MATHEMATICS – I

3110014

3rd Edition - 2020

Name: ____________________ Div. & Roll No.: ____________

Dedicated Faculty Committed Education

Darshan

Institute of Engineering & Technology

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UNIT 1

❖ INDETERMINATE FORMS:

✓ The following are indeterminate forms which we will study:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^0,\ \infty^0\ \ \&\ \ 1^\infty.$$

❖ L' HOSPITAL'S RULE:

If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ leads to the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$, provided the later limit exists.

✓ Procedure to find the limit using L' Hospital's rule:

(1). Differentiate numerator and denominator separately and apply the limit.

(2). If it again reduces to indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then again differentiate numerator and denominator separately and apply the limit.

(3). Continue this process till we get finite or infinite value of the limit.

✓ Remark: $\lim\limits_{x\to 0}\log x=-\infty$ & $\lim\limits_{x\to\infty}\log x=\infty$.

METHOD – 1: 0/0 TYPE INDETERMINATE FORM

C	1	Evaluate $\lim\limits_{x\to 0}\left(\frac{e^x-1-x}{x^2}\right)$. **Answer**: **1/2**.	
C	2	Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{2x-\pi}{\cos x}\right)$. **Answer**: **– 2**.	
H	3	Evaluate the examples: (1). $\lim\limits_{x\to 0}\left(\frac{x-\tan x}{x^3}\right)$ (2). $\lim\limits_{x\to 0}\frac{\sin x-x+\frac{x^3}{6}}{x^5}$. **Answer**: (1). **– 1/3** (2). **1/120**.	

T	**4**	Evaluate $\lim_{x\to 0} \frac{2x - x\cos x - \sin x}{2x^3}$. **Answer**: **1/3.**	
C	**5**	Evaluate $\lim_{x\to 0} \left(\frac{(\sin x)^2 - x^2}{x^2(\sin x)^2}\right)$. **Answer**: **– 1/3.**	
T	**6**	Evaluate the examples: (**1**). $\lim_{x\to 0} \left(\frac{\tan x - x}{\sin x - x}\right)$ (**2**). $\lim_{x\to 0} \left(\frac{\tan x - \sin x}{(\sin x)^3}\right)$ (**3**). $\lim_{x\to 0} \left(\frac{x(\cos x - 1)}{\sin x - x}\right)$ (**4**). $\lim_{x\to 0} \left(\frac{2\sqrt{1+x} - 2 - x}{2\sin^2 x}\right)$. **Answer**: (**1**). **– 2** (**2**). **1/2** (**3**). **3** (**4**). **– 1/8.**	
C	**7**	Evaluate $\lim_{x\to 0} \left(\frac{e^x + e^{-x} - x^2 - 2}{(\sin x)^2 - x^2}\right)$. **Answer**: **– 1/4.**	
H	**8**	Evaluate $\lim_{x\to 0} \left(\frac{e^x - e^{\sin x}}{x - \sin x}\right)$. **Answer**: **1.**	
T	**9**	Evaluate $\lim_{x\to \frac{1}{2}} \left(\frac{\cos^2 \pi x}{e^{2x} - 2ex}\right)$. **Answer**: $\boldsymbol{\pi^2/2e}$.	
C	**10**	Evaluate $\lim_{x\to y} \left(\frac{x^y - y^x}{x^x - y^y}\right)$. **Answer**: $(\mathbf{1 - \log y}) / (\mathbf{1 + \log y})$.	
H	**11**	Evaluate $\lim_{x\to 0} \left\{\frac{\ln \cos \sqrt{x}}{x}\right\}$. **Answer**: **– 1/2.**	
H	**12**	Evaluate $\lim_{x\to 0} \left(\frac{xe^x - \log(1+x)}{x^2}\right)$. **Answer**: **3/2.**	W-19 (4)

T	13	Evaluate $\lim_{x\to 0}\left(\frac{a^x - b^x}{x}\right)$. **Answer**: **log(a/b)**.	
T	14	Evaluate $\lim_{x\to 1}\left(\frac{x^x - x}{x-1-\log x}\right)$. **Answer**: **2**.	

METHOD – 2: ∞/∞ TYPE INDETERMINATE FORM

C	1	Evaluate $\lim_{x\to a}\left(\frac{\log(e^x - e^a)}{\log(x-a)}\right)$. **Answer**: **1**.	
H	2	Evaluate $\lim_{x\to 0}\left(\frac{\cot 2x}{\cot x}\right)$. **Answer**: **1/2**.	
T	3	Evaluate $\lim_{x\to \frac{\pi}{2}}\left(\frac{3\sec x}{1+\tan x}\right)$. **Answer**: **3**.	
C	4	Evaluate $\lim_{x\to \infty}\left(\frac{x^n}{e^x}\right), n > 1$. **Answer**: **0**.	
C	5	Evaluate $\lim_{x\to 0}\left(\log_{\tan x}\tan 2x\right)$. **Answer**: **1**.	
T	6	Evaluate the given examples: (**1**). $\lim_{x\to \infty}\left(\frac{\log x}{x^n}\right)$ (**2**). $\lim_{x\to \frac{\pi}{2}}\left(\frac{\ln(\cos x)}{\sec x}\right)$ (**3**). $\lim_{x\to \infty}\left(\frac{(\ln x)^2(3+\ln x)}{2x}\right)$. **Answer**: (**1**). **0** (**2**). **0** (**3**). **0**.	

❖ $0 \times \infty$ TYPE INDETERMINATE FORM:

In this case we write $f(x) \cdot g(x)$ as $\dfrac{f(x)}{\left\{\dfrac{1}{g(x)}\right\}}$ or $\dfrac{g(x)}{\left\{\dfrac{1}{f(x)}\right\}}$ which leads to the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, where L' Hospital's rule is applicable.

✓ Procedure to find the limit of indeterminate form $0 \times \infty$:

(1). Transform $f(x) \cdot g(x)$ into $\dfrac{f(x)}{\left\{\dfrac{1}{g(x)}\right\}}$ or $\dfrac{g(x)}{\left\{\dfrac{1}{f(x)}\right\}}$

i.e. transform $0 \times \infty$ into $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

(2). **Remember:** Don't put the logarithm function in the denominator in step (1).

(3). Apply L' Hospital's rule to find the value of given limit.

METHOD – 3: $0 \times \infty$ FORM

C	**1**	Evaluate $\lim\limits_{x \to \infty} \left\{\left(a^{\frac{1}{x}} - 1\right) x\right\}$. **Answer**: **log a.**	
H	**2**	Evaluate $\lim\limits_{x \to 1} \left\{(1-x) \ \tan\left(\dfrac{\pi x}{2}\right)\right\}$. **Answer**: $\mathbf{2/\pi}$.	
C	**3**	Evaluate $\lim\limits_{x \to \infty} \left\{\left(\sqrt{x+1} - \sqrt{x}\right) \log\left(\dfrac{1}{x}\right)\right\}$. **Answer**: **0.**	
H	**4**	Evaluate $\lim\limits_{x \to 0} \{(\sin x)(\ln x)\}$. **Answer**: **0.**	
T	**5**	Evaluate the given examples: (**1**). $\lim\limits_{x \to a} \left\{\ln\left(2 - \dfrac{x}{a}\right) \cot(x-a)\right\}$ (**2**). $\lim\limits_{x \to \frac{1}{2}} \left\{\ln\left(\dfrac{3}{2} - x\right) \cot\left(x - \dfrac{1}{2}\right)\right\}$. **Answer**: (**1**). $\mathbf{-1/a}$ (**2**). $\mathbf{-1}$.	

❖ ∞ - ∞ TYPE INDETERMINATE FORM:

In this case, we write $f(x)-g(x)=\dfrac{\left\{\dfrac{1}{g(x)}-\dfrac{1}{f(x)}\right\}}{\left\{\dfrac{1}{f(x)\,g(x)}\right\}}$ which leads to the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

where, L' Hospital's rule is applicable.

✓ Procedure to find the limit of indeterminate form $\infty-\infty$:

(1). Take L. C. M. in $f(x)-g(x)$ which will transform $\infty-\infty$ into $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

(2). Apply L' Hospital's rule to find the value of given limit.

METHOD – 4: ∞ – ∞ FORM

C	**1**	Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{1}{\sin x}-\dfrac{1}{x}\right\}$. **Answer: 0.**	
H	**2**	Evaluate $\lim\limits_{x\to 0}(\operatorname{cosec} x-\cot x)$. **Answer: 0.**	
T	**3**	State L'Hospital's Rule. Use it to evaluate $\lim\limits_{x\to 0}\left\{\dfrac{1}{x^2}-\dfrac{1}{(\sin x)^2}\right\}$. **Answer: $-1/3$.**	W-18 (4)
C	**4**	Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{1}{x^2}-(\cot x)^2\right\}$. **Answer: 2/3.**	
C	**5**	Use L'Hospital's rule to find the limit of $\lim\limits_{x\to 1}\left\{\dfrac{x}{x-1}-\dfrac{1}{\log x}\right\}$. **Answer: 1/2.**	S-19 (3)
H	**6**	Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{1}{x}-\dfrac{1}{e^x-1}\right\}$. **Answer: 1/2.**	
T	**7**	Evaluate $\lim\limits_{x\to 2}\left\{\dfrac{1}{x-2}-\dfrac{1}{\log(x-1)}\right\}$. **Answer: $-1/2$.**	

T	8	If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipodal moment P per unit volume is $P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$. Show that $\lim_{E \to 0^+} P(E) = 0$.	S-19 (3)

❖ 0^0, ∞^0 & 1^∞ TYPE INDETERMINATE FORM:

✓ In this case, we will write $y = \lim_{x \to *} \{ f(x)^{g(x)} \}$. After that we will take logarithm on both the sides, then we get $\log y = \lim_{x \to *} g(x) \cdot \log f(x)$, which leads to indeterminate form $0 \times \infty$ or $\infty \times 0$. We know that how to deal with it.

✓ **Remember:** It gives us value of $\log y = L$(any value), but we want the value of y. So, use $y = e^L$ to find the value of given limit i.e. y.

✓ Procedure to find the limit of indeterminate form 0^0, ∞^0 & 1^∞:

(1). Assume $y = \lim_{x \to *} \{ f(x)^{g(x)} \}$.

(2). Apply logarithm function both the side of above assumption.

(3). Simplify R.H.S. to convert function into indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

(4). Find the value of R.H.S. using above concept of indeterminate form i.e. value of $\log y$.

(5). Find the value of y i.e. value of given limit.

METHOD – 5: 0^0, ∞^0 & 1^∞ FORM

C	1	Evaluate $\lim_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$. **Answer**: **1**.	
H	2	Evaluate $\lim_{x \to 0} \left(\frac{1}{x}\right)^{1-\cos x}$. **Answer**: **1**.	

C	**3**	Evaluate $\lim_{x\to 0}\{(\cos x)^{\cot x}\}$. **Answer**: **1**.	
T	**4**	Evaluate given examples: (**1**). $\lim_{x\to 1}\{(x-1)^{x-1}\}$ (**2**). $\lim_{x\to\frac{\pi}{2}}\{(\sin x)^{\tan x}\}$ (**3**). $\lim_{x\to\frac{\pi}{2}}\{(\tan x)^{\cos x}\}$. **Answer**: (**1**). **1** (**2**). **1** (**3**). **1**.	
C	**5**	Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. **Answer**: $\mathbf{e^{1/3}}$.	
C	**6**	Evaluate $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{3x}}$. **Answer**: $\mathbf{(abc)^{1/9}}$.	
H	**7**	Evaluate the given examples: (**1**). $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+4^x}{4}\right)^{\frac{1}{x}}$ (**2**). $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}}$. **Answer**: (**1**). $\mathbf{24^{1/4}}$ (**2**). $\mathbf{(abcd)^{1/4}}$.	
T	**8**	Evaluate $\lim_{x\to 0}\left(\frac{e^x+e^{2x}+e^{3x}}{3}\right)^{\frac{1}{x}}$. **Answer**: $\mathbf{e^2}$.	
H	**9**	Evaluate $\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}$. **Answer**: **ae**.	
H	**10**	Evaluate $\lim_{x\to 1}(2-x)^{\left(\tan\frac{\pi x}{2}\right)}$. **Answer**: $\mathbf{e^{2/\pi}}$.	
T	**11**	Evaluate $\lim_{x\to\frac{\pi}{2}}(\cos x)^{\left(\frac{\pi}{2}-x\right)}$. **Answer**: **1**.	

T	12	Evaluate the examples: **(1).** $\lim_{x\to\frac{\pi}{2}} \left\{ (\operatorname{cosec} x)^{(\tan^2 x)} \right\}$ **(2).** $\lim_{x\to 0} \left(\cos\sqrt{x}\right)^{\frac{1}{x}}$. **Answer**: **(1).** $\sqrt{e}$ **(2).** $1/\sqrt{e}$.	
C	13	Evaluate $\lim_{x\to 0} x^{\left(\frac{1}{\ln(e^x-1)}\right)}$. **Answer**: **e.**	
T	14	Evaluate $\lim_{x\to 1} (1-x^2)^{\left(\frac{1}{\log(1-x)}\right)}$. **Answer**: **e.**	

❖ IMPROPER INTEGRALS:

- ✓ If the limit of integral is infinite (one or both) and/or integrand function is discontinuous for some value(s) on the interval of given integral then such integral is known as an improper integral.
- ✓ Types of Improper Integrals:
  - **(1).** Improper integral of first kind.
  - **(2).** Improper integral of second kind.
  - **(3).** Improper integral of third kind (combination of 1st & 2nd kind).

❖ IMPROPER INTEGRAL OF FIRST KIND:

For $\int_a^b f(x)dx$ either a or b or both (a and b) are infinite, then such integral is known as an improper integral of first kind.

- ✓ Sub-types of improper integrals of first kind are as follows:
  - **(1).** If f is continuous on $[a,\infty)$ then

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

  - **(2).** If f is continuous on $(-\infty, b]$ then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)dx.$$

(3). If f is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{t} f(x)dx + \int_{t}^{\infty} f(x)dx = \lim_{a\to-\infty} \int_{a}^{t} f(x)dx + \lim_{b\to\infty} \int_{t}^{b} f(x)dx.$$

METHOD – 6: IMPROPER INTEGRAL OF FIRST KIND

C	1	Evaluate $\int_{0}^{\infty} \frac{dx}{x^2+1}$. **Answer**: $\pi/2$.	W-19 (4)
H	2	Evaluate $\int_{1}^{\infty} \frac{1}{\sqrt{x}}\, dx$. **Answer**: ∞.	
T	3	Evaluate $\int_{0}^{\infty} \frac{x}{(1+x)^3}\, dx$. **Answer**: 1.	
T	4	Discuss type I & II improper integrals with example of each & find the value of $\int_{2}^{\infty} \frac{x+3}{(x-1)(x^2+1)}\, dx$. **Answer**: $(-\pi/2) + \log 5 + \tan^{-1} 2$.	
C	5	Evaluate $\int_{-\infty}^{0} e^{2x}\, dx$. **Answer**: $1/2$.	
T	6	Evaluate $\int_{-\infty}^{0} x \sin x\, dx$. **Answer**: $-\infty$.	

C	**7**	Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$ **Answer**: π.	
H	**8**	Evaluate $\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\,dx.$ **Answer**: **0**.	
T	**9**	Evaluate $\int_{-\infty}^{\infty} e^x\,dx.$ **Answer**: ∞.	
C	**10**	Evaluate $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx.$ **Answer**: $\pi/2$.	

❖ IMPROPER INTEGRAL OF SECOND KIND:

For $\int_a^b f(x)dx$, $f(x)$ become discontinuous (infinite) at $x = a$ or $x = b$ or at finite number of points in the interval (a, b), then such an integral is known as improper integral of second kind.

✓ Sub-types of improper integrals of second kind are as follows:

(1). If $x = a$ is the point of discontinuity for $f(x)$ then the integral is defined as

$$\int_a^b f(x)dx = \lim_{t \to a} \int_t^b f(x)\,dx.$$

(2). If $x = b$ is the point of discontinuity for $f(x)$ then the integral is defined as

$$\int_a^b f(x)dx = \lim_{t \to b} \int_a^t f(x)\,dx.$$

(3). If $x = c$ is the point of discontinuity for $f(x)$ then the integral is defined as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{t\to c}\int_a^t f(x)dx + \lim_{t\to c}\int_t^b f(x)dx.$$

METHOD – 7: IMPROPER INTEGRAL OF SECOND KIND

C	1	Evaluate the integrals $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ and $\int_a^b \frac{1}{(a-x)^2}\,dx$. **Answer**: $2\sqrt{3}$ & $-\infty$.	
H	2	Evaluate the integrals $\int_0^1 \frac{1}{x^2}\,dx$ and $\int_{-1}^1 \frac{1}{x^{2/3}}\,dx$. **Answer:** ∞ & 6.	
H	3	Evaluate $\int_0^{\frac{\pi}{2}} \sec x\,dx$. **Answer**: ∞.	
T	4	Evaluate $\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$. **Answer**: $2\sqrt{3}$.	
T	5	Can we solve the integral $\int_0^5 \frac{1}{(x-2)^2}\,dx$ directly? Give the reason. **Answer**: **No, given function is not continuous at** $x=2$.	
C	6	Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$. **Answer**: $3\left(2^{1/3}-(-1)^{1/3}\right)$ **OR** $3\left(2^{1/3}+1\right)$.	
T	7	Evaluate $\int_{-a}^a \frac{1}{\sqrt{a^2-x^2}}\,dx$. **Answer**: π.	

❖ CONVERGENCE OR DIVERGENCE OF IMPROPER INTEGRAL:

- ✓ If integral has finite value then we can say that given integral is convergent.
- ✓ If integral has infinite value then we can say that given integral is divergent.

❖ P-INTEGRAL TEST:

$\int_a^\infty \frac{1}{x^p}\,dx$ is convergent if $P > 1$ & divergent if $P \le 1$.

$\int_0^a \frac{1}{x^p}\,dx$ is convergent if $P < 1$ & divergent if $P \ge 1$.

❖ DIRECT COMPARISON TEST:

- ✓ If $f(x)$ and $g(x)$ are continuous on $[a,\infty)$ and $0 \le f(x) \le g(x)$ for all $x \ge a$ then

(1). If $\int_a^\infty g(x)dx$ is convergent then $\int_a^\infty f(x)dx$ is convergent.

(2). If $\int_a^\infty f(x)dx$ is divergent then $\int_a^\infty g(x)dx$ divergent.

❖ LIMIT COMPARISON TEST:

If $f(x)$ and $g(x)$ are positive function and continuous on $[a,\infty)$ and $L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$ then

(1). If $0 < L < \infty$ then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both convergent or divergent together.

(2). If $L = 0$ and $\int_a^\infty g(x)dx$ is convergent then $\int_a^\infty f(x)dx$ is convergent.

(3). If $L = \infty$ and $\int_a^\infty g(x)dx$ is divergent then $\int_a^\infty f(x)dx$ is divergent.

- ✓ Convergence of improper integral of second kind can be checked by proper changes in above Direct/Limit comparison test.

METHOD – 8: CONVERGENCE OF IMPROPER INTEGRAL OF FIRST KIND

C	**1**	Check the convergence of $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ and $\int_1^\infty \frac{1}{x^{\sqrt{2}}}\,dx$. **Answer: Divergent and convergent.**	
C	**2**	Check the convergence of $\int_0^\infty \frac{dx}{(1+x^2)(1+\tan^{-1}x)}$. **Answer: Convergent.**	
H	**3**	Check the convergence of $\int_1^\infty \frac{1}{x^p}\,dx$. **Answer: $p \le 1 \Rightarrow$ Divergent & $p > 1 \Rightarrow$ Convergent.**	
T	**4**	Check the convergence of $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ and $\int_1^\infty \frac{\sin^3 x}{x^3}\,dx$. **Answer: Convergent & Convergent.**	
C	**5**	Check the convergence of $\int_2^\infty \frac{1}{\log x}\,dx$. **Answer: Divergent.**	
H	**6**	Check the convergence of $\int_2^\infty \frac{x}{(1+x)^4}\,dx$ and $\int_1^\infty \frac{1}{\sqrt{x}\,(1+x)^2}\,dx$. **Answer: Convergent & Convergent.**	
T	**7**	Check the convergence of $\int_1^\infty e^{-x^2}\,dx$. **Answer: Convergent.**	
C	**8**	Investigate the convergence of $\int_5^\infty \frac{5x}{(1+x^2)^3}\,dx$. **Answer: 5/2704 & Convergent.**	W-18 (3)
H	**9**	Prove that $\int_1^\infty \frac{5}{e^x+3}\,dx$ is convergent.	

H	10	Check the convergence of $\int_{4}^{\infty} \frac{3x+5}{x^4+7}\,dx$. **Answer: Convergent.**	
T	11	Investigate the convergence of $\int_{0}^{\infty} \frac{x^{10}(1+x^5)}{(1+x)^{27}}\,dx$. **Answer: $2(10!)(5!)/26!$ & Convergent.**	W-18 (4)

METHOD – 9: CONVERGENCE OF IMPROPER INTEGRAL OF SECOND KIND

C	1	Check the convergence of $\int_{0}^{5} \frac{1}{x^2}\,dx$. **Answer: Divergent.**	
H	2	Determine whether the integral $\int_{0}^{3} \frac{dx}{x-1}$ converges or diverges. **Answer: Divergent.**	S-19 (3)
H	3	Determine $\int_{0}^{2} \frac{1}{(x-2)^2}\,dx$ and is it convergent or divergent? **Answer: $-\infty$ & Divergent.**	
T	4	Check the convergence of $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$. **Answer: $\pi/2$ & Convergent.**	
C	5	Define improper integral of both the kinds and check the convergence of $\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}}$. **Answer: $\pi/2$ & Convergent.**	
H	6	Check the convergence of $\int_{0}^{3} \frac{\cos 3x}{x^{5/2}}\,dx$. **Answer: Divergent.**	

T	7	Determine $\int_0^1 \ln x \, dx$ and is it convergent or divergent? **Answer**: -1 & **Convergent**.	
C	8	Check the convergence of $\int_{-2}^{2} \frac{dx}{x^2}$. **Answer**: **Divergent**.	

❖ GAMMA FUNCTION:

If $n > 0$, then Gamma function is defined by the integral $\int_0^\infty e^{-x} x^{n-1} dx$ and is denoted by $\Gamma(n)$. **i.e.** $\boldsymbol{\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx.}$

✓ Properties:

(1). Reduction formula for Gamma Function $\Gamma(n+1) = n\Gamma(n)$; where $n > 0$.

(2). If n is a positive integer, then $\Gamma(n+1) = n!$.

(3). Second Form of Gamma Function $\int_0^\infty e^{-x^2} x^{2m-1} dx = \frac{1}{2}\Gamma(m)$.

(4). $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n!\,4^n}$ for $n = 0,1,2,3,\ldots$

Examples: For $n = 0$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ For $n = 1$, $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

For $n = 2$, $\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$

❖ BETA FUNCTION:

If $m > 0, n > 0$, then Beta function is defined by the integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ and is denoted by $\beta(m,n)$ OR $B(m,n)$. **i.e.** $\boldsymbol{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx}$.

✓ Properties:

(1). Beta function is a symmetric function. i.e. $\beta(m,n) = \beta(n,m)$, where $m > 0, n > 0$.

(2). $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$.

(3). $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$.

(4). $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

(5). Relation Between Beta and Gamma Function, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

METHOD – 10: EXAMPLE ON GAMMA FUNCTION AND BETA FUNCTION

C	1	Find $\Gamma\left(\frac{13}{2}\right)$. **Answer**: $\mathbf{10395\sqrt{\pi}/64}$.	
C	2	Define Gamma function and evaluate $\int_0^\infty e^{-x^2} dx$. **Answer**: $\sqrt{\pi/2}$.	S-19 (4)
H	3	Evaluate $\int_0^\infty x^3 e^{-x} dx$. **Answer**: **6**.	
T	4	Evaluate $\int_0^\infty \frac{x^4}{4^x} dx$. **Answer**: $\mathbf{24/(\log 4)^5}$.	

C	**5**	Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$.	
H	**6**	Find $\beta(4,3)$. **Answer**: $\mathbf{1/60}$.	
T	**7**	Prove that $\dfrac{\beta(m+1,n)}{\beta(m,n)} = \dfrac{m}{m+n}$.	
C	**8**	Find $\beta\left(\frac{7}{2}, \frac{5}{2}\right)$. **Answer**: $\mathbf{3\pi/256}$.	W-19 (3)
H	**9**	Evaluate $\int_0^\infty \frac{x^5(1+x^5)}{(1+x)^{15}}\, dx$. **Answer**: $\mathbf{1/5005}$.	
C	**10**	Evaluate $\int_0^1 \frac{x}{\sqrt{1-x^5}}\, dx$. **Answer**: $\boldsymbol{\frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)}$.	

❖ VOLUME BY SLICING:

✓ A plane intersects a solid, then plane area of the cross section be $A(x)$ and if plane is perpendicular to x-axis at any point between $x = a$ and $x = b$, then volume of solid is

$$V = \int_{x=a}^{b} A(x)\, dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{Formula S1}$$

✓ A plane intersects a solid, then plane area of the cross section be $A(y)$ and if plane is perpendicular to y-axis at any point between $y = a$ and $y = b$, then volume of solid is

$$V = \int_{y=a}^{b} A(y)\, dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{Formula S2}$$

✓ Procedure for finding the volume of solid:

(1). Sketch the solid with cross-section.

(2). Find the area of that cross-section (i.e. $A(x)$ or $A(y)$).

(3). Find the limit of integration (i.e. "a" and "b").

(4). Integrate A(x) or A(y) using the standard formula.

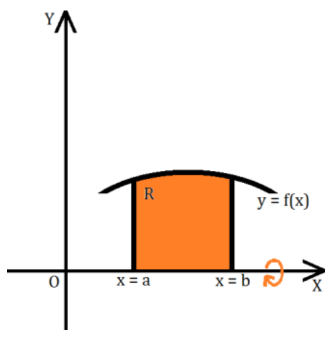
METHOD – 11: VOLUME BY SLICING METHOD

H	1	Using method of slicing, find volume of a cone with height 4cm and radius of base 4cm. **Answer**: $(64\pi/3)cm^3$.	
C	2	Using slicing method find the volume of a solid ball of radius 'a'. **OR** Using slicing method determine volume of a sphere of radius 'a'. **Answer**: $4\pi a^3/3$.	
C	3	Derive the formula for the volume of a right pyramid whose altitude is "h" and whose base is a square with sides of length 'a'. **Answer**: $a^2h/3$.	
H	4	A pyramid having 4 m height has a square base with sides of length 4m.The pyramid having a cross section at a distance x m down from the vertex and perpendicular to the altitude is a square with sides of length x m. Find the volume of the pyramid. **Answer**: $(64/3)m^3$.	

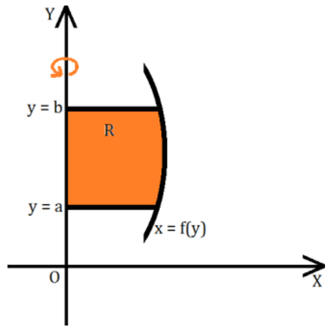
❖ VOLUME OF SOLID BY ROTATION USING DISK METHOD:

✓ The area bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and x-axis, which is revolved **about x-axis** then the volume of solid is given by

$$V = \pi \int_{x=a}^{b} y^2 \, dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{Formula R1}$$

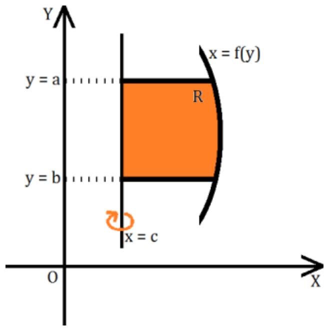


✓ The area bounded by the curve $x = f(y)$, the ordinates $y = a, y = b$ and y-axis, which is revolved **about y-axis** then the volume of solid is given by



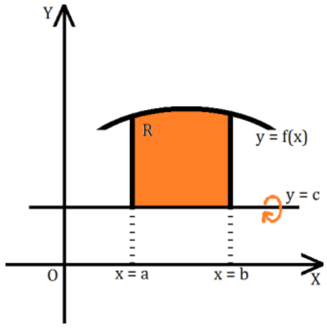
$$V = \pi \int_{y=a}^{b} x^2\, dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{Formula R2}$$

✓ The area bounded by the curve $x_1 = f(y)$, the ordinates $y = a, y = b$ and $x_2 = c$ (constant), which is revolved **about $x_2 = c$** then the volume of solid is given by



$$V = \pi \int_{y=a}^{b} (x_1 - x_2)^2\, dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{Formula R3}$$

✓ The area bounded by the curve $y_1 = f(x)$, the ordinates $x = a, x = b$ and $y_2 = c$ (constant), which is revolved **about $y_2 = c$** then the volume of solid is given by



$$\mathbf{V = \pi \int_{y=a}^{b} (y_1 - y_2)^2\, dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{Formula R4}$$

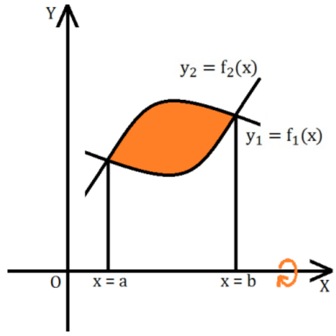
METHOD – 12: VOLUME OF SOLID BY ROTATION USING DISK METHOD

C	**1**	Find the volume of the solid generated by revolving the area bounded by the $2y = x^2, x = 4, y = 0$ and about **(1).** X-axis, **(2).** Y-axis. **Answer**: $\mathbf{256\pi/5}$ & $\mathbf{64\pi}$.	
T	**2**	The region between the curve $y = \sqrt{x}\,; 0 \le x \le 4$ and the X-axis is revolved about the X-axis to generate a solid. Find its volume. **Answer**: $\mathbf{8\pi}$.	
H	**3**	Find the volume of the solid formed by rotating the region between the graph of $y = 1 - x^2$ and $y = 0$ about the X-axis. **Answer**: $\mathbf{16\pi/15}$.	

H	**4**	Find the volume of solid of revolution; obtain by rotating the area bounded below the line $2x + 3y = 6$ in the first quadrant, about the X-axis. **Answer**: $\mathbf{4\pi}$.	
C	**5**	Find the volume of the solid generated by rotating the plane region bounded by $y = \frac{1}{x}, x = 1$ and $x = 3$ about the X axis. **Answer**: $\mathbf{2\pi/3}$.	W-19 (7)
T	**6**	The graph of $y = x^2$ between $x = 1$ and $x = 2$ is rotated around the X-axis. Find the volume of a solid so generated. **Answer**: $\mathbf{31\pi/5}$.	
H	**7**	A bowl has a shape that can generated by revolving the graph $y = x^2/2$ between $y = 0$ and $y = 5$ about the Y-axis. Find the volume of bowl. **Answer**: $\mathbf{25\pi}$.	
H	**8**	Find the volume of the solid generated by revolving the region between the Y-axis and the curve $x = 2\sqrt{y}$; $0 \le y \le 4$, about the Y-axis. **Answer**: $\mathbf{32\pi}$.	
C	**9**	Find the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 8$ and $x = 0$ about the Y-axis. **Answer**: $\mathbf{96\pi/5}$.	
C	**10**	Calculate the volume of solid generated by revolving the region between the parabola $2x = y^2 + 2$ and the line $x = 3$, about the line $x = 3$. **Answer**: $\mathbf{128\pi/15}$.	
T	**11**	Find the volume of the solid generated by revolving the arc bounded by the parabola $y^2 = 4ax$ & latus rectum about latus rectum, where $a > 0$. **Answer**: $\mathbf{32\pi a^3/15}$.	
C	**12**	Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1, x = 4$ about the line $y = 1$. **Answer**: $\mathbf{7\pi/6}$.	S-19 (4)

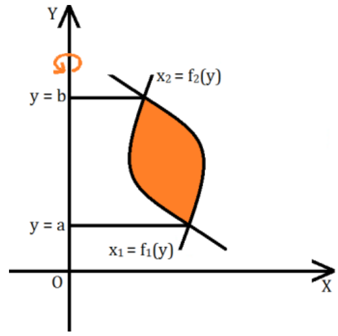
❖ VOLUME OF SOLID BY ROTATION USING WASHER METHOD:

✓ Let A be the area bounded by the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$. Let $x = a$ and $x = b$ be the x-coordinates of their point of intersection. The volume of the solid generated by area A rotating **about x-axis** is given by



$$V = \pi \int_{x=a}^{b} ({y_2}^2 - {y_1}^2)\, dx \quad ; \; y_2 \geq y_1 \dots\dots\dots\dots\dots \quad \text{Formula W1}$$

✓ Let A be the area bounded by the two curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. Let $y = a$ and $y = b$ be the y-coordinates of their point of intersection. The volume of the solid generated by area A rotating **about y-axis** is given by



$$V = \pi \int_{y=a}^{b} ({x_2}^2 - {x_1}^2)\, dy \quad ; \; x_2 \geq x_1 \dots\dots\dots\dots\dots \quad \text{Formula W2}$$

METHOD – 13: VOLUME OF SOLID BY ROTATION USING WASHER METHOD

H	**1**	Find the volume of the solid formed by rotating the region between the graphs of $y = 4 - x^2$ and $y = 6 - 3x$ about the X-axis. **Answer**: $\mathbf{8\pi/15}$.	
C	**2**	Find the volume generated by revolving the area cut off from the parabola $9y = 4(9 - x^2)$ by the lines $4x + 3y = 12$ about X-axis. **Answer**: $\mathbf{48\pi/5}$.	
H	**3**	Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the X-axis. **Answer**: $\mathbf{2\pi/15}$.	

T	4	Define volume of solid of revolution by washer's method and use it to find the volume of solid generated when the region between the graphs $y = \frac{1}{2} + x^2$ and $y = x$ over the interval $[0, 2]$ is revolved about X-axis. **Answer: $69\pi/10$.**	
H	5	Find the volume of the solid obtained by rotating about the Y-axis the region between $y = x$ and $y = x^2$. **Answer: $\pi/6$.**	
C	6	Find the volume of the solid that results when the region enclosed by the curves $y = x^2$ and $x = y^2$ is revolved about the Y-axis. **Answer: $3\pi/10$.**	
H	7	Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $x = -1$. **Answer: $7\pi/15$.**	
C	8	Find the volume of the solid that results when the region enclosed by $y = x^2$ and $y = x^3$ is revolved about the line $x = 1$. **Answer: $\pi/15$.**	
C	9	Find the volume of the solid obtained by rotating the region R enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$. **Answer: $8\pi/15$.**	W-18 (7)
H	10	Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$. **Answer: $153\pi/5$.**	

❖ LENGTH OF PLANE CURVES

✓ If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b (1 + [f'(x)]^2)^{\frac{1}{2}}\, dx\,.$$

METHOD – 14: LENGTH OF PLANE CURVES

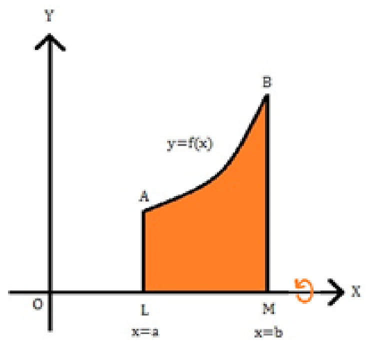
C	1	Using the arc length formula, find the length of the curve $y = x^{\frac{3}{2}}, 0 \le x \le 1$. **Answer**: $\frac{1}{27}[(13)^{\frac{3}{2}} - 8]$.	
H	2	Find the length of the curve $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$. **Answer**: **12.**	
H	3	Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to 2. **Answer**: $\frac{1}{2}(e^2 + e^{-2})$.	
C	4	Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$. **Answer**: $\frac{53}{6}$.	
C	5	Find the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$. **Answer**: $\frac{2}{27}[(10)^{\frac{3}{2}} - 1]$.	
T	6	Find the length of the arc of $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$. **Answer**: $\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$.	

