Unit-3. Ordinary D.E. of Higher order

Linear differential Eq. with Constant. Coefficient ⇒

A LDE with Constant Coeff. is that in which the dependent variable and its derivatives (may be of any order) occur only in first degree and are not multiplied together, and their coefficients are all constants.

Thus, the differential eq.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q \quad \text{—(1)}$$

where, $a_0, a_1, \ldots a_n \Rightarrow$ Constants

$Q \Rightarrow$ fn of x only (or Constant)

If we use $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, ...

then eq. (1) ⇒

$$\left[a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n\right] y = Q$$

$$\Rightarrow f(D) y = Q \quad \text{—— (2)}$$

$$f(D)y = \phi$$

When $\boxed{\phi = 0}$

$f(D)y = 0$ —— (3)

Called Homogeneous part of (2)

then sol. $\Rightarrow \boxed{y = CF}$

When $\boxed{\phi \neq 0}$

$f(D)y = \phi$ —— (4)

Called non-homogeneous part of (2)

then sol. $\Rightarrow \boxed{y = cf + PI}$

$\left\{\begin{array}{l} CF \rightarrow \text{Complementary } f^n \\ PI \rightarrow \text{Particular Integral} \end{array}\right.$

Obtaining C.F. of $f(D)y = \phi$

The C.F. of $f(D)y = \phi$ is obtained by putting $\phi = 0$.

then replace $D \rightarrow m$ and obtain Aux. Eq. in m

i.e. $f(m) = 0$. and then we find roots

Roots

Real & Distinct

let $m = m_1, m_2, m_3$

then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

Real and Equal

let $m = m_1, m_1, m_1$

then $y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$

Imaginary

let $m = a + ib,\ a - ib$

then $y = e^{ax}(c_1 \cos bx + c_2 \sin bx$

Q: Solve: $(D^3 - 6D^2 + 11D - 6)y = 0$

OR $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

Sol: Given: $(D^3 - 6D^2 + 11D - 6)y = 0$ $\quad D \equiv \dfrac{d}{dx}$

$\therefore$ A.E. $\Rightarrow m^3 - 6m^2 + 11m - 6 = 0$

$\Rightarrow (m-1)(m-2)(m-3) = 0 \quad \Rightarrow m = 1, 2, 3$

Hence Gen. sol. $\Rightarrow y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$.

Q: Solve: $(D-1)(D^2 - 6D + 25)y = 0$

Sol: A.E $\Rightarrow (m-1)(m^2 - 6m + 25) = 0$

$\Rightarrow m = 1,\ 3+4i,\ 3-4i$

Hence Gen sol. $\Rightarrow y = c_1 e^{x} + e^{3x}(c_2 \cos 4x + c_3 \sin 4x)$

Q: Solve: $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

Sol: $(D^2 - 4D + 1)y = 0$

A.E. $\Rightarrow m^2 - 4m + 1 = 0 \quad \Rightarrow m = 2 \pm \sqrt{3}$

Hence Gen. sol. $\Rightarrow e^{2x}(c_1 \cosh \sqrt{3}x + c_2 \sinh \sqrt{3}x)$

Q.(4) Solve: $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$

Sol: $(D^3 - 4D^2 + 5D - 2)y = 0$

A.E. $\Rightarrow m^3 - 4m^2 + 5m - 2 = 0$

$\Rightarrow (m-1)^2(m-2) = 0$

$\Rightarrow m = 1, 1, 2$

Hence sol. $\Rightarrow y = (c_1 + c_2 x)e^{x} + c_3 e^{2x}$

Q.(5) Solve: $\frac{d^4y}{dx^4} - 16y = 0$

Sol: $(D^4 - 16)y = 0$

A.E. $\Rightarrow m^4 - 16 = 0 \Rightarrow (m^2-4)(m^2+4) = 0$

$\Rightarrow m = \pm 2, \pm 2i$

Hence sol. $\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + (c_3 \cancel{e} \cos 2x + c_4 \sin 2x)$

Q.(6) Solve: $(D^4 - 8D^2 + 16)y = 0$

Sol: $m^4 + 8m^2 + 16 = 0 \Rightarrow (m^2+4)^2 = 0$

$\Rightarrow m = \pm 2i, \pm 2i$

Hence sol. $\Rightarrow y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

Q⑦ Solve: $(D^3 - 3D^2 + 4)y = 0$ (RTU 2023)

Marks - 2

Sol:

A.E. $\Rightarrow m^3 - 3m^2 + 4 = 0$

$\Rightarrow (m+1)(m-2)^2 = 0$

$\Rightarrow m = -1, 2, 2$

Hence sol. = $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$

Q⑧ Find the C.F. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \cos x$. (RTU-2021)

Sol:

$(D^2 + D - 2)y = \underbrace{x + \cos x}_{Q}$

for finding CF, $Q = 0$

$\therefore (D^2 + D - 2)y = 0$

A.E $\Rightarrow (m^2 + m - 2) = 0$

$\Rightarrow (m+2)(m-1) = 0 \Rightarrow m = 1, -2$

Hence CF $= c_1 e^{x} + c_2 e^{-2x}$

Particular Integral of $f(D)y = \phi$ ⇒

Given, $f(D)y = \phi$ $\quad$ [$\phi \to$ fn of x or a constant] ($\phi \neq 0$)

Its sol. ⇒ $y = CF + PI$

→ C.F. has already been discussed.

→ Now P.I.,

$$P.I = \frac{1}{f(D)}\phi$$

(P.I. is a fn of x free from arbitrary const.)

