BIA-652

Multivariate Data Analytics

Review of Probability & Random Variables

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First theme of the class

Individual	Age	Gender	Education	Weight	Height	Income

- Various practical methods of analyzing a “dataframe”
- Interpret the results

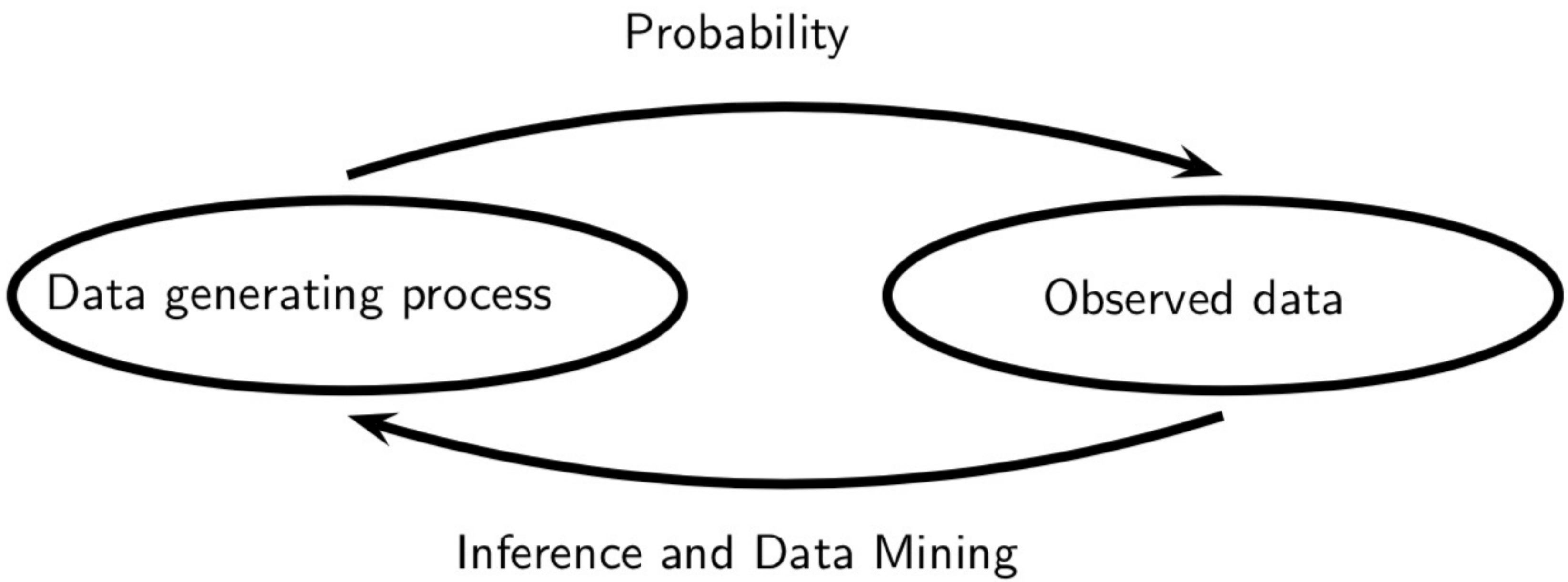
Second theme of the class

- Almost every statistics problem is an optimization problem.
- Learn how the optimization problem is formulated

  … and assume that we know how to solve the optimization problems.

Third theme of the class

- A probabilistic perspective



Why start with probability?
What about deep learning?

