Latent Dirichlet Allocation

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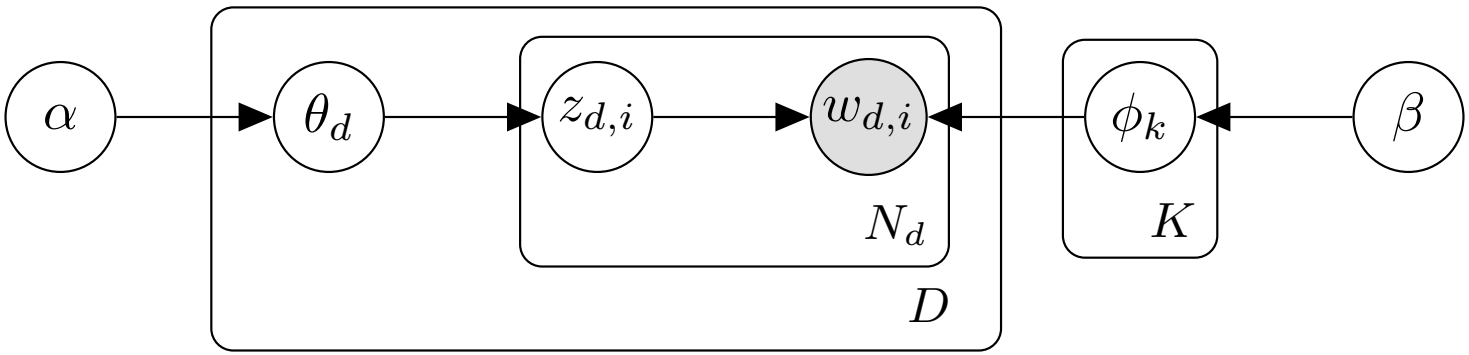
Introduction

Here is a PGM of LDA.



We want to sample the posterior distribution of the latent variable $z_{d,i}$ given $w_{d,i}$. One method is gibbs sampling, in every iteration, we sample z_i from distribution $p(z_i|z_{\neg i}, w)$. The joint distribution is

$$p(w, z|\alpha, \beta) = p(w|z, \beta)p(z|\alpha)$$

Posterior predictive of Dirichlet-Multinomial

Suppose we have dirichlet prior distribution,

$$p(\theta|\alpha) = \mathsf{Dir}(\theta|\alpha) \propto \prod_{k=1}^{K} \theta_i^{\alpha_i - 1}$$

Joint distribution I

The joint distribution is $p(w, z|\alpha, \beta) = p(w|z, \beta)p(z|\alpha)$. The first part is

$$p(w|z, \beta) = \int_{\phi_{1:K}} p(w|z, \phi_{1:K})p(\phi_{1:K}|\beta)d\phi_{1:K}$$

Let's see $p(w|z, \phi_{1:K})$ first,

$$p(w|z, \phi_{1:K}) = \prod_{i=1}^{W} \phi_{z_i, w_i}$$

Or we can rephrase it in another way, where we classify it by topic.

$$p(w|z, \phi_{1:K}) = \prod_{i=1}^{W} \phi_{z_i, w_i} = \prod_{k=1}^{K} \prod_{v=1}^{V} \phi_{k,v}^{n_k^{(v)}}$$

Joint distribution II

And

$$p(\phi_{1:K}|\beta) = \prod_{k=1}^{K} \prod_{v=1}^{V} \phi_{k,v}^{\beta_v - 1}$$

Adding these two together, we have

$$p(w|z,\beta) = \prod_{k=1}^{K} \frac{B(n_k + \beta)}{B(\beta)}$$

n_k represents the word apperance frequencies in topic k. The topic distribution $p(z|\alpha)$ can be derived similarly,

$$p(z|\alpha) = \prod_{d=1}^{D} \frac{B(n_d + \alpha)}{B(\alpha)}$$

Joint distribution III

So the conditional distribution is

$$p(z, w|\alpha, \beta) = \prod_{k=1}^{K} \frac{B(n_k + \beta)}{B(\beta)} \prod_{d=1}^{D} \frac{B(n_d + \alpha)}{B(\alpha)}$$

$$p(z_i = k|z_{\neg i}, w, \alpha, \beta) = \frac{p(w, z)|\alpha, \beta}{p(w, z_{\neg i}|\alpha, \beta)} \propto p(w, z|\alpha, \beta)$$

Thus we just sample the z_i according to the conditional distribution

$$p(z_i = k|z_{\neg i}, w, \alpha, \beta) \propto \prod_{k=1}^{K} \frac{B(n_k + \beta)}{B(\beta)} \prod_{d=1}^{D} \frac{B(n_d + \alpha)}{B(\alpha)}$$

