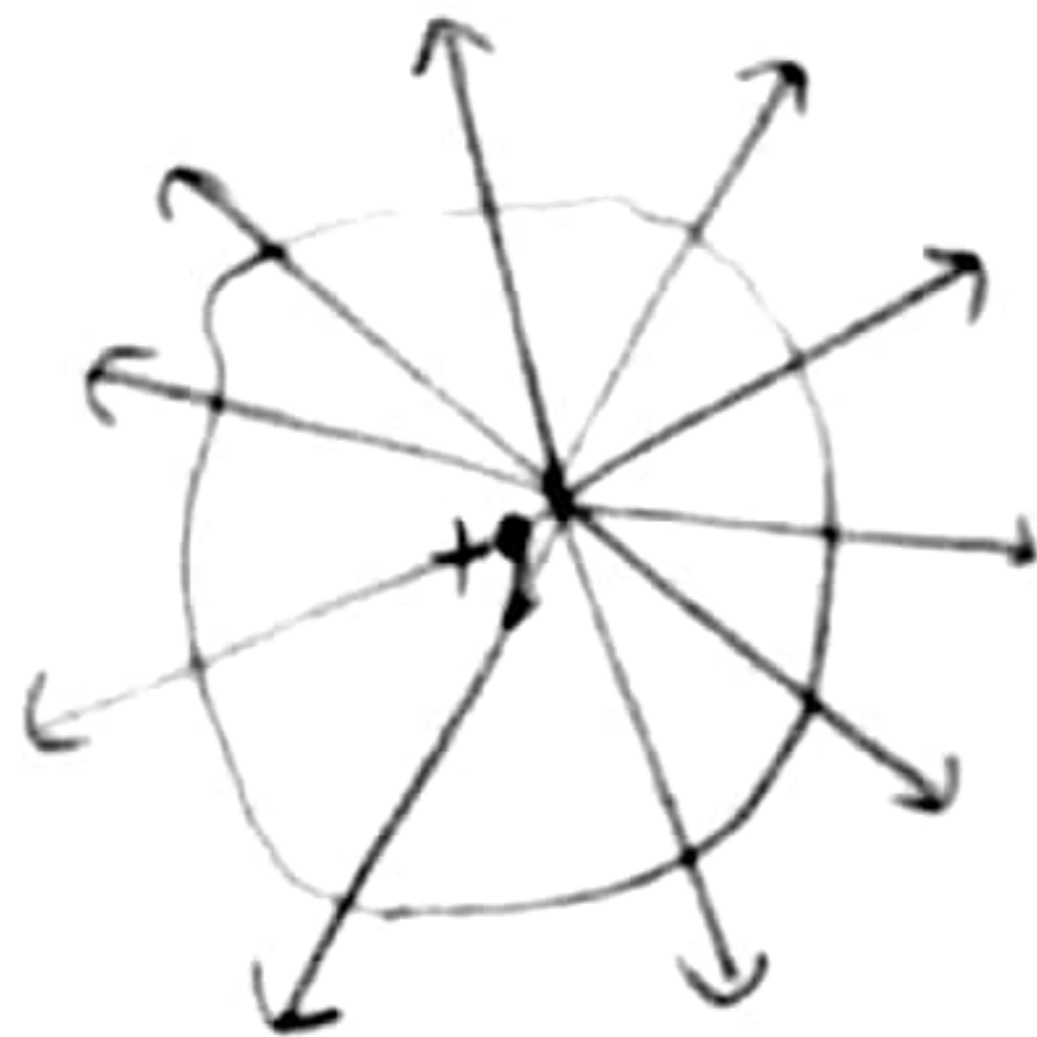
\* Divergence of a vector field is a scalar quantity

Physical significance of divergence:

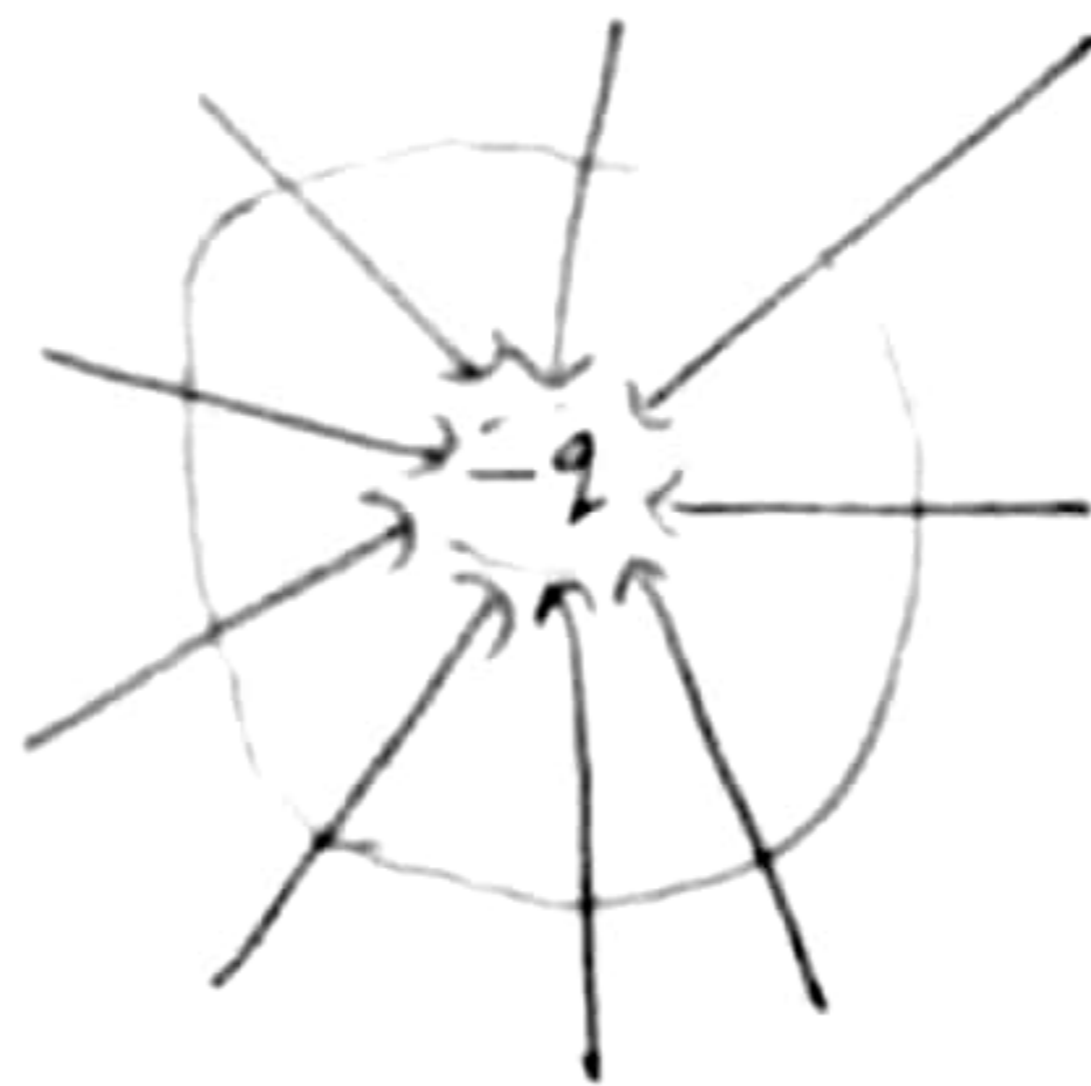
(a) For outgoing flux,  $\oint_S \vec{E} \cdot d\vec{s}$  will be positive i.e.

