LESSON 1: THE LIMIT OF A SEQUENCE

I. Theory

1. Finite Limit of a Sequence

1. Definition

We say that a sequence (u_n) **has a limit of 0** as n approaches positive infinity, if $|u_n|$ can be made smaller than any positive number, starting from a certain term onwards.

Notation: $\lim_{n\to+\infty} u_n = 0$ or $\lim u_n = 0$ or $u_n \to 0$ then $n \to +\infty$.

We say a sequence (v_n) has a finite limit a (or v_n approaches a) when $n \to +\infty$, if $\lim_{n\to+\infty}(v_n - a) = 0.$

Notation: $\lim_{n\to+\infty} v_n = a$ or $\lim v_n = a$ or $v_n \to a$ or $n \to +\infty$.

2. Some Basic Limits:

a) $\lim \frac{1}{n} = 0$; $\lim \frac{1}{n^k} = 0, \left(k \in \mathbb{N}^*\right)$;

a) $\lim \frac{c}{n} = 0$; $\lim \frac{c}{n^k} = 0, \left(k \in \mathbb{N}^*\right)$; c is a constant;

c) $\lim_{n\to+\infty} q^n = 0$ if $|q| < 1$;

d) The sequence (u_n) with $u_n = \left(1 + \frac{1}{n}\right)^n$ has a limit as an irrational number,and that limit is called e, $e = \lim\left(1 + \frac{1}{n}\right)^n$

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II. Theorem on the Finite Limit of a Sequence

a) if $\lim u_n = a$ and $\lim v_n = b$ and c is constant, then :

- $\lim(u_n + v_n) = a + b$ • $\lim(u_n - v_n) = a - b$
- $\lim(u_n . v_n) = a.b$ • $\lim \frac{u_n}{v_n} = \frac{a}{b}, (b \neq 0)$
- $\lim(c.u_n) = c.a$. • $\lim|u_n| = |a|$ and $\lim \sqrt[3]{u_n} = \sqrt[3]{a}$

b) if $u_n \geq 0$ for all n and $\lim u_n = a$ then $a \geq 0$ and $\lim \sqrt{u_n} = \sqrt{a}$.

III. Sum of an infinite Geometric Series

An infinite geometric sequence (u_n) with common ratio q, where $|q| < 1$ is called an infinite decreasing geometric series.

The sum of the infinite decreasing geometric series: $S = \frac{u_1}{1-q}$

IV. Infinite Limit

• We say that a sequence (u_n) has a limit of $+\infty$ when $n \to +\infty$, if u_n Can be greater than any positive number, starting from a certain term onwards.

Notation: $\lim u_n = +\infty$ or $u_n \to +\infty$ khi $n \to +\infty$.

• The sequence (u_n) has a limit of $-\infty$ then $n \to +\infty$, if $\lim(-u_n) = +\infty$.

Notation: $\lim u_n = -\infty$ or $u_n \to -\infty$ then $n \to +\infty$.

Remark: $u_n = +\infty \Leftrightarrow \lim(-u_n) = -\infty$.

Remark

a) $\lim n^k = +\infty$ with k positive integer;

b) $\lim q^n = +\infty$ if $q > 1$.

c) If $\lim u_n = a$ and $\lim v_n = \pm\infty$ then $\lim \frac{u_n}{v_n} = 0$.

d) If $\lim u_n = a > 0$, $\lim v_n = 0$ và $v_n > 0, \forall n > 0$ then $\lim \frac{u_n}{v_n} = +\infty$.

e) $\lim u_n = +\infty \Leftrightarrow \lim(-u_n) = -\infty$

e) If $\lim u_n = +\infty$ and $\lim v_n = a > 0$ then $\lim u_n.v_n = +\infty$.

Note :

Rule for Finding the Limit of a Product $\lim(\mathrm{u_n.v_n})$

If . $\lim \mathrm{u_n} = \mathrm{L}, \lim \mathrm{v_n} = +\infty\,(\mathrm{hay} - \infty)$ Then $\lim(\mathrm{u_n v_n})$

$\lim \mathrm{u_n} = \mathrm{L}$	$\lim \mathrm{v_n}$	$\lim(\mathrm{u_n v_n})$
$+$	$+\infty$	$+\infty$
$+$	$-\infty$	$-\infty$
$-$	$+\infty$	$-\infty$
$-$	$-\infty$	$+\infty$

Rule for Finding the Limit of a Quotient $\lim \frac{\mathrm{u_n}}{\mathrm{v_n}}$

$\lim \mathrm{u_n}$	$\lim \mathrm{v_n}$	Sign of $\mathrm{v_n}$	$\lim \frac{\mathrm{u_n}}{\mathrm{v_n}}$
L	$\pm\infty$	arbitrarily	0
$\mathrm{L} > 0$	0	$+$	$+\infty$
	0	$-$	$-\infty$
$\mathrm{L} < 0$	0	$+$	$-\infty$

	0	$-$	$+\infty$

Remark: We often use the rule for finding the limit of a product in problems involving the infinite of a sequence.