❖ AREA OF SURFACES BY REVOLUTION:

✓ Area of surface by revolution in Cartesian form:

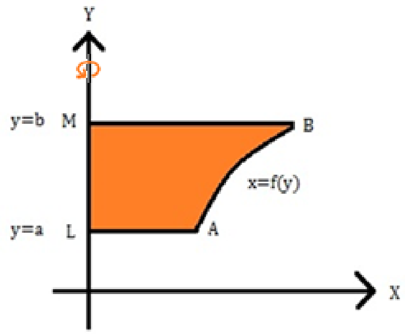
➢ If the curve $y = f(x)$ revolve about x-axis from $x = a$ to $x = b$ then the Area of the surface of a solid is

$$S = 2\pi \int_{x=a}^{b} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \ldots (1)$$



- If the curve $x = f(y)$ revolve about y-axis from $y = a$ to $y = b$ then the Area of the surface of a solid is

$$S = 2\pi \int_{y=a}^{b} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \dots (2)$$



✓ Area of surface by revolution in Polar form:

If the curve $r = f(\theta)$, then the Area of the surface generated by revolution from $\theta = \alpha$ to $\theta = \beta$,

(1). About $\theta = 0$ (Initial line) is

$$S = 2\pi \int_{\theta=\alpha}^{\beta} r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \dots (3)$$

(2). About $\theta = \frac{\pi}{2}$ is

$$S = 2\pi \int_{\theta=\alpha}^{\beta} r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \dots (4)$$

✓ Area of surface by revolution in Parametric form:

If the curve $x = f(t)$ and $y = g(t)$, are then the Area of the surface generated by revolution from $t = a$ to $t = b$,

(1). About x-axis is

$$S = 2\pi \int_{t=a}^{b} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \dots (5)$$

(2). About y-axis is

$$S = 2\pi \int_{t=a}^{b} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \dots (6)$$

METHOD – 15: AREA OF SURFACES BY REVOLUTION

C	1	Find the surface are of a sphere obtained by revolving the semi-circle $x^2 + y^2 = a^2, y > 0$. **Answer**: $\mathbf{4\pi a^2}$.	
H	2	Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$ from $1 \le x \le 2$ about the x-axis. **Answer**: $\mathbf{\frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2})}$.	
H	3	Find the area of the surface generated by revolving the curve $y = x^2$ from $x = 1$ to $x = 2$ about the y-axis. **Answer**: $\mathbf{\frac{\pi}{6}\left[(17)^{\frac{3}{2}} - (5)^{\frac{3}{2}}\right]}$.	
C	4	Find the surface area of the spindle shaped solid generated by revolving asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis. **Answer**: $\mathbf{\frac{12\pi a^2}{5}}$.	
T	5	Find the area of the surface generated by revolving the cardioid $r = a(1 + \cos\theta), a > 0$ about the initial line. **Answer**: $\mathbf{\frac{32\pi a^2}{5}}$.	
H	6	Determine the surface area of the curve $x = 2t, y = 4t, 0 \le t \le 2$ rotated about (i) x-axis, (ii) y-axis. **Answer**: $\mathbf{32\sqrt{5}\pi, 16\sqrt{5}\pi}$.	
C	7	Determine the surface area of the curve $x = 2\cos t, y = 2\sin t$ rotated about x-axis between $t = 0$ and $t = \pi$. **Answer**: $\mathbf{16\pi}$.	
T	8	Find the area of the surface generated by revolution of the loop at the curve $x = t^2, y = t - \frac{1}{3}t^3$ about x-axis. **Answer**: $\mathbf{3\pi}$.	

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UNIT 2

❖ SEQUENCE:

- ✓ A sequence is a set of numbers in a specific order.
- ✓ It is denoted by $\{a_n\}$ **OR** $\{a_n\}_{n=1}^{\infty}$ **OR** (a_n) **OR** $\{a_n\}_{n\geq 1}$.
- ✓ It is written as $\{a_n\} = a_1, a_2, \dots, a_n, \dots$

  Where, a_1 is called first term, a_2 is called second term and in general a_n is the n^{th} term of $\{a_n\}$. The individual numbers which form the sequence are called elements/terms/members of the sequence.
- ✓ Examples:

  (1). $2, 7, 12, 17, \dots = \{(5n-3)\}_1^{\infty}$

  (2). $1, 2, 3, 4, 5, \dots = \{n\}$

  (3). $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \frac{1}{2}, \frac{2}{3}, \dots, \frac{n}{n+1}, \dots$

  (4). $\left\{\frac{1}{4^n}\right\}_{n\geq 3} = \frac{1}{4^3}, \frac{1}{4^4}, \dots, \frac{1}{4^n}, \dots$

❖ BOUNDED AND UNBOUNDED SEQUENCE:

- ✓ A sequence $\{a_n\}$ is said to be bounded above if $\exists\, K \in \mathbb{R}$ such that $a_n \leq K, \forall n \in \mathbb{N}$. Here K is called an upper bound for $\{a_n\}$.
- ✓ A sequence $\{a_n\}$ is said to be bounded below if $\exists\, k \in \mathbb{R}$ such that $k \leq a_n, \forall n \in \mathbb{N}$. Here k is called a lower bound for $\{a_n\}$.
- ✓ A sequence $\{a_n\}$ is said to be bounded if it is both bounded above and bounded below.

  i.e., $\exists\, k, K \in \mathbb{R}$ such that $k \leq a_n \leq K, \forall n \in \mathbb{N}$.

❖ MONOTONIC SEQUENCE:

- ✓ A sequence $\{a_n\}$ is said to be an increasing sequence if $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$.
- ✓ A sequence $\{a_n\}$ is said to be a decreasing sequence if $a_n \geq a_{n+1}, \forall n \in \mathbb{N}$.
- ✓ A sequence $\{a_n\}$ is said to be a monotonic if it is either increasing or decreasing.

❖ THE SANDWICH THEOREM OR SQUEEZE THEOREM:

✓ Let $\{a_n\}, \{b_n\}$ & $\{c_n\}$ be the sequences of real numbers.

✓ If $a_n \le b_n \le c_n, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

Note: $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

METHOD – 1: CHARACTERISTICS OF SEQUENCE AND LIMIT

C	**1**	Check whether the following sequences are bounded or not: $(1). \{(2n-1)\}_1^\infty$ $(2). \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$ $(3). \{(-1)^n\}$ $(4). \{-n\}$. **Answer**: (1). **Bounded below** (2). & (3). **Bounded** (4). **Bounded above.**	
C	**2**	Are the following sequences monotonic? Justify your answer. $(1). \left\{\frac{3n+1}{n+1}\right\}$ $(2). \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$ $(3). \left\{\frac{n}{2^n}\right\}$ $(4). \left\{\frac{n}{n^2+1}\right\}$. **Answer**: (1). & (2). **Increasing** (3). & (4). **Decreasing.**	
C	**3**	Find the following limits: $(1). \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ $(2). \lim_{n\to\infty}\left(\frac{n\sum n}{2n^3}\right)$ $(3). \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)$ $(4). \lim_{n\to\infty}\left(\frac{\cos^{2014} n}{n}\right)$. **Answer**: (1). **0** (2). **1/4** (3). **0** (4). **0.**	
T	**4**	Find the following limits: $(1). \lim_{n\to\infty}\left(\frac{n^2}{n+5}\right)$ $(2). \lim_{n\to\infty}\left(\frac{3n}{n+7n^{\frac{1}{2}}}\right)$ $(3). \lim_{n\to\infty}\left\{\frac{(-1)^n}{3^n}\right\}$ $(4). \lim_{n\to\infty}\left(\frac{\sin n}{n}\right)$ $(5). \lim_{n\to\infty}\left(\log\left(\frac{2n+1}{n}\right)\right)$. **Answer**: (1). ∞ (2). **3** (3). **0** (4). **0** (5). **log 2.**	

❖ CONVERGENT SEQUENCE AND DIVERGENT SEQUENCE:

✓ A sequence $\{a_n\}$ is said to be convergent if $\lim_{n\to\infty} a_n = L$, where a_n is the n^{th} term of given sequence and 'L' is any finite real number.

- ✓ A sequence $\{a_n\}$ is said to be divergent if $\lim\limits_{n\to\infty} a_n = \infty$, where a_n is the n^{th} term of given sequence.

❖ OSCILLATING SEQUENCE:

- ✓ A sequence which is neither convergent nor divergent is said to be an oscillating sequence.

❖ NOTES:

- ✓ An increasing sequence which is bounded above is always convergent.
- ✓ A decreasing sequence which is bounded below is always convergent.
- ✓ Every bounded and monotonic sequence is convergent.

METHOD – 2: CONVERGENCE OF SEQUENCE

C	**1**	Check the convergence of following sequences: $(\mathbf{1}).\left\{\frac{n^2+1}{2n^2-1}\right\}$ $(\mathbf{2}).\{4^{n+1}\}$ $(\mathbf{3}).\{4-(-1)^n\}$. **Answer**: $(\mathbf{1})$. **Converges to 0.5** $(\mathbf{2})$. **Diverges to** ∞ $(\mathbf{3})$. **Finitely oscillate between 3 and 5.**	
H	**2**	Check the convergence of the sequences: $(\mathbf{1}).\{8+(-1)^n\}$ $(\mathbf{2}).\{n^2(-1)^n\}$. **Answer**: $(\mathbf{1})$. **Finitely oscillating between 7 & 9** $(\mathbf{2})$. **Infinitely oscillating between** $-\infty$ & ∞.	
C	**3**	Check the convergence of the sequence $\left\{\frac{2^n+5^n}{5^n+3^n}\right\}$. **Answer**: **Converges to 1.**	
C	**4**	Show that the sequence $\left\{\frac{\sin n}{n}\right\}$ converges to 0.	
H	**5**	Check the convergence of the following sequences: $(\mathbf{1}).\left\{\frac{\cos^{2012} n}{n}\right\}$ $(\mathbf{2}).\left\{\frac{\cos^2 n}{2^n}\right\}$ $(\mathbf{3}).\left\{\frac{1}{n}(-1)^{n+1}\right\}$. **Answer**: **All three sequences are converges to 0.**	

T	6	Check convergence of the sequence $\left\{\frac{n!}{n^n}\right\}$. **Answer**: **Converges to 0.**	
C	7	Define the convergence of a sequence (a_n) and verify whether the sequence whose n^{th} term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converges or not. **Answer**: **Converges to e^2.**	S-19 (3)
H	8	Check convergence of the sequence $\left\{\frac{\log n}{n}\right\}$. **Answer**: **Converges to 0.**	
H	9	Check the convergence of the following sequences: (**1**). $\left\{\frac{1}{3^n}\right\}$ (**2**). $\{\sqrt[n]{n}\}$ (**3**). $\{n^2 e^{-n}\}$. **Answer**: **All the sequences are convergent and it converges to 0, 1 & 0 respectively.**	
H	10	Check the convergence of the sequences $\left\{\frac{3^{2n}}{5^n}\right\}$ and $\left\{\frac{9^{3n}}{15^{2n}}\right\}$. **Answer**: **Both the sequences are divergent.**	
T	11	Check the convergence of the following sequences: (**1**). $\{\sqrt{n}\,(\sqrt{n+1}-\sqrt{n})\}$ (**2**). $\left\{\sqrt{n^2+1}-n\right\}$ (**3**). $\left\{\left(1+\frac{x}{n}\right)^n\right\}$ (**4**). $\left\{\left(\frac{n-2011}{n}\right)^n\right\}$. **Answer**: **All the sequences are convergent and it converges to 0.5, 0, e^x & e^{-2011} respectively.**	
C	12	For which values of r, $\{r^n\}$ is convergent? **Answer**: $-1 < r \le 1$.	
C	13	Check the convergence of the following sequences: (**1**). $\{0.3, 0.33, 0.333, \ldots\}$ (**2**). $\left\{\sqrt{6}, \sqrt{6\sqrt{6}}, \sqrt{6\sqrt{6\sqrt{6}}}, \ldots\right\}$. **Answer**: **Both the sequences are convergent and it converges to 1/3 & 6 respectively.**	

T	**14**	Show that the sequence $\{a_n\}$ is monotonic increasing and bounded, where n^{th} terms is $a_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$. Is it convergent? **Answer: Yes.**	

❖ CONTINUOUS FUNCTION THEOREM FOR SEQUENCES:

✓ Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ as $n \to \infty$ and if f is a continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

METHOD – 3: CONTINUOUS FUNCTION THEOREM

T	**1**	Show that $\left\{\sqrt{\frac{n+1}{n}}\right\} \to 1$ as $n \to \infty$.	
H	**2**	Check the convergence of the sequence $\left\{\frac{1}{2^n}\right\}$. **Answer: Converges to 0.**	
C	**3**	Check the convergence of the sequence $\left\{4^{\frac{1}{3n}}\right\}$. **Answer: Converges to 1.**	

❖ INFINITE SERIES:

✓ If $u_1, u_2, u_3, \dots, u_n, \dots$ is an infinite sequence of real numbers, then the sum of the terms of the sequence, $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is called an infinite series.

The infinite series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

❖ SEQUENCE OF PARTIAL SUMS:

✓ Let $\sum u_n$ be a given infinite series. Consider $S_1 = u_1, S_2 = u_1 + u_2, \dots, S_n = u_1 + u_2 + \cdots + u_n$. Then $\{S_n\}$ defined as above is called the sequence of partial sums of the given series.

❖ CONVERGENCE OF AN INFINITE SERIES:

✓ An infinite series $\sum u_n$ is said to be convergent if the sequence of partial sums $\{S_n\}$ converges to some finite number L.

i.e. $\lim\limits_{n\to\infty} S_n = L \Rightarrow \sum\limits_{n=1}^{\infty} u_n = L$, where L is the sum of series.

✓ If the sequence of partial sums $\{S_n\}$ does not converge, then the series is said to be divergent.

❖ POSITIVE TERM SERIES:

✓ If all terms after few negative terms in an infinite series are positive, such a series is called a positive terms series.

❖ TEST FOR GEOMETRIC SERIES:

An Infinite series of the form $\sum\limits_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$ is known as Geometric Series, where "r" is **common ratio** and "a" is **the 1st term**, then given series is

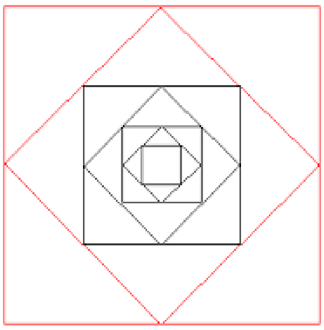
(1). **convergent** if $\mathbf{-1 < r < 1}$ and it's sum is given by $\mathbf{s = \dfrac{a}{1-r}}$.

(2). **divergent** if $\mathbf{r \geq 1}$.

(3). **oscillating** if $\mathbf{r \leq -1}$. (Finitely oscillating if $r = -1$ & Infinitely oscillating if $r < -1$).

METHOD – 4: GEOMETRIC SERIES

C	**1**	Test the convergence of $\sum\limits_{n=0}^{\infty} \dfrac{3^{2n}}{2^{3n}}$. **Answer: Divergent.**	
C	**2**	Define the Geometric series and find the sum of $\sum\limits_{n=1}^{\infty} \dfrac{3^{n-1}-1}{6^{n-1}}$. **Answer: 4/5.**	

H	3	Test the convergence of series $\sum_{n=1}^{\infty}\frac{4^n+5^n}{6^n}$. **Answer**: **Convergent.**	W-18 (3)
T	4	Check the convergence of (**1**). $\sum_{n=1}^{\infty}\frac{9^{2n}}{12^n}$ (**2**). $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)$. **Answer**: (**1**). **Divergent** (**2**). **Converges to 7/2.**	
C	5	Test the convergence for $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots$. **Answer**: **Convergent.**	
H	6	Prove that $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}\ldots$ is convergent and find it's sum. **Answer**: **3.**	
H	7	Check the convergence of $1+2\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^4+\cdots$. **Answer**: **Converges to 15/8.**	
T	8	Find the sum of the series $\sum_{n\geq 2}\frac{1}{4^n}$. **Answer**: **1/12.**	S-19 (2)
C	9	The figure shows the first seven of a sequence of squares. The outermost square has an area of $4m^2$. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of areas of all squares in the infinite sequence.  **Answer**: **8.**	
T	10	You drop a ball from 'a' meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where $0 < r < 1$. Find the total distance ball travels up and down when $a = 6$ m and $r = \frac{2}{3}$ m. **Answer**: **30m.**	

❖ TELESCOPING SERIES

✓ A telescoping series is a series in which many terms cancel in the partial sums.

✓ Procedure to check convergent/divergent of telescoping series:

(1). Find n^{th} term u_n of given series if it is not directly given.

(2). Using partial fraction separate it into two or three individual terms.

(3). Find S_n (sum of first n terms) using above individual terms.

(4). Find $\lim_{n\to\infty} S_n$. If above limit is finite, then the given series is converging to the limit otherwise divergent.

METHOD – 5: TELESCOPING SERIES

C	**1**	Check the convergence of $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots$. **Answer**: **Converges to 1/2.**	
H	**2**	Check the convergence of $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$. **Answer**: **Converges to 1.**	
H	**3**	Check the convergence of $\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\cdots$. **Answer**: **Converges to 1/4.**	
T	**4**	Find the sum of the series $\sum_{n\geq 1}\frac{4}{(4n-3)(4n+1)}$. **Answer**: **1.**	S-19 (2)
C	**5**	Check the convergence of $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots$. **Answer**: **Converges to 1.**	
H	**6**	Check the convergence of $\sum_{n=1}^{\infty}\frac{4}{n^2+2n}$. **Answer**: **Converges to 3.**	

C	7	Check the convergence of $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$. **Answer: Converges to $\pi/4$.**	
T	8	Check the convergence of $\sum_{n=1}^{\infty}[\tan^{-1}(n) - \tan^{-1}(n+1)]$. **Answer: Converges to $-\pi/4$.**	
C	9	Check the convergence of $\log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots$. **Answer: Divergent.**	
T	10	Check the convergence of $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots$. **Answer: Converges to $1/4$.**	

❖ ZERO TEST OR CAUCHY'S FUNDAMENTAL TEST FOR DIVERGENCE OR N^TH TERM TEST:

✓ Let $\sum u_n$ be given series where u_n is the n^{th} term of the series. If $\lim_{n\to\infty} u_n \neq 0$ or **does not exist** then the given series is **divergent.**

METHOD – 6: ZERO TEST

C	1	Check convergence of (1). $\sum_{n=1}^{\infty}\frac{n}{2n+5}$ (2). $\sum_{n=0}^{\infty}\frac{n^2+1}{3n^2-1}$ (3). $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+12}\right)^2$ (4). $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ (5). $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}$. **Answer: All the five series are Divergent.**	
C	2	Check the convergence of $\sum_{n=1}^{\infty} n\tan\left(\frac{1}{n}\right)$. **Answer: Divergent.**	
H	3	Check the convergence of $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots$. **Answer: Divergent.**	

T	4	Check the convergence of $\frac{1}{1+2^{-1}}+\frac{1}{1+2^{-2}}+\frac{1}{1+2^{-3}}+\cdots$. **Answer: Divergent.**	
C	5	Check the convergence of $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{4}{\sqrt{17}}+\frac{8}{\sqrt{65}}+\cdots$. **Answer: Divergent.**	
H	6	Check the convergence of $\sqrt{\frac{1}{3\cdot 2}}+\sqrt{\frac{2}{3\cdot 3}}+\sqrt{\frac{3}{3\cdot 4}}+\cdots$. **Answer: Divergent.**	
T	7	Check the convergence of $1+\sqrt{\frac{1}{2}}+\sqrt[3]{\frac{1}{3}}+\sqrt[4]{\frac{1}{4}}+\cdots$. **Answer: Divergent.**	

❖ COMBINING SERIES:

- ✓ Whenever we have two convergent series, we add them term by term, subtract term by term or multiply them by constant to make new convergent series.
- ✓ **Rules:** If $\sum a_n = a$ and $\sum b_n = b$ are convergent series then
  - **(1).** Sum rule: $\sum(a_n+b_n)=\sum a_n+\sum b_n = a+b$.
  - **(2).** Difference rule: $\sum(a_n-b_n)=\sum a_n-\sum b_n = a-b$.
  - **(3).** Constant multiple rule: $\sum ka_n = k\sum a_n = k\cdot a$.
- ✓ If $\sum a_n$ diverges, then $\sum ka_n$ also diverges.
- ✓ If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n+b_n)$ and $\sum(a_n-b_n)$ both diverges.
- ✓ It is not necessary that $\sum a_n$ and $\sum b_n$ both diverges $\Rightarrow \sum(a_n+b_n)$ diverges.

METHOD – 7: COMBINING SERIES

H	1	Give an example of sum and difference rule.	
H	2	Give an example of constant multiple rule.	

C	**3**	Give an example of two divergent series whose sum is convergent.	
T	**4**	Give an example of two divergent series whose sum is divergent.	

❖ INTEGRAL TEST:

For the positive term series $\sum_{n=a}^{\infty} u_n = \sum_{n=a}^{\infty} f(n)$, if $f(n)$ is **decreasing** and **integrable** and

(1). If $\int_a^{\infty} f(x)\ dx$ is **finite**, then the given series is **convergent**.

(2). If $\int_a^{\infty} f(x)\ dx$ is **infinite**, then the given series is **divergent**.

❖ P-SERIES:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is known as $p -$ series.

(1). $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is **convergent**, if $\mathbf{p > 1}$.

(2). $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is **divergent**, if $\mathbf{p \leq 1}$.

METHOD – 8: INTEGRAL TEST

C	**1**	Check the convergence of $\sum_{n=1}^{\infty} n \cdot e^{-n^2}$. **Answer: Convergent.**	
H	**2**	Check the convergence of $\sum_{n=1}^{\infty} e^{-n}$. **Answer: Convergent.**	

C	3	Test the convergence of $\sum_{n=1}^{\infty} \frac{2\tan^{-1} n}{1+n^2}$. **Answer: Convergent.**	
T	4	Check convergence of (1). $\sum \frac{e^n}{e^{2n}+1}$ (2). $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ (3). $\sum \frac{e^{\tan^{-1} n}}{n^2+1}$. **Answer: All the three series are convergent.**	
H	5	Show that the **Harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.	

❖ DIRECT COMPARISON TEST:

✓ If two given positive term series $\sum u_n$ and $\sum v_n$ with $0 \le u_n \le v_n, \forall n \in \mathbb{N}$ and

(1). If $\sum v_n$ is convergent, then $\sum u_n$ is also convergent.

(2). If $\sum u_n$ is divergent, then $\sum v_n$ is also divergent.

❖ REMARKS:

(1). If **bigger** series is **convergent,** then **smaller** series is also **convergent**.

(2). If **smaller** series is **divergent,** then **bigger** series is also **divergent**.

(3). For the series $\sum \frac{\log n}{n^p}$, if $p \le 1$ then $\frac{1}{n^p} \le \frac{\log n}{n^p}$.

(4). For the series $\sum \frac{\log n}{n^p}$, if $p > 1$ then $\frac{\log n}{n^p} \le \frac{1}{n^{\left(\frac{p+1}{2}\right)}}$.

METHOD – 9: DIRECT COMPARISION TEST

C	1	Check convergence of (1). $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ (2). $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-1}$ (3). $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$. **Answer:** (1). **Convergent** (2). **Divergent** (3). **Convergent.**	
H	2	Show that $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\cdots$ is convergent.	

H	3	Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n^2+4n+3}$ is convergent or divergent. **Answer**: **Convergent**.	
C	4	Check the convergence of $(\mathbf{1}).\ \sum_{n=1}^{\infty} \frac{\log n}{n}$ $(\mathbf{2}).\ \sum_{n=1}^{\infty} \frac{\log n}{n^2}$ $(\mathbf{3}).\ \sum_{n=1}^{\infty} \frac{\log n}{\sqrt{n}}$. **Answer**: (1). **Divergent** (2). **Convergent** (3). **Divergent**.	
T	5	Check the convergence of the series $\sum_{n=1}^{\infty} \frac{(\log n)^3}{n^3}$. **Answer**: **Convergent**.	S-19 (3)

❖ LIMIT COMPARISON TEST:

- ✓ If two positive term series $\sum u_n$ (given) & $\sum v_n$ (choose) be such that

  $L = \lim_{n\to\infty} \frac{u_n}{v_n}$ and

  (1). If L is **finite and non-zero,** then $\sum u_n$ & $\sum v_n$ **both** are **convergent or divergent** together.

  (2). If $\mathbf{L = 0}$ and $\sum v_n$ is **convergent,** then $\sum u_n$ is also **convergent**.

  (3). If $\mathbf{L} = \infty$ and $\sum v_n$ is **divergent,** then $\sum u_n$ is also **divergent**.

- ✓ Procedure to discuss about convergence of a series using limit comparison test:

  (1). Find u_n from given series if it is not directly given.

  (2). Derive v_n according to u_n.

  (3). Find the value of $L = \lim_{n\to\infty} \frac{u_n}{v_n}$.

  (4). Write the conclusion from limit comparison test.

METHOD – 10: LIMIT COMPARISION TEST

C	1	Check the convergence of $\sum_{n=1}^{\infty} \frac{3n^2+5n}{7+n^4}$. **Answer: Convergent.**	
H	2	State the $p-$ series test. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{n+1}{n^3-3n+2}$. **Answer: Convergent.**	W-18 (4)
C	3	Check convergence of $(1). \sum_{n=1}^{\infty} \frac{1}{2^n-3}$ $(2). \sum \frac{3+2\sin n}{n^3}$ $(3). \sum \left[10^{\frac{1}{n}}-1\right]$. **Answer: (1). Convergent (2). Convergent (3). Divergent.**	
H	4	Check the convergence of $(1). \sum \frac{7}{7n-2}$ $(2). \sum_{n=1}^{\infty} \frac{1}{5^n-1}$. **Answer: (1). Divergent (2). Convergent.**	
H	5	Check the convergence of $(1). \sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ $(2). \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}-2011}$. **Answer: (1). Convergent (2). Divergent.**	
H	6	Check convergence of $(1). \sum \frac{1}{n^2+1}$ $(2). \sum \frac{1}{\sqrt{n^2-1}}$ $(3). \sum \frac{1}{n(n+1)}$. **Answer: (1). Convergent (2). Divergent (3). Convergent.**	
T	7	Check the convergence of $(1). \sum \frac{n}{an^2+b}$ $(2). \sum \frac{n}{1+n\sqrt{n+1}}$. **Answer: Both the series are Divergent.**	
T	8	Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$. **Answer: Convergent.**	W-19 (4)
T	9	Determine convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$. **Answer: Divergent.**	

C	**10**	Check the convergence of $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$ & $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$. **Answer: Both the series are Convergent.**	
H	**11**	Check the convergence of (1). $\sum\left[\sqrt{n^2+1}-n\right]$ (2). $\sum_{n=1}^{\infty}\left[\sqrt{n^3+1}-\sqrt{n^3}\right]$. **Answer: (1). Divergent (2). Convergent.**	
T	**12**	Check the convergence of $\sum_{n=1}^{\infty}\frac{n^p}{\sqrt{n+1}+\sqrt{n}}$. **Answer:** $\mathbf{p \geq -1/2 \Rightarrow}$ **Divergent** & $\mathbf{p < -1/2 \Rightarrow}$ **Convergent.**	
C	**13**	Check the convergence of (1). $\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$ (2). $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$. **Answer: (1). Divergent (2). Convergent.**	
H	**14**	Check the convergence of (1). $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ (2). $\sum_{n=1}^{\infty}\sin^3\left(\frac{1}{n}\right)$. **Answer: (1). Divergent (2). Convergent.**	
C	**15**	Test the convergence of $\sum_{n=1}^{\infty}\frac{1}{1+2^2+3^2+\cdots+n^2}$. **Answer: Convergent.**	
C	**16**	Test the convergence of $\frac{1}{1^2-3}+\frac{3}{2^2-3}+\frac{5}{3^2-3}+\cdots$. **Answer: Divergent.**	
H	**17**	Check the convergence of (1). $2+\frac{3}{4}+\frac{4}{9}+\cdots$ & (2). $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\cdots$. **Answer: Both series are Divergent.**	
H	**18**	Test for convergence of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$. **Answer: Converges to 1.**	
H	**19**	Test the convergence of $\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\cdots$. **Answer: Convergent.**	

T	**20**	Test the convergence of the Series $\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\cdots$. **Answer**: **Convergent.**	
T	**21**	Check the convergence of $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots$. **Answer**: $\mathbf{p\le 2 \Rightarrow}$ **Divergent**, $\mathbf{p>2\Rightarrow}$ **Convergent.**	

❖ D'ALEMBERT'S RATIO TEST:

Let $\sum_{n=a}^{\infty} u_n$ be the given positive term series and $L=\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right|$ then

(1). If $\mathbf{L>1}$, then the given series is **covergent**.

(2). If $\mathbf{0\le L<1}$, then the given series is **divergent**.

(3). If $\mathbf{L=1}$, then the **ratio test fails**.

✓ Procedure to discuss about convergence using ratio test:

(1). Find u_n from given series if it is not directly given.

(2). Write u_{n+1} from u_n.

(3). Find the value of $L=\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right|$.

(4). Write the conclusion from the ratio test.

METHOD – 11: RATIO TEST

C	**1**	State D'Alembert's ratio test. Discuss the convergence of $\sum_{n=1}^{\infty}\frac{4^n(n+1)!}{n^{n+1}}$. **Answer**: **Divergent.**	W-18 (4)
C	**2**	Check the convergence of (**1**). $\sum\frac{2^n}{n^2}$ (**2**). $\sum_{n=0}^{\infty}\frac{2^n-1}{3^n}$ (**3**). $\sum_{n=1}^{\infty}\frac{n!\,2^n}{n^n}$. **Answer**: (**1**). **Divergent** (**2**). **Convergent** (**3**). **Convergent.**	

H	3	Check the convergence of $(1). \sum \frac{2^n}{n!}$ $(2). \sum \frac{n!}{n^2}$. **Answer**: (1). **Convergent** (2). **Divergent.**	
H	4	Check the convergence of $(1). \sum \frac{n^2}{3^n}$ $(2). \sum_{n=1}^{\infty} \frac{n}{e^{-n}}$. **Answer**: (1). **Convergent** (2). **Divergent.**	
H	5	Check the convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n^3+1}$. **Answer**: **Divergent.**	
T	6	Test for convergence of $\sum \frac{n^{10}}{10^n}$. **Answer**: **Convergent.**	
T	7	Check convergence of $(1). \sum \left(\frac{1}{1+n}\right)^n$ $(2). \sum \frac{n!}{n^n}$ $(3). \sum_{n=1}^{\infty} \frac{n\, 2^n (n+1)!}{3^n\, n!}$. **Answer**: **All the series are convergent.**	
C	8	Check the convergence of $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \cdots$. **Answer**: **Convergent.**	
C	9	Check the convergence of $\frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots$. **Answer**: **Convergent.**	
H	10	Check the convergence of $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots$. **Answer**: **Convergent.**	
H	11	Check the convergence of $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots$. **Answer**: **Convergent.**	
T	12	Check the convergence of $\frac{1}{10} + \frac{2!}{10^2} + \frac{3!}{10^3} + \cdots$. **Answer**: **Divergent.**	

C	13	Check convergence of (1). $\sum \frac{3^n+4^n}{4^n+5^n}$ & (2). $\sum \frac{5\cdot 9\cdot 13\cdot \ldots\cdot(4n+1)}{1\cdot 2\cdot 3 \ldots\cdot n}$. **Answer**: (1). **Convergent** (2). **Divergent.**	
C	14	Test the convergence of the series $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+\cdots$. **Answer**: $\mathbf{x>1 \Rightarrow}$ **Divergent** & $\mathbf{x\le 1\Rightarrow}$ **Convergent.**	W-19 (7)
H	15	Test the convergence of $\sqrt{\frac{1}{2}}x+\sqrt{\frac{2}{5}}x^2+\sqrt{\frac{3}{10}}x^3+\cdots\infty, x>0$. **Answer**: $\mathbf{x\ge 1\Rightarrow}$ **Divergent** & $\mathbf{0<x<1\Rightarrow}$ **Convergent.**	
T	16	Test the convergence of the series $\frac{1}{1\cdot 2\cdot 3}+\frac{x}{4\cdot 5\cdot 6}+\frac{x^2}{7\cdot 8\cdot 9}+$ $\frac{x^3}{10\cdot 11\cdot 12}\ldots, x\ge 0$. **Answer**: $\mathbf{x\ge 1\Rightarrow}$ **Convergent** & $\mathbf{0\le x<1\Rightarrow}$ **Divergent.**	W-18 (7)

❖ **RABBE'S TEST:**

Let $\sum_{n=a}^{\infty} u_n$ be the given positive term series and $L=\lim_{n\to\infty}\left\{n\cdot\left(\frac{u_n}{u_{n+1}}-1\right)\right\}$ then

(1). If $\mathbf{L>1}$, then the given series is **covergent.**

(2). If $\mathbf{L<1}$, then the given series is **divergent.**

(3). If $\mathbf{L=1}$, then **Rabbe test fails.**

✓ Procedure to discuss about convergence using Rabbe's test:

(1). Find u_n from given series if it is not directly given.

(2). Write u_{n+1} from u_n.

(3). Find the value of $L=\lim_{n\to\infty}\left\{n\cdot\left(\frac{u_n}{u_{n+1}}-1\right)\right\}$.

(4). Write the conclusion from the Rabbe's test.

METHOD – 12: RABBE'S TEST

C	1	Check the convergence of $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$. **Answer: Divergent.**	
C	2	Check the convergence of $\left(\frac{1}{2}\right)^3 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \cdots$. **Answer: Convergent.**	
H	3	Check the convergence of $\frac{2}{7} + \frac{2\cdot 5}{7\cdot 10} + \frac{2\cdot 5\cdot 8}{7\cdot 10\cdot 13} + \cdots$. **Answer: Convergent.**	
T	4	Check the convergence of $\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 + \left(\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\right)^2 + \cdots$. **Answer: Convergent.**	
C	5	Check the convergence of $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \ldots\ldots (2n-1)}{2\cdot 4\cdot 6 \ldots\ldots .2n} x^n,\ \ x > 0$. **Answer:** $\mathbf{x \geq 1 \Rightarrow}$ **Divergent** & $\mathbf{0 < x < 1 \Rightarrow}$ **Convergent.**	
H	6	Check the convergence of $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n,\ \ x > 0$. **Answer:** $\mathbf{x \geq 4 \Rightarrow}$ **Divergent** & $\mathbf{0 < x < 4 \Rightarrow}$ **Convergent.**	

❖ CAUCHY'S N^{TH} ROOT (RADICAL) TEST:

Let $\sum_{n=a}^{\infty} u_n$ be the given positive term series and $L = \lim_{n\to\infty} \left\{\sqrt[n]{u_n}\right\} = \lim_{n\to\infty} (u_n)^{\frac{1}{n}}$ then

(1). If $\mathbf{0 \leq L < 1}$, then the given series is **covergent.**

(2). If $\mathbf{L > 1}$, then the given series is **divergent.**

(3). If $\mathbf{L = 1}$, then the **root test fails.**

✓ Procedure to discuss about convergence using root test:

(1). Find u_n from given series if it is not directly given.

(2). Find the value of $L = \lim\limits_{n\to\infty} \{\sqrt[n]{u_n}\} = \lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}}$.

(3). Write the conclusion from the root test.

METHOD – 13: CAUCHY'S N^TH ROOT TEST

C	1	Check the convergence of $\sum\limits_{n=1}^{\infty} \dfrac{(n+\sqrt{n})^n}{3^n\, n^{n+1}}$. **Answer: Convergent.**	
H	2	Check the convergence of $(1).\ \sum\limits_{n=1}^{\infty} \left(\dfrac{n}{2n+5}\right)^n \quad (2).\ \sum\limits_{n=1}^{\infty} \dfrac{1}{(n+1)^n}$. **Answer: (1). Convergent (2). Convergent.**	
H	3	Check for the convergence of the series $\sum\limits_{n=1}^{\infty} \left(1+\dfrac{1}{n}\right)^{n^2}$. **Answer: Divergent.**	
T	4	State Cauchy's root test. Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{n}{(\log n)^n}$. **Answer: Convergent.**	W-18 (3)
C	5	Check convergence of $\left(\dfrac{2^2}{1^2}-\dfrac{2^1}{1^1}\right)^{-1} + \left(\dfrac{3^3}{2^3}-\dfrac{3^1}{2^1}\right)^{-2} + \left(\dfrac{4^4}{3^4}-\dfrac{4^1}{3^1}\right)^{-3} + \cdots$. **Answer: Convergent.**	
H	6	Test the convergence of the series $\dfrac{1}{3} + \left(\dfrac{2}{5}\right)^2 + \cdots + \left(\dfrac{n}{2n+1}\right)^n + \cdots$. **Answer: Convergent.**	W-19 (3)
T	7	Check the convergence of $\left(\dfrac{1}{2}\right)^{1^2} + \left(\dfrac{2}{3}\right)^{2^2} + \left(\dfrac{3}{4}\right)^{3^2} + \cdots$. **Answer: Convergent.**	
C	8	Check the convergence of $1 + \dfrac{2}{3}x + \left(\dfrac{3}{4}\right)^2 x^2 + \left(\dfrac{4}{5}\right)^3 x^3 + \cdots,\ \ x > 0$. **Answer:** $\mathbf{x \geq 1 \Rightarrow}$ **Divergent &** $\mathbf{0 < x < 1 \Rightarrow}$ **Convergent.**	
H	9	Check the convergence of $\sum\limits_{n=1}^{\infty} \left(\dfrac{n+1}{n+2}\right)^n x^n;\ \ x > 0$. **Answer:** $\mathbf{x \geq 1 \Rightarrow}$ **Divergent &** $\mathbf{0 \leq x < 1 \Rightarrow}$ **Convergent.**	

T	**10**	Check the convergence of $\frac{x}{\sqrt{1}}+\frac{x^2}{\sqrt[3]{2}}+\frac{x^3}{\sqrt[4]{3}}+\cdots,\ \ x>0.$ **Answer**: $\mathbf{x \geq 1 \Rightarrow}$ **Divergent** & $\mathbf{0<x<1 \Rightarrow}$ **Convergent**.	

❖ ALTERNATING SERIES :

✓ A series with alternate positive and negative terms is known as an alternating series.

❖ LEIBNITZ'S TEST FOR ALTERNATING SERIES:

The given alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\ u_n=u_1-u_2+u_3-u_4+\cdots$ is convergent

if $u_1 \geq u_2 \geq u_3 \geq u_4 \geq \cdots$ and $\lim\limits_{n\to\infty} u_n=0.$

✓ In above test, if $\lim\limits_{n\to\infty} u_n \neq 0$, then given series is oscillating series.

