Homogeneous Functions and Euler theorem

Consider the expression $f(x,y) = a_0x_n + a_1x_{n-1}y + a_2x_{n-2}y_2 + \cdots + a_ny_n$ ... (1)

The degree of each term in the above expression is 'n'. Such an expression is called a homogeneous function of degree 'n', Equation (i) can be written as $f(x,y) = x^n f\left(\frac{y}{x}\right)$.

Note:

A function $f(x,y)$ is said to be a homogeneous function in x and y of degree 'n' if

$$f(tx,ty) = t^n f(x,y)$$

For example,

$$f(x,y) = \frac{x^3+y^3}{x-y}$$

$$f(tx,ty) = \frac{t^3x^3+t^3y^3}{tx-ty}$$

$$= \frac{t^3(x^3+y^3)}{t(x-y)}$$

$$= \frac{t^2(x^3+y^3)}{x-y}$$

$$f(tx,ty) = t^2\ f(x,y)$$

$\therefore$ $f(x,y)$ is a homogeneous function in of degree '2'.

Euler's theorem on homogeneous function:

If u is a homogeneous function in x and y of degree 'n' then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Proof:

Given 'u' is a homogeneous function of degree 'n' in x and y

$$\therefore u(x,y) = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

and

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^{n-1}f'\left(\frac{y}{x}\right)$$

Hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1}f'\left(\frac{y}{x}\right) + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right) = nu$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Note:

(i) If 'u' is a homogeneous function of three variables x, y and z of degree 'n', then the Euler's theorem is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

(ii)Euler's extension theorem is $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Example:

Verify Euler's theorem for the function $\boldsymbol{u = \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)(x^n + y^n)}$

Solution:

Given $u(x,y) = \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)(x^n + y^n)$

$$\mathrm{U}(tx,ty) = \left(t^{\frac{1}{2}}x^{\frac{1}{2}} + t^{\frac{1}{2}}y^{\frac{1}{2}}\right)(t^n x^n + t^n y^n)$$

$$= t^{\frac{1}{2}}\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)t^n(x^n + y^n)$$

$$= t^{n+\frac{1}{2}}\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)(x^n + y^n$$

$$u(tx,ty) = t^{n+\frac{1}{2}}u(x,y)$$

$\therefore$ $u(x,y)$ is a homogeneous function in of degree '$n+\frac{1}{2}$'.

$\therefore$ Euler's theorem is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (n+\frac{1}{2})$u

Verification: Consider L.H.S = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}}(x^n + y^n) + nx^{n-1}(x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$x\frac{\partial u}{\partial x} = \frac{1}{2}x^{\frac{1}{2}}(x^n + y^n) + nx^n\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}}(x^n + y^n) + ny^{n-1}(x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$y\frac{\partial u}{\partial y} = \frac{1}{2}y^{\frac{1}{2}}(x^n + y^n) + ny^n\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) \ldots (2)$$

$(1) + (2) \Rightarrow$ L.H.S = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$= \frac{1}{2}x^{\frac{1}{2}}(x^n + y^n) + nx^n(x^{\frac{1}{2}} + y^{\frac{1}{2}}) + \frac{1}{2}y^{\frac{1}{2}}(x^n + y^n) + ny^n(x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$= \frac{1}{2}(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n) + n(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n)$$

$$= (n+\tfrac{1}{2})(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n)$$

$$= (n+\tfrac{1}{2})u = \text{R.H.S}$$

Hence Euler's theorem is verified.

Example:

Verify Euler's theorem for the function $u = sin^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)$

Solution:

Given $u(x, y) = sin^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)$

$$u(tx, ty) = sin^{-1}\left(\frac{tx}{ty}\right) + tan^{-1}\left(\frac{ty}{tx}\right)$$

$$= t^0[sin^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)]$$

$$u(tx, ty) = t^0\, u(x, y)$$

$\therefore$ $u(x, y)$ is a homogeneous function in of degree '0'.