$\oint_S \vec{E} \cdot d\vec{s} > 0$ . We conclude that charge inside the

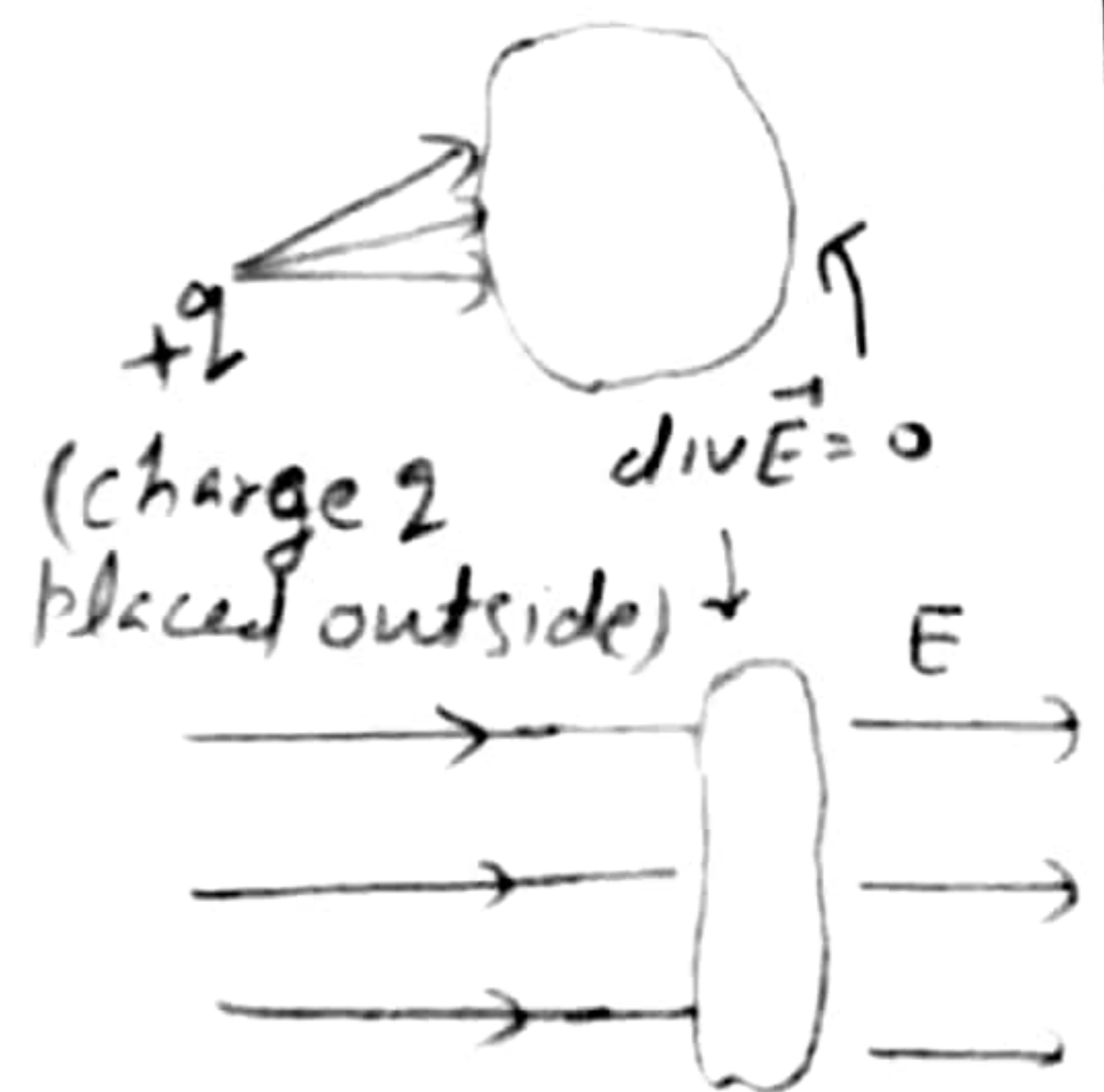
closed surface is positive. In fig 1(a), electric field acts outwards and divergence of electric field will be positive



$\text{div } \vec{E} = (+ve)$



$\text{div } \vec{E} = -ve$



(b) If outgoing flux from closed surface is less than incoming flux, then total flux will be negative