✓ Procedure to discuss about convergence of a series using Leibnitz's test:

(1). Check whether the given series is alternating or not.

(2). Check whether the sequence $\{u_n\}$ is decreasing or not.

(3). Check whether the value of $\lim\limits_{n\to\infty} u_n=0$ or not.

(4). If answer of above three step is yes then given series is convergent.

METHOD – 14: LEIBNITZ'S TEST

C	**1**	Check the convergence of $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\cdots.$ **Answer**: **Convergent**.	
H	**2**	Check the convergence of $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots.$ **Answer**: **Convergent**.	
T	**3**	Test the convergence of the series $\sum\frac{(-1)^{n+1}}{\log(n+1)}.$ **Answer**: **Convergent**.	

C	4	Check the convergence of $\frac{1}{1^2+1}-\frac{2}{2^2+1}+\frac{3}{3^2+1}+\cdots$. **Answer**: **Convergent.**	
H	5	Check the convergence of $1-\frac{1}{2}+\frac{2}{5}-\frac{3}{10}+\cdots$. **Answer**: **Convergent.**	
H	6	Check the convergence of $\frac{1}{1\cdot 2}-\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}-\frac{1}{7\cdot 8}+\cdots$. **Answer**: **Convergent.**	
H	7	Check the convergence of $\frac{3}{4}-\frac{5}{7}+\frac{7}{10}-\frac{9}{13}+\cdots$. **Answer**: **Oscillatory.**	
C	8	Test the convergence of the series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$. **Answer**: **Convergent.**	W-19 (3)
C	9	Check the convergence of $\left(\frac{1}{2}+\frac{1}{\ln 2}\right)-\left(\frac{1}{2}+\frac{1}{\ln 3}\right)+\left(\frac{1}{2}+\frac{1}{\ln 4}\right)-\cdots$. **Answer**: **Oscillatory.**	
T	10	Check the convergence of (**1**). $\sum \frac{(-1)^n n}{3n-2}$ (**2**). $\sum \frac{(-1)^{n-1} n}{2n-1}$. **Answer**: (**1**). **Oscillatory** (**2**). **Oscillatory.**	
C	11	Check the convergence of the series $\sum (-1)^n\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$. **Answer**: **Oscillatory.**	S-19 (4)
T	12	Check convergence of (**1**). $\sum \frac{(-1)^{n-1}}{n\sqrt{n}}$ (**2**). $\sum (-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$. **Answer**: (**1**). **Convergent** (**2**). **Convergent.**	

❖ ABSOLUTE CONVERGENT SERIES:

The given alternating series $\sum_{n=0}^{\infty} u_n$ is said to be absolute convergent, if $\sum |u_n|$ is convergent.

- ✓ Any absolute convergent series is **always convergent.**
- ✓ Procedure to discuss about absolute convergence:
  - **(1).** Write new series as $\sum w_n = \sum |u_n|$ from the given series.
  - **(2).** Discuss the convergence of $\sum w_n$ using any applicable test.
  - **(3).** If $\sum w_n$ is convergent, then the given series is absolute convergent.

METHOD – 15: ABSOLUTE CONVERGENT SERIES

C	1	Is series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ Convergent? Does it converge absolutely? **Answer: Convergent but not Absolute Convergent.**	
H	2	Determine the absolute convergence of $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}-\cdots$. **Answer: Absolute Convergent.**	
H	3	Determine the absolute convergence of $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots$. **Answer: Absolute Convergent.**	
C	4	Determine the absolute convergence of $1-2x+3x^2-4x^3+\cdots(0\le x<1)$. **Answer: Absolute Convergent.**	
T	5	Determine the absolute convergence of $\sum\frac{(-1)^n(n+1)^n}{(2\,n)^2}$. **Answer: Not Absolute Convergent.**	
C	6	Determine the absolute convergence of **(1).** $\sum\frac{(-1)^{n-1}}{n^p}\,(p>1)$ **(2).** $\sum(-1)^{n+1}\frac{\sin n}{n^2}$. **Answer:** **(1). Absolute Convergent** **(2). Absolute Convergent.**	
H	7	Determine absolute convergence of following series: **(1).** $\sum\frac{(-1)^{n+1}}{(n)^3}$ **(2).** $\sum\frac{(-1)^n5^n}{n!}$ **(3).** $\sum(-1)^{n+1}\frac{\cos n}{n^2}$. **Answer: All the series are Absolute Convergent.**	

T	8	Determine absolute convergence of following series: $(1). \sum (-1)^n \frac{2^{3n}}{3^{2n}}$ $(2). \sum \frac{\cos n\pi}{n^2+1}$. **Answer**: **Both the series are Absolute Convergent.**	

❖ CONDITIONALLY CONVERGENT SERIES:

The given alternating series $\sum_{n=0}^{\infty} u_n$ is said to be conditionally convergent, if

(1). $\sum u_n$ is **convergent**

(2). $\sum | u_n |$ is **divergent.**

✓ Procedure to discuss about conditionally convergence:

(1). Discuss convergence of given $\sum u_n$ using any applicable test.

(2). Write new series as $\sum w_n = \sum |u_n|$ from given series.

(3). Discuss convergence of $\sum w_n$ using any applicable test.

(4). If $\sum u_n$ is convergent and $\sum w_n$ is divergent, then given series is conditionally convergent.

✓ Infinite series can't be absolute convergent and conditionally convergent together.

METHOD – 16: CONDITIONALLY CONVERGENT SERIES

C	1	Check the absolute and conditional convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$. **Answer**: **Conditionally Convergent.**	
H	2	Determine the conditionally convergence of $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$. **Answer**: **Conditionally Convergent.**	
H	3	Determine the conditionally convergence of $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$. **Answer**: **Conditionally Convergent.**	

C	4	Determine the absolute or conditional convergence of series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$. **Answer: Conditionally Convergent.**	
T	5	Check for absolute/conditional convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$. **Answer: Conditionally Convergent.**	
T	6	Check the convergence of $-\frac{1}{1^p}+\frac{1}{2^p}-\frac{1}{3^p}+\frac{1}{4^p}-\frac{1}{5^p}+\cdots; p>0$. **Answer: $p>1 \Longrightarrow$ Absolute convergent,** $0<p\le 1 \Longrightarrow$ **Conditionally convergent.**	

❖ POWER SERIES:

- ✓ An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0(x-c)^0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

is known as power series in standard form, where a_n is the coefficient of the n^{th} term, c is a constant and x varies around c.

- ✓ If $c = 0$ in above series then it is called power series in terms of x.

i. e. $\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$.

❖ THEOREM: CONVERGENCE OF POWER SERIES

Let $\sum_{0}^{\infty} a_n x^n$ be a power series. Then exactly one of the following conditions hold.

(1). The series always converges at $x = 0$.

(2). The series converges for all x.

(3). There is some positive number R such that series converges for $|x| < R$ and diverges for $|x| > R$.

❖ RADIUS AND INTERVAL OF CONVERGENCE FOR POWER SERIES:

- ✓ An interval in which power series converges is called the interval of convergence and the half length of the interval of convergence is called the radius of convergence.
- ✓ If a power series is convergent for all values of x, then interval of convergence will be $(-\infty, \infty)$ and the radius of convergence will be ∞.
- ✓ In above theorem R is radius of convergence of the power series and it can be obtained from either of the formulas as follows:

$$\mathbf{R = \lim_{n \to \infty} \frac{1}{\left|\frac{a_{n+1}}{a_n}\right|} \quad or \quad R = \lim_{n \to \infty} \frac{1}{(a_n)^{\frac{1}{n}}}}.$$

- ✓ Procedure to discuss about convergence of power series:
  - **(1).** Using above formula of **R,** find interval of convergence.
  - **(2).** Check the convergence at the end point of interval individually.
  - **(3).** Update the interval of convergence.
  - **(4).** Find the radius i.e. half-length of the interval of convergence.
- ✓ **Note:** Any absolute convergent series is always convergent and infinite series can not absolute and conditionally convergent together.

METHOD – 17: POWER SERIES

C	**1**	Find the radius of convergence and interval of convergence. Also find for what values of x series is absolute convergent and conditionally convergent. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$. **Answer**: $\mathbf{1, \ (-1, 1], \ (-1, 1) \ \& \ x = 1}$.	
H	**2**	Find the radius of convergence and interval of convergence. Also find for what values of x series is absolute convergent and conditionally convergent. $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots$. **Answer**: $\mathbf{1, \ [-1, 1], \ [-1, 1]}$ & **for no values of x.**	

T	3	For the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$ find the series's radius and interval of convergence. For what values of x does the series converge absolutely, conditionally? **Answer**: $\mathbf{1}$, $(\mathbf{-3,-1}]$, $(\mathbf{-3,-1})$ & $\mathbf{x=-1}$.	
T	4	Find the radius of convergence and interval of convergence of the series $1-\frac{1}{2}(x-2)+\frac{1}{2^2}(x-2)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots.$ **Answer**: $\mathbf{\lvert x-2\rvert < 2 \Rightarrow 0 < x < 4}$.	
T	5	Find the radius of convergence and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n}$. For what values of x does the series converge absolutely? **Answer**: $\mathbf{1/3}$, $[\mathbf{1/3,1})$ & $(\mathbf{1/3,1})$.	S-19 (7)
C	6	For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$, find the radius and interval of convergence. **Answer**: $\mathbf{-1 \le x \le 1 \Longrightarrow}$ **Convergent** & **Radius of convergent is 1**.	
H	7	Test for the convergence of the series $x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots, x>0$. **Answer**: $\mathbf{x^2>1 \Longrightarrow}$ **Divergent** & $\mathbf{0<x^2\le 1 \Longrightarrow}$ **Convergent**.	
H	8	Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n.\ x^n}{\sqrt{n+1}}$. **OR** Find radius of convergence and interval of convergence of $\sum_{n=0}^{\infty} \frac{(-3)^n.\ x^n}{\sqrt{n+1}}$. **Answer**: $\mathbf{1/3}$ & $(\mathbf{-1/3,1/3}]$.	
T	9	Find the radius of convergence and interval of convergence of the series $\sum \frac{x^{2n-1}}{(2n-1)!}$. **Answer**: ∞ & $(-\infty,\infty)$.	

C	**10**	Test the convergence of $\sum_{n=1}^{\infty} \frac{[(n+1)x]^n}{n^{n+1}}$, $x \geq 0$. **Answer**: $\mathbf{x \geq 1 \Rightarrow}$ **Divergent** & $\mathbf{0 \leq x < 1 \Rightarrow}$ **Convergent.**	
H	**11**	Test for the convergence of the series $\frac{1}{1\cdot 2\cdot 3} + \frac{x}{4\cdot 5\cdot 6} + \frac{x^2}{7\cdot 8\cdot 9} + \cdots$. **Answer**: $\mathbf{\lvert x\rvert > 1 \Rightarrow}$ **Divergent** & $\mathbf{\lvert x\rvert \leq 1 \Rightarrow}$ **Convergent.**	
T	**12**	Test the convergence of the series and find radius of convergence. $\frac{x}{2\cdot 5} - \frac{x^2}{2^2\cdot 10} + \frac{x^3}{2^3\cdot 15} - \frac{x^4}{2^4\cdot 20} + \cdots$. **Answer**: $\mathbf{x \leq -2}$ & $\mathbf{2 < x \Rightarrow}$ **Divergent,** $\mathbf{-2 < x \leq 2 \Rightarrow}$ **Convergent** & **Radius of convergence = 2.**	
C	**13**	Test for convergence of the series $2 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \cdots$. **Answer**: $\mathbf{x \geq 1 \Rightarrow}$ **Divergent** & $\mathbf{0 \leq x < 1 \Rightarrow}$ **Convergent.**	
H	**14**	Find the radius of convergence and interval of convergence of the series $\frac{1}{2\sqrt{1}} - \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} - \frac{x^6}{5\sqrt{4}} + \cdots$. **Answer**: **1** & $\mathbf{[-1, 1]}$.	
T	**15**	Find the value of x for which the given series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots$ converge. **Answer**: $\mathbf{-1 \leq x \leq 1}$.	
T	**16**	Find the radius of convergence and the interval of convergence of the series $\frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots$. **Answer**: **2** & $\mathbf{(0, 4)}$.	

❖ TAYLOR'S SERIES: FIRST FORM IN POWER OF $(x-a)$

✓ If f(x) possesses derivatives of all order at point "a" then Taylor's series of given function f(x) at point "a" is given by

$$\mathbf{f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots}.$$

METHOD – 18: 1ST FORM OF TAYLOR'S SERIES

C	1	Expand e^x in power of $(x-1)$ up to first four terms. **Answer**: $\mathbf{e\left[1+(x-1)+\frac{1}{2!}(x-1)^2+\frac{1}{3!}(x-1)^3+\cdots\right]}$.	
C	2	Find the Taylor series generated by $f(x)=\frac{1}{x}$ at $a=2$. **Answer**: $\mathbf{\left[\frac{1}{2}-\frac{(x-2)}{2^2}+\frac{(x-2)^2}{2^3}-\cdots\right]}$.	
H	3	Express $f(x)=2x^3+3x^2-8x+7$ in terms of $(x-2)$. **Answer**: $\mathbf{19+28(x-2)+15(x-2)^2+2(x-2)^3}$.	W-19 (4)
H	4	Expand $3x^3+8x^2+x-2$ in powers of $x-3$. **Answer**: $\mathbf{154+130(x-3)+35(x-3)^2+3(x-3)^3}$.	
T	5	Find the Taylor's series expansion of order 3 generated by $f(x)=\sqrt{x}$ at $a=4$. **Answer**: $\mathbf{\left[2+\frac{x-4}{4}-\frac{(x-4)^2}{64}+\frac{(x-4)^3}{512}-\cdots\right]}$.	
C	6	Find the Taylor series expansion of $f(x)=x^3-2x+4$ at $a=2$. **Answer**: $\mathbf{[8+10(x-2)+6(x-2)^2+(x-2)^3]}$.	
C	7	Find the Taylor's series expansion of $f(x)=\tan x$ in power of $\left(x-\frac{\pi}{4}\right)$, showing at least four nonzero terms. Hence find the value of $\tan 46°$. **Answer**: $\mathbf{\left[1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3+\cdots\right]}$ & $\mathbf{1.0355}$.	
H	8	Expand $\log(\sin x)$ in power of $(x-2)$. **Answer**: $\mathbf{\left[\log(\sin 2)+(x-2)\cot 2-\frac{(x-2)^2\operatorname{cosec}^2 2}{2!}+\cdots\right]}$.	
T	9	Expand $\log(\cos x)$ about $\frac{\pi}{3}$ by Taylor's series. **Answer**: $\mathbf{\left[-\log 2-\sqrt{3}\left(x-\frac{\pi}{3}\right)-2\left(x-\frac{\pi}{3}\right)^2-\frac{4\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^3-\cdots\right]}$.	

❖ TAYLOR'S SERIES SECOND FORM OF $f(a+x)$

✓ If $f(x)$ possesses derivatives of all order at point "a", then the Taylor's series of given function $f(a+x)$ at point "a" is given by

$$\mathbf{f(a+x) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \cdots .}$$

✓ $\mathbf{1° = \frac{\pi}{180} rad = 0.0175\ rad.}$

METHOD – 19: 2ND FORM OF TAYLOR'S SERIES

C	1	Expand $\sin\left(\frac{\pi}{4}+x\right)$ in the power of x and hence find the value of $\sin 46°$. **Answer:** $\sin\left(\frac{\pi}{4}+x\right) = \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)$ & **0.7193.**	
H	2	Expand $\sin\left(\frac{\pi}{4}+x\right)$ in powers of x and hence find the value of $\sin 44°$ correct up to 4 decimal places. **Answer:** $\sin\left(\frac{\pi}{4}+x\right) = \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)$ & **0.6946.**	
H	3	Expand $\cos\left(\frac{\pi}{4}+x\right)$ in the power of x by Taylor series. Hence find the value of $\cos 46°$. **Answer:** $\cos\left(\frac{\pi}{4}+x\right) = \frac{1}{\sqrt{2}}\left(1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$ & **0.6946.**	
T	4	Find the expansion of $\tan\left(\frac{\pi}{4}+x\right)$ in ascending powers of x upto terms in x^4 and find approximately value of $\tan 43°$. **Answer:** $\tan\left(\frac{\pi}{4}+x\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \cdots$ & **0.9325.**	
T	5	Use Taylor's series to estimate $\sin 38°$. **Answer: 0.6156 approximately.**	S-19 (4)
C	6	Find the value of $\sqrt{18}$ using Taylor's series. **Answer: 4.2426 approximately.**	

H	7	Using Taylor's theorem to find $\sqrt{25.15}$. **Answer: 5.0149 approximately.**	
T	8	State Taylor's series for one variable and hence find $\sqrt{36.12}$. **Answer: 6.0099 approximately.**	
T	9	Find the value of $(82)^{\frac{1}{4}}$ using Taylor's series. **Answer: 3.0092 approximately.**	

❖ MACLAURIN'S SERIES:

- ✓ If f(x) possesses derivatives of all order at point "0" then the Maclaurin's series of given function f(x) at point "0" is given by

$$\mathbf{f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots .}$$

- ✓ If we take point $a = 0$ in the 1st form of Taylor's series then we get the Maclaurin's series.

METHOD – 20: MACLAURIN'S SERIES

C	1	Express $5 + 4(x-1)^2 - 3(x-1)^3 + (x-1)^4$ in ascending powers of x. **Answer: $13 - 21x + 19x^2 - 7x^3 + x^4$.**	
H	2	Express $2 + 5(x-1) + 2(x-1)^3 + (x-1)^4$ in ascending powers of x. **Answer: $-4 + 7x - 2x^3 + x^4$.**	
C	3	Using Maclaurin's series expand e^x and hence derive expansions of e^{-x}. **Answer: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ & $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$.**	
C	4	Using Maclaurin's series expand $\sin x$ and $\cos x$. **Answer: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ & $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$.**	
H	5	Using Maclaurin's series expand $\tan x$. **Answer: $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$.**	

T	6	Using Maclaurin's series expand sec x. **Answer:** $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} - \cdots$.	
T	7	Expand $x^2 \cos x$ in terms of x. **Answer:** $x^2 \cos x = x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \cdots$.	
C	8	Expand $e^{x \sin x}$ in power of x upto the terms containing x^6. **Answer:** $e^{x \sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \cdots$.	W-18 (3)
T	9	Expand $\frac{e^x}{\cos x}$ up to first four terms by Maclaurin's series. **Answer:** $\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3} + \cdots$.	
C	10	Expand $\tan^{-1} x$ up to the first four terms by Maclaurin's series and hence prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$. **Answer:** $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.	
T	11	Expand $\log_e\left(\frac{1+x}{1-x}\right)$ and then obtain approximate value of $\log_e\left(\frac{11}{9}\right)$. **Answer:** $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$ & **0.20067**.	
C	12	Using Maclaurin's series expand log (cos x). **Answer:** $\log(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots$.	
H	13	Expand log(sec x) in power of x. **Answer:** $\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$.	
T	14	Expand $\log(1 + e^x)$ in ascending power of x as far as term containing x^4. **Answer:** $\log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} - \cdots$.	

H	**15**	Using Maclaurin's series expand $\log(1+x)$ and hence derive expansions of $\log(1-x)$ and $\log 2$ and $\log y$. **Answer:** $\mathbf{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots}$ $\mathbf{\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots}$ $\mathbf{\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots}$ $\mathbf{\log y = (y-1) - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \cdots}$.	
H	**16**	Using Maclaurin's series expand $(1+x)^m$ and hence derive expansions of $(1-x)^m, (1+x)^{-1}, (1-x)^{-1}, (1+x)^{-2}$ and $(1-x)^{-2}$. **Answer:** $\mathbf{(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots}$ $\mathbf{(1-x)^m = 1 - mx + \frac{m(m-1)}{2!}x^2 - \cdots}$ $\mathbf{(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots}$ $\mathbf{(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots}$ $\mathbf{(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots}$ $\mathbf{(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots}$.	
C	**17**	Using Maclaurin's expansion, expand $(1+x)^{\frac{1}{x}}$ upto the term x^2. **Answer:** $\mathbf{(1+x)^{\frac{1}{x}} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \cdots\right)}$.	
T	**18**	Using Maclaurin's series expand $\frac{e^x}{e^x+1}$. **Answer:** $\mathbf{\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{1}{4}x - \frac{1}{48}x^3 + \cdots}$.	

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UNIT 3

❖ BASIC FORMULAE:

✓ When given function is polynomial function:

➢ **Leibnitz's formula** (consider polynomial function as "u")

$$\int \mathbf{u \cdot v \; dx = u\, v_1 - u'\, v_2 + u''\, v_3 - u'''\, v_4 + \cdots},$$

where $u', u'', \ldots$ are successive derivatives of u and $v_1, v_2, \ldots$ are successive integrals of v.

✓ When given function is exponential function:

$$\int \mathbf{e^{ax} \sin bx \; dx = \frac{e^{ax}}{a^2 + b^2}[a \sin bx - b \cos bx] + c}$$

$$\int \mathbf{e^{ax} \cos bx \; dx = \frac{e^{ax}}{a^2 + b^2}[a \cos bx + b \sin bx] + c}$$

✓ When given function is trigonometric function:

$$\mathbf{2 \sin a \cos b = \sin(a + b) + \sin(a - b)}$$

$$\mathbf{2 \cos a \sin b = \sin(a + b) - \sin(a - b)}$$

$$\mathbf{2 \cos a \cos b = \cos(a + b) + \cos(a - b)}$$

$$\mathbf{2 \sin a \sin b = -\cos(a + b) + \cos(a - b)}$$

❖ NOTE (FOR EVERY $n \in \mathbb{Z}$)

✓ $\cos n\pi = (-1)^n$ ✓ $\sin n\pi = 0$ ✓ $\cos(2n+1)\frac{\pi}{2} = 0$

✓ $\cos 2n\pi = (-1)^{2n} = 1$ ✓ $\sin 2n\pi = 0$ ✓ $\sin(2n+1)\frac{\pi}{2} = (-1)^n$

❖ DIRICHLET CONDITION FOR EXISTENCE OF FOURIER SERIES OF $f(x)$:

(1). $f(x)$ is bounded.

(2). $f(x)$ is single valued.

(3). $f(x)$ has finite number of maxima and minima in the interval.

(4). f(x) has finite number of discontinuities in the interval.

❖ NOTE:

✓ At a point of discontinuity, the sum of the series is equal to average of left and right hand limits of f(x) at the point of discontinuity, say x_0.

i.e. $f(x_0) = \dfrac{f({x_0}^-) + f({x_0}^+)}{2}$

❖ FOURIER SERIES

✓ Let f(x) be periodic function of period 2L defined in (0 , 2L) and satisfies Dirichlet's condition, then

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]},$$

where $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n$ are Fourier coefficients of series.

❖ ORTHOGONALITY OF TRIGONOMETRIC FOURIER SERIES

✓ **Orthogonality:** A collection of functions $\{f_0(x), f_1(x), \dots, f_m(x), \dots\}$ defined on $[a, b]$ is called orthogonal on $[a, b]$ if

$$(f_i, f_j) = \int_a^b f_i(x) f_j(x)\, dx = \begin{cases} 0 & ; i \neq j \\ c > 0 & ; i = j \end{cases}$$

e.g., The collection $\{1, \cos x, \cos 2x, \dots\}$ are orthogonal functions in $[-\pi, \pi]$.

✓ **Trigonometric Fourier series**: The set of functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$ are orthogonal in the interval $(c, c + 2L)$ for any value of c, where $n = 1,2,3, \dots$.

✓ i.e.,

$$\int_c^{c+2l} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \begin{cases} 0 & ; m \neq n \\ L & ; m = n \end{cases}$$

$$\int_c^{c+2l} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{c}^{c+2l} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0, \text{for all m, n.}$$

- ✓ Hence, any function f(x) can be represented in the terms of these orthogonal functions in the interval $(c, c + 2L)$ for any value of c.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

This series is known as trigonometric series.

❖ FOURIER SERIES IN THE INTERVAL $(c, c + 2L)$:

- ✓ The Fourier series for the function f(x) in the interval $(c, c + 2L)$ is defined by

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]}$$

Where the constants a_0, a_n and b_n are given by

$$a_0 = \frac{1}{L}\int_{c}^{c+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

❖ FOURIER SERIES IN THE INTERVAL $(0, 2L)$:

- ✓ The Fourier series for the function f(x) in the interval $(0,2L)$ is defined by

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]},$$

where the constants a_0, a_n and b_n are given by

$$a_0 = \frac{1}{L}\int_{0}^{2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{0}^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

METHOD – 1: FOURIER SERIES IN THE INTERVAL (0, 2L)

C	1	Find the Fourier series of 2π – periodic function $f(x) = x^2, 0 < x < 2\pi$ and hence deduce that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$. **Answer**: $f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right]$.	S-19 (7)
H	2	Find the Fourier series for $f(x) = x^2$ in $(0, 2)$. **Answer**: $f(x) = \frac{4}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\cos(n\pi x) - \frac{4}{n\pi}\sin(n\pi x)\right]$.	
H	3	Find the Fourier series to represent $f(x) = 2x - x^2$ in $(0, 3)$. **Answer**: $f(x) = \sum_{n=1}^{\infty}\left[-\frac{9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$.	
T	4	Find the Fourier series of $f(x) = \frac{\pi - x}{2}$ in the interval $(0, 2\pi)$. **Answer**: $f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$.	W-19 (7)
T	5	Show that $\pi - x = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{\sin 2nx}{n}$, when $0 < x < \pi$.	
C	6	Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2$. **Answer**: $f(x) = \frac{(1 - e^{-2})}{2} + \sum_{n=1}^{\infty}\frac{(1 - e^{-2})}{1 + n^2\pi^2}[\cos n\pi x + (n\pi)\sin n\pi x]$.	
H	7	Find Fourier Series for $f(x) = e^{-x}$, where $0 < x < 2\pi$. **Answer**: $f(x) = \frac{1 - e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{1 - e^{-2\pi}}{\pi(n^2 + 1)}[\cos nx + n\sin nx]$.	
C	8	Find the Fourier series of the periodic function $f(x) = \pi\sin\pi x$, where $0 < x < 1, p = 2l = 1$. **Answer**: $f(x) = 2 + \sum_{n=1}^{\infty}\frac{4}{1 - 4n^2}\cos(2n\pi x)$.	

C	9	Find the Fourier Series for the function f(x) given by $f(x)=\begin{cases}-\pi, & 0<x<\pi\\ x-\pi, & \pi<x<2\pi\end{cases}$. Hence show that $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$. **Answer**: $\mathbf{f(x)=-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{(1-(-1)^n)}{n^2\pi}\cos(nx)+\frac{(-1)^n-2}{n}\sin(nx)\right]}$.	
H	10	Find the Fourier series for periodic function with period 2 of $f(x)=\begin{cases}\pi x, & 0\le x\le 1\\ \pi(2-x), & 1\le x\le 2\end{cases}$. **Answer**: $\mathbf{f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2[(-1)^n-1]}{\pi n^2}\cos(n\pi x)}$.	
H	11	Find the Fourier series of $f(x)=\begin{cases}x^2 & ;\ 0<x<\pi\\ 0 & ;\pi<x<2\pi\end{cases}$. **Answer**: $\mathbf{f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left[\frac{2(-1)^n\cos nx}{n^2}+\frac{1}{\pi}\left\{-\frac{\pi^2(-1)^n}{n}+\frac{2(-1)^n}{n^3}-\frac{2}{n^3}\right\}\sin nx\right]}$.	
T	12	For the function $f(x)=\begin{cases}x & ;0\le x\le 2\\ 4-x & ;2\le x\le 4\end{cases}$, find its Fourier series. Hence show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$. **Answer**: $\mathbf{f(x)=1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{\pi^2n^2}\cos\left(\frac{n\pi x}{2}\right)}$.	

❖ ODD & EVEN FUNCTION:

✓ Let $f(x)$ be defined in $(-L, L)$, then

(1). A function $f(x)$ is said to be **Odd Function** if $\mathbf{f(-x)=-f(x)}$.

(2). A function $f(x)$ is said to be **Even Function** if $\mathbf{f(-x)=f(x)}$.

❖ NOTE:

(1). If $f(x)$ is an even function defined in $(-L, L)$, then $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$.

(2). If $f(x)$ is an odd function defined in $(-L, L)$, then $\int_{-L}^{L} f(x)\, dx = 0$.

❖ FOURIER SERIES FOR ODD & EVEN FUNCTION:

✓ Let, $f(x)$ be a periodic function defined in $(-L, L)$

$f(x)$ is even, $b_n = 0; n = 1,2,3, \dots$

$$\Rightarrow \mathbf{f(x)} = \frac{\mathbf{a_0}}{\mathbf{2}} + \sum_{n=1}^{\infty} \mathbf{a_n \cos\left(\frac{n\pi x}{L}\right)}$$

Where,

$$a_0 = \frac{2}{L}\int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$f(x)$ is odd, $a_0 = 0 = a_n; n = 1,2,3, \dots$

$$\Rightarrow \mathbf{f(x)} = \sum_{n=1}^{\infty} \mathbf{b_n \sin\left(\frac{n\pi x}{L}\right)}$$

Where,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Sr. No.	Type of Function	Examples
1.	Even Function	✓ $x^2, x^4, x^6, \dots$ i.e. x^n, where n is even. ✓ Any constant. e.g. $1, 2, \pi \dots$ ✓ Graph is symmetric about Y – axis. ✓ $\cos ax, \lvert x\rvert, \lvert x^3\rvert, \lvert\cos x\rvert, \dots$ ✓ $f(-x) = f(x)$
2.	Odd Function	✓ $x, x^3, x^5, \dots$ i.e. x^m, where m is odd. ✓ $\sin ax$ ✓ Graph is symmetric about Origin. ✓ $f(-x) = -f(x)$
3.	Neither Even nor Odd	✓ e^{ax} ✓ $ax^m + bx^n$; n is even & m is odd number.

METHOD – 2: FOURIER SERIES IN THE INTERVAL $(-L, L)$

C	1	Find the Fourier series of the periodic function $f(x) = 2x$, where $-1 < x < 1$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi x)$.	
H	2	Find the Fourier expansion for function $f(x) = x - x^3$ in $-1 < x < 1$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3} \sin n\pi x$.	
C	3	Expand $f(x) = x^2 - 2$ in $-2 < x < 2$ the Fourier series. **Answer**: $f(x) = -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$.	
H	4	Find the Fourier series for $f(x) = x^2$ in $-l < x < l$. **Answer**: $f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4\, l^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$.	
T	5	Obtain the Fourier series for $f(x) = x^2$ in the interval $-\pi < x < \pi$ and hence deduce that (1). $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (2). $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$. **Answer**: $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$.	
T	6	Find the Fourier series expansion of $f(x) = \lvert x \rvert; -\pi < x < \pi$. **Answer**: $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\,[(-1)^n - 1]}{\pi n^2} \cos nx$.	
C	7	Find the Fourier series of $f(x) = x^2 + x$, where $-2 < x < 2$. **Answer**: $f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \left[\frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) + \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]$.	
H	8	Find the Fourier series of $f(x) = x - x^2;\ -\pi < x < \pi$. Deduce that: $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$. **Answer**: $f(x) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n^2} \cos nx + \frac{2(-1)^{n+1}}{n} \sin nx\right]$.	

T	9	Find the Fourier series of $f(x) = x + \|x\|; -\pi < x < \pi$. **Answer**: $\mathbf{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2[(-1)^n - 1]}{\pi n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right]}$.	
C	10	Find the Fourier series to represent e^x in the interval $(-\pi, \pi)$. **Answer**: $\mathbf{f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{(e^{\pi} - e^{-\pi})(-1)^n}{\pi(n^2+1)}[\cos nx - n\sin nx]}$.	
C	11	Find the Fourier series for periodic function f(x) with period 2, where $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$. **Answer**: $\mathbf{f(x) = \sum_{n=1}^{\infty}\frac{2 - 2(-1)^n}{\pi n}\sin n\pi x}$.	
H	12	Find Fourier series expansion of the function given by $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$. **Answer**: $\mathbf{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{1 - (-1)^n}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]}$.	
T	13	Find Fourier series of $f(x) = \begin{cases} x, & -1 < x < 0 \\ 2, & 0 < x < 1 \end{cases}$. **Answer**: $\mathbf{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{1 + (-1)^{n+1}}{n^2\pi^2}\cos n\pi x + \frac{2 + 3(-1)^{n+1}}{n\pi}\sin n\pi x\right]}$.	W-18 (7)
C	14	Find the Fourier series expansion of the function $f(x) = \begin{cases} -\pi, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$. Deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. **Answer**: $\mathbf{f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{1 - 2(-1)^n}{n}\sin nx\right]}$.	

H	15	Obtain the Fourier Series for the function f(x) given by $f(x)=\begin{cases}0, & -\pi \le x \le 0\\ x^2, & 0 \le x \le \pi\end{cases}$. Hence prove $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots=\frac{\pi^2}{12}$. **Answer:** $f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left[\frac{2(-1)^n\cos nx}{n^2}+\frac{1}{\pi}\left\{-\frac{\pi^2(-1)^n}{n}+\frac{2(-1)^n}{n^3}-\frac{2}{n^3}\right\}\sin nx\right]$.	
T	16	Find the Fourier Series for the function f(x) given by $f(x)=\begin{cases}-\pi, & -\pi<x<0\\ x-\pi, & 0<x<\pi\end{cases}$. **Answer:** $f(x)=-\frac{3\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{\pi n^2}\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\right]$.	
C	17	If $f(x)=\begin{cases}\pi+x, & -\pi<x<0\\ \pi-x, & 0<x<\pi\end{cases}$ & $f(x)=f(x+2\pi)$, for all x then expand f(x) in a Fourier series. **Answer:** $f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}[1-(-1)^n]\cos nx.$	
H	18	Find Fourier series for 2π periodic function $f(x)=\begin{cases}-k, & -\pi<x<0\\ k, & 0<x<\pi\end{cases}$. Hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$. **Answer:** $f(x)=\sum_{n=1}^{\infty}\frac{2k[1-(-1)^n]}{n\pi}\sin nx.$	
T	19	Find the Fourier Series for the function f(x) given by $f(x)=\begin{cases}1+\frac{2x}{\pi}; & -\pi \le x \le 0\\ 1-\frac{2x}{\pi}; & 0 \le x \le \pi\end{cases}$ and prove $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$. **Answer:** $f(x)=\sum_{n=1}^{\infty}\frac{4}{\pi^2n^2}[1-(-1)^n]\cos nx.$	

❖ HALF-RANGE SERIES:

✓ If a function $f(x)$ is defined only on a half interval $(0, L)$ instead of $(c, c+2L)$, then it is possible to obtain a Fourier cosine or Fourier sine series.

❖ HALF-RANGE COSINE SERIES IN THE INTERVAL $(0, L)$:

✓ The half-range cosine series in the interval $(0, L)$ is defined by

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)},$$

where the constants a_0 and a_n are given by

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \quad \&$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

METHOD – 3: HALF-RANGE COSINE SERIES IN THE INTERVAL $(0, L)$

C	1	Find a cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$. **Answer**: $\mathbf{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^{n+1} + 1]}{\pi n^2} \cos nx.}$	
H	2	Find Fourier cosine series for $f(x) = x^2;\ 0 < x \le \pi$. **Answer**: $\mathbf{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx.}$	
T	3	Find half-range cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$. **Answer**: $\mathbf{f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos(n\pi x).}$	
C	4	Obtain the Fourier cosine series of the function $f(x) = e^x$ in the range $(0, L)$. **Answer**: $\mathbf{f(x) = \frac{e^L - 1}{L} + \sum_{n=1}^{\infty} \frac{2L[e^L(-1)^n - 1]}{n^2\pi^2 + L^2} \cos\left(\frac{n\pi x}{L}\right).}$	W-19 (4)
H	5	Find half-range cosine series for $f(x) = e^{-x}$ in $0 < x < \pi$. **Answer**: $\mathbf{f(x) = \frac{1 - e^{\pi}}{\pi} + \sum_{n=1}^{\infty} \frac{2[e^{-\pi}(-1)^{n+1} + 1]}{\pi(n^2 + 1)} \cos nx.}$	

C	6	Find half-range cosine series for $\sin x$ in $(0, \pi)$. **Answer:** $\mathbf{f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left[\frac{-(-1)^{1+n}+1}{1+n} + \frac{-(-1)^{1-n}+1}{1-n}\right]\cos nx, a_1 = 0.}$	
H	7	Find the half-range cosine series of $f(x) = \begin{cases} 2 \; ; -2 < x < 0 \\ 0 \; ; \; 0 < x < 2 \end{cases}$. **Answer:** $\mathbf{f(x) = 0.}$	S-19 (4)
T	8	Find half-range cosine series for $f(x) = \begin{cases} kx & ; \; 0 \le x < \frac{l}{2} \\ k(l-x) \; ; & \frac{l}{2} < x < l \end{cases}$. And hence deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. **Answer:** $\mathbf{f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right]\cos\left(\frac{n\pi x}{l}\right).}$	

❖ HALF-RANGE SINE SERIES IN THE INTERVAL $(0, L)$:

✓ The half-range sine series in the interval $(0, L)$ is defined by

$$\mathbf{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),}$$

where the constants b_n is given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

METHOD – 4: HALF-RANGE SINE SERIES IN THE INTERVAL $(0, L)$

C	1	Find the half-range sine series for $f(x) = 2x, 0 < x < 1$. **Answer:** $\mathbf{f(x) = \sum_{n=1}^{\infty}\frac{4}{n\pi}(-1)^{n+1}\sin n\pi x.}$	
H	2	Derive half-range sine series of $f(x) = \pi - x, 0 \le x \le \pi$. **Answer:** $\mathbf{f(x) = \sum_{n=1}^{\infty}\frac{2}{n}\sin nx.}$	W-18 (4)

T	3	Find half-range sine series of $f(x) = x^3, 0 < x < \pi$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \left[\frac{6}{n^2} - \pi^2\right] \sin nx$.	
T	4	Find half-range sine series for $f(x) = e^x$ in $0 < x < \pi$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{2n[e^{\pi}(-1)^{n+1} + 1]}{\pi(1 + n^2)} \sin nx$.	
C	5	Find the sine series $f(x) = \begin{cases} x & ; 0 < x < \frac{\pi}{2} \\ \pi - x & ; \frac{\pi}{2} < x < \pi \end{cases}$. **Answer**: $f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) \sin nx$.	
T	6	Find half-range sine series for $\cos 2x$ in $(0, \pi)$. **Answer**: $f(x) = \sum_{\substack{n=1 \\ n\neq 2}}^{\infty} \frac{2n[(-1)^{n+1} + 1]}{\pi(n^2 - 4)} \sin nx, b_2 = 0$.	

