CHAPTER 16

Multiobjective Decision Making

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EVALUATING AND PRIORITIZING PROJECTS AT NASA

More public pressure than ever before is on NASA to justify its choice of projects to undertake. There is demand for accountability, pressure to cut costs, and an increasing number of potential projects to choose from. In the past, a committee of 15 members from NASA met once a year to review the 30 to 50 proposals submitted by contractors and divisions with the Kennedy Space Center. The five voting members (the decision makers, or DMs) gave each proposal a score from 1 to 10, the scores were averaged over the five DMs, and the top scoring proposals were selected until the budget was exceeded. Because the process was viewed as intuitive, management expressed concern about its subjectivity and consistency. It wanted to replace this process with a more comprehensive and structured process. Tavana (2003) describes the system he developed to meet these needs. He calls it consensus-ranking organizational-support system (CROSS).

The selection of projects at NASA is clearly a multiobjective decision-making problem. As Tavana describes, there are a number of stakeholders for each project. Essentially, they are the different departments within NASA—including Safety, Systems Engineering, Reliability, and others—and each has its own criteria for a successful project. For example, Safety might be concerned about eliminating the possibility of death or serious injury, Systems Engineering might be concerned about eliminating reliance on identified obsolete technology, and Reliability might be concerned about increasing the mean time between failures. CROSS uses AHP (Analytic Hierarchy Process, discussed later in this chapter) to obtain the information each DM needs to obtain a score for each project. It then combines the DMs' scores to get an overall consensus ranking of projects. Finally, it uses this consensus ranking, along with project costs and the overall budget, to select the projects to be funded.

More specifically, the system first asks each DM to use AHP to evaluate the importance of the various stakeholders. For example, one DM might give Safety an importance weight of 0.5, whereas another might give Safety a weight of 0.4. In the next step, each stakeholder is asked to use AHP to evaluate the importance of its various criteria. This leads to a set of weights for each stakeholder-criterion combination. The stakeholders are also asked to estimate the probability that each potential project will be successful in satisfying each criterion. The system uses these probabilities to adjust the previous weights. Next, all of the weights from AHP are used to calculate a project-success factor for each project, as assessed by each DM, and these factors are used to obtain each DM's rankings of the projects. Finally, the system attempts to reach consensus in the rankings using another (non-AHP) methodology.

The system is now being used successfully to select NASA projects. As a measure of its perceived quality, 71 projects were submitted during the first two years of implementation of CROSS. Using this system, the DMs chose 21 projects of the 71, and management subsequently approved all 21 choices. ■

16.1 INTRODUCTION

In many of your classes, you have probably discussed how to make good decisions. Usually, you assume that the correct decision optimizes a *single* objective, such as profit maximization or cost minimization. In most situations you encounter in business and life, however, more than one relevant objective exists. For example, when you graduate, many of you will receive several job offers. Which should you accept? Before deciding which job offer to accept, you might consider how each job "scores" on several objectives, such as salary, interest in work, quality of life in the city you will live in, and nearness to family. In this situation, combining your multiple objectives into a single objective is difficult. Similarly, in determining an optimal investment portfolio, you want to maximize expected return, but you also want to minimize risk. How do you reconcile these conflicting objectives? In this chapter, we discuss three tools, goal programming, trade-off curves, and the Analytic Hierarchy Process, that decision makers can use to solve multi-objective problems. We show how to implement all three of these tools in a spreadsheet.

FUNDAMENTAL INSIGHT

Optimizing with Multiple Objectives

When there are multiple objectives, you can proceed in several fundamental ways. First, you can prioritize your objectives. This is done in goal programming, where the highest priority objective is optimized first, then the second, and so on. Second, you can optimize one objective while constraining the others to be no worse than specified values. This approach is used to find trade-off curves between the objectives. Finally, you can attempt to weight the objectives to measure their importance relative to one another. This is the approach taken by the Analytic Hierarchy Process. All of these approaches have their critics, but they can all be used to make difficult decision problems manageable.

16.2 GOAL PROGRAMMING

In many situations, a company wants to achieve several objectives. Given limited resources, it may prove impossible to meet all objectives simultaneously. If the company can prioritize its objectives, then **goal programming** can be used to make good decisions. The following media selection problem is typical of the situations in which goal programming is useful. This example presents a variation of the advertising model discussed in Chapters 4 and 7.

EXAMPLE 16.1 DETERMINING AN ADVERTISING SCHEDULE AT LEON BURNIT

The Leon Burnit Ad Agency is trying to determine a TV advertising schedule for a client. The client has three goals (listed here in descending order of importance) concerning whom it wants its ads to be seen by:

- Goal 1: at least 65 million high-income men (HIM)
- Goal 2: at least 72 million high-income women (HIW)
- Goal 3: at least 70 million low-income people (LIP)

Burnit can purchase several types of TV ads: ads shown on live sports shows, on game shows, on news shows, on sitcoms, on dramas, and on soap operas. At most $700,000 total can be spent on ads. The advertising costs and potential audiences (in millions of viewers) of a one-minute ad of each type are shown in Table 16.1. As a matter of policy, the client requires that at least two ads each be placed on sports shows, news shows, and dramas. Also, it requires that no more than 10 ads be placed on any single type of show. Burnit wants to find the advertising plan that best meets its client's goals.

Table 16.1 Data for the Advertising Example

Ad Type	HIM	HIW	LIP	Cost
Sports show	7	4	8	$120,000
Game show	3	5	6	$40,000
News	6	5	3	$50,000
Sitcom	4	5	7	$40,000
Drama	6	8	6	$60,000
Soap opera	3	4	5	$40,000

Objective To use goal programming to meet the company's goals of reaching various target audiences as much as possible, while staying within an advertising budget.

WHERE DO THE NUMBERS COME FROM?

As in previous advertising models, the company needs to estimate the number of viewers reached by each type of ad, and it needs to know the cost of each ad. Beyond this, however, management determines the goals. They can set whatever goals they believe are in the company's best interests, and they can prioritize these goals.

Solution

The variables and constraints for this advertising model are shown in Table 16.2. Most of this is the same as in optimization models in previous chapters. However, the objective is not obvious, and the table includes "deviations from goals" and "balances for goals." You

Table 16.1 Variables and Constraints for the Advertising Model

Input variables	Advertising data (potential audiences and cost for each type of ad), advertising budget, goals (lower limits) on various target audiences
Decision variables (changing cells)	Numbers of ads of various types, deviations from goals
Objective (target cell)	Multiple (see text)
Other output cells	Total cost of ads, balances for goals
Constraints	Ads on sports shows ≥ 2 Ads on news shows ≥ 2 Ads on dramas ≥ 2 Ads on any type of show ≤ 10 Total cost of ads $\leq$ Advertising budget Meet goals as well as possible

need to see how these fit into the goal programming methodology. You get there one step at a time. You first check whether the company can meet all of its goals simultaneously. To do so, set up a linear programming (LP) model with *no* objective. You simply want to see whether any solution satisfies all of the constraints, including the goals.

Developing the LP Model

The LP model that checks whether all goals can be met can be developed as follows. (See Figure 16.1 and the LP Model sheet of the file **Advertising Goals.xlsx**.)

Figure 16.1 Feasibility of Meeting All Goals

	A	B	C	D	E	F	G	H	I	J
1	**LP model - possible to meet all goals?**									
2				Note: All monetary values are in $1000s, and all exposures to ads are in millions of exposures.						
3										
4										
5	**Exposures to various groups per unit of advertising**								**Range names used:**	
6		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad		Budget	='LP Model'!$D$22
7	High-income men	7	3	6	4	6	3		Exposures	='LP Model'!$B$26:$B$28
8	High-income women	4	5	5	5	8	4		Goal	='LP Model'!$D$26:$D$28
9	Low-income people	8	6	3	7	6	5		Maximum_ads_allowed	='LP Model'!$B$19:$G$19
10									Minimum_ads_required	='LP Model'!$B$15:$G$15
11	Cost/unit	120	40	50	40	60	40		Number_purchased	='LP Model'!$B$17:$G$17
12									Total_cost	='LP Model'!$B$22
13	**Advertising plan**									
14		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad			
15	Minimum ads required	2	0	2	0	2	0			
16		<=	<=	<=	<=	<=	<=			
17	Number purchased	2.000	0.000	2.000	4.000	3.333	0.000			
18		<=	<=	<=	<=	<=	<=			
19	Maximum ads allowed	10	10	10	10	10	10			
20										
21	**Budget constraint**	Total cost		Budget						
22		$700	<=	$700	Use Solver, with no objective, to see whether all constraints, including goals, can					
23										
24	**Goals for numbers of exposures**									
25		Exposures		Goal						
26	High-income men	62.000	>=	65						
27	High-income women	64.667	>=	72						
28	Low-income people	70.000	>=	70						

1. **Inputs.** Enter all inputs in the blue ranges.
2. **Numbers of ads.** Enter *any* trial values for the numbers of ads in the Number_purchased range.
3. **Total cost.** Calculate the total amount spent on ads in cell B22 with the formula

=SUMPRODUCT(B11:G11,Number_purchased)

4 **Exposures obtained.** Calculate the number of people (in millions) in each group that the ads reach in the Exposures range. Specifically, enter the formula

=SUMPRODUCT(B7:G7,Number_purchased)

in cell B26 for the HIM group, and copy this to the rest of the Exposures range for the other two groups.

Using Solver

The completed Solver dialog box is shown in Figure 16.2. At this point, there is no objective to maximize or minimize. The goal at this point is to find *any* solution that meets all of the constraints. When you click on Solve, you get the message that there is no feasible solution because it is impossible to meet all of the client's goals and stay within the budget. To see how large the budget must be to meet all goals, you can run SolverTable with the Budget cell as the single input cell, varied from 700 to 850, and *any* cells as the output cells. (We chose the numbers of exposures cells as output cells.) The results appear in Figure 16.3. They show that unless the budget is greater than $750,000, it is impossible to meet all of the client's goals.

Figure 16.2
Solver Dialog Box for Finding a Feasible Solution

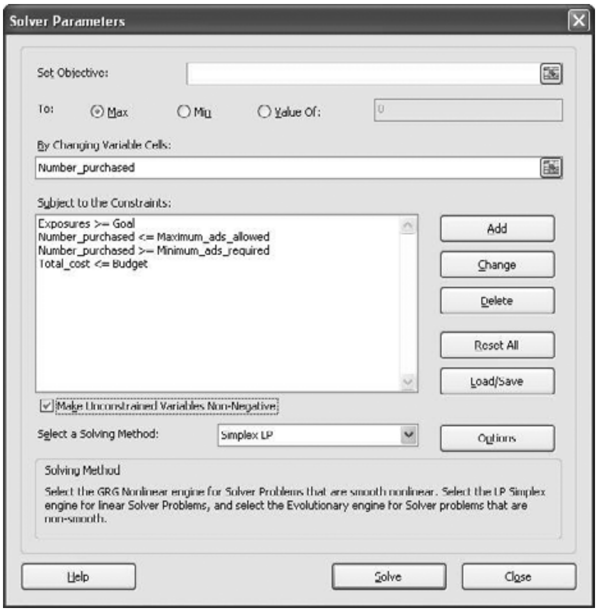


Figure 16.3
Checking How Large the Budget Must Be

	A	B	C	D	E	F
1	**Oneway analysis for Solver model in LP Model worksheet**					
2						
3	Budget (cell $D$22) values along side, output cell(s) along top					
4		Exposures_1	Exposures_2	Exposures_3		
5	$700	Not feasible				
6	$725	Not feasible				
7	$750	Not feasible				
8	$775	65.000	72.000	70.000		
9	$800	65.000	72.000	70.000		
10	$825	65.000	72.000	70.000		
11	$850	65.000	72.000	70.000		

Hard constraints must be satisfied. Soft constraints can be violated to some extent. In goal programming, the soft constraints are prioritized.

Now that you know that a \$700,000 budget is not sufficient to meet all of the client's goals, you can use goal programming to see how close Burnit can come to these goals. First, we introduce some terminology. The upper limits and lower limits on the ads of each type and the budget constraints are considered **hard constraints** in this model. This means that they cannot be violated under any circumstances. The goals on exposures, on the other hand, are considered **soft constraints.** The client certainly wants to satisfy these goals but is willing to come up somewhat short—in fact, it must because of the limited budget. In goal programming models, the soft constraints are prioritized. You first try to satisfy the goals with the highest priority (in this case, HIM exposures). If there is still any room to maneuver, you then try to satisfy the goals with the next highest priority (HIW exposures). If there is *still* room to maneuver, you move on to the goals with the third highest priority, and so on.

Developing the Goal Programming Model

In general, goal programming requires several consecutive Solver runs, one for each priority level. However, the model can be set up so that you can make these consecutive runs with only minor modifications from one run to the next. The procedure is illustrated in Figure 16.4. (See the GP Model sheet of the file **Advertising Goals.xlsx**.) To develop this model, first make a copy of the original LP Model sheet shown earlier in Figure 16.1. Then modify it using the following steps:

Figure 16.4 Minimizing Deviation for Highest Priority (HIM) Goal

	A	B	C	D	E	F	G	H	I	J
1	**Goal programming model**									
2				Note: All monetary values are in \$1000s, and all exposures to ads are in millions of exposures.						
3										
4										
5	**Exposures to various groups per unit of advertising**								**Range names used:**	
6		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad		Already_obtained	='GP Model'!$D$32:$D$34
7	High-income men	7	3	6	4	6	3		Amt_over_goal	='GP Model'!$D$26:$D$28
8	High-income women	4	5	5	5	8	4		Amt_under_goal	='GP Model'!$C$26:$C$28
9	Low-income people	8	6	3	7	6	5		Balance	='GP Model'!$E$26:$E$28
10									Budget	='GP Model'!$D$22
11	Cost/unit	120	40	50	40	60	40		Deviation_under	='GP Model'!$B$32:$B$34
12									Exposures	='GP Model'!$B$26:$B$28
13	**Advertising plan**								Goal	='GP Model'!$G$26:$G$28
14		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad		HIM_deviation	='GP Model'!$B$32
15	Minimum ads required	2	0	2	0	2	0		HIW_deviation	='GP Model'!$B$33
16		<=	<=	<=	<=	<=	<=		LIP_deviation	='GP Model'!$B$34
17	Number purchased	2.000	0.000	5.000	2.250	2.000	0.000		Maximum_ads_allowed	='GP Model'!$B$19:$G$19
18		<=	<=	<=	<=	<=	<=		Minimum_ads_required	='GP Model'!$B$15:$G$15
19	Maximum ads allowed	10	10	10	10	10	10		Number_purchased	='GP Model'!$B$17:$G$17
20									Total_cost	='GP Model'!$B$22
21	**Budget constraint**	Total cost		Budget						
22		\$700	<=	\$700						
23										
24	**Goals for numbers of exposures**									
25		Exposures	Amt under goal	Amt over goal	Balance		Goal			
26	High-income men	65.000	0	0	65.000	=	65			
27	High-income women	60.250	11.75	0	72.000	=	72			
28	Low-income people	58.750	11.25	0	70.000	=	70			
29										
30	**Deviations from goals (amounts below goals, or 0 if currently meeting goal)**									
31		Deviation under		Already obtained	Initially, enter large values in these cells (such as the original goals). Then, as high priority goals are met or partially met, enter the actual deviations obtained here (one at a time).					
32	HIM deviation	0.000	<=	65.000						
33	HIW deviation	11.750	<=	72.000						
34	LIP deviation	11.250	<=	70.000						

1 **New changing cells.** The exposure constraints are no longer shown as hard constraints. Instead, you need to introduce changing cells in the Amt_under_goal and Amt_over_goal ranges to indicate the amounts under or over each goal. These are the "deviations from goals" mentioned in Table 16.2. Enter any values in these ranges. (We entered 0s to get started.) Note that in the Solver solution, at least one of these two types of deviations will always be 0 for each goal—the solution will either be below the goal or above the goal, but not both.

