|| Maths ||

Unit 1

Beta & gamma function.

Gamma

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx$$

$\Gamma n = (n-1)!$; $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

$\Gamma n = (n-1)\Gamma (n-1)$

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta = \frac{\Gamma \frac{m+1}{2} \cdot \Gamma \frac{n+1}{2}}{2\Gamma \frac{m+n+2}{2}}$$

$$\frac{\Gamma n}{a^n} = \int_0^\infty x^{n-1} e^{-ax}\, dx$$

Beta

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

✓ $\beta(m,n) = \beta(n,m)$

or $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

or

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta$$

* Relation

$$\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma (m+n)}$$

* $$\Gamma p \cdot \Gamma (1-p) = \frac{\pi}{\sin(p\pi)}$$

Area

$$A = \int_a^b 2\pi y\, dl$$

* General

$$\boxed{A = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

(x Axis)

Parametric

$$A = \int_a 2\pi\, y(t) \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

Polar

$$A = \int_{\theta_1}^{\theta_2} 2\pi \underbrace{r\sin\theta}_{y} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(x Axis)

Volume

$$V = \int_a^b \pi y^2\, dx \quad (x\text{ Axis}) \; (y=0)$$

*

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

Unit 2

Hyperbolic function

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

Convergence

Sequence

$a_1, a_2, a_3, a_4 \dots$

Series

$S_n = a_1 + a_2 + a_3 + \dots$

$\lim_{n\to\infty} S_n = L$

Divergence

$\lim_{n\to\infty} S_n = \pm\infty$

Test for Convergence & Divergence

$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n-1} = \frac{1}{e}$

① GP test

If series is

$1 + x + x^2 + x^3 + \dots$

(i) $|x| < 1 \rightarrow C$

(ii) $x \geq 1 \rightarrow D$

otherwise oscillation.

② Auxillary / Harmonic / P Series

Series is $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} \dots$

(i) $p > 1 \rightarrow$ Convergence

(ii) $p \leq 1 \rightarrow$ Divergence

Imp if series $\rightarrow$ Convergence $\Rightarrow \lim_{n\to\infty} U_n = 0$

(Reverse is not true)

if $\lim_{n\to\infty} U_n \neq 0$ (finite) $\Rightarrow$ Series is Divergence

③ Comparision test

U_n = Given series

$\lim_{n\to\infty}\left(\frac{U_n}{V_n}\right) = l$ (let V_n)

if V_n is Convergence then U_n is C

if V_n is D then U_n is also D

④ Cauchy's n^{th} Root test

$\lim_{n\to\infty} (U_n)^{1/n} = l$ (finite)

$l > 1 \Rightarrow$ Divergence

$l < 1 \Rightarrow$ Convergence

$l = 1 \Rightarrow$ fail

⑤ Ratio test

$\lim_{n\to\infty}\left(\frac{U_n}{U_{n+1}}\right) = l$

$l > 1 \Rightarrow$ Convergence

$l < 1 \Rightarrow$ Divergence

$l = 1 \Rightarrow$ fail

⑥ Raabe's test

$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = l$ (finite)

$l > 1 \Rightarrow$ Convergence

$l < 1 \Rightarrow$ Divergence

$l = 1 \Rightarrow$ fail

(7) De Morgans (Bertrands) test

$$\lim_{n\to\infty}\left[n\left(\frac{U_n}{U_{n+1}}-1\right)-1\right]\log n = l$$

$l>1 \Rightarrow$ Convergence

$l<1 \Rightarrow$ Divergence

$l=1 \Rightarrow$ fail

(8) Logarithmic test

$$\lim_{n\to\infty} n\log\left(\frac{U_n}{U_{n+1}}\right) = l$$

$l>1 \Rightarrow$ Convergence

$l<1 \Rightarrow$ Divergence

$l=0 \Rightarrow$ fail

(9) Cauchy's Integral test

$f(x)$ = decreasing fun^{n}

$$\int_1^\infty f(x)\,dx = l$$

then series $\Rightarrow$ Convergence

otherwise $\Rightarrow$ Divergence.