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UNIT 4

❖ FUNCTION OF SEVERAL VARIABLES:

✓ A function of two or more than two variables is said to be function of several variables.

- The area of rectangle is a function of two variables.
- The volume of a rectangular box is a function of three variables.

❖ LIMIT OF FUNCTION $f(x, y)$ AT POINT (a, b):

✓ **Case-1**: $(a, b) = (0, 0)$

(1). Find $L_1 = \lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\}$

(2). Find $L_2 = \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\}$

(3). Find $L_3 = \lim_{x \to 0} \left\{ \lim_{y \to mx} f(x, y) \right\}$

(4). Find $L_4 = \lim_{x \to 0} \left\{ \lim_{y \to mx^2} f(x, y) \right\}$

- If $L_1 = L_2 = L_3 = L_4 = L$ then limit exist and the value of limit is L.
- If one of them is not equal to others then limit does not exist.

✓ **Case-2**: $(a, b) \neq (0, 0)$

(1). Let $L = \lim_{(x, y) \to (a, b)} f(x, y)$.

In above expression, apply the limit and we get L is finite number then limit exist, if not then take $x - a = h$ and $y - b = k$ and hence the limit $(h, k) \to (0, 0)$.

i.e. $L' = \lim_{(x, y) \to (a, b)} f(x, y) = \lim_{(h, k) \to (0, 0)} f(a + h, b + k)$

(2). Again, apply the limit in above expression and if we get L' is finite then limit exist, if not then check four steps of case-1 for L' and get the conclusion.

METHOD – 1: LIMIT OF FUNCTION OF TWO VARIABLES

C	1	Evaluate $\lim_{(x,\ y)\to(5,\ -2)} x^4 + 4x^3y - 5xy^2$. **Answer**: -475.	
H	2	Evaluate $\lim_{(x,\ y)\to(1,\ 2)} x^3y^3 - x^3y^2 + 3x + 2y$. **Answer**: 11.	
C	3	Find $\lim_{(x,\ y)\to(0,\ 0)} \frac{3x^2y}{x^2 + y^2}$ if it exist. **Answer**: 0.	
H	4	Evaluate $\lim_{(x,\ y)\to(0,\ 0)} \frac{x^3 - y^3}{x^2 + y^2}$. **Answer**: 0.	
C	5	Show that the function $f(x,y) = \frac{2x^2y}{x^4 + y^2}$ has no limit as (x,y) approaches to $(0,0)$.	S-19 (3)
H	6	Show that $\lim_{(x,\ y)\to(0,\ 0)} \frac{xy}{x^2 + y^2}$ does not exist.	
H	7	Show that $\lim_{(x,\ y)\to(0,\ 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.	
T	8	By considering different paths show that the function $f(x,y) = \frac{x^4 - y^2}{x^4 + y^2}$ has no limit as $(x,y) \to (0,0)$.	
T	9	Evaluate $\lim_{(x,\ y)\to(0,\ 0)} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$; $x \neq y$. **Answer**: 2.	

❖ CONTINUITY OF FUNCTION OF TWO VARIABLES:

✓ A function $f(x,y)$ is said to be continuous at point (a,b) if the following conditions are satisfied:

(1). $f(a,b)$ must be well define.

(2). $\lim\limits_{(x,\ y)\ \to\ (a,\ b)} f(x,y)$ must be exist.

(3). $\lim\limits_{(x,\ y)\ \to\ (a,\ b)} f(x,y) = f(a,b)$.

✓ Given function is continuous at every point except at the origin means we have to discuss continuity of that function at $(0,0)$.

Note: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ & $\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = 0$.

METHOD – 2: CONTINUITY OF FUNCTION OF TWO VARIABLES

C	**1**	Discuss the continuity of the function f defined as $f(x,y) = \dfrac{x^3 - y^3}{x^2 + y^2} \quad ; (x,y) \neq (0,0)$ $\quad = 0 \quad ; (x,y) = (0,0)$. **Answer**: **Continuous.**	W-18 (3)
H	**2**	Discuss the continuity of $f(x,y) = \dfrac{3x^2 y}{x^2 + y^2} \quad ; (x,y) \neq (0,0)$ $\quad = 0 \quad ; (x,y) = (0,0)$. **Answer**: **Continuous.**	
H	**3**	Show that following function is continuous at the origin. $f(x,y) = \dfrac{xy}{\sqrt{x^2 + y^2}} \quad ; (x,y) \neq (0,0)$ $\quad = 0 \quad ; (x,y) = (0,0)$.	
T	**4**	Examine for continuity at $(0,0)$ of the following function: $f(x,y) = \dfrac{xy}{xy + (x-y)} \quad ; (x,y) \neq (0,0)$ $\quad = 0 \quad ; (x,y) = (0,0)$. **Answer**: **Continuous.**	

C	**5**	Show that following function is not continuous at the origin. $f(x,y) = \frac{2x^2y}{x^3+y^3}$; $(x,y) \neq (0,0)$ $= 0$; $(x,y) = (0,0)$.	
H	**6**	Check the continuity for the following function at $(0,0)$ $f(x,y) = \frac{2xy}{x^2+y^2}$; $(x,y) \neq (0,0)$ $= 0$; $(x,y) = (0,0)$. **Answer**: **Discontinuous.**	
H	**7**	Discuss the continuity of the given function at $(0,0)$: $f(x,y) = \frac{y-x}{y+x}$; $(x,y) \neq (0,0)$ $= 0$; $(x,y) = (0,0)$. **Answer**: **Discontinuous.**	
H	**8**	Discuss the continuity of function at the origin: $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$; $(x,y) \neq (0,0)$ $= 2$; $(x,y) = (0,0)$. **Answer**: **Discontinuous.**	
T	**9**	Discuss the continuity of $f(x,y) = \frac{x^3-y^3}{x^2+y^2}$; $(x,y) \neq (0,0)$ $= 5$; $(x,y) = (0,0)$. **Answer**: **Discontinuous.**	
C	**10**	Determine the continuity of the function at origin $f(x,y) = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)$; $(x,y) \neq (0,0)$ $= 0$; $(x,y) = (0,0)$. **Answer**: **Continuous.**	

❖ PARTIAL DERIVATIVE:

✓ Let $f(x, y)$ be a function of two independent variables x and y.

➢ The **first order** partial derivative of **f** with respect to **x** is denoted by

$$\frac{\partial f}{\partial x} \text{ or } f_x \text{ or } \left(\frac{\partial f}{\partial x}\right)_{y\ \text{constant}} \quad \text{and it is defined as } \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x, y)}{h}.$$

➢ The **first order** partial derivative of **f** with respect to **y** is denoted by

$$\frac{\partial f}{\partial y} \text{ or } f_y \text{ or } \left(\frac{\partial f}{\partial y}\right)_{x\ \text{constant}} \quad \text{and it is defined as } \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

❖ HIGHER ORDER PARTIAL DERIVATIVE:

✓ If f is a function of two independent variables x and y i.e. $f(x, y)$, then the **second order** partial derivatives are as follow:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = f_{xy}, \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = f_{yx}.$$

➢ f_{xy} and f_{yx} are usually called mixed second order partial derivatives of f.

✓ If f is a function of two independent variables x and y then the **third order** partial derivatives are as bellow:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial^3 f}{\partial x^3} = f_{xxx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial^3 f}{\partial y^3} = f_{yyy},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial^3 f}{\partial y\, \partial x\, \partial x} = f_{xxy}, \qquad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial^3 f}{\partial x\, \partial y\, \partial y} = f_{yyx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial^3 f}{\partial y\, \partial y\, \partial x} = f_{xyy}, \qquad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial^3 f}{\partial x\, \partial x\, \partial y} = f_{yxx},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial^3 f}{\partial x\, \partial y\, \partial x} = f_{xyx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial^3 f}{\partial y\, \partial x\, \partial y} = f_{yxy}.$$

✓ On the similar way we can define partial derivatives of order greater than three.

✓ Number of second order partial derivatives of function of two independent variables is $\mathbf{4 = 2^2}$.

- ✓ Number of n^{th} order partial derivatives of function of **two** independent variables is $\mathbf{2^n}$.
- ✓ Number of n^{th} order partial derivatives of function of **k** independent variables is $\mathbf{k^n}$.

❖ MIXED DERIVATIVE THEOREM (CLAIRAUT'S THEOREM):

- ✓ If $f(x, y), f_x, f_y, f_{xy}, f_{yx}$ are defined throughout an open region containing a point (a, b) and f_x, f_y, f_{xy}, f_{yx} are all continuous at (a, b) then $\mathbf{f_{xy}(a, b) = f_{yx}(a, b)}$.

METHOD – 3: PARTIAL DERIVATIVES

C	1	If $f(x, y) = x^2y + xy^2$, then find $f_x(1, 2)$ and $f_y(1, 2)$ by definition. **Answer**: **8 & 5.**	
H	2	Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$, if $f(x, y) = x^2 + 3xy + y - 1$ by definition. **Answer**: $\mathbf{-7}$ & **13.**	
C	3	If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$. **Answer**: **16 & 8.**	
C	4	Find the second order partial derivative of $f(x, y) = x^3 + x^2y^3 - 2y^2$. **Answer**: $\mathbf{f_{xx} = 6x + 2y^3}$, $\mathbf{f_{xy} = 6xy^2}$, $\mathbf{f_{yx} = 6xy^2}$ & $\mathbf{f_{yy} = 6x^2y - 4}$.	
C	5	If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. **Answer**: $\mathbf{\cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}}$ & $\mathbf{-\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}}$.	
T	6	Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, if $f(x, y) = \frac{2y}{y + \cos x}$. **Answer**: $\mathbf{\frac{2y\sin x}{(y+\cos x)^2}}$ & $\mathbf{\frac{2\cos x}{(y+\cos x)^2}}$.	
T	7	Verify that the function $u = e^{-\alpha^2k^2t}\sin kx$ is a solution of heat conduction equation $u_t = \alpha^2u_{xx}$.	S-19 (3)

T	8	The amount of work done by the heart's main pumping chamber, the left ventricle is given by the equation: $W(P,V,\delta,v,g) = PV + \frac{V\delta v^2}{2g}$, where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ is the weight density of blood, v is the average velocity of the existing blood and g is the acceleration of the gravity which is constant. Find partial derivative of W with respect to each variable. **Answer**: $\frac{\partial W}{\partial P} = V$, $\frac{\partial W}{\partial V} = P + \frac{\delta v^2}{2g}$, $\frac{\partial W}{\partial \delta} = \frac{Vv^2}{2g}$ & $\frac{\partial W}{\partial v} = \frac{V\delta v}{g}$.	
C	9	If resistors of R_1, R_2 and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. Find the value of $\frac{\partial R}{\partial R_2}$ when $R_1 = 30, R_2 = 45$ and $R_3 = 90$ ohms. **Answer**: **1/9**.	
C	10	If $u = e^{3xyz}$ then show that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (3 + 27xyz + 27x^2y^2z^2)e^{3xyz}$.	
T	11	Calculate f_{xxyz} if $f(x,y,z) = \sin(3x + yz)$. **Answer**: $-9\cos(3x + yz) + 9yz\sin(3x + yz)$.	
C	12	If $u = \log(\tan x + \tan y + \tan z)$ then prove that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$.	
H	13	If $u = x^2y + y^2z + z^2x$ then find out $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$. **Answer**: $(x + y + z)^2$.	W-18 (3)
H	14	If $u = \log(x^2 + y^2)$, verify that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.	
H	15	For $u = \tan^{-1}\left(\frac{y}{x}\right)$. Show that (**a**) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & (**b**) $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.	
T	16	If $z = x + y^x$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.	

T	**17**	If $u=\dfrac{e^{x+y+z}}{e^x+e^y+e^z}$ then show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=2u$.	
T	**18**	If $u=r^2\cos 2\theta$ then show that $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0$.	
C	**19**	If $u=\log(x^3+y^3-x^2y-xy^2)$, prove that $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}\right)^2 u=-\dfrac{4}{(x+y)^2}$.	
H	**20**	If $u=x^2y+y^2z+z^2x$, prove that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=(x+y+z)^2$ and $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u=6(x+y+z)$.	

❖ TOTAL DIFFERENTIATION:

✓ If $u=f(x,y)$, then the total differentiation of u is given by

$$\mathbf{du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy.}$$

Here du, dx & dy are called the total differentials (or exact differentials) of u, x & y respectively.

✓ If $u=f(x,y,z)$, then the total differentiation of u is given by

$$\mathbf{du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz.}$$

✓ If $u=f(x_1,x_2,\ldots\ldots,x_n)$, then the total differential is given by

$$\mathbf{du=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+\cdots+\frac{\partial f}{\partial x_n}dx_n.}$$

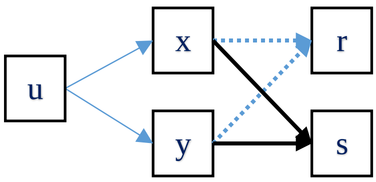
❖ CHAIN RULE:

✓ Case-1:

If $u=f(x,y)$ and $x=g(r,s)$ & $y=h(r,s)$, then

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

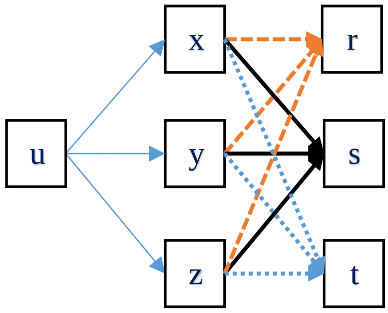
$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$



✓ Case-2:

If $u = f(x, y, z)$ and $x = g(r, s, t), y = h(r, s, t)$ & $z = k(r, s, t)$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$



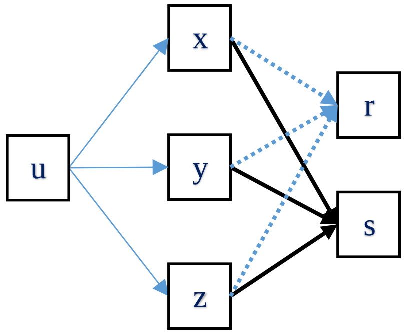
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

✓ Case-3:

If $u = f(x, y, z)$ and $x = g(r, s), y = h(r, s)$ & $z = k(r, s)$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

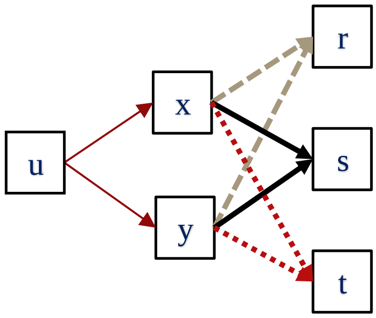


$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

✓ Case-4:

If $u = f(x, y)$ and $x = g(r, s, t)$ & $y = h(r, s, t)$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$



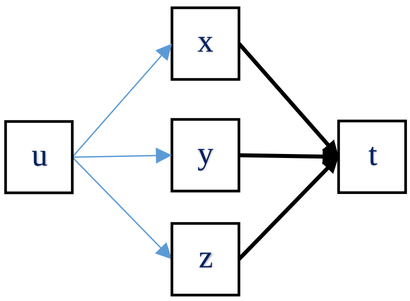
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

✓ Case-5:

If $u = f(x, y, z)$ and $x = g(t), y = h(t)$ & $z = k(t)$, the

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$



METHOD – 4: CHAIN RULE

C	1	Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s, if $w = x + 2y + z^2$, where $x = \frac{r}{s}$, $y = r^2 + \ln(s)$ & $z = 2r$. **Answer**: $\left(\frac{1}{s}\right) + 12r$ & $\left(\frac{2}{s}\right) - \left(\frac{r}{s^2}\right)$.	
H	2	If $z = xy^2 + x^2y, x = at^2, y = 2at$, find $\frac{dz}{dt}$. **Answer: $2a^3t^3\ (8 + 5t)$.**	
C	3	If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$. Find $\frac{dz}{dt}$ when $t = 0$. **Answer**: **6.**	
H	4	Let $u = x^4y + y^2z$, where $x = rse^t, y = rs^2e^{-t}$ & $z = r^2s\sin(t)$. Find the value of $\frac{\partial u}{\partial s}$, where $r = 2, s = 1$ & $t = 0$. **Answer**: **192.**	
T	5	For $z = \tan^{-1}\left(\frac{x}{y}\right), x = u\cos v, y = u\sin v$, evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $(1.3, \pi/6)$. **Answer**: $\frac{\partial z}{\partial u} = 0$ & $\frac{\partial z}{\partial v} = -1$.	
C	6	If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.	
H	7	If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.	
T	8	If $u = f(x-y, y-z, z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.	W-19 (7)
T	9	If $u = f(x^2 + 2yz, y^2 + 2zx)$, then prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.	

C	10	If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$.	
C	11	If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.	
T	12	Suppose f is a differential function of x and y and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the following table to calculate $g_u(0,0), g_v(0,0)$. (table below) **Answer**: **7, 2.**	S-19 (4)
C	13	Find $\frac{du}{dt}$ as total derivative for $u = 4x + xy - y^2, x = \cos 3t$ & $y = \sin 3t$. **Answer**: $\mathbf{-12\sin 3t + 3\cos 6t - 3\sin 6t}$.	
H	14	If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz. **Answer**: $\mathbf{(2x + 3y)dx + (3x - 2y)dy}$.	

Table for Question 12:

Point	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

❖ IMPLICIT FUNCTION:

- ✓ A function in which one variable cannot be expressed in terms of other variables is called implicit function.
- ✓ **First order** differential coefficient of an implicit function:

  Let $f(x, y) = c$ be an implicit function then

  $$\frac{dy}{dx} = -\frac{p}{q}, \text{ where } q \neq 0 \text{ and } p = \frac{\partial f}{\partial x} \;\; \& \;\; q = \frac{\partial f}{\partial y}.$$

- ✓ **Second order** differential coefficient of an implicit function:

  Let $f(x, y) = c$ be an implicit function then

$$\frac{d^2y}{dx^2} = -\left[\frac{q^2r - 2pqs + p^2t}{q^3}\right], \text{where } q \neq 0 \text{ and}$$

$$p = \frac{\partial f}{\partial x},\ q = \frac{\partial f}{\partial y},\ r = \frac{\partial^2 f}{\partial x^2},\ s = \frac{\partial^2 f}{\partial x\,\partial y}\ \ \&\ \ t = \frac{\partial^2 f}{\partial y^2}.$$

✓ Also, we can use following formula to find second order differential coefficient:

$$q^3\frac{d^2y}{dx^2} = 2pqs - p^2t - q^2r = \begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix}.$$

❖ PARTIAL DIFFERENTIATION OF IMPLICIT FUNCTION:

✓ If f(x, y, z)=0 is an implicit function then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z},\quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z},$$

$$\frac{\partial y}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y},\quad \frac{\partial y}{\partial z} = -\frac{\partial f/\partial z}{\partial f/\partial y},$$

$$\frac{\partial x}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial x}\ \ \&\ \ \frac{\partial x}{\partial z} = -\frac{\partial f/\partial z}{\partial f/\partial x}.$$

METHOD – 5: IMPLICIT FUNCTION

C	1	By using partial derivatives find the value of $\frac{dy}{dx}$ for $xe^y + \sin(xy) + y - \log 2 = 0$ at $(0, \log 2)$. **Answer**: $-(2 + \log 2)$.	
H	2	Find $\frac{dy}{dx}$ if $x^y + y^x = a^b$. **Answer**: $-\left[\frac{y\,x^{y-1} + y^x \log y}{x^y \log x + x\,y^{x-1}}\right]$.	
H	3	Find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$. **Answer**: $\left[\frac{y \tan x + \log(\sin y)}{\log(\cos x) - x \cot y}\right]$.	

T	4	Find $\frac{dy}{dx}$ when $y^{x^y} = \sin x$. **Answer**: $-\left[\frac{\mathbf{y\,x^{y-1}\log y - \cot x}}{\mathbf{x^y\,\{\log x \log y + (1/y)\}}}\right]$.	
T	5	Using partial differentiation find $\frac{dy}{dx}$ for $(\cos x)^y = x^{\sin y}$. **Answer**: $-\left[\frac{\mathbf{-y\sin x\,(\cos x)^{y-1} - \sin y\,(x)^{\sin y-1}}}{\mathbf{\ln\cos x\,(\cos x)^y - \ln x\cos y\,(x)^{\sin y}}}\right]$.	
C	6	If $x^3 + y^3 = 6xy$, then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **Answer**: $-\frac{\mathbf{x^2 - 2y}}{\mathbf{y^2 - 2x}}$ & $-\frac{\mathbf{16xy}}{\mathbf{(y^2 - 2x)^3}}$.	W-18 (4)
C	7	Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^6 + 2y^6 = 3a^2x^3$. **Answer**: $\frac{\mathbf{3a^2x^2 - 2x^5}}{\mathbf{4y^5}}$ & $-\left[\frac{\mathbf{27a^2x^4(3a^2 + 4x^3)}}{\mathbf{144y^{11}}}\right]$.	
H	8	Using partial derivatives find $\frac{d^2y}{dx^2}$ for $x^5 + y^5 = 5x^2$. **Answer**: $\frac{\mathbf{-6x^2(1 + x^3)}}{\mathbf{y^9}}$.	
C	9	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z defined implicitly as a function of x and y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$. **Answer**: $-\left[\frac{\mathbf{x^2 + 2yz}}{\mathbf{z^2 + 2xy}}\right]$ & $-\left[\frac{\mathbf{y^2 + 2xz}}{\mathbf{z^2 + 2xy}}\right]$.	
H	10	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial x}{\partial y}$ at $(1, 2, -3)$ if $x^2 + 2y^2 + z^2 = 16$. **Answer**: **1/3 & − 4.**	
T	11	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using partial derivatives for $z^3 - xy + yz + y^3 - 2 = 0$ at $(1, 1, 1)$. **Answer**: **1/4 & − 3/4.**	

❖ TANGENT PLANE TO THE GIVEN SURFACE AT GIVEN POINT:

- ✓ The equation of the tangent plane to the surface $f(x,y)=0$ at the point $P(x_1,y_1)$ is given by

$$(x-x_1)\left(\frac{\partial f}{\partial x}\right)_P+(y-y_1)\left(\frac{\partial f}{\partial y}\right)_P=0.$$

- ✓ The equation of the tangent plane to the surface $f(x,y,z)=0$ at the point $P(x_1,y_1,z_1)$ is given by

$$(x-x_1)\left(\frac{\partial f}{\partial x}\right)_P+(y-y_1)\left(\frac{\partial f}{\partial y}\right)_P+(z-z_1)\left(\frac{\partial f}{\partial z}\right)_P=0.$$

❖ NORMAL LINE TO THE GIVEN SURFACE AT GIVEN POINT:

- ✓ The equation of the normal line to the surface $f(x,y)=0$ at the point $P(x_1,y_1)$ is given by

$$\frac{x-x_1}{\left(\frac{\partial f}{\partial x}\right)_P}=\frac{y-y_1}{\left(\frac{\partial f}{\partial y}\right)_P}.$$

- ✓ The equation of the normal line to the surface $f(x,y,z)=0$ at the point $P(x_1,y_1,z_1)$ is given by

$$\frac{x-x_1}{\left(\frac{\partial f}{\partial x}\right)_P}=\frac{y-y_1}{\left(\frac{\partial f}{\partial y}\right)_P}=\frac{z-z_1}{\left(\frac{\partial f}{\partial z}\right)_P}.$$

METHOD – 6: TANGENT PLANE AND NORMAL LINE

C	1	Find the equations of tangent plane and normal line to the surface $2xz^2-3xy-4x=7$ at point $(1,-1,2)$. **Answer**: $\mathbf{7x-3y+8z=26}$ & $\frac{\mathbf{x-1}}{\mathbf{7}}=\frac{\mathbf{y+1}}{\mathbf{-3}}=\frac{\mathbf{z-2}}{\mathbf{8}}$.	
H	2	Find the equation of the tangent plane and normal line to the surface $x^3+2xy^2-7z^2+3y+1=0$ at $(1,1,1)$. **Answer**: $\mathbf{5x+7y-14z+2=0}$ & $\frac{\mathbf{x-1}}{\mathbf{5}}=\frac{\mathbf{y-1}}{\mathbf{7}}=\frac{\mathbf{z-1}}{\mathbf{-14}}$.	

H	3	Find the equation of the tangent plane and normal line at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$. **OR** Find the equation of the tangent plane and normal line of the surface $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$ at $(-2,1,-3)$. **Answer**: $\mathbf{3x-6y+2z+18=0}$ & $\mathbf{\frac{x+2}{-1}=\frac{y-1}{2}=\frac{3(z+3)}{-2}}$.	
H	4	Find the equation of the tangent plane and normal line to the surface $z=1-\frac{1}{10}(x^2+4y^2)$ at the point $(1,\ 1,\ 1/2)$. **Answer**: $\mathbf{x+4y+5z-\frac{15}{2}=0}$ & $\mathbf{\frac{5(x-1)}{1}=\frac{5(y-1)}{4}=\frac{2z-1}{2}}$.	
T	5	Find the equation of the tangent plane and normal line to the surface $x^2+y^2+z^2=3$ at the point $(1,1,1)$. **Answer**: $\mathbf{x+y+z-3=0}$ & $\mathbf{x-1=y-1=z-1}$.	W-19 (3)
T	6	Find the equation of the tangent plane and normal line at the point $(2,0,2)$ to the ellipsoid $2z-x^2=0$. **Answer**: $\mathbf{2x-z-2=0}$ & $\mathbf{\frac{x-2}{-4}=\frac{z-2}{2}}$.	
T	7	Find the equations of tangent plane and normal line to the surface $\cos\pi x-x^2y+e^{xz}+yz=4$ at point $P(0,1,2)$. **Answer**: $\mathbf{2x+2y+z-4=0}$ & $\mathbf{\frac{x}{2}=\frac{y-1}{2}=\frac{z-2}{1}}$.	
C	8	Show that the surfaces $z=xy-2$ and $x^2+y^2+z^2=3$ have the same tangent plane at $(1,1,-1)$. **Answer**: $\mathbf{x+y-z-3=0}$.	

❖ LOCAL EXTREME VALUES OF FUNCTION OF TWO VARIABLES:

✓ A point (a,b) is said to be a **stationary point** of $f(x,y)$,

$$\text{if } \left(\frac{\partial f}{\partial x}\right)_{(a,\ b)}=0 \ \ \& \ \ \left(\frac{\partial f}{\partial y}\right)_{(a,\ b)}=0.$$

- A point (a, b) is said to be a **critical point** of $f(x, y)$, if either point (a, b) is stationary point or at point (a, b) the derivative of f is not exist.
- A stationary point (a, b) is said to be a **saddle point** of given function, if at point (a, b) given function has neither local minima nor local maxima.
- A local maximum or local minimum values of a function is called its **local extreme values.**
- Procedure to find the local maxima and local minima of the function $f(x, y)$ is as follow:

(1). Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(2). Solve $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ to find the stationary point/ points $(a_1, b_1), (a_2, b_2), \dots$.

(3). Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ & $\frac{\partial^2 f}{\partial y^2}$.

(4). Find $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a_1,b_1)}$, $s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a_1,b_1)}$ & $t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a_1,b_1)}$.

(5). Find $\mathbf{rt - s^2}$ for stationary point (a_1, b_1).

(6). Conclusion for stationary point (a_1, b_1) from the following table.

Result:	**Conclusion:**
$rt - s^2 > 0$ & $r > 0$ (or $t > 0$)	Function has **local minimum** value.
$rt - s^2 > 0$ & $r < 0$ (or $t < 0$)	Function has **local maximum** value.
$rt - s^2 < 0$	Function has no local maxima or local minima (i.e. saddle point).
$rt - s^2 = 0$ **OR** $r = 0$	Nothing can be said about the local maxima or local minima. It requires further investigation.

(7). Repeat step (4), (5), (6) for all other stationary points.

(8). Substitute the point (a_i, b_i) in the given function to find the local maximum or local minimum value of the function at that point, if possible.

METHOD – 7: LOCAL EXTREME VALUES

C	1	Define saddle point. Find the local extreme values of $f(x,y) = x^2 - y^2 - xy - x + y + 6$, if possible. **Answer**: **Not possible.**	
C	2	Find the local extreme values of function $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. **Answer**: **Minimum** $= \mathbf{108}$ & **Maximum** $= \mathbf{112}$.	
H	3	Find the local extreme values of function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$. **Answer**: **Minimum** $= \mathbf{2}$ & **Maximum** $= \mathbf{38}$.	
H	4	Find the extreme values of $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$. **Answer**: **Minimum** $= \mathbf{0}$ & **Maximum** $= \mathbf{4}$.	W-18 (3)
T	5	Find the local extreme values of $f(x,y) = x^3 + y^3 - 3xy$. **Answer**: **Minimum** $= \mathbf{-1}$ & **Maximum does not exist.**	
T	6	Discuss the Maxima and Minima of the function $3x^2 - y^2 + x^3$. **Answer**: **Minimum does not exist** & **Maximum** $= \mathbf{4}$.	W-19 (4)
T	7	Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. **Answer**: $(\mathbf{0},\mathbf{0})$ = **further investigation is necessary**, $(\sqrt{2}, -\sqrt{2})$ = **minimum** & $(-\sqrt{2}, \sqrt{2})$ = **minimum.**	
C	8	For what values of the constant k does the second derivative test guarantee that $f(x,y) = x^2 + kxy + y^2$ will have a saddle point at $(0,0)$? A local minimum at $(0,0)$? **Answer:** $\mathbf{k} \in \mathbb{R}\backslash[\mathbf{-2},\mathbf{2}]$ & $\mathbf{k} \in (\mathbf{-2},\mathbf{2})$.	S-19 (4)

❖ LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS:

- ✓ This method is useful to find maxima and minima of given function with satisfying given condition or constraint.
- ✓ Procedure of Lagrange's method of undetermined multipliers for finding maxima or minima of the function $f(x,y)$ with satisfying the condition $\phi(x,y) = 0$ is as follow:

(1). Construct the new function $F(x,y)$ as $F(x,y) = f(x,y) + \lambda\phi(x,y)$.

(2). Solve the three equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$ & $\phi(x,y) = 0$ to find the stationary points$(x_1, y_1), (x_2, y_2) \dots$.

(3). Find the value of $f(x,y)$ at above stationary points & give conclusion about maxima or minima.

✓ Procedure of Lagrange's method of undetermined multipliers for finding maxima or minima of the function $f(x,y,z)$ with satisfying the condition $\phi(x,y,z) = 0$ is as follow:

(1). Construct the new function $F(x,y,z) = 0$ as $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$.

(2). Solve four equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ & $\phi(x,y,z) = 0$ to find the stationary points $(x_1, y_1, z_1), (x_2, y_2, z_2) \dots$.

(3). Find the value of $f(x,y,z)$ at above stationary points & give conclusion about maxima or minima.

❖ **NOTES:**

(1). The distance between two points $P(x,y,z)$ and $Q(a,b,c)$ is

$PQ = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.

(2). The area of open rectangular box is $xy + 2yz + 2zx$ and the area of the closed rectangular box is $2xy + 2yz + 2zx$.

(3). The volume of open/closed rectangular box is xyz.

(4). The drawback of this method is that it gives no information about the nature of stationary points i.e. about maxima and minima.

(5). λ is known as Lagrange's multiplier.

METHOD – 8: LAGRANGE'S MULTIPLIERS

T	1	Find the greatest and smallest values of the function $f(x, y) = xy$ on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$. **Answer**: **Greatest** $= \mathbf{2}$ & **Smallest** $= \mathbf{-2}$.	
C	2	Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange multipliers. **Answer**: **Maximum** $= \mathbf{5}$ & **Minimum** $= \mathbf{-5}$.	
H	3	Find the numbers x, y and z such that $xyz = 8$ and $xy + yz + zx$ is maximum, using Lagrange's method of undetermined multipliers. **Answer**: $\mathbf{x = y = z = 2}$.	
T	4	Using Lagrange's method of undetermined multipliers, find the maximum value of $f(x, y, z) = x^p y^q z^r$ subject to the condition $ax + by + cz = p + q + r$. **Answer**: $\mathbf{(p/a)^p \cdot (q/b)^q \cdot (r/c)^r}$.	
C	5	Divided a given number "a" into three positive points such that their sum is "a" & product is maximum. **Answer**: $\mathbf{(a/3,\ a/3,\ a/3)}$.	
C	6	Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$. **Answer**: $\mathbf{(3/2,\ 2,\ 5/2)}$.	
H	7	Find the point on the plane $2x + 3y - z = 5$ which is nearest to the origin, using Lagrange's method of undetermined multipliers. **Answer**: $\mathbf{(5/7,\ 15/14,\ -5/14)}$.	
C	8	Find the shortest and largest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$. **Answer**: **Shortest distance** $= \sqrt{\mathbf{6}}$ & **Largest distance** $= \sqrt{\mathbf{54}}$.	W-18 (7)
H	9	Find the maximum and minimum distance from the point $(1, 2, 2)$ to the sphere $x^2 + y^2 + z^2 = 36$. **Answer**: **Maximum** $= \mathbf{9}$ & **Minimum** $= \mathbf{3}$.	W-19 (7)

T	**10**	Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$. **Answer**: **Closest** $=\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right)$ & **Farthest** $=\left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right)$.	S-19 (7)
C	**11**	A rectangular box, open at the top, is to have a given capacity of 64 cm^3. Find the dimensions of the box requiring least material for its construction. **Answer**: $\left(4\cdot\sqrt[3]{2},\ 4\cdot\sqrt[3]{2},\ 2\cdot\sqrt[3]{2}\right)$.	
H	**12**	You are to construct an open rectangular box from 12 ft^2 of material. What dimensions will result in a box of maximum volume? **Answer**: $(2,\ 2,\ 1)$.	
T	**13**	A rectangular box, open at the top, is to have a volume 32 c.c. Find the dimensions of the box requiring least material for its construction. **Answer**: $(4,\ 4,\ 2)$.	

❖ VECTOR FUNCTION OF SINGLE VARIABLE:

✓ A vector function is function $\bar{r}: I \longrightarrow \mathbb{R}^n$, with $n=2,3,\ldots$ and $I\subset\mathbb{R}$.

- In short, a function whose domain is a set of real numbers and whose codomain is a set of vectors.
- For example: $\bar{r}(t)=(t^2, t+1, 2t)=t^2\,\hat{\imath}+(t+1)\,\hat{\jmath}+2t\,\hat{k}$.

✓ Motion in space motivates to define vector function.

✓ Given Cartesian coordinate in $\mathbb{R}^3$, the values of vector function can be written in components as $\bar{r}(t)=(x(t),\ y(t),\ z(t)), t\in I$ where $x(t),\ y(t),\ z(t)$ are the values of three scalar functions.

❖ SCALAR FUNCTION:

✓ A function whose codomain is scalars is called scalar function.

- For example: $f(x,y,z)=2x^2y+4z$ is scalar function.