2 Balance equations. To tie these new changing cells to the rest of the model, you create "balances" in column E that must logically equal the goals in column G. To do this, enter the formula

=B26+C26-D26

in cell E26 and copy it down. The balance equation for each group specifies that the actual number of exposures, plus the number under the goal, minus the number over the goal, *must* equal the goal.

The deviations are the key to goal programming. They indicate how far below or above the goals the current solution is.

3 Constraints on deviations under. The client is concerned only with too *few* exposures, not with too many. Therefore, you should set up constraints on the "under" deviations in rows 32 to 34. On the left side, in column B, enter links to the Amt_under_goals range by entering the formula

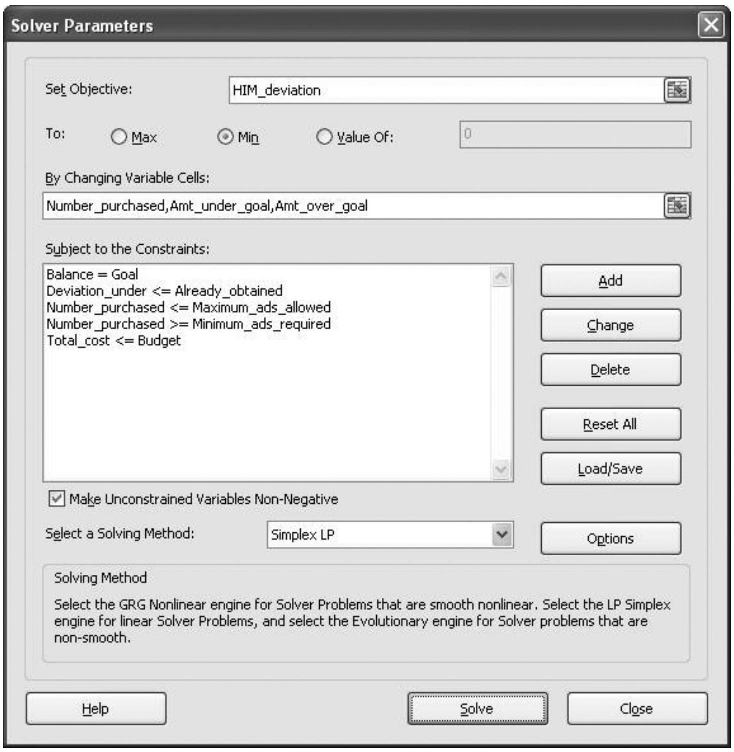
=C26

in cell B32 and copying down.

4 Highest priority goal. The first Solver run tries to achieve the highest priority goal (HIM exposures). To do so, you should minimize cell B32, the amount *under* the HIM goal. Do this as shown in Figure 16.4. Then set up the Solver dialog box as shown in Figure 16.5. The constraints include the hard constraints, the balance constraint, and the Deviation_under <= Already_obtained constraint. Note that the goals themselves have been entered in the Already_obtained range. Therefore, the Deviation_under <= Already_obtained constraint at this point is essentially redundant—the "under" deviations cannot possibly be greater than the goals themselves. This constraint is included because it becomes important in later Solver runs, which then require only minimal modifications. The solution from this Solver run is shown in Figure 16.4. It shows that Burnit can satisfy the HIM goal completely. However, the other two goals are not satisfied because their "under" deviations are positive.

Figure 16.5

Solver Dialog Box for the Highest Priority Goal



5 **Second highest priority goal.** Now we come to the key aspect of goal programming. After a high priority goal is satisfied as fully as possible, you move on to the next highest priority goal. However, you do not want to lose what you already gained with the high priority goal. Therefore, constrain its under deviation to be no greater than what has already been achieved. In this case, a deviation of 0 was already achieved in step 4, so enter 0 in cell D32 for the upper limit of the HIM under deviation. Then run Solver again, changing only one thing in the Solver dialog box—make cell B33 the target cell. Effectively, you are constraining the under deviation for the HIM group to remain at 0, and then minimizing the under deviation for the HIW group. The solution from this second Solver run appears in Figure 16.6. As promised, the HIM goal has not suffered at all, but the solution is now a little closer to the HIW goal than before. It was under by 11.75 before, and now it is under by only 11. The lowest priority goal (for the LIP group) essentially comes along for the ride in this step; it could either improve or get worse. It happened to get worse, moving from under by 11.25 to under by 18.

Figure 16.6 Minimizing Deviation for Second Priority Goal

	A	B	C	D	E	F	G	H	I	J
1	**Goal programming model**									
2				Note: All monetary values are in $1000s, and all exposures to ads are in millions of exposures.						
3										
4										
5	**Exposures to various groups per unit of advertising**								**Range names used:**	
6		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad		Already_obtained	='GP Model'!$D$32:$D$34
7	High-income men	7	3	6	4	6	3		Amt_over_goal	='GP Model'!$D$26:$D$28
8	High-income women	4	5	5	5	8	4		Amt_under_goal	='GP Model'!$C$26:$C$28
9	Low-income people	8	6	3	7	6	5		Balance	='GP Model'!$E$26:$E$28
10									Budget	='GP Model'!$D$22
11	Cost/unit	120	40	50	40	60	40		Deviation_under	='GP Model'!$B$32:$B$34
12									Exposures	='GP Model'!$B$26:$B$28
13	**Advertising plan**								Goal	='GP Model'!$G$26:$G$28
14		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad		HIM_deviation	='GP Model'!$B$32
15	Minimum ads required	2	0	2	0	2	0		HIW_deviation	='GP Model'!$B$33
16		<=	<=	<=	<=	<=	<=		LIP_deviation	='GP Model'!$B$34
17	Number purchased	2.000	0.000	5.000	0.000	3.500	0.000		Maximum_ads_allowed	='GP Model'!$B$19:$G$19
18		<=	<=	<=	<=	<=	<=		Minimum_ads_required	='GP Model'!$B$15:$G$15
19	Maximum ads allowed	10	10	10	10	10	10		Number_purchased	='GP Model'!$B$17:$G$17
20									Total_cost	='GP Model'!$B$22
21	**Budget constraint**	Total cost		Budget						
22		$700	<=	$700						
23										
24	**Goals for numbers of exposures**									
25		Exposures	Amt under goal	Amt over goal	Balance		Goal			
26	High-income men	65.000	0	0	65.000	=	65			
27	High-income women	61.000	11	0	72.000	=	72			
28	Low-income people	52.000	18	0	70.000	=	70			
29										
30	**Deviations from goals (amounts below goals, or 0 if currently meeting goal)**				Initially, enter large values in these cells (such as the original goals). Then, as high priority goals are met or partially met, enter the actual deviations obtained here (one at a time).					
31		Deviation under		Already obtained						
32	HIM deviation	0.000	<=	0.000						
33	HIW deviation	11.000	<=	72.000						
34	LIP deviation	18.000	<=	70.000						

6 **Lowest priority goal.** You can probably guess the last step by now. You minimize cell B34, the deviation for the LIP group, while ensuring that the two higher priority goals are achieved as fully as in steps 4 and 5. As the model is set up, only two changes are necessary—enter 11 in cell D33 and change the Solver objective cell to cell B34. When you run Solver this time, however, you will find no room left to maneuver. The solution remains exactly the same as in Figure 16.6. This occurs frequently in goal programming models. After satisfying the first goal or two as fully as possible, there is often no room to improve later goals.

Discussion of the Solution

To summarize Burnit's situation, the budget of $700,000 allows it to satisfy the client's HIM goal, miss the HIW goal by 11 million, and miss the LIP goal by 18 million. Given the priorities on these three goals, this is the best possible solution. Note that all of the hard constraints are satisfied, as they must be. For example, no more than 10 ads of any type are used, and the budget is not exceeded. Note also that the amounts *over* the goals are all 0. This is not guaranteed to happen, but it did in this example.

Sensitivity Analysis

Sensitivity analysis should be a part of goal programming just as it is for previous models we have discussed. However, there is no quick way to do it. SolverTable works on only a *single* objective, whereas goal programming requires a sequence of objectives. Therefore, if you want to see how the solution to Burnit's model changes with different budgets, say, you need to go through the preceding steps several times and keep track of the results manually. This is certainly possible, but it is tedious.

Effect of Changing Priorities

With three goals, six orderings of the goals are possible. The goal programming solutions corresponding to these orderings are listed in Figure 16.7. Row 4 corresponds to the ordering used in the example. Clearly, the solution can change if the priorities of the goals change. For example, if you give the HIW goal the highest priority (rows 6 and 7), *none* of the goals is achieved completely. (Problem 1 asks you to verify the details.)

Figure 16.7 Effect of Changing Priorities

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Results from changing priorities**											
2												
3	Priority 1	Priority 2	Priority 3	HIM deviation	HIW deviation	LIP deviation	Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad
4	HIM	HIW	LIP	0	11	18	2	0	5	0	3.5	0
5	HIM	LIP	HIW	0	11.75	11.25	2	0	5	2.25	2	0
6	HIW	HIM	LIP	3	6	12	2	0	2	0	6	0
7	HIW	LIP	HIM	3	6	12	2	0	2	0	6	0
8	LIP	HIM	HIW	1.956	9.304	0	2	0	3.043	4.696	2	0
9	LIP	HIW	HIM	3	7.333	0	2	0	2	4	3.333	0

Modeling Issues

1. The results for the Burnit model are based on allowing the numbers of ads to have noninteger values. They could easily be constrained to integer values, and the solution method would remain exactly the same. However, the goals might not be met as fully as before because of the extra integer constraints.
2. Each priority level in the Burnit model contains exactly one goal. It is easy to generalize to the case where a given priority level can have multiple goals, each modeled with a certain deviation from a target. When you run Solver for this priority level, you use a weighted average of these deviations as the objective to minimize, where the decision maker can choose appropriate weights.
3. All of the deviations in the objectives of the Burnit model are *under* deviations. However, it is certainly possible to include *over* deviations as objectives. For example, if the budget constraint were treated as a soft constraint, you would try to minimize its over deviation to stay as little over the budget as possible. It is even possible for *both* the under and over deviations of some goal to be included as objectives. This occurs in situations where you want to come as close as possible to some target value—neither under nor over.

4. The use of changing cells for the under and over deviations might not be intuitive, but it serves two purposes. First, it provides exactly the information needed for the objectives in goal programming. Second, it keeps the model linear. If you used an IF function instead (without the under and over cells) to capture the under deviations, the model would be nonlinear and nonsmooth, and Evolutionary Solver would be necessary. ■

PROBLEMS

Solutions for problems whose numbers appear within a colored box can be found in the Student Solutions Files. Refer to this book's preface for purchase information.

Skill-Building Problems

1. For each set of priorities of goals, solve the Burnit problem and verify that the values in Figure 16.7 are correct.

2. Gotham City must determine how to allocate ambulances during the next year. It costs $5000 per year to run an ambulance. Each ambulance must be assigned to one of two districts. Let x_i be the number of ambulances assigned to district i, $i = 1, 2$. The average time (in minutes) it takes for an ambulance to respond to a call from district 1 is $40 - 3x_1$; for district 2, the time is $50 - 4x_2$. Gotham City has three goals (listed in order of priority):
   - Goal 1: At most $100,000 per year should be spent on ambulance service.
   - Goal 2: Average response time in district 1 should be at most five minutes.
   - Goal 3: Average response time in district 2 should be at most five minutes.

   a. Use goal programming to determine how many ambulances to assign to each district.

   b. How does your answer change if goal 2 has the highest priority, then goal 3, and then goal 1?

3. Fruit Computer Company is ready to make its annual purchase of computer chips. Fruit can purchase chips (in lots of 100) from three suppliers. Each chip's quality is rated as excellent, good, or mediocre. During the coming year, Fruit needs 5000 excellent chips, 3000 good chips, and 1000 mediocre chips. The characteristics of the chips purchased from each supplier are shown in the file **P16_03.xlsx**. Each year, Fruit has budgeted $28,000 to spend on chips. If Fruit does not obtain enough chips of a given quality, it can special-order additional chips at $10 per excellent chip, $6 per good chip, and $4 per mediocre chip. Fruit assesses a penalty of $1 for each dollar it goes over the annual budget (in payments to suppliers). Determine how Fruit can minimize the penalty associated with meeting the annual chip requirements. Also use goal programming to determine a purchasing strategy. Let the budget constraint have the highest priority, followed in order by the restrictions on excellent, good, and mediocre chips.

4. Hiland Appliance must determine how many TVs and Blu-Ray disc players to stock. It costs Hiland $1000 to purchase a TV and $400 to purchase a Blu-Ray player. A TV requires three square yards of storage space, and a Blu-ray disc player requires one square yard. The sale of a TV earns Hiland a profit of $150, and each Blu-ray disc player sale earns a profit of $100. Hiland has set the following goals (listed in order of importance):
   - Goal 1: A maximum of $60,000 can be spent on purchasing TVs and Blu-ray disc players.
   - Goal 2: Highland should earn at least $7,000 profit from the sale of TVs and Blu-ray disc players.
   - Goal 3: TVs and Blu-ray disc players should not use up more than 200 square yards of storage space.

   Use a goal programming model to determine how many TVs and Blu-ray disc players Hiland should order. How can you modify the model if Hiland's second goal is to have a profit of *exactly* $7,000?

5. Each week, Stockco produces two products. Relevant information for each product is shown in the file **P16_05.xlsx**. Stockco has a goal of $4800 in weekly profit and incurs a $1 penalty for each dollar it falls short of this goal. A total of 3200 hours of labor are available. A $2 penalty is incurred for each hour of overtime (labor over 3200 hours) used, and a $1 penalty is incurred for each hour of available labor that is unused. Marketing considerations require that at least 700 units of product 1 be produced and at least 1000 units of product 2 be produced. For each unit (of either product) by which production falls short of demand, a penalty of $5 is assessed.

   a. Determine how to minimize the total penalty incurred by Stockco.

   b. Suppose the company sets (in order of importance) the following goals:
   - Goal 1: Make the required profit.
   - Goal 2: Avoid underuse of labor.
   - Goal 3: Meet demand for product 1.
   - Goal 4: Meet demand for product 2.
   - Goal 5: Do not use any overtime.

Use goal programming to determine an optimal production schedule.