- https://www.youtube.com/watch?v=x7psGHgatGM
- (starts from 11:05)

Goal of the class

- NOT to teach you every multivariate technique
- Provide you the knowledge to
  - Read more advanced statistics/ML books
  - Know how to read the manual of a statistical package/software

Prerequisite of the class

- Have taken a calculus class (derivative and integration)
- Know some basic Linear Algebra (vector and matrix operation)
- Prior Python knowledge is not required

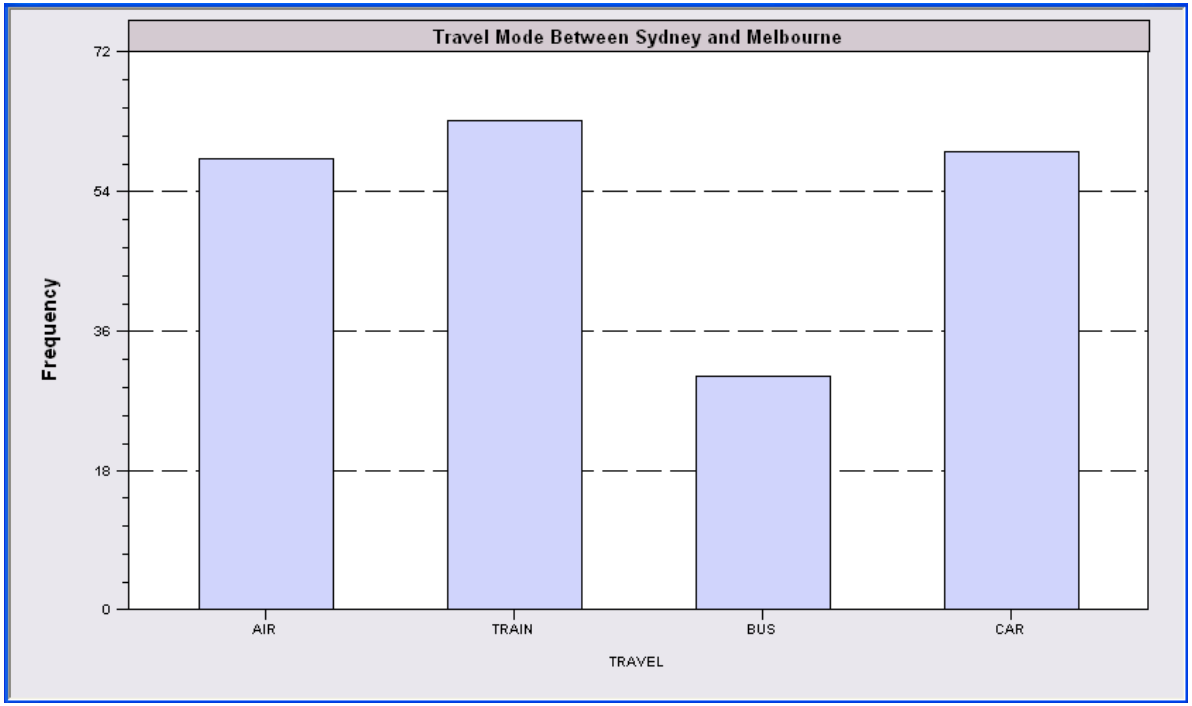
Syllabus

1- Probability

The Sample Space

- Collection of all possible outcomes
  - Exclusive
  - Exhaustive
  - Ω = {the set of possible outcomes}
- Random outcomes: The result of a process
  - Sequence of events,
  - Number of events,
  - Measurement of a length of time, space, etc.
- Experiments, outcomes, and sample spaces

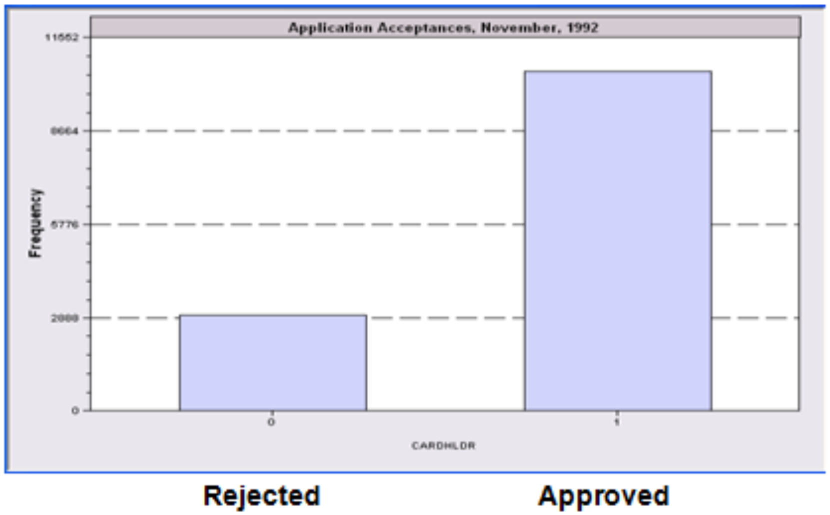
Consumer Choice:
4 possible ways a randomly chosen traveler might have traveled between Sydney and Melbourne



