Exercises
Computational Intelligence Lab
SS 2018

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Series 09 Solutions (Dictionary Learning and Compressed Sensing)

Problem 1 (Sparse coding with overcomplete dictionary):

Given a signal $\mathbf{x} = (3, 1, -2) \in \mathbb{R}^3$ and an overcomplete dictionary $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4] \in \mathbb{R}^{3\times 4}$,

$$\mathbf{U} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

find the sparse representation $\mathbf{z}$ of the signal $\mathbf{x}$ with $\|\mathbf{z}\|_0 \leq 2$.

a. Find the atom $\mathbf{u}^{(1)}$ that minimize the reconstruction error $\|\mathbf{x} - \hat{\mathbf{x}}^{(0)}\|_2^2$ where $\hat{\mathbf{x}}^{(0)} = z^{(1)}\mathbf{u}^{(1)}$, and compute the residual $\mathbf{r}^{(1)} = \mathbf{x} - \hat{\mathbf{x}}^{(0)}$.
Solution: The atom $\mathbf{u}^{(1)}$ that minimizes the reconstruction error $\|\mathbf{x} - z^{(1)}\mathbf{u}^{(1)}\|$ is the atom that is best correlated $\mathbf{x}$. The correlation between the signal and the atoms in the dictionary are as following,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{u}_1 \rangle &= \frac{2}{\sqrt{3}} \\ \langle \mathbf{x}, \mathbf{u}_2 \rangle &= -\frac{4}{\sqrt{3}} \\ \langle \mathbf{x}, \mathbf{u}_3 \rangle &= 0 \\ \langle \mathbf{x}, \mathbf{u}_4 \rangle &= 2\sqrt{3}. \end{aligned}$$

Since the absolute correlation coefficient between the atom $\mathbf{u}_4$ and the signal $\mathbf{x}$ has the largest value, $\hat{\mathbf{x}}^{(0)} = \langle \mathbf{x}, \mathbf{u}_4 \rangle \cdot \mathbf{u}_4 = 2\sqrt{3} \cdot \mathbf{u}_4$ minimizes $\|\mathbf{x} - \hat{\mathbf{x}}^{(0)}\|_2^2$. And the residual $\mathbf{r}^{(1)} = \mathbf{x} - \hat{\mathbf{x}}^{(0)} = (1, -1, 0)$

b. Find the atom $\mathbf{u}^{(2)}$ that minimize the reconstruction error $\|\mathbf{r}^{(1)} - \hat{\mathbf{x}}^{(1)}\|_2^2$ where $\hat{\mathbf{x}}^{(1)} = z^{(2)}\mathbf{u}^{(2)}$.
Solution: Similarly, we want to find the atom best correlated with $\mathbf{r}^{(1)}$ among the remaining atoms $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. The correlation coefficients between the atoms and the residual are

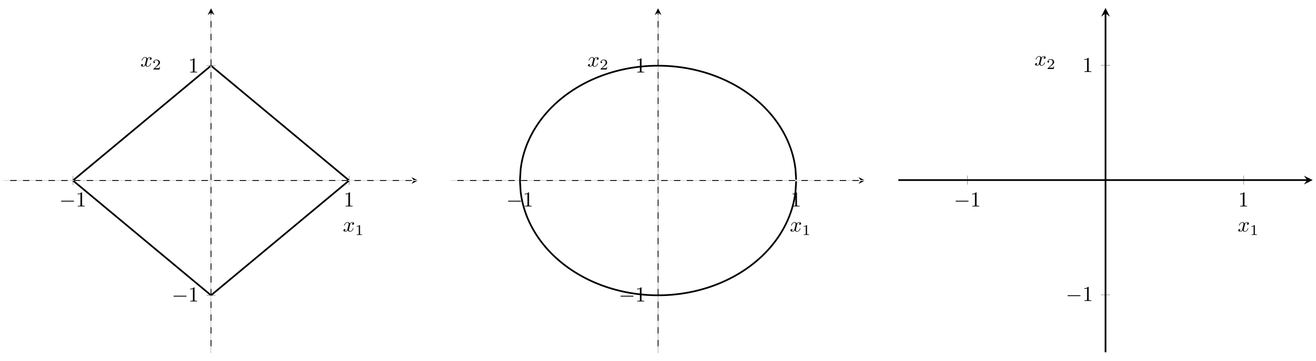
$$\begin{aligned} \langle \mathbf{r}^{(1)}, \mathbf{u}_1 \rangle &= 0 \\ \langle \mathbf{r}^{(1)}, \mathbf{u}_2 \rangle &= -\frac{2}{\sqrt{3}} \\ \langle \mathbf{r}^{(1)}, \mathbf{u}_3 \rangle &= \frac{2}{\sqrt{3}}. \end{aligned}$$

