CHAPTER 4

Introduction to Probability

CONTENTS

PROBABILITY TO THE RESCUE*

On August 5, 2010, the San José copper and gold mine suffered a cave-in. Thirty-three men were trapped more than 2000 feet underground in the Atacama Desert near Copiapó, Chile. While most feared that these men's prospects were grim, several attempts were made to locate the miners and determine if they were still alive. Seventeen days later, rescuers reached the men with a 5½-inch borehole and ascertained that they were still alive.

While it was important to bring these men safely to the surface as quickly as possible, it was imperative that the rescue effort proceed cautiously. "The mine is old and there is concern of further collapses," Murray & Roberts Cementation Managing Director Henry Laas said in an interview with the *Santiago Times*. "The rescue methodology therefore has to be carefully designed and implemented."

The Chilean government asked NASA to consult on the rescue operation. In response, NASA sent a four-person team consisting of an engineer (Clinton Cragg), two physicians (Michael Duncan and J. D. Polk), and a psychologist (Al Holland). When asked why a space agency was brought in to consult on the rescue of trapped miners, Duncan stated, "We brought our experience in vehicle design and long duration confinement to our Chilean counterparts."

The probability of failure was prominent in the thoughts of everyone involved. "We were thinking that the rescue vehicle would have to make over 40 round trips, so in consideration of the probability of part failures, we suggested the rescue team have three rigs and several sets of replacement parts available," said Cragg. "We also tried to increase the probability of success by placing spring-loaded rollers on the sides of the cage so the cage itself would not be damaged through direct contact with the rock wall as it moved through the rescue portal."

While it was important to consider probabilities of various events, precise estimates of probabilities could not be made. Probabilities could only be estimated subjectively based on similar circumstances NASA scientists had experienced during space flights. Duncan explained, "While we and the Chileans would have preferred to have

NASA scientists based probabilities on similar circumstances experienced during space flights. © Hugo Infante/Government of Chile/Handout/Reuters.

precise estimates of various probabilities based on historical data, the uniqueness of the situation made this infeasible. For example, a miner had to stand virtually straight up in the cage on an ascent that was originally estimated to last two to four hours per miner, so we had to be concerned about fainting. All we could do was consider what we thought to be the facts and apply what we had learned from astronauts' experiences in their returns from short- and long-duration space missions." Duncan then concluded, "It actually took 15 minutes to bring the cage up from the bottom of the mine, so our estimates in this case were very conservative. Considering the risk involved, that is exactly what we wanted."

Ultimately the rescue approach designed by the Chileans in consultation with NASA was successful. On October 13 the last of the miners emerged; the 13-foot-long, 924-pound steel Fénix 2 rescue capsule withstood over 40 trips into the mine, and no miner fainted on ascent.

The use of subjective probabilities in unique situations is common for NASA. With limited space available on a space vehicle, assessing probability of failure for various components and the risks associated with these potential failures becomes critical to how NASA decides which spare components to include on a space flight. NASA also employs probability to estimate the likelihood of crew health and performance issues arising on space exploration missions.

*The authors are indebted to Dr. Michael Duncan and Clinton Cragg of NASA for providing input for this Statistics in Practice.

Managers often base their decisions on an analysis of uncertainties such as the following:

1. What are the chances that sales will decrease if we increase prices?
2. What is the likelihood a new assembly method will increase productivity?
3. How likely is it that the project will be finished on time?
4. What is the chance that a new investment will be profitable?

Some of the earliest work on probability originated in a series of letters between Pierre de Fermat and Blaise Pascal in the 1650s.

Probability is a numerical measure of the likelihood that an event will occur. Thus, probabilities can be used as measures of the degree of uncertainty associated with the four events previously listed. If probabilities are available, we can determine the likelihood of each event occurring.

Probability values are always assigned on a scale from 0 to 1. A probability near zero indicates an event is unlikely to occur; a probability near 1 indicates an event is almost certain to occur. Other probabilities between 0 and 1 represent degrees of likelihood that an event will occur. For example, if we consider the event "rain tomorrow," we understand that when the weather report indicates "a near-zero probability of rain," it means almost no chance of rain. However, if a .90 probability of rain is reported, we know that rain is likely to occur. A .50 probability indicates that rain is just as likely to occur as not. Figure 4.1 depicts the view of probability as a numerical measure of the likelihood of an event occurring.

4.1 Experiments, Counting Rules, and Assigning Probabilities

In discussing probability, we define an **experiment** as a process that generates well-defined outcomes. On any single repetition of an experiment, one and only one of the possible experimental outcomes will occur. Several examples of experiments and their associated outcomes follow.

Experiment	Experimental Outcomes
Toss a coin	Head, tail
Select a part for inspection	Defective, nondefective
Conduct a sales call	Purchase, no purchase
Roll a die	1, 2, 3, 4, 5, 6
Play a football game	Win, lose, tie

By specifying all possible experimental outcomes, we identify the **sample space** for an experiment.

FIGURE 4.1   PROBABILITY AS A NUMERICAL MEASURE OF THE LIKELIHOOD OF AN EVENT OCCURRING
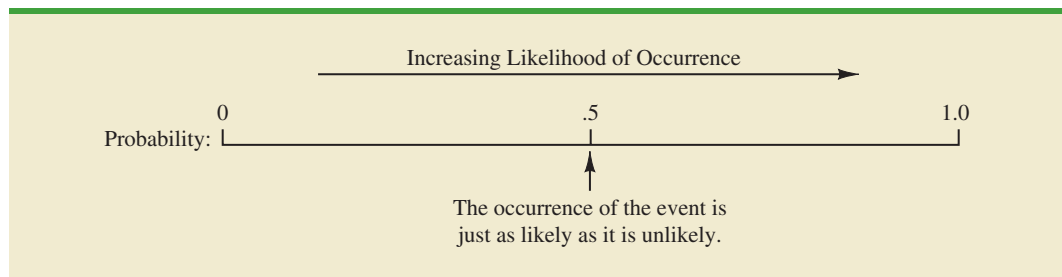
SAMPLE SPACE

The sample space for an experiment is the set of all experimental outcomes.

Experimental outcomes are also called sample points.

An experimental outcome is also called a **sample point** to identify it as an element of the sample space.

Consider the first experiment in the preceding table—tossing a coin. The upward face of the coin—a head or a tail—determines the experimental outcomes (sample points). If we let S denote the sample space, we can use the following notation to describe the sample space.

$$S = \{\text{Head, Tail}\}$$

The sample space for the second experiment in the table—selecting a part for inspection—can be described as follows:

$$S = \{\text{Defective, Nondefective}\}$$

Both of the experiments just described have two experimental outcomes (sample points). However, suppose we consider the fourth experiment listed in the table—rolling a die. The possible experimental outcomes, defined as the number of dots appearing on the upward face of the die, are the six points in the sample space for this experiment.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Counting Rules, Combinations, and Permutations

Being able to identify and count the experimental outcomes is a necessary step in assigning probabilities. We now discuss three useful counting rules.

Multiple-step experiments The first counting rule applies to **multiple-step experiments**. Consider the experiment of tossing two coins. Let the experimental outcomes be defined in terms of the pattern of heads and tails appearing on the upward faces of the two coins. How many experimental outcomes are possible for this experiment? The experiment of tossing two coins can be thought of as a two-step experiment in which step 1 is the tossing of the first coin and step 2 is the tossing of the second coin. If we use H to denote a head and T to denote a tail, (H, H) indicates the experimental outcome with a head on the first coin and a head on the second coin. Continuing this notation, we can describe the sample space (S) for this coin-tossing experiment as follows:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

Thus, we see that four experimental outcomes are possible. In this case, we can easily list all the experimental outcomes.

The counting rule for multiple-step experiments makes it possible to determine the number of experimental outcomes without listing them.

COUNTING RULE FOR MULTIPLE-STEP EXPERIMENTS

If an experiment can be described as a sequence of k steps with n_1 possible outcomes on the first step, n_2 possible outcomes on the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2) \cdots (n_k)$.

Viewing the experiment of tossing two coins as a sequence of first tossing one coin ($n_1 = 2$) and then tossing the other coin ($n_2 = 2$), we can see from the counting rule that $(2)(2) = 4$ distinct experimental outcomes are possible. As shown, they are $S = \{(H, H),$

FIGURE 4.2 TREE DIAGRAM FOR THE EXPERIMENT OF TOSSING TWO COINS
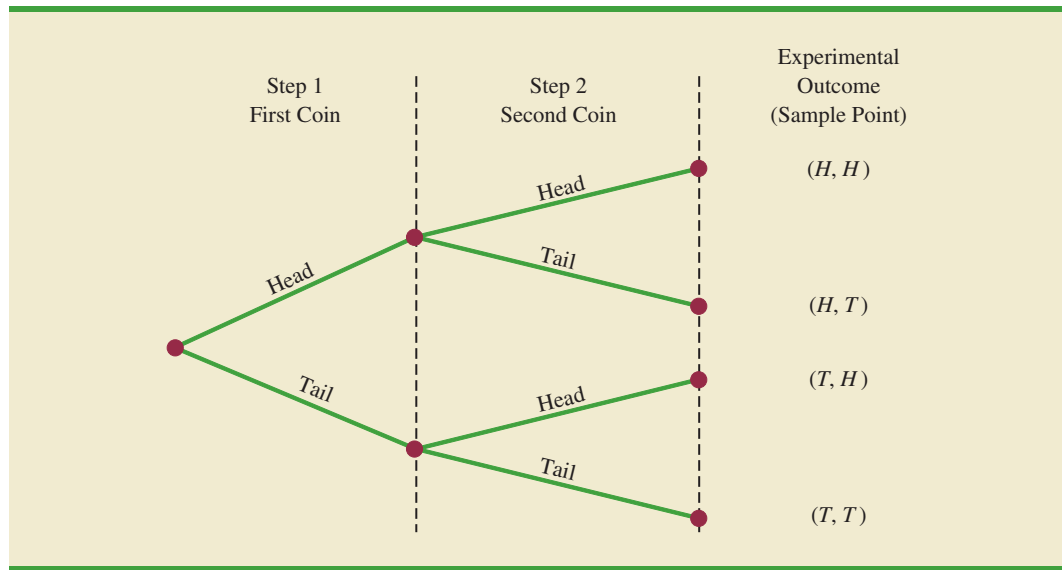
$(H, T), (T, H), (T, T)\}$. The number of experimental outcomes in an experiment involving tossing six coins is $(2)(2)(2)(2)(2)(2) = 64$.

Without the tree diagram, one might think only three experimental outcomes are possible for two tosses of a coin: 0 heads, 1 head, and 2 heads.

A **tree diagram** is a graphical representation that helps in visualizing a multiple-step experiment. Figure 4.2 shows a tree diagram for the experiment of tossing two coins. The sequence of steps moves from left to right through the tree. Step 1 corresponds to tossing the first coin, and step 2 corresponds to tossing the second coin. For each step, the two possible outcomes are head or tail. Note that for each possible outcome at step 1 two branches correspond to the two possible outcomes at step 2. Each of the points on the right end of the tree corresponds to an experimental outcome. Each path through the tree from the leftmost node to one of the nodes at the right side of the tree corresponds to a unique sequence of outcomes.

Let us now see how the counting rule for multiple-step experiments can be used in the analysis of a capacity expansion project for the Kentucky Power & Light Company (KP&L). KP&L is starting a project designed to increase the generating capacity of one of its plants in northern Kentucky. The project is divided into two sequential stages or steps: stage 1 (design) and stage 2 (construction). Even though each stage will be scheduled and controlled as closely as possible, management cannot predict beforehand the exact time required to complete each stage of the project. An analysis of similar construction projects revealed possible completion times for the design stage of 2, 3, or 4 months and possible completion times for the construction stage of 6, 7, or 8 months. In addition, because of the critical need for additional electrical power, management set a goal of 10 months for the completion of the entire project.

