Time Synchronization in WSNs with Random Bounded Communication Delays

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***Abstract*—Recently, it was pointed out that many popular consensus-based time synchronization algorithms like Average TimeSynch (ATS) *et al.* are all divergent in a network with arbitrarily small random communication delays under any interconnection topology. This note proposes a least square estimation based time synchronization (LSTS for short) algorithm. This algorithm modifies the ATS algorithm in relative drift estimator, low-pass filter and consensus protocol. By analyzing mathematic model of the algorithm, it is proved that LSTS ensures asymptotic convergence of the drift estimation and boundedness of the global time synchronization error under random bounded communication delays if the network topology satisfies some connectedness conditions.**

***Index Terms*—time synchronization, convergence, wireless sensor network, consensus.**

I. INTRODUCTION

Consensus-based time synchronization (CBTS) protocols use some consensus algorithms to estimate drift and/or offset of a common virtual clock [1], [2], [3], [4], or directly estimate a common virtual time [5]. CBTS protocols do not use any special node such as root or gateway as a reference, and all nodes run exactly the same algorithm. Therefore, they belong to the category of distributed global time synchronization protocols. Compared with spanning-tree based protocols such as Timing-Sync Protocols for Sensor Networks (TPSN) [6] and Flooding Time Synchronization Protocol (FTSP) [7], CBTS algorithms have a number of advantages including robustness against node failure and new node appearance.

How to maintain good performance and convergence of CBTS algorithms under measurement noises and communication delays is a challenging problem. Average TimeSynch (ATS) protocol proposed in [8] estimates relative clock drifts using a distributed low-pass filter to depress measurement noises and compensates both clock drifts and offsets using an average consensus algorithm. It is proved in [8] that ATS is exponentially convergent in an ideal circumstance without communication delays. Liao and Barooah consider the average consensus based algorithm for drift and offset estimations with noisy relative measurements in networks with undirected Markovian jumping topologies [9]. By assuming that the relative measurement noises are symmetric, they prove that the estimation error is mean-square convergent. Regarding communication delays, it is shown in [10] that the clock cannot be synchronized precisely if there are asymmetric time delays in a network. Yet, when the delays are time-invariant, the clock drift can be precisely determined, and the time synchronization error is caused just by the estimation error of clock offset and thus is bounded [10]. He *et al.* propose a weighted maximum time synchronization (WMTS) algorithm to deal with random communication delays [11], and prove that the expectation of the drift estimation error of WMTS is finite-time convergent, and the drift estimation is asymptotically mean-square convergent in time-invariant connected networks.

However, it is pointed out recently in [12] that for a network with random delays the asymptotic convergence of drift estimation is only a necessary condition for the boundedness of the time synchronization error, and the time synchronization error can be unbounded even if drift estimations of all the nodes converge to a common value asymptotically in a converging rate slower than $O(1/k)$, where k represents the discrete time. It is also shown in [12] that many popular CBTS algorithms including ATS and WMTS are divergent in networks with arbitrarily small random communication delays under any interconnection topology. This gives a strong motivation to design CBTS algorithms with bounded convergence. The author proposes a Least Square estimation based Time Synchronization (LSTS) algorithm in the conference version of this note [13] without proof of convergence. This note analyzes the convergence of LSTS and provides sufficient conditions to ensure that the drift estimate in LSTS converges in a rate of $O(1/k)$, and the time synchronization error is bounded as time goes to infinity.

Notations of this note are collected here. $\mathbb{N}^+$ denotes the set of all nonnegative integers. $[\cdot]_{ij}$ denotes the (i,j)th entry of a matrix $[\cdot]$. For $u \in \mathbb{R}^n$, we use $[u]_\vee$ and $[u]_\wedge$ to denote the smallest and the largest element of u, respectively. For $u_k \in \mathbb{R}^n, v_k \in \mathbb{R}^n, k \in \mathbb{N}^+$, $v_k = o(u_k)$ implies that $\frac{\|v_k\|}{\|u_k\|} \to 0$ as $k \to \infty$; $v_k = O(u_k)$ implies that there exist constants $\alpha > 0$ and $k_0 > 0$ such that $\|v_k\| \le \alpha \|u_k\|$ for all $k > k_0$; $v_k = \Theta(u_k)$ implies that $v_k = O(u_k)$ and $u_k = O(v_k)$. The interconnection topology of a WSN at time instant k is represented by a digraph $G(k) = (\mathcal{N}, \mathcal{E}(k))$, where $\mathcal{N} = \{1, 2, \cdots, n\}$ is the node set, $\mathcal{E}(k)$ is the edge set which defines all the available communication links at k. The symbol $\mathcal{N}_j$ represents the set of neighbors of node j. For two digraphs $G_1 = (\mathcal{N}, \mathcal{E}_1)$ and $G_2 = (\mathcal{N}, \mathcal{E}_2)$, their union is denoted by $G_1 \cup G_2 := G(\mathcal{N}, \mathcal{E}_1 \cup \mathcal{E}_2)$; and their composition is denoted by $G_2 \circ G_1 := G(\mathcal{N}, \mathcal{E}_c)$, where $\mathcal{E}_c$ is defined in such a way so that $(i,j) \in \mathcal{E}_c$ just in case

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there is a node v such that $(i, v) \in \mathcal{E}_1$ and $(v, j) \in \mathcal{E}_2$ [14]. $\bigodot_{m=0}^{\kappa_{l+1}-\kappa_l-1} G(\kappa_l + m) = G(\kappa_{l+1} - 1) \circ \cdots \circ G(\kappa_l + 1) \circ G(\kappa_l)$ and $\bigodot_{m=0}^{0} G(\kappa_l + m) = G(\kappa_l)$.

