Unit - 1 Matrices

Matrix → means arrangement or rectangular array.

→ A matrix is a rectangular array of $m \times n$ numbers (or f^ns) arranged in m horizontal lines (→) called rows and n vertical lines (↓) called columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

→ a_{ij} → called its elements

i → indicates the row

j → indicates the column.

→ $A = [a_{ij}]_{m \times n}$ (representation)

eg ⇒

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \\ -3 & 4 \end{bmatrix}_{3 \times 2}$$

↑ No. of rows ↑ No. of columns

Types of Matrix →

① **Row matrix** → A Matrix has only one row and any no. of columns.

Eg. $A = \begin{bmatrix} 1 & 3 & 5 & -4 \end{bmatrix}_{1\times 4}$

② **Coloumn Matrix** → A matrix has only one coloumn and any no. of row.

Eg. $A = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}_{3\times 1}$

③ **Null Matrix / Zero Matrix** ⇒ Each element of Matrix is 0

Eg. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

④ **Square Matrix** ⇒ An $m\times n$ matrix for which $m = n$.

→ $A = [a_{ij}]_{n\times n}$ [order ⇒ n]

→ $a_{11}, a_{22}, \dots a_{nn}$ ⇒ diagonal elements and the line along which they lie is called the principal diagonal.

Eg. ⇒ $A = \begin{bmatrix} \boxed{1} & 2 & 4 & 9 \\ -4 & \boxed{0} & 8 & 1 \\ 1 & 4 & \boxed{9} & -2 \\ 6 & -2 & 0 & \boxed{1} \end{bmatrix}_{4\times 4}$ ⇒ order = 4

⑤ Diagonal Matrix ⇒ A square matrix in which all the non-diagonal elements are zero. i.e. $a_{ij} = 0$ for $(i \neq j)$.

Eg ⇒ $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{3\times 3}$

⑥ Scalar Matrix ⇒ A diagonal matrix whose diagonal elements are identical i.e $a_{ii} = k$ for every i, where k is a constant.

Eg ⇒ $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{3\times 3}$

⑦ Unit Matrix/Identity Matrix ⇒ A square matrix of which all the diagonal elements are unity and non-diagonal elements are zero.

Eg ⇒ $A = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

⑧ Transpose of a matrix ⇒ Interchange the rows & coloumns of a given matrix A, then new matrix is called transpose of a matrix, represented by AT.

Eg. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 6 & -3 \end{bmatrix}_{3\times 2}$ then $A^T = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 4 & -3 \end{bmatrix}_{2\times 3}$

(9) Symmetric Matrix $\Rightarrow$ $A = A^T$

RTU-2021
Define symmetric & skew-symmetric matrices.

Eg. $\Rightarrow$ $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 5 \\ 0 & 5 & 6 \end{bmatrix}_{3\times3} = A^T$

(10) Skew-Symmetric Matrix $\Rightarrow$ $A^T = -A$

Eg. $\Rightarrow$ $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} = -A$

* Here diagonal elements $\Rightarrow$ "0"

(11) Trace of Matrix $\Rightarrow$ Sum of elements of diagonal in a square matrix.

Eg. $A = \begin{bmatrix} 1 & 2 & 4 \\ -6 & 7 & 9 \\ 4 & 0 & 1 \end{bmatrix}$

then $tr(A) = 1 + 7 + 1 = 9$

(12) Triangular Matrix $\Rightarrow$ A square matrix A, whose element below the principal diagonal are zero. $\Rightarrow$ Upper T.M.

& above $\Rightarrow$ lower T.M.

Eg. $\Rightarrow$ $A = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow$ Upper

$B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 7 & 1 & 9 \end{bmatrix} \Rightarrow$ Lower

(13) Equal Matrices ⇒ Two matrices are said to be equal if ⇒ (i) They are of the same size (order).

(ii) The corresponding elements of both matrices are same.

Eg. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = B = \begin{bmatrix} 2 & 3 \\ 4 & -3 \end{bmatrix}$

then $a = 2, \quad b = 3, \quad c = 4, \quad d = -3.$

(14) orthogonal Matrix ⇒ (RTU-2024) → Define orthogonal Matrix

If $AA^T = A^TA = I$

or $A^T = A^{-1}$

Eg. ⇒ $A = \begin{bmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{bmatrix}$

$A^T = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

⇒ A → orthogonal Matrix.