$\therefore$ Euler's theorem is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = 0.u = 0

Verification: Consider L.H.S = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} \times \left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2+x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2+x^2}} - \frac{xy}{x^2+y^2} \; ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2+x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2+x^2}} + \frac{xy}{x^2+y^2} \; ...(2)$$

$$(1) + (2) \Rightarrow \; L.H.S = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \frac{x}{\sqrt{y^2+x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2+x^2}} + \frac{xy}{x^2+y^2}$$

= 0 = R.H.S

Hence Euler's theorem is verified.

Example:

If $u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u$

Solution:

Given $u(x,y) = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$tanu = \frac{x^3+y^3}{x-y}$$

Let $z = tanu = \frac{x^3+y^3}{x-y}$

Consider $z(x,y) = \frac{x^3+y^3}{x-y}$

$$z(tx,ty) = \frac{t^3x^3+t^3y^3}{tx-ty} = t^2 z(x,y)$$

$\therefore$ $z(x,y)$ is a homogeneous function in of degree '2'.

$\therefore$ By Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2.z$

Put $z = tanu$

$$x\frac{\partial(tanu)}{\partial x} + y\frac{\partial(tanu)}{\partial y} = 2tanu$$

$$xsec^2u\frac{\partial u}{\partial x} + ysec^2u\frac{\partial u}{\partial y} = 2tanu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{sinu}{cosusec^2u}$$

$$= \frac{2\frac{sinu}{cosu}}{\frac{1}{cos^2u}}$$

$$= 2sinu\,cosu$$

$$= sin2u$$

Hence proved.

Example:

If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x^2\frac{\partial^2u}{\partial x^2} + 2xy\frac{\partial^2u}{\partial x\partial y} + y^2\frac{\partial^2u}{\partial y^2} = \frac{-sinucos2u}{4cos^3u}$

Solution:

Given $u(x,y) = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$sinu = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $z = sinu = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Consider $z(x,y) = \frac{tx+ty}{\sqrt{t}\sqrt{x}+\sqrt{t}\sqrt{y}}$

$$z(tx,ty) = t^{\frac{1}{2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = t^{\frac{1}{2}}z(x,y)$$

$\therefore$ $z(x,y)$ is a homogeneous function in of degree $\frac{1}{2}$.

∴ By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$

Put $z = sinu \Rightarrow x\frac{\partial(sinu)}{\partial x} + y\frac{\partial(sinu)}{\partial y} = \frac{1}{2}sinu$

$$xcosu\frac{\partial u}{\partial x} + ycosu\frac{\partial u}{\partial y} = \frac{1}{2}sinu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{sinu}{cosu}$$

$$= \frac{1}{2}tanu = f(u)$$

By Euler's extension theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = f(u)[f'(u) - 1]$$

$$= \frac{1}{2}tanu(\frac{1}{2}sec^2u - 1)$$

$$= \frac{1}{2}\frac{sinu}{cosu}(\frac{1}{2cos^2u} - 1)$$

$$= \frac{1}{2}\frac{sinu}{cosu}(\frac{1-2cos^2u}{2cos^2u})$$

$$= \frac{1}{4}\frac{sinu(-cos2u)}{cos^3u}$$

$$= \frac{-sinucos2u}{4cos^3u}$$

Hence proved.

Example:

If $u = (x-y)f\left(\frac{y}{x}\right)$, then find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ [AU May 2001,Dec2014]

Solution:

Given $u(x,y) = (x-y)f\left(\frac{y}{x}\right)$

$$u(tx,ty) = (tx - ty)f\left(\frac{ty}{tx}\right)$$

$$= t^1u(x,y)$$

∴ $u(x,y)$ is a homogeneous function in of degree '1'.

∴ By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u = f(u)$

By Euler's extension theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = f(u)[f'(u) - 1]$$

$$= u(1-1) = u.0 = 0$$