NOTE → ① $\frac{1}{D}$ stands for integration

$$\Rightarrow \frac{1}{D}\phi = \int \phi \, dx$$

② $$\frac{1}{D-\alpha}\phi = e^{\alpha x}\int e^{-\alpha x}\phi \, dx$$

$$\frac{1}{D+\alpha}\phi = e^{-\alpha x}\int e^{\alpha x}\phi \, dx$$

③ $$\frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)}\phi =$$

$$A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x}\phi \, dx + \cdots + A_n e^{\alpha_n x}\int e^{-\alpha_n x}\phi \, dx$$

Case-1. To find P.I., when $Q = e^{ax}$

where $a \to$ Constant and $f(a) \neq 0$

$$f(D)y = e^{ax}$$

$$P.I. = \frac{1}{f(D)} e^{ax}$$

$$P.I. = \frac{e^{ax}}{f(a)} \qquad [f(a) \neq 0]$$

Q: Solve: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

[RTU-2024] $(D^2 - 3D + 2)y = e^x$

Sol: $(D^2 - 3D + 2)y = e^{5x}$

A.E. $\Rightarrow m^2 - 3m + 2 = 0 \quad \Rightarrow m = 1, 2$

$$C.F. = C_1 e^x + C_2 e^{2x}$$

Now, $P.I. = \frac{1}{D^2 - 3D + 2} e^{5x}$

$$= \frac{1}{(5)^2 - 3(5) + 2} e^{5x} \qquad (\text{Put } D = 5 = a)$$

$$= \frac{e^{5x}}{12}$$

Hence sol. $\Rightarrow y = CF + P.I$

$$= C_1 e^x + C_2 e^{2x} + \frac{e^{5x}}{12} \quad \text{Ans.}$$

Case-2 To find P.I., when $\phi = e^{ax}$ and $f(a) = 0$ →

$$f(D)y = \phi \qquad \text{and} \quad f(a) = 0$$

$$P.I. = \frac{1}{f(D)} e^{ax} \quad \cancel{= x e^{ax}}$$

* If $(D-a)^r y = e^{ax}$

$$P.I. = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$$

Q. Solve: $(D^2 + 2D + 1)y = e^{-x}$

Sol: $(D^2 + 2D + 1)y = e^{-x}$

AE ⇒ $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

CF ⇒ $(C_1 + C_2 x)e^{-x}$

$$PI = \frac{1}{D^2 + 2D + 1} e^{-x}$$

$$= \frac{1}{(D+1)^2} e^{-x}$$

Here $(r = 2)$

$$= \frac{x^2}{2!} e^{-x} = \frac{1}{2} x^2 e^{-x}$$

Hence sol ⇒ $y = CF + PI$

$$y = (C_1 + C_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$$

Case - 3. To find P. I., when $Q = \sin ax$ or $\cos ax$ and $f(-a^2) \neq 0$

$f(D)y = \sin ax$ OR $f(D)y = \cos ax$

$$P.I. = \frac{1}{f(D)} \sin ax \qquad \frac{1}{f(D)} \cos ax$$

Replace $D^2 \longrightarrow (-a^2)$

Q: Solve: $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x$

Sol: $(D^2 + D + 1)y = \sin 2x$

A.E $\Rightarrow$ $m^2 + m + 1 = 0$ $\Rightarrow m = \dfrac{-1 \pm \sqrt{3}i}{2}$

$$C.F = e^{-\frac{x}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2} x \right)$$

now, $P.I. = \dfrac{1}{D^2 + D + 1} \sin 2x$

Replace $D^2 \rightarrow (-2^2) = -4$

$$= \frac{1}{-4 + D + 1} \sin 2x$$

$$= \frac{1}{D - 3} \sin 2x$$

$$= \frac{D + 3}{D^2 - 9} \sin 2x$$

$D^2 \rightarrow -4$

$$= \frac{(D+3)\sin 2x}{-4 - 9}$$

$$P.I = \frac{-1}{13}(2\cos 2x + 3\sin 2x)$$

Sol $\Rightarrow y = C.F. + P.I.$

Case-4 To find P. I., when $Q = \sin ax$ or $\cos ax$ & $f(-a^2) = 0$

$f(D)\,y = \sin ax$ $\qquad$ $f(D)\,y = \cos ax$

$$PI = \frac{1}{f(D)} \sin ax \qquad PI = \frac{1}{f(D)} \cos ax$$

let $f(D) = D^2 + a^2$

$$PI = \frac{1}{D^2+a^2} \sin ax \qquad PI = \frac{1}{D^2+a^2} \cos ax$$

$$= \frac{-x}{2a} \cos ax \qquad = \frac{+x}{2a} \sin ax$$

Q: Solve: $(D^2+4)y = \sin 2x$

Sol: A. E $\Rightarrow$ $m^2 + 4 = 0$ $\Rightarrow$ $m = \pm 2i$

CF $\Rightarrow$ $C_1 \cos 2x + C_2 \sin 2x$

$$PI = \frac{1}{D^2+4} \sin 2x$$

$$= \frac{-x}{2(2)} \cos 2x$$

$$= \frac{-x}{4} \cos 2x$$

Complete sol $\Rightarrow$ $y = CF + PI$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{x}{4} \cos 2x.$$

Case-5. To find P.I., when $Q = x^m$, where $m \to$ +ve int.

$$f(D)y = x^m \qquad m \to \text{+ve int.}$$

$$P.I. = \frac{1}{f(D)} x^m$$

$$= \frac{1}{a_0}\left[1 + \phi(D)\right]^{-n} x^m$$

Now using Binomial Theorem,

expand $[1+\phi(D)]^{-n}$ in ascending powers of D

upto the term containing D^m.

and then differentiate.