(15) Idempotent Matrix $\Rightarrow$

If $A^2 = A$

(16) Involuntary Matrix $\Rightarrow$

If $A^2 = I$.

Algebra of Matrices ⇒

① Addition / subtraction ⇒

If A and B be two matrices of same order, then $(A \pm B)$ is defined as matrix of that order.

Eg. $A = \begin{bmatrix} 2 & -1 & 4 \\ 6 & 0 & 8 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 5 \end{bmatrix}_{2\times 3}$

$$A + B = \begin{bmatrix} -2 & 5 & 7 \\ 6 & 1 & 13 \end{bmatrix}_{2\times 3}$$

$$A - B = \begin{bmatrix} 6 & -7 & 1 \\ 6 & -1 & 3 \end{bmatrix}_{2\times 3}$$

② Multiplication ⇒

ⓐ scalar multiple of Matrix ⇒

Eg. $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 3 & 1 \end{bmatrix}$

then $4A = \begin{bmatrix} 12 & 8 & -4 \\ 16 & 12 & 4 \end{bmatrix}$

↑
scalar.

ⓑ Product of two matrices ⇒

$A = [a_{ij}]_{m\times n}$ & $B = [b_{jk}]_{n\times p}$

$C = AB \neq BA$ (Not always)

↳ order ⇒ $m \times p$

Eg. $\Rightarrow$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}_{3\times 4}$$

$$A \times B = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}_{3\times 4}$$

Determinant $\Rightarrow$

Determinant of order 2 $\Rightarrow$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

↑ value of determinant

Determinant of order 3 $\Rightarrow$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant of order 4 $\Rightarrow$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Q: Find the value of determinant

$$A = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 2 & 1 & 2 \end{vmatrix} \qquad B = \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix}$$

Ans
$A = 17$
$B = abcd.$

Minors and Cofactors ⇒

Eg. ⇒ $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 4 & -1 \\ 5 & 6 & 8 \end{bmatrix}$

Minor of $a_{11} = M_{11} = \begin{vmatrix} 4 & -1 \\ 6 & 8 \end{vmatrix} = 38$

— " — $a_{12} = M_{12} = \begin{vmatrix} 1 & -1 \\ 5 & 8 \end{vmatrix} = 13$

$M_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 6 \end{vmatrix} = -14$

$M_{21} = \begin{vmatrix} 3 & 2 \\ 6 & 8 \end{vmatrix} = 12$

Similarly $M_{22} = 6$, $M_{23} = -3$, $M_{31} = -11$

$M_{32} = -4$ $M_{33} = 5$.

Now,

Cofactor of $a_{11} = A_{11} = (-1)^{1+1} \times M_{11} = 38$

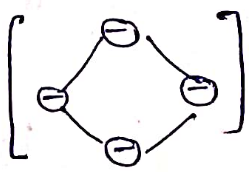
$A_{12} = (-1)^{1+2} M_{12} = -13$

$A_{13} = (-1)^{1+3} M_{13} = -14$

$A_{21} = -12$, $A_{22} = 6$, $A_{23} = 3$

$A_{31} = -11$ $A_{32} = 4$ $A_{33} = -5$

NOTE ⇒



Adjoint of a square Matrix ⇒

let $A = [a_{ij}]_{n \times n}$ and $A_{ij} \rightarrow$ cofactor of a_{ij}

then $\text{adj}(A) = [A_{ij}]^{T}_{n \times n}$

Eg. ⇒ $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

$A_{11} = 6, \quad A_{12} = -2 \quad A_{13} = -3$

$A_{21} = 1 \quad A_{22} = -5 \quad A_{23} = 3$

$A_{31} = -5 \quad A_{32} = 4 \quad A_{33} = -1$

$$\text{adj}(A) = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^{T} = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

Property of adjoint Matrix ⇒

$$(\text{adj}\, A) A = A(\text{adj}\, A) = |A|\, I_n$$

where $I_n \rightarrow$ unit matrix of order $n \times n$.

Singular and Non-singular Matrices ⇒ $A \rightarrow$ square Matrix

If $|A| = 0 \Rightarrow$ singular, otherwise non-singular.