Because this project has three possible completion times for the design stage (step 1) and three possible completion times for the construction stage (step 2), the counting rule for multiple-step experiments can be applied here to determine a total of $(3)(3) = 9$ experimental outcomes. To describe the experimental outcomes, we use a two-number notation; for instance, (2, 6) indicates that the design stage is completed in 2 months and the construction stage is completed in 6 months. This experimental outcome results in a total of $2 + 6 = 8$ months to complete the entire project. Table 4.1 summarizes the nine experimental outcomes for the KP&L problem. The tree diagram in Figure 4.3 shows how the nine outcomes (sample points) occur.

The counting rule and tree diagram help the project manager identify the experimental outcomes and determine the possible project completion times. From the information in

TABLE 4.1   EXPERIMENTAL OUTCOMES (SAMPLE POINTS) FOR THE KP&L PROJECT

Completion Time (months)			
Stage 1 Design	Stage 2 Construction	Notation for Experimental Outcome	Total Project Completion Time (months)
2	6	(2, 6)	8
2	7	(2, 7)	9
2	8	(2, 8)	10
3	6	(3, 6)	9
3	7	(3, 7)	10
3	8	(3, 8)	11
4	6	(4, 6)	10
4	7	(4, 7)	11
4	8	(4, 8)	12

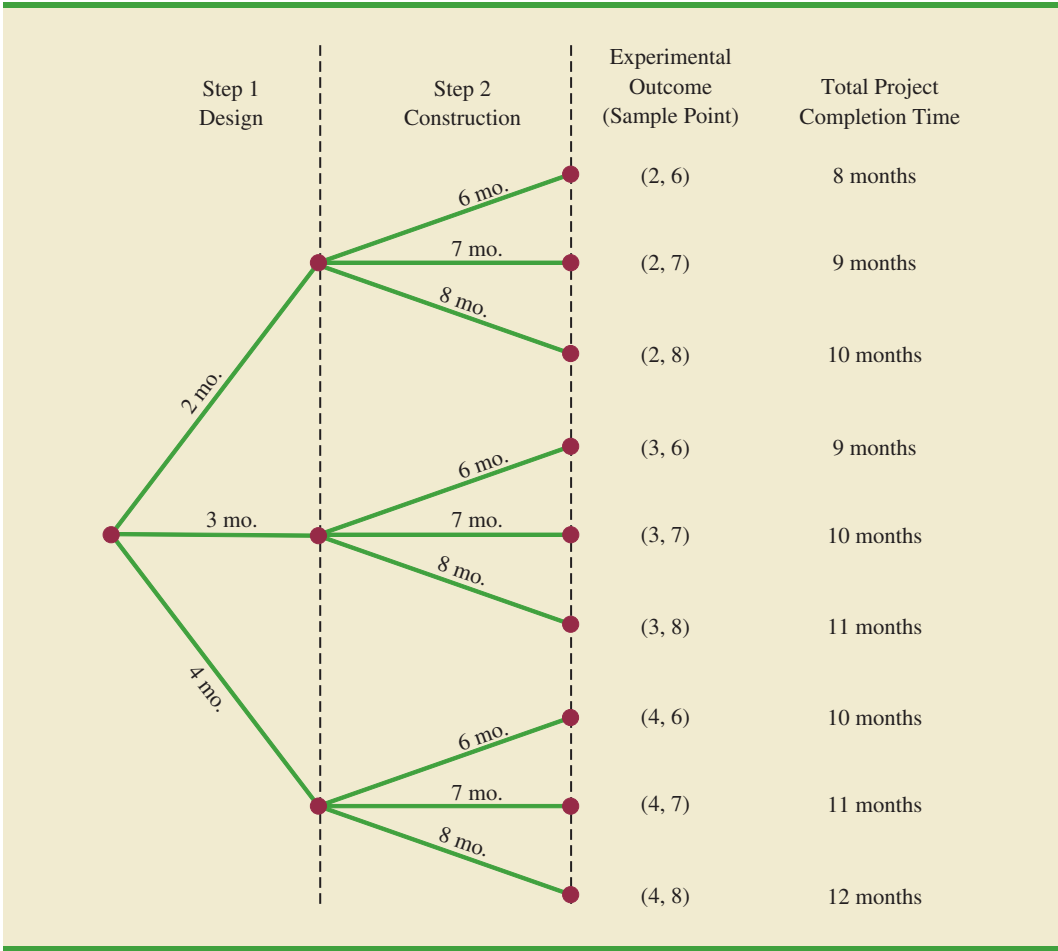**FIGURE 4.3**   TREE DIAGRAM FOR THE KP&L PROJECT

Figure 4.3, we see that the project will be completed in 8 to 12 months, with six of the nine experimental outcomes providing the desired completion time of 10 months or less. Even though identifying the experimental outcomes may be helpful, we need to consider how probability values can be assigned to the experimental outcomes before making an assessment of the probability that the project will be completed within the desired 10 months.

Combinations A second useful counting rule allows one to count the number of experimental outcomes when the experiment involves selecting n objects from a set of N objects. It is called the counting rule for **combinations**.

> COUNTING RULE FOR COMBINATIONS
>
> The number of combinations of N objects taken n at a time is
>
> $$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{4.1}$$
>
> where
> $$N! = N(N-1)(N-2)\cdots(2)(1)$$
> $$n! = n(n-1)(n-2)\cdots(2)(1)$$
>
> and, by definition, $0! = 1$

In sampling from a finite population of size N, the counting rule for combinations is used to find the number of different samples of size n that can be selected.

The notation ! means *factorial;* for example, 5 factorial is $5! = (5)(4)(3)(2)(1) = 120$.

As an illustration of the counting rule for combinations, consider a quality control procedure in which an inspector randomly selects two of five parts to test for defects. In a group of five parts, how many combinations of two parts can be selected? The counting rule in equation (4.1) shows that with $N = 5$ and $n = 2$, we have

$$C_2^5 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{(5)(4)(3)(2)(1)}{(2)(1)(3)(2)(1)} = \frac{120}{12} = 10$$

Thus, 10 outcomes are possible for the experiment of randomly selecting two parts from a group of five. If we label the five parts as A, B, C, D, and E, the 10 combinations or experimental outcomes can be identified as AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE.

As another example, consider that the Florida lottery system uses the random selection of 6 integers from a group of 53 to determine the weekly winner. The counting rule for combinations, equation (4.1), can be used to determine the number of ways 6 different integers can be selected from a group of 53.

$$\binom{53}{6} = \frac{53!}{6!(53-6)!} = \frac{53!}{6!47!} = \frac{(53)(52)(51)(50)(49)(48)}{(6)(5)(4)(3)(2)(1)} = 22{,}957{,}480$$

The counting rule for combinations shows that the chance of winning the lottery is very unlikely.

The counting rule for combinations tells us that almost 23 million experimental outcomes are possible in the lottery drawing. An individual who buys a lottery ticket has 1 chance in 22,957,480 of winning.

Permutations A third counting rule that is sometimes useful is the counting rule for **permutations**. It allows one to compute the number of experimental outcomes when n objects are to be selected from a set of N objects where the order of selection is

important. The same n objects selected in a different order are considered a different experimental outcome.

COUNTING RULE FOR PERMUTATIONS

The number of permutations of N objects taken n at a time is given by

$$P_n^N = n! \binom{N}{n} = \frac{N!}{(N-n)!} \qquad \textbf{(4.2)}$$

The counting rule for permutations closely relates to the one for combinations; however, an experiment results in more permutations than combinations for the same number of objects because every selection of n objects can be ordered in $n!$ different ways.

As an example, consider again the quality control process in which an inspector selects two of five parts to inspect for defects. How many permutations may be selected? The counting rule in equation (4.2) shows that with $N = 5$ and $n = 2$, we have

$$P_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{(5)(4)(3)(2)(1)}{(3)(2)(1)} = \frac{120}{6} = 20$$

Thus, 20 outcomes are possible for the experiment of randomly selecting two parts from a group of five when the order of selection must be taken into account. If we label the parts A, B, C, D, and E, the 20 permutations are AB, BA, AC, CA, AD, DA, AE, EA, BC, CB, BD, DB, BE, EB, CD, DC, CE, EC, DE, and ED.

Assigning Probabilities

Now let us see how probabilities can be assigned to experimental outcomes. The three approaches most frequently used are the classical, relative frequency, and subjective methods. Regardless of the method used, two **basic requirements for assigning probabilities** must be met.

BASIC REQUIREMENTS FOR ASSIGNING PROBABILITIES

1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively. If we let E_i denote the ith experimental outcome and $P(E_i)$ its probability, then this requirement can be written as

$$0 \le P(E_i) \le 1 \text{ for all } i \qquad \textbf{(4.3)}$$

2. The sum of the probabilities for all the experimental outcomes must equal 1.0. For n experimental outcomes, this requirement can be written as

$$P(E_1) + P(E_2) + \cdots + P(E_n) = 1 \qquad \textbf{(4.4)}$$

The **classical method** of assigning probabilities is appropriate when all the experimental outcomes are equally likely. If n experimental outcomes are possible, a probability of $1/n$ is assigned to each experimental outcome. When using this approach, the two basic requirements for assigning probabilities are automatically satisfied.

For an example, consider the experiment of tossing a fair coin; the two experimental outcomes—head and tail—are equally likely. Because one of the two equally likely outcomes is a head, the probability of observing a head is 1/2, or .50. Similarly, the probability of observing a tail is also 1/2, or .50.

As another example, consider the experiment of rolling a die. It would seem reasonable to conclude that the six possible outcomes are equally likely, and hence each outcome is assigned a probability of 1/6. If $P(1)$ denotes the probability that one dot appears on the upward face of the die, then $P(1) = 1/6$. Similarly, $P(2) = 1/6$, $P(3) = 1/6$, $P(4) = 1/6$, $P(5) = 1/6$, and $P(6) = 1/6$. Note that these probabilities satisfy the two basic requirements of equations (4.3) and (4.4) because each of the probabilities is greater than or equal to zero and they sum to 1.0.

The **relative frequency method** of assigning probabilities is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times. As an example, consider a study of waiting times in the X-ray department for a local hospital. A clerk recorded the number of patients waiting for service at 9:00 A.M. on 20 successive days and obtained the following results.

Number Waiting	Number of Days Outcome Occurred
0	2
1	5
2	6
3	4
4	3
Total	20

These data show that on 2 of the 20 days, zero patients were waiting for service; on 5 of the days, one patient was waiting for service; and so on. Using the relative frequency method, we would assign a probability of $2/20 = .10$ to the experimental outcome of zero patients waiting for service, $5/20 = .25$ to the experimental outcome of one patient waiting, $6/20 = .30$ to two patients waiting, $4/20 = .20$ to three patients waiting, and $3/20 = .15$ to four patients waiting. As with the classical method, using the relative frequency method automatically satisfies the two basic requirements of equations (4.3) and (4.4).

The **subjective method** of assigning probabilities is most appropriate when one cannot realistically assume that the experimental outcomes are equally likely and when little relevant data are available. When the subjective method is used to assign probabilities to the experimental outcomes, we may use any information available, such as our experience or intuition. After considering all available information, a probability value that expresses our *degree of belief* (on a scale from 0 to 1) that the experimental outcome will occur is specified. Because subjective probability expresses a person's degree of belief, it is personal. Using the subjective method, different people can be expected to assign different probabilities to the same experimental outcome.