We shall check out what is the same and what is different for different k where $z_i = k$. As a result, it can be shown that

$$p(z_i = k|z_{\neg i}, w, \alpha, \beta) \propto \Gamma(n_{k,w_i}^{\neg i})\Gamma(n_{d,k}^{\neg i})$$

Joint distribution IV

The symbol here is quite messy, if you have question, please feel free to contact me. $n_{k,w_i}^{\neg i}$ means except for this word, the frequency of word w_i in topic k. And $n_{d,k}^{\neg i}$ means that except for this word, the frequency of word in document d in topic k. This is collapsed Gibbs sampling. In implementing this, **what we need to do is keeps sampling** z, and that is enough. And after the sampling is done, the multinomial parameters can be derived as follows

$$p(\theta_d|w, z, \alpha) \sim \text{Dir}(\theta_m|n_m + \alpha)$$

$$p(\phi_d|w, z, \beta) \sim \text{Dir}(\phi_d|n_d + \beta)$$

Evidence Lower Bound(ELBO) I

$$\ln(p(X)) = \ln(\frac{p(X,Z)}{q(Z)} - \ln(\frac{p(Z|X)}{q(Z)})$$

Taking the expecttion w.r.t $q(Z)$ on both sides

$$\begin{aligned}\ln(P(X)) &= \int q(Z) \ln(\frac{p(X,Z)}{q(Z)} dZ - \int q(Z) \ln(\frac{p(Z|X)}{q(Z)}) dZ \\ &= \mathcal{L}(q) + KL(q||p)\end{aligned}$$

Evidence Lower Bound(ELBO) II

Alternative derivation using Jensen's inequality:

$$\begin{aligned}
\ln(X) &= \ln \int_Z p(X, Z) dZ \\
&= \ln \int_Z p(X, Z) \frac{q(Z)}{q(Z)} dZ \\
&= \ln \mathbb{E}_{Z \sim q(Z)} [\frac{p(X, Z)}{q(Z)}] \\
&\geq \ln \mathbb{E}_{Z \sim q(Z)} [\frac{p(X, Z)}{q(Z)}] \\
&= \int q(Z) \ln(\frac{p(X, Z)}{q(Z)}) dZ
\end{aligned}$$

Which is $\mathcal{L}(q)$. We are interested in finding the lower bound of $\ln(X)$. And when the selected $q(Z)$ is closest to $p(Z|X)$, the KL divergence is 0, therefore, minimum is achieved. There are two typical approaches of get $q(Z)$,

Evidence Lower Bound(ELBO) III

- Assume that $q(Z)$ has some factorization $q(Z) = \prod_{i=1}^{M} q_i(Z_i)$.
- Assume $q(Z)$ is in some families of distribution(e.g. exponential families).

Factorization form

Assume that $q(z)$ can be factorized as follows, $q(z) = \sum_{i=1}^{M} q_i(z_i)$

$$\begin{aligned}\mathcal{L}(q) &= \int q(z) \ln p(x,z) dz - \int q(z) \ln q(z) dz \\ &= \int \prod_{i=1}^{M} q_i(z_i) \ln p(x,z) dz - \int \prod_{i=1}^{M} q_i(z_i) \sum_{i=1}^{M} \ln q_i(z_i) dz\end{aligned}$$

If we just optimize $q_i(z_i)$ one at a time as follows,(Note that it is a iterative process)

$$\begin{aligned}\mathcal{L}(q) &= \int q_i(z_i) \mathbb{E}_{z_{\neg i}}[\ln p(x,z)] dz_i - \int q_i(z_i) \ln q_i(z_i) dz_i \\ &= \int q_i(z_i) \ln \tilde{p}(x,z_i) dz_i - \int q_i(z_i) \ln q_i(z_i) dz_i \\ &= \int q_i(z_i) \ln \frac{\tilde{p}(x,z_i)}{q_i(z_i)} = KL[\tilde{p}(x,z_i) || q_i(z_i)]\end{aligned}$$

Exponential family distribution I

Exponential family distribution is a rich family of distribution which computational tractable which is suitable for variational inference.

$$p(x) = h(x)e^{\eta^\top T(x) - A(\eta)}$$

Probability distribution in exponential family has several advantages. We can see that it is very easy to get the maximum likelihood estimation.

$$\ln \prod_{i=1}^{N} p(x_i|\eta) = \sum_{i=1}^{N} \ln h(x_i) + \eta^\top (\sum_{i=1}^{N} T(x_i)) - N A(\eta)$$