Eg. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\qquad B = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

$|A| = 1$ ↳ Non-singular

$|B| = 0$ ↳ singular

Inverse of a Matrix $\Rightarrow$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A \qquad (|A| \neq 0)$$

only for non-singular Matrices

Eg. $\Rightarrow$ $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$|A| = 1 \neq 0 \quad \Rightarrow A$ is invertible.

$$\operatorname{adj} A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Properties of invertible Matrix $\Rightarrow$

A & B $\rightarrow$ invertible Matrices.

① $(AB)^{-1} = B^{-1}A^{-1}$

② $(A^T)^{-1} = (A^{-1})^T$

Q: Find A^{-1}, if $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Sol:

$|A| = -2$

$$\operatorname{adj} A = \begin{bmatrix} -1 & +1 & -1 \\ +8 & -6 & +2 \\ -5 & +3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Rank of a Matrix ⇒

→ Rank of Matrix A $= \rho(A) = r \rightarrow$ Non-negative integer

→ There is at-least one square sub-matrix of A of order r, whose determinant is non-zero.

→ Rank of null matrix $= 0$

→ Rank of id. matrix $=$ order of Matrix.

→ for Matrix $(m \times n) \Rightarrow \rho(A) \leq \min(m,n)$

→ for square matrix $(n \times n) \Rightarrow \rho(A) = n$, if $|A| \neq 0$

$\Rightarrow \rho(A) < n$, if $|A| = 0$

Q: Find rank.

① $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ ② $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

Sol:

① Here $|A| = 0 \Rightarrow \rho(A) < 3$

Further A has ~~has~~ at least one non-zero minor of order 2, say, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, thus $\rho(A) = 2$.

② minor $\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, det of minor $\neq 0$

$\Rightarrow \rho(A) = 2$.

Echelon form of Matrix ⇒

A Mat. A is Called echelon form if

1.> A row of M which has all its elements zero occurs below a row which has a non-zero element.

2.> The no. of zeros before the first non-zero element in a row is less than the no. of such zeros in the next row.

Note - ① $\rho(A)$ in echelon form = No. of non-zero rows.

② To find rank, reduce in echelon form by elementary transformation,

The no. of non zero rows = $\rho(A)$.

Q. Determine Rank of Mat. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 6 & 5 \end{bmatrix}$ (AKTU-2023)

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 6 & 5 \end{bmatrix}.$$

$$|A| = 8 \neq 0 \Rightarrow \rho(A) = 3.$$

Q: Rank = ?

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & -12 & 13 \end{bmatrix}$$

Sol.

$R_2 \to 3R_2 - 4R_1 \qquad R_3 \to 3R_3 - 5R_1 \qquad R_4 \to 3R_4 - 10R_1$

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -7 & -14 & -21 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2 \qquad R_4 \to R_4 - 7R_2$

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \therefore \rho(A) = \cancel{3}\,2$$

Q: Rank = ?

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

sol:

$R_2 \rightarrow R_2 + 2R_1$

$R_3 \rightarrow R_3 + R_1$

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 16 & 4 & 12 & 15 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$R_4 \rightarrow R_4 - R_2$

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 2$ (no. of non-zero rows)

Q: Rank = ?

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

sol: Interchanging R_1 and R_2 $\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$R_4 \rightarrow R_4 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$R_4 \rightarrow R_4 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

Q: Rank = ?

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

Sol:

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - R_1$ $\Rightarrow$ $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$R_3 \to R_3 - \frac{1}{3}R_2$ $\Rightarrow$ $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now, A is an Echelon form

So, Rank of A = $\rho(A)$ = No. of non-zero rows

$\rho(A) = 2$

Q: Rank = ?

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$$

Sol:

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + R_1$ $\Rightarrow$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 10 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\Rightarrow$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow$ Echelon form.

$\rho(A) = 2$ (No. of Non-zero rows)

Normal form of a Matrix ⇒

Let $\rho(A) = r$ (>0), then by applying elementary transformation A can be reduced to any one of below forms:

$$I_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} I_r & 0 \end{bmatrix}, \quad \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

Here $I_r \rightarrow$ Id. matrix

Called its normal form.