6. Based on Steuer (1984). Deancorp produces sausage by blending beef head, pork chuck, mutton, and water. The cost per pound, fat per pound, and protein per pound for these ingredients are listed in the file **P16_06.xlsx**. Deancorp needs to produce 1000 pounds of sausage and has set the following goals, listed in order of priority:

- Goal 1: Sausage should consist of at least 15% protein.
- Goal 2: Sausage should consist of at most 8% fat.
- Goal 3: Cost per pound of sausage should not exceed \$0.06.

Use a goal programming model to determine the composition of sausage.

7. Based on Welling (1977). The Touche Young accounting firm must complete three jobs during the next month. Job 1 will require 500 hours of work, job 2 will require 300 hours, and job 3 will require 100 hours. At present, the firm consists of five partners, five senior employees, and five junior employees, each of whom can work up to 40 hours per week. The dollar amount (per hour) that the company can bill depends on the type of accountant assigned to each job, as shown in the file **P16_07.xlsx**. (The "X" indicates that a junior employee does not have enough experience to work on job 1.) All jobs must be completed. Touche Young has also set the following goals, listed in order of priority:

- Goal 1: Monthly billings should exceed \$74,000.
- Goal 2: At most one partner should be hired.
- Goal 3: At most three senior employees should be hired.
- Goal 4: At most one junior employee should be hired.

Use goal programming to help Touche solve its problem.

8. There are four teachers in the Faber College Business School. Each semester, 200 students take each of the following courses: Marketing, Finance, Production, and Statistics. The "effectiveness" of each teacher in teaching each course is given in the file **P16_08.xlsx**. Each teacher can teach a total of 200 students during the semester. The dean has set a goal of obtaining an average teaching effectiveness level of at least 6 in each course. Deviations from this goal in any course are considered equally important. Determine the semester's teaching assignments.

9. The city of Bloomington has 17 neighborhoods. The number of high school students in each neighborhood and the time required to drive from each neighborhood to each of the city's two high schools (North and South) are listed in the file **P16_09.xlsx**. The Bloomington Board of Education needs to determine how to assign students to high schools. All students in a given neighborhood must be assigned to the same high school. The Board has set (in order of priority, from highest to lowest) the following goals:

- Goal 1: Ensure that the difference in enrollment at the two high schools differs by at most 50.
- Goal 2: Ensure that average student travel time is at most 13 minutes.
- Goal 3: Ensure that at most 4% of the students must travel at least 25 minutes to school.

a. Determine an optimal assignment of students to high schools.

b. If the enrollment at the two high schools can differ by at most 100 (a change in goal 1), how does your answer change?

Skill-Extending Problems

10. Based on Lee and Moore (1974). Faber College is admitting students for the class of 2007. Data on its applicants are shown in the file **P16_10.xlsx**. Each row indicates the number of in-state or out-of-state applicants with a given SAT score who plan to be business or nonbusiness majors. For example, 1900 of its in-state applicants have a 700 SAT score, and 1500 of these applicants plan to major in business. Faber has set four goals for this class, listed in order of priority:

- Goal 1: The entering class should include at least 5000 students.
- Goal 2: The entering class should have an average SAT score of at least 640.
- Goal 3: The entering class should consist of at least 25% out-of-state students.
- Goal 4: At least 2000 members of the entering class should not be business majors.

Use goal programming to determine how many applicants of each type to admit. Assume that all applicants who are admitted will decide to attend Faber.

11. During the next four quarters, Wivco faces the following demands for globots: quarter 1, 13; quarter 2, 14; quarter 3, 12; quarter 4, 15. Globots can be produced by regular-time labor or by overtime labor. Production capacity (number of globots) and production costs during the next four quarters are shown in the file **P16_11.xlsx**. Wivco has set the following goals in order of importance:

- Goal 1: Each quarter's demand should be met on time.
- Goal 2: Inventory at the end of each quarter should not exceed three units.
- Goal 3: Total production cost should be no greater than \$250.

Use a goal programming model to determine Wivco's production schedule for the next four quarters. Assume that at the beginning of the first quarter, one globot is in inventory.

12. Lucy's Music Store at present employs five full-time employees and three part-time employees. The normal workload is 40 hours per week for full-time employees and 20 hours per week for part-time employees. Each full-time employee is paid \$6 per hour for work up to 40 hours per week and can sell five recordings per hour. A full-time employee who works overtime is paid \$10 per hour. Each part-time employee is paid \$3 per hour and can sell three recordings per hour. It costs Lucy \$6 to buy a recording, and each recording sells for \$9. Lucy has weekly fixed expenses of \$500. She has established the following weekly goals, in order of priority:

- Goal 1: Sell at least 1600 recordings per week.
- Goal 2: Earn a profit of at least \$2200 per week.
- Goal 3: Full-time employees should work at most 100 hours of overtime.
- Goal 4: To promote a sense of job security, the number of hours by which each full-time employee fails to work 40 hours should be minimized.

Use a goal programming model to determine how many hours per week each employee should work.

13. Based on Taylor and Keown (1984). Gotham City is trying to determine the type and location of recreational facilities to build during the next decade. Four types of facilities are under consideration: golf courses, swimming pools, gymnasiums, and tennis courts. Six sites are under consideration. If a golf course is built, it must be built at either site 1 or site 6. Other facilities can be built at sites 2 through 5. The amounts of available land (in thousands of square feet) at sites 2 through 5 are given in the file **P16_13.xlsx**. The cost of building each facility (in thousands of dollars), the annual maintenance cost (in thousands of dollars) for each facility, and the land (in thousands of square feet) required for each facility are also given in the same file. The number of user-days (in thousands) for each type of facility, also shown in this file, depends on where it is built.

a. Consider the following set of priorities:
- Priority 1: The amount of land used at each site should be no greater than the amount of land available.
- Priority 2: Construction costs should not exceed \$1.2 million.
- Priority 3: User-days should exceed 200,000.
- Priority 4: Annual maintenance costs should not exceed \$200,000.

For this set of priorities, use goal programming to determine the type and location of recreation facilities in Gotham City.

b. Consider the following set of priorities:
- Priority 1: The amount of land used at each site should be no greater than the amount of land available.
- Priority 2: User-days should exceed 200,000.
- Priority 3: Construction costs should not exceed \$1.2 million.
- Priority 4: Annual maintenance costs should not exceed \$200,000.

For this set of priorities, use goal programming to determine the type and location of recreation facilities in Gotham City.

14. A small aerospace company is considering eight projects:
- Project 1: Develop an automated test facility.
- Project 2: Bar code all inventory and machinery.
- Project 3: Introduce a CAD/CAM system.
- Project 4: Buy a new lathe and deburring system.
- Project 5: Institute an FMS (Flexible Manufacturing System).
- Project 6: Install a LAN (Local Area Network).
- Project 7: Develop an AIS (Artificial Intelligence Simulation).
- Project 8: Set up a TQM (Total Quality Management) program.

Each project has been rated on five attributes: return on investment (ROI), cost, productivity improvement, workforce requirements, and degree of technological risk. These ratings are given in the file **P16_14.xlsx**. The company has set the following five goals (listed in order of priority):
- Goal 1: Achieve an ROI of at least \$3250.
- Goal 2: Limit cost to \$1300.
- Goal 3: Achieve a productivity improvement of at least 6.
- Goal 4: Limit workforce use to 108.
- Goal 5: Limit technological risk to a total of 4.

Use goal programming to determine which projects should be undertaken.

15. A new president has just been elected and has set the following economic goals (listed from highest to lowest priority):
- Goal 1: Balance the budget (this means revenues are at least as large as costs).
- Goal 2: Cut spending by at most \$150 billion.
- Goal 3: Raise at most \$550 billion in taxes from the upper class.
- Goal 4: Raise at most \$350 billion in taxes from the lower class.

Currently the government spends \$1 trillion per year. Revenue can be raised in two ways: through a gas tax and through an income tax. You must determine G, the per-gallon tax rate (in cents); T_1, the tax rate charged on the first \$30,000 of income; T_2, the tax rate charged on any income earned over \$30,000; and C, the cut in spending (in billions). If the government chooses G, T_1, and T_2, then we assume that the revenue given in the file **P16_15.xlsx** (in billions of dollars) is raised.

Of course, the tax rate on income over \$30,000 must be at least as large as the tax rate on the first \$30,000 of income. Use goal programming to help the president meet his goals.

16. The HAL computer must determine which of eight research and development (R&D) projects to undertake. For each project, four quantities are of interest: (1) the net present value (NPV, in millions of dollars) of the project; (2) the annual growth rate in sales generated by the project; (3) the probability that the project will succeed; and (4) the cost (in millions of dollars) of the project. The relevant information is given in the file **P16_16.xlsx**. HAL has set the following four goals:

- Goal 1: The total NPV of all chosen projects should be at least \$200 million.
- Goal 2: The average probability of success for all projects chosen should be at least 0.75.
- Goal 3: The average growth rate of all projects chosen should be at least 15%.
- Goal 4: The total cost of all chosen projects should be at most \$1 billion.

For the following sets of priorities, use (integer) goal programming to determine the projects that should be selected.

a. Goal 2, Goal 4, Goal 1, Goal 3.
b. Goal 1, Goal 3, Goal 4, Goal 2.

17. Based on Klingman and Phillips (1984). The Marines need to fill three types of jobs in two cities (Los Angeles and Chicago). The numbers of jobs of each type that must be filled in each city are shown in the file **P16_17.xlsx**. The Marines available to fill these jobs have been classified into six groups according to the types of jobs each person is capable of doing, the type of job each person prefers, and the city in which each person prefers to live. The data for each of these six groups are also listed in this file. The Marines have the following three goals, listed from highest priority to lowest priority:

- Goal 1: Ensure that all jobs are filled by qualified workers.
- Goal 2: Ensure that at least 8000 employees are assigned to the jobs they prefer.
- Goal 3: Ensure that at least 8000 employees are assigned to their preferred cities.

Determine how the Marines should assign their workers. (*Note:* You may allow fractional assignments of workers.)

18. Based on Vasko et al. (1987). Bethlehem Steel can fill orders using five different types of steel molds. Up to three different molds of each type can be purchased. Each individual mold can be used to fill up to 100 orders per year. Six different types of orders must be filled during the coming year. The waste (in tons) incurred if a type of mold is used to fill an order is shown in the file **P16_18.xlsx** (where an “x” indicates that a type of mold cannot be used to fill an order). The number of each order type that must be filled during the coming year is also shown in this file. Bethlehem Steel has the following two goals, listed in order of priority.

- Goal 1: Because molds are very expensive, Bethlehem wants to use at most five molds.
- Goal 2: Bethlehem wants to have at most 600 tons of total waste.

Use goal programming to determine how Bethlehem should fill the coming year’s orders.

16.3 PARETO OPTIMALITY AND TRADE-OFF CURVES

In a multiobjective problem with no uncertainty, it is common to search for Pareto optimal solutions. We assume that the decision maker has exactly two objectives and that the set of feasible points under consideration must satisfy a prescribed set of constraints.

First, we need to define some terms. A solution (call it A) to a multiobjective problem is called **Pareto optimal** if no other feasible solution is at least as good as A with respect to every objective and strictly better than A with respect to at least one objective. A related concept is *domination.* A feasible solution B *dominates* a feasible solution A to a multiobjective problem if B is at least as good as A on every objective and is strictly better than A on at least one objective. From this definition, it follows that Pareto optimal solutions are feasible solutions that are not dominated.

If the “score” of all Pareto optimal solutions is graphed in the x–y plane with the x-axis score being the score on objective 1 and the y-axis score being the score on objective 2, the graph is called a **trade-off curve.** It is also called the **efficient frontier.** To illustrate, suppose that the set of feasible solutions for a multiobjective problem is the shaded region bounded by the curve AB and the axes in Figure 16.8. If the goal is to maximize both

Figure 16.8
Trade-off Curve for Maximizing Two Objectives

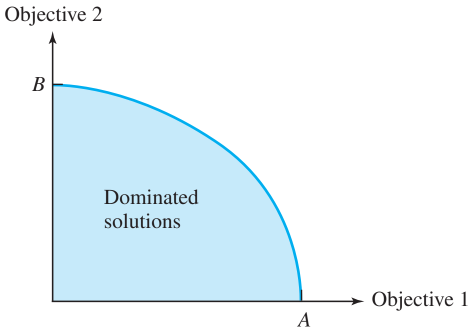
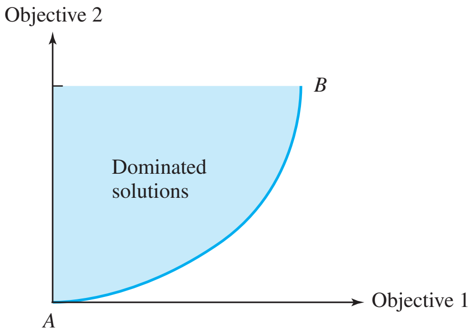


Figure 16.9
Trade-off Curve for Maximizing Objective 1 and Minimizing Objective 2



objectives 1 and 2, then the curve *AB* is the set of Pareto optimal points. All points below the *AB* curve are dominated by points on the curve.

As another illustration, suppose the set of feasible solutions for a multiple-objective problem is all shaded points in the first quadrant bounded from below by the curve *AB* in Figure 16.9. If the goal is to maximize objective 1 and minimize objective 2, then the curve *AB* is the set of Pareto optimal points. In this case, all points to the left of the curve are dominated by points on the curve.

Finding a Trade-off Curve

To find a trade-off curve, you can proceed according to the following steps.

1. Choose an objective, say objective 1, and determine its best attainable value V_1. For the solution attaining V_1, find the value of objective 2 and label it V_2. Then (V_1, V_2) is a point on the trade-off curve.
2. For values V of objective 2 that are better than V_2, solve the optimization problem in step 1 with the additional constraint that the value of objective 2 is at least as good as V. Varying V (over values of V preferred to V_2) yields other points on the trade-off curve.
3. Step 1 located one endpoint of the trade-off curve. Now determine the best value of objective 2 that can be attained, to obtain the other endpoint of the trade-off curve.

We illustrate the concept of Pareto optimality (and how to determine Pareto optimal solutions) with the following example.

EXAMPLE 16.2 Maximizing Profit and Minimizing Pollution at Chemcon

Chemcon plans to produce eight products. The profit per unit, the labor and raw material used per unit produced, and the pollution emitted per unit produced are given in Table 16.3. This table also includes lower and upper limits on production that Chemcon has imposed. Currently 1300 labor hours and 1000 units of raw material are available. Chemcon's two objectives are to maximize profit and minimize pollution produced. Chemcon wants to graph the trade-off curve for this problem.

Table 16.3 Data for the Chemcon Example

Product	1	2	3	4	5	6	7	8
Labor hrs/unit	5	5	1	4	3.5	4	2	3.5
Raw material/unit	3	4.5	5	5	4.5	2	3.5	3
Pollution/unit	25	29	35	26	17	25	28	6
Profit/unit	53	69	73	69	51	49	71	40
Min production	0	30	0	10	20	50	30	0
Max production	190	110	140	140	190	190	110	150

Objectives To find the trade-off curve between pollution and profit by solving a number of LP problems.

Where Do the Numbers Come From?

The required data here is basically the same as in the product mix problems from Chapter 3. Of course, the company also needs to find how much pollution each product is responsible for, which requires some scientific investigation.