The rest of this note is organized as follows. In Section II we state the time synchronization problem under random communication delays. The LSTS algorithm is also presented in this section. Section III provides rigorous theoretical analysis of the convergence of LSTS under random bounded communication delays. Simulation results are presented in Section IV and conclusion is drawn in Section V.

II. Time synchronization over WSN with random delays

Each node in the network has its own local clock whose dynamics are given by an affine function of time:

$$\tau_i(t) = \alpha_i t + \beta_i, \tag{1}$$

where t represents the absolute time which is unknown to all the nodes in the network, τ_i is the local clock reading, α_i is the local clock drift and β_i is the local clock offset. Note that α_i and β_i may have different values for different clocks but they are assumed to be time-invariant in this note. Denote by α_{ji} the relative drift of node j with respect to node i, which is defined by $\alpha_{ji} = \frac{\alpha_i}{\alpha_j}$.

In a WSN with more than two nodes, to achieve time synchronization each node may construct its virtual clock by

$$\widehat{\tau}_i(t) = \widehat{\alpha}_i \tau_i(t) + \widehat{\beta}_i, \tag{2}$$

where $\widehat{\alpha}_i$ and $\widehat{\beta}_i$ are called drift compensation and offset compensation, respectively, of node i, which are to be designed. It is said that a bounded time synchronization is realized in WSN if there exist constants $\varepsilon > 0$ and $t_f > 0$ such that $|\widehat{\tau}_i(t) - \widehat{\tau}_j(t)| < \varepsilon$ for all $i, j \in \mathcal{N}$ and all $t > t_f$. Fig. 1 illustrates the notations of local clocks and virtual clocks.

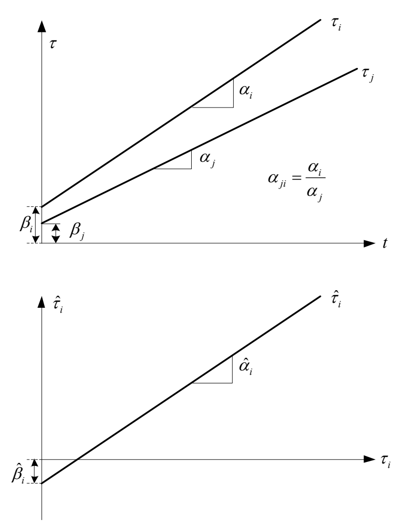

Fig. 1. Local reading vs absolute time and virtual clock vs local reading

In the process of time synchronization, each node broadcasts its packet including the packet order l ($l = 0, 1, \cdots$), local time $\tau_i(t_l^i)$, and drift compensation $\widehat{\alpha}_i(t_l^i)$ periodically, where t_l^i represents the lth broadcast instant of node i in the absolute reference time. Let the broadcast period be T. Obviously, T is pseudo-periodic, and the real sending period of node i in the absolute reference time is $\widehat{T}_i = \frac{T}{\alpha_i}$. Suppose that node j receives the lth packet of node i at its local time $\tau_j(t_l^{\rm id})$. As shown by Fig. 2, the time-delay of the lth packet from node i to node j in the absolute time is denoted by

$$d_{ji}(t_l^{\rm id}) = t_l^{\rm id} - t_l^i, \tag{3}$$

which might be a random number in most practical cases. Each node updates its virtual clock as soon as it receives a packet from any other node. To avoid ordering chaos, we make the following assumption.

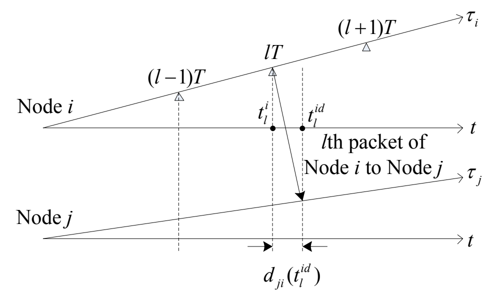

Fig. 2. Time delay between node i and node j

Assumption 1. There exists a positive real number $M_d \in (0, \min_{i \in \mathcal{N}} \widehat{T}_i)$ such that

$$0 < d_{ji}(t_l^{\rm id}) \le M_d \tag{4}$$

for all $i, j \in \mathcal{N}$. Each node does not update its virtual clock in a time slot (dormancy slot) of width M_d just after its each broadcast instant, and drops the packets received from its neighbors in any dormancy slot.