❖ DERIVATIVE OF A VECTOR FUNCTION:

✓ A vector function $\overline{V}(t)=(v_1(t),\ v_2(t),\ v_3(t))$ is said to be differentiable at a point t if following limit exists:

$$\overline{V}'(t) = \lim_{\Delta t \to 0} \frac{\overline{V}(t+\Delta t) - \overline{V}(t)}{\Delta t}.$$

- ✓ The vector $\overline{V}'(t)$ is called the derivative of $\overline{V}(t)$.
- ✓ $\overline{V}(t)$ is differentiable at point t **iff** it's all components are differentiable at t.
- ✓ The derivative $\overline{V}'(t)$ is obtained by differentiating each component separately as
  $\overline{V}'(t) = (\, v_1'(t),\ \ v_2'(t),\ \ v_3'(t)\,) = \frac{\partial v_1}{\partial t}\,\hat{\imath} + \frac{\partial v_2}{\partial t}\,\hat{\jmath} + \frac{\partial v_3}{\partial t}\,\hat{k}.$
- ✓ Properties of derivative for vector function:
  - **(1).** $(C \cdot \overline{V})' = C \cdot \overline{V}'.$
  - **(2).** $(\overline{U} + \overline{V})' = \overline{U}' + \overline{V}'.$
  - **(3).** $(\overline{U} \cdot \overline{V})' = (\overline{U}' \cdot \overline{V}) + (\overline{U} \cdot \overline{V}').$
  - **(4).** $(\overline{U} \cdot \overline{V} \cdot \overline{W})' = (\overline{U}' \cdot \overline{V} \cdot \overline{W}) + (\overline{U} \cdot \overline{V}' \cdot \overline{W}) + (\overline{U} \cdot \overline{V} \cdot \overline{W}').$
  - **(5).** $(\overline{U} \times \overline{V})' = (\overline{U}' \times \overline{V}) + (\overline{V}' \times \overline{U}).$

❖ GRADIENT OF A SCALAR FUNCTION:

- ✓ The gradient of a scalar function $f(x,y,z)$ is denoted by grad f/ ∇ f and defined as
  $\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}$ where del operator $= \nabla(\text{nabla}) = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}.$
- ✓ The gradient of scalar function is a vector.
- ✓ The gradient of a scalar field $f(x,y,z)$ is a vector normal to the surface $f(x,y,z) = c$ and has a magnitude equal to the rate of change of $f(x,y,z)$ along this normal.

METHOD – 9: GRADIENT

T	**1**	Find gradient of $\phi = 2z^3 - 3(x^2 + y^2)z + \tan^{-1}(xz)$ at $(1,1,1)$. **Answer**: $(\mathbf{-11/2},\ \ \mathbf{-6},\ \ \mathbf{1/2})$.	
H	**2**	Find gradient of $f(x,y,z) = 2x + yz - 3y^2$. **Answer**: $(\mathbf{2},\ \ \mathbf{z-6y},\ \ \mathbf{y})$.	
C	**3**	Find $\nabla\phi$ at $(1,-2,-1)$, if $\phi = 3x^2y - y^3z^2$. **Answer**: $(\mathbf{-12},\ \ \mathbf{-9},\ \ \mathbf{-16})$.	

T	4	Find grad(ϕ) if $\phi = \log(x^2 + y^2 + z^2)$ at the point $(1,0,-2)$. **Answer**: $(2/5,\ 0,\ -4/5)$.	
C	5	Evaluate ∇e^{r^2}, where $r^2 = x^2 + y^2 + z^2$. **Answer**: $2e^{r^2}(x,y,z)$.	
T	6	If vector $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $r = \lvert\bar{r}\rvert$ then show that $\nabla r^n = n\, r^{n-2}\,\bar{r}$. Hence evaluate $\nabla r^3, \nabla\left(\frac{1}{r}\right)$. **Answer**: $3r\bar{r},\ -\frac{\bar{r}}{r^3}$.	

❖ DIRECTIONAL DERIVATIVE:

- ✓ The directional derivative of $f(x,y)$ at (a,b) in the direction of a unit vector $\bar{u}$ is denoted by $D_{\bar{u}} f$ and it is defined as "rate of change of f in the direction of $\bar{u} = (u_1, u_2)$ at point (a,b)".

  $$\therefore D_{\bar{u}} f = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}.$$

- ✓ Directional derivative in terms of gradient vector:

  $$D_{\bar{u}} f = (\text{grad } f)_{(a,b)} \cdot \frac{\bar{u}}{|\bar{u}|} \text{ OR } D_{\bar{u}} f = (\nabla f)_{(a,b)} \cdot \frac{\bar{u}}{|\bar{u}|}.$$

- ✓ The directional derivative of $f(x,y)$ at (a,b) in the direction $\bar{u} = (u_1, u_2)$ is

  $$D_{\bar{u}} f(a,b) = f_x(a,b)\frac{u_1}{|\bar{u}|} + f_y(a,b)\frac{u_2}{|\bar{u}|}.$$

- ✓ The directional derivative of $f(x,y,z)$ at (x_0,y_0,z_0) in the direction $\bar{u} = (u_1,u_2,u_3)$ is

  $$D_{\bar{u}} f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\frac{u_1}{|\bar{u}|} + f_y(x_0,y_0,z_0)\frac{u_2}{|\bar{u}|} + f_z(x_0,y_0,z_0)\frac{u_3}{|\bar{u}|}.$$

- ✓ The directional derivative of $f(x,y)$ in direction of unit vector $\bar{u} = (u_1,u_2)$ which makes an angle α with positive $x-$axis and at point (x_0,y_0) is

  $$D_{\alpha} f(x_0,y_0) = \cos\alpha\ f_x(x_0,y_0) + \sin\alpha\ f_y(x_0,y_0).$$

METHOD – 10: DIRECTIONAL DERIVATIVE

C	1	Find derivative of $f(x,y) = xe^y + \cos xy$ at the point $(2,0)$ in direction of $\overline{A} = 3\,\hat{\imath} - 4\,\hat{\jmath}$. **Answer**: -1.	
H	2	Find derivative of $f(x,y) = x^2\sin 2y$ at the point $(1,\pi/2)$ in the direction of $\bar{v} = 3\,\hat{\imath} - 4\,\hat{\jmath}$. **Answer**: $8/5$.	
C	3	Define gradient of a function. Use it to find directional derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P(1,1,0)$ in the direction of $\bar{a} = 2\,\hat{\imath} - 3\,\hat{\jmath} + 6\,\hat{k}$. **Answer**: $4/7$.	W-18 (4)
H	4	Find the directional derivatives of $f = xy^2 + yz^2$ at the point $(2,-1,1)$ in the direction $\hat{\imath} + 2\,\hat{\jmath} + 2\,\hat{k}$. **Answer**: -3.	W-19 (3)
H	5	The temperature T at any point in space is given by $T = xy + yz + zx$. Determine the directional derivative of T in the direction of $\overline{V} = 3\,\hat{\imath} - 4\,\hat{k}$ at $(1,1,1)$. **Answer**: $-2/5$.	
H	6	Find the directional derivative of $g(x,y,z) = 3e^x \cos yz$ at $P(0,0,0)$ in the direction of $\overline{A} = 2\,\hat{\imath} + \hat{\jmath} - 2\,\hat{k}$. **Answer**: 2.	
H	7	Find the directional derivatives of $f(x,y,z) = xyz$ at the point $P(-1,1,3)$ in the direction of the vector $\bar{a} = \hat{\imath} - 2\,\hat{\jmath} + 2\,\hat{k}$. **Answer**: $7/3$.	
T	8	Find the directional derivative of $\phi = 4xz^3 - 3x^2\,y^2z$ at the point $(2,-1,2)$ in the direction $(2, 3, 6)$. **Answer**: $664/7$.	
T	9	Find the directional derivative of $f(x,y,z,) = 2x^2 + 3y^2 + z^2$ at the point $P(2,1,3)$ in the direction of $\bar{a} = (1,\ 0,-2)$. **Answer**: $-4/\sqrt{5}$.	

C	**10**	Find the directional derivative of $x^2y^2z^2$ at $(1,1,-1)$ along a direction equally inclined with co-ordinate axes. **Answer**: $\mathbf{2/\sqrt{3}}$.	
T	**11**	Find the directional derivative of $\phi(x,y,z) = xy^2 + yz^2 + zx^2$ at point $(1,1,1)$ along the tangent to the curve $x = t,\ y = t^2,\ z = t^3$ at $t = 1$. **Answer**: $\mathbf{18/\sqrt{14}}$.	
C	**12**	Find the directional derivative $D_u f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and u is the unit vector given by angle $\theta = \frac{\pi}{6}$. What is $D_u f(1,2)$? **Answer**: $\mathbf{\frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y),\ \ \frac{13 - 3\sqrt{3}}{2}}$.	S-19 (3)
T	**13**	Find $D_{(\alpha)} f(x_0,y_0),\ \alpha = 30°$ for $f(x,y) = x^2 - y^2$ at $(x_0,y_0) = (1,2)$. **Answer**: $\mathbf{D_{(30°)} f(1,2) = \sqrt{3} - 2}$.	

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UNIT 5

❖ MULTIPLE INTEGRALS:

✓ The multiple integral is a generalization of the definite integral to function of more than one variable.

❖ DOUBLE INTEGRALS:

✓ Integral of function of two variables over a region R^2 is called double integrals.

✓ Properties of double integrals: Let $f(x, y)$ and $g(x, y)$ be two continuous functions in region R then

(1). $\iint_R [f(x,y) \pm g(x,y)]\, dR = \iint_R f(x,y)\, dR \pm \iint_R g(x,y)\, dR.$

(2). $\iint_R k\, f(x,y)\, dR = k \iint_R f(x,y)\, dR$, where k is constant. (Linearity property).

(3). $\iint_R f(x,y)\, dR \geq 0$, if $f(x,y) \geq 0$ on R.

(4). $\iint_R f(x,y)\, dR \geq \iint_R g(x,y)\, dR$, if $f(x,y) \geq g(x,y), \forall\, (x,y) \in R$. (Inequality property).

(5). If $R = R_1 \cup R_2$ and R_1 and R_2 are non over lapping sub domain of region R then

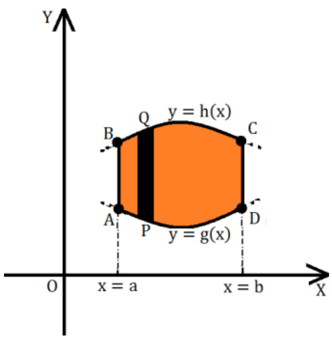
$$\iint_R f(x,y)\, dR = \iint_{R_1} f(x,y)\, dR + \iint_{R_2} f(x,y)\, dR. \text{ (Additivity of region).}$$

❖ FUBINI'S THEOREM FOR EVALUATING DOUBLE INTEGRALS:

✓ There are two cases to evaluate a double integral known as Fubini's Theorem.

Case-I: The region R (i. e. ABCD) is bounded by

$$y = g(x),\ y = h(x) \text{ and two lines } x = a,\ x = b.$$



✓ Procedure for evaluating double integrals of case-I:

(1). Draw a **strip** PQ parallel to y-axis. (i.e. Vertical strip).

(2). According to the strip PQ, the **lower limit** of **inner** integral is obtained from the curve where the strip starts i.e. $y = g(x)$ and the **upper limit** is obtained from the curve where it ends i.e. $y = h(x)$.

(3). The lower limit of **outer** integral is the **left most** point i.e. $x = a$ of the region and upper limit is the **right most** point i.e. $x = b$ of the region.

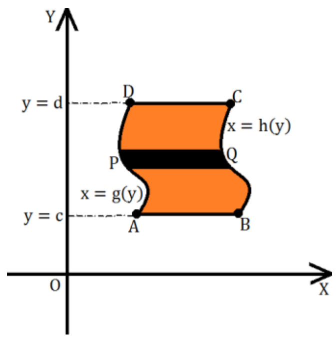
(4). Hence, we get the double integral

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left\{ \int_{g(x)}^{h(x)} f(x,y)\,dy \right\} dx.$$

(5). Solve the above inner integral first then outer integral to find the value of given double integrals.

Case II: The region R (i. e. ABCD) is bounded by

$x = g(y),\ x = h(y)$ and two lines $y = c,\ y = d$.



✓ Procedure for evaluating double integral of Case-II:

(1). Draw the **strip** PQ parallel to x-axis. (i.e. Horizontal strip).

(2). According to the strip PQ, the **lower limit** of the **inner** integral is obtained from the curve where the strip starts i.e. $x = g(y)$ and **upper limit** is obtained from the strip where it ends i.e. $x = h(y)$.

(3). The lower limit of outer integral is the **lowest point** i.e. $y = c$ of the region and upper limit is the **highest point** i.e. $y = d$ of the region.

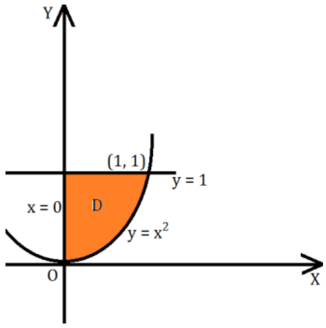
(4). Hence, we get the double integral

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left\{ \int_{g(y)}^{h(y)} f(x,y)\,dx \right\} dy.$$

(5). Solve the above inner integral first then outer integral to find the value of given double integrals.

❖ NORMAL DOMAIN R^2:

- ✓ Any domain D in R^2 is known as normal domain if the **projection** of domain D on either x-axis or y-axis is **bounded** by the two lines (perpendicular to x or y axis).



- ✓ If the domain D is normal with respect to the x-axis and $f : D \to R$ is a continuous function, then

$$\iint_D f(x,y)\ dx\,dy = \int_{x=a}^{x=b} dx * \int_{y=g(x)}^{y=h(x)} dy$$

Where g(x) and h(x) are define between the two lines (parallel to y-axis) which contains the domain.

- ✓ If the domain D is normal with respect to the y-axis and $f : D \to R$ is a continuous function, then

$$\iint_D f(x,y)\ dx\,dy = \int_{y=a}^{y=b} dy * \int_{x=g(y)}^{x=h(y)} dx$$

Where g(y) and h(y) are define between the two lines (parallel to x-axis) which contains the domain.

METHOD – 1: DOUBLE INTEGRALS BY DIRECT INTEGRATION

H	**1**	Evaluate: (**1**). $\int_{-1}^{1}\int_{0}^{2}(1-6x^2y^2)\,dx\,dy$ (**2**). $\int_{0}^{2}\int_{0}^{2}(x^2+y^2)\,dx\,dy$ (**3**). $\int_{0}^{1}\int_{2}^{3}y^2\,dx\,dy$ (**4**). $\int_{0}^{1}\int_{2}^{1}(x+y)\,dx\,dy$. **Answer**: (**1**). $\mathbf{-20/3}$ (**2**). $\mathbf{32/3}$ (**3**). $\mathbf{1/3}$ (**4**). $\mathbf{-2}$.	
H	**2**	Evaluate: $\int_{-1}^{1}\int_{0}^{2}(1-6x^2y)\,dx\,dy$. **Answer**: **4**.	

C	3	Evaluate $\int_0^1 \int_1^2 xy\, dy\, dx.$ **Answer**: **3/4.**	W-19 (3)
H	4	Evaluate: (**1**). $\int_0^1 \int_0^x (x^2+y^2)\, dy\, dx$ (**2**). $\int_0^1 \int_x^{x^2} (x^2+3y+2)\, dy\, dx$ (**3**). $\int_0^1 \int_0^x e^{\frac{y}{x}}\, dy\, dx$ (**4**). $\int_0^1 \int_0^x e^{x+y}\, dy\, dx.$ **Answer**: (**1**). $\mathbf{1/3}$ (**2**). $\mathbf{-7/12}$ (**3**). $\mathbf{(e-1)/2}$ (**4**). $\mathbf{(e-1)^2/2}$.	
C	5	Evaluate $\int_1^2 \int_0^1 \frac{1}{(2x+5y)^2}\, dx\, dy.$ **Answer**: $\mathbf{(-0.1)(\log 12 - \log 7 - \log 2)}$.	
C	6	Evaluate $\int_0^1 \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\, dx\, dy.$ **Answer**: $\mathbf{\pi^2/4}$.	
T	7	Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}.$ **Answer**: $\mathbf{\pi\ \log\left(1+\sqrt{2}\right)/4}$.	
H	8	Evaluate $\int_1^4 \int_{2x^2}^{3x^2} \left(x\, e^{x^2+y}\right) dy\, dx.$ **Answer**: $\mathbf{(e^{64}-e^4)/8 - (e^{48}-e^3)/6}$.	
H	9	Evaluate: (**1**). $\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{x} \cos(x+y)\, dy\, dx$ (**2**). $\int_0^1 \int_0^{1-y} \left(y-\sqrt{x}\right) dx\, dy$ (**3**). $\int_0^1 \int_0^y xye^{-x^2}\, dx\, dy.$ **Answer**: (**1**). **0** (**2**). $\mathbf{-1/10}$ (**3**). $\mathbf{1/4e}$.	

H	**10**	Evaluate: (1). $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\ dr\,d\theta$ (2). $\int_0^{2\pi}\int_{a\sin\theta}^{a} r\ dr\,d\theta$. **Answer**: (1). $3\pi a^2/4$ (2). $\pi a^2/2$	
H	**11**	Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\ dr\,d\theta$. **Answer**: $\pi - 2/8$.	
T	**12**	Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\ dr\,d\theta$. **Answer**: $\pi/4$.	W-19 (3)
C	**13**	Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{1-\sin\theta} r^2\cos\theta\ dr\,d\theta$. **Answer**: $1/12$.	
H	**14**	Evaluate $\int_0^{\pi}\int_0^{1-\cos\theta} r^2\sin\theta\ dr\,d\theta$. **Answer**: $4/3$.	
T	**15**	Evaluate: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\ dr\,d\theta$ & $\int_0^{a}\int_0^{\cos^{-1}\left(\frac{r}{a}\right)} r\sin\theta\ d\theta\,dr$. **Answer**: $32/9$ & $a^2/6$.	

❖ TRIPLE INTEGRALS:

- ✓ Integral of a function of three variables over a region R^3 is called triple integral.
- ✓ Properties of triple Integrals: Let $f(x,y,z)$ and $g(x,y,z)$ be two continuous functions in V then,

(1). $$\iiint_V [f(x,y,z) \pm g(x,y,z)]\ dV = \iiint_V f(x,y,z)\ dV \pm \iiint_V g(x,y,z)\ dV.$$

(2). $\iiint\limits_V k\, f(x,y,z)\, dV = k \iiint\limits_V f(x,y,z)\, dV$, where k is a constant.

(3). $\iiint\limits_V f(x,y,z)\, dV \geq 0$, if $f(x,y,z) \geq 0$ on V.

(4). $\iiint\limits_V f(x,y,z)\, dV \geq \iiint\limits_V g(x,y,z)\, dV$, if $f(x,y,z) \geq g(x,y,z), \forall (x,y,z) \in V$.

(5). If $V = V_1 \cup V_2$ and V_1 and V_2 are non over lapping sub domain of V then

$$\iiint\limits_V f(x,y,z)\, dV = \iiint\limits_{V_1} f(x,y,z)\, dV + \iiint\limits_{V_2} f(x,y,z)\, dV.$$

METHOD – 2: TRIPLE INTEGRALS BY DIRECT INTEGRATION

H	**1**	Evaluate: (1). $\int_1^2 \int_2^3 \int_0^1 xyz\, dz\, dx\, dy$ (2). $\int_0^1 \int_0^{1-y} \int_0^2 dx\, dz\, dy$ (3). $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$. **Answer**: (1). **15/8** (2). **1** (3). **7/6**.	
C	**2**	Evaluate $\int_{-1}^1 \int_0^2 \int_0^1 xz - y^3\, dz\, dy\, dx$. **Answer**: **− 8**.	
C	**3**	Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$. **Answer**: **1/24**.	
H	**4**	Evaluate $\int_0^1 \int_0^{1-y} \int_0^{1-y-z} z\, dx\, dz\, dy$. **Answer**: **1/24**.	

T	5	Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$. **Answer**: **1/48**.	
T	6	Evaluate $\int_0^1 \int_0^x \int_0^{\sqrt{x+y}} z\, dz\, dy\, dx$. **Answer**: **1/4**.	
T	7	Evaluate $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx$. **Answer**: $\mathbf{8abc(a^2 + b^2 + c^2)/3}$.	
H	8	Evaluate $\int_0^2 \int_1^2 \int_0^{yz} xyz\, dx\, dy\, dz$. **Answer**: **15/2**.	
H	9	Evaluate $\int_0^2 \int_1^z \int_0^{yz} xyz\, dx\, dy\, dz$. **Answer**: **7/2**.	
H	10	Evaluate $\int_0^1 \int_0^{\pi} \int_0^{\pi} y \sin z\, dx\, dy\, dz$. **Answer**: $\boldsymbol{\pi^3 (1 - \cos 1)/2}$.	
H	11	Evaluate $\int_1^3 \int_{\frac{1}{x}}^{1} \int_0^{\sqrt{xy}} xyz\, dz\, dy\, dx$. **Answer**: **(13/9) – { (log3)/6 }**.	
C	12	Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$. **Answer**: **{ (8 log 2)/3 } – (19/9)**.	

H	**13**	Evaluate $\int_0^1 \int_0^y \int_0^{x+2y} (x+y+z)^2 \, dz\, dx\, dy$. **Answer**: **257/60.**	
H	**14**	Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \, dz\, dy\, dx$. **Answer**: $\boldsymbol{\pi^2/8}$.	
C	**15**	Evaluate the integral $\int_1^e \int_1^{\log y} \int_1^{e^x} (x^2+y^2) \, dz\, dx\, dy$. **Answer**: $\boldsymbol{(-3e^4+4e^3-9e^2+84e-172)/36}$.	W-18 (4)
C	**16**	Evaluate $\iiint_V dv$, where v is the solid region bounded by $1 \le x \le 2$, $2 \le y \le 4$ & $2 \le z \le 5$. **Answer**: **6.**	
C	**17**	Evaluate $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2\cos^2\theta + z^2)\, r \, d\theta\, dr\, dz$. **Answer**: $\boldsymbol{\pi/3}$.	
C	**18**	Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^7 \sin^3\theta \cos\theta \sin\phi \cos\phi \, dr\, d\theta\, d\phi$. **Answer**: $\boldsymbol{a^8/64}$.	
H	**19**	$\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dz\, dr\, d\theta$. **Answer**: $\boldsymbol{5\pi a^3/64}$.	

❖ **DOUBLE INTEGRALS OVER RECTANGLES AND GENERAL REGION IN CARTESIAN COORDINATES:**

✓ Procedure for evaluating double integrals over general region:

(1). **Draw** all the curves and identify required region.

(2). Find the **point(s) of intersection** of all the curves if required.

(3). Identify the **region** bounded by all the curves.

(4). Take the **strip** parallel to x-axis or y-axis.

(5). If it is required then **divide** the region into two parts.

(6). Find the limit of inner integral from the strip.

(7). Find the limit of the outer integral from axis.

(8). Calculate the example as the example of double integrals.

METHOD – 3: D.I. OVER GENERAL REGION IN CARTESIAN COORDINATES

C	1	Evaluate $\iint_R (x-3y^2)\,dA$, where $R=(x,\ y),\ \ 0\le x\le 2,\ \ 1\le y\le 2.$ **Answer**: $-\mathbf{12}$.	
H	2	Evaluate $\iint_R y\sin(xy)\,dA$, where R is the region bounded by $x=1,\ \ x=2,\ \ y=0$ and $y=\frac{\pi}{2}$. **Answer**: **1**.	W-18 (3)
H	3	Evaluate $\iint_R \sin x \sin y\,dA$, where $R=(x,\ y),\ \ 0\le x\le\frac{\pi}{2},\ \ 0\le y\le\frac{\pi}{2}$. **Answer**: **1**.	
H	4	Evaluate $\iint_R (x+y)\ \ dy\,dx$, where R is the region bounded by $x=0,\ \ x=2,$ $y=x,\ \ y=x+2.$ **Answer**: **12**.	
C	5	Evaluate $\iint_R e^{2x+3y}\ \ dx\,dy$, where R is the triangle bounded by $x=0,\ \ y=0,$ $x+y=1.$ **Answer**: $\mathbf{(2e^3-3e^2+1)/6}$.	

H	6	Evaluate $\iint_R (x^2 - y^2)\, dx\, dy$ over the triangle with the vertices $(0,1), (1,1)$ & $(1,2)$. **Answer**: $-2/3$.	W-19 (4)
H	7	Evaluate $\iint_S \sqrt{xy - y^2}\, dx\, dy$, where S is a triangle with vertices $(0,0), (10,1)$ & $(1,1)$. **Answer**: 6.	
C	8	Evaluate $\iint_R (2x - y^2)\, dA$; over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$ and $y = 3$. **Answer**: $-68/3$.	
C	9	Evaluate $\iint_R x^2\, dA$, where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$. **Answer**: 448.	
T	10	Evaluate $\iint_R \frac{xy}{\sqrt{1 - y^2}}\, dx\, dy$ over positive quadrant of circle $x^2 + y^2 = 1$. **Answer**: $1/6$.	
C	11	Sketch the region of integration and evaluate the integral $\iint_R (y - 2x^2)\, dA$, where R is the region inside the square $\lvert x\rvert + \lvert y\rvert = 1$. **Answer**: $-2/3$.	S-19 (4)
T	12	Evaluate $\iint (6x^2 + 2y)\, dx\, dy$ over the region R bounded between $y = x^2$ and $y = 4$. **Answer**: $512/5$.	

T	**13**	Evaluate $\iint\limits_R y\,dx\,dy$, where R is the region in the first quadrant bounded by $x = 0,\ y = x^2,\ x + y = 2$. **Answer**: **16/15.**	
C	**14**	Evaluate $\iint\limits_R xy\,dx\,dy$ over the region bounded by the parabolas $x^2 = y$ and $y^2 = -x$. **Answer**: $\mathbf{-1/12}$.	
H	**15**	Evaluate $\iint\limits_R xy\,dA$, where R is the region bounded by $x -$ axis, ordinate $x = 2a$ & the curve $x^2 = 4ay$. **Answer**: $\mathbf{a^4/3}$.	
H	**16**	Evaluate $\iint\limits_R xy\,dx\,dy$, where R is the region bounded by $x^2 + y^2 - 2x = 0$, $y^2 = 2x$ and $y = x$. **Answer**: **7/12.**	
T	**17**	Evaluate $\iint\limits_R \sqrt{4x^2 - y^2}\,dx\,dy$ over the area of triangle $y = x$, $y = 0$ and $x = 1$. **Answer**: $\boldsymbol{\pi/9 + \sqrt{3}/6}$.	
H	**18**	Evaluate $\iint\limits_R y\,dx\,dy$ over the ellipse $4x^2 + 9y^2 = 36$ in the positive quadrant. **Answer**: **4.**	
C	**19**	Evaluate $\iint\limits_D (x + 2y)\,dA$, where D is the region bounded by the parabolas $y = 2x^2,\ y = 1 + x^2$. **Answer**: **32/15.**	

H	**20**	Evaluate $\iint\limits_{D} xy\, dA$, where D is the region bounded by the line $y = x - 1$ over the parabola $y^2 = 2x + 6$. **Answer**: **36**.	
T	**21**	Evaluate $\iint\limits_{R} \frac{\sin x}{x}\, dA$, where R is the triangle in $xy -$ plane bounded by $x -$ axis, $y = x$ and $x = 1$. **Answer**: **0.46**.	
H	**22**	Sketch the region of integration and evaluate $\int_{1}^{4}\int_{0}^{\sqrt{x}} \frac{3}{2}\, e^{\frac{y}{\sqrt{x}}}\, dy\, dx$. **Answer**: $\mathbf{7(e-1)}$.	

❖ DOUBLE INTEGRALS AS VOLUMES:

- ✓ When $f(x, y)$ is a positive function over a rectangular region R in the xy – plane, we may interpret the double integral of f over R as the volume of the 3 – dimensional solid region over the $xy -$ plane bounded below by R and above by the surface $z = f(x, y)$. We define this volume as

$$\text{Volume} = \iint\limits_{R} f(x, y)\, dA.$$

- ✓ When the solid region lies between the two surfaces, say lower surface is $z_1 = f_1(x, y)$ and upper surface $z_2 = f_2(x, y)$ and R is the projection on $xy -$ plane, then the volume of a solid region by double integral define as

$$\text{Volume} = \iint\limits_{R} [f_2(x, y) - f_1(x, y)]\, dA.$$

- ✓ In polar coordinates, the volume can be obtained by the integral

$$\text{Volume} = \iint\limits_{R} z \cdot r\, dr d\theta.$$

METHOD – 4: DOUBLE INTEGRALS AS VOLUMES

C	**1**	Find the volume of the prism whose base is the triangle in the xy – plane bounded by the x – axis and the lines $y = x$ and $x = 1$ and whose top in the plane $z = f(x, y) = 3 - x - y$. **Answer**: **1**.	
H	**2**	Find the volume under the plane $x + y + z = 6$ and above the triangle in xy – plane bounded by $2x = 3y$, $y = 0$ and $x = 3$. **Answer**: **10**.	
H	**3**	Find the volume of the region bounded by the surface $x = 0, y = 0, z = 0$ and $2x + 3y + z = 6$. **Answer**: **6**.	
T	**4**	Find the volume bounded by the cylinder $x^2 + y^2 = 4$ between the planes $y + z = 3$ and $z = 0$. **Answer**: $\mathbf{12\pi}$.	
C	**5**	Find the volume below the surface $z = x^2 + y^2$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 2y$. **Answer**: $\mathbf{3\pi/2}$.	W-18 (7)

❖ DOUBLE INTEGRALS BY CHANGE OF ORDER OF INTEGRATION:

- ✓ Sometimes, change of order of integration makes calculation of integral easier.
- ✓ Procedure for evaluating double integrals by change of order of integration:

  (1). Draw the **region** from given data.

  (2). Find the intersection point(s).

  (3). Draw the **strip** parallel to x-axis if we want to integrate first with respect to x or Draw the strip parallel to y-axis if we want to integrate first with respect to y.

  (4). Using Fubini's theorem find the **new limit points** of both the integrals.

  (5). Calculate the example as the example of double integrals.

METHOD – 5: D.I. BY CHANGE OF ORDER OF INTEGRATION

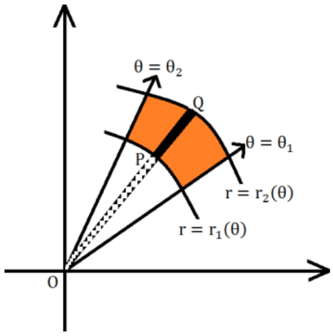
H	1	Change the order only of $\int_{0}^{3}\int_{1-x}^{1+x} f(x,y)\,dy\,dx$. **Answer**: $\int_{1}^{4}\int_{y-1}^{3} f(x,y)\,dx\,dy + \int_{-2}^{1}\int_{1-y}^{3} f(x,y)\,dx\,dy.$	
T	2	Change the order only of $\int_{0}^{3}\int_{\frac{y^2}{9}}^{\sqrt{10-y^2}} f(x,y)\,dx\,dy$. **Answer**: $\int_{0}^{1}\int_{0}^{3\sqrt{x}} f(x,y)\,dy\,dx + \int_{1}^{\sqrt{10}}\int_{0}^{\sqrt{10-x^2}} f(x,y)\,dy\,dx.$	
H	3	Change the order only of $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$. **Answer**: $\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy.$	
C	4	Change the order only of $\int_{1}^{2}\int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} f(x,y)\,dy\,dx.$ **Answer**: $\int_{0}^{2}\int_{1}^{1+\sqrt{2y-y^2}} f(x,y)\,dx\,dy.$	
H	5	Change the order of integrations and hence evaluate $\int_{0}^{1}\int_{4y}^{4} e^{x^2}\,dx\,dy.$ **Answer**: $(e^{16}-1)/8.$	
T	6	Change the order of integrations and hence evaluate $\int_{0}^{\infty}\int_{x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx.$ **Answer**: $1.$	

C	**7**	Evaluate $\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\, dy\, dx.$ **Answer**: $(\mathbf{e-1})$.	S-19 (7)
H	**8**	Change the order of integrations and hence evaluate $\int_0^1 \int_{2x}^2 e^{y^2}\, dy\, dx.$ **Answer**: $(\mathbf{e^4-1})/\mathbf{4}$.	
T	**9**	Change the order of integration and evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx.$ **Answer**: $(\mathbf{1-\cos 1})/\mathbf{2}$ **or** $\mathbf{0.23}$.	W-19 (7)
H	**10**	By changing the order of integration, evaluate $\int_0^3 \int_y^3 \frac{x}{x^2+y^2}\, dx\, dy.$ **Answer**: $\mathbf{3\pi/4}$.	W-18 (4)
T	**11**	Change the order of integration and evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx.$ **Answer**: $\mathbf{1-1/\sqrt{2}}$.	
C	**12**	Change the order of integrations and hence evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\, dy\, dx.$ **Answer**: $(\boldsymbol{\pi}/\mathbf{2})\,\mathbf{log}\{\,(\mathbf{e+1})/\mathbf{2}\,\}$.	
T	**13**	Change the order of integrations and hence evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy \quad \& \quad \int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} y^2\, dA.$ **Answer**: $\mathbf{241/60}$ & $\mathbf{a^4(1+\pi/2)/16}$.	

H	**14**	Find $\int_{0}^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$ by changing the order of integration. **Answer**: $\pi a^2/2$.	
H	**15**	Change the order of integrations and hence evaluate $\int_{0}^{2}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\,dy$. **Answer**: 2π.	
H	**16**	Change the order of integration and hence evaluate $\int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$. **Answer**: $16a^2/3$.	
C	**17**	Change the order of integration and evaluate $\int_{0}^{4}\int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\,dx$. **Answer**: $16/3$.	
H	**18**	Change the order of integrations and hence evaluate $\int_{0}^{2}\int_{0}^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$. **Answer**: $(e^8-1)/4$.	
C	**19**	Express as single integral and hence evaluate $\int_{0}^{1}\int_{0}^{y}(x^2+y^2)\,dx\,dy+\int_{1}^{2}\int_{0}^{2-y}(x^2+y^2)\,dx\,dy$. **Answer**: $\int_{0}^{1}\int_{x}^{2-x}(x^2+y^2)\,dy\,dx$ & $4/3$.	

❖ DOUBLE INTEGRALS OVER GENERAL REGION IN POLAR COORDINATES:

- ✓ In Cartesian coordinate, we have x-axis & y-axis (in two dimension). While in polar coordinate, we have r-axis & θ-axis.
- ✓ Procedure for evaluating double integral over general region in Polar co-ordinates:



(1). Draw the given curves to find the required **region**.

(2). Draw the **strip** from **origin** within the region.

(3). According to the strip PQ, the **lower limit** of **inner** integral is obtained from the curve where the strip starts i.e. $r = r_1(\theta)$ and the **upper limit** is obtained from the curve where the strip ends i.e. $r = r_2(\theta)$.

(4). The **lower limit** of **outer** integral is obtained from the angle where the region starts i.e. $\theta = \theta_1$ and the **upper limit** is obtained from the angle where the region ends i.e. $\theta = \theta_2$.

(5). Hence, we get the double integral

$$\iint f(r,\ \theta)\, dr\, d\theta = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\ \theta)\, dr\, d\theta.$$

(6). Calculate the example as the example of double integrals.

METHOD – 6: D.I. OVER GENERAL REGION IN POLAR COORDINATES

C	**1**	Evaluate $\iint_R r^3 \sin 2\theta\, dr\, d\theta$ over the area bounded in the first quadrant between the circles $r = 2,\ r = 4$. **Answer**: **60**.	
T	**2**	Evaluate $\iint_D r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$. **Answer**: $\mathbf{45\pi/2}$.	

H	**3**	Evaluate $\iint r\sqrt{a^2 - r^2}\ dr\, d\theta$ over upper half of the circle $r = a\cos\theta$. **Answer**: $\{(3\pi - 4)a^3\}/18$.	
C	**4**	Evaluate $\iint r\sin\theta\ dr\, d\theta$ over the area of the cardioid $r = a(1 + \cos\theta)$ above the initial line. **Answer**: $4a^2/3$.	
T	**5**	Evaluate integral $\iint_R \frac{r dr d\theta}{\sqrt{a^2 + r^2}}$ over the region R which is one loop of $r^2 = a^2 \cos 2\theta$. **Answer**: $a(4 - \pi)/2$.	W-18 (3)
H	**6**	Find by double integrals, the volume of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $y + z = 2$. **Answer**: 8π.	

❖ AREA BY DOUBLE INTEGRATION IN CARTESIAN COORDINATES:

✓ The area A of a region R in the $xy-$plane bounded by the curves $y = f_1(x),\ y = f_2(x)$ and the lines $x = a$ and $x = b$ is given by:

$$\text{Area} = \iint_R dx\, dy = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx.$$

✓ The area A of a region R in the $xy-$plane bounded by the curves $x = f_1(y),\ x = f_2(y)$ and the lines $y = a$ and $y = b$ is given by:

$$\text{Area} = \iint_R dx\, dy = \int_a^b \int_{f_1(y)}^{f_2(y)} dx\, dy.$$

METHOD – 7: AREA BY DOUBLE INTEGRATION IN CARTESIAN COORDINATES

H	1	Find the area bounded by $x - 2y + 4 = 0,\ x + y - 5 = 0\ \ \&\ \ y = 0$. **Answer**: $\mathbf{27/2}$.	
C	2	Find the area of the region bounded by the curves $y = \sin x,\ \ y = \cos x$ and the lines $x = 0$ and $x = \frac{\pi}{4}$. **Answer**: $\mathbf{\sqrt{2} - 1}$.	S-19 (4)
T	3	Using the double integration, find the area of the region common to the parabolas $y^2 = 8x$ and $x^2 = 8y$. **Answer**: $\mathbf{64/3}$.	
H	4	Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$. **Answer**: $\mathbf{9/2}$.	
C	5	Find the area of the area included between the curve $y^2(2a - x) = x^3$ and its asymptote. **Answer**: $\mathbf{3\pi a^2}$.	