$$\Omega = \{\text{Air, Train, Bus, Car}\}$$

Market Behavior: outcome of credit card application

13,444 Applicants for a Credit Card (November, 1992)



Experiment = A randomly picked application.

Let X = 0 if Rejected

Let X = 1 if Accepted

X is DISCRETE (Binary). This is called a Bernoulli random variable.

$$\Omega = \{\text{Reject, Accept}\}$$

Lifetimes of light bulbs

- A box of light bulbs states "Average life is 1500 hours"
- Outcome = length of time until failure (lifetime) of a randomly chosen light bulb

$$\Omega = \{\text{lifetime} \mid \text{lifetime} \geq 0\}$$

Events

- Events are defined as
  - Subsets of sample space, such as empty set
- It can be
  - Empty
  - Intersection of related events
  - Complements such as “A” and “not A”
  - Disjoint sets such as (train, bus),(air, car)

1.1 Example. If we toss a coin twice then $\Omega = \{HH, HT, TH, TT\}$. The event that the first toss is heads is $A = \{HH, HT\}$. ■

Summary of Terminology

Ω	sample space
ω	outcome (point or element)
A	event (subset of Ω)
A^c	complement of A (not A)
$A \bigcup B$	union (A or B)
$A \bigcap B$ or AB	intersection (A and B)
$A - B$	set difference (ω in A but not in B)
$A \subset B$	set inclusion
$\emptyset$	null event (always false)
Ω	true event (always true)

Mutually Exclusive (Disjoint) Events and Partition

We say that $A_1, A_2, \ldots$ are **disjoint** or are **mutually exclusive** if $A_i \bigcap A_j = \emptyset$ whenever $i \neq j$. For example, $A_1 = [0, 1), A_2 = [1, 2), A_3 = [2, 3), \ldots$ are disjoint. A **partition** of Ω is a sequence of disjoint sets $A_1, A_2, \ldots$ such that $\bigcup_{i=1}^{\infty} A_i = \Omega$. Given an event A, define the **indicator function of** A by

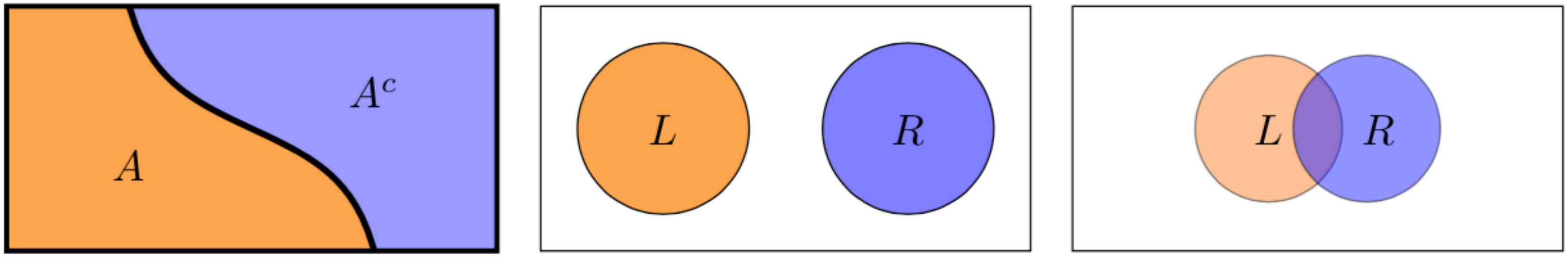
$$I_A(\omega) = I(\omega \in A) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Probability