Solution

Get the two extreme points on the trade-off curve by maximizing profit, ignoring pollution, and then minimizing pollution, ignoring profit.

The model itself is a straightforward version of the product mix models from Chapter 3. The objective is to find the product mix that stays within the lower and upper production limits, uses no more labor or raw material than are available, keeps pollution low, and keeps profit high. None of the formulas in the spreadsheet model (see Figure 16.10 and the file **Pollution Tradeoff.xlsx**) presents anything new, so we focus instead on the solution procedure.

Referring to the general three-step procedure for finding the trade-off curve, let profit be objective 1 and pollution be objective 2. To obtain one endpoint of the curve (step 1), you maximize profit and *ignore* pollution. That is, you maximize the Profit cell and delete the constraint indicated in row 26 from the Solver dialog box. You can check that the solution has profit \$20,089 and pollution level 9005.[1] (This is *not* the solution shown in the figure.) At the other end of the spectrum (step 3), you minimize the pollution in cell B26 and ignore any constraint on profit. You can check that this solution has pollution level 3560 and profit \$8360. In other words, profit can get as high as \$20,089 by ignoring pollution or as low as \$8360, and pollution can get as low as 3560 or as high as 9005. These establish the extremes. Now you can search for points in between (step 2).

[1]Actually, this is not quite true, as one user pointed out. If you maximize profit and *ignore* pollution, the resulting pollution level is 8980. To find the maximum possible pollution level, you need to *maximize* pollution. The resulting pollution level is 9005. Surprisingly, the profit from this solution is *less* than the maximum profit, \$20,089.

Figure 16.10 The Chemcon Model

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Chemcon profit versus pollution model**											
2												
3	**Input data**										**Range names used:**	
4	Product	1	2	3	4	5	6	7	8		Actual_pollution	=Model!$B$26
5	Labor hours/unit	5	5	1	4	3.5	4	2	3.5		Max_production	=Model!$B$17:$I$17
6	Raw material/unit	3	4.5	5	5	4.5	2	3.5	3		Min_production	=Model!$B$13:$I$13
7											Pollution_upper_bound	=Model!$D$26
8	Pollution/unit	25	29	35	26	17	25	28	6		Profit	=Model!$B$29
9	Profit/unit	$53	$69	$73	$69	$51	$49	$71	$40		Resources_available	=Model!$D$21:$D$22
10											Resources_used	=Model!$B$21:$B$22
11	**Production plan**										Units_produced	=Model!$B$15:$I$15
12	Product	1	2	3	4	5	6	7	8			
13	Min production	0	30	0	10	20	50	30	0			
14		<=	<=	<=	<=	<=	<=	<=	<=			
15	Units produced	0.0	30.0	0.0	10.0	21.1	50.0	48.6	150.0			
16		<=	<=	<=	<=	<=	<=	<=	<=			
17	Max production	190	110	140	140	190	190	110	150			
18												
19	**Constraints on resources**											
20		Resources used		Resources available								
21	Labor hours	1086.0	<=	1300								
22	Raw material	1000.0	<=	1000								
23												
24	**Constraint on pollution**											
25		Actual pollution		Pollution upper bound								
26		5000.0	<=	5000								
27												
28	**Objective to maximize**											
29	**Profit**	$15,738										

Get other points on the trade-off curve by maximizing profit, constraining pollution with varying upper bounds.

Fortunately, SolverTable is the perfect tool. According to step 2, you need to constrain pollution to various degrees and see how large profit can be. This is indicated in Figure 16.10, where the objective is to maximize profit with an upper limit on pollution. (You could get the same effect by minimizing pollution and putting a *lower* limit on profit.) The only upper limits on pollution you need to consider are those between the extremes, 3560 and 9005. Therefore, you can use SolverTable with the setup shown in Figure 16.11. Note that we have used the option to enter nonequally spaced inputs: 3560, 4000, 4500,

Figure 16.11
SolverTable Dialog Box

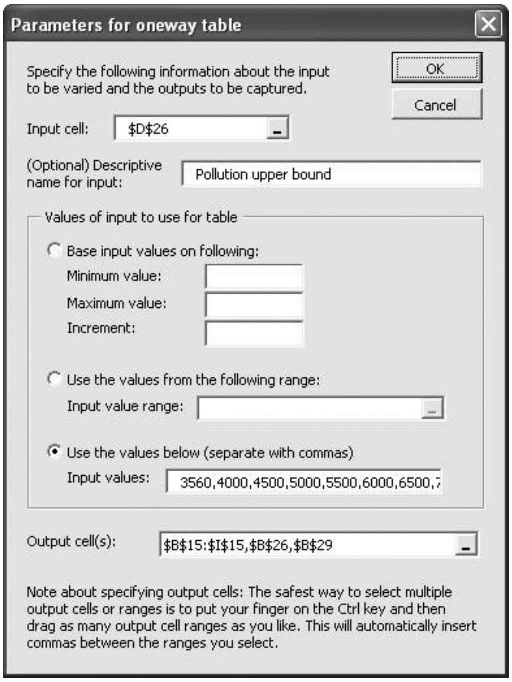


Figure 16.12 SolverTable Results

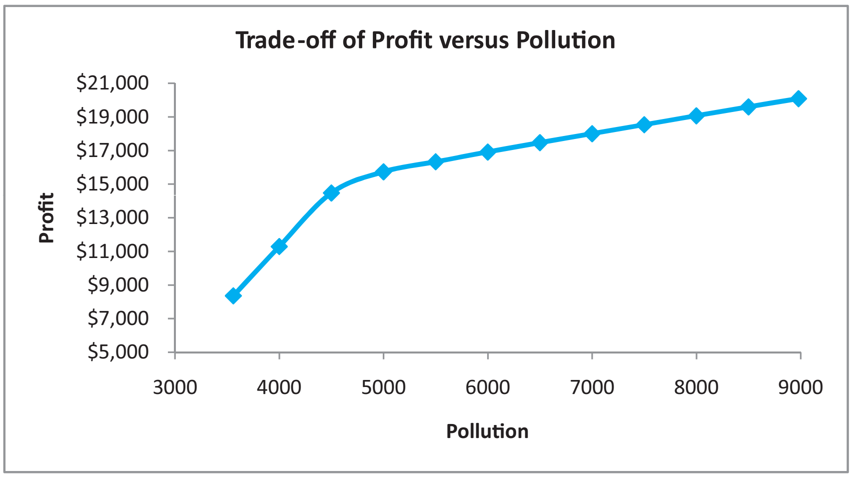
	A	B	C	D	E	F	G	H	I	J	K
1	**Oneway analysis for Solver model in Model worksheet**										
2											
3	Pollution upper bound (cell $D$26) values along side, output cell(s) along top										
4		Units_produced_1	Units_produced_2	Units_produced_3	Units_produced_4	Units_produced_5	Units_produced_6	Units_produced_7	Units_produced_8	Actual_pollution	Profit
5	3560	0.0	30.0	0.0	10.0	20.0	50.0	30.0	0.0	3560.0	$8,360
6	4000	0.0	30.0	0.0	10.0	20.0	50.0	30.0	73.3	4000.0	$11,293
7	4500	0.0	30.0	0.0	10.0	22.4	50.0	30.0	150.0	4500.0	$14,480
8	5000	0.0	30.0	0.0	10.0	21.1	50.0	48.6	150.0	5000.0	$15,738
9	5500	0.0	30.0	0.0	10.0	20.0	50.0	72.9	123.3	5500.0	$16,336
10	6000	0.0	30.0	0.0	10.0	20.0	50.0	96.7	95.6	6000.0	$16,916
11	6500	0.0	30.0	0.0	10.0	20.0	60.5	110.0	73.0	6500.0	$17,474
12	7000	0.0	30.0	0.0	10.0	20.0	84.3	110.0	57.1	7000.0	$18,006
13	7500	0.0	30.0	0.0	10.0	20.0	108.1	110.0	41.3	7500.0	$18,537
14	8000	0.0	30.0	0.0	10.0	20.0	131.9	110.0	25.4	8000.0	$19,069
15	8500	0.0	30.0	0.0	10.0	20.0	155.7	110.0	9.5	8500.0	$19,601
16	9005	0.0	30.0	0.0	10.0	20.0	190.0	98.6	0.0	8980.0	$20,089

and so on, ending with 9005. Alternatively, equally spaced inputs could be used. All that is required is a representative set of values between the extremes. The results appear in Figure 16.12.

Discussion of the Solution

These results show that as you allow more pollution, profit increases. Also, the product mix shifts considerably. Product 8, a low polluter with a low profit margin, eventually leaves the mix when pollution is allowed to increase, which makes sense. It is less clear why the level of product 6 increases so dramatically. Product 6 is only a moderate polluter and has a moderate profit margin, so the key is evidently that it requires low levels of labor and raw materials. The trade-off curve is created as a scatter chart (with the points connected) directly from columns J and K of the table. This curve appears in Figure 16.13. It

Figure 16.13
Trade-off Curve for Profit versus Pollution



indicates that profit indeed increases as Chemcon allows more pollution, but at a decreasing rate. For example, when pollution is allowed to increase from 4000 to 4500, Chemcon can make an extra \$3187 in profit. However, when pollution is allowed to increase from 8000 to 8500, the extra profit is only \$532. All points below the curve are dominated—for a given level of pollution, the company can achieve a larger profit—and all points above the curve are unattainable. ■

Trade-off curves are not limited to linear models. The following example illustrates a trade-off curve in a situation where the objective is a nonlinear function of the changing cells.

EXAMPLE 16.3 Trade-Offs Between Exposures to Men and Women at Leon Burnit

This example is a modification of the Burnit advertising example in Example 16.1. Now we assume that Burnit's client is concerned only with *two* groups of people, men and women. Also, the number of exposures to these groups is now a nonlinear square root function of the number of ads placed of any particular type. This implies a marginal decreasing effect of ads—each extra ad of a particular type reaches fewer extra people than the previous ad of this type.[2]

The data for this problem appear in Tables 16.4 and 16.5. The first of these specifies the proportionality constants for the square root exposure functions. For example, if five ads are placed in sports shows, this will achieve $15\sqrt{5} = 33.541$ million exposures to men, but only $5\sqrt{5} = 11.180$ million exposures to women. Evidently, what works well for men does not work so well for women, and vice versa. Given a budget of \$1.5 million, find the trade-off curve for exposures to men versus exposures to women.

Table 16.4 Proportionality Constants for Square Root Exposure Functions

	Sports Show	Game Show	News Show	Sitcom	Drama	Soap Opera
Men	15	3	7	7	8	1
Women	5	5	6	10	9	4

Table 16.5 Data on Ads for the Burnit Example

	Sports Show	Game Show	News Show	Sitcom	Drama	Soap Opera
Cost/ad (\$1000s)	120	40	50	40	60	20
Lower limit	2	0	2	0	2	0
Upper limit	10	5	10	5	10	5

Objective To find the trade-off curve for exposures to men versus exposures to women by solving a number of NLP problems.

[2]The square root function is an alternative to the exponential advertising response function we used in Example 7.5 of Chapter 7. Each increases at a decreasing rate.

Where Do the Numbers Come From?

We have discussed these same types of numbers in previous examples. Specifically, the parameters in Table 16.4 can be estimated from historical data, exactly as described in Example 7.5 of Chapter 7.

Solution

Again, the model itself is straightforward, as shown in Figure 16.14. (See the file **Advertising Tradeoff.xlsx**.) You calculate the exposures achieved in rows 22 and 23 by entering the formula

=B8*SQRT(B$17)

in cell B22 and copying it to the range B22:G23. You then sum these in cells B30 and B33, and calculate the total cost in the usual way with the SUMPRODUCT function.

For the three-step trade-off curve procedure, let exposures to men be objective 1 and exposures to women be objective 2. For step 1, you maximize exposures to men and ignore women. That is, you do *not* include the constraint in row 30 in the Solver dialog box. You can check that the corresponding solution achieves 89.515 million exposures to men and 79.392 million exposures to women. Reversing the roles of men and women (step 3), you can check that if you maximize exposures to women and ignore men, the solution achieves 89.220 million exposures to women and only 84.899 million exposures to men.

All other points on the trade-off curve are between these two extremes, and they can again be found easily with SolverTable. You now set up Solver to maximize exposures to men, and you include the lower limit constraint on exposures to women in the Solver dialog box. (Do you see why it is a *lower* limit constraint in this example, whereas it was

Figure 16.14 The Advertising Trade-off Model

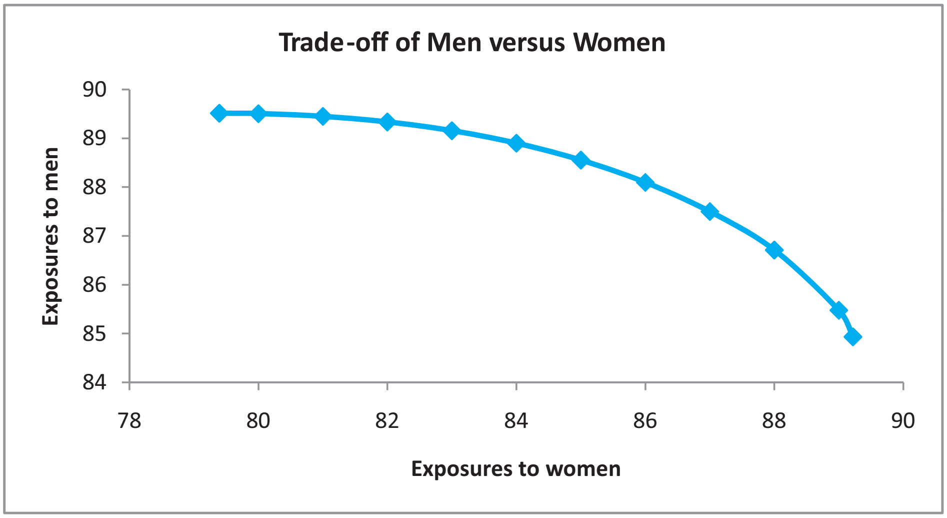
	A	B	C	D	E	F	G	H	I	J
1	**Burnit nonlinear advertising model**									
2			Assumption: The number of exposures (in millions) to each group is proportional to the square root of the number of ads of a particular type shown.							
3										
4										
5										
6	**Proportionality constants for exposure functions**								**Range names used:**	
7		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ads		Budget	=Model!$D$26
8	Exposures to men	15	3	7	7	8	1		Exposures_to_men	=Model!$B$33
9	Exposures to women	5	5	6	10	9	4		Exposures_to_women	=Model!$B$30
10									Maximum_ads_allowed	=Model!$B$19:$G$19
11	Cost/ad ($1,000s)	120	40	50	40	60	20		Minimum_ads_required	=Model!$B$15:$G$15
12									Number_purchased	=Model!$B$17:$G$17
13	**Advertising plan**								Total_cost	=Model!$B$26
14		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad		Women_lower_bound	=Model!$D$30
15	Minimum ads required	2	0	2	0	2	0			
16		<=	<=	<=	<=	<=	<=			
17	Number purchased	2.000	5.000	5.387	5.000	8.177	5.000			
18		<=	<=	<=	<=	<=	<=			
19	Maximum ads allowed	10	5	10	5	10	5			
20										
21	**Exposures obtained**	Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad			
22	Men	21.213	6.708	16.247	15.652	22.877	2.236			
23	Women	7.071	11.180	13.926	22.361	25.737	8.944			
24										
25	**Budget constraint**	Total cost		Budget						
26		1500.000	<=	1500						
27										
28	**Constraint on minimal exposures to women**									
29		Exposures to women		Women lower bound						
30		89.219	>=	89.219						
31										
32	**Objective to maximize**									
33	Exposures to men	84.934								

an upper limit constraint in the previous example? There the objective was to make pollution low. Here the objective is to make exposures to women high.) The lower limit cell (D30) becomes the single input cell for SolverTable, which can vary from (slightly greater than) 79.392 to (slightly less than) 89.220 with suitable values in between. The results appear in table form in Figure 16.15 and in graphical form in Figure 16.16.