The subjective method requires extra care to ensure that the two basic requirements of equations (4.3) and (4.4) are satisfied. Regardless of a person's degree of belief, the probability value assigned to each experimental outcome must be between 0 and 1, inclusive, and the sum of all the probabilities for the experimental outcomes must equal 1.0.

Consider the case in which Tom and Judy Elsbernd make an offer to purchase a house. Two outcomes are possible:

$$E_1 = \text{their offer is accepted}$$
$$E_2 = \text{their offer is rejected}$$

Judy believes that the probability their offer will be accepted is .8; thus, Judy would set $P(E_1) = .8$ and $P(E_2) = .2$. Tom, however, believes that the probability that their offer will be accepted is .6; hence, Tom would set $P(E_1) = .6$ and $P(E_2) = .4$. Note that Tom's probability estimate for E_1 reflects a greater pessimism that their offer will be accepted.

Bayes' theorem (see Section 4.5) provides a means for combining subjectively determined prior probabilities with probabilities obtained by other means to obtain revised, or posterior, probabilities.

Both Judy and Tom assigned probabilities that satisfy the two basic requirements. The fact that their probability estimates are different emphasizes the personal nature of the subjective method.

Even in business situations where either the classical or the relative frequency approach can be applied, managers may want to provide subjective probability estimates. In such cases, the best probability estimates often are obtained by combining the estimates from the classical or relative frequency approach with subjective probability estimates.

Probabilities for the KP&L Project

To perform further analysis on the KP&L project, we must develop probabilities for each of the nine experimental outcomes listed in Table 4.1. On the basis of experience and judgment, management concluded that the experimental outcomes were not equally likely. Hence, the classical method of assigning probabilities could not be used. Management then decided to conduct a study of the completion times for similar projects undertaken by KP&L over the past three years. The results of a study of 40 similar projects are summarized in Table 4.2.

After reviewing the results of the study, management decided to employ the relative frequency method of assigning probabilities. Management could have provided subjective probability estimates but felt that the current project was quite similar to the 40 previous projects. Thus, the relative frequency method was judged best.

In using the data in Table 4.2 to compute probabilities, we note that outcome (2, 6)—stage 1 completed in 2 months and stage 2 completed in 6 months—occurred six times in the 40 projects. We can use the relative frequency method to assign a probability of $6/40 = .15$ to this outcome. Similarly, outcome (2, 7) also occurred in six of the 40 projects, providing a $6/40 = .15$ probability. Continuing in this manner, we obtain the probability assignments for the sample points of the KP&L project shown in Table 4.3. Note that $P(2, 6)$ represents the probability of the sample point (2, 6), $P(2, 7)$ represents the probability of the sample point (2, 7), and so on.

TABLE 4.2   COMPLETION RESULTS FOR 40 KP&L PROJECTS

Completion Time (months)		Sample Point	Number of Past Projects Having These Completion Times
Stage 1 Design	Stage 2 Construction		
2	6	(2, 6)	6
2	7	(2, 7)	6
2	8	(2, 8)	2
3	6	(3, 6)	4
3	7	(3, 7)	8
3	8	(3, 8)	2
4	6	(4, 6)	2
4	7	(4, 7)	4
4	8	(4, 8)	6
		Total	40

TABLE 4.3 PROBABILITY ASSIGNMENTS FOR THE KP&L PROJECT BASED ON THE RELATIVE FREQUENCY METHOD

Sample Point	Project Completion Time	Probability of Sample Point
(2, 6)	8 months	$P(2, 6) = 6/40 = .15$
(2, 7)	9 months	$P(2, 7) = 6/40 = .15$
(2, 8)	10 months	$P(2, 8) = 2/40 = .05$
(3, 6)	9 months	$P(3, 6) = 4/40 = .10$
(3, 7)	10 months	$P(3, 7) = 8/40 = .20$
(3, 8)	11 months	$P(3, 8) = 2/40 = .05$
(4, 6)	10 months	$P(4, 6) = 2/40 = .05$
(4, 7)	11 months	$P(4, 7) = 4/40 = .10$
(4, 8)	12 months	$P(4, 8) = 6/40 = \underline{.15}$
		Total 1.00

NOTES AND COMMENTS

1. In statistics, the notion of an experiment differs somewhat from the notion of an experiment in the physical sciences. In the physical sciences, researchers usually conduct an experiment in a laboratory or a controlled environment in order to learn about cause and effect. In statistical experiments, probability determines outcomes. Even though the experiment is repeated in exactly the same way, an entirely different outcome may occur. Because of this influence of probability on the outcome, the experiments of statistics are sometimes called *random experiments*.

2. When drawing a random sample without replacement from a population of size N, the counting rule for combinations is used to find the number of different samples of size n that can be selected.

Exercises

Methods

1. An experiment has three steps with three outcomes possible for the first step, two outcomes possible for the second step, and four outcomes possible for the third step. How many experimental outcomes exist for the entire experiment?

2. How many ways can three items be selected from a group of six items? Use the letters A, B, C, D, E, and F to identify the items, and list each of the different combinations of three items.

3. How many permutations of three items can be selected from a group of six? Use the letters A, B, C, D, E, and F to identify the items, and list each of the permutations of items B, D, and F.

4. Consider the experiment of tossing a coin three times.
   a. Develop a tree diagram for the experiment.
   b. List the experimental outcomes.
   c. What is the probability for each experimental outcome?

5. Suppose an experiment has five equally likely outcomes: E_1, E_2, E_3, E_4, E_5. Assign probabilities to each outcome and show that the requirements in equations (4.3) and (4.4) are satisfied. What method did you use?

6. An experiment with three outcomes has been repeated 50 times, and it was learned that E_1 occurred 20 times, E_2 occurred 13 times, and E_3 occurred 17 times. Assign probabilities to the outcomes. What method did you use?

7. A decision maker subjectively assigned the following probabilities to the four outcomes of an experiment: $P(E_1) = .10$, $P(E_2) = .15$, $P(E_3) = .40$, and $P(E_4) = .20$. Are these probability assignments valid? Explain.

Applications

8. In the city of Milford, applications for zoning changes go through a two-step process: a review by the planning commission and a final decision by the city council. At step 1 the planning commission reviews the zoning change request and makes a positive or negative recommendation concerning the change. At step 2 the city council reviews the planning commission's recommendation and then votes to approve or to disapprove the zoning change. Suppose the developer of an apartment complex submits an application for a zoning change. Consider the application process as an experiment.
   a. How many sample points are there for this experiment? List the sample points.
   b. Construct a tree diagram for the experiment.

SELF test

9. Simple random sampling uses a sample of size *n* from a population of size *N* to obtain data that can be used to make inferences about the characteristics of a population. Suppose that, from a population of 50 bank accounts, we want to take a random sample of four accounts in order to learn about the population. How many different random samples of four accounts are possible?

SELF test

10. Many students accumulate debt by the time they graduate from college. Shown in the following table is the percentage of graduates with debt and the average amount of debt for these graduates at four universities and four liberal arts colleges (*U.S. News and World Report, America's Best Colleges,* 2008).

University	% with Debt	Amount($)	College	% with Debt	Amount($)
Pace	72	32,980	Wartburg	83	28,758
Iowa State	69	32,130	Morehouse	94	27,000
Massachusetts	55	11,227	Wellesley	55	10,206
SUNY—Albany	64	11,856	Wofford	49	11,012

   a. If you randomly choose a graduate of Morehouse College, what is the probability that this individual graduated with debt?
   b. If you randomly choose one of these eight institutions for a follow-up study on student loans, what is the probability that you will choose an institution with more than 60% of its graduates having debt?
   c. If you randomly choose one of these eight institutions for a follow-up study on student loans, what is the probability that you will choose an institution whose graduates with debts have an average debt of more than $30,000?
   d. What is the probability that a graduate of Pace University does not have debt?
   e. For graduates of Pace University with debt, the average amount of debt is $32,980. Considering all graduates from Pace University, what is the average debt per graduate?

11. The National Occupant Protection Use Survey (NOPUS) was conducted to provide probability-based data on motorcycle helmet use in the United States. The survey was conducted by sending observers to randomly selected roadway sites where they collected data on motorcycle helmet use, including the number of motorcyclists wearing a Department of Transportation (DOT)-compliant helmet (National Highway Traffic Safety Administration website, January 7, 2010). Sample data consistent with the most recent NOPUS are shown below.

Region	Type of Helmet	
	DOT-Compliant	**Noncompliant**
Northeast	96	62
Midwest	86	43
South	92	49
West	76	16
Total	350	170

a. Use the sample data to compute an estimate of the probability that a motorcyclist wears a DOT-compliant helmet.
b. The probability that a motorcyclist wore a DOT-compliant helmet five years ago was .48, and last year this probability was .63. Would the National Highway Traffic Safety Administration be pleased with the most recent survey results?
c. What is the probability of DOT-compliant helmet use by region of the country? What region has the highest probability of DOT-compliant helmet use?

12. The Powerball lottery is played twice each week in 28 states, the Virgin Islands, and the District of Columbia. To play Powerball a participant must purchase a ticket and then select five numbers from the digits 1 through 55 and a Powerball number from the digits 1 through 42. To determine the winning numbers for each game, lottery officials draw five white balls out of a drum with 55 white balls, and one red ball out of a drum with 42 red balls. To win the jackpot, a participant's numbers must match the numbers on the five white balls in any order and the number on the red Powerball. Eight coworkers at the ConAgra Foods plant in Lincoln, Nebraska, claimed the record $365 million jackpot on February 18, 2006, by matching the numbers 15-17-43-44-49 and the Powerball number 29. A variety of other cash prizes are awarded each time the game is played. For instance, a prize of $200,000 is paid if the participant's five numbers match the numbers on the five white balls (Powerball website, March 19, 2006).
a. Compute the number of ways the first five numbers can be selected.
b. What is the probability of winning a prize of $200,000 by matching the numbers on the five white balls?
c. What is the probability of winning the Powerball jackpot?

13. A company that manufactures toothpaste is studying five different package designs. Assuming that one design is just as likely to be selected by a consumer as any other design, what selection probability would you assign to each of the package designs? In an actual experiment, 100 consumers were asked to pick the design they preferred. The following data were obtained. Do the data confirm the belief that one design is just as likely to be selected as another? Explain.

Design	Number of Times Preferred
1	5
2	15
3	30
4	40
5	10

4.2 Events and Their Probabilities

In the introduction to this chapter we used the term *event* much as it would be used in every-day language. Then, in Section 4.1 we introduced the concept of an experiment and its associated experimental outcomes or sample points. Sample points and events provide the foundation for the study of probability. As a result, we must now introduce the formal definition of an **event** as it relates to sample points. Doing so will provide the basis for determining the probability of an event.

> **EVENT**
>
> An event is a collection of sample points.

For an example, let us return to the KP&L project and assume that the project manager is interested in the event that the entire project can be completed in 10 months or less. Referring to Table 4.3, we see that six sample points—(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), and (4, 6)—provide a project completion time of 10 months or less. Let C denote the event that the project is completed in 10 months or less; we write

$$C = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (4, 6)\}$$

Event C is said to occur if *any one* of these six sample points appears as the experimental outcome.

Other events that might be of interest to KP&L management include the following.

L = The event that the project is completed in *less* than 10 months
M = The event that the project is completed in *more* than 10 months

Using the information in Table 4.3, we see that these events consist of the following sample points.

$$L = \{(2, 6), (2, 7), (3, 6)\}$$
$$M = \{(3, 8), (4, 7), (4, 8)\}$$

A variety of additional events can be defined for the KP&L project, but in each case the event must be identified as a collection of sample points for the experiment.