In the rest of this section, we briefly introduce the LSTS algorithm proposed in the conference version of this note [13], and explain why it avoids the divergence conditions presented in [12].

(1) *Longspan relative drift estimator*

Suppose that node j gets successively messages from node i and records in its memory the triples $\big(0, \tau_j(t_0^{\rm id}), \tau_i(t_0^i)\big)$, $\big(1, \tau_j(t_1^{\rm id}), \tau_i(t_1^i)\big)$, $\cdots$, $\big(l, \tau_j(t_l^{\rm id}), \tau_i(t_l^i)\big)$, $\cdots$. Note that l is the sending order of agent i instead of the receiving order of agent j. In ATS and most other CBTS algorithms, α_{ij}, the inverse relative drift of node j with respect to node i, is estimated by two successive messages. In LSTS, a longspan estimator for this parameter is proposed as:

$$\hat{\alpha}_{ij}(l) = \frac{\tau_j(t_l^{\rm id}) - \tau_j(t_0^{\rm id})}{\tau_i(t_l^i) - \tau_i(t_0^i)}, \ l = 1, 2, \cdots. \tag{5}$$

According to the local clock (1), the pseudo-periodic broadcast protocol, and the delay expression (3), we have

$$\begin{aligned}
&\tau_i(t_l^i) - \tau_i(t_0^i) = lT = \alpha_i l \widehat{T}_i, \\
&\tau_j(t_l^{\rm id}) - \tau_j(t_0^{\rm id}) = \alpha_j t_l^{\rm id} + \beta_j - (\alpha_j t_0^{\rm id} + \beta_j) \\
&= \alpha_j (t_l^i + d_{ji}(t_l^{\rm id})) - \alpha_j (t_0^i + d_{ji}(t_0^{\rm id})) \\
&= \alpha_j (l\widehat{T}_i + d_{ji}(t_l^{\rm id}) - d_{ji}(t_0^{\rm id})).
\end{aligned}$$

Hence, (5) can be rewritten as

$$\hat{\alpha}_{ij}(l) = \alpha_{ij}(1+\hat{\delta}_{ij}(l)), \tag{6}$$

where $\hat{\delta}_{ij}(l) = \frac{d_{ji}(t_l^{\mathrm{id}})-d_{ji}(t_0^{\mathrm{id}})}{l\widehat{T}_i}$. By (4), we have $|\hat{\delta}_{ij}(l)| \le \frac{M_d}{l\widehat{T}_i}$. Therefore, $\hat{\delta}_{ij}(l)$ decays in a rate of $O(1/l)$.

(2) *Low-pass filter design by the least square principle*

To reduce the influence of noises, ATS uses a time-invariant low-pass filter to recursively estimate the relative drift. Instead, LSTS uses an optimal low-pass filter based on the least square principle. Let $\hat{\alpha}^*_{ij}(l)$ be the estimator of the inverse relative drift to be designed for time t_l^{id}. Denote by $e_k = \hat{\alpha}_{ij}(k)\tau_i(t_k^i) - \hat{\alpha}^*_{ij}(l)\tau_i(t_k^i), k=1,2,\cdots,l$. We propose a least square index as $J = \sum_{k=1}^{l} e_k^2$. Then, $\frac{\partial J}{\partial \hat{\alpha}^*_{ij}(l)} = 0$ with consideration of $\tau_i(t_k^i) = kT$ yields the optimal estimator as follows

$$\hat{\alpha}^*_{ij}(l) = \frac{\hat{\alpha}_{ij}(1) + 2^2\hat{\alpha}_{ij}(2) + \cdots + l^2\hat{\alpha}_{ij}(l)}{1+2^2+\cdots+l^2}. \tag{7}$$

Equivalently, $\hat{\alpha}^*_{ij}(l)$ can be also calculated recursively as follows

$$\hat{\alpha}^*_{ij}(l) = (1-\rho^*(l))\hat{\alpha}^*_{ij}(l-1) + \rho^*(l)\frac{\tau_j(t_l^j)-\tau_j(t_0^j)}{\tau_i(t_l^i)-\tau_i(t_0^i)}, \tag{8}$$

where

$$\rho^*(l) = \frac{l^2}{1+2^2+\cdots+l^2}, \tag{9}$$

and the initial condition is set as $\hat{\alpha}^*_{ij}(0) = 1$.

Substituting (5) into (7) yields

$$\hat{\alpha}^*_{ij}(l) = \alpha_{ij}(1+\hat{\delta}^*_{ij}(l)), \tag{10}$$

where

$$\hat{\delta}^*_{ij}(l) = \frac{\sum_{m=1}^{l} m(d_{ji}(t_m^{\mathrm{id}}) - d_{ji}(t_0^{\mathrm{id}}))}{\frac{l(l+1)(2l+1)}{6}\widehat{T}_i}. \tag{11}$$

By (4), we have $|\hat{\delta}^*_{ij}(l)| \le \frac{M_d \sum_{m=1}^{l} m}{\frac{l(l+1)(2l+1)}{6}\widehat{T}_i} = \frac{3M_d}{(2l+1)\widehat{T}_i}$, which implies that the decay rate of $\hat{\delta}^*_{ij}(l)$ is still $O(1/l)$. This decay rate ensures that LSTS avoids the divergence conditions given by [12].