Figure 16.15 SolverTable Results for the Advertising Trade-off Model

	A	B	C	D	E	F	G	H	I
1	**Oneway analysis for Solver model in Model worksheet**								
2									
3	Women lower bound (cell $D$30) values along side, output cell(s) along top								
4		Number_purchased_1	Number_purchased_2	Number_purchased_3	Number_purchased_4	Number_purchased_5	Number_purchased_6	Exposures_to_women	Exposures_to_men
5	79.393	4.839	1.744	6.072	5.000	5.508	0.776	79.393	89.515
6	80	4.715	1.835	6.100	5.000	5.620	0.928	80.000	89.506
7	81	4.503	1.994	6.143	5.000	5.807	1.215	81.000	89.449
8	82	4.280	2.163	6.178	5.000	5.997	1.555	82.000	89.336
9	83	4.048	2.347	6.204	5.000	6.186	1.954	83.000	89.156
10	84	3.801	2.538	6.220	5.000	6.383	2.421	84.000	88.900
11	85	3.540	2.745	6.228	5.000	6.578	2.969	85.000	88.554
12	86	3.262	2.976	6.217	5.000	6.777	3.604	86.000	88.096
13	87	2.964	3.225	6.189	5.000	6.979	4.357	87.000	87.500
14	88	2.600	3.580	6.173	5.000	7.269	5.000	88.000	86.713
15	89	2.057	4.276	6.207	5.000	7.863	5.000	89.000	85.478
16	89.219	2.000	5.000	5.387	5.000	8.177	5.000	89.219	84.934

Figure 16.16
Trade-off Curve for the Advertising Example



Discussion of the Solution

As you look down the table (or to the right in the chart), more exposures to women are required, which has an increasingly negative effect on exposures to men. Not surprisingly, the corresponding solutions place more ads in the shows watched predominantly by women (game shows, dramas, and soaps) and fewer ads in sports and news shows. The upper limit placed on sitcom ads prevents you from seeing how the number of sitcom ads would change if it were not constrained. It would probably change fairly dramatically, given that these ads are relatively cheap and they tend to reach more women than men.

Technical Note

We ran into two problems that you might experience. First, depending on the starting solution, one of the changing cells might become slightly negative (due to numerical roundoff), in which case the SQRT function is undefined, and you get an error message. To remedy this, you can add a constraint such as Ads>=0.0001. Second, when we ran SolverTable, it indicated "no feasible solution" to the problem in row 49 of Figure 16.15, although we know there is a feasible solution. This can sometimes occur with nonlinear models, depending on the starting solution used. SolverTable uses the solution from the previous problem as the starting solution for the next problem. This seems reasonable, but it *can* produce this error. If it does, try running the Solver on this particular problem again with your own initial solution (such as all 0s). This is what we did to get the values in row 49. ■

MODELING ISSUES

1. A trade-off curve is useful because it gives the ultimate decision maker many undominated solutions to choose from. However, it does *not* specify a "best" solution. The decision maker still has to make the difficult decision of which solution from the trade-off curve to implement. This can be done subjectively or with the help of a *multiattribute utility function.* However, estimating these types of functions is difficult, so their use in real-world applications has been limited.
2. These trade-off models can be generalized to a situation where there are more than two objectives by constructing trade-off curves between each *pair* of objectives. ■

PROBLEMS

Skill-Building Problems

19. Widgetco produces two types of widgets. Each widget is made of steel and aluminum and is assembled with skilled labor. The resources used and the per-unit profit contribution (ignoring cost of overtime labor purchased) for each type of widget are given in the file **P16_19.xlsx**. At present, 200 pounds of steel, 300 pounds of aluminum, and 300 hours of labor are available. Extra overtime labor can be purchased for $10 per hour. Construct a trade-off curve between the objectives of maximizing profit and minimizing overtime labor.

20. Plantco produces three products. Three workers work for Plantco, and the company must determine which product(s) each worker should produce. The number of units each worker would produce if he or she spent the whole day producing each type of product is given in the file **P16_20.xlsx**. The company is also interested in maximizing the happiness of its workers. The amount of happiness "earned" by a worker who spends the entire day producing a given product is also given in this file. Construct a trade-off curve between the objectives of maximizing total units produced daily and total worker happiness.

21. If a company spends a on advertising (measured in thousands of dollars) and charges a price of p dollars per unit, then it can sell $1000 - 10p + 20a^{1/2}$ units of the product. The cost per unit of producing the product is $6. Construct a trade-off curve between the objectives of maximizing profit and maximizing the number of units sold.

22. GMCO produces three types of cars: compact, medium, and large. The variable cost per car and production capacity (per year) for each type of car are given in the file **P16_22.xlsx**. The annual demand for each type of car depends on the prices of the three types of cars, also given in this file. In this latter table, P_C is the price charged for a compact car (in thousands of dollars). The variables P_M and P_L are defined similarly for medium and large cars. Suppose that each compact car gets 30 mpg, each medium car gets 25 mpg, and each large car gets 18 mpg. GMCO wants to keep the planet pollution free, so in addition to maximizing profit, it wants to maximize the average miles per gallon attained by the cars it sells. Construct a trade-off curve between these two objectives.

23. In the capital budgeting example from Chapter 6 (see Example 6.1), we maximized NPV for a given budget. Now find a trade-off curve for NPV versus budget. Specifically, minimize the amount invested, with a lower bound constraint on the NPV obtained. What lower bounds should you use? Do you get the same trade-off curve as in Figure 6.4?

24. The portfolio optimization example from Chapter 7 (see Example 7.9) found the efficient frontier by minimizing portfolio variance, with a lower bound constraint on the expected return. Do it the opposite way. That is, calculate the efficient frontier by maximizing the expected return, with an *upper* bound on the portfolio standard deviation. Do you get the same results as in Example 7.9?

16.4 THE ANALYTIC HIERARCHY PROCESS (AHP)

When multiple objectives are important to a decision maker, choosing between alternatives can be difficult. For example, if you are choosing a job, one job might offer the highest starting salary but rate poorly on other objectives such as quality of life in the city where the job is located and the nearness of the job to your family. Another job offer might rate highly on these latter objectives but have a relatively low starting salary. In this case, it can be difficult for you to choose between job offers. The **Analytic Hierarchy Process (AHP),** developed originally by Thomas Saaty, is a powerful tool that can be used to make decisions in situations where multiple objectives are present. We present an example to illustrate such a case.[3] (*Note:* Matrix notation and matrix multiplication are used in this section. You may need to review the discussion of matrices in section 7.7.)

EXAMPLE 16.4 Using AHP to Select a Job

Jane is about to graduate from college and is trying to determine which job to accept. She plans to choose among the offers by determining how well each job offer meets the following four objectives:

- Objective 1: High starting salary
- Objective 2: Quality of life in city where job is located
- Objective 3: Interest of work
- Objective 4: Nearness of job to family

Objective To use the AHP method to help Jane select a job that is best in terms of the various job criteria.

Where Do the Numbers Come From?

As discussed shortly, Jane must make a number of trade-offs during the implementation of AHP. In this case, the decision maker supplies the data.

[3]The leading software package for implementing AHP is Expert Choice, developed by Expert Choice Inc.

Solution

To illustrate how AHP works, suppose that Jane is facing three job offers and must determine which offer to accept. In this example, there are four objectives, as listed previously. For each objective, AHP generates a weight (by a method to be described shortly). By convention, the weights are always chosen so that they sum to 1. Suppose that Jane's weights are $w_1 = 0.5115$, $w_2 = 0.0986$, $w_3 = 0.2433$, and $w_4 = 0.1466$. These weights indicate that a high starting salary is the most important objective, followed by interest of work, nearness to family, and quality of life.

AHP is essentially a process of rating the importance of each objective and then rating how well each possible decision meets each objective. The result is a score for each possible decision, with higher scores preferred.

Next, suppose that Jane determines (again by a method to be described shortly) how well each job "scores" on each objective. For example, suppose these scores are those listed in Table 16.6. You can see from this table that job 1 best meets the objective of a high starting salary, but scores worst on all other objectives. Note that the scores of the jobs on each objective are normalized, which means that for each objective, the sum of the scores of the jobs on that objective is 1.

Given the weights for the objectives and the scores shown in Table 16.6, Jane can now determine which job offer to accept. Specifically, for each job, she calculates an overall score that is a weighted sum of the scores for that job, using the w's as weights. For example, the overall score for job 1 weights the scores in the first row of Table 16.6:

$$\begin{aligned}\text{Job 1 score} &= 0.5115(0.5714) + 0.0986(0.1593) + 0.2433(0.0882) + 0.1466(0.0824)\\ &= 0.3415\end{aligned}$$

Table 16.6 Job Scores on Objectives in the AHP Example

	Salary	Quality of Life	Interest of Work	Nearness to Family
Job 1	0.5714	0.1593	0.0882	0.0824
Job 2	0.2857	0.2519	0.6687	0.3151
Job 3	0.1429	0.5889	0.2431	0.6025

Similarly, the overall scores for jobs 2 and 3 are obtained by weighting the scores in the second and third rows of Table 16.6:

$$\begin{aligned}\text{Job 2 score} &= 0.5115(0.2857) + 0.0986(0.2519) + 0.2433(0.6687) + 0.1466(0.3151)\\ &= 0.3799\end{aligned}$$

$$\begin{aligned}\text{Job 3 score} &= 0.5115(0.1429) + 0.0986(0.5889) + 0.2433(0.2431) + 0.1466(0.6025)\\ &= 0.2786\end{aligned}$$

Because the overall score for job 2 is the largest, AHP suggests that Jane should accept this job.

The following discussion on how AHP actually works is technical and is not really necessary for *using* the method. We have included the file **Choosing Jobs with VBA.xlsm** that implements AHP as a decision support system with macros. We urge you to try it out, especially if you are currently making a job decision. You don't need to understand the details behind AHP to run the application. You simply need to make a number of pairwise comparisons, as indicated in a number of dialog boxes. However, if you really *do* want to understand how AHP works, then read on. By the way, the term *criterion* is commonly used instead of objective when discussing AHP. The file **Choosing Jobs with VBA.xlsm** uses this term consistently.

Pairwise Comparison Matrices

To obtain the weights for the various objectives, you begin by forming a matrix A, known as the pairwise comparison matrix. The entry in row i and column j of A, labeled a_{ij}, indicates

Table 16.7 Interpretation of Values in the Pairwise Comparison Matrix

Value of a_{ij}	Interpretation
1	Objectives i and j are equally important.
3	Objective i is slightly more important than j.
5	Objective i is strongly more important than j.
7	Objective i is very strongly more important than j.
9	Objective i is absolutely more important than j.

how much more (or less) important objective i is than objective j to the decision maker. "Importance" is measured on an integer-valued scale from 1 to 9, with each number having the interpretation shown in Table 16.7. The phrases in this table, such as "strongly more important than," are suggestive only. They simply indicate discrete points on a continuous scale that can be used to compare the relative importance of any two objectives.

For example, if $a_{13} = 3$, then objective 1 is slightly more important to Jane than objective 3. If $a_{ij} = 4$, a value not in the table, then objective i is somewhere between slightly and strongly more important than objective j. If objective i is *less* important to Jane than objective j, the reciprocal of the appropriate index is used. For example, if objective i is slightly less important than objective j, then $a_{ij} = 1/3$. Finally, for all objectives i, the convention is to set $a_{ii} = 1$.

For consistency, it is necessary to set $a_{ji} = 1/a_{ij}$. For example, if $a_{13} = 3$, then it is necessary to have $a_{31} = 1/3$. This simply states that if objective 1 is slightly more important than objective 3, then objective 3 is slightly less important than job 1. It is usually easier to determine all a_{ij}'s that are greater than 1 and then use the relationship $a_{ji} = 1/a_{ij}$ to determine the remaining entries in the pairwise comparison matrix.

To illustrate, suppose that Jane has identified the following pairwise comparison matrix for her four objectives:

$$A = \begin{bmatrix} 1 & 5 & 2 & 4 \\ 1/5 & 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1 & 2 \\ 1/4 & 2 & 1/2 & 1 \end{bmatrix}$$

The rows and columns of A each correspond to Jane's four objectives: salary, quality of life, interest of work, and nearness to family. Considering the first row, for example, she believes that salary is more important, in various degrees, than quality of life, interest of work, and nearness to family.

The entries in this matrix have built-in pairwise consistency because we require $a_{ij} = 1/a_{ij}$ for each i and j. However, they might not be consistent when three (or more) alternatives are considered simultaneously. For example, Jane claims that salary is strongly more important than quality of life ($a_{12} = 5$) and that salary is very slightly more important than interesting work ($a_{13} = 2$). But she also says that interesting work is very slightly more important than quality of life ($a_{32} = 2$). The question is whether these ratings are all consistent with one another. They are not, at least not exactly. It can be shown that some of Jane's pairwise comparisons are slightly inconsistent. When a person is asked to make a number of pairwise comparisons, slight inconsistencies are common and fortunately do not cause serious difficulties. An index that can be used to measure the consistency of Jane's preferences is discussed later in this section.

Determining the Weights

Although the ideas behind AHP are fairly intuitive, the mathematical reasoning required to derive the weights for the objectives is advanced. Therefore, we simply describe how it is done.

Starting with the pairwise comparison matrix A, you find the weights for Jane's four objectives using the following two steps:

1. For each of the columns of A, divide each entry in the column by the sum of the entries in the column. This yields a new matrix (call it A_{norm}, for "normalized") in which the sum of the entries in each column is 1. For Jane's pairwise comparison matrix, this step yields

$$A_{\text{norm}} = \begin{bmatrix} 0.5128 & 0.5000 & 0.5000 & 0.5333 \\ 0.1026 & 0.1000 & 0.1250 & 0.0667 \\ 0.2564 & 0.2000 & 0.2500 & 0.2667 \\ 0.1282 & 0.2000 & 0.1250 & 0.1333 \end{bmatrix}$$

2. Estimate w_i, the weight for objective i, as the average of the entries in row i of A_{norm}. For Jane's matrix this yields

$$w_1 = \frac{0.5128 + 0.5000 + 0.5000 + 0.5333}{4} = 0.5115$$

$$w_2 = \frac{0.1026 + 0.1000 + 0.1250 + 0.0667}{4} = 0.0986$$

$$w_3 = \frac{0.2564 + 0.2000 + 0.2500 + 0.2667}{4} = 0.2433$$

$$w_4 = \frac{0.1282 + 0.2000 + 0.1250 + 0.1333}{4} = 0.1466$$

Intuitively, why does w_1 approximate the weight for objective 1 (salary)? Here is the reasoning. The proportion of weight that salary is given in pairwise comparisons of each objective to salary is 0.5128. Similarly, 0.50 represents the proportion of total weight that salary is given in pairwise comparisons of each objective to quality of life. Therefore, each of the four numbers averaged to obtain w_1 represents a measure of the total weight attached to salary. Averaging these numbers should give a good estimate of the proportion of the total weight given to salary.