Given the probabilities of the sample points shown in Table 4.3, we can use the following definition to compute the probability of any event that KP&L management might want to consider.

PROBABILITY OF AN EVENT

The probability of any event is equal to the sum of the probabilities of the sample points in the event.

Using this definition, we calculate the probability of a particular event by adding the probabilities of the sample points (experimental outcomes) that make up the event. We can now compute the probability that the project will take 10 months or less to complete. Because this event is given by $C = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (4, 6)\}$, the probability of event C, denoted $P(C)$, is given by

$$P(C) = P(2, 6) + P(2, 7) + P(2, 8) + P(3, 6) + P(3, 7) + P(4, 6)$$

Refer to the sample point probabilities in Table 4.3; we have

$$P(C) = .15 + .15 + .05 + .10 + .20 + .05 = .70$$

Similarly, because the event that the project is completed in less than 10 months is given by $L = \{(2, 6), (2, 7), (3, 6)\}$, the probability of this event is given by

$$P(L) = P(2, 6) + P(2, 7) + P(3, 6)$$
$$= .15 + .15 + .10 = .40$$

Finally, for the event that the project is completed in more than 10 months, we have $M = \{(3, 8), (4, 7), (4, 8)\}$ and thus

$$P(M) = P(3, 8) + P(4, 7) + P(4, 8)$$
$$= .05 + .10 + .15 = .30$$

Using these probability results, we can now tell KP&L management that there is a .70 probability that the project will be completed in 10 months or less, a .40 probability that the project will be completed in less than 10 months, and a .30 probability that the project will be completed in more than 10 months. This procedure of computing event probabilities can be repeated for any event of interest to the KP&L management.

Any time that we can identify all the sample points of an experiment and assign probabilities to each, we can compute the probability of an event using the definition. However, in many experiments the large number of sample points makes the identification of the sample points, as well as the determination of their associated probabilities, extremely cumbersome, if not impossible. In the remaining sections of this chapter, we present some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities.

NOTES AND COMMENTS

1. The sample space, S, is an event. Because it contains all the experimental outcomes, it has a probability of 1; that is, $P(S) = 1$.
2. When the classical method is used to assign probabilities, the assumption is that the experimental outcomes are equally likely. In such cases, the probability of an event can be computed by counting the number of experimental outcomes in the event and dividing the result by the total number of experimental outcomes.

Exercises

Methods

14. An experiment has four equally likely outcomes: E_1, E_2, E_3, and E_4.
    a. What is the probability that E_2 occurs?
    b. What is the probability that any two of the outcomes occur (e.g., E_1 or E_3)?
    c. What is the probability that any three of the outcomes occur (e.g., E_1 or E_2 or E_4)?

15. Consider the experiment of selecting a playing card from a deck of 52 playing cards. Each card corresponds to a sample point with a 1/52 probability.
    a. List the sample points in the event an ace is selected.
    b. List the sample points in the event a club is selected.
    c. List the sample points in the event a face card (jack, queen, or king) is selected.
    d. Find the probabilities associated with each of the events in parts (a), (b), and (c).

16. Consider the experiment of rolling a pair of dice. Suppose that we are interested in the sum of the face values showing on the dice.
    a. How many sample points are possible? (*Hint:* Use the counting rule for multiple-step experiments.)
    b. List the sample points.
    c. What is the probability of obtaining a value of 7?
    d. What is the probability of obtaining a value of 9 or greater?
    e. Because each roll has six possible even values (2, 4, 6, 8, 10, and 12) and only five possible odd values (3, 5, 7, 9, and 11), the dice should show even values more often than odd values. Do you agree with this statement? Explain.
    f. What method did you use to assign the probabilities requested?

Applications

17. Refer to the KP&L sample points and sample point probabilities in Tables 4.2 and 4.3.
    a. The design stage (stage 1) will run over budget if it takes 4 months to complete. List the sample points in the event the design stage is over budget.
    b. What is the probability that the design stage is over budget?
    c. The construction stage (stage 2) will run over budget if it takes 8 months to complete. List the sample points in the event the construction stage is over budget.
    d. What is the probability that the construction stage is over budget?
    e. What is the probability that both stages are over budget?

18. To investigate how often families eat at home, Harris Interactive surveyed 496 adults living with children under the age of 18 (*USA Today,* January 3, 2007). The survey results are shown in the following table.

Number of Family Meals per Week	Number of Survey Responses
0	11
1	11
2	30
3	36
4	36
5	119
6	114
7 or more	139

For a randomly selected family with children under the age of 18, compute the following.
    a. The probability the family eats no meals at home during the week.
    b. The probability the family eats at least four meals at home during the week.
    c. The probability the family eats two or fewer meals at home during the week.

19. Do you think the government protects investors adequately? This question was part of an online survey of investors under age 65 living in the United States and Great Britain (*Financial Times*/Harris Poll, October 1, 2009). The numbers of investors from the United States and Great Britain who answered Yes, No, or Unsure to this question are provided below.

Response	United States	Great Britain
Yes	187	197
No	334	411
Unsure	256	213

    a. Estimate the probability that an investor in the United States thinks the government is not protecting investors adequately.
    b. Estimate the probability that an investor in Great Britain thinks the government is not protecting investors adequately or is unsure the government is protecting investors adequately.
    c. For a randomly selected investor from these two countries, estimate the probability that the investor thinks the government is not protecting investors adequately.
    d. Based on the survey results, does there appear to be much difference between the perceptions of investors in the United States and investors in Great Britain regarding the issue of the government protecting investors adequately?

20. *Fortune* magazine publishes an annual list of the 500 largest companies in the United States. The following data show the five states with the largest number of *Fortune* 500 companies (*The New York Times Almanac,* 2006).

State	Number of Companies
New York	54
California	52
Texas	48
Illinois	33
Ohio	30

Suppose a *Fortune* 500 company is chosen for a follow-up questionnaire. What are the probabilities of the following events?
a. Let N be the event the company is headquartered in New York. Find $P(N)$.
b. Let T be the event the company is headquartered in Texas. Find $P(T)$.
c. Let B be the event the company is headquartered in one of these five states. Find $P(B)$.

21. Data on U.S. work-related fatalities by cause follow (*The World Almanac,* 2012).

Cause of Fatality	Number of Fatalities
Transportation incidents	1795
Assaults and violent acts	837
Contact with objects and equipment	741
Falls	645
Exposure to harmful substances or environments	404
Fires and explosions	113

Assume that a fatality will be randomly chosen from this population.
a. What is the probability the fatality resulted from a fall?
b. What is the probability the fatality resulted from a transportation incident?
c. What cause of fatality is least likely to occur? What is the probability the fatality resulted from this cause?

4.3 Some Basic Relationships of Probability

Complement of an Event

Given an event A, the **complement of A** is defined to be the event consisting of all sample points that are *not* in A. The complement of A is denoted by A^c. Figure 4.4 is a diagram, known as a **Venn diagram**, which illustrates the concept of a complement. The rectangular area represents the sample space for the experiment and as such contains all possible sample points. The circle represents event A and contains only the sample points that belong to A. The shaded region of the rectangle contains all sample points not in event A and is by definition the complement of A.

In any probability application, either event A or its complement A^c must occur. Therefore, we have

$$P(A) + P(A^c) = 1$$

FIGURE 4.4   COMPLEMENT OF EVENT *A* IS SHADED
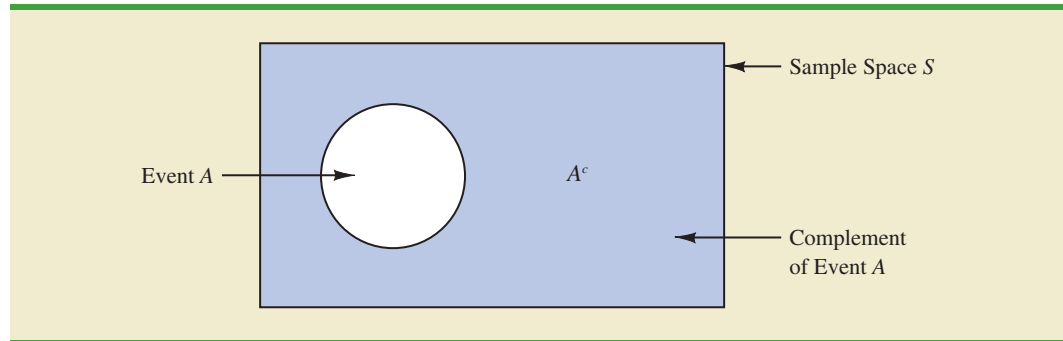
Solving for $P(A)$, we obtain the following result.

COMPUTING PROBABILITY USING THE COMPLEMENT

$$P(A) = 1 - P(A^c) \tag{4.5}$$

Equation (4.5) shows that the probability of an event *A* can be computed easily if the probability of its complement, $P(A^c)$, is known.

    As an example, consider the case of a sales manager who, after reviewing sales reports, states that 80% of new customer contacts result in no sale. By allowing *A* to denote the event of a sale and A^c to denote the event of no sale, the manager is stating that $P(A^c) = .80$. Using equation (4.5), we see that

$$P(A) = 1 - P(A^c) = 1 - .80 = .20$$

We can conclude that a new customer contact has a .20 probability of resulting in a sale.

    In another example, a purchasing agent states a .90 probability that a supplier will send a shipment that is free of defective parts. Using the complement, we can conclude that there is a $1 - .90 = .10$ probability that the shipment will contain defective parts.

Addition Law

The addition law is helpful when we are interested in knowing the probability that at least one of two events occurs. That is, with events *A* and *B* we are interested in knowing the probability that event *A* or event *B* or both occur.

    Before we present the addition law, we need to discuss two concepts related to the combination of events: the *union* of events and the *intersection* of events. Given two events *A* and *B*, the **union of *A* and *B*** is defined as follows.

UNION OF TWO EVENTS

The *union* of *A* and *B* is the event containing *all* sample points belonging to *A or B or both*. The union is denoted by $A \cup B$.

The Venn diagram in Figure 4.5 depicts the union of events *A* and *B*. Note that the two circles contain all the sample points in event *A* as well as all the sample points in event *B*.

FIGURE 4.5   UNION OF EVENTS *A* AND *B* IS SHADED
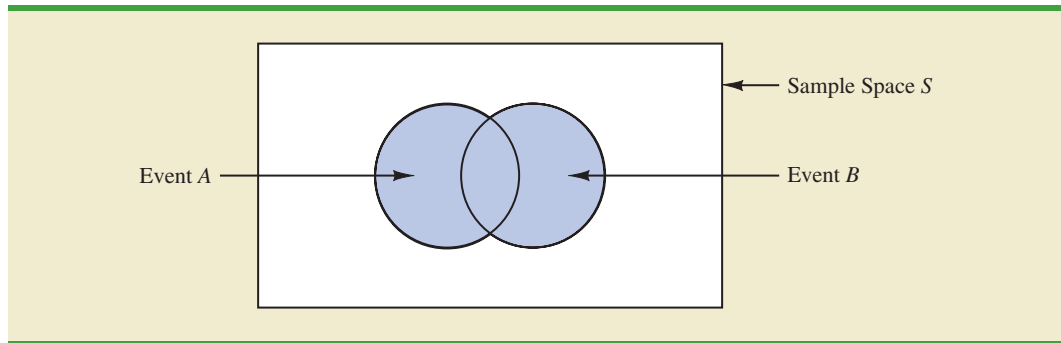
The fact that the circles overlap indicates that some sample points are contained in both *A* and *B*.

The definition of the **intersection of *A* and *B*** follows.

> INTERSECTION OF TWO EVENTS
>
> Given two events *A* and *B*, the *intersection* of *A* and *B* is the event containing the sample points belonging to *both A and B*. The intersection is denoted by $A \cap B$.