(3) *Consensus-based compensation with a decaying factor*

At time instant t_l^i node i broadcasts its lth packet including the drift compensation $\widehat{\alpha}_i$ and clock reading $\widehat{\tau}_i$. Its neighboring node j receives this packet at time t_l^{id}. By Assumption 1, node i will not make new updating in time period $(t_l^i, t_l^{\mathrm{id}}]$. So, $\widehat{\alpha}_i(t_l^i) = \widehat{\alpha}_i(t_l^{\mathrm{id}})$ and $\widehat{\tau}_i(t_l^i) = \widehat{\tau}_i(t_l^{\mathrm{id}})$.

Based on the received information from node i, at time instant t_l^{id} node j updates its drift compensation as follows

$$\begin{aligned}\widehat{\alpha}_j(t^+) &= (1-a(t)\rho_a)\widehat{\alpha}_j(t) + a(t)\rho_a(\hat{\alpha}^*_{ij}(l))^{-1}\widehat{\alpha}_i(t),\\ &\quad i \in \mathcal{N}_j(t),\ t = t_l^{\mathrm{id}},\end{aligned} \tag{12}$$

where

$$a(t) = \frac{1}{(1+l)^{\mu}},\ 0<\mu<1. \tag{13}$$

is a piece-wide constant function depending on l. Obviously, $\widehat{\alpha}_j(t)$ is its old drift compensation, $(\hat{\alpha}^*_{ij}(l))^{-1}\widehat{\alpha}_i(t)$ is an estimation of node j's drift compensation based on node j's relative drift estimation and node i's drift compensation. At time instant t_l^{id} node j's offset compensation is updated as follows

$$\begin{aligned}\widehat{\beta}_j(t^+) &= \widehat{\beta}_j(t) + \rho_b(\widehat{\tau}_i(t_l^i) - \widehat{\tau}_j(t_l^{\mathrm{id}})),\\ &\quad i \in \mathcal{N}_j(t),\ t = t_l^{\mathrm{id}}.\end{aligned} \tag{14}$$

In (12) and (14), ρ_a and ρ_b are arbitrary positive constants in the interval $(0,1)$; $\widehat{\alpha}_i(t) = \widehat{\alpha}_i(t_l^{\mathrm{id}}) = \widehat{\alpha}_i(t_l^i)$ is the drift compensation of node i at t_l^i; the initial conditions for all the nodes are set as $\widehat{\alpha}_j(0) = 1$, and $\widehat{\beta}_j(0) = 0$.

This part of the algorithm is almost the same as that of ATS [8] except the introduction of a decaying factor $a(t)$ in the drift compensation. The decaying factor is a well known technique for depressing additive noises in stochastic approximation and consensus control [15], [16]. Recently, it was applied to a time synchronization algorithm called DiSync in a logarithm form by [17]. Here, we introduce the decaying factor to further speed up the decay of the influence of the relative drift estimation error, which is caused by random communication delays.

A combination of (8), (9), (12), (13) and (14) is referred to as the Least Square estimation based Time (LSTS) algorithm.

III. Convergence analysis

A. *Consensus under multiplicative uncertainty*

As a starting point of our convergence analysis, we consider the consensus problem for the following discrete-time system with multiplicative uncertainty:

$$x(k+1) = (I - (L_{G(k)} + \Delta(k)))x(k), \tag{15}$$

where $x(k) \in R^n$ is the state vector, $L_{G(k)} \in R^{n\times n}$ is the Laplacian matrix of a time-varying digraph $G(k)$, $\Delta(k) \in R^{n\times n}$ represents some multiplicative perturbation. The system is said to reach a consensus asymptotically if

$$\lim_{k\to\infty} |x_i(k) - x_j(k)| = 0 \tag{16}$$

for all $i,j \in \mathcal{N}$.

According to [18], we say that the time-varying digraph $G(k)$ is K-uniformly composed strongly rooted (K-UCSR) if there exist a positive integer $K<\infty$ and an integer sequence $0=\kappa_0<\kappa_1<\cdots<\kappa_l\cdots$, such that $1 \le \kappa_{l+1}-\kappa_l \le K$, and $G_{Cl} = \bigodot_{m=0}^{\kappa_{l+1}-\kappa_l-1} G(\kappa_{l+m})$ are strongly rooted for all $l \in \mathbb{N}^+$. Obviously, a strongly rooted digraph is 1-UCSR.

The following result proved in [18] is crucial to our further analysis.