Q: Solve: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$

Sol: $(D^2 - 4D + 4)y = x^2$

A.E. $\Rightarrow m^2 - 4m + 4 = 0 \quad \Rightarrow m = 2, 2$.

$$C.F. = (C_1 + C_2 x)e^{2x}$$

$$P.I = \frac{1}{D^2 - 4D + 4} x^2$$

$$= \frac{1}{(D-2)^2} x^2$$

$$= \frac{1}{4\left[1 - \frac{D}{2}\right]^2} x^2$$

$$= \frac{1}{4}\left[1-\frac{D}{2}\right]^{-2} x^2$$

$$= \frac{1}{4}\left[1+D+\frac{3}{4}D^2\right] x^2$$

$$= \frac{1}{4}\left[x^2 + Dx^2 + \frac{3}{4}D^2 x^2\right]$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]$$

$\therefore$ The sol. $\Rightarrow$ $y = CF + PI$

$$y = (C_1 + C_2 x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

<u>NOTE</u> $\Rightarrow$

* $$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

* $$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

* $$(1+x)^{n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

* $$(1-x)^{n} = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Q: Solve: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^x + x^2 - \sin x$ [FTU-2023] (4-Marks)

Sol:
$$(D^2 + 2D + 1)y = e^x + x^2 - \sin x$$

A.E $\Rightarrow m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

C.F. $= (C_1 + C_2 x)e^{-x}$

P.I. $= \frac{1}{(D^2 + 2D + 1)}(e^x + x^2 - \sin x)$

$$= \underbrace{\frac{1}{D^2 + 2D + 1}e^x}_{(D=1)} + \underbrace{\frac{1}{D^2 + 2D + 1}x^2}_{\text{B.T.}} - \underbrace{\frac{1}{D^2 + 2D + 1}\sin x}_{D^2 = -1}$$

=

$$= \frac{1}{1^2 + 2.1 + 1}e^x + \frac{1}{(D+1)^2}x^2 - \frac{1}{(-1 + 2D + 1)}\sin x$$

$$= \frac{1}{4}e^x + (1+D)^{-2}x^2 - \frac{1}{2}\frac{1}{D}\sin x$$

$$= \frac{1}{4}e^x + (1 - 2D + 3D^2 + \dots)x^2 + \frac{1}{2}\cos x$$

$$= \frac{1}{4}e^x + x^2 - 4x + 6 + \frac{1}{2}\cos x$$

Complete sol. $\Rightarrow$ $y = CF + PI$

$$y = (C_1 + C_2 x)e^{-x} + \frac{1}{4}e^x + x^2 - 4x + 6 + \frac{1}{2}\cos x$$

Case-6 ⇒

To find P.I., When $Q = e^{\alpha x} \times V$, where $V \to f^n$ of x ⇒

$$\frac{1}{f(D)}\left[e^{\alpha x} V\right] = e^{\alpha x}\frac{1}{f(D+\alpha)} V$$

Q: Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$

Sol: Given eq. ⇒ $(D^2 - 2D + 1)y = x^2 e^{3x}$

The A.E ⇒ $m^2 - 2m + 1 = 0$ ⇒ $m = 1, 1$

$\therefore$ CF $= (C_1 + C_2 x)e^x$

Now, $P.I. = \frac{1}{(D-1)^2} x^2 e^{3x}$

$= e^{3x}\frac{1}{(D+3-1)^2} \times x^2$ replacing $D \to (D+3)$

$= e^{3x}\frac{1}{(D+2)^2} x^2$

$= e^{3x} \cdot \frac{1}{4}\left[\frac{1}{\left(1+\frac{D}{2}\right)^2}\right] x^2$

$= \frac{e^{3x}}{4}\left[1+\frac{D}{2}\right]^{-2} x^2$

$= \frac{e^{3x}}{4}\left[1 - D + \frac{3}{4}D^2\right] x^2$ [neglecting higher powers]

$= \frac{e^{3x}}{4}\left[x^2 - Dx^2 + \frac{3}{4}D^2 x^2\right]$

$= \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$

$\therefore$ Sol. ⇒ $y = CF + PI = (C_1 + C_2 x)e^x + \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{2}\right)$

Q: Solve: $(D^2+3D+2)y = e^{2x}\sin x$ [RTU- 2024] (5-Marks)

Q: Solve: $(D^2-4D+13)y = 18e^{2x}\sin 3x$ [RTU - 2021] (5-Marks)

Case-7 ⇒ To find P.I., when $Q = xV$, where $V = f(x)$. ⇒

$$\frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V$$

Q: Solve: $(D^2 + 2D + 1)y = x \sin x$

Sol:

A.E. ⇒ $m^2 + 2m + 1 = 0$ ⇒ $m = -1, -1$

$CF = (C_1 + C_2 x)e^{-x}$

$$P.I = \frac{1}{D^2 + 2D + 1} x \sin x$$

$$= x \frac{1}{D^2 + 2D + 1} \sin x - \frac{2D + 2}{(D^2 + 2D + 1)^2} \sin x$$

$$= x \frac{1}{-1 + 2D + 1} \sin x - \frac{2(D+1)}{(-1 + 2D + 1)^2} \sin x \qquad [D^2 = -1]$$

$$= \frac{x}{2} \frac{1}{D} \sin x - \frac{2(D+1)}{4D^2} \sin x$$

$$= \frac{-x}{2} \cos x - \frac{1}{2} \frac{(D+1)}{(-1)} \sin x \qquad [D^2 = -1]$$

$$= \frac{-x}{2} \cos x + \frac{1}{2} (D+1) \sin x$$

$$= \frac{-x}{2} \cos x + \frac{1}{2} (\cos x + \sin x)$$

Complete sol. $y = CF + PI = (C_1 + C_2 x)e^{-x} - \frac{x}{2} \cos x + \frac{1}{2} (\cos x + \sin x)$

Simultaneous Linear Differential Equations $\Rightarrow$

Method of Solving Simultaneous LDE $\Rightarrow$

Q: Solve: $\frac{dx}{dt} + 2x = t$ and $\frac{dy}{dt} - 2x = \frac{1}{t}$

Given that $x(1) = 0 = y(1)$.