❖ **AREA BY DOUBLE INTEGRATION IN POLAR COORDINATES:**

✓ The area A of a region R bounded by the curves $r = f_1(\theta), r = f_2(\theta)$ and the radii vector $\theta = \theta_1$ and $\theta = \theta_2$ is given by:

$$\text{Area} = \iint\limits_R dx\,dy = \int\limits_{\theta_1}^{\theta_2} \int\limits_{f_1(\theta)}^{f_2(\theta)} r\ dr\,d\theta.$$

METHOD – 8: AREA BY DOUBLE INTEGRATION IN POLAR COORDINATES

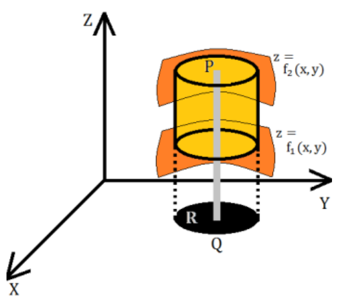
T	1	Find the area common to the circles $r = a$ and $r = 2a\cos\theta$ using double integrals. **Answer**: $\mathbf{a^2\left\{(2\pi/3) - (\sqrt{3}/2)\right\}}$.	
C	2	Find the area common to both of the circles $r = \cos\theta$ and $r = \sin\theta$. **Answer**: $\mathbf{(\pi/8) - (1/4)}$.	

H	3	Find the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$, using the double integration. **Answer**: 3π.	
C	4	Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ by double integration. **Answer**: $2 + (\pi/4)$.	
H	5	Find area outside the circle $r = 2a\cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$. **Answer**: $a^2\{2 - (\pi/4)\}$.	
T	6	Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$. **Answer**: $a^2(5\pi/4 - 2)$.	
H	7	Using double integrals, find area enclosed by the cardioid $r = a(1 + \cos\theta)$. **Answer**: $3a^2\pi/2$.	
C	8	Find the total area enclosed by lemniscate $r^2 = a^2\cos 2\theta$. **Answer**: a^2.	
T	9	Use double integrals in polar to find the area enclosed by three petaled rose $r = \sin 3\theta$. **Answer**: $\pi/4$.	

❖ TRIPLE INTEGRALS OVER GENERAL REGION IN CARTESIAN COORDINATES:

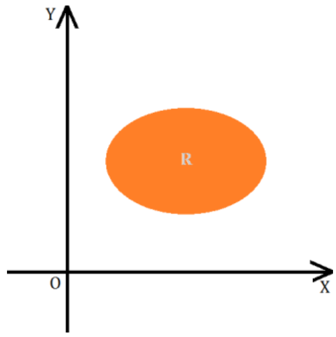
✓ Procedure for evaluating triple integrals over general region in cartesian coordinates:

(1). Suppose the solid region D in three-dimensional space is bounded below by the surface $z = f_1(x, y)$ and bounded above by the surface $z = f_2(x, y)$ as shown in the figure.



(2). Draw a **strip** PQ parallel to z-axis.

(3). According to the strip **lower limit** of **inner integral** is obtained from the curve where the strip starts i.e., $z = f_1(x, y)$ and the **upper limit** is obtained from the curve where the strip ends i.e., $z = f_2(x, y)$.



(4). Draw a **separate figure** showing region R in xy-plane as shown in the figure.

(5). By Fubini's theorem **find limits** for both the inner integrals.

(6). Calculate the example as the example of triple integrals.

METHOD – 9: T.I. OVER GENERAL REGION IN CARTESIAN COORDINATES

C	**1**	Find the volume of tetrahedron bounded by the plane $x + y + z = 2$ and the planes $x = 0,\ y = 0$ and $z = 0$. **Answer**: **4/3**.	
H	**2**	Evaluate $\iiint x^2\, y\, z\ dx\, dy\, dz$ over the region bounded by the planes $x = 0$, $y = 0,\ z = 0$ and $x + y + z = 1$. **Answer**: **1/2520**.	

❖ TRIPLE INTEGRALS OVER GENERAL REGION IN CYLINDRICAL COORDINATES:

✓ Cylindrical coordinates r, θ, z are used to evaluate the integral for the region which is bounded by cylinder along z-axis, planes passing through z-axis and planes perpendicular to the z-axis.

✓ Procedure for evaluating the triple integrals over general region in cylindrical coordinates:

(1). Substitute $x = r\cos\theta, y = r\sin\theta$ & $z = z$.

$$\iiint f(x, y, z)dx\, dy\, dz = \iiint f(r\cos\theta\,, r\sin\theta\,, z)\ |J|\ dr\, d\theta\, dz.$$

(2). Calculate Jacobian J, where $J = \dfrac{\partial(x,\ y,\ z)}{\partial(r,\ \theta,\ z)} = r.$

$$\iiint f(x,y,z)dx\,dy\,dz == \iiint f(r\cos\theta\,, r\sin\theta\,, z)\; r\,dr\,d\theta\,dz.$$

METHOD – 10: T.I. OVER GENERAL REGION IN CYLINDRICAL COORDINATES

T	**1**	Evaluate $\iiint xyz\ dx\,dy\,dz$ over the region bounded by the planes $x = 0$, $y = 0, z = 0, z = 1$ and the cylinder $x^2 + y^2 = 1$. **Answer**: $\mathbf{1/16}$.	
C	**2**	Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 3$ and $z = 0$. **Answer**: $\mathbf{12\pi}$.	
H	**3**	Use triple integration to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ between the planes $z = 1$ and $x + z = 1$. **Answer**: $\mathbf{0}$.	

❖ **TRIPLE INTEGRALS OVER GENERAL REGION IN SPHERICAL COORDINATES:**

✓ Spherical coordinates r, θ, ϕ are used to evaluate the integral for the region bounded by the

(1). Sphere with center at origin.

(2). Cone with vertices at origin and z-axis is as an axis of cone.

✓ Procedure for evaluating the triple integrals in spherical coordinates:

(1). Substitute $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi\ \&\ z = r\cos\theta$.

$$\iiint f(x,y,z)dx\,dy\,dz = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\ |J|\ dr\,d\theta\,d\phi.$$

(2). Calculate Jacobian J, where $J = \dfrac{\partial(x,\ y, z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

$$\iiint f(x,y,z)dx\,dy\,dz = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\ r^2\sin\theta\ dr\,d\theta\,d\phi.$$

✓ **Remark:** When the given region is sphere $x^2 + y^2 + z^2 = a^2$ with centre $(0,0,0)$ and radius a, then limits of r, θ, ϕ are as follows:

Integrated region	**r**	**θ**	**ϕ**
For Positive octant	$r = 0$ to $r = a$	$\theta = 0$ to $\theta = \frac{\pi}{2}$	$\phi = 0$ to $\phi = \frac{\pi}{2}$
For Hemisphere	$r = 0$ to $r = a$	$\theta = 0$ to $\theta = \frac{\pi}{2}$	$\phi = 0$ to $\phi = 2\pi$
For Complete Sphere	$r = 0$ to $r = a$	$\theta = 0$ to $\theta = \pi$	$\phi = 0$ to $\phi = 2\pi$

✓ Procedure for evaluating triple integrals:

(1). Draw given curves and **identify the region** of integration.

(2). **Draw a volume** for parallel to axis (y-axis or x-axis).

(3). The **lower limit** for z is the starting point of the volume and **upper limit** is the point where it ends.

(4). Draw the region **of projection** in any plane (xy, zx or yz plane).

(5). Follow the steps of double integration to find the limit of x and y (z, x or y & z).

METHOD – 11: T.I. OVER GENERAL REGION IN SPHERICAL COORDINATES

C	**1**	Evaluate $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = 4$. **Answer**: **4/3**.	W-19 (7)
T	**2**	Evaluate $\iiint_E 2x\,dV$, where E is the region under the plane $2x + 3y + z = 6$ that lies in first octant. **Answer**: **9**.	

❖ **JACOBIAN:**

✓ If u and v be the function of two independent variables x and y then the Jacobian of u, v with respect to x, y is denoted by $J\left(\frac{u,\ v}{x,\ y}\right)$ or $\frac{\partial(u,\ v)}{\partial(x,\ y)}$ or $J(u,\ v)$ and it is defined as:

$$J = J\left(\frac{u,\ v}{x,\ y}\right) = \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Same way we can define $J = J\left(\frac{x,\ y}{u,\ v}\right) = \frac{\partial(x,\ y)}{\partial(u,\ v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$

If u, v and w are the functions of three independent variables x, y and z then the Jacobian of u, v, w W. R. T. x, y, z is denoted by $J\left(\frac{u,\ v,\ w}{x,\ y,\ z}\right)$ or $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ and it is defined as:

$$J = J\left(\frac{u,\ v,\ w}{x,\ y,\ z}\right) = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}.$$

Same way we can define $J = J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right) = \frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}.$

❖ RESULTS:

If $J = \frac{\partial(u,\ v)}{\partial(x,\ y)}$ then $J' = \frac{\partial(x,\ y)}{\partial(u,\ v)}$ **or** If $J = \frac{\partial(x,\ y)}{\partial(u,\ v)}$ then $J' = \frac{\partial(u,\ v)}{\partial(x,\ y)}$.

If $J = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ then $J' = \frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}$ **or** If $J = \frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}$ then $J' = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$.

Multiplication of Jacobian J and J' is 1. **or** $J \cdot J' = 1$.

❖ CHAIN RULE OF JACOBIAN:

✓ If u, v are functions of r, s and r, s are functions of x, y then the Jacobian of u, v with respect to x, y is given by the following chain rule:

$$J = J\left(\frac{u,\ v}{x,\ y}\right) = \frac{\partial(u,\ v)}{\partial(x,\ y)} = \frac{\partial(u,\ v)}{\partial(r,\ s)} \cdot \frac{\partial(r,\ s)}{\partial(x,\ y)}.$$

✓ If u, v, w are functions of r, s, t and r, s, t are functions of x, y, z then the Jacobian of u, v, w with respect to x, y, z is given by the following chain rule:

$$J = J\left(\frac{u,\ v,\ w}{x,\ y,\ z}\right) = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \frac{\partial(u,\ v,\ w)}{\partial(r,\ s,\ t)} \cdot \frac{\partial(r,\ s,\ t)}{\partial(x,\ y,\ z)}.$$

METHOD – 12: JACOBIAN

H	1	Find $J = \dfrac{\partial(u,\ v)}{\partial(x,\ y)}$ where $u = x^2 - y^2, v = 2xy$. **Answer**: $\mathbf{4(x^2 + y^2)}$.	
H	2	If $x = a(u+v),\ \ y = b(u-v)$ and $u = r^2\cos 2\theta, v = r^2 \sin 2\theta$, find $\dfrac{\partial(x,\ y)}{\partial(r,\ \theta)}$. **Answer**: $\mathbf{-8abr^3}$.	
H	3	If $v = 2xy,\ \ u = x^2 - y^2$ and $x = r\cos\theta,\ \ y = r\sin\theta$, find $\dfrac{\partial(u,\ v)}{\partial(r,\ \theta)}$. **Answer**: $\mathbf{4r^3}$.	
H	4	If $x = u\cos v,\ \ y = u\sin v$ then prove that $\dfrac{\partial(u,\ v)}{\partial(x,\ y)} \cdot \dfrac{\partial(x,\ y)}{\partial(u,\ v)} = 1$.	
T	5	If $x = r\cos\theta$ and $y = r\sin\theta$, show that $J \cdot J' = 1$.	
C	6	If $x = e^v \sec u$ & $y = e^v \tan u$, find the values of J and J' to show that $J \cdot J' = 1$. **Answer**: $\mathbf{-xe^v}$ & $\mathbf{-1/xe^v}$.	
H	7	The transformation from $r\theta z$ – space to xyz – space is given by the equation $x = r\cos\theta,\ y = r\sin\theta,\ z = z$, find $J(r,\ \theta,\ z)$. **Answer**: $\mathbf{1/r}$.	
C	8	If $x = \rho\sin\emptyset\cos\theta,\ \ y = \rho\sin\emptyset\sin\theta,\ \ z = \rho\cos\emptyset$ then obtain Jacobian $J = \dfrac{\partial(x,\ y,\ z)}{\partial(\rho,\ \emptyset,\ \theta)}$. **OR** Find the Jacobian of transformation $x = \rho\sin\emptyset\cos\theta,\ \ y = \rho\sin\emptyset\sin\theta$, $z = \rho\cos\emptyset$. **Answer**: $\mathbf{\rho^2 \sin\emptyset}$.	
C	9	If $x = \sqrt{vw},\ \ y = \sqrt{wu},\ \ z = \sqrt{uv}$ and $u = r\sin\theta\cos\emptyset,\ v = r\sin\theta\sin\emptyset, w = r\cos\theta$ find $\dfrac{\partial(x,\ y,\ z)}{\partial(r,\ \theta,\ \emptyset)}$. **Answer**: $\mathbf{(r^2/4)\sin\theta}$.	

C	**10**	If $x+y+z=u,\ \ y+z=uv,\ \ \ z=uvw,$ find $\dfrac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}$. **Answer**: $\mathbf{u^2v}$.	

❖ DOUBLE INTEGRALS BY CHANGE OF VARIABLE OF INTEGRATION IN CARTESIAN COORDINATES:

The variables x, y in $\iint\limits_R h(x,y)\,dx\,dy$ are changed into u and v with the relations $x=f(u,v)$ and $y=g(u,v)$ then the integral is transformed into $\iint\limits_G h\{f(u,v),g(u,v)\}\,|J|\,du\,dv.$

- ✓ where,

$$J=\frac{\partial(x,\ y)}{\partial(u,\ v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$$ and G is the region in uv − plane corresponding to the region R in xy-plane.

- ✓ Procedure for evaluating double integrals by change of variable of integration:

(1). Find **x and y** in the form of u and v using given relations.

(2). Find **Jacobian** $J(x,y)$.

(3). Draw the **region** in xy-plane and from that draw the region in uv-plane.

(4). Find **limit points** for u and v.

(5). Replace the **area element** dx dy by | J |du dv. Hence, we get

$$\iint\limits_R h(x,y)\,dx\,dy=\iint\limits_G h\{f(u,v),g(u,v)\}\,|\,J\,|\,du\,dv.$$

(6). Calculate the example as example of double integrals.

METHOD – 13: D.I. BY CHANGE OF VARIABLE IN CARTESIAN COORDINATES

H	1	Evaluate $\iint_R (x+y)^2\,dx\,dy$ by changing variables, where R is the region bounded by $x+y=0,\ x+y=1,\ 2x-y=0\ \&\ 2x-y=3$ using transformations $u=x+y\ \&\ v=2x-y$. **Answer**: $\mathbf{1/3}$.	
C	2	Evaluate the integral $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx$ by changing the variables $x+y=u$ and $y=uv$. **Answer**: $\mathbf{(e-1)/2}$.	
T	3	Evaluate $\iint_R (x^2+y^2)\,dA$ by changing the variables, where R is the region lying in the first quadrant and bounded by the hyperbolas $x^2-y^2=9$, $x^2-y^2=1,\ xy=2$ and $xy=4$. **Answer**: $\mathbf{8}$.	
H	4	Evaluate $\iint_R (y-x)\,dx\,dy$ over the region R in XY − plane bounded by the straight lines $y=x-3,\ y=x+1,\ 3y+x=5\ \&\ 3y+x=7$. **Answer**: $\mathbf{-2}$.	
C	5	Evaluate $\iint_R (x+y)\,dA$, where R is the trapezoidal region with vertices $(0,0),(5,0),\left(\frac{5}{2},\frac{5}{2}\right)\ \&\ \left(\frac{5}{2},-\frac{5}{2}\right)$ using the transformations $x=2u+3v$ and $y=2u-3v$. **Answer**: $\mathbf{125/4}$.	
T	6	Evaluate $\int_0^4\int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$ by applying transformations $u=\frac{2x-y}{2}\ \&\ v=\frac{y}{2}$. Draw both the regions. **Answer**: $\mathbf{2}$.	

T	7	Given that $x + y = u,\ y = uv$. Evaluate by change the variables to u, v in the integral $\iint [\,xy\,(1-x-y)\,]^{\frac{1}{2}}\,dx\,dy$ taken over the area of the triangle with sides $x = 0,\ y = 0\ \&\ x + y = 1$. **Answer**: $2\pi/105$.	
H	8	Evaluate $\iint_R x\,dx\,dy$, where R is the region bounded by triangle with vertices at $(0,0), (1,0)$ and $(1,1)$, using the transformations $x = u$ and $y = uv$. **Answer**: $1/3$.	
C	9	Evaluate $\iint_R (x^2 - y^2)^2\,dA$ over the area bounded by lines $\lvert x\rvert + \lvert y\rvert = 1$ using the transformations $x + y = u$ and $x - y = v$. **Answer**: $2/9$.	

❖ DOUBLE INTEGRALS BY CHANGE OF VARIABLE IN POLAR COORDINATES:

✓ Procedure for evaluating double integrals by changing variables form cartesian to polar coordinates:

(1). **Substitute** $x = r\cos\theta$ and $y = r\sin\theta$.

(2). Find **Jacobian** $J = r$.

(3). **Replace** $dx\,dy$ by $\lvert J\rvert\,dr\,d\theta$.

(4). Find **r and θ** in terms of x and y.

(5). Draw the **region**. Hence, we get

$$\iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

METHOD – 14: D.I. BY CHANGE OF VARIABLE IN POLAR COORDINATES

C	1	By changing into polar co – ordinates, evaluate $\int_0^1\int_0^1 dx\,dy$. **Answer**: 1.	

H	2	By changing into polar coordinates, evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$. Also sketch the regions. **Answer**: $\boldsymbol{\pi/4}$.	
C	3	Evaluate the integral $\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy$. **Answer**: $\boldsymbol{\pi/4}$.	S-19 (3)
T	4	Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$ by changing into polar coordinates. **Answer**: $\boldsymbol{2\pi}$.	W-18 (4)
T	5	Change into polar co − ordinates, evalute $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dy\,dx$. **Answer**: $\boldsymbol{\left(1-e^{-a^2}\right)\pi/4}$.	
H	6	Evaluate the intrgral $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{(x^2+y^2)}\,dy\,dx$ by changing into polar co − ordinates. **Answer**: $\boldsymbol{a^5\pi/20}$.	
C	7	By changing into polar co−ordinates, evaluate the integral $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$. **Answer**: $\boldsymbol{3\pi a^4/4}$.	
H	8	By change into polar coordinates, evaluate $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$. **Answer**: $\boldsymbol{a^2\pi}$.	

H	9	Evaluate the integral by change into the polar coordinates. $\int_0^1 \int_0^x \sqrt{x^2+y^2}\ dy\,dx.$ **Answer**: $\left\{\frac{1}{\sqrt{2}}+\log\left(1+\sqrt{2}\right)\right\}/6.$	
T	10	By transforming into polar coordinates, evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\ dxdy.$ **Answer**: $a\pi/4.$	
C	11	Integrate $f(x,y)=\frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}$ over the region $1 \le x^2+y^2 \le e$ by changing into polar coordinates. **Answer**: $2\pi(2-\sqrt{e}).$	
H	12	By transforming into polar co − ordinates, find $\iint \frac{x^2y^2}{x^2+y^2}\ dx\,dy$ over the annular region between the circles $x^2+y^2=a^2$ & $x^2+y^2=b^2 (a<b).$ **Answer**: $\pi(b^4-a^4)/16.$	
H	13	Evaluate $\iint_R (x^2+y^2)\ dA$, where R is the annular region between the two circles $x^2+y^2=1$ and $x^2+y^2=5$, by changing into polar coordinate. **Answer**: $12\pi.$	
T	14	Evaluate the integral $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\ dx\,dy$ by transforming into polar co − ordinates. **Answer**: $8a^2\{(\pi/2)-(5/3)\}.$	

❖ TRIPLE INTEGRALS BY CHANGE OF VARIABLE OF INTEGRATION:

✓ Procedure for evaluating triple integrals by change of variable:

(1). Find **x, y, and z** in terms of u, v and w by transformation.

(2). Find **Jacobian** $J(u, v, w)$.

(3). Find **limits** for u, v and w.

(4). The **new integral** form is

$$\iiint f(x, y, z) dx\, dy\, dz = \iiint f(g_1, g_2, g_3)|J|\, du\, dv\, dw,$$

where $g_1 = g_1(u, v, w), \quad g_2 = g_2(u, v, w), \quad g_3 = g_3(u, v, w)$.

(5). Calculate the example as the example of triple integrals.

METHOD – 15: T.I. BY CHANGE OF VARIABLE OF INTEGRATION

H	**1**	Change into spherical polar coordinate and hence evaluate $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} dz\, dy\, dx.$$ **Answer**: $\pi^2 a^2$.	
C	**2**	Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\ dz\, dy\, dx$. **Answer**: $8\pi/3$.	

★ ★ ★ ★ ★ ★ ★ ★ ★ ★

UNIT 6

❖ ELEMENTARY ROW TRANSFORMATION/OPERATION ON A MATRIX:

- ✓ R_{ij} or $R_i \leftrightarrow R_j$ means interchange of i^{th} and j^{th} rows.
- ✓ $k \cdot R_i$ means multiplication of all the elements of i^{th} row by non-zero scalar k.
- ✓ $R_{ij}(k)$ or $R_j + k \cdot R_i$ means multiplication of all the elements of i^{th} row by nonzero scalar k and add it into j^{th} row.

❖ ROW ECHELON FORM OF MATRIX:

- ✓ Procedure to convert the matrix into Row Echelon Form is as follow:

  (1). Every zero row of the matrix occurs below the non-zero rows.

  (2). Arrange all the rows in decreasing order.

  (3). Make all the entries zero below the leading (first non-zero entry of the row) element of 1^{st} row.

  (4). Repeat step-3 for each row.

❖ REDUCED ROW ECHELON FORM OF MATRIX:

- ✓ Procedure to convert the matrix into Reduced Row Echelon Form is as follow:

  (1). Convert the given matrix into Row Echelon Form.

  (2). Make all the leading elements 1(one).

  (3). Make all the entries zero above the leading element 1(one) of each row.

❖ RANK OF A MATRIX:

- ✓ Let A be an $m \times n$ matrix. Then the positive integer r is said to be the rank of A if

  (1). There is at least one minor of order r which is non-zero.

  (2). Every $(r + 1)^{th}$ order minor of A is zero.

- ✓ Rank of matrix A is denoted by $\rho(A)$ **OR** $\text{Rank}(A)$.

❖ NOTES:

- ✓ $\rho(AB) \le \rho(A)$ **OR** $\rho(AB) \le \rho(B)$.

- ✓ $\rho(A) = \rho(A^T)$.
- ✓ For a matrix A of order $m \times n$, $\rho(A) \le \min\{m, n\}$.
- ✓ Procedure to find the Rank of given matrix is as follow:
  - (1). Reduced the given matrix in Row Echelon Form.
  - (2). Number of non-zero raw in Row-Echelon Form is the Rank of given matrix.

METHOD – 1: ECHELON FORM AND RANK OF MATRIX

C	**1**	Determine whether the following matrices are in Row-Echelon Form, Reduced Row-Echelon Form, both or none. $(1).\,[3]\quad (2).\,[1\quad 2]\quad (3).\begin{bmatrix}0&0\\0&0\end{bmatrix}\quad (4).\begin{bmatrix}1&1&0\\0&1&0\\0&0&0\end{bmatrix}\quad (5).\begin{bmatrix}1&0&0\\0&1&0\\0&2&0\end{bmatrix}$ $(6).\begin{bmatrix}1&2&3\\0&0&0\\0&0&1\end{bmatrix}\quad (7).\begin{bmatrix}1&0&0&4\\0&1&0&7\\0&0&1&-1\end{bmatrix}\quad (8).\begin{bmatrix}1&4&-3&7\\0&1&6&2\\0&0&1&5\end{bmatrix}$ $(9).\begin{bmatrix}0&1&2&6&0\\0&0&1&-1&0\\0&0&0&0&1\end{bmatrix}\quad (10).\begin{bmatrix}0&1&-2&0&1\\0&0&0&1&3\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}.$ **Answer**: **(1). Echelon Form (2). & (3). Both (4). Echelon Form (5). None (6). None (7). Both (8). & (9). Echelon Form (10). Both.**	
H	**2**	Is the matrix $\begin{bmatrix}1&0&4&6\\0&1&5&7\\0&0&0&1\end{bmatrix}$ in Row Echelon Form or Reduced Row Echelon Form? **Answer**: **Echelon Form.**	
C	**3**	Find Reduced Row Echelon Form of $A = \begin{bmatrix}2&4&3&4\\1&2&1&3\\1&3&2&2\\3&7&4&8\end{bmatrix}$. **Answer**: $\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{3}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{-2}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}$.	

H	**4**	Find Reduced Row Echelon Form of $A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$. **Answer**: $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{3} & \mathbf{0} & \mathbf{7} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} \end{bmatrix}$.	
T	**5**	Convert $\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$ into its equivalent Reduced Row Echelon Form. **Answer**: $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} & \mathbf{4} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1/3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.	
C	**6**	Reduce matrix $A = \begin{bmatrix} 1 & 5 & 3 & -2 \\ 2 & 0 & 4 & 1 \\ 4 & 8 & 9 & -1 \end{bmatrix}$ to Row Echelon Form and find it's Rank. **Answer**: $\begin{bmatrix} \mathbf{1} & \mathbf{5} & \mathbf{3} & \mathbf{-2} \\ \mathbf{0} & \mathbf{-10} & \mathbf{-2} & \mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{-3/5} & \mathbf{1} \end{bmatrix}$ & $\boldsymbol{\rho(A) = 3}$.	W-18 (3)
H	**7**	Find the Rank of the following matrices: $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 4 & 8 & 12 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$, $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 2 & 4 & 0 \end{bmatrix}$ & $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$. **Answer**: **1, 1, 2, 2, 2, 3, 3 & 3.**	
T	**8**	Define the rank of a matrix and find the rank of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$. **Answer**: $\boldsymbol{\rho(A) = 3}$.	S-19 (3)

C	**9**	Convert the following matrix in to Reduced Row Echelon Form and hence find the Rank of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$. **Answer**: $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1/2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ & $\boldsymbol{\rho(A) = 2}$.	
H	**10**	Find the Rank of the matrix $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$. **Answer**: $\boldsymbol{\rho(A) = 2}$.	
H	**11**	Find the Rank of the matrix $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$. **Answer**: $\boldsymbol{\rho(A) = 2}$.	
H	**12**	Find the Rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. **Answer**: $\boldsymbol{\rho(A) = 2}$.	
T	**13**	Find the Rank of the matrix $A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$. **Answer**: $\boldsymbol{\rho(A) = 2}$.	
C	**14**	Find the Rank of the matrix $A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$. **Answer**: $\boldsymbol{\rho(A) = 2}$.	
H	**15**	Obtain the Reduced Row Echelon Form of the matrix $A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 8 \end{bmatrix}$ and hence find the Rank of the matrix A. **Answer**: $\boldsymbol{\rho(A) = 3}$.	
C	**16**	Find the Rank of $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix}$ in terms of determinants. **Answer**: $\boldsymbol{\rho(A) = 2}$.	

H	**17**	Find the Rank of the matrix $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 2 & 4 & 0 \end{bmatrix}$ in terms of determinants. **Answer**: $\rho(A) = 3$.	

❖ MATRIX EQUATION:

✓ Let us consider **system** of m linear equations with n unknowns(variables):

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

✓ The matrix representation of above system is $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{mxn}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}_{mx1} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{nx1}.$$

✓ The matrix A is called the co-efficient matrix of above system.

✓ The matrix X is called the variable matrix of above system.

✓ The matrix B is called the constant matrix of above system.

❖ AUGMENTED MATRIX:

✓ The augmented matrix of the above linear system is defined as follow:

$$[A|B] \textbf{ OR } [A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \vdots & b_2 \\ a_{31} & a_{32} & a_{33} & & a_{3n} & \vdots & b_3 \\ & \vdots & & \ddots & & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}_{mx(n+1)}.$$

❖ CONSISTENT AND INCONSISTENT SYSTEM:

✓ If the system of linear equations $AX = B$ has solution (unique or infinitely many), then the system is called **consistent** system and if system has no solution, then the system is called **inconsistent** system.

- In Gauss Elimination and Gauss-Jordan method if $\rho(A) = \rho(A : B)$ then system is consistent otherwise it is inconsistent.

❖ HOMOGENEOUS AND NON-HOMOGENEOUS SYSTEM

- A linear system $AX = B$ is called **Homogeneous** linear system if matrix $\mathbf{B} = \mathbf{0}$ (null or zero column matrix). Otherwise it is called **Non-homogeneous** linear system.
- Non-homogeneous linear system (i. e. $B \neq \mathbf{0}$) has three types of solutions.

  (1). No solution, (2). Unique solution & (3). Infinitely many solutions.
- Homogeneous linear system (i. e. $B = \mathbf{0}$) has two types of solutions.

  (1). Unique solution (trivial solution: (0, 0,…,0))

  (2). Infinitely many solutions (non-trivial solution)
- For the linear system in which no. of equations = no. of unknown (square system),

  No solution $\Longrightarrow$ $[0 \quad 0 \quad \ldots \quad 0 \quad : \quad \#]$

  Unique solution $\Longrightarrow$ $[0 \quad 0 \quad \ldots \quad \# \quad : \quad 0]$ or $[0 \quad 0 \quad \ldots \quad \# \quad : \ast]$

  Infinite solutions $\Longrightarrow$ $[0 \quad 0 \quad \ldots \quad 0 \quad : \quad 0]$
- Solution by rank:

  $\rho(A) = \rho(A : B) = n \rightarrow$ Unique solution,

  $\rho(A) = \rho(A : B) < n \rightarrow$ Infinitely many solution,

  $\rho(A) \neq \rho(A : B) \rightarrow$ No solution, where $n =$ no. of unknowns.

❖ NOTES:

- In the case of infinitely many solutions:
  - The number of free variables $= n - \rho(A)$, where $n =$ no. of unknowns.
- If the Non-Homogeneous system of equations $AX = B$ is square system (no. of unknown = no. of equations) then find the determinant of co-efficient matrix A.
  - If $\det(A) \neq 0$, then the system of linear equations has unique solution.
  - If $\det(A) = 0$, then the system of linear equations has either no solution or infinitely many solutions.

- ✓ If the homogeneous system of equations $AX = \mathbf{0}$ is square system (no. of unknown = no. of equations) then find the determinant of co-efficient matrix A.
  - ➢ If $\det(A) \neq 0$, then the system of linear equations has unique solution It means trivial solution.
  - ➢ If $\det(A) = 0$, then the system of linear equations has infinitely many solutions.

❖ NOTE:

- ✓ Since the homogeneous system $AX = \mathbf{0}$ contains trivial solution, this system is always consistent.

❖ GAUSS ELIMINATION METHOD TO SOLVE SYSTEM OF LINEAR EQUATIONS:

- ✓ Procedure to solve the given system of linear equations using Gauss Elimination Method:

  (1). Start with augmented matrix $[A : B]$.

  (2). Convert $[A : B]$ into Row Echelon Form with leading element of each row is one (1).

  (3). Apply back substitution for getting equations.

  (4). Solve the equations and find the unknown variables (i.e. solution).

METHOD – 2: GAUSS ELIMINATION METHOD

C	**1**	Using Gauss Elimination method solve the following system $-x + 3y + 4z = 30,\ \ 3x + 2y - z = 9,\ \ 2x - y + 2z = 10.$ **Answer**: $(\mathbf{2}, \mathbf{4}, \mathbf{5})$.	W-19 (7)
H	**2**	Solve by using Gauss Elimination Method: $(\mathbf{1}).\, x_1 + x_2 + 2x_3 = 8,\ \ -x_1 - 2x_2 + 3x_3 = 1,\ \ 3x_1 - 7x_2 + 4x_3 = 10.$ $(\mathbf{2}).\, x_1 + 2x_2 + 3x_3 = 4,\ \ 2x_1 + 5x_2 + 3x_3 = 5,\ \ x_1 + 8x_3 = 9.$ **Answer**: $(\mathbf{1}).\,(\mathbf{3}, \mathbf{1}, \mathbf{2})$ & $(\mathbf{2}).\,(\mathbf{1}, \mathbf{0}, \mathbf{1})$.	
H	**3**	Solve the given system of linear equations by using Gauss Elimination Method: $2x + y - z = 4,\ \ x - y + 2z = -2,\ \ -x + 2y - z = 2.$ **Answer**: $(\mathbf{1}, \mathbf{1}, -\mathbf{1})$.	

H	**4**	Solve system of linear equations by using Gauss Elimination method, if solution exists. $x+y+2z=9,\ \ 2x+4y-3z=1,\ \ 3x+6y-5z=0.$ **Answer**: $(\mathbf{1},\mathbf{2},\mathbf{3})$.	W-18 (4)
C	**5**	Solve by using Gauss Elimination Method: $-\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30,\ \ \frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9,\ \ \frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10.$ **Answer**: $(\mathbf{1/2}\,,\mathbf{1/4}\,,\mathbf{1/5})$.	
T	**6**	Solve the given system of linear equations by using Gauss Elimination Method: $x+y+z=3,\ \ x+2y-z=4,\ \ x+3y+2z=4.$ **Answer**: $(\mathbf{13/5},\mathbf{3/5},-\mathbf{1/5})$.	
T	**7**	Solve the given system of linear equations using Gauss Elimination Method: $2x_1+x_2+2x_3+x_4=6,\ \ 6x_1-x_2+6x_3+12x_4=36,$ $4x_1+3x_2+3x_3-3x_4=1,\ \ 2x_1+2x_2-x_3+x_4=10.$ **Answer**: $(\mathbf{7/4},\mathbf{3/2},-\mathbf{5/6},\mathbf{8/3})$.	
C	**8**	Solve by using Gauss Elimination Method: $-2b+3c=1,\ \ 3a+6b-3c=-2,\ \ 6a+6b+3c=5.$ **Answer**: **No solution.**	
H	**9**	Solve by using Gauss Elimination Method: **(1).** $-2y+3z=1,\ \ 3x+6y-3z=-2,\ \ 6x+6y+3z=5.$ **(2).** $3x+y-3z=13,\ \ 2x-3y+7z=5,\ \ 2x+19y-47z=32.$ **Answer**: **No solution for each system.**	
C	**10**	Solve the given system of linear equations by using Gauss Elimination Method: $x_1-2x_2+3x_3=-2,\ \ -x_1+x_2-2x_3=3,\ \ 2x_1-x_2+3x_3=-7.$ **Answer**: $\{(-\mathbf{4}-\mathbf{t},\mathbf{t}-\mathbf{1},\mathbf{t})/\mathbf{t}\in\mathbb{R}\}$.	