Lemma 1. Suppose that (1) $G(k)$ is K-UCSR; (2)$a(k)>0$ for all $k \in \mathbb{N}^+$, $a(k) = \Theta\left(\frac{1}{(1+k)^{\mu}}\right)$, with $\mu \in [0,1]$ satisfying $K\mu < 1$, and $I - a(k)L_{G(k)}$ is a stochastic matrix; (3) there exist non-negative constants $\hat{k}$, $\bar{b}$ and $\gamma > 1$, such that $(1+k)^{\gamma}a(k)|[\Delta(k)]_{ij}| \le \bar{b}$ for all $i,j \in \mathcal{N}$ and $k \ge \hat{k}$. Then, system (15) reaches a consensus at a constant asymptotically, and the convergence rate of $|x_i(k)-x_j(k)|$ is $O\left(\frac{1}{k^{\gamma-K\mu}}\right)$ for all $i,j \in \mathcal{N}$.

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B. Consensus of drift estimations

Now, let us substitute (10) into (12) and multiply two sides of (12) by α_j,

$$\begin{aligned}&\widehat{\alpha}_j(t^+)\alpha_j = (1-a(t)\rho_a)\widehat{\alpha}_j(t)\alpha_j\\&+a(t)\rho_a\left(1+\hat{\delta}^*_{ij}(l)\right)^{-1}\widehat{\alpha}_i(t)\alpha_i,\ i\in\mathcal{N}_j(t),\ t=t_l^{\rm id}.\end{aligned} \tag{17}$$

Suppose that node j makes the kth updating counted in the entire WSN at instant $t_l^{\rm id}$. Then, we denote the time instant t_l^{id} as t_k. And, subsequently, we rewrite l as $l_{ji}(k)$, which represents the sending order of the packet received by agent j from node i at t_k. Let us introduce a discrete-time state variable as $x_j(k)=\widehat{\alpha}_j(t_k)\alpha_j$ with a little abuse of notation $k=t_k$. Note that $x_j(k)$ is actually the drift estimation of virtual clock $\widehat{\tau}_j$ at time k.

Since it has been assumed that each node does not updates its state during the dormancy slot, $\widehat{\alpha}_i(t)$ is invariant in the time interval $[t_l^i, t_l^{\rm id}]$, hence $\widehat{\alpha}_i(t_l^i)\alpha_i = x_i(k)$.Then, by (17), $x_j(k)$ can be expressed by the following discrete-time model

$$\begin{aligned}x_j(k+1) &= (1-a(k))\rho_a)x_j(k)\\&+a(k)\left(\rho_a-\delta_{ji}(k)\right)x_i(k),\ i\in\mathcal{N}_j(k)\end{aligned} \tag{18}$$

where

$$\delta_{ji}(k)=\frac{\rho_a\hat{\delta}^*_{ij}(k)}{1+\hat{\delta}^*_{ij}(k)} \tag{19}$$

and

$$a(k)=\frac{1}{(1+l_{ji}(k))^\mu},\quad \mu\in(0,1). \tag{20}$$

For notation convenience we use J_k to denote the set of nodes which update estimations at t_k. Obviously, if node $j\notin J_k$, then $x_j(k+1)=x_j(k)$, and $\delta_{ji}(k)=0$.

Now, let $x(k)=[x_1(k),x_2(k),\cdots,x_n(k)]^T$. Under Assumption 1, the dynamics of drift estimations of all the nods in a WSN can be expressed in a vector form exactly as (15), in which $L_{G(k)}\in\mathbb{R}^{n\times n}$ is given by

$$\begin{cases}[L_{G(k)}]_{ji}=\begin{cases}-\rho_a,\ i\in\mathcal{N}_j(k), i\neq j,\\ 0,\ i\notin\mathcal{N}_j(k), i\neq j,\\ \rho_a,\ i=j\end{cases} & \text{if } j\in J_k,\\ [L_{G(k)}]_{ji}=0,\ \forall i\in\mathcal{N}, & \text{if } j\notin J_k;\end{cases} \tag{21}$$

and $\Delta(k)=\in\mathbb{R}^{n\times n}$ is given by

$$\begin{cases}[\Delta(k)]_{ji}=\begin{cases}\delta_{ji}(k), i\in\mathcal{N}_j(k), i\neq j,\\ 0,\ \text{otherwise},\end{cases} & \text{if } j\in J_k,\\ [\Delta(k)]_{ji}=0,\ \forall i\in\mathcal{N}, & \text{if } j\notin J_k.\end{cases} \tag{22}$$

Obviously, $I-a(k)L_{G(k)}$ is a stochastic matrix.

Theorem 1. If the digraph $G(k)$ is K-UCSR and $K\mu<1$, then, under Assumption 1 the LSTS algorithm ensures that drift estimations of all the nodes in a WSN asymptotically converge to a common constant, and the convergence rate of the drift estimation error $|\widehat{\alpha}_i(k)\alpha_i-\widehat{\alpha}_j(k)\alpha_j|$ is $O\left(\frac{1}{k^{1+\mu-K\mu}}\right)$ for all $i,j\in\mathcal{N}$.