II. Exercise

Question 1: (Remembering) Given $\lim u_n = a$ and $\lim v_n = b$. Find $\lim(u_n + v_n)$.

Question 2: (Remembering) Given $\lim u_n = a$ và $\lim v_n = b$. Find $\lim(u_n - v_n)$.

Question 3: (Remembering) Given $\lim u_n = a$ và $\lim v_n = b$. Find $\lim(u_n . v_n)$.

Question 4: (Remembering) Give the sequence (u_n) with $u_n = \frac{1}{n}$. Calculate $\lim u_n$

Question 5: (Understanding) Give the sequence (u_n) with $u_n = \frac{1}{n^k}$ with $k \in \mathbb{N}^*$. Calculate $\lim u_n$

Question 6: (Understanding) Give an infinite geometric sequence (u_n) has a denominator q, with $|q| < 1$. Calculate the sum of all the terms of the given geometric sequence.

Question 7: (Applying) Given $\lim u_n = 3$ và $\lim v_n = 1$. Find $\lim(u_n - v_n)$.

Question 8: (Applying) Given $\lim u_n = 5$ và $\lim v_n = 2$. Find $\lim(u_n . v_n)$.

Question 9: (Applying) Give the sequence (u_n) with $u_n = \frac{1}{n+1}$. Calculate $\lim u_n$

Question 10: (Applying) Give the sequence (u_n) with $u_n = \left(\frac{2}{5}\right)^n$. Calculate $\lim u_n$

Question 11: (Applying) Given a geometric sequence with $u_1 = 2$ and has a denominator $q = \frac{1}{4}$. Calculate the sum of all the terms of the given geometric sequence.

Question 12: (Applying) Give the sequence (u_n) with $u_n = \frac{1}{n^3 + 2}$. Calculate $\lim u_n$

Question 13: (Analyzing) Calculate $\lim\left(\mathrm{n}^4 - 2\mathrm{n}^2 + 3\right)$

Question 14: (Analyzing) Calculate $\lim\left(-2\mathrm{n}^3 + 3\mathrm{n} - 1\right)$

Question 15: (Analyzing) Calculate $\lim\left(-2\mathrm{n}^2 + 4\right)^3$

Question 16: (Analyzing) Sum of an Infinite Decreasing Geometric Series $S = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \ldots$ results in:

Question 17: (Evaluating) Give the sequence (u_n) with $u_n = \frac{\sqrt{n+1}}{n+2}$. Calculate $\lim u_n$

Question 18: (Evaluating) Give the sequence (u_n) with $u_n = \frac{-2n^3 + 3n^2 + 4}{n^4 + 4n^3 + n}$. Calculate $\lim u_n$

Question 19: (Evaluating) Calculate $\lim \frac{-4n^2 + n + 2}{2n^2 + n + 1}$

Question 20: (Evaluating) Calculate $\lim \frac{2n^3 - 2n + 3}{1 - 4n^3}$

Question 21: **(Evaluating)** Give the sequence (u_n) with $u_n = \frac{(-1)^n .2^{5n+1}}{3^{5n+2}}$. Calculate $\lim u_n$

Question 22: **(Creating)** Calculate $\lim\left(\sqrt{n^2+3n+5}-n\right)$.

Question 23: **(Creating)** Calculate $\lim\left(\sqrt{9n^2+3n-4}-3n+2\right)$.

Question 24: **(Creating)** Give the sequence (u_n) with $u_n = \sqrt{n^2+1}-n$. Calculate $\lim u_n$

Question 25: **(Creating)** Calculate $\lim\frac{2n-\sqrt{4n^2+n}}{n+\sqrt[3]{4n^2-n^3}}$.

Question 26: **(Creating)** Calculate $\lim\frac{\sqrt{n^4+2n+2}}{n^2+1}$

Question 27: **(Creating)** Calculate $\lim(\sqrt{n^2-4n}-n)$

Question 28: **(Creating)** Calculate $\lim\left(\sqrt{n^2+2n}-n\right)$

Question 29: **(Creating)** Calculate $\lim\left(\sqrt{n^2+4}+\sqrt{n^2-3}\right)$

Question 30: **(Creating)** In square $ABCD$ with a side length of 1. we inscribe a second square whose vertices are the midpoints of the sides of the first square. We continue this process, inscribing squares within squares. Find the sum of the perimeters of these squares.