So if we want to get the maximum likelihood estimation,

$$\frac{dA(\eta)}{d\eta} = \frac{\sum_{i=1}^{N} T(x_i)}{N}$$

Exponential family distribution II

Let's see an example of one-dimensional Gaussian distribution,

$$\mathcal{N}(x|\mu,\sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

We transform it into the exponential family distribution,

$$p(x|\eta) = h(x)e^{\eta^\top T(x) - A(\eta)} = \exp(\begin{bmatrix} x \\ x^2 \end{bmatrix}^\top \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} - (\frac{-\eta_1^2}{4\eta_2} + \frac{1}{2}\ln\eta_2) - \frac{1}{2}\ln 2\pi)$$

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{bmatrix}$$

and

$$A(\eta) = \frac{-\eta_1^2}{4\eta_2} + \frac{1}{2}\ln\eta_2 \qquad T(x) = \begin{bmatrix} x \\ x^2 \end{bmatrix} \qquad \theta = \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} = \begin{bmatrix} \frac{-\eta_1}{2\eta_2} \\ \frac{-1}{2\eta_2} \end{bmatrix}$$

Exponential family distribution III

Now we use the previous conclusion to get the maximum likelihood estimation.

$$\frac{dA(\eta)}{d\eta} = \frac{\sum_{i=1}^{N} T(x_i)}{N}$$

$$A(\eta) = \frac{-\eta_1^2}{4\eta_2} + \frac{1}{2}\ln \eta_2$$

$$\begin{bmatrix} -\frac{\eta_1}{2\eta_2} \\ \frac{\eta_1^2}{4\eta_2^2} + \frac{1}{2\eta_2} \end{bmatrix} = \begin{bmatrix} \frac{\sum_{i=1}^{N} x_i}{N} \\ \frac{\sum_{i=1}^{N} x_i^2}{N} \end{bmatrix}$$

With

$$\theta = \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} = \begin{bmatrix} \frac{-\eta_1}{2\eta_2} \\ \frac{-1}{2\eta_2} \end{bmatrix}$$

it is easy to solve μ and σ^2. Moreover, conjugate distribution in exponential familiy distribution are easy to compute.

Exponential family distribution IV

We now show one important property of exponential family distribution that, for

$$p(x|\lambda) = h(x)\exp(T(x)^\top \lambda - A_g(\lambda))$$

We have

$$\nabla_\lambda A_g(\lambda) = \mathbb{E}_{p(x|\lambda)}[T(x)]$$

From

$$\int_x p(x|\lambda)dx = \int_x h(x)\exp(T(x)^\top \lambda - A_g(\lambda))dx = 1$$

Exponential family distribution V

$$\nabla_\lambda \int_x h(x) \exp(T(x)^\top \lambda - A_g(\lambda)) dx = 0$$

$$\int_x \nabla_\lambda h(x) \exp(T(x)^\top \lambda - A_g(\lambda)) dx = 0$$

$$\int_x \nabla_\lambda h(x) \exp(T(x)^\top \lambda - A_g(\lambda)) dx = 0$$

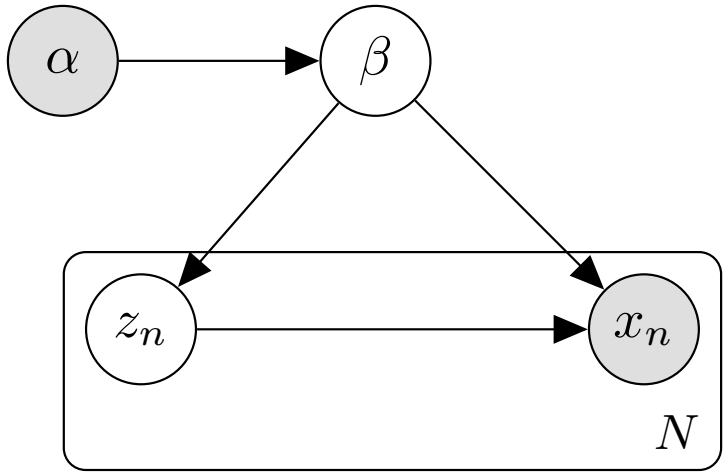
$$\int_x p(x|\lambda)(T(x) - \nabla A_g(\lambda)) dx = 0$$

$$\nabla_\lambda A_g(\lambda) = \mathbb{E}_{p(x|\lambda)}[T(x)]$$

Thus the equation holds.

Variational Inference I

Now let's see how to implement do variational inference for the following model.