C	11	Solve by using Gauss Elimination Method: $x_1 + 3x_2 - 2x_3 + 2x_5 = 0,\ \ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1,$ $5x_3 + 10x_4 + 15x_6 = 5,\ \ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6.$ **Answer**: $\mathbf{x_1 = -3r - 4s - 2t,\ \ x_2 = r,\ \ x_3 = -2s,\ \ x_4 = s,\ \ x_5 = t,}$ $\mathbf{x_6 = 1/3;\ \text{where}\ r, s, t \in \mathbb{R}.}$	
H	12	Solve by using Gauss Elimination Method: $x_1 - 2x_2 - x_3 + 3x_4 = 1, 2x_1 - 4x_2 + x_3 = 5,\ \ x_1 - 2x_2 + 2x_3 - 3x_4 = 4.$ **Answer**: $\mathbf{x_1 = 2 + 2t_2 - t_1, x_2 = t_2, x_3 = 1 + 2t_1, x_4 = t_1,\ \ t_1, t_2 \in \mathbb{R}.}$	
T	13	Solve by using Gauss Elimination Method: **(1).** $5x + 3y + 7z = 4,\ \ 3x + 26y + 2z = 9,\ \ 7x + 2y + 10z = 5.$ **(2).** $2x_1 + 2x_2 + 2x_3 = 0,\ \ -2x_1 + 5x_2 + 2x_3 = 1,\ \ 8x_1 + x_2 + 4x_3 = -1.$ **Answer**: (1). $\{((\mathbf{7 - 16t})/\mathbf{11}, (\mathbf{3 + t})/\mathbf{11}, \mathbf{t})/\mathbf{t} \in \mathbb{R}\}$ & (2). $\{\mathbf{x_1 = (-1 - 3t)/7, x_2 = (1 - 4t)/7, x_3 = t;\ \text{where}\ t} \in \mathbb{R}\}.$	
C	14	The augmented matrix of a linear system has the form $\begin{bmatrix} -2 & 3 & 1 & : & a \\ 1 & 1 & -1 & : & b \\ 0 & 5 & -1 & : & c \end{bmatrix}.$ Determine when the linear system is consistent. Determine when the linear system is inconsistent. Does the linear system have a unique solution or infinitely many solutions? **Answer**: $\mathbf{a + 2b - c = 0,\ \ a + 2b - c \neq 0}$ & **Infintely many solutions.**	
H	15	Determine when the given augmented matrix represents a consistent linear system. $\begin{bmatrix} 1 & 0 & 2 & : & a \\ 2 & 1 & 5 & : & b \\ 1 & -1 & 1 & : & c \end{bmatrix}.$ **Answer**: $\mathbf{b + c - 3a = 0.}$	
H	16	For what choices of parameter λ the following system is consistent? $x_1 + x_2 + 2x_3 + x_4 = 1,\ \ x_1 + 2x_3 = 0,\ \ 2x_1 + 2x_2 + 3x_3 = \lambda,$ $x_2 + x_3 + 3x_4 = 2\lambda.$ **Answer**: **For $\boldsymbol{\lambda = 1}$ given system is consistent.**	

T	17	Determine the values of k, for which the equations $3x-y+2z=1, -4x+2y-3z=k, 2x+z=k^2$ possesses solution. Find solutions in each case. **Answer**: $\mathbf{k=2 \Rightarrow \{((4-t)/2, (10+t)/2, t)/t \in \mathbb{R}\}}$ & $\mathbf{k=-1 \Rightarrow \{((1-t)/2, (1+t)/2, t)/t \in \mathbb{R}\}}$.	
C	18	For which value of λ and k the following system has **(i)** No solution, **(ii)** Unique solution, **(iii)** An infinite no. of solution. $x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=k.$ **Answer**: (i) $\boldsymbol{\lambda=3,\ k\neq 10 \Rightarrow}$ **no solution**, (ii) $\boldsymbol{\lambda\neq 3,\ k\in\mathbb{R}\Rightarrow}$ **unique solution** & (iii) $\boldsymbol{\lambda=3,\ k=10\Rightarrow}$ **infinite solutions.**	
H	19	For which values of 'a' will the following system has **(i)** no solution, **(ii)** unique solution, **(iii)** Infinitely many solutions. $x+2y-3z=4,\ 3x-y+5z=2,\ 4x+y+(a^2-14)z=a+2.$ **Answer**: (i) $\mathbf{a=-4\Rightarrow}$ **no solution**, (ii) $\mathbf{a\in\mathbb{R}-\{-4,4\}\Rightarrow}$ **unique solution** & (iii) $\mathbf{a=4\Rightarrow}$ **infinite solutions.**	
T	20	Determine the value of k so that the system of homogeneous equations $2x+y+2z=0,\ x+y+3z=0,\ 4x+3y+kz=0$ has **(a)** Trivial solution. **(b)** Non-trivial solution. Also find non-trivial solution. **Answer**: (a)**If** $\mathbf{k\neq 8\Rightarrow}$ **trivial solution. i. e.** $\mathbf{(0,0,0)}$ & (b) **If** $\mathbf{k=8\Rightarrow x=k; y=-4k; z=k}$, **where k is a parameter**.	
C	21	Find real value of λ for which the equations $(1-\lambda)x-y+z=0,\ 2x+(1-\lambda)y=0,\ 2y-(1+\lambda)z=0$ have solution other than $x=y=z=0$. Also find the solution for each value of λ. **Answer**: $\boldsymbol{\lambda=1\Longrightarrow x=0;\ y=k;\ z=k}$, **where** $\mathbf{k\in\mathbb{R}}$ **is arbitrary**.	

❖ **GAUSS-JORDAN ELIMINATION METHOD TO SOLVE THE SYSTEM OF LINEAR EQUATIONS:**

✓ Procedure to solve the system of linear equations by Gauss-Jordan Elimination Method:

(1). Start with the augmented matrix $[A : B]$.

(2). Convert [A : B] into reduced row echelon form.

(3). Find the unknowns, which is required solutions.

METHOD – 3: GAUSS - JORDAN METHOD

C	1	Solve the following system of linear equations by Gauss-Jordan Method. $x+y+2z=8,\quad -x-2y+3z=1,\quad 3x-7y+4z=10.$ **Answer**: $(\mathbf{3},\mathbf{1},\mathbf{2})$.	
H	2	Solve by using Gauss-Jordan Method: **(1).** $x+y+z=6,\quad x+2y+3z=14,\quad 2x+4y+7z=30.$ **(2).** $x_1+2x_2+3x_3=1,\quad 2x_1+5x_2+3x_3=6,\quad x_1+8x_3=-6.$ **(3).** $x_1+2x_2+3x_3=4,\quad 2x_1+5x_2+3x_3=5,\quad x_1+8x_3=9.$ **Answer**: **(1).** $(\mathbf{0},\mathbf{4},\mathbf{2})$ **(2).** $(\mathbf{2},\mathbf{1},-\mathbf{1})$ **(3).** $(\mathbf{1},\mathbf{0},\mathbf{1})$.	
H	3	Solve the system of linear equations by using Gauss-Jordan Elimination Method. $x+4y-3z=0,\qquad -x-3y+5z=-3,\qquad 2x+8y-5z=1.$ **Answer**: $(\mathbf{23},-\mathbf{5},\mathbf{1})$.	
H	4	Solve by using Gauss-Jordan Method: **(1).** $2x-y-3z=0,\quad -x+2y-3z=0,\quad x+y+4z=0.$ **(2).** $2x_1+x_2+3x_3=0,\quad x_1+2x_2=0,\quad x_2+x_3=0.$ **Answer**: **Trivial solution** $(\mathbf{0},\mathbf{0},\mathbf{0})$ **for both system**.	
T	5	Solve by using Gauss-Jordan Method, where $0\le\alpha\le2\pi, 0\le\beta\le2\pi, 0\le\gamma\le\pi.$ $2\sin\alpha-\cos\beta+3\tan\gamma=3,\quad 4\sin\alpha+2\cos\beta-2\tan\gamma=2,$ $6\sin\alpha-3\cos\beta+\tan\gamma=9.$ **Answer**: $(\mathbf{1},-\mathbf{1},\mathbf{0})$.	
C	6	Solve by using Gauss-Jordan Method: $-2y+3z=1,\quad 3x+6y-3z=-2,\quad 6x+6y+3z=5.$ **Answer**: **Inconsistent(no solution)**.	

C	**7**	Use Gauss-Jordan algorithm to solve the system of linear equations: $2x_1 + 2x_2 - x_3 + x_5 = 0, \quad -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0,$ $x_1 + x_2 - 2x_3 - x_5 = 0, \; x_3 + x_4 + x_5 = 0.$ **Answer**: $\{(\mathbf{-s-t, s, -t, 0, t})/\ \mathbf{t, s} \in \mathbb{R}\}.$	S-19 (4)
H	**8**	Solve the system of linear equations $v + 3w - 2x = 0, \quad 2u + v - 4w + 3x = 0, \quad 2u + 3v + 2w - x = 0, -4u - 3v + 5w - 4x = 0.$ **Answer**: $\mathbf{u = (7s - 5t)/2, v = 2t - 3s, w = s, x = t,}$ **where** $\mathbf{s, t} \in \mathbb{R}.$	
H	**9**	Solve $x_1 + x_2 + 2x_3 - 5x_4 = 3, \; 2x_1 + 5x_2 - x_3 - 9x_4 = -3,$ $2x_1 + x_2 - x_3 + 3x_4 = -11, x_1 - 3x_2 + 2x_3 + 7x_4 = -5$ using Gauss-Jordan Method. **Answer**: $\{(\mathbf{-5 - 2t, 2 + 3t, 3 + 2t, t})/\mathbf{t} \in \mathbb{R}\}.$	
T	**10**	Solve by using Gauss-Jordan Method: **(1).** $x_1 + 3x_2 + x_4 = 0, \; x_1 + 4x_2 + 2x_3 = 0, \; -2x_2 - 2x_3 - x_4 = 0.$ **(2).** $2x_1 - 4x_2 + x_3 + x_4 = 0, \; x_1 - 2x_2 - x_3 + x_4 = 0.$ **Answer**: $(1).\{(\mathbf{5t, -2t, 3t/2, t})/\mathbf{t} \in \mathbb{R}\},$ $(2).\{((\mathbf{6t_2 - 2t_1})/\mathbf{3, t_2, t_1/3, t_1})/\mathbf{t_1}\ \&\ \mathbf{t_2} \in \mathbb{R}\}.$	
T	**11**	Solve using by Gauss-Jordan Method: **(i).** $3x - y - z = 0, \; x + y + 2z = 0, \; 5x + y + 3z = 0$ & **(ii).** $x + y - z + w = 0, \; x - y + 2z - w = 0, \; 3x + y + w = 0.$ **Answer**: **(i).** $\left\{\left(\mathbf{x = -\frac{1}{4}t, y = -\frac{7}{4}t, z = t,}\ \textbf{where}\ \mathbf{t} \in \mathbb{R}\right)\right\},$ **(ii).** $\left\{\left(\mathbf{x = -\frac{1}{2}t_1, y = \frac{3}{2}t_1 - t_2, z = t_1, w = t_2\,;\ t_1, t_2} \in \mathbb{R}\right)\right\}.$	

❖ INVERSE BY GAUSS - JORDAN METHOD:

✓ Procedure to find the inverse of Matrix by Gauss-Jordan Method is as follow:

(1). Start with augmented matrix $[A : I]$, where I is the identity matrix of the same size of matrix A.

(2). Convert $[A : I]$ into reduced row echelon form.

(3). The inverse of the matrix A is I with respective changes. i.e. $[I : A^{-1}]$.

METHOD – 4: INVERSE BY GAUSS - JORDAN METHOD

C	**1**	Find the inverse of the given matrix by Gauss-Jordan Method $A=\begin{bmatrix}1 & -2\\ 3 & -1\end{bmatrix}$. **Answer**: $\mathbf{A^{-1}=\frac{1}{5}\begin{bmatrix}-1 & 2\\ -3 & 1\end{bmatrix}}$.	
H	**2**	If possible then find the inverse of the given matrix by Gauss-Jordan Method $A=\begin{bmatrix}6 & -4\\ -3 & 2\end{bmatrix}$. **Answer**: **Not possible because** $\mathbf{\det(A)=0}$.	
C	**3**	Find the inverse of matrix $A=\begin{bmatrix}2 & 1 & 3\\ 3 & 1 & 2\\ 1 & 2 & 3\end{bmatrix}$ by Gauss-Jordan Method. **Answer**: $\mathbf{A^{-1}=\frac{1}{6}\begin{bmatrix}-1 & 3 & -1\\ -7 & 3 & 5\\ 5 & -3 & -1\end{bmatrix}}$.	
H	**4**	If possible then find the inverse of matrix by Gauss-Jordan Method $A=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & -1\\ 0 & 1 & 0\end{bmatrix}$. **Answer**: **Not possible because** $\mathbf{\det(A)=0}$.	
H	**5**	Find the inverse of the given matrix by Gauss-Jordan Method $A=\begin{bmatrix}2 & 6 & 6\\ 2 & 7 & 6\\ 2 & 7 & 7\end{bmatrix}$. **Answer**: $\mathbf{A^{-1}=\begin{bmatrix}7/2 & 0 & -3\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}}$.	
T	**6**	Find the inverse of matrix $A=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$ using Gauss-Jordan Method. **Answer**: $\mathbf{\begin{bmatrix}1/2 & -1/2 & 1/2\\ 0 & 0 & 1\\ 1/2 & 1/2 & -1/2\end{bmatrix}}$.	
T	**7**	Find the inverse of matrix $A=\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}$ by Gauss-Jordan Method. **Answer**: $\mathbf{A^{-1}=\frac{1}{2}\begin{bmatrix}1 & -1 & 1\\ -8 & 6 & -2\\ 5 & -3 & 1\end{bmatrix}}$.	

C	8	Using Gauss − Jordan method find A^{-1} for $A=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}$. **Answer**: $A^{-1}=\begin{bmatrix}\mathbf{-40}&\mathbf{16}&\mathbf{9}\\\mathbf{13}&\mathbf{-5}&\mathbf{-3}\\\mathbf{5}&\mathbf{-2}&\mathbf{-1}\end{bmatrix}$.	W-19 (7)
H	9	Find the inverse of matrix $A=\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}$ using row operations. **Answer**: $A^{-1}=\begin{bmatrix}\mathbf{7}&\mathbf{-3}&\mathbf{-3}\\\mathbf{-1}&\mathbf{1}&\mathbf{0}\\\mathbf{-1}&\mathbf{0}&\mathbf{1}\end{bmatrix}$.	
H	10	Find A^{-1} for $A=\begin{bmatrix}0&1&-1\\3&1&1\\1&2&-1\end{bmatrix}$, if exists. **Answer**: $A^{-1}=\begin{bmatrix}\mathbf{3}&\mathbf{1}&\mathbf{-2}\\\mathbf{-4}&\mathbf{-1}&\mathbf{3}\\\mathbf{-5}&\mathbf{-1}&\mathbf{3}\end{bmatrix}$.	
H	11	Find the inverse of $A=\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$ using row operations. **Answer**: $A^{-1}=\begin{bmatrix}\mathbf{1}&\mathbf{-1}&\mathbf{0}\\\mathbf{-2}&\mathbf{3}&\mathbf{-4}\\\mathbf{-2}&\mathbf{3}&\mathbf{-3}\end{bmatrix}$.	
T	12	Find the inverse of following matrices by Gauss-Jordan Method $\begin{bmatrix}1&0&0&0\\1&3&0&0\\1&3&5&0\\1&3&5&7\end{bmatrix}$ & $\begin{bmatrix}1&-1&0&2\\0&1&1&-1\\2&1&2&1\\3&-2&1&6\end{bmatrix}$. **Answer**: $\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{-1/3}&\mathbf{1/3}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{-1/5}&\mathbf{1/5}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{-1/7}&\mathbf{1/7}\end{bmatrix}$ & $\begin{bmatrix}\mathbf{2}&\mathbf{-1}&\mathbf{1}&\mathbf{-1}\\\mathbf{-5}&\mathbf{-3}&\mathbf{1}&\mathbf{1}\\\mathbf{2}&\mathbf{3}&\mathbf{-1}&\mathbf{0}\\\mathbf{-3}&\mathbf{-1}&\mathbf{0}&\mathbf{1}\end{bmatrix}$.	
T	13	Find the inverse of $\begin{bmatrix}1&2&3&1\\1&3&3&2\\2&4&3&3\\1&1&1&1\end{bmatrix}$ using Gauss-Jordan Method. **Answer**: $\begin{bmatrix}\mathbf{1}&\mathbf{-2}&\mathbf{1}&\mathbf{0}\\\mathbf{1}&\mathbf{-2}&\mathbf{2}&\mathbf{-3}\\\mathbf{0}&\mathbf{1}&\mathbf{-1}&\mathbf{1}\\\mathbf{-2}&\mathbf{3}&\mathbf{-2}&\mathbf{3}\end{bmatrix}$.	

❖ EIGENVALUES, EIGENVECTORS & EIGEN SPACE:

- ✓ Let A be an $n \times n$ matrix, then the scalar λ is called an Eigen Value of A, if there exist a nonzero vector X such that $AX = \lambda X$.
- ✓ The non-zero vector X is called an Eigen Vector of A corresponding to λ.
- ✓ The solution space of system $[A - \lambda I]X = O$ is called Eigen Space of matrix A corresponding to λ.

❖ NOTES:

- ✓ Let A be an $n \times n$ matrix then

  (1). An Eigen Value of A is a scalar λ such that $\det(A - \lambda I) = 0$.

  (2). The Eigen Vectors of A corresponding to λ are the non-zero solutions of $(A - \lambda I)\,X = 0$. Here $\det(A - \lambda I) = 0$ is called the Characteristic Equation of A.

- ✓ Eigen value is also known as characteristic root/latent value/proper roots and Eigen Vector is also known as characteristic vector/latent vector/proper vector corresponding to the eigen value λ of the matrix.
- ✓ The set of all eigenvalues of matrix A is called Spectrum of A.
- ✓ The characteristic equation of 2×2 matrix A is $\boldsymbol{\lambda^2 - S_1\lambda + S_2 = 0}$.

  Where, S_1=Sum of the principal diagonal elements (Trace(A)) and S_2= Determinant of A.

- ✓ The characteristic equation of 3×3 matrix A is $\boldsymbol{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$.

  Where S_1= Sum of the principal diagonal elements or Trace(A),

  S_2= Sum of the minors of the principal diagonal elements &

  S_3= Determinant of A.

- ✓ Procedure for finding eigen values, eigen vectors and eigen spaces for 3×3 matrix A.

  (1). Find the characteristic equation $\det(A - \lambda I) = 0$. It will be a polynomial equation of degree 3 in the variable λ.

  (2). Find the roots of the characteristic equation. These are the Eigen Values of A, say $\lambda_1, \lambda_2, \lambda_3$.

(3). For each Eigen Value λ_i, find the Eigen Vectors corresponding to λ_i by solving the Homogeneous system $[A - \lambda_i I] \cdot X = O$. (Here the solution space of this system is Eigen Space corresponding to λ_i).

- ✓ The procedure remains same for the matrices of order 2.

❖ PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

- ✓ Trace(A)=Sum of the eigen values of matrix A.
- ✓ Det(A)=Product of the eigen values of matrix A.
- ✓ The eigen values of triangular matrix are the elements on its principal diagonal.
- ✓ If one of the eigen value is zero then det(A)=0, hence A^{-1} does not exist.
- ✓ The square matrix A and A^T have the same eigen values but eigen vectors need not to be same.
- ✓ If λ is an eigen value of a non-singular matrix A and X is an eigen vector correspond to λ then $\frac{1}{\lambda}$ is an eigen value of A^{-1} and X is an eigen vector correspond to $\frac{1}{\lambda}$.
- ✓ If λ is an eigen value of matrix A and X is an eigen vector correspond to λ then λ^k is an eigen value of A^k and X is an eigen vector.
- ✓ If λ is an eigen value of matrix A then $\lambda \pm k$ is an eigen value of $A \pm kI$.
- ✓ If λ an is eigen value of matrix A then $k\lambda$ is an eigen value of kA.

METHOD – 5: EIGEN VALUES, EIGEN VECTORS AND EIGEN SPACE

C	**1**	Define the eigen value, eigen vector and eigen space.	
C	**2**	Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ then find the eigen values of $A,\ A^{25},\ 3A,\ A^{-1},\ A^T,\ A + 2I\ \ \&\ \ A^3 - 5I$. **Answer**: $\mathbf{-1, 3 \quad -1, 3^{25} \quad -3, 9 \quad -1, 1/3 \quad -1, 3 \quad 1, 5 \quad \& \quad -6, 22}$.	
H	**3**	Find the eigen values of A and A^{-1}, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$. **Answer**: $\mathbf{1, 4 \quad \& \quad 1, 1/4}$.	W-18 (2)

C	**4**	Find the eigen values of $A=\begin{bmatrix}-5 & 4 & 34\\ 0 & 0 & 4\\ 0 & 0 & 4\end{bmatrix}$. Is A invertible? **Answer:** $-5,0,4$ & no.	
H	**5**	If $A=\begin{bmatrix}1 & 3 & 4\\ 0 & 0 & 5\\ 0 & 0 & 9\end{bmatrix}$, then find the eigen values of the matrix A^T. Is A an invertible? **Answer:** $0,1,9$ & no.	
H	**6**	If $A=\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & 3\\ 0 & 0 & 2\end{bmatrix}$ then find the eigen values of A^T and 5A. **Answer:** $1,2,2$ & $5,10,10$.	
H	**7**	Find the eigen values of $A=\begin{bmatrix}1 & 0 & 0\\ 5 & 2 & 0\\ 12 & 15 & 3\end{bmatrix}$ and hence deduce eigen values of A^5 and A^{-1}. **Answer:** $1,2,3$ $\quad 1,2^5,3^5$ & $1,1/2,1/3$.	
H	**8**	Find the eigen values of A^9 for $A=\begin{bmatrix}1 & 3 & 7 & 11\\ 0 & 0.5 & 3 & 8\\ 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 2\end{bmatrix}$. **Answer:** $1,0.5^9,0,2^9$.	
T	**9**	Find the eigen values of the following matrices: (1). $\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ (2). $\begin{bmatrix}5 & 3\\ 1 & 3\end{bmatrix}$ (3). $\begin{bmatrix}10 & -9\\ 4 & -2\end{bmatrix}$ (4). $\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & 3\\ 0 & 0 & 2\end{bmatrix}$ (5). $\begin{bmatrix}4 & 0 & 1\\ -2 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}$ (6). $\begin{bmatrix}3 & 0 & -5\\ 1/5 & -1 & 0\\ 1 & 1 & -2\end{bmatrix}$. **Answer:** (1). $1,2$ (2). $2,6$ (3). $4,4$ (4). $1,2,2$ (5). $1,2,3$ (6). $0,-\sqrt{2},\sqrt{2}$.	
C	**10**	Let $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, then find the eigen values, eigen vectors corresponding to each eigen values of A. Also write the eigen vector space for each eigen values. **Answer:** 1 & -1, $\begin{bmatrix}1\\ 1\end{bmatrix}$ & $\begin{bmatrix}-1\\ 1\end{bmatrix}$, $E_1=\{t\begin{bmatrix}1\\ 1\end{bmatrix}/t\in\mathbb{R}\}$, $E_{-1}=\{t\begin{bmatrix}-1\\ 1\end{bmatrix}/t\in\mathbb{R}\}$.	

H	11	Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. **Answer**: $\mathbf{6, -1}$ & $\mathbf{(1,1)^T, (-2/5, 1)^T}$.	
T	12	Show that if $0 < \theta < \pi$, then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has no real eigen values and consequently no eigen vectors.	
C	13	Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$. **Answer**: $\mathbf{2, 3, 5}$ & $\mathbf{(-1,1,0)^T, (1,0,0)^T, (3,2,1)^T}$.	
H	14	Find the eigen values and basis for eigen space for the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$. **Answer**: $\mathbf{1, 3, 4}$ & $\{\mathbf{(1,2,1)^T, (1,-1,1)^T, (-1,0,1)^T}\}$.	
T	15	Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$, then find eigen values, eigen vectors corresponding to each eigen values of A & write eigen space for each eigen values. **Answer**: $\mathbf{1, 2, 3}$ & $\begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{-1} \\ \mathbf{\frac{1}{2}} \\ \mathbf{1} \end{bmatrix}, \begin{bmatrix} -\mathbf{\frac{1}{2}} \\ \mathbf{\frac{1}{2}} \\ \mathbf{1} \end{bmatrix}$ & $\mathbf{E_1} = \left\{ \mathbf{t} \begin{bmatrix} -1 \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} / \mathbf{t} \in \mathbb{R} \right\}$, $\mathbf{E_2} = \left\{ \mathbf{t} \begin{bmatrix} \mathbf{-1} \\ \mathbf{1/2} \\ \mathbf{1} \end{bmatrix} / \mathbf{t} \in \mathbb{R} \right\}$, $\mathbf{E_3} = \left\{ \mathbf{t} \begin{bmatrix} \mathbf{-1/2} \\ \mathbf{1/2} \\ \mathbf{1} \end{bmatrix} / \mathbf{t} \in \mathbb{R} \right\}$.	
C	16	Find the eigen values and bases for the eigen space of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. **Answer**: $\mathbf{-1, -1, 2}$ & $\{\mathbf{(-1,1,0)^T, (-1,0,1)^T, (1,1,1)^T}\}$.	
H	17	Find the eigen values and corresponding eigen vectors for the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$. **Answer**: $\mathbf{1, 2, 2}$ & $\mathbf{(-1,-1,1)^T, (2,1,0)^T}$.	W-18 (7)

H	**18**	Find the eigen values and bases for the eigen space of the matrix $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 2 & 0\\ -3 & 5 & 2\end{bmatrix}$. **Answer**: $\mathbf{1,2,2}$ & $\{(\mathbf{1,-1,8})^{T},(\mathbf{0,0,1})^{T}\}$.	
T	**19**	Find the eigen values and eigen vectors of $A=\begin{bmatrix}2 & 1 & 2\\ 0 & 2 & -1\\ 0 & 1 & 0\end{bmatrix}$. **Answer**: $\mathbf{1,1,2}$ & $(\mathbf{-3,1,1})^{T},(\mathbf{1,0,0})^{T}$.	
C	**20**	Find the eigen values and corresponding eigen vectors of the matrix $A=\begin{bmatrix}5 & 0 & 1\\ 1 & 1 & 0\\ -7 & 1 & 0\end{bmatrix}$. **Answer**: $\mathbf{2,2,2}$ & $(\mathbf{-1/3,-1/3,1})^{T}$.	
H	**21**	Find the eigen values and eigen vectors of the matrix $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix}$. **Answer**: $\mathbf{1,1,1}$ & $(\mathbf{1,\ 1,\ 1})^{\mathbf{T}}$.	W-19 (4)

❖ ALGEBRAIC AND GEOMETRIC MULTIPLICITY OF AN EIGENVALUE:

- ✓ If eigenvalue of matrix A is repeated n-times, then n is called the Algebraic Multiplicity (A.M.) of λ.

  Example: If $\lambda = 2, 2, 3$ then A.M. of $\lambda = 2$ is 2 and A.M. of $\lambda = 3$ is 1.

- ✓ The number of linearly independent eigenvectors corresponding to λ is called the Geometric Multiplicity (G.M.) of λ. i.e., G.M.= No. of variables$-$Rank $(A - \lambda I)$.
- ✓ $\text{G. M.} \le \text{A. M.}$ for each eigen value.
- ✓ Procedure for finding A.M. and G.M. is as follow:

  (1). Find eigen values of given matrix.

  (2). Find A.M. using definition.

  (3). Find eigen vectors for each eigen values.

  (4). Find G.M. using definition.

METHOD – 6: ALGEBRAIC AND GEOMETRIC MULTIPLICITY

C	1	Explain the algebraic multiplicity and geometric multiplicity with example.	
H	2	Find the geometric and algebraic multiplicity of each eigen values of $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$. **Answer**: $\mathbf{A.M.\ of\ 2 = 2\ \ \&\ \ G.M.\ of\ 2 = 1.}$	
C	3	Find the algebraic and geometric multiplicity of each of eigen value of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. **Answer**: $\mathbf{A.M.(-1\ \&\ 2) = 2\ \&\ 1\ \ \&\ \ G.M.(-1\ \&\ 2) = 2, 1.}$	
C	4	Find A. M. and G. M. for each eigen values of $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$. **Answer**: $\mathbf{A.M.(1) = 3\ \ \&\ \ G.M.(1) = 1.}$	
H	5	Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Also find algebraic and geometric multiplicity of each eigen values. **Answer**: $\mathbf{5, 1, 1,\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ \ A.M. = 1, 2\ \ \&\ \ G.M. = 1, 2.}$	
T	6	Find the eigen values and eigen vectors of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. Also find algebraic and geometric multiplicity of each eigen values. **Answer**: $\mathbf{0, 3, 15,\ \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},\ \ A.M. = 1, 1, 1\ \ \&\ \ G.M. = 1, 1, 1.}$	

❖ **DIAGONALIZATION:**

✓ A matrix A is said to be diagonalizable if it is similar to diagonal matrix.

✓ It means a matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix known as spectral matrix and P is known as modal matrix.

- ✓ An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.
- ✓ If the eigenvalues of an $n \times n$ matrix are all distinct then the matrix is always diagonalizable.
- ✓ The necessary and sufficient condition for a square matrix to be diagonalizable is that the geometric multiplicity of each eigenvalues is same as its algebraic multiplicity.
- ✓ Procedure to check whether the given matrix is diagonalizable or not:
  - (1). Find the eigen values of a given matrix.
  - (2). If all the eigen values are distinct then the given matrix is diagonalizable, otherwise follow step (3).
  - (3). Find A.M. and G.M. for each eigen value.
  - (4). If both the multiplicities are same for all eigen value then the given matrix is diagonalizable.
- ✓ Procedure for finding a matrix P and $P^{-1}AP$ is as follow:
  - (1). Find the eigen values and eigen vectors of a given matrix.
  - (2). Construct matrix P by writing all the eigen vectors as its column vectors.
  - (3). Find P^{-1}.
  - (4). Find $P^{-1}AP$ and verify that $P^{-1}AP$ is diagonal matrix or not. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues on its main diagonal.
- ✓ Note that the order of the eigen vectors used to form P will determine the order in which eigen values appear on the main diagonal of D.
- ✓ $P^{-1}AP = D \Longrightarrow A = PDP^{-1}$.
- ✓ Let A be a diagonalizable matrix and P is an invertible matrix such that $A = PDP^{-1}$ then $A^n = PD^nP^{-1}, \forall n \in \mathbb{N}$.
- ✓ If $D = \mathrm{Diag}[a \ \ b \ \ c]$ then $D^n = \mathrm{Diag}[a^n \ \ b^n \ \ c^n]$.
- ✓ Every diagonal matrix is diagonalizable.

METHOD – 7: DIAGONALIZATION

C	1	Check whether the following matrices are diagonalizable or not: $\begin{bmatrix}10 & -9\\4 & -2\end{bmatrix}, \begin{bmatrix}0 & 3\\4 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\1 & 2 & 0\\-3 & 5 & 2\end{bmatrix}$ & $\begin{bmatrix}4 & 0 & 1\\-2 & 1 & 0\\-2 & 0 & 1\end{bmatrix}$. **Answer**: **No, yes, no & yes.**	
H	2	Check whether the following matrices are diagonalizable or not: $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 2 & 2\\0 & 2 & 1\\-1 & 2 & 2\end{bmatrix}$ & $\begin{bmatrix}0 & 0 & -2\\1 & 2 & 1\\1 & 0 & 3\end{bmatrix}$. **Answer**: **Yes, yes, no & yes.**	
T	3	Check whether the following matrices are diagonalizable or not: $\begin{bmatrix}3 & 0 & -5\\1/5 & -1 & 0\\1 & 1 & -2\end{bmatrix}, \begin{bmatrix}-2 & 0 & 1\\-6 & -2 & 0\\19 & 5 & -4\end{bmatrix}, \begin{bmatrix}-1 & 0 & 1\\-1 & 3 & 0\\-4 & 13 & -1\end{bmatrix}$ & $\begin{bmatrix}5 & 0 & 1\\1 & 1 & 0\\-7 & 1 & 0\end{bmatrix}$. **Answer**: **Yes, yes, yes & no.**	
T	4	Check whether the following matrices are diagonalizable or not: $\begin{bmatrix}0 & 0 & 2 & 0\\1 & 0 & 1 & 0\\0 & 1 & -2 & 0\\0 & 0 & 0 & 1\end{bmatrix}$ & $\begin{bmatrix}-1/3 & 0 & 0 & 0\\0 & -1/3 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1/2\end{bmatrix}$. **Answer**: **Yes & yes.**	
T	5	If $A = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}$ and $B = \begin{bmatrix}-1 & 1 & 0\\0 & -1 & 1\\0 & 0 & 2\end{bmatrix}$ then show that A is diagonalizable but B is not diagonalizable.	
C	6	Let $A = \begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}$. Find a matrix P such that $P^{-1}AP$ is diagonal matrix. **Answer**: $\mathbf{P = \begin{bmatrix}0 & 1/2\\1 & 1\end{bmatrix}}$.	
H	7	Find the matrix P that diagonalize $A = \begin{bmatrix}1 & 0\\6 & -1\end{bmatrix}$. Also determine $P^{-1}AP$. **Answer**: $\mathbf{P = \begin{bmatrix}1/3 & 0\\1 & 1\end{bmatrix}}$ & $\mathbf{P^{-1}AP = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}$.	

T	8	For the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$, find the nonsingular matrix P and the diagonal matrix D such that $D = P^{-1}AP$. **Answr**: $\mathbf{P} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ & $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.	
C	9	Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find an orthonormal matrix P such that $P^{-1}AP$ is diagonal matrix. **Answer**: $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$.	
H	10	Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal matrix. **Answer**: $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.	
H	11	Find a matrix P that diagonalize $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Also determine $P^{-1}AP$. **Answer**: $\mathbf{P} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $\mathbf{P^{-1}AP} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.	
T	12	Find a matrix P that diagonalize $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$. Also determine $P^{-1}AP$. **Answer**: $\mathbf{P} = \begin{bmatrix} 1 & 2/3 & 1/4 \\ 1 & 1 & 3/4 \\ 1 & 1 & 1 \end{bmatrix}$ & $\mathbf{P^{-1}AP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.	
C	13	Prove that $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ is diagonalizable and use it to find A^{13}. **Answer**: $\mathbf{P} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ & $\mathbf{A^{13}} = \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$.	S-19 (7)

T	**14**	Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Find matrix P such that $P^{-1}AP$ is diagonal matrix and find A^5. **Answer**: $\mathbf{P} = \begin{bmatrix} \mathbf{-1/2} & \mathbf{1/2} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ & $\mathbf{A^5} = \begin{bmatrix} \mathbf{21856} & \mathbf{-10912} & \mathbf{10912} \\ \mathbf{-10912} & \mathbf{5488} & \mathbf{-5456} \\ \mathbf{10912} & \mathbf{-5456} & \mathbf{5488} \end{bmatrix}$.	

THE CAYLEY-HAMILTON THEOREM:

- Every square matrix satisfies its own characteristic equation. i.e., if A is an $n \times n$ matrix, and $P_A(\lambda) = 0$ is its charateristic equation, then $P_A(A) = \mathbf{O}_n$.
- That means the characteristic equation is satisfied by A.
- By using the Cayley-Hamilton theorem we can find A^{-1} and integer power of A. i.e., A^n.
- Procedure to verify Cayley-Hamilton theorem for 2×2 matrix is as follow:
  - **(1).** Find the characteristic equation of given matrix A.
  - **(2).** Put $\lambda = A$ in the characteristic equation and give it equation (1).
  - **(3).** Find A^2 using given matrix.
  - **(4).** Put the values of A^2, A, I in equation (1) and verify it.
  - **(5).** If you want to find A^{-1} using Cayley-Hamilton theorem then multiply equation (1) both sides by A^{-1} and substitute the values of A and I.
- Procedure to verify Cayley-Hamilton theorem for 3×3 matrix is as follow:
  - **(1).** Find the characteristic equation of given matrix A.
  - **(2).** Put $\lambda = A$ in the characteristic equation and give it equation (1).
  - **(3).** Find A^3 & A^2 using given matrix.
  - **(4).** Put the values of A^3, A^2, A, I in equation (1) and verify it.
  - **(5).** If you want to find A^{-1} using Cayley-Hamilton theorem then multiply equation (1) both sides by A^{-1} and substitute the values of A^2, A and I. By the similar way we can find A^{-2}, A^2, A^4 etc ...

METHOD – 8: THE CAYLEY - HAMILTON THEOREM

C	1	State Cayley-Hamilton theorem.	W-18 (1)
C	2	Using Cayley-Hamilton theorem find A^{-1} for $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$. **Answer**: $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.	W-19 (4)
T	3	Using Cayley-Hamilton theorem, find A^{-1}, A^{-2} for $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. **Answer**: $A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ & $A^{-2} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$.	
C	4	Verify Cayley-Hamilton theorem and find inverse for $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. **Answer**: $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.	
H	5	State Cayley-Hamilton theorem and verify it for the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.	S-19 (7)
T	6	Using Cayley-Hamilton theorem, find inverse of $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. **Answer**: $\frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$.	
C	7	Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ and hence find A^4. **Answer**: $\begin{bmatrix} 193 & 160 & 144 \\ 224 & 177 & 160 \\ 272 & 224 & 193 \end{bmatrix}$.	
H	8	Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{bmatrix}$ and hence find A^4. **Answer**: $\begin{bmatrix} 2176 & -520 & 1400 \\ -1920 & 776 & -1400 \\ 1920 & 520 & 2696 \end{bmatrix}$.	

C	**9**	Determine A^{-1} by using Cayley-Hamilton theorem for matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \\ -2 & 1 & 5 \end{bmatrix}$. Hence find the matrix represented by $A^8 - 5A^7 - A^6 + 37A^5 + A^4 - 5A^3 - 3A^2 + 41A + 3I$. **Answer**: $\mathbf{A^{-1} = \frac{1}{37}\begin{bmatrix} 9 & 13 & -14 \\ 8 & -9 & 4 \\ 2 & 7 & 1 \end{bmatrix}}$ & $\mathbf{\begin{bmatrix} 13 & 8 & -40 \\ 16 & -11 & -16 \\ 16 & 8 & -27 \end{bmatrix}}$.	
T	**10**	Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$. **Answer**: $\mathbf{\begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}}$.	

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FORMULAE, RESULTS & SYMBOLS

STANDARD SYMBOLS:

α alpha	ξ xu	σ sigma
β beta	η eta	τ tau
γ gamma	ζ zeta	ω, Ω omega
δ delta	λ lamda	Γ capital gamma
ε epsilon	∂ del	Δ capital delta
θ, Θ theta	μ mu	Σ capital sigma
ϕ, Φ phi	π pi	∞ infinity
ψ psi	ρ rho	∇ capital del

BASIC FORMULAE:

(1). $(a+b)^2 = a^2 + 2ab + b^2$ & $(a-b)^2 = a^2 - 2ab + b^2$.

(2). $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$.

(3). $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a-b)$.

(4). $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ & $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

(5). $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

(6). $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$.