The Venn diagram depicting the intersection of events *A* and *B* is shown in Figure 4.6. The area where the two circles overlap is the intersection; it contains the sample points that are in both *A* and *B*.

Let us now continue with a discussion of the addition law. The **addition law** provides a way to compute the probability that event *A* or event *B* or both occur. In other words, the addition law is used to compute the probability of the union of two events. The addition law is written as follows.

> ADDITION LAW
> $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{4.6}$$

FIGURE 4.6   INTERSECTION OF EVENTS *A* AND *B* IS SHADED
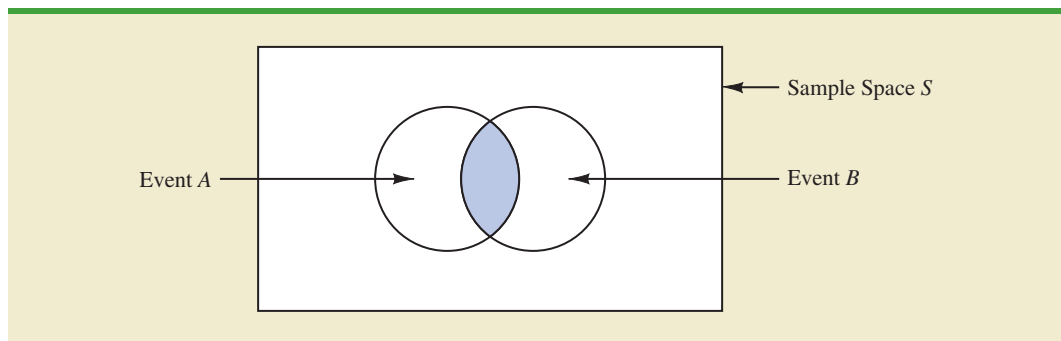
To understand the addition law intuitively, note that the first two terms in the addition law, $P(A) + P(B)$, account for all the sample points in $A \cup B$. However, because the sample points in the intersection $A \cap B$ are in both A and B, when we compute $P(A) + P(B)$, we are in effect counting each of the sample points in $A \cap B$ twice. We correct for this over-counting by subtracting $P(A \cap B)$.

As an example of an application of the addition law, let us consider the case of a small assembly plant with 50 employees. Each worker is expected to complete work assignments on time and in such a way that the assembled product will pass a final inspection. On occasion, some of the workers fail to meet the performance standards by completing work late or assembling a defective product. At the end of a performance evaluation period, the production manager found that 5 of the 50 workers completed work late, 6 of the 50 workers assembled a defective product, and 2 of the 50 workers both completed work late *and* assembled a defective product.

Let

$$L = \text{the event that the work is completed late}$$
$$D = \text{the event that the assembled product is defective}$$

The relative frequency information leads to the following probabilities.

$$P(L) = \frac{5}{50} = .10$$

$$P(D) = \frac{6}{50} = .12$$

$$P(L \cap D) = \frac{2}{50} = .04$$

After reviewing the performance data, the production manager decided to assign a poor performance rating to any employee whose work was either late or defective; thus the event of interest is $L \cup D$. What is the probability that the production manager assigned an employee a poor performance rating?

Note that the probability question is about the union of two events. Specifically, we want to know $P(L \cup D)$. Using equation (4.6), we have

$$P(L \cup D) = P(L) + P(D) - P(L \cap D)$$

Knowing values for the three probabilities on the right side of this expression, we can write

$$P(L \cup D) = .10 + .12 - .04 = .18$$

This calculation tells us that there is a .18 probability that a randomly selected employee received a poor performance rating.

As another example of the addition law, consider a recent study conducted by the personnel manager of a major computer software company. The study showed that 30% of the employees who left the firm within two years did so primarily because they were dissatisfied with their salary, 20% left because they were dissatisfied with their work assignments, and 12% of the former employees indicated dissatisfaction with *both* their salary and their work assignments. What is the probability that an employee who leaves within

two years does so because of dissatisfaction with salary, dissatisfaction with the work assignment, or both?

Let

$$S = \text{the event that the employee leaves because of salary}$$
$$W = \text{the event that the employee leaves because of work assignment}$$

We have $P(S) = .30$, $P(W) = .20$, and $P(S \cap W) = .12$. Using equation (4.6), the addition law, we have

$$P(S \cup W) = P(S) + P(W) - P(S \cap W) = .30 + .20 - .12 = .38.$$

We find a .38 probability that an employee leaves for salary or work assignment reasons.

Before we conclude our discussion of the addition law, let us consider a special case that arises for **mutually exclusive events**.

MUTUALLY EXCLUSIVE EVENTS

Two events are said to be mutually exclusive if the events have no sample points in common.

Events A and B are mutually exclusive if, when one event occurs, the other cannot occur. Thus, a requirement for A and B to be mutually exclusive is that their intersection must contain no sample points. The Venn diagram depicting two mutually exclusive events A and B is shown in Figure 4.7. In this case $P(A \cap B) = 0$ and the addition law can be written as follows.

ADDITION LAW FOR MUTUALLY EXCLUSIVE EVENTS
$$P(A \cup B) = P(A) + P(B)$$

FIGURE 4.7   MUTUALLY EXCLUSIVE EVENTS
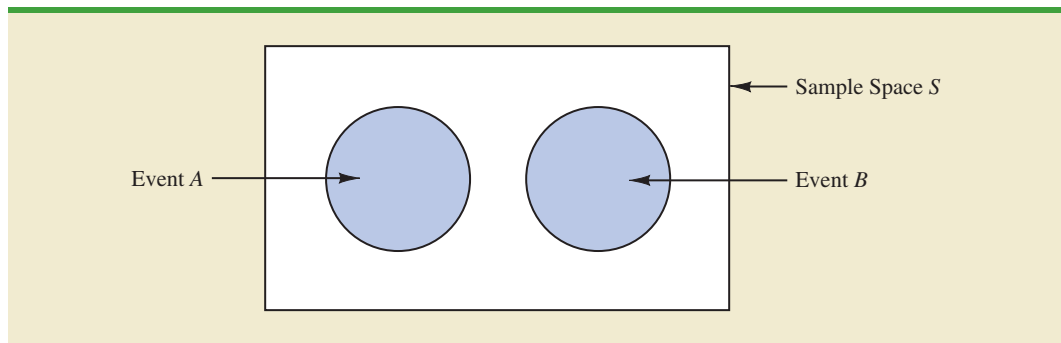
Exercises

Methods

22. Suppose that we have a sample space with five equally likely experimental outcomes: E_1, E_2, E_3, E_4, E_5. Let

$$A = \{E_1, E_2\}$$
$$B = \{E_3, E_4\}$$
$$C = \{E_2, E_3, E_5\}$$

   a. Find $P(A)$, $P(B)$, and $P(C)$.
   b. Find $P(A \cup B)$. Are A and B mutually exclusive?
   c. Find A^c, C^c, $P(A^c)$, and $P(C^c)$.
   d. Find $A \cup B^c$ and $P(A \cup B^c)$.
   e. Find $P(B \cup C)$.

23. Suppose that we have a sample space $S = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$, where $E_1, E_2, \ldots,$ E_7 denote the sample points. The following probability assignments apply: $P(E_1) = .05$, $P(E_2) = .20$, $P(E_3) = .20$, $P(E_4) = .25$, $P(E_5) = .15$, $P(E_6) = .10$, and $P(E_7) = .05$. Let

$$A = \{E_1, E_4, E_6\}$$
$$B = \{E_2, E_4, E_7\}$$
$$C = \{E_2, E_3, E_5, E_7\}$$

   a. Find $P(A)$, $P(B)$, and $P(C)$.
   b. Find $A \cup B$ and $P(A \cup B)$.
   c. Find $A \cap B$ and $P(A \cap B)$.
   d. Are events A and C mutually exclusive?
   e. Find B^c and $P(B^c)$.

Applications

24. Clarkson University surveyed alumni to learn more about what they think of Clarkson. One part of the survey asked respondents to indicate whether their overall experience at Clarkson fell short of expectations, met expectations, or surpassed expectations. The results showed that 4% of the respondents did not provide a response, 26% said that their experience fell short of expectations, and 65% of the respondents said that their experience met expectations.
   a. If we chose an alumnus at random, what is the probability that the alumnus would say their experience *surpassed* expectations?
   b. If we chose an alumnus at random, what is the probability that the alumnus would say their experience met or surpassed expectations?

25. The U.S. Census Bureau provides data on the number of young adults, ages 18–24, who are living in their parents' home.[1] Let

$$M = \text{the event a male young adult is living in his parents' home}$$
$$F = \text{the event a female young adult is living in her parents' home}$$

   If we randomly select a male young adult and a female young adult, the Census Bureau data enable us to conclude $P(M) = .56$ and $P(F) = .42$ (*The World Almanac,* 2006). The probability that both are living in their parents' home is .24.
   a. What is the probability at least one of the two young adults selected is living in his or her parents' home?
   b. What is the probability both young adults selected are living on their own (neither is living in their parents' home)?

---

[1]The data include single young adults who are living in college dormitories because it is assumed these young adults will return to their parents' home when school is not in session.

26. Information about mutual funds provided by Morningstar Investment Research includes
     the type of mutual fund (Domestic Equity, International Equity, or Fixed Income) and the
     Morningstar rating for the fund. The rating is expressed from 1-star (lowest rating) to
     5-star (highest rating). A sample of 25 mutual funds was selected from *Morningstar Funds
     500* (2008). The following counts were obtained:
     - Sixteen mutual funds were Domestic Equity funds.
     - Thirteen mutual funds were rated 3-star or less.
     - Seven of the Domestic Equity funds were rated 4-star.
     - Two of the Domestic Equity funds were rated 5-star.

     Assume that one of these 25 mutual funds will be randomly selected in order to learn more
     about the mutual fund and its investment strategy.
     a. What is the probability of selecting a Domestic Equity fund?
     b. What is the probability of selecting a fund with a 4-star or 5-star rating?
     c. What is the probability of selecting a fund that is both a Domestic Equity fund *and* a
         fund with a 4-star or 5-star rating?
     d. What is the probability of selecting a fund that is a Domestic Equity fund *or* a fund
         with a 4-star or 5-star rating?

27. What NCAA college basketball conferences have the higher probability of having a
     team play in college basketball's national championship game? Over the last 20 years,
     the Atlantic Coast Conference (ACC) ranks first by having a team in the championship
     game 10 times. The Southeastern Conference (SEC) ranks second by having a team in
     the championship game 8 times. However, these two conferences have both had teams
     in the championship game only one time, when Arkansas (SEC) beat Duke (ACC)
     76–70 in 1994 (NCAA website, April 2009). Use these data to estimate the following
     probabilities.
     a. What is the probability the ACC will have a team in the championship game?
     b. What is the probability the SEC will have team in the championship game?
     c. What is the probability the ACC and SEC will both have teams in the championship
         game?
     d. What is the probability at least one team from these two conferences will be in the
         championship game? That is, what is the probability a team from the ACC or SEC will
         play in the championship game?
     e. What is the probability that the championship game will not a have team from one of
         these two conferences?

28. A survey of magazine subscribers showed that 45.8% rented a car during the past 12 months
     for business reasons, 54% rented a car during the past 12 months for personal reasons, and
     30% rented a car during the past 12 months for both business and personal reasons.
     a. What is the probability that a subscriber rented a car during the past 12 months for
         business or personal reasons?
     b. What is the probability that a subscriber did not rent a car during the past 12 months
         for either business or personal reasons?