NOTE ⇒ Zero matrix is its own normal form.

Q: Reduce the Mat. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (RTU-2023, 2008)

in its normal form & hence find its rank.

Sol:

$R_1 \leftrightarrow R_2$
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_4 \rightarrow R_4 - (R_1 + R_2)$

$R_3 \rightarrow R_3 - 3R_1$
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 3C_2$

$C_4 \to C_4 + C_2$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_1$

$C_4 \to C_4 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ A is in its normal form with $r = 2$

Hence $\rho(A) = 2$

Q: Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ (RTU-2024)

to normal form, and Hence find the rank.

Sol:

Nullity of a Matrix ⇒

Let $AX = 0$ be a system of homogeneous equations. Its solution vector X constitute a vector space X called the null space of A.

The dimension of this space, denoted by N_A is called the nullity of A.

Rank - Nullity Th. ⇒ (RTU. 2024)

Let A be an $m \times n$ matrix of rank r_A and nullity N_A, then

$$r_A + N_A = n$$

i.e. rank of A + Nullity of A = No. of columns in A.

Q: Find the nullity of the Mat. $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

Sol. Since $\rho(A) = r_A = 2$

and No. of columns $= 4$

$$\therefore N_A = n - \rho = 4 - 2 = 2.$$

System of Linear Simultaneous Equations ⇒

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

or $\quad AX = B$

where, $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m\times n}$ called Coefficient matrix.

$X = [x_1 \ x_2 \ \dots \ x_n]^T$ called solution (vector) matrix

$B = [b_1 \ b_2 \ \dots \ b_n]^T$ called coloumn matrix.

System of Linear Sim. Eq.

Homogeneous syst.

↓

[all b_i ($i = 1$ to m) $= 0$]

- $\boxed{\text{If } D \neq 0}$ Trival sol. ↓ $\rho(A) = \rho(A:B) = n$
- $\boxed{\text{If } D = 0}$ Non-trival sol. or Infinite sol. ↓ $\rho(A) = \rho(A:B) < n$

Non-homogeneous syst.

↓

[at least one b_i ($i = 1$ to m) $\neq 0$.]

Non-homog. system.

- Consistent syst. (have ~~sol.~~ at least one sol.)
  - Unique sol. → $\rho(A) = \rho(A:B) = n$
  - ∞ sol. → $\rho(A) = \rho(A:B) < n$
- Inconsistent (No sol.) → $\rho(A) \neq \rho(A:B)$

Where $\rho(A:B)$. $\Rightarrow$ Rank of Augmented Matrix.

(1) Solution of system of n- simultaneous non- homogeneous L.E. in n unknowns $\Rightarrow$

M-1 Matrix Inversion method $\Rightarrow$

Consider, $AX = B$. ($A \to$ Non- singular Matrix)

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B \Rightarrow X = A^{-1}B$$

Q: solve the following by matrix inversion method —

$x_1 + x_2 + 2x_3 = 4, \quad 2x_1 + 5x_2 - 2x_3 = 3, \quad x_1 + 7x_2 - 7x_3 = 5$

Sol: let $A = IA$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -6 \\ 0 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$

$R_3 \to R_3 - 2R_2$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A$

$R_2 \to \frac{R_2}{3}$, $R_3 \to \frac{R_3}{3}$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ 1 & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} A$

$R_2 \to R_2 + 2R_3 \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & -1 & 2/3 \\ 1 & -2/3 & 1/3 \end{bmatrix} A$

$R_1 \to R_1 - R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & 1 & -2/3 \\ 4/3 & -1 & 2/3 \\ 1 & -2/3 & 1/3 \end{bmatrix} A$

$R_1 \to R_1 - 2R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7/3 & 7/3 & -4/3 \\ 4/3 & 1 & 2/3 \\ 1 & -2/3 & 1/3 \end{bmatrix} A$

$\Rightarrow A^{-1} = \begin{bmatrix} -7/3 & 7/3 & -4/3 \\ 4/3 & 1 & 2/3 \\ 1 & -2/3 & 1/3 \end{bmatrix}$

$\therefore X = A^{-1} B \xrightarrow{\text{Solving}} \begin{bmatrix} -9 \\ 17/3 \\ 11/3 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -9 \\ x_2 &= 17/3 \\ x_3 &= 11/3 \end{aligned}$

M-2 **Gauss Elemination Method** ⇒

→ In this method, the coefficient matrix is reduced into Echelon form by elementary row transformations.