Leibnitz's test

(for Alternating series)

Series is convergence : if

(i) $U_1 \geq U_2 \geq U_3 \geq U_4 \geq \dots$

(ii) $\lim_{n\to\infty} U_n = 0$

Otherwise Divergence

Power Series

$a_0 + a_1x + a_2x^2 \dots$

(i) $$f(x+h) = f(x) + \frac{h f'(x)}{1!} + \frac{h^2 f''(x)}{2!} + \frac{h^3 f'''(x)}{3!} + \dots$$

(ii) about point a

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots$$

(iii) $x \to 0$

$$f(h) = f(0) + \frac{h f'(0)}{1!} + \frac{h^2 f''(0)}{2!} + \frac{h^3 f'''(0)}{3!} \dots$$

[maclaurin's series]

test

$$\lim_{n\to\infty}\left(\frac{U_{n+1}}{U_n}\right) = xL \Rightarrow$$

$|xL|<1 \Rightarrow$ Convergence

$|xL|>1 \Rightarrow$ Divergence

Unit 3

Even function

$f(x) = f(-x)$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Symmetric about y-axis

Odd function

$f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = 0$$

Symmetric about origin

* Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

here

$$a_0 = \frac{1}{2\pi}\int_{c}^{c+2\pi} f(x)\,dx$$

→ Euler's formula

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx \quad \Big| \quad b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

* if function is even then ⇒ $b_n = 0$

odd then ⇒ a_0 & $a_n = 0$

* Half Range Series :-

if we want only sine or only cosine series

f^n ⇒ even ⇒ $b_n = 0$

Cosine Series

f^n ⇒ odd ⇒ $a_n = a_0 = 0$

Sine series.

Change of Interval

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

- Parseval theorem

$f(x) \Rightarrow$ Periodic function
(Period = $2l$)

then

$$\frac{1}{2l}\int_{c}^{c+2l} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Unit 4 Multivariable curves

Evaluating limits

Unit 4 (Multivariable Calculus)

Limit

function $f(x,y)$

$L_1 = x \to y$

$L_2 = y \to x$

(if $L_1 = L_2$ then limit exist)

when $L_1 = L_2 = 0$

then check

$L_3\ (x \to mx)$ & $L_4\ (x \to mx^2)$

Continuity

if $L_1 = L_2 = f(x_1, y_1)$ then at x_1 & y_1

$f(x,y)$ is continious.

Partial Differentation :-

$Z = f(x,y)$

$\frac{\delta z}{\delta x} = fx$ | $\frac{\delta z}{\delta y} = f(y)$

$$\frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta y \delta x} = fyx$$

Eulare theorem

$U = f(x,y)$ [homo fun]

then

$$\boxed{x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} = nf}$$

&

$$\left(x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} + z\frac{\delta f}{\delta z} = nf\right)$$

&

$$\boxed{x^2\frac{\delta^2 f}{\delta x^2} + 2xy\frac{\delta^2 f}{\delta x \delta y} + y^2\frac{\delta^2 f}{\delta y^2} = n(n-1)f}$$

Homogenous function

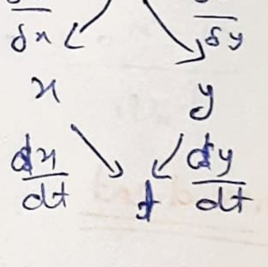
$$f(x,y) = a_0 x^n y^0 + a_1 x^{n-1} y^1 + \dots + a_n x^0 y^n$$

$$\boxed{f(x,y) = x^n \phi(y/x)}$$

Total Derivative

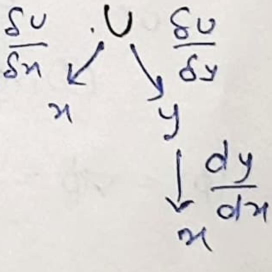
1. $U = f(x,y)$

$$\frac{dU}{dt} = \frac{\delta U}{\delta x} \times \frac{dx}{dt} + \frac{\delta U}{\delta y} \times \frac{dy}{dt}$$



(2) $U = f(x,y)$

$$\frac{dU}{dx} = \frac{\delta U}{\delta x} + \frac{\delta U}{\delta y} \times \frac{dy}{dx}$$