Determining the Score of Each Decision Alternative on Each Objective

Now that the weights for the various objectives have been determined, the next step is to determine how well each job scores on each objective. To determine these scores, you use the same scale described in Table 16.7 to construct a pairwise comparison matrix for each objective. Consider the salary objective, for example. Suppose that Jane assesses the following pairwise comparison matrix. We denote this matrix as A_1 because it reflects her comparisons of the three jobs with respect to the first objective, salary.

$$A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$$

The rows and columns of this matrix correspond to the three jobs. For example, the first row means that Jane believes job 1 is superior to job 2 (and even more superior to job 3) in terms of salary. To find the relative scores of the three jobs on salary, the *same* two-step procedure as previously discussed is applied to the salary pairwise comparison matrix A_1. That is, you first divide each column entry by the column sum to obtain

$$A_{1,\text{norm}} = \begin{bmatrix} 0.5714 & 0.5714 & 0.5714 \\ 0.2857 & 0.2857 & 0.2857 \\ 0.1429 & 0.1429 & 0.1429 \end{bmatrix}$$

Then you average the numbers in each row to obtain the vector of scores for the three jobs on salary, denoted by S_1:

$$S_1 = \begin{bmatrix} 0.5714 \\ 0.2857 \\ 0.1429 \end{bmatrix}$$

That is, the scores for jobs 1, 2, and 3 on salary are 0.5714, 0.2857, and 0.1429. In terms of salary, job 1 is clearly the favorite.

Next, these calculations are repeated for Jane's other objectives. Each of these objectives requires a pairwise comparison matrix, which we denote as A_2, A_3, and A_4. Suppose that Jane's pairwise comparison matrix for quality of life is

$$A_2 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 1/3 \\ 3 & 3 & 1 \end{bmatrix}$$

Then the corresponding normalized matrix is

$$A_{2,\text{norm}} = \begin{bmatrix} 0.1667 & 0.1111 & 0.2000 \\ 0.3333 & 0.2222 & 0.2000 \\ 0.5000 & 0.6667 & 0.6000 \end{bmatrix}$$

and by averaging,

$$S_2 = \begin{bmatrix} 0.1593 \\ 0.2519 \\ 0.5889 \end{bmatrix}$$

Here, job 3 is the clear favorite. However, this does not have much impact because Jane puts relatively little weight on quality of life.

For interest of work, suppose the pairwise comparison matrix is

$$A_3 = \begin{bmatrix} 1 & 1/7 & 1/3 \\ 7 & 1 & 3 \\ 3 & 1/3 & 1 \end{bmatrix}$$

Then the same types of calculations show that the scores for jobs 1, 2, and 3 on interest of work are

$$S_3 = \begin{bmatrix} 0.0882 \\ 0.6687 \\ 0.2431 \end{bmatrix}$$

Finally, suppose the pairwise comparison matrix for nearness to family is

$$A_4 = \begin{bmatrix} 1 & 1/4 & 1/7 \\ 4 & 1 & 1/2 \\ 7 & 2 & 1 \end{bmatrix}$$

In this case, the scores for jobs 1, 2, and 3 on nearness to family are

$$S_4 = \begin{bmatrix} 0.0824 \\ 0.3151 \\ 0.6025 \end{bmatrix}$$

Determining the Best Alternative

Let's summarize what has been determined so far. Jane first assesses a pairwise comparison matrix A that measures the relative importance of each of her objectives to one another. From this matrix, she obtains a vector of weights that summarizes the relative importance of the objectives. Next, Jane assesses a pairwise comparison matrix A_i for each objective i. This matrix measures how well each job compares to other jobs with regard to this objective. For each matrix A_i, she obtains a vector of scores S_i that summarizes how the jobs compare in terms of achieving objective i.

The final step is to combine the scores in the S_i vectors with the weights in the w vector. Actually, this has already been done. Note that the columns of Table 16.6 are the S_i vectors just obtained. If you form a matrix S of these score vectors and multiply this matrix by w, you obtain a vector of overall scores for each job, as shown here:

$$Sw = \begin{bmatrix} 0.5714 & 0.1593 & 0.0882 & 0.0824 \\ 0.2857 & 0.2519 & 0.6687 & 0.3151 \\ 0.1429 & 0.5889 & 0.2431 & 0.6025 \end{bmatrix} \times \begin{bmatrix} 0.5115 \\ 0.0986 \\ 0.2433 \\ 0.1466 \end{bmatrix} = \begin{bmatrix} 0.3415 \\ 0.3799 \\ 0.2786 \end{bmatrix}$$

These are the same overall scores listed earlier. As before, the largest of these overall scores is for job 2, so AHP suggests that Jane should accept this job. Job 1 follows closely behind, with job 3 somewhat farther behind.

Checking for Consistency

As mentioned earlier, any pairwise comparison matrix can suffer from inconsistencies. We now describe a procedure to check for inconsistencies. We illustrate this on the A matrix and its associated vector of weights w. The same procedure can be used on any of the A_i matrices and their associated weights vector S_{i}:

1. Calculate Aw. For the example,

$$Aw = \begin{bmatrix} 1 & 5 & 2 & 4 \\ 1/5 & 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1 & 2 \\ 1/4 & 2 & 1/2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.5115 \\ 0.0986 \\ 0.2433 \\ 0.1466 \end{bmatrix} = \begin{bmatrix} 2.0774 \\ 0.3958 \\ 0.9894 \\ 0.5933 \end{bmatrix}$$

2. Find the ratio of each element of Aw to the corresponding weight in w and average these ratios. For the example, this calculation is

$$\frac{\dfrac{2.0774}{0.5115} + \dfrac{0.3958}{0.0986} + \dfrac{0.9894}{0.2433} + \dfrac{0.5933}{0.1466}}{4} = 4.0477$$

3. Calculate the consistency index (labeled CI) as

$$CI = \frac{(\text{Step 2 result}) - n}{n - 1}$$

where n is the number of objectives. For the example this is $CI = \dfrac{4.0477 - 4}{4 - 1} = 0.0159$.

4. Compare CI to the random index (labeled RI) in Table 16.8 for the appropriate value of n.

Table 16.8 Random Indices for Consistency Check for the AHP Example

n	2	3	4	5	6	7	8	9	10
RI	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.51

To be a perfectly consistent decision maker, each ratio in step 2 should equal n. This implies that a perfectly consistent decision maker has $CI = 0$. The values of RI in Table 16.8 give the average value of CI if the entries in A were chosen at random (subject to the constraints that a_{ii}'s must equal 1, and $a_{ij} = 1/a_{ji}$). If the ratio of CI to RI is sufficiently small, the decision maker's comparisons are probably consistent enough to be useful. Saaty suggested that if $CI/RI < 0.10$, the degree of consistency is satisfactory, whereas if $CI/RI > 0.10$, serious inconsistencies exist and AHP may not yield meaningful results. In Jane's example, $CI/RI = 0.0159/0.90 = 0.0177$, which is much less than 0.10. Therefore, Jane's pairwise comparison matrix A does not exhibit any serious inconsistencies. (You can check that the same is true of her other pairwise comparison matrices A_1 through A_4.)

Developing the Spreadsheet Model

We now show how to implement AHP on a spreadsheet. (See Figure 16.17 and the file **Choosing Jobs.xlsx**.)

Figure 16.17 The AHP Job Selection Model

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Job selection using analytical hierarchy process**											
2												
3	**Pairwise comparisons among objectives**						**Normalized matrix**					**Weights**
4		Salary	Life quality	Work interest	Near family							
5	Salary	1	5	2	4		0.5128	0.5000	0.5000	0.5333		0.5115
6	Life quality	1/5	1	1/2	1/2		0.1026	0.1000	0.1250	0.0667		0.0986
7	Work interest	1/2	2	1	2		0.2564	0.2000	0.2500	0.2667		0.2433
8	Near family	1/4	2	1/2	1		0.1282	0.2000	0.1250	0.1333		0.1466
9												
10	**Pairwise comparisons among jobs on salary**						**Normalized matrix**					**Scores**
11		Job 1	Job 2	Job 3								
12	Job 1	1	2	4			0.5714	0.5714	0.5714			0.5714
13	Job 2	1/2	1	2			0.2857	0.2857	0.2857			0.2857
14	Job 3	1/4	1/2	1			0.1429	0.1429	0.1429			0.1429
15												
16	**Pairwise comparisons among jobs on quality of life**						**Normalized matrix**					**Scores**
17		Job 1	Job 2	Job 3								
18	Job 1	1	1/2	1/3			0.1667	0.1111	0.2000			0.1593
19	Job 2	2	1	1/3			0.3333	0.2222	0.2000			0.2519
20	Job 3	3	3	1			0.5000	0.6667	0.6000			0.5889
21												
22	**Pairwise comparisons among jobs on interest of work**						**Normalized matrix**					**Scores**
23		Job 1	Job 2	Job 3								
24	Job 1	1	1/7	1/3			0.0909	0.0968	0.0769			0.0882
25	Job 2	7	1	3			0.6364	0.6774	0.6923			0.6687
26	Job 3	3	1/3	1			0.2727	0.2258	0.2308			0.2431
27												
28	**Pairwise comparisons among jobs on nearness to family**						**Normalized matrix**					**Scores**
29		Job 1	Job 2	Job 3								
30	Job 1	1	1/4	1/7			0.0833	0.0769	0.0870			0.0824
31	Job 2	4	1	1/2			0.3333	0.3077	0.3043			0.3151
32	Job 3	7	2	1			0.5833	0.6154	0.6087			0.6025
33												
34	**Determining best job**											
35	Matrix of scores						Weighted scores					
36		Salary	Life quality	Work interest	Near family							
37	Job 1	0.5714	0.159	0.088	0.082		0.3415		Job 2 has the			
38	Job 2	0.2857	0.252	0.669	0.315		0.3799		highest score			
39	Job 3	0.1429	0.589	0.243	0.602		0.2786					

1 **Inputs.** Enter the pairwise comparison matrices in the shaded ranges. (Note that you can enter fractions such as 1/7 in cell C24, and have them appear as fractions by formatting the cells with the Fraction option.)

2 **Normalized matrix.** Calculate the normalized matrix for the first pairwise comparison matrix in the range G5:J8. This can be done quickly as follows. Starting with the cursor in cell G5, highlight the range G5:J8. Then type the formula

=B5/SUM(B$5:B$8)

and press Control+Enter (both keys at once). We introduced this really useful shortcut in an earlier chapter as a quick way to enter the same formula in an entire range.

3 **Weights of objectives.** In the range L5:L8, calculate the weights for each objective. Again, do this the quick way. Starting with the cursor in cell L5, highlight the range L5:L8. Then type the formula

=AVERAGE(G5:J5)

and press Control+Enter.

4 **Scores for jobs on objectives.** Repeat the same calculations in steps 2 and 3 for the other pairwise comparison matrices to obtain the normalized matrices in columns G through I and scores vectors in column L.

5 **Overall job scores.** In the range B37:E39, form a matrix of job scores on the various objectives. To get the score vector in the range L12:L14 into the range B37:B39, for example, highlight this latter range, type the formula

=L12

and press Control+Enter. Do likewise for the other three scores vectors in column L. Then to obtain the overall job scores (from the matrix product Sw), highlight the range G37:G39, type the formula

=MMULT(B37:E39,L5:L8)

and press Control+Shift+Enter. (Remember that Control+Shift+Enter is used to enter a matrix function. In contrast, Control+Enter is equivalent to copying a formula to a highlighted range.) Again you can see that job 2 is the most preferred of the three jobs.

Calculating the Consistency Index

We now show how to compute the consistency index CI for each of the pairwise comparison matrices. (See Figure 16.18, which is also part of the file **AHPJobs.xlsx**. Note that columns G through K have been hidden to save space. These contain the normalized matrices from step 2 in the previous section.) The following steps are relevant for the first pairwise comparison matrix. The others are done in analogous fashion.

1 **Product of comparison matrix and vector of weights (or scores).** Calculate the product of the first pairwise comparison matrix and the weights vector in the range N5:N8 by highlighting this range, typing

=MMULT(B5:E8,L5:L8)

and pressing Ctrl+Shift+Enter.

2 **Ratios.** In cell O5, calculate the ratio of the two cells to its left with the formula

=N5/L5

and copy this to the range O6:O8.

Figure 16.18 Checking for Consistency

	A	B	C	D	E	F	L	M	N	O
1	**Job selection using analytical hierarchy process**									
2										
3	**Pairwise comparisons among objectives**						**Weights**		**Product**	**Ratios**
4		Salary	Life quality	Work interest	Near family					
5	Salary	1	5	2	4		0.5115		2.0774	4.0611
6	Life quality	1/5	1	1/2	1/2		0.0986		0.3958	4.0161
7	Work interest	1/2	2	1	2		0.2433		0.9894	4.0672
8	Near family	1/4	2	1/2	1		0.1466		0.5933	4.0459
9									CI	0.0159
10	**Pairwise comparisons among jobs on salary**						**Scores**		CI/RI	0.0176
11		Job 1	Job 2	Job 3						
12	Job 1	1	2	4			0.5714		1.7143	3
13	Job 2	1/2	1	2			0.2857		0.8571	3
14	Job 3	1/4	1/2	1			0.1429		0.4286	3
15									CI	0
16	**Pairwise comparisons among jobs on quality of life**						**Scores**		CI/RI	0.0000
17		Job 1	Job 2	Job 3						
18	Job 1	1	1/2	1/3			0.1593		0.4815	3.0233
19	Job 2	2	1	1/3			0.2519		0.7667	3.0441
20	Job 3	3	3	1			0.5889		1.8222	3.0943
21									CI	0.0270
22	**Pairwise comparisons among jobs on interest of work**						**Scores**		CI/RI	0.0465
23		Job 1	Job 2	Job 3						
24	Job 1	1	1/7	1/3			0.0882		0.2648	3.0018
25	Job 2	7	1	3			0.6687		2.0154	3.0139
26	Job 3	3	1/3	1			0.2431		0.7306	3.0054
27									CI	0.0035
28	**Pairwise comparisons among jobs on nearness to family**						**Scores**		CI/RI	0.0061
29		Job 1	Job 2	Job 3						
30	Job 1	1	1/4	1/7			0.0824		0.2473	3.0005
31	Job 2	4	1	1/2			0.3151		0.9460	3.0019
32	Job 3	7	2	1			0.6025		1.8096	3.0035
33									CI	0.0010
34	**Determining best job**								CI/RI	0.0017

3 Consistency index. Calculate the consistency index *CI* in cell O9 with the formula

=(AVERAGE(O5:O8)–4)/3

Then in cell O10, calculate the ratio of *CI* to *RI* (for $n = 4$) with the formula

=O9/0.90

(The 0.90 comes from Table 16.8 earlier in the chapter. For the other four pairwise comparison matrices in Figure 16.18, you should use $n = 3$ and $RI = 0.58$.)