Q: Solve the following equations –

$x - y + 2z = 3, \quad x + 2y + 3z = 5, \quad 3x - 4y - 5z = -13$

Sol:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -22 \end{bmatrix}$$

$R_3 \to R_3 + \frac{R_2}{3}$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -32/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -64/3 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x - y + 2z &= 3 \\ 3y + z &= 2 \\ -\frac{32}{3} z &= \frac{-64}{3} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -1 \\ y &= 0 \\ z &= 2 \end{aligned}$$

Ans.

Q: Test Consistency of the following system of equations –

$x_1 + x_2 - x_3 = 0, \quad 2x_1 - x_2 + x_3 = 3, \quad 4x_1 + 2x_2 - 2x_3 = 2$

Sol: Augmented Matrix $\Rightarrow$

$$C = \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\Rightarrow C = \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{array}\right]$$

$R_3 \to R_3 - \frac{2}{3} R_2$

$$\Rightarrow C = \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 = \rho(C) < 3$ (No. of unknowns)

Thus the system is Consistent and has ∞ no. of sol.

Q: For what values of k the equation (RTU-2023, 2012)

$x + y + z = 1, \quad 2x + y + 4z = k, \quad 4x + y + 10z = k^2$

have a sol, and solve them Completely in each case.

↳ unique/∞

Sol: Augmented Matrix

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$ $\Rightarrow$ $C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & -3 & 6 & k^2-6 \end{array}\right]$

$R_3 \to R_3 - 3R_2$ $\Rightarrow$ $C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-4-3(k-2) \end{array}\right]$

for a ~~unique~~ sol. $\Rightarrow$ $\rho(A) = \rho(C)$

$\Rightarrow$ $k^2 - 3k + 2 = 0$ $\Rightarrow$ $k = 1, 2$

<u>for $k=1$</u> $\Rightarrow$ $C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ $y = 2z + 1$, $x = -3z$

<u>for $k=2$</u> $\Rightarrow$ $C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ $y = 2z$, $x = 1 - 3z$

Q: Check the Consistency and if possible solve the system of eq.

(1) $2x + 3y + 4z = 11$; $x + 5y + 7z = 15$; $3x + 11y + 3z = 25$ (RTU-2008)

(2) $4x - 2y + 6z = 8$; $x + y - 3z = -1$; $15x - 3y + 9z = 21$ (RU-2000)

(3) $x + y + z = 6$; $2x + y + 3z = 13$; $5x + 2y + z = 12$

$2x + 3y - 2z = -10$ (RU-2002, 2006, RTU-2009)

<u>Ans</u> - (1) $x = 2, y = -3, z = 4$ (Consistent / unique sol.)

(2) $x = 1$, $y = 3k - 2$, $z = k$ (Consisten / ∞ sol.)

(3) $x = 1$, $y = 2$, $z = 3$ (Consistent / unique sol.)

(2) Homogeneous System $\Rightarrow$

It is of the form $AX = 0$ $\quad A = [\ ]_{m \times n}$

NOTE $\Rightarrow$ (1) If $|A| \neq 0 \Rightarrow \rho(A) = n \Rightarrow$ Trival sol.

$\Rightarrow x_1 = x_2 = \cdots = x_n = 0$

(2) If $|A| = 0 \Rightarrow \rho(A) = r < n \Rightarrow$ Non-trival sol

$\Rightarrow$ (at least one $x_1/x_2/\cdots \neq 0$)

Q: Determine the value of k so that the equations

$2x + y + 2z = 0$, $x + y + 3z = 0$, $4x + 3y + kz = 0$

has a non-zero sol.

Sol:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$$

for a non-zero sol. $\Rightarrow |A| = 0$

$\Rightarrow \boxed{k = 8}$ A

Q: Investigate the values of λ and μ so that equations

$2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$

have (i) no sol. (ii) unique sol. (iii) ∞ no. of sol.