3. $U = f(x, y)$

$$\begin{tikzcd} & U \arrow[dl, "\frac{\delta U}{\delta x}"'] \arrow[dr, "\frac{\delta u}{\delta y}"] & \\ x \arrow[dr, "\frac{\delta x}{\delta s}"'] & & y \arrow[dl, "\frac{dy}{\delta s}"] \\ & s & \end{tikzcd} \quad \& \quad \begin{tikzcd} & U \arrow[dl, "\frac{\delta u}{\delta x}"'] \arrow[dr, "\frac{\delta u}{\delta y}"] & \\ x \arrow[dr, "\frac{dx}{\delta t}"'] & & y \arrow[dl, "\frac{\delta y}{\delta t}"] \\ & t & \end{tikzcd}$$

$$\frac{dU}{dS} = \frac{\delta U}{\delta x} \times \frac{\delta x}{\delta S} + \frac{\delta U}{\delta y} \times \frac{\delta U}{\delta S}.$$

& $$\frac{dU}{dt} = \frac{\delta U}{\delta x} \times \frac{\delta x}{\delta t} + \frac{\delta y}{\delta y} \times \frac{\delta y}{\delta t}$$

& $f(x,y) \rightarrow$ implicit function $\quad \frac{dy}{dx} = -\frac{\delta f / \delta x}{\delta f / \delta y}$

$f(x,y) = 0$

δ

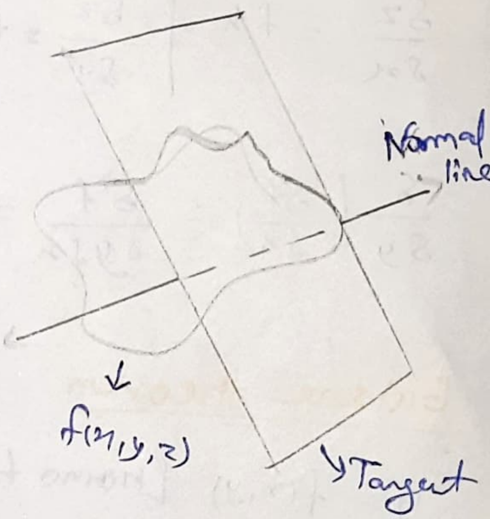
* Tanagent Plane & Normal

here $f'(x) = f_x = \frac{\delta f}{\delta x}$

$f'(y) = f_y = \frac{\delta f}{\delta y}$

Tanagent plane

$(x - x_0) f_x(P) + (y - y_0) f_y(P) + (z - z_0) f_z(P)$



Normal line

$$\frac{x - x_0}{f_x(P)} = \frac{y - y_0}{f_y(P)} = \frac{z - z_0}{f_z(P)}$$

Gradient

$$\nabla = \hat{i} \frac{\delta}{\delta x} + \hat{j} \frac{\delta}{\delta y} + \hat{k} \frac{\delta}{\delta z}$$

(Direction)

$$\text{Grad}(\phi) = \nabla\phi = \frac{\delta \phi}{\delta x}\hat{i} + \frac{\delta \phi}{\delta y}\hat{j} + \frac{\delta \phi}{\delta z}\hat{k}$$

Direction Derivative

for $\vec{a}$

$$\boxed{DD = \nabla\phi \cdot \hat{a} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}}$$

Divergence

(Dot product)

$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$\text{div}\,\vec{f} = \vec{\nabla}\cdot\vec{f}$

$\text{div}\,\vec{f} = \frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y} + \frac{\delta f_3}{\delta z}$

if $\vec{\nabla}\cdot\vec{f} = 0$

(Solenoidal)

Curl

(Cross Product)

$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$\text{Curl}\,\vec{F} = \vec{\nabla}\times\vec{f}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

if $\vec{\nabla}\times\vec{f} = 0$

(Irrotational)

- **Minima & Maxima**

Single variable

If

$f'(\alpha) = f''(\alpha) = f'''(\alpha) = \dots = f^{n-1}(\alpha) = 0$

& $f^n(\alpha) \neq 0$

then

n	even	$f^n(\alpha) > 0$	local max
n	even	$f^n(\alpha) < 0$	local min
n	odd	$f^n(\alpha) > 0$	PoI & increasing
n	odd	$f^n(\alpha) < 0$	PoI & Decreasing

PoI = Saddle Point point of Inflaxtion.