- Probability is a measure defined on all subsets of Ω
- Probability is a function P that assigns a real number P(A) to each event A
- The three Axioms of Probability
  - $A \subset \Omega \Rightarrow P(A) \geq 0$
  - $P(\Omega) = 1$
  - If $A \cap B = \{\varnothing\}$, $P(A \cup B) = P(A) + P(B)$
- Interpretation
  - Frequency
  - Degree-of-belief

Implications of the Axioms

- $P(A^C) = 1 – P(A)$ as $A \cup A^C = \Omega$
- $P(\varnothing) = 0$
- $A \subset B \Rightarrow P(A) \leq P(B)$
- **Addition rule**: $P(A \cup B) = P(A) + P(B) – P(A \cap B)$



$\Omega = A \cup A^c$, no overlap $\quad L \cup R$, no overlap $\quad L \cup R$, overlap $= L \cap R$

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Calculating Probability: Counting Rules

- If Ω is finite and if each outcome is equally likely, then

assigning probability: **'Size' of an event relative to size of sample space**.

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|},$$

- Counting rules for equally likely discrete outcomes
  - How many ways for an outcome to be included in A?
  - Using multi-step experiment, combinations and permutations to count elements

Counting Rule for Multiple-Step Experiments

- If an experiment consists of a sequence of k steps in which there are n_1 possible results on the first step, n_2 possible results on the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2) \ldots (n_k)$.
- **Example:** Flipping 3 coins. How many outcomes are possible?

  Step 1 Toss coin 1 $n_1 = 2$

  Step 2 Toss coin 2 $n_2 = 2$

  Step 3 Toss coin 3 $n_3 = 2$

Total Number of Outcomes: $n_1 n_2 n_3 = 2*2*2 = 8$

S={HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}

Counting Rule for Combinations

Another useful counting rule enables us to count the number of different experimental outcomes when k objects are to be selected from a set of n objects.

- The number of combinations of n objects taken k at a time is (n choose k)

$$\boxed{C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

where n! = n(n - 1)(n - 2) . . . (2)(1) (n factorial)

k! = k(k - 1)(k - 2) . . . (2)(1)

0! = 1

Counting Rule for Permutations

k objects are to be selected from a set of *n* objects, where the *order* of selection is important.

Number of <u>Permutations</u> of *n* Objects Taken *k* at a Time

$$P_k^n = k!\binom{n}{k} = \frac{n!}{(n-k)!}$$

- where: $n! = n(n-1)(n-2)\ldots(2)(1)$
- $k! = k(k-1)(k-2)\ldots(2)(1)$
- $0! = 1$

Bivariate Probabilities: Marginal, Joint, and Conditional

- Outcomes for bivariate events:

	B_1	B_2	. . .	B_k
A_1	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	. . .	$P(A_1 \cap B_k)$
A_2	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	. . .	$P(A_2 \cap B_k)$
. . .	. . .	. . .	. . .	. . .
A_h	$P(A_h \cap B_1)$	$P(A_h \cap B_2)$	. . .	$P(A_h \cap B_k)$

Joint and Marginal Probabilities

- The probability of a joint event, A ∩ B:

$$P(A \cap B) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

- Computing a marginal probability:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

Bivariate Probabilities:
Coffee preference and gender

Coffee Preference

Gender	Coffee	Mocha	Espresso	Latte	Tea	Total
Male	15	9	13	7	6	50
Female	11	14	5	9	11	50
Total	26	23	18	16	17	100

Joint Frequency

Marginal Frequency

Conditional Probability

- Probability of event A given that event B occurs.
- $P(A|B) = P(A \cap B)/P(B)$
  $=$ Size of A relative to B (a subset of Ω)

- Multiplication rule:

$p(A \cap B) = p(A|B)\, p(B)$ (follows from the definition)

