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FinKont0: The Final Project

The task for the bachelor project is to answer the following questions – and to compose a fluent report (in Danish or English as you see fit) that can be read on a stand-alone basis.

The deadline is 12:00 CET on Friday June 10, 2022. You must hand in via Digital Eksamen. Give your project the title "FinKont0: `[Anything you want]`"

Empirical analysis of a linear SDE

Consider a linear stochastic differential equation (SDE)

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dW(t).$$

It is also known as a Gaussian Ornstein-Uhlenbeck process or the Vasicek model.

Show that

$$X(t+u)|\mathcal{F}_t \sim N\left(X(t)e^{-\kappa u} + \theta(1 - e^{-\kappa u}), \frac{\sigma^2(1 - e^{-2\kappa u})}{2\kappa}\right)$$

Hint: Look at the dynamics of the process defined by $Z(t) = e^{\kappa t}X(t)$.

(You are not asked to prove the following – it is a small lesson stochastic calculus and theoretical statistics.) The Markov-property of X ensures that it does not matter whether we condition on all information ($\mathcal{F}_t$) or just on X_t. This means that with Δt-equidistant observations of X ($n+1$ of them; we denote by x_i the observation of $X_{i\Delta t}$ and call this our data) the likelihood function is

$$L(\text{data}; \theta, \kappa, \sigma) = \prod_{i=1}^{n} \phi\left(x_i; x_{i-1}e^{-\kappa\Delta t} + \theta(1 - e^{-\kappa\Delta t}), \frac{\sigma^2(1 - e^{-2\kappa\Delta t})}{2\kappa}\right),$$

where

$$\phi(y; \mu, s^2) = \frac{1}{s\sqrt{2\pi}}e^{-\frac{(y-\mu)^2}{2s^2}}$$

is the density function for the normal distribution with mean μ and variance s^2. Estimating parameters by maximizing the likelihood function is a good idea. The best there is, in fact.

Derive closed-form expressions for the maximum likelihood estimators of θ, κ, and σ. Hint: Reparameterizing $e^{-\kappa\Delta t} \rightsquigarrow a$, $\theta(1 - e^{-\kappa\Delta t}) \rightsquigarrow b$, $\sigma^2(1 - e^{-2\kappa\Delta t})/(2\kappa) \rightsquigarrow v$ seems convenient. **Does this** work if observations are not equidistant?

How would you estimate standard errors of estimators, find confidence bands? Hint: We find/estimate the observed information by finding (numerically) the second derivative (matrix) of the log-likelihood functiion and the use general asymptotic theory for maximum likelihood estimation. It comes down to getting the scaling and inversion right! **Conduct** a simulation experiment to verify your calculations and investigate small sample behaviour of estimators.

Download and plot daily data (as far back as you can go) on the volatility index called VIX (eg. from Yahoo Finance). You can read something about what VIX is on Wiki and many, many other places – but that is not particularly important right now. **Estimate** parameters and give 95% confidence bands for two model specifications: One where VIX is assumed to follow a linear SDE, and on where the logarithm of VIX is assumed to follow a linear SDE. **Which** model specification would you say fits the data best – and why?

Pricing and hedging in the Black-Scholes model

Let us first look at a generic Geometric Brownian motion (ie. no finance context yet) X, which we write as:

$$dX_t = \beta X_t dt + \xi X_t dW_t.$$

Let $K > 0$, $T \geq 0$, and α be constants. **Prove** the very useful result that

$$\begin{aligned} e^{-\alpha T}\mathbf{E}((X(T)-K)^+) = {} & e^{(\beta-\alpha)T}X(0)\Phi\left(\frac{\ln(X(0)/K)+(\beta+\xi^2/2)T}{\xi\sqrt{T}}\right) \\ & -e^{-\alpha T}K\Phi\left(\frac{\ln(X(0)/K)+(\beta-\xi^2/2)T}{\xi\sqrt{T}}\right). \end{aligned}$$

Hint: Look at $Y(t) = e^{(\beta-\alpha)(T-t)}X(t)$, find $dY(t)$, and reuse the Black-Scholes call formula calculation.

We now go to the Black-Scholes model: Interest rate r, stock price P-dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P,$$

and Q denoting the martingale (or risk-neutral measure).

Show that the time t-Δ for a call option (ie. the derivative of the option price wrt. the stock price) is indeed

$$\Delta(S_t, t) = \Phi\left(\frac{\ln(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Consider now a digital (or binary) option, ie. a option that pays $\mathbf{1}_{S_T \leq K}$ at time T, where $\mathbf{1}$ means the indicator function. **Calculate** the arbitrage-free time t-price of the digital option – and **do likewise** for its Δ.

Give a detalied description of the discrete Δ-hedging experiment consider in class – where we used used this R-code. Use the code to investigate (in all cases: explain your results):

- What happens to the terminal hedge error for the call option if we use $\mu \neq r$?
- What happens to the terminal hedge error for the call option if we put `sigma` and `sigma_hedge` to different values in the code?
- How does the standard deviation of the terminal hedge error depend on the number of hedge points, `Nhedge` is the code?
- What if we consider the digital option instead?

Consider now so-called no-touch version of the digital option, ie. an option whose payoff at time T is $\mathbf{1}_{M_T \leq K}$, where M denotes the maximal stock price,

$$M(t) = \max_{0 \leq u \leq t} S(u).$$

Derive a closed-form expression for the time 0-price of the no-touch digital. Hint: Use arguments like these. Suppose you want to calculate/verify the no-touch digital price by simulation. **What** would you do and which complications could you encounter?

Empirical hedging

Download and plot daily data on the S&P500 index (eg. from Yahoo Finance again). **Make** a dataset where the S&P500 data lines up with the VIX data from the first part of the project. **Plot** the two times series together (in an appropriate way). **What** do you see?

Suppose the S&P500 index follows a Geometric Brownian motion. Using the whole full time series, **what** do your esitmate of the volatility be? (Let's call this σ_∞.) **What** if you estmate using some rolling window (or/and exponentially weighted moving average)? **How** does this comapre to the VIX (divided by 100)?

Now split you data into non-overlapping blocks of 21 consecutive days – a stylized business month. (So about 400 blocks.) Consider an experiment – like in the second part of the project – where we want to do daily Δ-hedging of one-month, at-the-money options. (And where we put the interest rate to 0.) Using the S&P500 block paths rather the simulated paths and σ_∞ as hedge volatility, **what** do hedge errors look like (report descriptive statistics, plot histograms, show time series, ...)? **What** would they look like if Geometric Brownian motion were the true model?

Does it improve hedge performance if instead you use a moving window estimate as hedge volatility, or the VIX(/100), or ? (There is room for innovation/creativity here – but be sure to describe what you do in a precise way.)