Sol: Given eqs. $\Rightarrow$ $(D+2)x = t$ —— ①

and $-2x + Dy = \frac{1}{t}$ —— ②

on solving ① $m+2=0 \Rightarrow m=-2$

$$CF = C_1 e^{-2t}$$

$$P.I. = \frac{1}{D+2} t$$

$$= \frac{1}{2}(1+D)^{-1} t$$

$$= \frac{1}{2}(1-D+D^2)t$$

$$= \frac{1}{2}(t-1) = \frac{t}{2} - \frac{1}{4}$$

$$x = x(t) = C_1 e^{-2t} + \frac{t}{2} - \frac{1}{4} \quad \text{—— ③}$$

from ② & ③ $\Rightarrow$ $Dy = \frac{1}{t} + 2\left[C_1 e^{-2t} + \frac{t}{2} - \frac{1}{4}\right]$

$$Dy = 2C_1 e^{-2t} + t - \frac{1}{2} + \frac{1}{t}$$

on integrating, we get –

$$y = C_1 e^{-2t} + \frac{t^2}{2} - \frac{t}{2} + \log t + C_2 \quad \text{—— ④}$$

Given $x(1) = 0$

$\therefore$ from (3) $\Rightarrow$ $x(1) = c_1 e^{-2t} + \frac{1}{2} - \frac{1}{4} = 0$

$$\Rightarrow c_1 = -\frac{1}{4} e^{2}$$

Also, $y(1) = 0$

so, from (4) $\Rightarrow$ $y(1) = -c_1 e^{-2} + \frac{1}{2} - \frac{1}{2} + \overset{0}{\cancel{\log}} + c_2 = 0$

$$\Rightarrow c_2 = c_1 e^{-2} = \frac{-1}{4}$$

$$\Rightarrow x(t) = -\frac{1}{4} e^{2(1-t)} + \frac{t}{2} - \frac{1}{4}$$

and $y(t) = \frac{1}{4} e^{2(1-t)} - \frac{1}{4} + \frac{t^2}{2} - \frac{t}{2} + \log t$.

Solution of Simultaneous Equations of type $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Method - 1 Grouping Method ⇒

Take any two members say $\frac{dx}{P} = \frac{dy}{Q}$ & integrate

Next take other two members and integrate.

Q: Solve: $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ —— ①

Sol:

Taking first two ratios,

$$\frac{dx}{yz} = \frac{dy}{zx}$$

$$\Rightarrow x\,dx = y\,dy$$

$$\Rightarrow x^2 = y^2 + c_1$$

Taking last two ratios

$$\frac{dy}{zx} = \frac{dz}{xy}$$

$$\Rightarrow y\,dy = z\,dz$$

$$\Rightarrow y^2 = z^2 + c_2$$

Hence General sol. of eq. ①

$$\phi(c_1, c_2) = 0$$

$\phi(x^2 - y^2, y^2 - z^2) = 0$, where $\phi \rightarrow$ arbitary fn.

Method - 2 ⇒ Multiplier Method ⇒

Let there be some multiplier l, m, n chosen in such a way that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

So that $lP + mQ + nR = 0$

and hence $l\,dx + m\,dy + n\,dz = 0$

It is an exact d.e., then on integration, we get one part of the general solution.

Similarly, we may choose another set of multipliers & proceed as above to get another part of gen. sol.

The two parts thus obtained together form the complete solution.

Method - 3. ⇒ Combined method ⇒

In this method, we use jointly, the above two methods to obtain the solution of simultaneous L.D.E.

Q: Solve: $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ —— (1) (RTU-2012)

Sol: Taking first two members, $\frac{dx}{y+z} = \frac{dy}{z+x}$

applying component and dividendo,

$$= \frac{dx-dy}{(y+z)-(z+x)}$$

$$= \frac{dx-dy}{y-x} \quad \text{—— (2)}$$

Similarly taking first & last,

$$= \frac{dz-dx}{x-z} \quad \text{—— (3)}$$

from (2) & (3), $\frac{dx-dy}{y-x} = \frac{dz-dx}{x-z}$

or $\frac{dx-dy}{x-y} = \frac{dz-dx}{z-x}$

$$\Rightarrow \log(x-y) = \log(z-x) + \log c_1$$

$$\Rightarrow \frac{x-y}{z-x} = c_1$$

now taking multipliers as (1,1,1), we have

$$\frac{dx+dy+dz}{2(x+y+z)}$$

Again, $\dfrac{dx-dy}{y-x} = \dfrac{dx+dy+dz}{2(x+y+z)}$

$$\Rightarrow \frac{dx+dy+dz}{x+y+z} + 2\,\frac{dx-dy}{x-y} = 0$$

$$\Rightarrow \log(x+y+z) + 2\log(x-y) = \log C_2$$

$$\Rightarrow (x+y+z)(x-y)^2 = C_2$$

$\therefore$ Sol. of given d.e. $\Rightarrow$

$$\phi\left(\frac{x-y}{z-x},\ (x+y+z)(x-y)^2\right) = 0 \quad \underline{\underline{A}}.$$

where $\phi \Rightarrow$ an arbitrary fn.