$= p(B|A)\, p(A)$

- Factorization (chain rule)

$P(a \cap b \cap c \ldots y \cap z) = P(a \mid b, c, \ldots.\, y, z)\, P(b \mid c, \ldots\, y, z)\, P(c| \ldots\, y,z) \ldots P(y|z)\, P(z)$

Bayes Theorem

1.16 Theorem (The Law of Total Probability). *Let $A_1, \ldots, A_k$ be a partition of Ω. Then, for any event B,*

$$\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(B|A_i)\mathbb{P}(A_i).$$

1.17 Theorem (Bayes' Theorem). *Let $A_1, \ldots, A_k$ be a partition of Ω such that $\mathbb{P}(A_i) > 0$ for each i. If $\mathbb{P}(B) > 0$ then, for each $i = 1, \ldots, k$,*

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_j \mathbb{P}(B|A_j)\mathbb{P}(A_j)}. \tag{1.5}$$

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \qquad \text{Target}$$

$$= \frac{P(B \mid A)P(A)}{P(B)} \qquad \text{Theorem}$$

$$= \frac{P(B \mid A)P(A)}{P(A,B) + P(notA,B)} \qquad \text{Definition}$$

$$= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid notA)P(notA)} \qquad \text{Computation}$$

Conditional Prob and Bayes Theorem Example: Color Blindness

- Inherited color blindness has different prevalence rates in men and women. Women usually carry the defective gene and men usually inherit it.

- **Experiment: pick an individual at random from the population.**
  CB = has inherited color blindness
  MALE = gender, Not-Male = FEMALE

- **Marginal**: $P(CB) = 2.75\%$
  $P(MALE) = 50.0\%$

- **Joint**: $P(CB \cap MALE) = 2.5\%$
  $P(CB \cap FEMALE) = 0.25\%$

- **Conditional**: $P(CB|MALE) = 5.0\%$ (1 in 20 men)
  $P(CB|FEMALE) = 0.5\%$ (1 in 200 women)

- $P(CB|Male) = .05$
- What is $P(Male|CB)$?

- $P(M|CB) = P(M \cap CB)/P(CB)$
  $= P(CB|M)P(M)/P(CB) = .05*.5 / P(CB)$

- $P(CB) = P(CB \cap M) + P(CB \cap F)$
  $= P(CB|M)P(M) + P(CB|F)P(F)$
  $= .05(.5) + .005(.5) = .0275$ (as we knew)

- $P(M|CB) = .025 / .0275 = .909$ (i.e., 91% of colorblind people are male.

A drilling company has estimated a 40% chance of striking oil for their new well. A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.

Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Independent events

- Definition: $P(A|B) = P(A)$

- Multiplication rule: $P(A \cap B) = P(A)P(B)$ if A and B are independent

- Two random men are both color blind =
  .05 (Men 1) x .05 (Men 2) = .0025

Random Variables

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Random Variable

- Definition: Maps elements of the sample space to a variable.
  - Assigns a value to $\omega \in \Omega$
- Types of random variables
  - Discrete: Coin toss
  - Continuous: Lightbulb lifetimes

Market Behavior: outcome of credit card application

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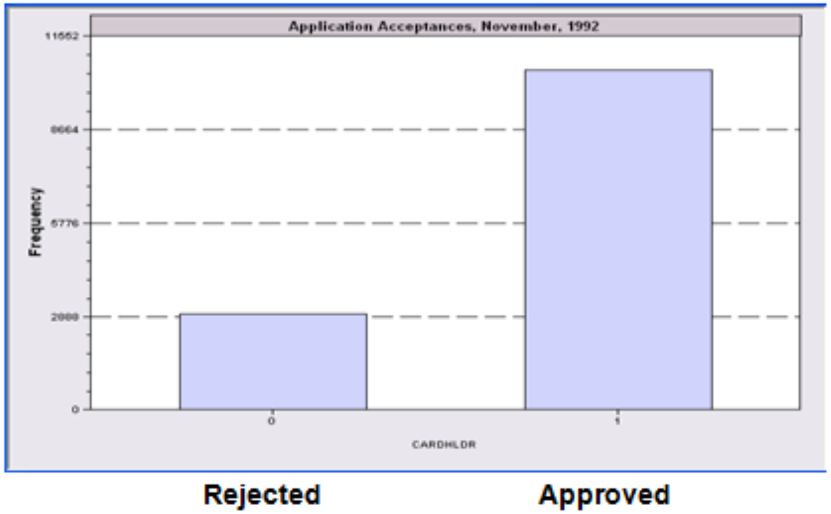