As Figure 16.18 illustrates, all of the pairwise comparison matrices are sufficiently consistent—the CI/RI ratio for each is well less than 0.10. ■

Modeling Issues

1. In Jane's job selection example, suppose that quality of life depends on two subobjectives: recreational facilities and educational facilities. Then we need a pairwise comparison matrix to calculate the proportion of the quality of life score that is determined by recreational facilities and the proportion that is determined by educational facilities. Next, we need to determine how each job scores (separately) on recreational facilities and educational facilities. Then we can again determine a quality of life score for each job and proceed with AHP as before. Using this idea, AHP can

handle a *hierarchy* of objectives and subobjectives—hence the term "hierarchy" in the name of the procedure.

2. Although the finished version of the **Choosing Jobs.xlsx** file can be used as a template for other AHP problems, it is clear by now that typical users would not want to go to all this trouble to create a spreadsheet model, certainly not from scratch. If you intend to make any real decisions with AHP, you will want to acquire special-purpose software such as Expert Choice. Alternatively, you can use the file **Choosing Jobs with VBA.xlsm** mentioned earlier. ■

ADDITIONAL APPLICATIONS

Automated Manufacturing Decisions Using AHP

Weber (1993) reports the successful use of AHP in deciding which of several technologies to purchase for automated manufacturing. As he discusses, these decisions can have several types of impacts: quantitative financial (such as purchase cost), quantitative nonfinancial (such as throughput, cycle time, and scrap, which are difficult to translate directly into dollars), and qualitative (such as product quality and manufacturing flexibility, which are also difficult to translate into dollars). When the decision maker is trying to rate the different technologies along nonmonetary criteria, then he or she should use the method discussed in this section. (For example, how much more do you prefer technology 1 to technology 2 in the area of product quality?) However, he advises that when quantitative financial data are available (for example, technology 1 costs twice as much as technology 2), then this objective information should be used in the AHP preference matrices. Weber developed a software package called AutoMan to implement the AHP method. This software has been purchased by more than 800 customers since its first release in 1989.

AHP in Saudi Arabia

Bahurmoz (2003) designed and implemented a system based on AHP to select the best candidates to send overseas to do graduate studies and eventually become teachers at the Dar Al-Hekma women's college in Saudi Arabia.

Other Applications of AHP

AHP has been used by companies in many areas, including accounting, finance, marketing, energy resource planning, microcomputer selection, sociology, architecture, and political science. See Zahedi (1986), Golden et al. (1989), and Saaty (1988) for a discussion of applications of AHP. ■

PROBLEMS

Skill-Building Problems

25. Each professor's annual salary increase is determined by his or her performance in three areas: teaching, research, and service to the university. The administration has assessed the pairwise comparison matrix for these objectives as shown in the file **P16_25.xlsx**. The administration has compared two professors with regard to their teaching, research, and service over the past year. The pairwise comparison matrices are also shown in this file.
    a. Which professor should receive a bigger raise?
    b. Does AHP indicate how large a raise each professor should be given?
    c. Check the pairwise comparison matrix for consistency.

26. Your company is about to purchase a new PC. Three objectives are important in determining which computer you should purchase: cost, user friendliness, and software availability. The pairwise comparison matrix

for these objectives is shown in the file **P16_26.xlsx**. Three computers are being considered for purchase. The performance of each computer with regard to each objective is indicated by the pairwise comparison matrices also shown in this file.

a. Which computer should you purchase?

b. Check the pairwise comparison matrices for consistency.

27. You are ready to select your mate for life and have determined that physical attractiveness, intelligence, and personality are key factors in selecting a satisfactory mate. Your pairwise comparison matrix for these objectives is shown in the file **P16_27.xlsx**. Three people (Chris, Jamie, and Pat) are begging to be your mate. (This problem attempts to be gender-neutral.) Your view of these people's attractiveness, intelligence, and personality is given in the pairwise comparison matrices also shown in this file.

a. Who should you choose as your lifetime mate?

b. Evaluate all pairwise comparison matrices for consistency.

28. In determining where to invest your money, two objectives, expected rate of return and degree of risk, are considered to be equally important. Two investments (1 and 2) have the pairwise comparison matrices shown in the file **P16_28.xlsx**.

a. How would you rank these investments?

b. Now suppose another investment (investment 3) is available. The pairwise comparison matrices for these investments are also shown in this file. (Observe that the entries in the comparison matrices for investments 1 and 2 have not changed.) How would you now rank the investments? Contrast your ranking of investments 1 and 2 with your answer from part **a**.

29. You are trying to determine which MBA program to attend. You have been accepted at two schools: Indiana and Northwestern. You have chosen three attributes to use in helping you make your decision: cost, starting salary for graduates, and ambience of school (can we party there?). Your pairwise comparison matrix for these attributes is shown in the file **P16_29.xlsx**. For each attribute, the pairwise comparison matrix for Indiana and Northwestern is also shown in this file. Which MBA program should you attend?

30. You are trying to determine which of two secretarial candidates (John or Sharon) to hire. The three objectives that are important to your decision are personality, typing ability, and intelligence. You have assessed the pairwise comparison matrix for the three objectives in the file **P16_30.xlsx**. The score of each employee on each objective is also shown in this file. If you follow the AHP method, which employee should you hire?

Skill-Extending Problems

31. A consumer is trying to determine which type of frozen dinner to eat. She considers three attributes to be important: taste, nutritional value, and price. Nutritional value is considered to be determined by cholesterol and sodium level. Three types of dinners are under consideration. The pairwise comparison matrix for the three attributes is shown in the file **P16_31.xlsx**. Among the three frozen dinners, the pairwise comparison matrix for each attribute is also shown in this file. To determine how each dinner rates on nutrition, you will need the pairwise comparison matrix for cholesterol and sodium also shown in this file. Which frozen dinner would the consumer prefer? (*Hint:* The nutrition score for a dinner equals the score of the dinner on sodium multiplied by the weight for sodium plus the score for the dinner on cholesterol multiplied by the weight for cholesterol.)

32. Based on Lin et al. (1984). You have been hired by Arthur Ross to determine which of the following accounts receivable methods should be used in an audit of the Keating Five and Dime Store: analytic review (method 1), confirmations (method 2), or test of subsequent collections (method 3). The three criteria used to distinguish among the methods are reliability, cost, and validity. The pairwise comparison matrix for the three criteria is shown in the file **P16_32.xlsx**. The pairwise comparison matrices of the three accounting methods for the three criteria are shown in this file. Use AHP to determine which auditing procedure should be used. Also check the first pairwise comparison matrix for consistency.

16.5 CONCLUSION

Whenever you face a problem with multiple competing objectives, as is the case in many real-world problems, you are forced to make trade-offs among these objectives. This is usually a very difficult task, and not all management scientists agree on the best way to proceed. When the objectives are very different in nature, no method can disguise the inherent complexity of comparing "apples to oranges." Although one method, finding Pareto optimal solutions and drawing the resulting trade-off curve, locates solutions that are not dominated by any others, you still face the problem of choosing one of the (many) Pareto optimal solutions to implement. The other two methods discussed in this chapter, goal programming and AHP, make trade-offs and ultimately locate an "optimal" solution. These methods have their critics, but when they are used carefully, they have the potential to help solve some difficult and important real-world problems.

Summary of Key Management Science Terms

Term	Explanation	Page
Goal programming	Optimization method that prioritizes multiple objectives (goals); tries to achieve higher priority goals before considering lower priority goals	16-3
Hard constraint	A constraint that *must* be satisfied	16-6
Soft constraint	A constraint you would like to satisfy but don't absolutely have to satisfy	16-6
Pareto optimal solution	Solution that is not *dominated,* that is, no other solution is at least as good on all objectives and better on at least one objective	16-13
Trade-off curve, Efficient frontier	Curve showing Pareto optimal solutions, used primarily to show the trade-offs between two competing objectives	16-13
Analytical Hierarchy Process (AHP)	Method used to find best decision when a decision maker faces multiple criteria; requires a series of pairwise comparisons between criteria and between alternative decisions for each criterion	16-22

PROBLEMS

Skill-Building Problems

33. The Pine Valley Board of Education must hire teachers for the coming school year. The types of teachers and the salaries that must be paid are given in the file **P16_33.xlsx**. For example, 20 teachers who are qualified to teach history and science have applied for jobs, and each of these teachers must be paid an annual salary of $21,000. Each teacher who is hired teaches the two subjects he or she is qualified to teach. Pine Valley needs to hire 35 teachers qualified to teach history, 30 teachers qualified to teach science, 40 teachers qualified to teach math, and 32 teachers qualified to teach English. The board has $1.4 million to spend on teachers' salaries. A penalty cost of $1 is incurred for each dollar the board goes over budget. For each teacher by which Pine Valley's goals are unmet, the following costs are incurred (because of the lower quality of education): science, $30,000; math, $28,000; history, $26,000; and English, $24,000. Determine how the board can minimize its total cost due to unmet goals.

34. Stockco fills orders for three products for a local warehouse. Stockco must determine how many of each product should be ordered at the beginning of the current month. This month, 400 units of product 1, 500 units of product 2, and 300 units of product 3 will be demanded. The cost and space taken up by one unit of each product are shown in the file **P16_34.xlsx**. If Stockco runs out of stock before the end of the month, the stockout costs also shown in this file are incurred. Stockco has $17,000 to spend on ordering products and has 3700 square feet of warehouse space. A $1 penalty is assessed for each dollar spent over the budget limit, and a $10 cost is assessed for every square foot of warehouse space needed.

a. Determine Stockco's optimal ordering policy.

b. Suppose that Stockco has set the following goals, listed in order of priority:

- Goal 1: Spend at most $17,000.
- Goal 2: Use at most 3700 square feet of warehouse space.
- Goal 3: Meet demand for product 1.
- Goal 4: Meet demand for product 2.
- Goal 5: Meet demand for product 3.

Develop a goal programming model for Stockco.

35. BeatTrop Foods is trying to choose one of three companies to merge with. Seven factors are important in this decision:

- Factor 1: Contribution to profitability

- Factor 2: Growth potential
- Factor 3: Labor environment
- Factor 4: R&D ability of company
- Factor 5: Organizational fit
- Factor 6: Relative size
- Factor 7: Industry commonality

The pairwise comparison matrix for these factors is shown in the file **P16_35.xlsx**. The three contenders for merger have the pairwise comparison matrices for each factor also shown in this file.

Use AHP to determine the company that BeatTrop should merge with.

36. Productco produces three products. Each product requires labor, lumber, and paint. The resource requirements, unit price, and variable cost (exclusive of labor, lumber, and paint) for each product are given in the file **P16_36.xlsx**. At present, 900 labor hours, 1550 gallons of paint, and 1600 board feet of lumber are available. Additional labor can be purchased at \$6 per hour. Additional paint can be purchased at \$2 per gallon. Additional lumber can be purchased at \$3 per board foot. For the following two sets of priorities, use goal programming to determine an optimal production schedule. For set 1:
- Priority 1: Obtain profit of at least \$10,500.
- Priority 2: Purchase no additional labor.
- Priority 3: Purchase no additional paint.
- Priority 4: Purchase no additional lumber.

For set 2:
- Priority 1: Purchase no additional labor.
- Priority 2: Obtain profit of at least \$10,500.
- Priority 3: Purchase no additional paint.
- Priority 4: Purchase no additional lumber.

Skill-Extending Problems

37. A hospital outpatient clinic performs four types of operations. The profit per operation, as well as the minutes of X-ray time and laboratory time used, are given in the file **P16_37.xlsx**. The clinic has 500 private rooms and 500 intensive care rooms. Type 1 and type 2 operations require a patient to stay in an intensive care room for one day, whereas type 3 and type 4 operations require a patient to stay in a private room for one day. Each day, the hospital is required to perform at least 100 operations of each type. The hospital has set the following goals (listed in order of priority):
- Goal 1: Earn a daily profit of at least \$100,000.
- Goal 2: Use at most 50 hours daily of X-ray time.
- Goal 3: Use at most 40 hours daily of laboratory time.

Use goal programming to determine the types of operations that should be performed.

38. Jobs at Indiana University are rated on three factors:
- Factor 1: Complexity of duties
- Factor 2: Education required
- Factor 3: Mental and/or visual demands

For each job at IU, the requirement for each factor has been rated on a scale of 1 to 4, with a 4 in factor 1 representing high complexity of duty, a 4 in factor 2 representing high educational requirement, and a 4 in factor 3 representing high mental and/or visual demands. IU wants to determine a formula for grading each job. To do this, it will assign a point value to the score for each factor that a job requires. For example, suppose that level 2 of factor 1 yields a point total of 10, level 3 of factor 2 yields a point total of 20, and level 3 of factor 3 yields a point total of 30. Then a job with these requirements has a point total of $10 + 20 + 30 = 60$. A job's hourly salary equals half its point total. IU has two goals (listed in order of priority) in setting up the points given to each level of each job factor.
- Goal 1: When increasing the level of a factor by 1, the points should increase by at least 10. For example, level 2 of factor 1 should earn at least 10 more points than level 1 of factor 1. Goal 1 is to minimize the sum of deviations from this requirement.
- Goal 2: For the benchmark jobs referred to in the file **P16_38.xlsx**, the actual point total for each job should come as close as possible to the point total listed in the table. Goal 2 is to minimize the sum of the absolute deviations of the point totals from the desired scores.

Use goal programming to find appropriate point totals. What salary should a job with skill levels of 3 for each factor be paid?

39. You are trying to determine which city to live in. New York and Chicago are under consideration. Four objectives will determine your decision: housing cost, cultural opportunities, quality of schools and universities, and crime level. The weight for each objective is in the file **P16_39.xlsx**. For each objective (except for quality of schools and universities), New York and Chicago scores are also given in this file. Suppose that the score for each city on the quality of schools and universities depends on two things: a score on public school quality and a score on university quality. The pairwise comparison matrix for public school and university quality is also shown in this file. To see how each city scores on public school quality and university quality, use the pairwise comparison matrices also shown in this file. You should be able to derive a score for each city on the quality of schools and universities objective. Then use AHP to determine where you should live.

40. At Lummins Engine Corporation, production employees work 10 hours per day, four days per week. Each day of the week, at least the following number of employees must be working: Monday through Friday, seven employees; Saturday and Sunday, three employees. Lummins has set the following goals, listed in order of priority:

- Goal 1: Meet employee requirements with 11 workers.
- Goal 2: The average number of weekend days off per employee should be at least 1.5 days.
- Goal 3: The average number of consecutive days off an employee gets during the week should not exceed 2.8 days.

Use goal programming to determine how to schedule Lummins employees.