Q: Solve: $\dfrac{dx}{z-y} = \dfrac{dy}{x-z} = \dfrac{dz}{y-x}$ —(1) [RTU-2024] (5-Marks)

Sol: Using 1, 1, 1 as multiplier,

$$\frac{dx+dy+dz}{z-y+x-z+y-x} = \frac{dx+dy+dz}{0}$$

$$\Rightarrow dx+dy+dz = 0 \quad \Rightarrow x+y+z = C_1 \quad \text{—(2)}$$

Using x, y, z as multiplier,

$$\frac{x\,dx+y\,dy+z\,dz}{x(z-y)+y(x-z)+z(y-x)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2 \quad \text{—— (3)}$$

Hence general sol. $\Rightarrow$

$$\phi\left(x+y+z, \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right) = 0$$

$\phi \rightarrow$ an arbitrary const.

Q: Solve: $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$ —— (1)

Sol: Using 1, 1, 1 as multiplier, we get

$$= \frac{dx + dy + dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{dx + dy + dz}{0}$$

$$\Rightarrow dx + dy + dz = 0$$

$$\Rightarrow x + y + z = c_1 \quad \text{—— (2)}$$

Again taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier, we get

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \log x + \log y + \log z = \log c_2 \Rightarrow xyz = c_2 \quad \text{—— (3)}$$

Hence general sol. $\Rightarrow$ $\phi(x+y+z, xyz) = 0$

$\phi \rightarrow$ an arbitrary fn.

Second order differential Equations with variable Coefficients ⇒

The general form of L.D.E. of second order is –

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where, P, Q, R are the fn of x (or a constant).

It is an ordinary d.e. with variable coefficients.

When a Part of C.F. is known ⇒

L.D.E of 2nd order ⇒ $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

P, Q, R ⇒ fn of x

P Coeff. of $\frac{d^2y}{dx^2}$ is unity (always).

S. NO.	Condition satisfied	An integral of C.F. 'u'
* 1.	$1 + P + Q = 0$	$y = e^{x}$
* 2.	$1 - P + Q = 0$	$y = e^{-x}$
3.	$1 + \frac{P}{a} + \frac{Q}{a^2} = 0$	$y = e^{ax}$
* 4.	$P + Qx = 0$	$y = x$
5.	$2 + 2Px + Qx^2 = 0$	$y = x^2$
6.	$m(m-1) + Pmx + Qx^2 = 0$	$y = x^m$

Q: Solve: $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x) y = e^x \sin x$

Sol: Here, $P = -\cot x$, $Q = -(1-\cot x)$, $R = e^x \sin x$

Here, $1 + P + Q = 0$

Therefore, $y = e^x$ is a part of C.F.

Let $y = uv$ is complete solution of the given eq.

$\therefore$ $y = v.e^x$ —①

$\Rightarrow \frac{dy}{dx} = ve^x + e^x \frac{dv}{dx}$ —②

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} e^x + 2\frac{dv}{dx} e^x + ve^x$ —③

Putting eq. ①, ② and ③ in given eq.

we, get $\frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x$

Let $\frac{dv}{dx} = z$

$\Rightarrow \frac{dz}{dx} + (2-\cot x) z = \sin x$ (Linear d.e. in z)

$\therefore$ I.F. $= e^{\int (2-\cot x)dx} = \frac{e^{2x}}{\sin x}$

Hence the sol. is :

$$z \frac{e^{2x}}{\sin x} = \int \sin x . \frac{e^{2x}}{\sin x} dx + C$$

$$= \frac{e^{2x}}{2} + c$$

$$\Rightarrow \quad z = \frac{\sin x}{2} + c e^{-2x} \sin x$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{\sin x}{2} + c e^{-2x} \sin x$$

on integrating by separating the variables,

$$\Rightarrow \quad v = -\frac{\cos x}{2} - \frac{c e^{-2x}}{5} (\cos x + 2\sin x) + c_1$$

Hence, the complete sol. is :

$$y = v . e^x$$

$$y = -\frac{e^x}{2}\cos x - \frac{c e^{-2x}}{5}(\cos x + 2\sin x) + c_1 e^x . \quad \underline{\underline{A}}.$$

Q: Solve : $\quad x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$

Sol: Given d.e. $\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{2}{x}(1+x)\frac{dy}{dx} + \frac{2(1+x)}{x^2} y = x \quad$ — (1)

Here, $\quad P = -\frac{2}{x}(1+x), \quad Q = \frac{2(1+x)}{x^2}, \quad R = x$

So, $\quad P + Qx = 0$

Therefore, $\quad y = x$ is a part of C.F.

let $\quad y = vx \quad$ —— (2)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{—— (3)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2} \quad \text{———} (4)$$

Substitute eq. (2), (3), (4) in eq.(1),

we get $$\frac{d^2v}{dx^2} + \left[\frac{2}{x} - \frac{2}{x}(1+x)\right]\frac{dv}{dx} = 1$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1$$

put $$\frac{dv}{dx} = z$$

$$\Rightarrow \frac{dz}{dx} - 2z = 1 \qquad (\text{L.d.e. in } z)$$

$$\therefore \text{I.F.} = e^{-2\int dx} = e^{-2x}$$

$$\therefore \text{The sol is} \Rightarrow z e^{-2x} = \int e^{-2x} dx + c$$

$$\Rightarrow z = -\frac{1}{2} + ce^{2x}$$

$$\Rightarrow \frac{dv}{dx} = -\frac{1}{2} + ce^{2x}$$

on integrating,

$$\Rightarrow v = -\frac{1}{2}x + \frac{c}{2}e^{2x} + c_1$$

Hence, the Complete sol. : $y = vx$

$$\Rightarrow y = -\frac{x^2}{2} + \frac{cx}{2}e^{2x} + c_1 x. \quad \text{Ans.}$$

Change of Dependent Variable

or (Removal of first derivative / Reduction to Normal form) ⇒

When "u" can not be evaluated from the part of C.F in integral, then to get "u" we change the given eq. form of a second order d.e. by substituting $y = uv$ in which first order derivative is absent.