Proof: We use Lemma 1 to prove this theorem. We have checked that $I-a(k)L_{G(k)}$ is a stochastic matrix. Obviously, $a(k)>0$ since $a(k)=\frac{1}{(1+l_{ji}(k))^\mu}$, $0<\mu<1$ in LSTS. By (19), (20), (11) and (4), we have

$$\begin{aligned}&|a(k)\delta_{ji}(k)|\\ \leq\ & \frac{\frac{(l_{ji}(k)+1)l_{ji}(k)}{2(l_{ji}(k)+1)^\mu}\rho_a M_d}{\frac{l_{ji}(k)(l_{ji}(k)+1)(2l_{ji}(k)+1)}{6}\widehat{T}_i-\frac{(l_{ji}(k)+1)l_{ji}(k)}{2}\widehat{T}_i}\\ =\ & \frac{3\rho_a M_d}{(l_{ji}(k)+1)^\mu(2l_{ji}(k)-1)\widehat{T}_i}.\end{aligned}$$

According to our communication protocol we have

$$\begin{aligned}&(1-\eta)n\left(l_{ji}(k)\min_{i,j\in\mathcal{N}}\left(\hat{T}_i/\hat{T}_j\right)-1\right)\leq k\\ &\leq nl_{ji}(k)\max_{i,j\in\mathcal{N}}\left(\hat{T}_i/\hat{T}_j\right),\end{aligned} \tag{23}$$

for $j\in J_k, i\in\mathcal{N}_j, k\in\mathbb{N}^+$, where $\eta\in(0,1)$ denotes the maximum packet-dropping rate in the network. Denote $P_u=\max_{i,j\in\mathcal{N}}\left(\hat{T}_i/\hat{T}_j\right)$. Then, by (23), we know that

$$(1+k)^{1+\mu}|a(k)\delta_{ji}(k)|\leq\frac{3\rho_a M_d(nP_u)^{1+\mu}}{\widehat{T}_i}:=\bar{b}, \tag{24}$$

when $k>nP_u+1$.

Noticing that

$$1+l_{ji}(k)\leq 1+l_{ji}(k+1)P_u\leq(1+l_{ji}(k+1))P_u,$$

we have $a(k+1)\leq P_u^\mu a(k)$, for all $k\in\mathbb{N}^+$, which implies that $a(k+1)=O(a(k))$. Yet, from (23) it follows that

$$\frac{(1-\eta)^\mu n^\mu P_l^\mu}{(1+k)^\mu}\leq\frac{1}{(1+l_{ji}(k))^\mu}\leq\frac{n^\mu P_u^\mu}{(1+k)^\mu}, \tag{25}$$

for all $k>\frac{(1-\eta)n(1+P_l)-1}{1-P_l}$, where $P_l=\min_{i,j\in\mathcal{N}}\left(\hat{T}_i/\hat{T}_j\right)$. This implies that $a(k)=\Theta\left(\frac{1}{(1+k)^\mu}\right)$.

Up to now, we have checked that conditions (2) and (3) of Lemma 1 are all satisfied for the LSTS algorithm. Since condition (1) is assumed, we have arrived at the conclusion of Theorem 1. □

C. Boundedness of time synchronization errors

Let $z_j(t)=\widehat{\beta}_j(t)+\widehat{\alpha}_j(t)\beta_j$ be the offset estimation of virtual clock $\widehat{\tau}_j$ at time t. Then, by (14) we get

$$\begin{aligned}&z_j(t^+)=(1-\rho_b)z_j(t)+\beta_j(\widehat{\alpha}_j(t^+)-\widehat{\alpha}_j(t))\\&+\rho_b\left(z_i(t)+(\widehat{\alpha}_i(t)\alpha_i-\widehat{\alpha}_j(t)\alpha_j)t_l^i-\widehat{\alpha}_j(t)\alpha_j d_{ji}(t_l^{id})\right),\end{aligned}$$

where $t=t_l^{\rm id}$. Similar to the modeling of drift estimation, we denote the time instant $t_l^{\rm id}$ as t_k. Noticing that $t_l^i=\widehat{T}_i l_{ji}(k)$, we have

$$\begin{aligned}&z_j(k+1)=(1-\rho_b)z_j(k)+\rho_b z_i(k)\\&+\frac{\beta_j}{\alpha_j}(x_j(k+1)-x_j(k))\\&+\rho_b\left((x_i(k)-x_j(k))\widehat{T}_i l_{ji}(k)-x_j(k)d_{ji}(k)\right).\end{aligned}$$