Experiment = A randomly picked application.

Let X = 0 if Rejected

Let X = 1 if Accepted

X is DISCRETE (Binary). This is called a Bernoulli random variable.

Ω= {Reject, Accept}

X = 0=reject, 1=accept

2.9 Definition. *X is* **discrete** *if it takes countably[3] many values $\{x_1, x_2, \ldots\}$. We define the* **probability function** *or* **probability mass function** *for X by $f_X(x) = \mathbb{P}(X = x)$.*

- A set is countable if it is finite or it can be put in a one-to-one correspondence with the integers. The even numbers, the odd numbers, and the rationales are countable; the set of real numbers between 0 and 1 is not countable.

Features of Random Variables

- Probability mass function (PMF)

  $f(x) = \text{Prob}(X=x)$ -- For discrete random variables

- Cumulative distribution (density) function (CDF)

  $F(x) = \text{Prob}(X \leq x)$

- Quantiles: x such that $F(x) = Q$
  - Median: $x = \text{median}, Q = 0.5$.

Properties of CDF

1. F is non-decreasing. That is, its graph never goes down, or symbolically if $a \leq b$ then $F(a) \leq F(b)$.
2. $0 \leq F(a) \leq 1$.
3. $\lim_{a \to \infty} F(a) = 1, \qquad \lim_{a \to -\infty} F(a) = 0.$

Discrete Random Variables

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Bernoulli

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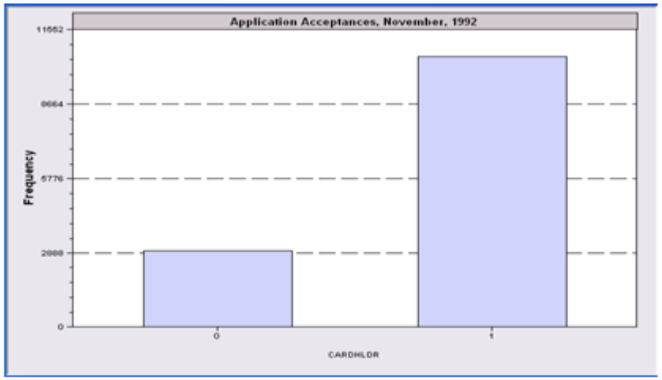


Rejected Approved

Experiment = A randomly picked application.

Let X = 0 if Rejected

Let X = 1 if Accepted

X is DISCRETE (Binary). This is called a Bernoulli random variable.

X = {0=reject, 1=accept} or X = 1[Accepted]

THE BERNOULLI DISTRIBUTION. Let X represent a binary coin flip. Then $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$ for some $p \in [0, 1]$. We say that X has a Bernoulli distribution written $X \sim \text{Bernoulli}(p)$. The probability function is $f(x) = p^x(1-p)^{1-x}$ for $x \in \{0, 1\}$.

Binomial: X = Number of successes in n trials

THE BINOMIAL DISTRIBUTION. Suppose we have a coin which falls heads up with probability p for some $0 \leq p \leq 1$. Flip the coin n times and let X be the number of heads. Assume that the tosses are independent. Let $f(x) = \mathbb{P}(X = x)$ be the mass function. It can be shown that

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Sum of n Bernoulli trials

Family has 4 children.

Prob[4 daughters] $= \binom{4}{4}\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^0 = \frac{1}{16}$ What is P(2 daughters)?