Given eq. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

After change of dependent variable $y \to v$

$$\Rightarrow \frac{d^2v}{dx^2} + Iv = S \quad \text{(Normal form)}$$

where, $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

$S = \frac{R}{u}$ where, $u = e^{-\frac{1}{2}\int P\,dx}$

Q: Solve the d.e. by method of change of dependent variable.

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x \sec x$$ [RTU-2023] (5-Marks)

Sol: Here, $P = -2\tan x$, $Q = 5$, $R = e^x \sec x$

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{+\frac{1}{2}\int 2\tan x\,dx} = e^{\log \sec x} = \sec x.$$

$$\therefore S = \frac{R}{u} = \frac{e^x \sec x}{\sec x} = e^x.$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= 5 - \frac{1}{2}(-2\sec^2 x) - \frac{1}{4}(4\tan^2 x)$$

$$= 5 + \sec^2 x - \tan^2 x = 6$$

$\therefore$ Normal form $\Rightarrow$ $\frac{d^2v}{dx^2} + 6v = e^x$

A.E. $\Rightarrow$ $m^2 + 6 = 0$ $\Rightarrow m = \pm i\sqrt{6}$

C.F $\Rightarrow$ $C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}x$

and P.I. $= \frac{1}{D^2+6} e^x = \frac{e^x}{7}$ $[D^2 = 1]$

$\therefore$ $V = CF + PI$

$$= C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x + \frac{e^x}{7}$$

Hence, Complete sol. $\Rightarrow$ $y = uv$

$$\Rightarrow y = \sec x\left(C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x + \frac{e^x}{7}\right)$$

Q: Solve: $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x.$

Sol: $P = -4x$, $Q = (4x^2-1)$, $R = -3e^{x^2}\sin 2x$

Let $y = uv$ is the Complete sol.

$$u = e^{-\frac{1}{2}\int P dx} = e^{x^2}$$

$\therefore$ $S = \frac{R}{u} = -3\sin 2x$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= (4x^2-1) - \frac{1}{2}(-4) + \frac{16x^2}{4}$$

$$I = 1$$

$\therefore$ Normal form $\Rightarrow$ $\frac{d^2v}{dx^2} + v = -3\sin 2x$

AE $\Rightarrow$ $m^2+1=0$ $\Rightarrow$ $m = \pm i$

C.F. $\Rightarrow$ $C_1\cos x + C_2 \sin x$

Now, $PI = -\frac{1}{D^2+1} 3\sin 2x$

$= \sin 2x$ $\quad [D^2 = -(2)^2]$ $\cancel{(D^2=2)}$

$\therefore$ the sol. $\Rightarrow$ $v = CF + PI$

$= C_1\cos x + C_2\sin x + \sin 2x$

Hence, Complete sol. $\Rightarrow$ $y = uv$

$\Rightarrow y = e^{x^2}(C_1\cos x + C_2\sin x + \sin 2x)$ Ans.

Change of Independent variable ⇒

Given D.E ⇒ $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

After change of independent variable $x \to z$ $[z = f(x)]$

$$\Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

Where, $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Q: Solve: $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$

[RTU-2008, 06, 03, 01]

Sol:

$P = \tan x, \quad Q = -2\cos^2 x, \quad R = 2\cos^4 x$

Now changing independent variable x to z by a relation $z = f(x)$ and the given eq. into

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{—— (2)}$$

$$\cancel{P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}} = \cancel{\frac{-\sin x + \tan x \cos x}{\cos^2 x}}$$

Taking $Q_1 = -2$ $\qquad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

$$\Rightarrow -2 = \frac{-2\cos^2 x}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \frac{dz}{dx} = \cos x \qquad \Rightarrow z = \sin x.$$

Now, finding, $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \dfrac{-\sin x + \tan x \cos x}{\cos^2 x} = 0$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x$$

Hence, eq. ② becomes $\Rightarrow \dfrac{d^2y}{dz^2} - 2y = 2\cos^2 x$

$$= 2(1-z^2)$$

$$\Rightarrow (D^2-2)y = 2(1-z^2).$$

A.E. $\Rightarrow m^2 - 2 = 0 \quad \Rightarrow m = \pm\sqrt{2}$

C.F. $\Rightarrow C_1 e^{\sqrt{2}z} + C_2 e^{-\sqrt{2}z}$

$= C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x}$

P.I. $= \dfrac{1}{D^2-2} 2(1-z^2) = -\left[1 - \dfrac{D^2}{2}\right]^{-1}(1-z^2)$

$= -\left(1 + \dfrac{D^2}{2}\right)(1-z^2) = -(1 - z^2 - 1)$

$= z^2 = \sin^2 x$

$\therefore$ Complete sol. $\Rightarrow \boxed{y = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x} + \sin^2 x}$ A.

Homogeneous Linear Differential Equation OR (Euler - Cauchy Differential Eq.) ⇒

A differential equation of the form

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = Q . \quad \text{——} (1)$$

Where, $a_0, a_1, a_2 \Rightarrow$ Constants

and $Q \Rightarrow f^n$ of x / Constant,

is known as Euler - Cauchy eq. or Homog. diff. eq. of II order.