29. High school seniors with strong academic records apply to the nation's most selective
     colleges in greater numbers each year. Because the number of slots remains relatively stable,
     some colleges reject more early applicants. Suppose that for a recent admissions class, an Ivy
     League college received 2851 applications for early admission. Of this group, it admitted
     1033 students early, rejected 854 outright, and deferred 964 to the regular admission pool for
     further consideration. In the past, this school has admitted 18% of the deferred early admis-
     sion applicants during the regular admission process. Counting the students admitted early
     and the students admitted during the regular admission process, the total class size was 2375.
     Let E, R, and D represent the events that a student who applies for early admission is admitted
     early, rejected outright, or deferred to the regular admissions pool.
     a. Use the data to estimate $P(E)$, $P(R)$, and $P(D)$.
     b. Are events E and D mutually exclusive? Find $P(E \cap D)$.

c.   For the 2375 students who were admitted, what is the probability that a randomly selected student was accepted during early admission?

d.   Suppose a student applies for early admission. What is the probability that the student will be admitted for early admission or be deferred and later admitted during the regular admission process?

4.4   Conditional Probability

Often, the probability of an event is influenced by whether a related event already occurred. Suppose we have an event A with probability $P(A)$. If we obtain new information and learn that a related event, denoted by B, already occurred, we will want to take advantage of this information by calculating a new probability for event A. This new probability of event A is called a **conditional probability** and is written $P(A \mid B)$. We use the notation | to indicate that we are considering the probability of event A *given* the condition that event B has occurred. Hence, the notation $P(A \mid B)$ reads "the probability of A given B."

As an illustration of the application of conditional probability, consider the situation of the promotion status of male and female officers of a major metropolitan police force in the eastern United States. The police force consists of 1200 officers, 960 men and 240 women. Over the past two years, 324 officers on the police force received promotions. The specific breakdown of promotions for male and female officers is shown in Table 4.4.

After reviewing the promotion record, a committee of female officers raised a discrimination case on the basis that 288 male officers had received promotions, but only 36 female officers had received promotions. The police administration argued that the relatively low number of promotions for female officers was due not to discrimination, but to the fact that relatively few females are members of the police force. Let us show how conditional probability could be used to analyze the discrimination charge.

Let

$$M = \text{event an officer is a man}$$
$$W = \text{event an officer is a woman}$$
$$A = \text{event an officer is promoted}$$
$$A^c = \text{event an officer is not promoted}$$

Dividing the data values in Table 4.4 by the total of 1200 officers enables us to summarize the available information with the following probability values.

$$P(M \cap A) = 288/1200 = .24 \text{ probability that a randomly selected officer is a man } and \text{ is promoted}$$
$$P(M \cap A^c) = 672/1200 = .56 \text{ probability that a randomly selected officer is a man } and \text{ is not promoted}$$

TABLE 4.4   PROMOTION STATUS OF POLICE OFFICERS OVER THE PAST TWO YEARS

	Men	Women	Total
Promoted	288	36	324
Not Promoted	672	204	876
Total	960	240	1200

TABLE 4.5   JOINT PROBABILITY TABLE FOR PROMOTIONS

Joint probabilities appear in the body of the table.	Men (M)	Women (W)	Total
Promoted (A)	.24	.03	.27
Not Promoted (A^c)	.56	.17	.73
Total	.80	.20	1.00

Marginal probabilities appear in the margins of the table.

$$P(W \cap A) = 36/1200 \; = .03 \text{ probability that a randomly selected officer is a woman } and \text{ is promoted}$$

$$P(W \cap A^c) = 204/1200 = .17 \text{ probability that a randomly selected officer is a woman } and \text{ is not promoted}$$

Because each of these values gives the probability of the intersection of two events, the probabilities are called **joint probabilities**. Table 4.5, which provides a summary of the probability information for the police officer promotion situation, is referred to as a *joint probability table.*

The values in the margins of the joint probability table provide the probabilities of each event separately. That is, $P(M) = .80$, $P(W) = .20$, $P(A) = .27$, and $P(A^c) = .73$. These probabilities are referred to as **marginal probabilities** because of their location in the margins of the joint probability table. We note that the marginal probabilities are found by summing the joint probabilities in the corresponding row or column of the joint probability table. For instance, the marginal probability of being promoted is $P(A) = P(M \cap A) + P(W \cap A) = .24 + .03 = .27$. From the marginal probabilities, we see that 80% of the force is male, 20% of the force is female, 27% of all officers received promotions, and 73% were not promoted.

Let us begin the conditional probability analysis by computing the probability that an officer is promoted given that the officer is a man. In conditional probability notation, we are attempting to determine $P(A \mid M)$. To calculate $P(A \mid M)$, we first realize that this notation simply means that we are considering the probability of the event A (promotion) given that the condition designated as event M (the officer is a man) is known to exist. Thus $P(A \mid M)$ tells us that we are now concerned only with the promotion status of the 960 male officers. Because 288 of the 960 male officers received promotions, the probability of being promoted given that the officer is a man is $288/960 = .30$. In other words, given that an officer is a man, that officer had a 30% chance of receiving a promotion over the past two years.

This procedure was easy to apply because the values in Table 4.4 show the number of officers in each category. We now want to demonstrate how conditional probabilities such as $P(A \mid M)$ can be computed directly from related event probabilities rather than the frequency data of Table 4.4.

We have shown that $P(A \mid M) = 288/960 = .30$. Let us now divide both the numerator and denominator of this fraction by 1200, the total number of officers in the study.

$$P(A \mid M) = \frac{288}{960} = \frac{288/1200}{960/1200} = \frac{.24}{.80} = .30$$

We now see that the conditional probability $P(A \mid M)$ can be computed as .24/.80. Refer to the joint probability table (Table 4.5). Note in particular that .24 is the joint probability of

A and *M*; that is, $P(A \cap M) = .24$. Also note that .80 is the marginal probability that a randomly selected officer is a man; that is, $P(M) = .80$. Thus, the conditional probability $P(A \mid M)$ can be computed as the ratio of the joint probability $P(A \cap M)$ to the marginal probability $P(M)$.

$$P(A \mid M) = \frac{P(A \cap M)}{P(M)} = \frac{.24}{.80} = .30$$

The fact that conditional probabilities can be computed as the ratio of a joint probability to a marginal probability provides the following general formula for conditional probability calculations for two events *A* and *B*.

CONDITIONAL PROBABILITY

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{4.7}$$

or

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \tag{4.8}$$

The Venn diagram in Figure 4.8 is helpful in obtaining an intuitive understanding of conditional probability. The circle on the right shows that event *B* has occurred; the portion of the circle that overlaps with event *A* denotes the event $(A \cap B)$. We know that once event *B* has occurred, the only way that we can also observe event *A* is for the event $(A \cap B)$ to occur. Thus, the ratio $P(A \cap B)/P(B)$ provides the conditional probability that we will observe event *A* given that event *B* has already occurred.

Let us return to the issue of discrimination against the female officers. The marginal probability in row 1 of Table 4.5 shows that the probability of promotion of an officer is $P(A) = .27$ (regardless of whether that officer is male or female). However, the critical issue in the discrimination case involves the two conditional probabilities $P(A \mid M)$ and $P(A \mid W)$. That is, what is the probability of a promotion *given* that the officer is a man, and what is the probability of a promotion *given* that the officer is a woman? If these two probabilities are equal, a discrimination argument has no basis because the chances of a promotion are the same for male and female officers. However, a difference in the two conditional probabilities will support the position that male and female officers are treated differently in promotion decisions.

FIGURE 4.8   CONDITIONAL PROBABILITY $P(A \mid B) = P(A \cap B)/P(B)$
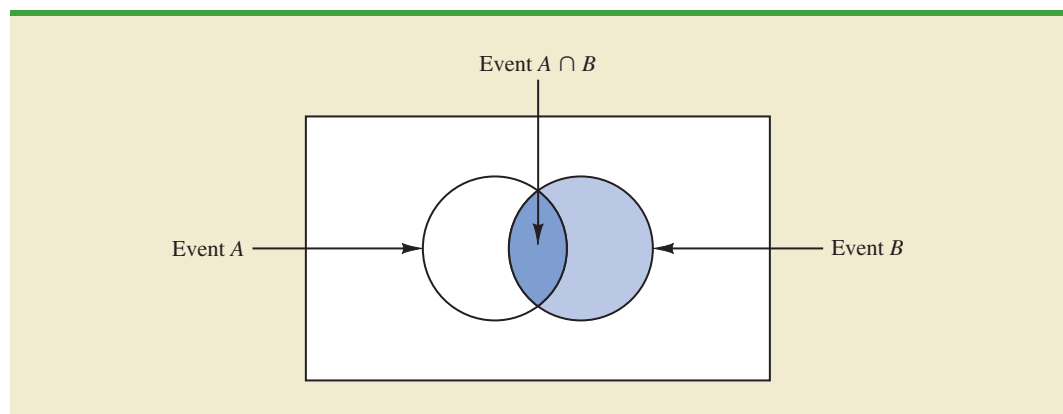
We already determined that $P(A \mid M) = .30$. Let us now use the probability values in Table 4.5 and the basic relationship of conditional probability in equation (4.7) to compute the probability that an officer is promoted given that the officer is a woman; that is, $P(A \mid W)$. Using equation (4.7), with W replacing B, we obtain

$$P(A \mid W) = \frac{P(A \cap W)}{P(W)} = \frac{.03}{.20} = .15$$

What conclusion do you draw? The probability of a promotion given that the officer is a man is .30, twice the .15 probability of a promotion given that the officer is a woman. Although the use of conditional probability does not in itself prove that discrimination exists in this case, the conditional probability values support the argument presented by the female officers.

Independent Events

In the preceding illustration, $P(A) = .27$, $P(A \mid M) = .30$, and $P(A \mid W) = .15$. We see that the probability of a promotion (event A) is affected or influenced by whether the officer is a man or a woman. Particularly, because $P(A \mid M) \neq P(A)$, we would say that events A and M are dependent events. That is, the probability of event A (promotion) is altered or affected by knowing that event M (the officer is a man) exists. Similarly, with $P(A \mid W) \neq P(A)$, we would say that events A and W are *dependent events*. However, if the probability of event A is not changed by the existence of event M—that is, $P(A \mid M) = P(A)$—we would say that events A and M are **independent events**. This situation leads to the following definition of the independence of two events.

INDEPENDENT EVENTS

Two events A and B are independent if

$$P(A \mid B) = P(A) \qquad \textbf{(4.9)}$$

or

$$P(B \mid A) = P(B) \qquad \textbf{(4.10)}$$

Otherwise, the events are dependent.

Multiplication Law

Whereas the addition law of probability is used to compute the probability of a union of two events, the multiplication law is used to compute the probability of the intersection of two events. The multiplication law is based on the definition of conditional probability. Using equations (4.7) and (4.8) and solving for $P(A \cap B)$, we obtain the **multiplication law**.

MULTIPLICATION LAW

$$P(A \cap B) = P(B)P(A \mid B) \qquad \textbf{(4.11)}$$

or

$$P(A \cap B) = P(A)P(B \mid A) \qquad \textbf{(4.12)}$$

To illustrate the use of the multiplication law, consider a newspaper circulation department where it is known that 84% of the households in a particular neighborhood subscribe to the daily edition of the paper. If we let D denote the event that a household subscribes to the daily edition, $P(D) = .84$. In addition, it is known that the probability that a household that already holds a

daily subscription also subscribes to the Sunday edition (event S) is .75; that is, $P(S \mid D) = .75$. What is the probability that a household subscribes to both the Sunday and daily editions of the newspaper? Using the multiplication law, we compute the desired $P(S \cap D)$ as

$$P(S \cap D) = P(D)P(S \mid D) = .84(.75) = .63$$

We now know that 63% of the households subscribe to both the Sunday and daily editions.