Weapons system has 20 components. It will fail if 2 or more break down.

Prob any component fails is 0.2. Component failures are independent.

Prob(system breaks down) $= \text{Prob}[X \geq 2] = \sum_{x=2}^{20} \binom{20}{x} .2^2 .8^{20-x}$

$= 1 - \text{Prob}[X < 2]$

$= 1 - \text{Prob}(X=0) - \text{Prob}(X=1)$

$= 1 - (1).2^0.8^{20} - (20).2^1.8^{19} = 1 - 0.011529 - 0.57646$

$= 1 - 8^{19}(.8 + (20).2)) = 0.93082$

Notation Reminder

- X is a random variable
- x denotes a particular value of the random variable
- *n* and *p* are parameters, that is, fixed real numbers.
- The parameter p is usually unknown and must be estimated from data.
- Be aware of the sample space and the support.

Poisson

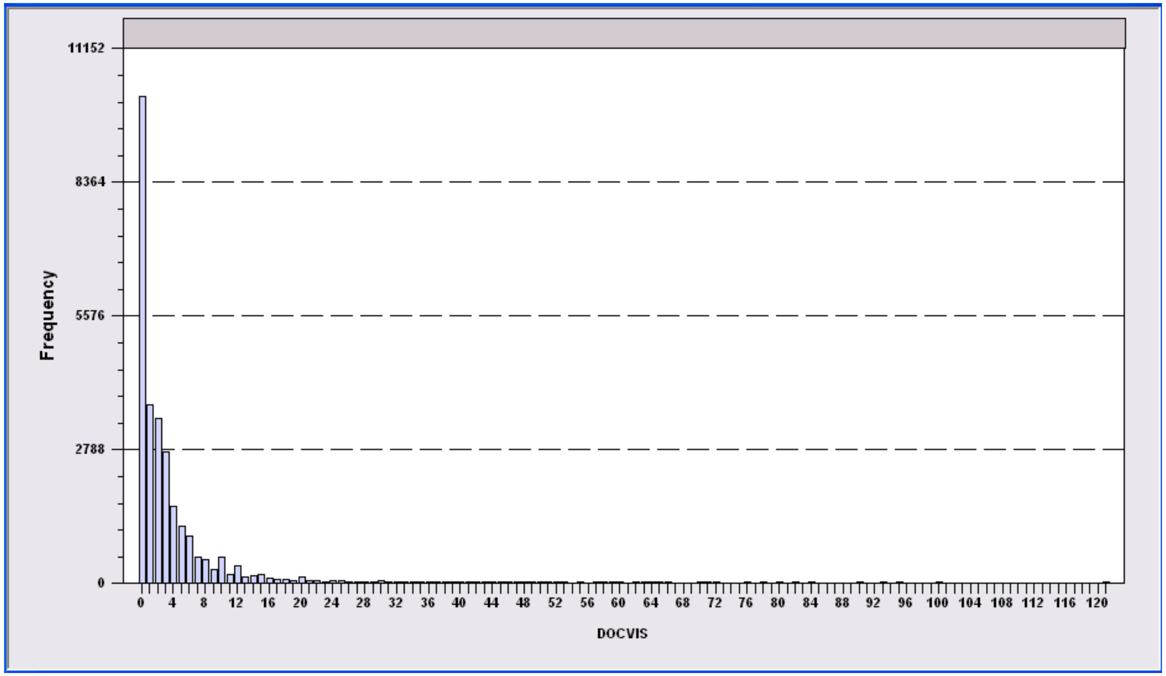
The **Poisson frequency function** with parameter λ ($\lambda > 0$) is

$$P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}, \qquad k = 0, 1, 2, \ldots$$

- Two Interpretations:
  - Can be considered as an approximation to binomial
  - General model for a type of process

    (number of arrivals in a time period)

A Poisson Process: Doctor visits in the survey year by people in a sample of 27,326. $\lambda = .8$



Thank you!

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