41. You are the mayor of Gotham City and you must determine a tax policy for the city. Five types of taxes are used to raise money:

- Property taxes. Let *p* be the property tax rate.
- A sales tax on all items except food, drugs, and durable goods. Let *s* be the sales tax rate.
- A sales tax on durable goods. Let *d* be the durable goods sales tax rate.
- A gasoline sales tax. Let *g* be the gasoline sales tax rate.
- A sales tax on food and drugs. Let *f* be the sales tax on food and drugs.

The city consists of three groups of people: low income (LI), middle income (MI), and high income (HI). The amount of revenue (in millions of dollars) raised from each group by setting a particular tax at a 1% level is given in the file **P16_41.xlsx**. For example, a 3% tax on durable good sales will raise 360 million dollars from low-income people. Your tax policy must satisfy the following restrictions:

- Restriction 1: The tax burden on MI people cannot exceed $2.8 billion.
- Restriction 2: The tax burden on HI people cannot exceed $2.4 billion.
- Restriction 3: The total revenue raised must exceed the current level of $6.5 billion.
- Restriction 4: *s* must be between 1% and 3%.

Given these restrictions, the city council has set the following three goals (listed in order of priority):

- Goal 1: Limit the tax burden on LI people to $2 billion.
- Goal 2: Keep the property tax rate under 3%.
- Goal 3: If their tax burden becomes too high, 20% of the LI people, 20% of the MI people, and 40% of the HI people may consider moving to the suburbs. Suppose that this will happen if their total tax burden exceeds $1.5 billion. To discourage this exodus, goal 3 is to keep the total tax burden on these people below $1.5 billion.

Use goal programming to determine an optimal tax policy.

42. Based on Sartoris and Spruill (1974). Wivco produces two products, which it sells for both cash and credit. Revenues from credit sales will not have been received but are included in determining profit earned during the current six-month period. Sales during the next six months can be made either from units produced during the next six months or from beginning inventory. Relevant information about products 1 and 2 is as follows.

- During the next six months, at most 150 units of product 1 can be sold on a cash basis, and at most 100 units of product 1 can be sold on a credit basis. It costs $35 to produce each unit of product 1, and each sells for $40. A credit sale of a unit of product 1 yields $0.50 less profit than a cash sale (because of delays in receiving payment). Two hours of production time are needed to produce each unit of product 1. At the beginning of the six-month period, 60 units of product 1 are in inventory.
- During the next six months, at most 175 units of product 2 can be sold on a cash basis, and at most 250 units of product 2 can be sold on a credit basis. It costs $45 to produce each unit of product 2, and each sells for $52.50. A credit sale of a unit of product 2 yields $1.00 less profit than a cash sale. Four hours of production time are needed to produce each unit of product 2. At the beginning of the six-month period, 30 units of product 2 are in inventory.
- During the next six months, Wivco has 1000 hours for production available. At the end of the next six months, Wivco incurs a 10% holding cost on the value of ending inventory (measured relative to production cost). An opportunity cost of 5% is also assessed against any cash on hand at the end of the six-month period.

a. Develop and solve an LP model that yields Wivco's maximum profit during the next six months. What is Wivco's ending inventory position? Assuming an initial cash balance of $0, what is Wivco's ending cash balance?

b. Because an ending inventory and cash position of $0 is undesirable (for ongoing operations), Wivco is considering other options. At the beginning of the six-month period, Wivco can obtain a loan (secured by ending inventory) that incurs an interest cost equal to 5% of the value of the loan. The maximum value of the loan is 75% of the value of the ending inventory. The loan will be repaid one year from now. Wivco has the following goals (listed in order of priority):

  - Goal 1: Make the ending cash balance of Wivco come as close as possible to $75.
  - Goal 2: Make profit come as close as possible to the profit level obtained in part **a**.

- Goal 3: At any time, Wivco's current ratio is defined to be

$$\text{Current ratio} = \frac{\text{Wivco's assets}}{\text{Wivco's liabilities}}$$

Assuming initially that current liabilities equal \$150, six months from now Wivco's current ratio will equal

$$\text{Current ratio} = \frac{\text{CR} + \text{AR} + \text{EI}}{150 + \text{Size of loan}}$$

where CB is the ending cash balance, AR is the value of accounts receivable, and EI is the value of the ending inventory. Six months from now, Wivco wants the current ratio to be as close as possible to 2.

Use goal programming to determine Wivco's production and financial strategy.

Modeling Problems

43. How might you use goal programming to help Congress balance the budget?

44. A company is considering buying up to five other businesses. Given knowledge of the company's view of the trade-off between risk and return, how could trade-off curves be used to determine the companies that should be purchased?

45. How would you use AHP to determine the greatest sports record of all time? (Many believe it is Joe DiMaggio's 56-game hitting streak.)

46. You are planning to renovate a hospital. How would you use AHP to help determine what improvements to include in the renovation?

47. You are planning to overhaul a hospital computer system. How would you use AHP to determine the type of computer system to install?

48. You have been commissioned to assign 100 remedial education teachers to the 40 schools in the St. Louis School System. What are some objectives you might consider in assigning the teachers to schools?

49. You have been hired as a consultant to help design a new airport in northern Indiana that will supplant O'Hare as Chicago's major airport. Discuss the objectives you consider important in designing and locating the airport.

50. In the Indiana MBA program we need to divide a class of 60 students into 10 six-person teams. In the interest of diversity, we have the following goals (listed in descending order of importance):

- At least one woman per team
- At least one member of a minority per team
- At least one student with a financial or accounting background per team
- At least one engineer per team

Explain how you could use the material in this chapter to develop a model to assign students to teams.

CASE 16.1 PLAY TIME TOY COMPANY

Play Time Toy faces a highly seasonal pattern of sales. In the past, Play Time has used a *seasonal* production schedule, where the amount produced each month matches the sales for that month. Under this production plan, inventory is maintained at a constant level. The production manager, Thomas Lindop, is proposing a switch to a *level*, or constant, production schedule. This schedule would result in significant savings in production costs but would have higher storage and handling costs, fluctuating levels of inventories, and implications for financing. Jonathan King, president of Play Time Toy, has been reviewing pro forma income statements, cash budgets, and balance sheets for the coming year under the two production scenarios. Table 16.9 shows the pro forma analysis under seasonal production, and Table 16.10 shows the pro forma analysis under level production.

Table 16.9 Seasonal Production

Annual net profit **237** **Play Time Toy Company**

	Actual Dec 2010	Projected for 2011 Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Total
Production (sales value)	850	108	126	145	125	125	125	145	1,458	1,655	1,925	2,057	1,006	9000
Inventory (sales value)	813	813	813	813	813	813	813	813	813	813	813	813	813	

INCOME STATEMENT	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Total
Net sales	108	126	145	125	125	125	145	1,458	1,655	1,925	2,057	1,006	9,000
Cost of goods sold													
Materials and regular wages	70	82	94	81	81	81	94	950	1,079	1,254	1,340	656	5,865
Overtime wages	0	0	0	0	0	0	0	61	91	131	151	0	435
Gross profit	38	44	51	44	44	44	51	447	486	539	565	350	2,700
Operating expenses	188	188	188	188	188	188	188	188	188	188	188	188	2,256
Inventory cost	0	0	0	0	0	0	0	0	0	0	0	0	0
Profit before interest and taxes	(150)	(144)	(137)	(144)	(144)	(144)	(137)	259	298	351	377	162	444
Net interest payments	10	2	1	1	2	2	2	3	7	18	19	19	86
Profit before taxes	(160)	(146)	(138)	(146)	(146)	(147)	(140)	256	290	333	359	144	358
Taxes	(55)	(50)	(47)	(50)	(50)	(50)	(48)	87	99	113	122	49	122
Net profit	(106)	(97)	(91)	(96)	(97)	(97)	(92)	169	192	220	237	95	237

BALANCE SHEET	Actual Dec 2010	Projected for 2011 Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Cash	175	782	1,365	1,116	934	808	604	450	175	175	175	175	175
Accts receivable	2,628	958	234	271	270	250	250	270	1,603	3,113	3,580	3,982	3,063
Inventory	530	530	530	530	530	530	530	530	530	530	530	530	530
Net P/E	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070
Total Assets	4,403	3,340	3,199	2,987	2,804	2,658	2,454	2,320	3,378	4,888	5,355	5,757	4,838
Accts payable	255	32	38	44	38	38	38	44	437	497	578	617	302
Notes payable	680	0	0	0	0	0	0	0	408	1,600	1,653	1,656	966
Accrued taxes	80	25	(24)	(151)	(232)	(282)	(363)	(411)	(324)	(256)	(143)	(21)	(4)
Long term debt	450	450	450	450	450	450	425	425	425	425	425	425	400
Equity	2,938	2,832	2,736	2,644	2,548	2,452	2,355	2,263	2,431	2,623	2,843	3,080	3,175
Total liability and equity	4,403	3,340	3,199	2,987	2,804	2,658	2,454	2,320	3,378	4,888	5,355	5,757	4,838

Table 16.10 Level Production

Annual net profit 373 Play Time Toy Company

	Actual Dec 2010	Projected for 2011												
		Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Total
Production (sales value)	850	750	750	750	750	750	750	750	750	750	750	750	750	9000
Inventory (sales value)	813	1455	2079	2684	3309	3934	4559	5164	4456	3551	2376	1069	813	

INCOME STATEMENT	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Total
Net sales	108	126	145	125	125	125	145	1,458	1,655	1,925	2,057	1,006	9,000
Cost of goods sold													
Materials and regular wages	70	82	94	81	81	81	94	950	1,079	1,254	1,340	656	5,865
Overtime wages	0	0	0	0	0	0	0	0	0	0	0	0	0
Gross profit	38	44	51	44	44	44	51	508	576	671	717	350	3,135
Operating expenses	188	188	188	188	188	188	188	188	188	188	188	188	2,256
Inventory cost	0	2	6	10	13	17	20	16	11	4	0	0	100
Profit before interest and taxes	(150)	(147)	(143)	(154)	(158)	(161)	(158)	304	377	478	529	162	779
Net interest payments	10	3	2	5	10	15	21	26	32	37	31	22	214
Profit before taxes	(160)	(149)	(146)	(159)	(168)	(177)	(179)	277	346	441	498	141	565
Taxes	(55)	(51)	(50)	(54)	(57)	(60)	(61)	94	118	150	169	48	192
Net profit	(106)	(99)	(96)	(105)	(111)	(117)	(118)	183	228	291	329	93	373

BALANCE SHEET	Actual Dec 2010	Projected for 2011											
		Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Cash	175	556	724	175	175	175	175	175	175	175	175	175	175
Accts receivable	2,628	958	234	271	270	250	250	270	1,603	3,113	3,580	3,982	3,063
Inventory	530	948	1,355	1,749	2,157	2,564	2,971	3,365	2,904	2,314	1,549	697	530
Net P/E	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070
Total Assets	4,403	3,533	3,383	3,265	3,672	4,059	4,466	4,880	5,752	6,672	6,374	5,924	4,838
Accts payable	255	225	225	225	225	225	225	225	225	225	225	225	225
Notes payable	680	0	0	108	704	1,259	1,900	2,493	3,087	3,693	2,953	2,005	836
Accrued taxes	80	25	(25)	(155)	(240)	(297)	(389)	(450)	(355)	(269)	(119)	50	66
Long term debt	450	450	450	450	450	450	425	425	425	425	425	425	400
Equity	2,938	2,832	2,734	2,637	2,533	2,422	2,305	2,187	2,370	2,599	2,890	3,218	3,311
Total liability and equity	4,403	3,533	3,383	3,265	3,672	4,059	4,466	4,880	5,752	6,672	6,374	5,924	4,838

Greg Cole, chief financial officer of Play Time, prepared the two tables. He explained that the pro forma analyses in Tables 16.9 and 16.10 take fully into account the 11% interest payments on the unsecured loan from Bay Trust Company and the 3% interest received from its cash account. An interest charge of 11%/12 on the balance of the loan at the end of a month must be paid the next month. Similarly, an interest payment of 3%/12 on the cash balance at the end of a month is received in the next month.

The inventory available at the end of December 2010 is $530,000 (measured in terms of cost to produce). Mr. Cole assumed that this inventory represents a sales value of $530,000/0.651667 = $813,300.

Table 16.11 Play Time Cost Information

- **Gross margin.** The cost of goods sold (excluding overtime costs) is 65.1667% of sales under any production schedule. Materials costs are 30% of sales. All other nonmaterial costs, including regular wages but excluding overtime wages, are 35.1667% of sales.
- **Overtime cost.** Running at capacity but without using any overtime, the plant can produce $1,049,000 of monthly sales. Units produced in excess of this capacity in a month incur an additional overtime cost of 15% of sales. (The monthly production capacity of the plant running on full overtime is $2,400,000 of sales. Since November has the maximum level of projected sales at $2,057,000, the capacity on full overtime should never pose a problem.)
- **Inventory cost.** The plant has a limited capacity to store finished goods. It can store $1,663,000 worth of sales at the plant. Additional units must be moved and stored in rented warehouse space. The cost of storage, handling, and insurance of finished goods over this capacity is 7% of the sales value of the goods per year, or 7%/12 per month.

The inventory and overtime costs in Tables 16.9 and 16.10 are based on the cost information developed by Mr. Lindop. This information is summarized in Table 16.11.

Mr. Cole further explained how the cost information was used in the pro forma analyses. For example, in Table 16.9, the production in August is $1,458,000. The overtime cost in August is therefore calculated to be $61,000 (= 0.15 × (1,458,000 − 1,049,000)). Play Time uses LIFO (last-in, first-out) accounting, so overtime costs are always charged in the month that they occur.[4] The annual overtime cost for the seasonal production plan is $435,000. In Table 16.10, under level production, finished goods worth $5,164,000 are in inventory at the end of July. The inventory cost for the month is $20,000 (= 0.07/12 × (5,164,000 − 1,663,000)). The annual-inventory cost for the level production plan is $100,000.

Mr. Lindop felt that a minimum of $813,300 of inventory (measured in terms of sales value, or $530,000 measured in terms of cost to produce) must be kept on hand at the end of each month. This inventory level represents a reasonable safety stock, which is required because orders do not occur uniformly during a month.

Mr. King was impressed at the possible increase in profit from $237,000 under the seasonal production plan to $373,000 under level production. While studying the pro forma projections, Mr. King realized that some combination of the two production plans might be even better. He asked Mr. Lindop to try to find a production plan with a higher profit than the seasonal and level plans.

Mr. Lindop proceeded to develop a spreadsheet-based LP model to maximize annual net profit.

Question

Note: Mr. Lindop's model is contained in the file. **Play Time.xlsx**. The spreadsheet is ready to be optimized, but it has not been optimized yet.

1. Run the optimization model in this file. What is the optimal production plan? What is the optimal annual net profit? How does this optimal production plan compare to the seasonal and level production plans?
2. Suppose that Play Time's bankers will not extend any credit over $1.9 million—in other words, the loan balance in any month cannot exceed $1.9 million. Modify the spreadsheet model to take into account this restriction. What is the optimal production plan in this case? What is the optimal annual net profit?
3. Annual profit is a measure of reward for Play Time Toy. The maximum loan balance is a measure of risk for the bank. Construct a trade-off curve between optimal annual profit and the maximum loan balance. ■

[4]This assumes that overtime production is used only to satisfy current demand and not to build up inventory.