Double variable

$f(x,y) = f^n$

$A = \begin{bmatrix} r & s \\ s & t \end{bmatrix}$

$\left.\frac{\delta f}{\delta x}\right|_{P(a,b)} = 0$

$\left.\frac{\delta f}{\delta y}\right|_{P(a,b)} = 0$

(i) $A_1 = |r|$

(ii) $A_2 = \begin{vmatrix} r & s \\ s & t \end{vmatrix}$

$\frac{\delta^2 f}{\delta x^2} = r$ & $\frac{\delta^2 f}{\delta y^2} = t$ & $\frac{\delta^2 f}{\delta x \delta y} = s$

Case 1

$A_1 \to (+)$
$A_2 \to (+)$ } (+) definite at (a, b) (Minima)

Case 2

$A_1 \to (-)$
$A_2 \to (+)$ } (-) definite at (a, b) (Maxima)

Case 3

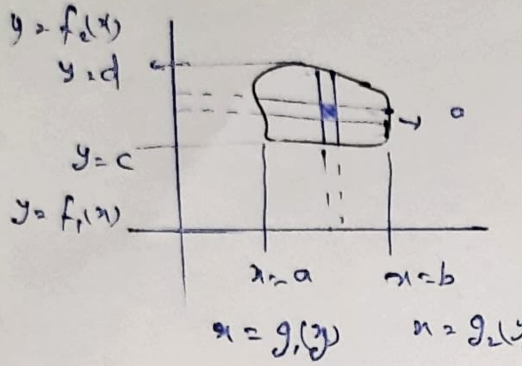
$A_1 \to 0$ & other than (i) & (ii) case $\Rightarrow$ Saddle Point at (a, b)
$A_2 \to 0$

Trick

$A_1 = +$	$A_1 = -$
$A_2 = +$	$A_2 = +$
$A_3 = +$	$A_3 = -$
Positive (definite) Minima	$\downarrow$ negative (maxima)

Unit 5 Multivariable Calculus

Change of Integration.



$$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)\, dx\, dy = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy$$

Area & Volume

$$A = \iint_R dx\, dy \quad \text{or} \quad \iint_R r\, dr\, d\theta$$

$$V = \iint_R f(x,y)\, dx\, dy \quad \text{or} \quad \iint_R f_1(r,\theta)\, r\, dr\, d\theta.$$

Triple Integration

$$\iiint_S f(x,y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z)\, dx\, dy\, dz$$

volume by Triple Integration,

$$Vol = \iiint_S dx\, dy\, dz$$

Line Integral

$$\text{line Integral} = \int_C \vec{f} \cdot d\vec{r}.$$

$$(d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k})$$

* Surface Integral

$$\iint \vec{f} \cdot d\vec{s} = \iint_R \vec{f} \cdot \hat{n}\, ds$$

$$\hat{n} = \frac{\vec{\nabla \phi}}{|\vec{\nabla \phi}|}$$

(here $\phi \Rightarrow$ Surface)

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \frac{dy\,dz}{|\hat{n} \cdot \hat{i}|} = \frac{dx\,dz}{|\hat{n} \cdot \hat{j}|}$$

* Gauss Divergence Theorem

$$\iint \vec{f} \cdot \hat{n}\, ds = \iiint_S \operatorname{div} \vec{f}\, dv$$

$(dv = dx\,dy\,dz)$

Centre of Gravity & COM

Density $\rho = f(x, y)$

total mass

$$m = \iint_A \rho\, dx\,dy$$

& COM

$$\bar{x} = \frac{\iint \rho x\, dx\,dy}{\iint \rho\, dx\,dy}$$

$$\bar{y} = \frac{\iint \rho y\, dx\,dy}{\iint \rho\, dx\,dy}$$

* Stoke's theorem

$$\int \vec{f} \cdot d\vec{r} = \iint_C \operatorname{Curl} \vec{f} \cdot \hat{n}\, ds$$

$$= \iint_C (\vec{\nabla} \times \vec{f}) \cdot \hat{n}\, ds$$

* Greens theorem

$$\oint_C \vec{f} \cdot d\vec{r} = \oint_C M\, dx + N\, dy$$

$$= \iint_S \left[\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right] dx\,dy$$