Sol: The argumented Matrix for the given system

$$C = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$R_2 \rightarrow 2R_2 - 7R_1$

$R_3 \rightarrow R_3 - R_1$

$$C = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

(i) for no sol. $\Rightarrow$ $\rho(A) \neq \rho(C)$

$\therefore$ $\lambda - 5 = 0$ and $\mu - 9 \neq 0$

$\underline{\lambda = 5}$ $\quad$ $\underline{\mu \neq 9}$

(ii) for unique sol. $\Rightarrow$ $\rho(A) = \rho(C)$

$\therefore$ $\lambda - 5 \neq 0$ and μ can have any value.

$\underline{\lambda \neq 5}$

(iii) for ∞ sol. $\Rightarrow$ $\rho(A) = \rho(C) < 3$

$\lambda - 5 = 0$ and $\mu - 9 = 0$

$\underline{\lambda = 5}$ $\quad$ $\underline{\mu = 9}$

Q: Check for trival sol./ Non-trival sol.

$x + y - 2z + 3w = 0$

$x - 2y + z - w = 0$

$4x + y - 5z + 8w = 0$

$5x - 7y + 2z - w = 0$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - 4R_1$
$R_4 \rightarrow R_4 - 5R_1$

$R_4 \rightarrow R_4 - 4R_2$
$R_3 \rightarrow R_3 - R_2$

$\rho(A) = 2 < 4$

$\Rightarrow \infty$ sol.

$\Rightarrow$ Non-trival sol.

Symmetric Matrix ⇒ $A = A^T$ or $A^T = A$

Skew-symmetric Matrix ⇒ $A^T = -A$

orthogonal Matrix → $AA^T = I$

Linearly Dependent (LD) & Linearly Independent (LI) vectors ⇒

let $x_1, x_2, \ldots x_n$ → n-vector

and Scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ → scalar (not all zero)

is said to be LD → if linear Combination

such that $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$

otherwise it is L.I.

→ $\rho(A)$ = No. of variables ⇒ L.I.

→ $\rho(A) <$ No. of variables ⇒ L.D.

NOTE ⇒ If a vector is L.D., then ~~...~~. a can be written as in form of other Combination.

Q: Are the following vectors linearly dependent? If so, express one of these as a linear combination of other two.

Ex $x_1 = (1, 3, 4, 2)$ $\quad x_2 = (3, -5, 2, 2)$ $\quad x_3 = (2, -1, 3, 2)$

Sol:

$$A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 3 & -5 & 2 & 2 \\ 2 & -1 & 3 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -14 & -10 & -4 \\ 0 & -7 & -5 & -2 \end{bmatrix}$$

Applying $R_3 \to R_3 - \frac{R_2}{2}$

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -14 & -10 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 2$ (No. of non-zero rows in Echelon form)

$\Rightarrow$ Two vectors are L.I.

Thus the given set of vectors is L.D.

Let $x_1 = \lambda x_2 + \mu x_3$

$(1, 3, 4, 2) = \lambda(3, -5, 2, 2) + \mu(2, -1, 3, 2)$

$$\Rightarrow \left.\begin{aligned} 3\lambda + 2\mu &= 1 \\ -5\lambda - \mu &= 3 \\ 2\lambda + 3\mu &= 4 \end{aligned}\right\} \Rightarrow \quad \lambda = -1, \ \mu = 2$$

$\Rightarrow x_1 + x_2 - 2x_3 = 0$

Which is required linear combination.

Eigen values and Eigen vectors →

Let $A = [a_{ij}]_{n \times n}$ be any n-square matrix

and λ → be any scalar.

Then the matrix $[A - \lambda I]$ is called the Characteristic Matrix of A.

$|A - \lambda I|$ → when expended will give a polynomial in λ of degree n, called Characteristic Polynomial of A.

If Polynomial = 0 ⇒ $|A - \lambda I| = 0$

called Characteristic Equation of A.

Eigen values ⇒

Root of Char. Eq. → called Char. Roots. (Eigen values)

Properties of Eigen values ⇒

(1) A and A^T ⇒ have same Eigen values.