STANDARD SERIES:

(1). $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \sum_{k=1}^{n} k.$$

(2). $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \sum_{k=1}^{n} k^2.$$

(3). $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \sum_{k=1}^{n} k^3.$$

(4). $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty=\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}.$

TRIGONOMETRIC IDENTITIES AND FORMULAE:

(1). $\cos^2\theta+\sin^2\theta=1$

(2). $1+\tan^2\theta=\sec^2\theta$

(3). $1+\cot^2\theta=\mathrm{cosec}^2\theta$

(4). $\sin 2\theta=2\sin\theta\cos\theta=\frac{2\tan\theta}{1+\tan^2\theta}$

(5). $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

(6). $\cos 2\theta=\cos^2\theta-\sin^2\theta$

(7). $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$

(8). $\cos^2\theta=\frac{1+\cos 2\theta}{2}$

(9). $\sin^2\theta=\frac{1-\cos 2\theta}{2}$

(10). $\tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$

(11). $\cot^2\theta=\frac{1+\cos 2\theta}{1-\cos 2\theta}$

(12). $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

(13). $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

(14). $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

(15). $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

(16). $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

(17). $\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

(18). $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

(19). $\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\alpha-\cot\beta}$

(20). $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(21). $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(22). $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(23). $\cos C-\cos D=$
$-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(24). $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

(25). $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$

(26). $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$

(27). $2\cos\alpha\cos\beta$
$=\cos(\alpha+\beta)+\cos(\alpha-\beta)$

VALUES OF CIRCULAR FUNCTIONS:

(1). $\sin n\pi = 0, \ \forall\, n \in \mathbb{Z}$

(2). $\cos n\pi = (-1)^n, \ \forall\, n \in \mathbb{Z}$

(3). $\sin(2n-1)\frac{\pi}{2} = (-1)^{n+1}, \ \forall\, n \in \mathbb{N}$

(4). $\cos(2n-1)\frac{\pi}{2} = 0, \ \forall\, n \in \mathbb{N}$

(5). If $\sin\theta = 0$ then $\theta = n\pi, \ \forall\, n \in \mathbb{Z}$

(6). If $\cos\theta = 1$ then $\theta = 2n\pi, \ \forall\, n \in \mathbb{Z}$

(7). If $\sin\theta = 1$ then $\theta = (4n+1)\frac{\pi}{2}, \ \forall\, n \in \mathbb{Z}^+$

(8). If $\cos\theta = 0$ then $\theta = (2n+1)\frac{\pi}{2}, \ \forall\, n \in \mathbb{Z}$

INVERSE TRIGONOMETRIC FUNCTIONS:

(1). $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x} = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(2). $\cos^{-1}x = \sec^{-1}\frac{1}{x} = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

(3). $\tan^{-1}x = \cot^{-1}\frac{1}{x} = \sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(4). $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(5). $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

(6). $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(7). $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{\pi}{2}\right); \ xy = 1$

(8). $\sin^{-1}(-x) = -\sin^{-1}x$

(9). $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(10). $\tan^{-1}(-x) = -\tan^{-1}x$

(11). $\cot^{-1}(-x) = \pi - \cot^{-1}x$

(12). $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(13). $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$

(14). $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

(15). $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-y^2}\sqrt{1-x^2}\right)$

DIFFERENTIATION:

(1). $\frac{d}{dx}\, x^n = nx^{n-1}$

(2). $\frac{d}{dx}\, a^x = a^x \log a, \quad a \in \mathbb{R}^+ - \{1\}$

(3). $\frac{d}{dx}\, e^{ax} = a\, e^{ax}$

(4). $\frac{d}{dx}\log x = \frac{1}{x}\,, \quad x \neq 0$

(5). $\frac{d}{dx}\, \sin x = \cos x$

(6). $\frac{d}{dx}\, \cos x = -\sin x$

(7). $\frac{d}{dx}\, \tan x = \sec^2 x$

(8). $\frac{d}{dx}\, \cot x = -\mathrm{cosec}^2 x$

(9). $\frac{d}{dx}\, \sec x = \sec x \tan x$

(10). $\frac{d}{dx}\, \mathrm{cosec}\, x = -\mathrm{cosec}\, x \cot x$

(11). $\frac{d}{dx}\, \sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

(12). $\frac{d}{dx}\, \cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

(13). $\frac{d}{dx}\, \tan^{-1}x = \frac{1}{1+x^2}$

(14). $\frac{d}{dx}\, \cot^{-1}x = \frac{-1}{1+x^2}$

(15). $\frac{d}{dx}\, \sec^{-1}x = \frac{1}{|x|\,\sqrt{x^2-1}}\,, \quad |x| > 1$

(16). $\frac{d}{dx}\, \mathrm{cosec}^{-1}x = \frac{-1}{|x|\,\sqrt{x^2-1}}\,, \quad |x| > 1$

(17). $\frac{d}{dx}(u \cdot v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$

(18). $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$

INTEGRATION:

(1). $\int x^n\, dx = \frac{x^{n+1}}{n+1}\,, \quad n \neq -1$

(2). $\int \frac{1}{x}\, dx = \log|\,x\,|$

(3). $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)}\,, \qquad n \neq -1$

(4). $\int \frac{dx}{(ax+b)} = \frac{1}{a} \log|ax+b|$

(5). $\int \frac{dx}{x(ax+b)} = \frac{1}{b} \log\left(\frac{x}{ax+b}\right)$

(6). $\int \sqrt{(ax+b)^n}\, dx = \frac{2}{a} \frac{\sqrt{(ax+b)^{n+2}}}{n+2}, \quad n \neq -2$

(7). $\int \frac{\sqrt{ax+b}}{x}\, dx = 2\sqrt{ax+b} + b \int \frac{dx}{x\sqrt{ax+b}}$

(8). $\int \frac{dx}{(a^2+x^2)}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

(9). $\int \frac{dx}{(a^2-x^2)} = \frac{1}{2a} \log\left|\frac{x+a}{x-a}\right|$

(10). $\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3} \tan^{-1}\left(\frac{x}{a}\right)$

(11). $\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3} \log\left|\frac{x+a}{x-a}\right|$

(12). $\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right)$

(13). $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \log\left|x+\sqrt{x^2-a^2}\right|$

(14). $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

(15). $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) = \log\left(x+\sqrt{a^2+x^2}\right)$

(16). $\int x^2\sqrt{a^2+x^2}\, dx = \frac{x(a^2+2x^2)\sqrt{a^2+x^2}}{8} - \frac{a^4}{8} \sinh^{-1}\left(\frac{x}{a}\right)$

(17). $\int \frac{x^2}{\sqrt{a^2+x^2}}\, dx = -\frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{2}$

(18). $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a} \log\left(\frac{a+\sqrt{a^2+x^2}}{x}\right)$

(19). $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(20). $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

(21). $\int \frac{x^2}{\sqrt{a^2-x^2}}\, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{x}{2}\sqrt{a^2-x^2}$

(22). $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

(23). $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \log\left|x+\sqrt{x^2-a^2}\right|$

(24). $\displaystyle\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{x^2-a^2}$

(25). $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right)$

(26). $\displaystyle\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right)$

(27). $\displaystyle\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)$

(28). $\displaystyle\int \frac{x}{\sqrt{2ax-x^2}}\,dx = a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax-x^2}$

(29). $\displaystyle\int \sin ax\;dx = -\frac{1}{a}\cos ax$ **(30).** $\displaystyle\int \cos ax\;dx = \frac{1}{a}\sin ax$

(31). $\displaystyle\int \sin^n ax\;dx = -\frac{(\sin^{n-1} ax)\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$

(32). $\displaystyle\int \cos^n ax\;dx = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$

(33). $\displaystyle\int \sin ax\cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}, \qquad a^2 \neq b^2$

(34). $\displaystyle\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$

(35). $\displaystyle\int \cos ax\cos bx\;dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$

(36). $\displaystyle\int \sin ax\cos ax\,dx = -\frac{\cos 2ax}{4a}$ **(37).** $\displaystyle\int \cot ax\;dx = \frac{1}{a}\log(\sin ax)$

(38). $\displaystyle\int \sin^n ax\cos ax\;dx = \frac{\sin^{n+1} ax}{(n+1)a}, \qquad n \neq -1$

(39). $\displaystyle\int \cos^n ax\sin ax\;dx = -\frac{\cos^{n+1} ax}{(n+1)a}, \qquad n \neq -1$

(40). $\int \tan ax \ dx = -\frac{1}{a} \log(\cos ax) = \frac{1}{a} \log(\sec ax)$

(41). $\int \sin^n ax \cos^m ax \ dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(m+n)} + \frac{n-1}{m+n} \int \sin^{n-2} ax \cos^m ax \ dx, \ n \neq -m$

(42). $\int \frac{dx}{b + c \sin ax} = -\frac{2}{a\sqrt{b^2 - c^2}} \tan^{-1}\left[\sqrt{\frac{b-c}{b+c}} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right], \quad b^2 > c^2$

$= -\frac{1}{a\sqrt{c^2 - b^2}} \log\left[\frac{c + b \sin ax + \sqrt{c^2 - b^2} \cos ax}{b + c \sin ax}\right], \quad b^2 < c^2$

(43). $\int \frac{dx}{1 + \sin ax} = -\frac{1}{a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

(44). $\int \frac{dx}{1 - \sin ax} = \frac{1}{a} \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

(45). $\int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan\left(\frac{ax}{2}\right)$

(46). $\int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot\left(\frac{ax}{2}\right)$

(47). $\int \tan^2 ax \ dx = \frac{1}{a} \tan ax - x$

(48). $\int \cot^2 ax \ dx = -\frac{1}{a} \cot ax - x$

(49). $\int \tan^n ax \ dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int \tan^{n-2} ax \ dx, \quad n \neq 1$

(50). $\int \cot^n ax \ dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int \cot^{n-2} ax \ dx, \ n \neq 1$

(51). $\int \sec ax \ dx = \frac{1}{a} \log|\sec ax + \tan ax|$

(52). $\int \operatorname{cosec} ax \ dx = -\frac{1}{a} \log|\operatorname{cosec} ax - \cot ax|$

(53). $\int \sec^2 ax \ dx = \frac{1}{a} \tan ax$

(54). $\int \operatorname{cosec}^2 ax \ dx = -\frac{1}{a} \cot ax$

(55). $\int \sec^n ax \ dx = \frac{\sec^{n-2} ax \ \tan ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax \ dx, \quad n \neq 1$

(56). $\int \operatorname{cosec}^n ax \ dx = -\frac{\operatorname{cosec}^{n-2} ax \ \cot ax}{a(n-1)} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} ax \ dx, \quad n \neq 1$

(57). $\int \sec^n ax \ \tan ax \, dx = \frac{\sec^n ax}{na}$

(58). $\int \operatorname{cosec}^n ax \ \cot ax \, dx = -\frac{\operatorname{cosec}^n ax}{na}$

(59). $\int \sin^{-1} ax \ dx = x \sin^{-1} ax + \frac{1}{a}\sqrt{1 - a^2x^2}$

(60). $$\int \cos^{-1} ax \; dx = x \cos^{-1} ax - \frac{1}{a}\sqrt{1 - a^2x^2}$$

(61). $$\int \tan^{-1} ax \; dx = x \tan^{-1} ax - \frac{1}{2a} \log|\, 1 + a^2x^2 \,|$$

(62). $$\int e^{ax} \; dx = \frac{e^{ax}}{a}$$

(63). $$\int b^{ax} \; dx = \frac{1}{a}\,\frac{b^{ax}}{\log b}, \quad b > 0, b \neq 1$$

(64). $$\int e^{ax} \sin bx \; dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$$

(65). $$\int e^{ax} \; \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \; (a \cos bx + b \sin bx)$$

(66). $$\int x^n \log ax \; dx = \frac{x^{n+1}}{n+1} \log ax - \frac{x^{n+1}}{(n+1)^2}, \qquad n \neq 1$$

(67). $$\int \log ax \; dx = x \log ax - x$$

(68). $$\int uv \, dx = u \int v \, dx - \int \left\{ \frac{du}{dx} * \int v \; dx \right\} dx$$

When u is a polynomial function then it is better to use following rule:

(69). $$\int uv \, dx = (u)(v_1) - (u')(v_2) + (u'')(v_3) - (u''')(v_4) + \cdots$$

where dashes of u denotes the derivatives and the subscripts of v denotes integrations.

(70). $$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \; dx = \int_0^{\frac{\pi}{2}} \cos^n x \; dx = \frac{n-1}{n} \; I_{n-2}, \text{where } I_0 = \frac{\pi}{2}, \;\; I_1 = 1$$

(71). $$\int_0^a f(x) \, dx = 2 \int_0^{\frac{a}{2}} f(x) \, dx \qquad \text{if } f(a - x) = f(x)$$

$$= 0 \qquad \text{if } f(a - x) = -f(x)$$

(72). $$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx \qquad \text{if } f(x) \text{ is an even function}$$

$$= 0 \qquad \text{if } f(x) \text{ is an odd function}$$

(73). $$\int [f(x)]^{\,n} \; f'(x) \, dx = \frac{[f(x)]^{\,n+1}}{n+1} + c, \qquad n \neq -1$$

(74). $$\int \frac{f'(x)}{f(x)} \, dx = \log|f(x)| + c$$

(75). $$\int f'(x) \, e^{f(x)} \; dx = e^{f(x)}$$

RELATION WITH CARTESIAN COORDINATES:

(1). Polar coordinates (r, θ) : $x = r\cos\theta$, $y = r\sin\theta$

: $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$

(2). Cylindrical coordinates (r, θ, z): $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

(3). Spherical coordinates $(r, \theta, \emptyset)$: $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$, $z = r\cos\theta$

HYPERBOLIC FUNCTIONS:

(1). $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(2). $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(3). $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(4). $\operatorname{cosech} x = \dfrac{1}{\sin h\, x} = \dfrac{2}{e^x - e^{-x}}$

(5). $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(6). $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

(7). $\sinh(-x) = -\sinh x$

(8). $\cosh(-x) = \cosh x$

(9). $\cosh^2 x - \sinh^2 x = 1$

FREQUENTLY USED LIMIT:

(1). $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

(2). $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$

(3). $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a$

(4). $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a}$

$= na^{n-1}$; n is rational no.

(5). $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

(6). $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$

(7). $\lim\limits_{n\to\infty} (n)^{\frac{1}{n}} = 1$

(8). $\lim\limits_{n\to\infty} \dfrac{\log n}{n} = 0$

(9). $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$

(10). $\lim\limits_{n\to\infty} (x)^{\frac{1}{n}} = 1, x > 0$

(11). $\lim_{n\to\infty} \frac{x^n}{n!} = 0, x > 0$

(12). $\lim_{n\to\infty} x^n = 0, |x| < 1$

LOGARITHM RULES:

(1). $\log 1 = 0$

(2). $\log e = 1$

(3). $\log(xy) = \log(x) + \log(y)$

(4). $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

(5). $\log(x^y) = y \log x$

(6). $\log_y x = \frac{\log_z x}{\log_z y}$

AREA:

Sr. No.	Type	Perimeter	Area	Remarks
(1).	Square	$4a$	a^2	Side “a”
(2).	Rectangle	$2a + 2b$	$a\,b$	Sides “a” & “b”
(3).	Triangle	$a + b + c$	$(b\,h)/2$	Altitude “h” & Base “b”
(4).	Circle	$2\pi r$	$\pi\, r^2$	Radius “r”
(5).	Sector of Circle	---	$(r^2\,\theta)/2$	Radius “r” & Angle “θ”

VOLUME:

Sr. No.	Type	Volume	Surface Area	Remarks
(1).	Cube	a^3	$6a^2$	Side “a”
(2).	Cuboid	$a\,b\,c$	$2(ab + bc + ca)$	Sides “a”, “b” & “c”
(3).	Right circular Cylinder	$\pi\, r^2 h$	$2\,\pi\, r\, h$	Altitude “h” & Radius “r”
(4).	Right circular Cone	$(\pi\, r^2 h)/3$	$\pi\, r\sqrt{r^2 + h^2}$	Altitude “h” & Radius “r”
(5).	Sphere	$(4\,\pi\, r^3)/3$	$4\,\pi\, r^2$	Radius “r”

VALUE OF SOME CONSTANTS:

(1). $e = 2.71828$

(2). $\pi = 3.14159$

(3). Log 1 = 0 (any base $\neq 1$)

(4). $\pi^R = 180°$

(5). $1^\circ = \left(\frac{\pi}{180}\right)^R = 0.017453^R$

(6). $1^R = \left(\frac{180}{\pi}\right)^\circ$

(7). $1^\circ = 60'$

(8). $1' = 60''$

Here $^\circ$ = degree, R = radian, $'$ = minute and $''$ = second

TRIGONOMETRIC TABLE:

θ°	0°	30°	45°	60°	90°	180°	270°	360°
θ^R	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0
$\operatorname{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

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LIST OF ASSIGNMENTS

ASSIGNMENT	UNIT	METHOD-NO. (EXAMPLE NO.)
1	1	M-1 (Ex – 4, 6, 8, 11, 13, 14)
		M-2 (Ex – 3, 6)
		M-4 (Ex – 3, 7, 8)
		M-5 (Ex – 4, 8, 10, 12, 14)
		M-8 (Ex – 4, 7, 11)
		M-9 (Ex – 4, 6, 7)
2	2	M-4 (Ex – 3, 4, 7, 8, 10)
		M-9 (Ex – 2, 5)
		M-10 (Ex – 5, 7, 9, 12, 20, 21)
		M-11 (Ex – 4, 6, 7, 12, 16)
		M-13 (Ex – 3, 4, 7, 10)
		M-17 (Ex – 3, 4, 9, 12, 15, 16)
		M-19 (Ex – 3, 4, 7, 9)
		M-20 (Ex – 5, 6, 7, 9, 11, 14, 18)
3	3	M-1 (Ex – 4, 5, 10, 12)
		M-2 (Ex – 2, 5, 6, 9, 13, 16, 19)
		M-3 (Ex – 3, 7, 8)
		M-4 (Ex – 3, 4, 6)

4	4	M-2 (Ex – 3, 4, 9, 10)
		M-3 (Ex – 2, 6, 8, 11, 14, 16, 17, 18, 20)
		M-4 (Ex – 2, 5, 8, 9, 12, 14)
		M-6 (Ex – 3, 5, 6, 7)
		M-7 (Ex – 5, 6, 7)
		M-8 (Ex – 3, 4, 7, 10, 13)
		M-10 (Ex – 5, 8, 9, 11, 13)
5	5	M-1 (Ex – 2, 7, 8, 13, 14)
		M-2 (Ex – 4, 5, 7, 10, 11, 13, 14, 16,19)
		M-3 (Ex – 2, 6, 7, 10, 12, 15, 18, 22)
		M-5 (Ex – 1, 5, 10, 12, 14, 17, 18)
6	6	M-1 (Ex – 5, 8, 10, 13, 16, 17)
		M-2 (Ex – 3, 6, 7, 8, 13, 17, 19)
		M-3 (Ex – 3, 5, 9)
		M-4 (Ex – 2, 6, 7, 13)
		M-5 (Ex – 3, 8, 11, 12, 15, 19, 21)
		M-7 (Ex – 2, 4, 5, 8, 12, 14)
		M-8 (Ex – 1, 3, 6, 10)

GUJARAT TECHNOLOGICAL UNIVERSITY

Bachelor of Engineering
Subject Code: 3110014
SUBJECT NAME: Mathematics-1
1^{st} Year

Type of course: Basic Science Course

Prerequisite: Algebra, Trigonometry, Geometry

Rationale: The study of rate of changes, understanding to compute area, volume and express the function in terms of series, to apply matrix algebra**.**

Teaching and Examination Scheme:

Teaching Scheme			Credits	Examination Marks				Total Marks
				Theory Marks		Practical Marks		
L	T	P	C	ESE (E)	PA (M)	ESE (V)	PA (I)	
3	2	0	5	70	30	0	0	100

Content:

Sr. No.	**Content**	**Total Hrs**	**% Weightage**
01	Indeterminate Forms and L'Hôspital's Rule.	01	15 %
	Improper Integrals, Convergence and divergence of the integrals, Beta and Gamma functions and their properties.	03	
	Applications of definite integral, Volume using cross-sections, Length of plane curves, Areas of Surfaces of Revolution	03	
02	Convergence and divergence of sequences, The Sandwich Theorem for Sequences, The Continuous Function Theorem for Sequences, Bounded Monotonic Sequences, Convergence and divergence of an infinite series, geometric series, telescoping series, □□□ term test for divergent series, Combining series, Harmonic Series, Integral test, The p - series, The Comparison test, The Limit Comparison test, Ratio test, Raabe's Test, Root test, Alternating series test, Absolute and Conditional convergence, Power series, Radius of convergence of a power series, Taylor and Maclaurin series.	08	20 %
03	Fourier Series of *2□* periodic functions, Dirichlet's conditions for representation by a Fourier series, Orthogonality of the trigonometric system, Fourier Series of a function of period *2□*, Fourier Series of even and odd functions, Half range expansions.	04	10 %
04	Functions of several variables, Limits and continuity, Test for non existence of a limit, Partial differentiation, Mixed derivative theorem, differentiability, Chain rule, Implicit differentiation, Gradient, Directional derivative, tangent plane and normal line, total differentiation, Local extreme values, Method of Lagrange Multipliers.	08	20 %
05	Multiple integral, Double integral over Rectangles and general regions, double integrals as volumes, Change of order of integration, double integration in polar coordinates, Area by double integration, Triple integrals in rectangular, cylindrical and spherical coordinates, Jacobian, multiple integral by substitution.	08	20 %
06	Elementary row operations in Matrix, Row echelon and Reduced row echelon forms, Rank by echelon forms, Inverse by Gauss-Jordan method,	07	15%

	Solution of system of linear equations by Gauss elimination and Gauss-Jordan methods. Eigen values and eigen vectors, Cayley-Hamilton theorem, Diagonalization of a matrix.		

Suggested Specification table with Marks (Theory):

Distribution of Theory Marks					
R Level	U Level	A Level	N Level	E Level	C Level
10	25	35	0	0	0

Legends: R: Remembrance; U: Understanding; A: Application, N: Analyze and E: Evaluate C: Create and above Levels (Revised Bloom's Taxonomy).

Reference Books:

(1) Maurice D. Weir, Joel Hass, Thomas' Calculus, Early Transcendentals, 13e, Pearson, 2014.
(2) Howard Anton, Irl Bivens, Stephens Davis, Calculus, 10e, Wiley, 2016.
(3) James Stewart, Calculus: Early Transcendentals with Course Mate, 7e, Cengage, 2012.
(4) Anton and Rorres, Elementary Linear Algebra, Applications version,, Wiley India Edition.
(5) T. M. Apostol, Calculus, Volumes 1 & 2,, Wiley Eastern.
(6) Erwin Kreyszig, Advanced Engineering Mathematics, Wiley India Edition.
(7) Peter O'Neill, Advanced Engineering Mathematics, 7th Edition, Cengage.

Course Outcomes

The objective of this course is to familiarize the prospective engineers with techniques in calculus, multivariate analysis and matrices. It aims to equip the students with standard concepts and tools at an intermediate to advanced level that will serve them well towards tackling more advanced level of mathematics and applications that they would find useful in their disciplines.

Sr. No.	Course Outcomes	Weightage in %
1	To apply differential and integral calculus to improper integrals and to determine applications of definite integral. Apart from some other applications they will have a basic understanding of indeterminate forms, Beta and Gamma functions.	15
2	To apply the various tests of convergence to sequence, series and the tool of power series and fourier series for learning advanced Engineering Mathematics.	30
3	To compute directional derivative, maximum or minimum rate of change and optimum value of functions of several variables.	20
4	To compute the areas and volumes using multiple integral techniques.	20
5	To perform matrix computation in a comprehensive manner.	15

List of Open Source Software/learning website:

Scilab, MIT Opencourseware.

Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE –SEMESTER 1&2(NEW SYLLABUS)EXAMINATION- WINTER 2018

Subject Code: 3110014 **Date: 07-01-2019**

Subject Name: Mathematics - I

Time: 10:30 am to 01:30 pm **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

				Marks
Unit-6	**Q.1**	**(a)**	State Cayley– Hamilton theorem. Find eigen values of A and A^{-1}, where $A=\begin{bmatrix}1 & 2\\ 0 & 4\end{bmatrix}$	**03**
Unit-1		**(b)**	State L' Hospital's Rule. Use it to evaluate $\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right]$	**04**
Unit-1		**(c)**	Investigate convergence of the following integrals: (i) $\int_5^\infty \frac{5x}{\left(1+x^2\right)^3}dx$ (ii) $\int_0^\infty \frac{x^{10}\left(1+x^5\right)}{(1+x)^{27}}dx$	**07**
Unit-2	**Q.2**	**(a)**	Test the convergence of series $\sum_{n=1}^{\infty}\frac{4^n+5^n}{6^n}$	**03**
Unit-2		**(b)**	State the p-series test. Discuss the convergence of the series $\sum_{n=2}^{\infty}\frac{n+1}{n^3-3n+2}$	**04**
Unit-2		**(c)**	State D'Alembert's ratio test and Cauchy's root test. Discuss the convergence of the following series: (i) $\sum_{n=1}^{\infty}\frac{4^n\left(n+1\right)!}{n^{n+1}}$ (ii) $\sum_{n=2}^{\infty}\frac{n}{\left(\log n\right)^n}$	**07**
			OR	
Unit-2		**(c)**	Test the convergence of the series $\frac{1}{1\cdot 2\cdot 3}+\frac{x}{4\cdot 5\cdot 6}+\frac{x^2}{7\cdot 8\cdot 9}+\frac{x^3}{10\cdot 11\cdot 12}+\cdots;$ $x\ge 0$	**07**
Unit-6	**Q.3**	**(a)**	Reduce matrix $A=\begin{bmatrix}1 & 5 & 3 & -2\\ 2 & 0 & 4 & 1\\ 4 & 8 & 9 & -1\end{bmatrix}$ to row echelon form and find its rank.	**03**
Unit-3		**(b)**	Derive half range sine series of $f\left(x\right)=\pi-x,\ 0\le x\le\pi$	**04**
Unit-6		**(c)**	Find the eigen values and corresponding eigen vectors for the matrix A where $A=\begin{bmatrix}1 & 2 & 2\\ 0 & 2 & 1\\ -1 & 2 & 2\end{bmatrix}$	**07**

OR

Unit-2	**Q.3**	**(a)**	Expand $e^{x\sin(x)}$ in power of x up to the terms containing x^6.	**03**
Unit-6		**(b)**	Solve system of linear equation by Gauss Elimination method, if solution exists. $x+y+2z=9;\ 2x+4y-3z=1;\ 3x+6y-5z=0$	**04**
Unit-3		**(c)**	Find Fourier series of $f(x)=\begin{cases} x, & -1<x<0 \\ 2, & 0<x<1 \end{cases}$	**07**
Unit-4	**Q.4**	**(a)**	Discuss the continuity of the function f defined as $f(x,y)=\begin{cases} \dfrac{x^3-y^3}{x^2+y^2}; & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$	**03**
Unit-4		**(b)**	Define gradient of a function. Use it to find directional derivative of $f(x,y,z)=x^3-xy^2-z$ at $P(1,1,0)$ in the direction of $\bar{a}=2\hat{i}-3\hat{j}+6\hat{k}$.	**04**
Unit-4		**(c)**	Find the shortest and largest distance from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2=24$.	**07**

OR

Unit-4	**Q.4**	**(a)**	Find the extreme values of $x^3+3xy^2-3x^2-3y^2+4$	**03**
Unit-5		**(b)**	Evaluate $\int_0^2\int_0^{\sqrt{4-x^2}}\left(x^2+y^2\right)dydx$ by changing into polar coordinates.	**04**
Unit-4		**(c)**	(i) If $u=x^2y+y^2z+z^2x$ then find out $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}$ (ii) If $x^3+y^3=6xy$ then find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$	**07**
Unit-5	**Q.5**	**(a)**	Evaluate $\iint_R y\sin(xy)\,dA$, where R is the region bounded by $x=1$, $x=2$, $y=0$ and $y=\dfrac{\pi}{2}$.	**03**
Unit-5		**(b)**	By changing the order of integration, evaluate $\int_0^3\int_y^3 \dfrac{xdxdy}{x^2+y^2}$	**04**
Unit-5		**(c)**	Find the volume below the surface $z=x^2+y^2$, above the plane $z=0$, and inside the cylinder $x^2+y^2=2y$.	**07**

OR

Unit-5	**Q.5**	**(a)**	Evaluate integral $\iint_R \dfrac{rdrd\theta}{\sqrt{a^2+r^2}}$ over the region R which is one loop of $r^2=a^2\cos 2\theta$	**03**
Unit-5		**(b)**	Evaluate the integral $\int_1^e\int_1^{\log y}\int_1^{e^x}\left(x^2+y^2\right)dzdxdy$.	**04**
Unit-1		**(c)**	Find the volume of the solid obtained by rotating the region R enclosed by the curves $y=x$ and $y=x^2$ about the line $y=2$.	**07**

Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–I &II (NEW) EXAMINATION – SUMMER-2019

Subject Code: 3110014 **Date: 06/06/2019**

Subject Name: Mathematics – I

Time: 10:30 AM TO 01:30 PM **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

				Marks
Unit-1	**Q.1**	**(a)**	Use L'Hospital's rule to find the limit of $\lim\limits_{x\to1}\left(\frac{x}{x-1}-\frac{1}{lnx}\right)$.	**03**
Unit-1		**(b)**	Define Gamma function and evaluate $\int_0^{\infty} e^{-x^2}dx$.	**04**
Unit-5		**(c)**	Evaluate $\int_0^3\int_{\frac{\sqrt{x}}{3}}^1 e^{y^3}dydx$.	**07**
Unit-2	**Q.2**	**(a)**	Define the convergence of a sequence (a_n) and verify whether the sequence whose n^{th} term is $a_n=\left(\frac{n+1}{n-1}\right)^n$ converges or not.	**03**
Unit-5		**(b)**	Sketch the region of integration and evaluate the integral $\iint_R (y-2x^2)dA$ where R is the region inside the square $\lvert x\rvert+\lvert y\rvert=1$.	**04**
Unit-2		**(c)**	**(i)** Find the sum of the series $\sum_{n\geq2}\frac{1}{4^n}$ and $\sum_{n\geq1}\frac{4}{(4n-3)(4n+1)}$. **(ii)** Use Taylor's series to estimate $sin38°$.	**07**
			OR	
Unit-5		**(c)**	Evaluate the integrals $\int_0^{\infty}\int_0^{\infty}\frac{1}{(1+x^2+y^2)^2}dxdy$ and $\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{ln3}\frac{\pi e^{2x}sin\pi y^2}{y^2}dxdydz$.	**07**
Unit-1	**Q.3**	**(a)**	If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipode moment P per unit volume is $P(E)=\frac{e^E+e^{-E}}{e^E-e^{-E}}-\frac{1}{E}$. Show that $\lim\limits_{E\to0^+}P(E)=0$.	**03**
Unit-4		**(b)**	For what values of the constant k does the second derivative test guarantee that $f(x,y)=x^2+kxy+y^2$ will have a saddle point at (0,0)? A local minimum at (0,0)?	**04**
Unit-2		**(c)**	Find the series radius and interval of convergence for $\sum_{n=0}^{\infty}\frac{(3x-2)^n}{n}$. For what values of x does the series converge absolutely?	**07**
			OR	
Unit-1	**Q.3**	**(a)**	Determine whether the integral $\int_0^3\frac{dx}{x-1}$ converges or diverges.	**03**
Unit-1		**(b)**	Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1, x=4$ about the line $y=1$.	**04**

Unit-2		(c)	Check the convergence of the series $\sum_{n=1}^{\infty}\frac{(ln n)^3}{n^3}$ and $\sum_{n=0}^{\infty}(-1)^n(\sqrt{n+\sqrt{n}}-\sqrt{n})$.	07
Unit-4	Q.4	(a)	Show that the function $f(x,y)=\frac{2x^2y}{x^4+y^2}$ has no limit as (x,y) approaches to $(0,0)$.	03
Unit-4		(b)	Suppose f is a differentiable function of x and y and $g(u,v)=f(e^u+sinv, e^u+cosv)$. Use the following table to calculate $g_u(0,0), g_v(0,0)$, ~~$g_u(1,2)$ and $g_v(1,2)$.~~	04

	f	g	f_x	f_y
(0,0)	3	6	4	8
(1,2)	6	3	2	5

Unit-3		(c)	Find the Fourier series of 2π −periodic function $f(x)=x^2, 0<x<2\pi$ and hence deduce that $\frac{\pi^2}{6}=\sum_{n=0}^{\infty}\frac{1}{n^2}$.	07

OR

Unit-4	Q.4	(a)	Verify that the function $u=e^{-\alpha^2k^2t}\cdot sinkx$ is a solution f the heat conduction euation $u_t=\alpha^2u_{xx}$.	03
Unit-3		(b)	Find the half-range cosine series of the function $f(x)=\begin{cases}2, & -2<x<0\\ 0, & 0<x<2\end{cases}$.	04
Unit-4		(c)	Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$.	07
Unit-4	Q.5	(a)	Find the directional derivative $D_uf(x,y)$ if $f(x,y)=x^3-3xy+4y^2$ and u is the unit vector given by angle $\theta=\frac{\pi}{6}$. What is $D_uf(1,2)$?	03
Unit-5		(b)	Find the area of the region bounded y the curves $y=sinx, y=cosx$ and the lines $x=0$ and $x=\frac{\pi}{4}$.	04
Unit-6		(c)	Prove that $A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}$ is diagonalizable and use it to find A^{13}.	07

OR

Unit-6	Q.5	(a)	Define the rank of a matrix and find the rank of the matrix $A=\begin{bmatrix}2&-1&0\\4&5&-3\\1&-4&7\end{bmatrix}$.	03
Unit-6		(b)	Use Gauss-Jordan algorithm to solve the system of linear equations $2x_1+2x_2-x_3+x_5=0$ $-x_1-x_2+2x_3-3x_4+x_5=0$ $x_1+x_2-2x_3-x_5=0$ $x_3+x_4+x_5=0$	04
Unit-6		(c)	State Cayley-Hamilton theorem and verify if for the matrix $A=\begin{bmatrix}4&0&1\\-2&1&0\\-2&0&1\end{bmatrix}$.	07

Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER– I & II (NEW) EXAMINATION – WINTER 2019

Subject Code: 3110014 **Date: 17/01/2020**

Subject Name: Mathematics – I

Time: 10:30 AM TO 01:30 PM **Total Marks: 70**

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

				MARKS
Unit-4	**Q.1**	**(a)**	Find the equations of the tenagent plane and normal line to the surface $x^2+y^2+z^2 = 3$ at the point (1,1,1).	**03**
Unit-1		**(b)**	Evaluate $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$.	**04**
Unit-6		**(c)**	Using Gauss Elimination method solve the following system. -x+3y+4z =30 3x+2y-z =9 2x-y+2z =10	**07**
Unit-2	**Q.2**	**(a)**	Test the convergence of the series $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots+\left(\frac{n}{2n+1}\right)^n+\cdots$.	**03**
Unit-4		**(b)**	Discuss the Maxima and Minima of the function $3x^2-y^2+x^3$.	**04**
Unit-3		**(c)**	Find the fourier series of $f(x)=\frac{(\pi-x)}{2}$ in the interval (0,2π).	**07**
			OR	
Unit-5		**(c)**	Change the order of integration and evaluate $\int_0^1\int_x^1 \sin y^2\, dy\, dx$.	**07**
Unit-1	**Q.3**	**(a)**	Find the value of $\beta\left(\frac{7}{2},\frac{5}{2}\right)$.	**03**
Unit-3		**(b)**	Obtain the fourier cosine series of the function f(x) = e^x in the range (0,l).	**04**
Unit-4		**(c)**	Find the maximum and minimum distance fom the point (1,2,2) to the sphere $x^2+y^2+z^2 = 36$.	**07**
			OR	
Unit-2	**Q.3**	**(a)**	Test the convergence of the series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots.$	**03**
Unit-5		**(b)**	Evaluate $\iint(x^2-y^2)\,dx\,dy$ over the triangle with the vertices (0,1), (1,1), (1,2).	**04**
Unit-1		**(c)**	Find the volume of the solid generated by rotating the plane region bounded by $y=\frac{1}{x}$, x=1 and x=3 about the X axis.	**07**
Unit-5	**Q.4**	**(a)**	Evaluate $\int_0^\pi\int_0^{\sin\theta} r\,dr\,d\theta$.	**03**
Unit-2		**(b)**	Express f(x) = $2x^3+3x^2-8x+7$ in terms of (x-2).	**04**

Unit-6		(c)	Using Gauss-Jordan method find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$	07

OR

Unit-6	**Q.4**	**(a)**	Using Cayley-Hamilton Theorem find A^{-1} for $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$	**03**
Unit-1		**(b)**	Evaluate $\int_0^\infty \frac{dx}{x^2+1}$	**04**
Unit-2		**(c)**	Test the convergence of the series $\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \cdots$	**07**
Unit-5	**Q.5**	**(a)**	Evaluate $\int_0^1 \int_1^2 xy\, dy\, dx$.	**03**
Unit-6		**(b)**	Find the eigen values and eigenvectors of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$	**04**
Unit-4		**(c)**	If u = f (x-y, y-z, z-x) then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.	**07**

OR

Unit-4	**Q.5**	**(a)**	Find the directional derivatives of f = $xy^2 + yz^2$ at the point (2,-1,1) in the direction of i+2j+2k.	**03**
Unit-2		**(b)**	Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$	**04**
Unit-5		**(c)**	Evaluate $\iiint xyz\, dx\, dy\, dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = 4$.	**07**

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