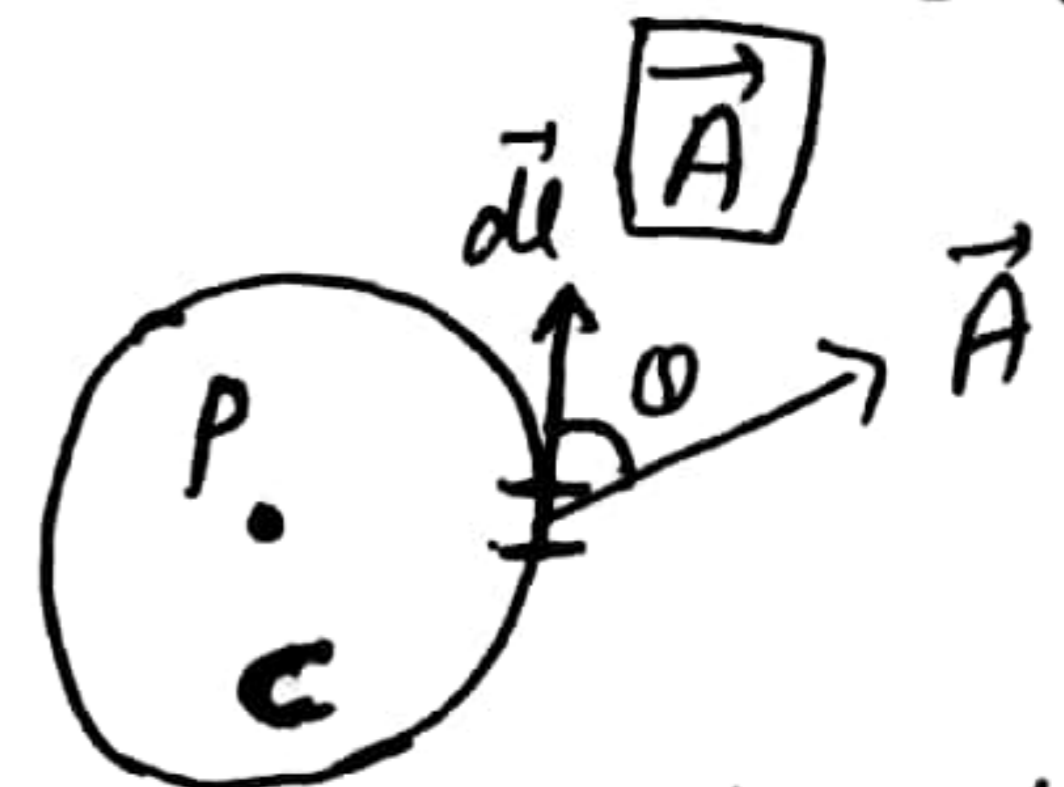
$\oint_S \vec{E} \cdot d\vec{s} < 0$ . In this position direction of electric field will be inward and nature of charge will be negative

In fig 1(b), divergence of  $E$  will be obtained negative

(iii) If outgoing flux from closed surface is zero i.e.  $\oint_S \vec{E} \cdot d\vec{S} = 0$ , then there will be no source to produce field inside the surface i.e., source may lie outside the surface. Hence the Divergence of vector field will be zero fig 1(c).

### Curl of a vector field:

- \* Curl of any vector field is related to line integral of that vector
- \* Curl of a vector field is a vector quantity
- \* To understand the term line integral, let us consider a point  $P$  in vector field  $\vec{A}$  and path  $C$  enclosing the point. Let us divide this path into small length element  $d\vec{l}$ . If the surface area enclosed by path  $C$  is small, means vector  $\vec{A}$  remains same on each point of this path, then the term  $\oint_C \vec{A} \cdot d\vec{l}$  is called line integral of  $\vec{A}$  over closed path  $C$ .



[ $da$  = area bound by closed path]

So, curl of any vector field is defined as the line integral per unit infinitesimal small surface area enclosed by the path.

$$\text{Curl } \vec{A} = \lim_{da \rightarrow 0} \left[ \frac{\oint_c \vec{A} \cdot d\vec{l}}{da} \right] \hat{n}$$

It's direction is perpendicular to the plane in which the line integral along the boundary of surface element is maximum.