Because the aboslute correlation coefficient values of $\mathbf{u}_2$ and $\mathbf{u}_3$ are the same, either $\hat{\mathbf{x}}^{(1)} = -\frac{2}{\sqrt{3}}\mathbf{u}_2$ or $\hat{\mathbf{x}}^{(1)} = \frac{2}{\sqrt{3}}\mathbf{u}_3$ can minimize $\|\mathbf{r}^{(1)} - \hat{\mathbf{x}}^{(1)}\|_2^2$.

c. Write down the sparse representation $\mathbf{z}$ of signal $\mathbf{x}$.
Solution: The sparse representations $\mathbf{z}$ that satisfies $\|\mathbf{z}\|_0 \leq 2$ are $(0, 0, 0, 2\sqrt{3})$, $(0, 0, \frac{2}{\sqrt{3}}, 2\sqrt{3})$ and $(0, -\frac{2}{\sqrt{3}}, 0, 2\sqrt{3})$.

Problem 2 (Compressed Sensing):

a. Map each of the three equations $\|\mathbf{x}\|_2 = 1$, $\|\mathbf{x}\|_1 = 1$, and $\|\mathbf{x}\|_0 = 1$ to a plot among a., b., or c. on the following figure. Note that $\mathbf{x}$ is s 2D vector with coordinates x_1 and x_2 (i.e. $\mathbf{x} = \begin{bmatrix} x_1, x_2 \end{bmatrix}$).



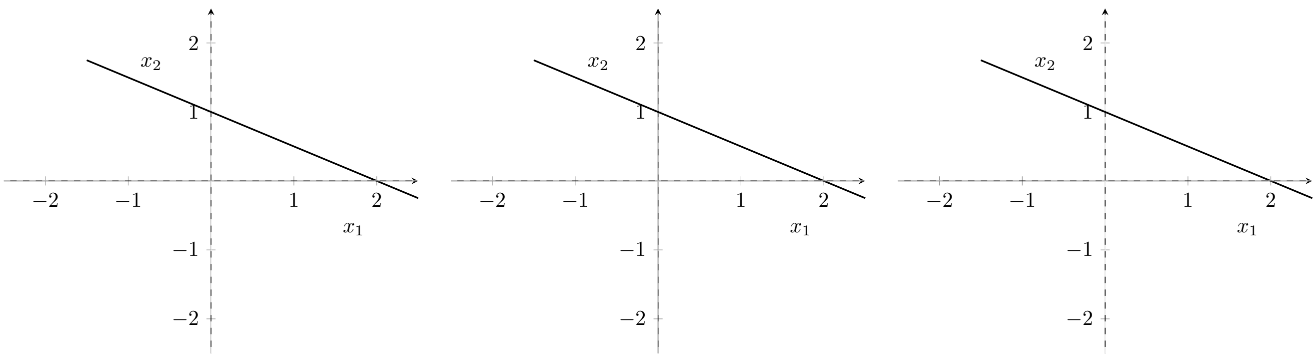
a. $\|\mathbf{x}\|_1$ b. $\|\mathbf{x}\|_2$ c. $\|\mathbf{x}\|_0$

b. Show the solution of each optimization problem on plots a., b., and c. of the following figure.

$\min \|\mathbf{x}\|_2$ Subject to $\frac{1}{2}x_1 + x_2 = 1$

$\min \|\mathbf{x}\|_1$ Subject to $\frac{1}{2}x_1 + x_2 = 1$

$\min \|\mathbf{x}\|_0$ Subject to $\frac{1}{2}x_1 + x_2 = 1$



a. b. c.

Solution:

a. The answer will be the closest point on the line to the origin, i.e.

$$\frac{1}{2}\mathbf{x}_1 + \mathbf{x}_2 = 1 \leftrightarrow \mathbf{x}_1 = 2 - 2\mathbf{x}_2 \tag{1}$$

$$\min\{\mathbf{x}_1^2 + \mathbf{x}_2^2\} = \min\{(2 - 2\mathbf{x}_2)^2 + \mathbf{x}_2^2\} \tag{2}$$

$$\frac{d}{d\mathbf{x}_2}\left[(2 - 2\mathbf{x}_2)^2 + \mathbf{x}_2^2\right] \stackrel{!}{=} 0 \leftrightarrow \mathbf{x}_2 = 0.8, \mathbf{x}_1 = 0.4 \tag{3}$$

b. $\mathbf{x}_1 = 0, \mathbf{x}_2 = 1$ c. two solutions $[\mathbf{x}_1 = 0, \mathbf{x}_2 = 1], [\mathbf{x}_1 = 2, \mathbf{x}_2 = 0]$

c. We can formulate the above three optimization problem as

$$\min \|\mathbf{x}\|_p$$
$$\text{subject to } \frac{1}{2}x_1 + x_2 = 1,$$

where $p \in \{0, 1, 2\}$. Mark the right sentence using your previous answers.

[ ] Solutions of the constrained problems have intersection for $p = 1$ and $p = 0$.

[ ] Solutions of the constrained problems have intersection for $p = 2$ and $p = 0$.

Solution:

Solutions of the constrained problems have intersection for $p = 1$ and $p = 0$.