Method to solve ⇒

Transform the given eq. into linear by transforming the independent variable x to z by relation

$$x = e^z \quad \Rightarrow z = \log x.$$

Substitute

$$x \frac{dy}{dz} = Dy$$

$$x^2 \frac{d^2y}{dz^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dz^3} = D(D-1)(D-2)y$$

$$\vdots$$

$$x^n \frac{d^ny}{dz^n} = \{D(D-1)(D-2) \ldots [D-(n-1)]\}y \quad \text{in eq. (1).}$$

Now new eq. can be solved easily.

Q: Solve: $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$.

Sol: Here, $(1+x) = e^z$

$\log(1+x) = z$

Substitute $(1+x)\frac{dy}{dx} = Dy$

$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y$

on substituting,

$[D(D-1) + D + 1] = 4\cos z$

$(D^2+1)y = 4\cos z$

A.E. $\Rightarrow m^2+1 = 0 \Rightarrow m = \pm i$

C.F. $\Rightarrow C_1\cos z + C_2 \sin z$

and $P.I. = \frac{1}{D^2+1} 4\cos z = 4 \times \frac{z}{2}\sin z$

$= 2z\sin z$

∴ The complete ~~general~~ sol. $\Rightarrow y = CF + PI$

$= C_1 \cos z + C_2\sin z + 2z\sin z$

$= C_1\cos\{\log(1+x)\} + C_2\sin\{\log(1+x)\}$
$+ 2\log(1+x)\sin\{\log(1+x)\}$. Ans.

Q: Solve: $x^2 \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 12\frac{\log x}{x}$

Sol: Given eq. is $\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$ —— ①

Here, let $x = e^z$, $z = \log x$.

Substitute

$\therefore x^2\frac{d^2y}{dx^2} = D(D-1)y$

$x\frac{dy}{dx} = Dy$ in eq. ①.

we get

$\Rightarrow [D(D-1)+D]y = 12z$

$\Rightarrow D^2y = 12z$

A.E. $\Rightarrow m^2 = 0 \Rightarrow m = 0, 0$

C.F. $= C_1 + C_2 z$

Now, $PI = \frac{1}{D^2} \times 12z = 12\frac{1}{D}\left(\frac{z^2}{2}\right) = 12 \times \frac{1}{2} \times \frac{z^3}{3} = 2z^3$

So, Gen. sol. $\Rightarrow y = CF + PI$

$y = C_1 + C_2 z + 2z^3$

$y = C_1 + C_2\log x + 2(\log x)^3$ Ans.

Exact Differential Equation ⇒

A second order d.e. is known as exact differential equation if it can be derived by direct differentiation (without any further process) from a first order d.e.

Let a second order d.e.

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q \quad \text{——} (1)$$

where, P_0, P_1, P_2 and Q are f^n of x.

If $P_2 - P_1' + P_0'' = 0 \Rightarrow$ Exact d.e.

then the primitive of the given equation is –

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int Q\,dx + C. \quad \text{——} (2)$$

Now eq. (2) is a first order d.e., it can be solved easily.

Q: Solve: $(2x^2+3x)\frac{d^2y}{dx^2} + (6x+3)\frac{dy}{dx} + 2y = (x+1)e^x$

Sol: $P_0 = (2x^2+3x) \qquad P_1 = (6x+3) \qquad P_2 = 2.$

Here, $P_2 - P_1' + P_0'' = 0 \Rightarrow$ Exact d.e.

∴ the primitive of the given eq. is –

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int Q\,dx + C$$

$$\Rightarrow (2x^2+3x)\frac{dy}{dx} + \{(6x+3)-(4x+3)\}y = \int (x+1)e^x + c$$

$$\Rightarrow (2x^2+3x)\frac{dy}{dx} + 2xy = xe^x + C$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{2x+3}y = \frac{e^x}{2x+3} + \frac{c}{(2x+3)x}$$

↳ L.d.E. in y

$$\therefore \text{ I.F.} = e^{\int \frac{2}{2x+3}dx} = e^{\log(2x+3)} = 2x+3.$$

Hence sol. ⇒

$$y(2x+3) = C_1 + \int \frac{(2x+3)e^x}{2x+3}dx + c\int \frac{2x+3}{x(2x+3)}dx$$

$$= C_1 + e^x + c\log x.$$

$$\Rightarrow \quad y(2x+3) = C_1 + e^x + c\log x. \quad \underline{\text{Ay}}.$$

<u>Q:</u> Solve: $\quad x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$

<u>Sol:</u> $\quad P_0 = x^2, \quad P_1 = 3x, \quad P_2 = 1$

$\therefore \quad P_0'' = 2, \quad P_1' = 3.$

So, $\quad P_0'' - P_1' + P_2 = 1-3+2 = 0$

$\therefore$ Primitive of the eq. ⇒

$$P_0\frac{dy}{dx} + (P_1 - P_0')y = \int Q\,dx + C$$

$$\Rightarrow x^2 \frac{dy}{dx} + (3x - 2x)y = \int \frac{1}{(1-x)^2} dx + c$$

$$\Rightarrow x^2 \frac{dy}{dx} + xy = \frac{1}{1-x} + c$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x^2(1-x)} + \frac{c}{x^2}$$

↳ L.d.e. in y.

$$\therefore \quad IF = e^{\int \frac{1}{x} dx} = x$$

Hence, the sol. is

$$yx = \int x \left[\frac{1}{x^2(1-x)} + \frac{c}{x^2} \right] dx + c_1$$

$$= \int \frac{dx}{x(1-x)} + c \int \frac{1}{x} dx + c_1$$

$$= \int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx + c \log x + c_1$$

$$xy = \log x - \log(1-x) + c \log x + c_1$$

$$xy = \log \frac{x}{1-x} + c \log x + c_1 \qquad \underline{\text{Ans}}.$$

Method of variation of parameters ⇒

Given eq. ⇒ $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —— ①

Working Procedure ⇒

① Find value of P, Q, R from given eq. It is noted that the coeff. of $\frac{d^2y}{dx^2}$ should be unity.