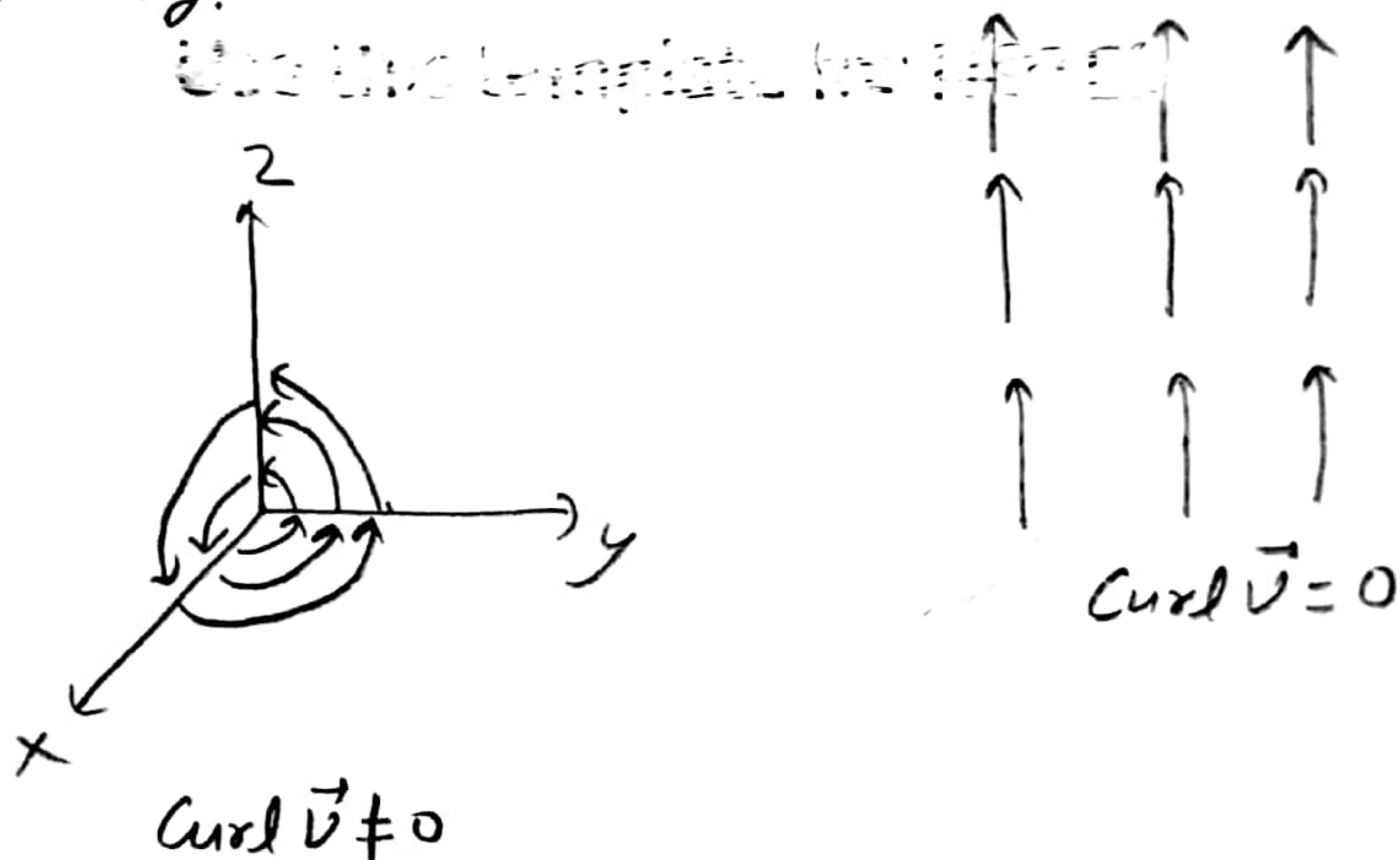
$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = \left[ \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \times [A_x \hat{i} + A_y \hat{j} + A_z \hat{k}]$$

$$\text{Curl}(\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Physical Significance:

The term "curl" is a measure of how much the vector  $\vec{A}$  'curls around' the point of observation. Curl of any vector field comes from its rotational motion.

If liquid flows in a particular direction then ~~curl~~ curl of it's velocity is zero, unless it has rotational motion ( $\uparrow \rightleftarrows$  or  $\downarrow \rightleftarrows$ ). In the bottom of any water tub, if there is a hole then water outgoes with rotational motion i.e.  $\neq$  it means for this case ( $\text{Curl } \vec{V} = 0$ ) As shown in fig.



Solenoidal vector: A vector field  $\vec{A}$  is such that

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = 0$$

then  $\vec{A}$  is called solenoidal vector

Example:  $\text{div } \vec{B} = 0$  so  $\vec{B}$  (Magnetic field intensity) is a solenoidal vector.

Irrrotational vector: A vector field  $\vec{A}$ , such that

$$\text{Curl } \vec{A} = 0 = \nabla \times \vec{A}$$

then  $\vec{A}$  is called Irrrotational vector field

Example: For static field

$\nabla \times \vec{E} = 0$ ,  $\vec{E}$  (Electric field intensity) is irrrotational vector.

~~Equation (1) is called for Poisson's~~

### Poisson and Laplace Equation

This equation is used to solve the boundary value problems. When electrostatic conditions at some boundaries are given and we have to find the  $E$  and  $V$  throughout the region, these problems are known as boundary value problem

From Gauss law

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{--- (1)} \quad [\rho = \text{Volume charge density}]$$

As we know  $\vec{E} = -\vec{\nabla} V$  --- (2)

Put eq (2) into eq. (1)

$$\vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \quad \text{--- (3)}$$

This equation is called Poisson equation

For a charge free region  $\rho = 0$

$$\boxed{\nabla^2 V = 0} \quad - (4)$$

Above equation known as Laplace equation, Poisson equation is for inhomogeneous medium and Laplace equation is for homogeneous medium with  $\rho = 0$ .