(2) Sum of Eigen values = tr.(A)

(3) Product of Eigen values = $|A|$

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then

the eigenvalues of

ⓐ KA are $\Rightarrow K\lambda_1, K\lambda_2, \ldots, K\lambda_n$

ⓑ A^m are $\Rightarrow \lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$

ⓒ A^{-1} are $\Rightarrow \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

(5) The set of eigen values of A is called spectrum of A

Q: Find the eigen values of Matrix (KTU-2021) (4-Marks)

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol: Char. Eq. of $A = |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \lambda = 2, 2, 8 \Rightarrow \text{Eigen Values.}$$

Q: If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ then find the eigen values of $3A^3 + 5A^2 - 6A + 2I$.

Sol:

$|A - \lambda I| = 0 \Rightarrow \lambda = 1, 3, -2$

Eigen values of $A^3 \Rightarrow 1^3, 3^3, (-2)^3 \Rightarrow 1, 27, -8$

—— " —— $A^2 \Rightarrow 1, 9, 4$

—— " —— $A \Rightarrow 1, 3, -2$

—— " —— $I \Rightarrow 1, 1, 1$

Eigen values of given eq.

$1^{st} = 3(1) + 5(1) - 6(1) + 2(1) = 4$

$2^{nd} = 3(27) + 5(9) - 6(3) + 2(1) = 110$

$3^{rd} = 3(-8) + 5(4) - 6(-2) + 2(1) = 10$

Hence, required Eigen $\Rightarrow 4, 110, 10$

Eigen vectors (Char. vector) ⇒

If λ is an eigen value of n-square matrix A, then a non-zero vector x which satisfies the matrix equation

$$AX = \lambda X$$

$$\Rightarrow (A - \lambda I) X = 0$$

Called ⇒ Char. vector / Eigen vector of A.

Properties ⇒

① The eigen vector x of a matrix A is not unique.

② If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues then $x_1, x_2, \ldots x_n \rightarrow$ form L. I.

③ Two eigen-vectors x_1 and x_2 are called orthogonal if $x_1^T x_2 = x_2^T x_1 = 0$.

Q: Find the eigen values and eigen-vectors of the matrix

$$A = \begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$

(RTU-2007)

Sol: $|A - \lambda I| = 0 \Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\Rightarrow (\lambda-1)(\lambda+1)(\lambda-2) = 0$

$\Rightarrow \lambda = 1, -1, 2$

for $\lambda = 1$

$(A - I)X = 0$

$$\begin{bmatrix} -3 & 1 & 1 \\ -11 & 3 & 5 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{11}{3} R_1$

$R_3 \to R_3 - \frac{1}{3} R_1 \Rightarrow$

$R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2/3 & 4/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3x_1 + x_2 + x_3 = 0$

$-\frac{2}{3}x_2 + \frac{4}{3}x_3 = 0$

or $x_2 = 2x_3$

$\Rightarrow$ let $x_3 = 1$

$\Rightarrow x_2 = 2$

$x_1 = 1$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Similarly for $\lambda = -1$ $\quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

for $\lambda = 2$ $\quad X_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Diagonalization of a Matrix ⇒

Theorem ⇒ If a square Matrix A of order n has n linearly independent eigen vectors, then a matrix B can be found such that $B^{-1}AB$ is a diagonal matrix, whose diagonal elements are the eigen values of A.

Q: Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. If so, obtain the matrix P such that $P^{-1}AP$ is a diagonal matrix.

[RTU-2022]

Sol.

Char. Eq. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + 22\lambda + 45 = 0 \quad \Rightarrow \lambda = -3, -3, 5$$

(Eigen-values)

for $\lambda = -3$ ⇒ $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$$

Let $x_1 = 1,\ x_2 = 1,\ x_3 = 1$ $\qquad P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = P_2$

for $\lambda = 5$,
$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$R_2 \to R_2 + 2R_1$
$$\begin{bmatrix} -7 & 2 & -3 \\ -12 & 0 & -12 \\ -4 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 3R_3 - R_2$
$$\begin{bmatrix} -7 & 2 & -3 \\ -12 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$
$$-12x_1 + 12x_3 = 0 \quad \Rightarrow x_1 = -x_3$$

$$\Rightarrow -7x_1 + 2x_2 + 3x_1 = 0 \quad \Rightarrow 4x_1 = 2x_2$$
$$\Rightarrow 2x_1 = x_2$$

Let $x_2 = 2$, $x_1 = 1$, $x_3 = -1$ $\Rightarrow \beta_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\therefore P = [P_1 \ P_2 \ P_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Here $|P| = 0 \Rightarrow$ Matrix A is not diagnalizable

Caley - Hamilton Theorem ⇒

statement ⇒ Every square matrix satisfies its own C.E.