The joint distribution is

$$p(x, z, \beta|\alpha) = p(\beta|\alpha) \prod_{n=1}^{N} p(x_n, z_n|\beta)$$

Variational Inference II

We make the following assumptions, the conditional distribution belongs to exponential family distribution,

$$p(\beta|x,z,\alpha) = h(\beta)\exp(\eta_g(x,z,\alpha)^\top T(\beta) - A_g(\eta_g(x,z,\alpha)))$$

and

$$p(z_{nj}|x_n,z_{n,\neg j},\beta)=h(z_{nj})\exp(\eta_l(x_n,z_{n,\neg j},\beta)^\top T(z_{nj})-A_l(\eta_l(x_n,z_{n,\neg j},\beta)))$$

The evidence lower bound is

$$\mathcal{L}(q) = \mathbb{E}_q[\log p(x,z,\beta)] - \mathbb{E}_q[\log q(z,\beta)]$$

We select the distribution q such that it is both expressive and easier to optimize, (Note that we only restrict the function form for p, but have not specifies the factorization form of p.)

$$q(z,\beta) = q(\beta|\lambda)\prod_{n=1}^{N}\prod_{j=1}^{J} q(z_{nj}|\phi_{nj})$$

Variational Inference III

Other than the factorization of $q(z, \beta)$, we restrict $q(z, \beta)$ and $q(z_{nj}|\phi_{nj})$.

$$q(\beta|\lambda) = h(\beta)\exp(\lambda^\top T(\beta) - A_g(\lambda))$$

$$q(z_{nj}|\phi_{nj}) = h(z_{nj})\exp(\phi_{nj}^\top T(z_{nj}) - A_l(\phi_{nj}))$$

Now let's alter the parameter λ, ϕ to maximize the ELBO. Rather than $\mathcal{L}(q)$, we can write it as $\mathcal{L}(\lambda, \phi)$. It turns out we can optimize λ and ϕ in turn. We first fix ϕ and optimize λ.

Variational Inference IV

$$
\begin{aligned}
\mathcal{L}(q) &= \mathbb{E}_q[\log p(x, z, \beta)] - \mathbb{E}_q[\log q(z, \beta)] \\
&= \mathbb{E}_q[\log p(\beta|x, z)] + \underbrace{\mathbb{E}_q[\log p(x, z)]}_{\text{not related to } \beta} - \mathbb{E}_q[\log q(z, \beta)] \\
&= \mathbb{E}_q[\log h(\beta)] + \mathbb{E}_q[\eta_g(x, z)]^\top \mathbb{E}_q[T(\beta)] - \mathbb{E}_q[A_g(\eta_g(x, z))] \\
&\quad - \mathbb{E}_q[\log h(\beta)] - \mathbb{E}_q[T(\beta)^\top \lambda] + \mathbb{E}[A_g(\lambda)] \\
&= \mathbb{E}_q[\eta_g(x, z)]^\top \mathbb{E}_q[T(\beta)] - \mathbb{E}_q[A_g(\eta_g(x, z))] - \mathbb{E}_q[T(\beta)^\top \lambda] \\
&\quad + \mathbb{E}[A_g(\lambda)] \\
&= \mathbb{E}_q[\eta_g(x, z)]^\top \mathbb{E}_q[T(\beta)] - \mathbb{E}_q[T(\beta)]^\top \lambda + A_g(\lambda) \\
&= \nabla_\lambda A_g(\lambda)^\top (\mathbb{E}_{q(z|\phi)}[\eta_g(x, z)] - \lambda) + A_g(\lambda)
\end{aligned}
$$

Maximize w.r.t $\mathcal{L}(\lambda)$ a.k.a $\mathcal{L}(q)$, $\frac{d\mathcal{L}(\lambda)}{d\lambda} = 0$, we have

$$
\nabla^2_\lambda A_g(\lambda)^\top (\mathbb{E}_{q(z|\phi)}[\eta_g(x, z)] - \lambda) = 0
$$

Variational Inference V

Assume that $\nabla^2_\lambda A_g(\lambda) \neq 0$, so we have

$$\mathbb{E}_{q(z|\phi)}[\eta_g(x, z)] - \lambda = 0$$

That's cool! Right? What have we done so far?

- We first specifies a graphical model, and specifies the latent variables e.g. z, β.
- We define the distribution so that the conditional distribution for latent varaibels given other variables is always exponential family distribution.
- For joint distribution of latent variable e.g. $q(z, \beta)$, we assume that $q(z, \beta)$ factorizes into $q(z|\lambda)q(\beta|\phi)$.

Variational Inference VI

- Then, we have a very nice conclusion from our previous lengthy derivation that if we optimize the parameters e.g. λ, ϕ one by one, we just do it in this way. Repeat between $\lambda = \mathbb{E}_{q(z|\phi)}[\eta_g(x, z)]$ and $\phi = \mathbb{E}_{q(z|\lambda)}[\eta_l(x, \beta)]$ which is **set the parameter to be the expectation of natural prameter conditioning on all other latent variables**.