### Gauss Divergence Theorem

Surface integration of any vector field over a closed surface, is equal to the volume integration of its divergence. This is called Gauss's divergence theorem.

$$\oint_S \vec{A} \cdot d\vec{a} = \int_V \text{div} \vec{A} \, dV = \int_V (\vec{\nabla} \cdot \vec{A}) \, dV$$

This theorem is used to convert surface integral to volume integral or vice-versa.

### Stokes Curl Theorem

Stokes Curl theorem states that line integration of any vector field along the closed path is equal to the surface integration of curl of that vector, over the surface bounded by closed path. This is called

# Maxwell Equation in Electromagnetism

There are four fundamental equations in electromagnetism which governs all phenomenon based on electricity and magnetism are called Maxwell equations

Maxwell equations based on four fundamental laws of electricity and magnetism

- (I) Gauss law of Electrostatic
- (II) Gauss law of Magnetostatic
- (III) Faraday's law of Electro magnetic induction (EMI)
- (IV) Ampere's circuital law

1 Maxwell I equation  
(Gauss law of electrostatic)

Gauss law states that net electric flux passes through closed surface is equal to  $\frac{1}{\epsilon_0}$  times total charge enclosed by closed surface.

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \quad \text{--- (1)}$$

As we know, A closed surface always bound volume

$$q = \int_V \rho \, dV \quad [\rho = \text{volume charge density}]$$

Put value of  $q$  in eq. (1) so

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_V \rho \, dV \quad \text{--- (2)}$$

eq. (2) is called Integral form of Maxwell I equation.

Now apply Gauss-divergence theorem

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{E}) \, dV$$

use above relation in eq. (2)

$$\int_V (\nabla \cdot \vec{E}) \, dV = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

$$\int_V \left[ \nabla \cdot \vec{E} - \frac{\rho}{\epsilon_0} \right] dV = 0$$

$$\Rightarrow \boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \quad \text{--- (3)}$$

Above eq. is called differential form of Maxwell I eq.

Rewrite eq. (3)

$$\nabla \cdot (\epsilon_0 \vec{E}) = \rho$$

$$\boxed{\nabla \cdot \vec{D} = \rho} \quad \left[ \vec{D} = \epsilon_0 \vec{E} = \text{Displacement Vector} \right]$$

Another form of Maxwell I equation.

Maxwell II equation

(Gauss law of Magnetostatic)

Gauss law states that net magnetic flux passes through a closed surface is always equal to zero.

$$\oint_S \vec{B} \cdot d\vec{a} = 0 \quad \text{--- (4)}$$

Above eq. is called integral form of Maxwell II eq.

Now apply Gauss divergence theorem

$$\oint_S \vec{B} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{B}) dV$$

use in eq. (4)

$$\int_V (\nabla \cdot \vec{B}) dV = 0$$

$$\Rightarrow \boxed{\nabla \cdot \vec{B} = 0} \quad \text{--- (5)} \quad \left[ \text{Differential form of Maxwell II equation} \right]$$

Above equation signifies that Magnetic monopole does not exist

Maxwell III equation

[ Faraday's law of electromagnetic Induction (EMI) ]

Faraday law states that an emf is induced in the closed loop when magnetic flux passes through the loop is changing

$$\text{emf} = \mathcal{E} = -\frac{\partial}{\partial t} \left[ \int_S \vec{B} \cdot d\vec{a} \right] \quad (6)$$

[ -ve sign due to Lenz's law ]

As we know

$$\text{emf} = \mathcal{E} = \left[ \oint_L \vec{E} \cdot d\vec{l} \right] \quad (7)$$

Put eq. (7) in eq. (6)

$$\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \left[ \int_S \vec{B} \cdot d\vec{a} \right]$$

$$\oint_L \vec{E} \cdot d\vec{l} = - \left[ \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \right] \quad (8)$$

Above eq. is integral form of Maxwell III<sup>rd</sup> equation

Apply Stokes and Curl theorem

$$\oint_L \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{a}$$

use above relation in eq. (8)

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{a} = - \left[ \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \right]$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}} \quad \text{--- (9)}$$

Differential form of Maxwell III equation

For static field  $\frac{\partial \vec{B}}{\partial t} = 0$

$$\boxed{\nabla \times \vec{E} = 0}$$

Maxwell IV equation

(Ampere Circuital law)

Ampere circuital law states that the line integral of magnetic field around any closed loop is equal to the product of  $\mu_0$  times sum of the current passing through the closed loop