Let $z(k)=[z_1(k),z_2(k),\cdots,z_n(k)]^T$. Then, the dynamics of offset estimations of all the nods in a WSN can be expressed in a vector form as follows

$$z(k+1)=(I-L_{G(k)})z(k)+v(k), \tag{26}$$

where $L_{G(k)} = \{l_{ji}^G(k)\} \in \mathbb{R}^{n\times n}$ is given by

$$\begin{cases} l_{ji}^G(k) = \begin{cases} -\rho_b, \ i \in \mathcal{N}_j(k), i \neq j, \\ 0, \ i \notin \mathcal{N}_j(k), i \neq j, \\ \rho_b, \ i = j \end{cases} & \text{if } j \in J_k, \\ l_{ji}^G(k) = 0, \ \forall i \in \mathcal{N}, & \text{if } j \notin J_k; \end{cases} \tag{27}$$

and $v(k) = [v_1(k), \cdots, v_j(k), \cdots, v_n(k)]^T$ is given by

$$\begin{aligned} v_j(k) &= \frac{\beta_j}{\alpha_j}(x_j(k+1) - x_j(k)) \\ &\quad +\rho_b\left(\widehat{T}_i(x_i(k) - x_j(k))l_{ji}(k) - x_i(k)d_{ji}(k)\right), \\ &\quad i \in \mathcal{N}_j(k), \end{aligned} \tag{28}$$

if $j \in J_k$, otherwise, $v_j(k) = 0$.

Next theorem indicates that LSTS guarantees that the global steady-state time synchronization error is bounded.

Theorem 2. If the digraph $G(k)$ is 1-UCSR, then, under Assumption 1 the LSTS algorithm ensures that there exist $k^* > 0$ and $W > 0$ such that

$$|z_i(k) - z_j(k)| \leq W \max_{j\in\mathcal{N}} \widehat{T}_j/(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j) + \bar{\alpha}M_d, \tag{29}$$

and

$$|\widehat{\tau}_i(k) - \widehat{\tau}_j(k)| \leq 2W \max_{j\in\mathcal{N}} \widehat{T}_j/(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j) + \bar{\alpha}M_d \tag{30}$$

for all $i, j \in \mathcal{N}$ and all $k > k^*$, where $\bar{\alpha}$ is the consensus value of the drift estimation.

Proof: Let $V_k = [z(k)]_\wedge - [z(k)]_\vee$. Then,

$$\begin{aligned} V_{k+1} &= [z(k+1)]_\wedge - [z(k+1)]_\vee \\ &\leq \left[(I - L_{G(k)})z(k)\right]_\wedge - \left[(I - L_{G(k)})z(k)\right]_\vee \\ &\quad +[v(k)]_\wedge - [v(k)]_\vee. \end{aligned}$$

Since $P(k) = I - L_{G(k)}$ is strongly rooted, by Lemma 2 of [8] we have

$$[(I - L_{G(k)})z(k)]_\wedge - [(I - L_{G(k)})z(k)]_\vee \leq (1-\rho_b)V_k. \tag{31}$$

Now, let us look at (28). By applying Theorem 1 with $K = 1$ it is easy to get that $x_j(k+1) - x_j(k) \to 0$, $x_i(k) - x_j(k) \to 0$, and $x_j(k) \to \bar{\alpha}$ as $k \to \infty$, where $\bar{\alpha} \in \mathbb{R}$ is a constant; and there exists a constant $W \in \mathbb{R}$ such that

$$\begin{aligned} &(x_i(k) - x_j(k))l_{ji}(k) \leq \\ &\left(\frac{W}{1+k} + o\left(\frac{1}{1+k}\right)\right)\frac{k + (1-\eta)n}{(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j)}, \\ &\text{(using (23))} \\ &\leq W/(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j) + \delta(k), \end{aligned}$$

where $\lim_{k\to\infty} \delta(k) = 0$. Note that $0 < d_{ji}(k) \leq M_d$. Hence, we have

$$[v(k)]_\wedge - [v(k)]_\vee \leq \sigma(k) + \frac{\rho_b(\max_{i\in\mathcal{N}} \widehat{T}_i)W}{(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j)} + \rho_b\bar{\alpha}M_d, \tag{32}$$

where $\sigma(k)$ denotes the sum of decay items in $[v(k)]_\wedge - [v(k)]_\vee$, i.e., $\lim_{k\to\infty} \sigma(k) = 0$. Summarizing (31) and (32) we have

$$V_{k+1} \leq (1-\rho_b)V_k + \sigma(k) + \bar{v}, \tag{33}$$

where $\bar{v} = \rho_b(\max_{i\in\mathcal{N}} \widehat{T}_i)W/(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j) + \rho_b\bar{\alpha}M_d$. Set $u_k = V_k - \bar{v}/\rho_b$. By applying Lemma 3 in [19] (Page 45) to (33) we get $\limsup_{k\to\infty}(V_k - \bar{v}/\rho_b) \leq 0$, which implies that there exists a $k_1 > 0$ such that

$$|z_i(k) - z_j(k)| \leq \bar{v}/\rho_b, \ \forall i, j \in \mathcal{N}, \forall k > k_1,$$

which is just (29).