(RTU-2023)

If $|A-\lambda I| = (-1)^n \lambda^n + \sigma_1 \lambda^{n-1} + \sigma_2 \lambda^{n-2} + \dots + \sigma_n = 0$

be C.E. of Matrix A, then it is satisfied

by $\lambda = A$ i.e.

$$(-1)^n A^n + \sigma_1 A^{n-1} + \sigma_2 A^{n-2} + \dots + \sigma_n I = 0$$

Inverse by C-I Theorem ⇒

Q: verify C-H theorem for matrix —

(RTU-2024)

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

and hence find its inverse.

Sol: C.E. of A $= |A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) - 2 \times 2(2-\lambda) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 - 8 + 4\lambda = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 15\lambda - 14 = 0 \quad \text{——— (1)}$$

We need to show that A satisfies eq.(1) i.e.

$$A^3 - 6A^2 + 15A - 14 = 0 \quad \text{——} \quad (2)$$

Now,
$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \Big[$$

$$A^3 = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} =$$

$$\therefore \; A^3 - 6A^2 + 15A - 14 = 0$$

$$= \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} - 6 \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} + 15 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 14 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Further, mul. eq(2) by A^{-1} $\Rightarrow A^2 - 6A + 15 - 14A^{-1} = 0$

$$\Rightarrow A^{-1} = \frac{1}{14}\left[A^2 - 6A + 15\right]$$

$$= \frac{1}{14}\left\{ \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 15 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$A^{-1} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$

Ans.

Now, eq. ② $\times A^{-1}$

$$\Rightarrow A^{-1} = \frac{-1}{2}\left(A^2 - 3A - 8I\right)$$

$$= \frac{-1}{2}\left\{\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Q: Verify C-H theorem for the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Hence find A^{-1}.

Sol:

The C.E. of A $\Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$

~~$\Rightarrow \lambda = 1, 2, 3.$~~

$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)$

or $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ —— ①

Now, $A^3 - 6A^2 + 11A - 6I$ —— ②

$$= \begin{bmatrix} 8 & 7 & 7 \\ -19 & 8 & -19 \\ 19 & -7 & 20 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix} + 11\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \text{Verified C-H}$$

Now, eq. ② $\times A^{-1}$ $\Rightarrow A^{-1} = \frac{1}{6}\left(A^2 - 6A + 11I\right)$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & -1/2 \\ 1/6 & 1/2 & 1/6 \\ -1/6 & 1/2 & 5/6 \end{bmatrix}$$

Q: State C-H theorem, verify it for the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$ Hence find A^{-1}.

[RTU- 2023, 2021]

[MNIT- 2006, 2005]

Sol: St. ⇒ Every square matrix A satisfies its own C.E.

C.E ⇒ $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 8\lambda + 2 = 0 \quad \text{—— (1)}$$

Now, $A^2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$

$A^3 = \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix}$

Now, $A^3 - 3A^2 - 8A + 2I_3$ —— (2)

$= \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix} - 3\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} - 8\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ⇒ Hence A satisfies C.E.

∴ Theorem → verified.

Orthogonal Transformation ⇒

A linear transformation $y = AX$ is called orthogonal if its matrix A is orthogonal.

Here $Y = [y_1 \ y_2 \ \dots \ y_n]^T$

$X = [x_1 \ x_2 \ \dots \ x_n]^T$

and $A \rightarrow$ coeff. of matrix of order n.

Q: Show that the transformation

$y_1 = \frac{2}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3$, $y_2 = \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3$,

$y_3 = \frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3$ is orthogonal.

Sol: The transformation can be written as

$$Y = AX$$

where $Y = [y_1 \ y_2 \ y_3]^T$, $X = [x_1 \ x_2 \ x_3]^T$

and $A = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$

or $A = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

$A^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix}$

Now, $AA^T = \frac{1}{9}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$AA^T = I$$

$\Rightarrow$ A is orthogonal.

Hence the transformation $Y = AX$ is orthogonal.