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 \left[ \sum I \right] \quad \text{--- (10)}$$

└─ Sum over all current

Total current can be written in terms of current density ( $\vec{J}$ )

$$\Sigma I = \int_S \vec{J} \cdot d\vec{a} \quad \text{--- (11)}$$

Put ( $\Sigma I$ ) in eq. (10), so we get

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 \left[ \int_S \vec{J} \cdot d\vec{a} \right] \quad \text{--- (12) [Integral form]}$$

Now use Stokes curl theorem

$$\oint_L \vec{B} \cdot d\vec{l} = \int_S (\nabla \times \vec{B}) \cdot d\vec{a}$$

use in eq. (12)

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \left[ \int_S \vec{J} \cdot d\vec{a} \right]$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}} \quad \text{--- (13)}$$

Above relation is called differential form of Ampere's law.

But above relation is hold good only for static field

Taking both side divergence on eq. (13)

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 (\nabla \cdot \vec{J})$$

Use vector identity  $\nabla \cdot (\nabla \times \vec{B}) = 0 = \text{div}(\text{curl } \vec{B})$

so  $\mu_0 (\nabla \cdot \vec{J}) = 0$

$\mu_0 \neq 0$  so  $\boxed{\nabla \cdot \vec{J} = 0}$  — (14)

Now use equation of continuity

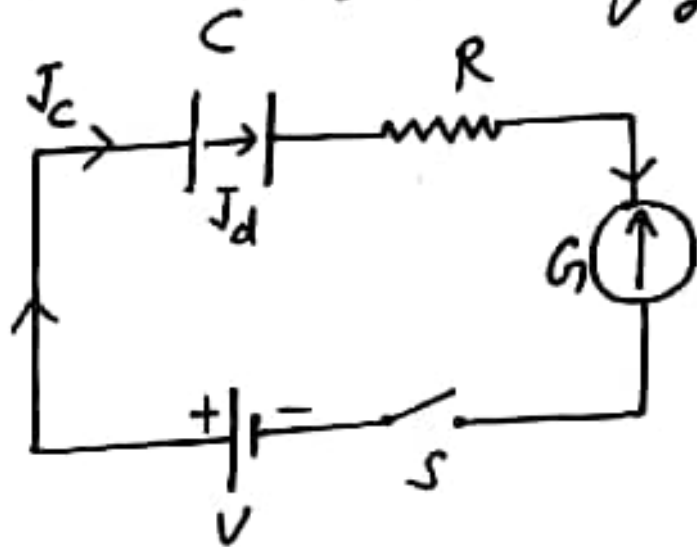
$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$  — (15)  $\left[ \begin{array}{l} \vec{J} = \text{Current density} \\ \rho = \text{Volume charge density} \end{array} \right.$

Now compare eq. (14) and (15)

$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \boxed{\rho = \text{constant}}$

Above relation show that eq. (13) only hold for static field which does not change with time. Eq. (13) does not work for time varying field. This does not explain the current passes between two plates of capacitor.

Maxwell modify above eq. (13) & introduce the concept of displacement current so ~~eq. (13)~~ <sup>it <sup>does</sup> work</sup> also work in time varying field.



At the instant, switch  $S$  is open or close, the galvanometer show deflection. But as we know current always passes in close circuit

So how we can explain the current passes between two plate of capacitor?

To explain this current, Maxwell argued that as change in magnetic flux induced current [Faraday's law] similarly change in electric flux also induced current. This current is called displacement current which passes between two plate of capacitor due to changing electric flux or the current passes through vacuum.

So total current density

$$\vec{J} = \vec{J}_c + \vec{J}_d$$

eq. (14)

$$\nabla \cdot \vec{J} = 0$$

$$\nabla \cdot (\vec{J}_c + \vec{J}_d) = 0$$

$$\nabla \cdot \vec{J}_c = -\nabla \cdot \vec{J}_d \quad (16)$$

$J_c$  = Conduction Current density due to flow of free electron in Conductor  
 $J_d$  = Displacement Current due to change in electric field between two plate of capacitor at the instant switch ON/OFF

Now eq. (15) Rewritten as

$$\nabla \cdot \vec{J}_c = -\frac{\partial \rho}{\partial t} \quad (17) \quad \left[ \text{In eq. (15) } \vec{J} \text{ is } \vec{J}_c \right]$$