Before concluding this section, let us consider the special case of the multiplication law when the events involved are independent. Recall that events A and B are independent whenever $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$. Hence, using equations (4.11) and (4.12) for the special case of independent events, we obtain the following multiplication law.

> MULTIPLICATION LAW FOR INDEPENDENT EVENTS
>
> $$P(A \cap B) = P(A)P(B) \qquad \textbf{(4.13)}$$

To compute the probability of the intersection of two independent events, we simply multiply the corresponding probabilities. Note that the multiplication law for independent events provides another way to determine whether A and B are independent. That is, if $P(A \cap B) = P(A)P(B)$, then A and B are independent; if $P(A \cap B) \neq P(A)P(B)$, then A and B are dependent.

As an application of the multiplication law for independent events, consider the situation of a service station manager who knows from past experience that 80% of the customers use a credit card when they purchase gasoline. What is the probability that the next two customers purchasing gasoline will each use a credit card? If we let

$$A = \text{the event that the first customer uses a credit card}$$
$$B = \text{the event that the second customer uses a credit card}$$

then the event of interest is $A \cap B$. Given no other information, we can reasonably assume that A and B are independent events. Thus,

$$P(A \cap B) = P(A)P(B) = (.80)(.80) = .64$$

To summarize this section, we note that our interest in conditional probability is motivated by the fact that events are often related. In such cases, we say the events are dependent and the conditional probability formulas in equations (4.7) and (4.8) must be used to compute the event probabilities. If two events are not related, they are independent; in this case neither event's probability is affected by whether the other event occurred.

NOTES AND COMMENTS

Do not confuse the notion of mutually exclusive events with that of independent events. Two events with nonzero probabilities cannot be both mutually exclusive and independent. If one mutually exclusive event is known to occur, the other cannot occur; thus, the probability of the other event occurring is reduced to zero. They are therefore dependent.

Exercises

Methods

SELF test

30. Suppose that we have two events, A and B, with $P(A) = .50$, $P(B) = .60$, and $P(A \cap B) = .40$.
    a. Find $P(A \mid B)$.
    b. Find $P(B \mid A)$.
    c. Are A and B independent? Why or why not?

31. Assume that we have two events, A and B, that are mutually exclusive. Assume further that we know $P(A) = .30$ and $P(B) = .40$.
    a. What is $P(A \cap B)$?
    b. What is $P(A \mid B)$?
    c. A student in statistics argues that the concepts of mutually exclusive events and independent events are really the same, and that if events are mutually exclusive they must be independent. Do you agree with this statement? Use the probability information in this problem to justify your answer.
    d. What general conclusion would you make about mutually exclusive and independent events given the results of this problem?

Applications

32. The automobile industry sold 657,000 vehicles in the United States during January 2009 (*The Wall Street Journal,* February 4, 2009). This volume was down 37% from January 2008 as economic conditions continued to decline. The Big Three U.S. automakers—General Motors, Ford, and Chrysler—sold 280,500 vehicles, down 48% from January 2008. A summary of sales by automobile manufacturer and type of vehicle sold is shown in the following table. Data are in thousands of vehicles. The non-U.S. manufacturers are led by Toyota, Honda, and Nissan. The category Light Truck includes pickup, minivan, SUV, and crossover models.

		Type of Vehicle	
		Car	**Light Truck**
Manufacturer	**U.S.**	87.4	193.1
	Non-U.S.	228.5	148.0

    a. Develop a joint probability table for these data and use the table to answer the remaining questions.
    b. What are the marginal probabilities? What do they tell you about the probabilities associated with the manufacturer and the type of vehicle sold?
    c. If a vehicle was manufactured by one of the U.S. automakers, what is the probability that the vehicle was a car? What is the probability it was a light truck?
    d. If a vehicle was not manufactured by one of the U.S. automakers, what is the probability that the vehicle was a car? What is the probability it was a light truck?
    e. If the vehicle was a light truck, what is the probability that it was manufactured by one of the U.S. automakers?
    f. What does the probability information tell you about sales?

SELF test

33. Students taking the Graduate Management Admissions Test (GMAT) were asked about their undergraduate major and intent to pursue their MBA as a full-time or part-time student. A summary of their responses follows.

		Undergraduate Major			
		Business	**Engineering**	**Other**	**Totals**
Intended Enrollment Status	**Full-Time**	352	197	251	800
	Part-Time	150	161	194	505
	Totals	502	358	445	1305

    a. Develop a joint probability table for these data.
    b. Use the marginal probabilities of undergraduate major (business, engineering, or other) to comment on which undergraduate major produces the most potential MBA students.

c.   If a student intends to attend classes full-time in pursuit of an MBA degree, what is the probability that the student was an undergraduate engineering major?

d.   If a student was an undergraduate business major, what is the probability that the student intends to attend classes full-time in pursuit of an MBA degree?

e.   Let A denote the event that the student intends to attend classes full-time in pursuit of an MBA degree, and let B denote the event that the student was an undergraduate business major. Are events A and B independent? Justify your answer.

34. The U.S. Department of Transportation reported that during November, 83.4% of Southwest Airlines' flights, 75.1% of US Airways' flights, and 70.1% of JetBlue's flights arrived on time (*USA Today,* January 4, 2007). Assume that this on-time performance is applicable for flights arriving at concourse A of the Rochester International Airport, and that 40% of the arrivals at concourse A are Southwest Airlines flights, 35% are US Airways flights, and 25% are JetBlue flights.

a.   Develop a joint probability table with three rows (airlines) and two columns (on-time arrivals vs. late arrivals).

b.   An announcement has just been made that Flight 1424 will be arriving at gate 20 in concourse A. What is the most likely airline for this arrival?

c.   What is the probability that Flight 1424 will arrive on time?

d.   Suppose that an announcement is made saying that Flight 1424 will be arriving late. What is the most likely airline for this arrival? What is the least likely airline?

35. According to the Ameriprise Financial Money Across Generations study, 9 out of 10 parents with adult children ages 20 to 35 have helped their adult children with some type of financial assistance ranging from college, a car, rent, utilities, credit-card debt, and/or down payments for houses (*Money,* January 2009). The following table with sample data consistent with the study shows the number of times parents have given their adult children financial assistance to buy a car and to pay rent.

		Pay Rent	
		Yes	No
Buy a Car	Yes	56	52
	No	14	78

a.   Develop a joint probability table and use it to answer the remaining questions.

b.   Using the marginal probabilities for buy a car and pay rent, are parents more likely to assist their adult children with buying a car or paying rent? What is your interpretation of the marginal probabilities?

c.   If parents provided financial assistance to buy a car, what it the probability that the parents assisted with paying rent?

d.   If parents did not provide financial assistance to buy a car, what is the probability the parents assisted with paying rent?

e.   Is financial assistance to buy a car independent of financial assistance to pay rent? Use probabilities to justify your answer.

f.   What is the probability that parents provided financial assistance for their adult children by either helping buy a car or pay rent?

36. Jamal Crawford of the National Basketball Association's Portland Trail Blazers is the best free-throw shooter on the team, making 93% of his shots (ESPN website, April 5, 2012). Assume that late in a basketball game, Jamal Crawford is fouled and is awarded two shots.

a.   What is the probability that he will make both shots?

b.   What is the probability that he will make at least one shot?

c.   What is the probability that he will miss both shots?

d.   Late in a basketball game, a team often intentionally fouls an opposing player in or-der to stop the game clock. The usual strategy is to intentionally foul the other team's worst free-throw shooter. Assume that the Portland Trail Blazers' center makes 58% of his free-throw shots. Calculate the probabilities for the center as shown in parts (a), (b), and (c), and show that intentionally fouling the Portland Trail Blazers' center is a better strategy than intentionally fouling Jamal Crawford. Assume as in parts (a), (b), and (c) that two shots will be awarded.

37.   Visa Card USA studied how frequently young consumers, ages 18 to 24, use plastic (debit and credit) cards in making purchases (Associated Press, January 16, 2006). The results of the study provided the following probabilities.
   •   The probability that a consumer uses a plastic card when making a purchase is .37.
   •   Given that the consumer uses a plastic card, there is a .19 probability that the con-sumer is 18 to 24 years old.
   •   Given that the consumer uses a plastic card, there is a .81 probability that the consumer is more than 24 years old.

   U.S. Census Bureau data show that 14% of the consumer population is 18 to 24 years old.
   a.   Given the consumer is 18 to 24 years old, what is the probability that the consumer uses a plastic card?
   b.   Given the consumer is over 24 years old, what is the probability that the consumer uses a plastic card?
   c.   What is the interpretation of the probabilities shown in parts (a) and (b)?
   d.   Should companies such as Visa, MasterCard, and Discover make plastic cards avail-able to the 18 to 24 year old age group before these consumers have had time to es-tablish a credit history? If no, why? If yes, what restrictions might the companies place on this age group?

38.   Students in grades 3 through 8 in New York State are required to take a state mathematics exam. To meet the state's proficiency standards, a student must demonstrate an under-standing of the mathematics expected at his or her grade level. The following data show the number of students tested in the New York City school system for grades 3 through 8 and the number who met and did not meet the proficiency standards on the exam (New York City Department of Education website, January 16, 2010).

Grade	Met Proficiency Standards?	
	Yes	No
3	47,401	23,975
4	35,020	34,740
5	36,062	33,540
6	36,361	32,929
7	40,945	29,768
8	40,720	31,931

   a.   Develop a joint probability table for these data.
   b.   What are the marginal probabilities? What do they tell about the probabilities of a stu-dent meeting or not meeting the proficiency standards on the exam?
   c.   If a randomly selected student is a third grader, what is the probability that the student met the proficiency standards? If the student is a fourth grader, what is the probabil-ity that the student met the proficiency standards?
   d.   If a randomly selected student is known to have met the proficiency standards on the exam, what is the probability that the student is a third grader? What is the probability if the student is a fourth grader?

Bayes' Theorem

In the discussion of conditional probability, we indicated that revising probabilities when new information is obtained is an important phase of probability analysis. Often, we begin the analysis with initial or **prior probability** estimates for specific events of interest. Then, from sources such as a sample, a special report, or a product test, we obtain additional information about the events. Given this new information, we update the prior probability values by calculating revised probabilities, referred to as **posterior probabilities**. **Bayes' theorem** provides a means for making these probability calculations. The steps in this probability revision process are shown in Figure 4.9.

As an application of Bayes' theorem, consider a manufacturing firm that receives shipments of parts from two different suppliers. Let A_1 denote the event that a part is from supplier 1 and A_2 denote the event that a part is from supplier 2. Currently, 65% of the parts purchased by the company are from supplier 1 and the remaining 35% are from supplier 2. Hence, if a part is selected at random, we would assign the prior probabilities $P(A_1) = .65$ and $P(A_2) = .35$.

The quality of the purchased parts varies with the source of supply. Historical data suggest that the quality ratings of the two suppliers are as shown in Table 4.6. If we let G denote the event that a part is good and B denote the event that a part is bad, the information in Table 4.6 provides the following conditional probability values.

$$P(G \mid A_1) = .98 \quad P(B \mid A_1) = .02$$
$$P(G \mid A_2) = .95 \quad P(B \mid A_2) = .05$$

The tree diagram in Figure 4.10 depicts the process of the firm receiving a part from one of the two suppliers and then discovering that the part is good or bad as a two-step experiment. We see that four experimental outcomes are possible; two correspond to the part being good and two correspond to the part being bad.