To prove (30), we just note that

$$\begin{aligned} \widehat{\tau}_i(k) &= \widehat{\alpha}_i(k)\tau(k) + \widehat{\beta}_i(k) \\ &= \widehat{\alpha}_i(k)(\alpha_i t_k + \beta_i) + \widehat{\beta}_i(k) \\ &= x_i(k)t_k + z_i(k). \end{aligned}$$

Therefore, by applying (29) and Theorem 1 again, we know that there exists $k^* > 0$ such that

$$\begin{aligned} |\widehat{\tau}_i(k) - \widehat{\tau}_j(k)| &= |(x_i(k) - x_j(k))t_k + z_i(k) - z_j(k)| \\ &\leq |(x_i(k) - x_j(k))t_k| + |z_i(k) - z_j(k)| \\ &\leq \frac{W}{1+k}t_k + \bar{v}/\rho_b + o\left(\frac{1}{1+k}\right) \end{aligned}$$

for all $k > k^*$. Note that $t_k \leq t_{l+1}^i = (l_{ji}(k)+1)\widehat{T}_i \leq \frac{k+2(1-\eta)n}{(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j)} \max_{j\in\mathcal{N}} \widehat{T}_j$. Then, we get

$$|\widehat{\tau}_i(k) - \widehat{\tau}_j(k)| \leq W \max_{j\in\mathcal{N}} \widehat{T}_j/(1-\eta)n(\min_{i,j\in\mathcal{N}} \hat{T}_i/\hat{T}_j) + \bar{v}/\rho_b$$

as $t \to \infty$. The theorem is thus proved. □

Theorem 2 also indicates that LSTS guarantees the global steady-state time synchronization error is affine to the bound of communication delays.

IV. Simulation

In numerical experiments, we consider a WSN with 35 nodes shown by Fig. 1. The local clock drifts $\alpha_i (i = 1, \cdots, 35)$ are randomly selected in $[0.999, 1.001]$, the local clock offsets $\beta_i (i = 1, \cdots, 35)$ are randomly selected in $[0, 5]$(s), the pseudo-periodic T is 35s. The communicative delays between each node and its neighbors are pseudo-random numbers uniformly distributed in $[0, 0.01]$(s).

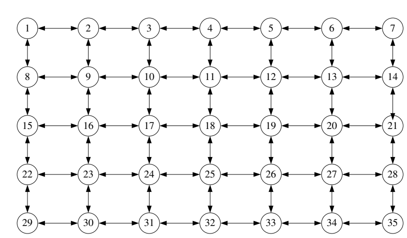

Fig. 3. A WSN with 35 nodes

Firstly, we apply the ATS algorithm with the same parameters as given in [8]. Fig. 4 presents the extremal value of synchronization errors in the WSN and the drift estimation

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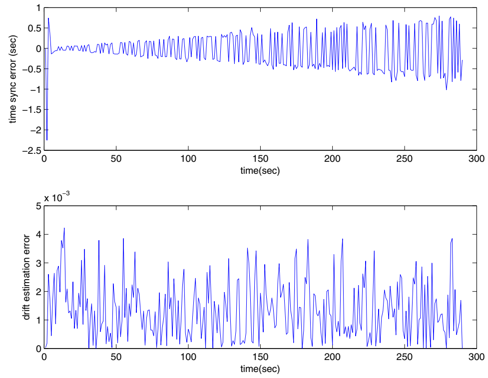

Fig. 4. Synchronization error and drift estimation error of ATS algorithm.

error between node 1 and node 8. It shows that, as time goes on, the extremal time synchronization error is growing, and the drift estimation error is oscillating. Then, we apply the LSTS algorithm to the same WSN. The parameters of the LSTS algorithm are chosen as follows: $\mu = 0.3$, $\rho_a = 0.5$, $\rho_b = 0.5$. In this case, besides communication delays, we introduce additional pseudo-random noises uniformly distributed in $[-0.005, 0.005]$(s) into time stamps and broadcast periods. The simulation results are presented in Fig. 5. The results show that LSTS is quite robust against the noises. As time goes on, the extremal time synchronization error in the network is bounded by 10^{-2}s after $t > 200$s, and the drift estimation error is approaching to zero.

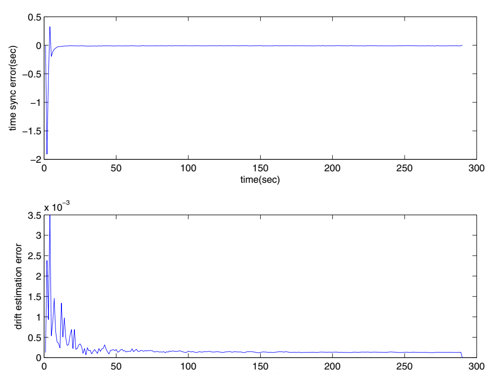

Fig. 5. Synchronization error and drift estimation error of LSTS algorithm.

V. Conclusion

Unlike [12] which discusses divergence conditions by analyzing the structure and parameters of CBTS algorithms, this note provides a topological condition for bounded convergence of LSTS - a CBTS algorithm. It is proved that under the assumption that topology digraph $G(k)$ is 1-UCSR the LSTS algorithm ensures that drift estimation errors converge in a rate of $O(1/k)$, the global time-synchronization error is bounded in networks with random bounded communication delays; and the global steady-state time synchronization error is affine to the bound of communication delays.

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