Compare eq. (16) and eq. (17)

$$\nabla \cdot \vec{J}_d = \frac{\partial \rho}{\partial t} \quad (18)$$

Maxwell I equation

$$\nabla \cdot \vec{D} = \rho$$

Put  $\rho = \nabla \cdot \vec{D}$  in eq.(18)

$$\nabla \cdot \vec{J}_d = \frac{\partial}{\partial t} (\nabla \cdot \vec{D})$$

$$\nabla \cdot \vec{J}_d = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad [\vec{D} = \epsilon_0 \vec{E}]$$

So Total Current density becomes

$$\vec{J} = \vec{J}_c + \vec{J}_d$$

$$\vec{J} = \vec{J}_c + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

eq.(13) become

$$\nabla \times \vec{B} = \mu_0 \left[ \vec{J}_c + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}_c + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Differential form of Maxwell IV equation

## Poynting Theorem

An electromagnetic wave [EM wave] propagate through space from one point to another and transport energy with it. Poynting theorem establish a relationship between rate of flow of energy and amplitude of time varying electric and magnetic field. EM wave stored energy in form of electric and Magnetic field.

Energy density in form of electric field ( $U_E$ ) =  $\frac{1}{2} \epsilon_0 E^2$

Energy density in form of magnetic field ( $U_M$ ) =  $\frac{B^2}{2\mu_0}$

So, total energy density stored in EM waves =

$$U = \frac{1}{2} \epsilon_0 E^2 + \frac{B^2}{2\mu_0}$$

Total energy transport by EM waves ( $W$ ) =  $\int_V \left( \frac{1}{2} \epsilon_0 E^2 + \frac{B^2}{2\mu_0} \right) dV$

Rate of flow of energy passes through a given surface area is



Put eq.(3) and eq(4) in eq. (2)

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot \left( -\mu_0 \frac{\partial \vec{H}}{\partial t} \right) - \vec{E} \cdot \left( \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla \cdot (\vec{E} \times \vec{H}) = - \left[ \mu_0 \left( \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \right) + \epsilon_0 \left( \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right) \right] \quad (5)$$

$$(\star \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) = \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} = 2(\vec{E} \cdot \frac{\partial \vec{E}}{\partial t})) \quad [\text{use } \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}]$$

$$\therefore \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t} \quad [\vec{E} \cdot \vec{E} = E^2]$$

$$\text{similar } \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \frac{\partial H^2}{\partial t}$$

Put above value in eq. (5)

$$\nabla \cdot (\vec{E} \times \vec{H}) = - \left[ \mu_0 \frac{1}{2} \frac{\partial H^2}{\partial t} + \epsilon_0 \frac{1}{2} \frac{\partial E^2}{\partial t} \right]$$

$$= - \frac{\partial}{\partial t} \left[ \frac{1}{2} \mu_0 H^2 + \frac{1}{2} \epsilon_0 E^2 \right]$$

$$\nabla \cdot (\vec{E} \times \vec{H}) = - \frac{\partial}{\partial t} \left[ \frac{B^2}{2\mu_0} + \frac{1}{2} \epsilon_0 E^2 \right] \quad (\text{use } \vec{B} = \mu_0 \vec{H})$$

taking both side volume integral

$$\int_V \nabla \cdot (\vec{E} \times \vec{H}) dV = - \int_V \frac{\partial}{\partial t} \left[ \frac{B^2}{2\mu_0} + \frac{1}{2} \epsilon_0 E^2 \right] dV \quad (6)$$

As time and volume are independent so interchange the position of  $\int_V$  &  $\frac{\partial}{\partial t}$  and also use Gauss Divergence theorem

$$\int_V \nabla \cdot (\vec{E} \times \vec{H}) dV = \oint_S (\vec{E} \times \vec{H}) \cdot d\vec{a}$$

So eq. (6) become

$$\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{a} = -\frac{\partial}{\partial t} \left[ \int_V \left[ \frac{B^2}{2\mu_0} + \frac{1}{2} \epsilon_0 E^2 \right] dV \right] \quad - (7)$$

use eq. (1)

$$-\frac{\partial W}{\partial t} = -\frac{\partial}{\partial t} \left[ \int_V \left( \frac{B^2}{2\mu_0} + \frac{\epsilon_0 E^2}{2} \right) dV \right] \quad - (8)$$

Now compare eq. (7) and eq. (8)

$$-\frac{\partial W}{\partial t} = \oint_S (\vec{E} \times \vec{H}) \cdot d\vec{a}$$

$\vec{E} \times \vec{H}$  = Flow of energy per unit area per unit time is called Poynting vector.