② Find C.F. of given d.e.

let it is $C.F. = C_1 u + C_2 v$ $C_1, C_2 \Rightarrow$ Const.

③ let $y = Au + Bv$ be the complete sol.

④ Solve the simultaneous eq.s. $u\frac{dA}{dx} + v\frac{dB}{dx} - 0 = 0$

and $\frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} - R = 0$

by cross multiplication method, and find $\frac{dA}{dx}$ & $\frac{dB}{dx}$.

⑤ Integrate $\frac{dA}{dx}$ and $\frac{dB}{dx}$ and find A and B.

⑥ Putting these values of A and B in $y = Au + Bv$, the complete solution of given d.e. is obtained.

Q: Solve by the method of variation of Parameters

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$ [RTU-2024] (10-Marks)

Sol: Given: $(D^2-1)y = \frac{2}{1+e^x}$ —①

A.E. $\Rightarrow m^2-1=0 \Rightarrow m = \pm 1$

$\therefore$ C.F. $= c_1e^x + c_2e^{-x}$ —②

let us assume $y = Ae^x + Be^{-x}$ complete sol of eq.①
—③

Here, ~~$A=e^x$~~ ~~and~~ ~~$B=e^{-x}$~~ $u=e^x$, $v=e^{-x}$

Now, ~~A+B~~ $e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0$ —④

and $e^x\frac{dA}{dx} - e^{-x}\frac{dB}{dx} - \frac{2}{1+e^x} = 0$ —⑤

solving ④ and ⑤,

$$\frac{\left(\frac{dA}{dx}\right)}{-e^{-x}\times\frac{2}{1+e^x}} = \frac{\left(\frac{dB}{dx}\right)}{\frac{2e^x}{1+e^x}} = \frac{1}{-e^x.e^{-x} - e^x.e^{-x}}$$

$$\frac{\left(\frac{dA}{dx}\right)}{\frac{-2e^{-x}}{1+e^x}} = \frac{\left(\frac{dB}{dx}\right)}{\frac{2e^x}{1+e^x}} = \frac{-1}{2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{e^{-x}}{1+e^x} \quad \Rightarrow A = \log\left(\frac{1+e^x}{e^x}\right) - e^{-x}$$

$$\text{and } \frac{dB}{dx} = \frac{-e^{+x}}{1+e^x} \quad \Rightarrow B = -\log(1+e^x)$$

Hence, sol. $\Rightarrow$

$$y = C_1 e^x + C_2 e^{-x} + e^x \log\left(\frac{1+e^x}{e^x}\right) - 1 - e^{-x}\log(1+e^x). \quad \underline{A}.$$

Q: Solve: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$ by the method of variation of parameters.

[RTU-2023]
(Marks-10)

Sol: Given eq. is homogeneous,

$\therefore$ ~~Changing~~ substitute $x\frac{dy}{dx} = Dy$

and $x^2\frac{d^2y}{dx^2} = D(D-1)y$.

$x = e^z$
$\Rightarrow z = \log x.$
(Changing the independent var. from $x \to z$)

$\therefore$ Eq. become $\Rightarrow [D(D-1) + D - 1]y = e^{2z} e^{e^z}$

$$\Rightarrow (D^2-1)y = e^{2z} e^{e^z}$$

for C.F. $\Rightarrow$ A.E $\Rightarrow m^2 - 1 = 0 \quad \Rightarrow m = \pm 1$

$$\therefore CF = C_1 e^z + C_2 e^{-z} = C_1 x + \frac{C_2}{x}$$

Let $Ae^z + Be^{-z} \Rightarrow$ Complete sol.

or $Ax + \frac{B}{x}$

~~Now~~,
Here, $u = x$ and $v = \frac{1}{x}$

Now,

$$x\frac{dA}{dx} + \frac{1}{x}\frac{dB}{dx} - 0 = 0$$

$$\left\{\begin{array}{l} u\frac{dA}{dx} + v\frac{dB}{dx} - 0 = 0 \\ u'\frac{dA}{dx} + v'\frac{dB}{dx} - R = 0 \end{array}\right.$$

and $$1.\frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} - e^x = 0$$

Here $R = e^x$

$$\frac{\left(\frac{dA}{dx}\right)}{-\frac{e^x}{x}} = \frac{\left(\frac{dB}{dx}\right)}{+xe^x} = \frac{1}{-\frac{1}{x} - \frac{x}{x^2}}$$

$$\frac{dA}{dx} = \frac{1}{2}e^x \quad \text{and} \quad \frac{dB}{dx} = -\frac{1}{2}x^2e^x$$

$$A = \frac{1}{2}\int e^x dx \qquad B = -\frac{1}{2}\int x^2e^x dx$$

$$A = \frac{1}{2}e^x \qquad = \left(\frac{-x^2}{2} + x - 1\right)e^x$$

Hence, the complete sol. ⇒ ~~y = CF + PI~~

$$y = C_1x + \frac{C_2}{x} + \frac{1}{2}e^x . x + \frac{1}{x}\left(\frac{-x^2}{2} + x - 1\right)e^x$$

$$y = C_1x + \frac{C_2}{x} + e^x - x^{-1}e^x$$