Each of the experimental outcomes is the intersection of two events, so we can use the multiplication rule to compute the probabilities. For instance,

$$P(A_1, G) = P(A_1 \cap G) = P(A_1)P(G \mid A_1)$$
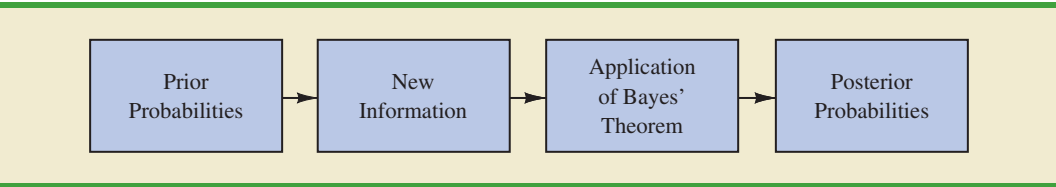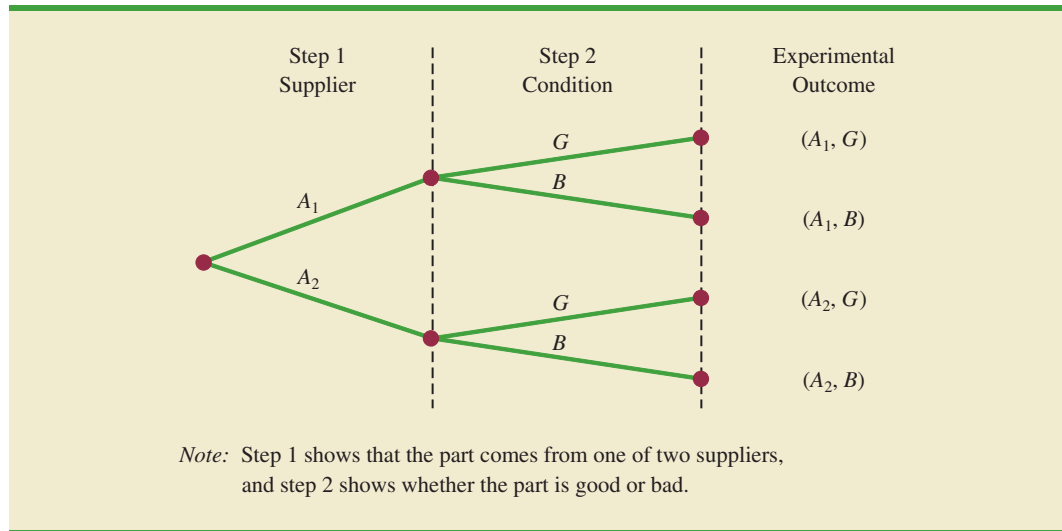
FIGURE 4.9   PROBABILITY REVISION USING BAYES' THEOREM

TABLE 4.6   HISTORICAL QUALITY LEVELS OF TWO SUPPLIERS

	Percentage Good Parts	Percentage Bad Parts
Supplier 1	98	2
Supplier 2	95	5

FIGURE 4.10 TREE DIAGRAM FOR TWO-SUPPLIER EXAMPLE

Note: Step 1 shows that the part comes from one of two suppliers,
and step 2 shows whether the part is good or bad.

The process of computing these joint probabilities can be depicted in what is called a probability tree (see Figure 4.11). From left to right through the tree, the probabilities for each branch at step 1 are prior probabilities and the probabilities for each branch at step 2 are conditional probabilities. To find the probabilities of each experimental outcome, we simply multiply the probabilities on the branches leading to the outcome. Each of these joint probabilities is shown in Figure 4.11 along with the known probabilities for each branch.

Suppose now that the parts from the two suppliers are used in the firm's manufacturing process and that a machine breaks down because it attempts to process a bad part. Given the information that the part is bad, what is the probability that it came from supplier 1 and what is the probability that it came from supplier 2? With the information in the probability tree (Figure 4.11), Bayes' theorem can be used to answer these questions.

FIGURE 4.11 PROBABILITY TREE FOR TWO-SUPPLIER EXAMPLE
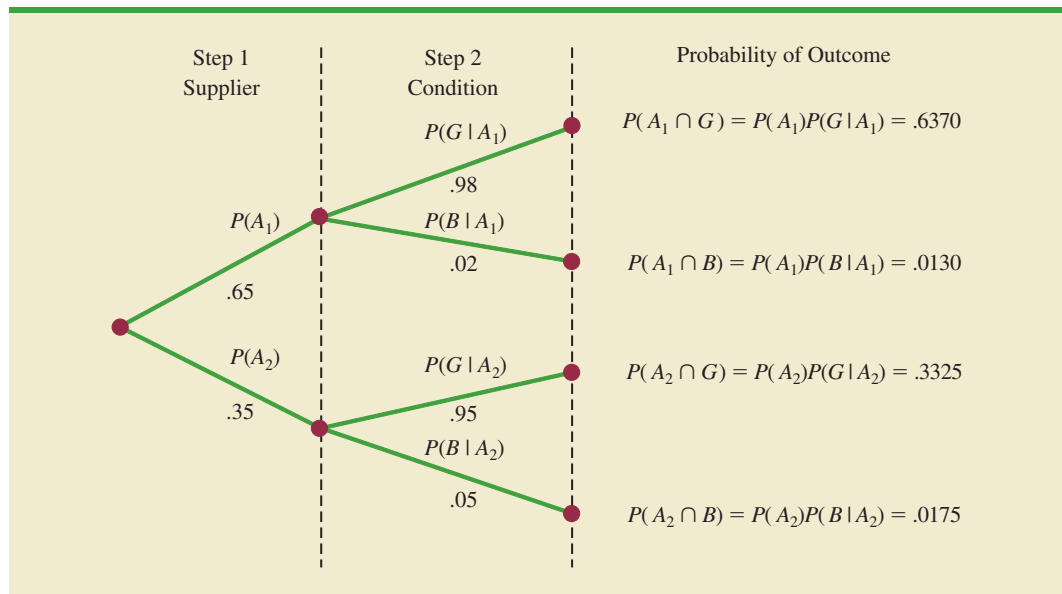
Letting B denote the event that the part is bad, we are looking for the posterior probabilities $P(A_1 \mid B)$ and $P(A_2 \mid B)$. From the law of conditional probability, we know that

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} \qquad (4.14)$$

Referring to the probability tree, we see that

$$P(A_1 \cap B) = P(A_1)P(B \mid A_1) \qquad (4.15)$$

To find $P(B)$, we note that event B can occur in only two ways: $(A_1 \cap B)$ and $(A_2 \cap B)$. Therefore, we have

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) \\ &= P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) \end{aligned} \qquad (4.16)$$

Substituting from equations (4.15) and (4.16) into equation (4.14) and writing a similar result for $P(A_2 \mid B)$, we obtain Bayes' theorem for the case of two events.

The Reverend Thomas Bayes (1702–1761), a Presbyterian minister, is credited with the original work leading to the version of Bayes' theorem in use today.

BAYES' THEOREM (TWO-EVENT CASE)

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)} \qquad (4.17)$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)} \qquad (4.18)$$

Using equation (4.17) and the probability values provided in the example, we have

$$\begin{aligned} P(A_1 \mid B) &= \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)} \\ &= \frac{(.65)(.02)}{(.65)(.02) + (.35)(.05)} = \frac{.0130}{.0130 + .0175} \\ &= \frac{.0130}{.0305} = .4262 \end{aligned}$$

In addition, using equation (4.18), we find $P(A_2 \mid B)$.

$$\begin{aligned} P(A_2 \mid B) &= \frac{(.35)(.05)}{(.65)(.02) + (.35)(.05)} \\ &= \frac{.0175}{.0130 + .0175} = \frac{.0175}{.0305} = .5738 \end{aligned}$$

Note that in this application we started with a probability of .65 that a part selected at random was from supplier 1. However, given information that the part is bad, the probability that the part is from supplier 1 drops to .4262. In fact, if the part is bad, it has better than a 50–50 chance that it came from supplier 2; that is, $P(A_2 \mid B) = .5738$.

Bayes' theorem is applicable when the events for which we want to compute posterior probabilities are mutually exclusive and their union is the entire sample space.[2] For the case of n mutually exclusive events $A_1, A_2, \ldots, A_n$, whose union is the entire sample space, Bayes' theorem can be used to compute any posterior probability $P(A_i \mid B)$ as shown here.

BAYES' THEOREM

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)} \quad \textbf{(4.19)}$$

With prior probabilities $P(A_1), P(A_2), \ldots, P(A_n)$ and the appropriate conditional probabilities $P(B \mid A_1), P(B \mid A_2), \ldots, P(B \mid A_n)$, equation (4.19) can be used to compute the posterior probability of the events $A_1, A_2, \ldots, A_n$.

Tabular Approach

A tabular approach is helpful in conducting the Bayes' theorem calculations. Such an approach is shown in Table 4.7 for the parts supplier problem. The computations shown there are done in the following steps.

Step 1. Prepare the following three columns:
  Column 1—The mutually exclusive events A_i for which posterior probabilities are desired
  Column 2—The prior probabilities $P(A_i)$ for the events
  Column 3—The conditional probabilities $P(B \mid A_i)$ of the new information B given each event

Step 2. In column 4, compute the joint probabilities $P(A_i \cap B)$ for each event and the new information B by using the multiplication law. These joint probabilities are found by multiplying the prior probabilities in column 2 by the corresponding conditional probabilities in column 3; that is, $P(A_i \cap B) = P(A_i)P(B \mid A_i)$.

Step 3. Sum the joint probabilities in column 4. The sum is the probability of the new information, $P(B)$. Thus we see in Table 4.7 that there is a .0130 probability that the part came from supplier 1 and is bad and a .0175 probability that the part came from supplier 2 and is bad. Because these are the only two ways in which a bad part can be obtained, the sum .0130 + .0175 shows an overall

TABLE 4.7   TABULAR APPROACH TO BAYES' THEOREM CALCULATIONS
FOR THE TWO-SUPPLIER PROBLEM

(1) Events A_i	(2) Prior Probabilities $P(A_i)$	(3) Conditional Probabilities $P(B \mid A_i)$	(4) Joint Probabilities $P(A_i \cap B)$	(5) Posterior Probabilities $P(A_i \mid B)$
A_1	.65	.02	.0130	.0130/.0305 = .4262
A_2	.35	.05	.0175	.0175/.0305 = .5738
	1.00		$P(B) = .0305$	1.0000

[2]If the union of events is the entire sample space, the events are said to be collectively exhaustive.

probability of .0305 of finding a bad part from the combined shipments of the two suppliers.

Step 4. In column 5, compute the posterior probabilities using the basic relationship of conditional probability.

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

Note that the joint probabilities $P(A_i \cap B)$ are in column 4 and the probability $P(B)$ is the sum of column 4.

NOTES AND COMMENTS

1. Bayes' theorem is used extensively in decision analysis. The prior probabilities are often subjective estimates provided by a decision maker. Sample information is obtained and posterior probabilities are computed for use in choosing the best decision.

2. An event and its complement are mutually exclusive, and their union is the entire sample space. Thus, Bayes' theorem is always applicable for computing posterior probabilities of an event and its complement.

Exercises

Methods

SELF test

39. The prior probabilities for events A_1 and A_2 are $P(A_1) = .40$ and $P(A_2) = .60$. It is also known that $P(A_1 \cap A_2) = 0$. Suppose $P(B \mid A_1) = .20$ and $P(B \mid A_2) = .05$.
    a. Are A_1 and A_2 mutually exclusive? Explain.
    b. Compute $P(A_1 \cap B)$ and $P(A_2 \cap B)$.
    c. Compute $P(B)$.
    d. Apply Bayes' theorem to compute $P(A_1 \mid B)$ and $P(A_2 \mid B)$.

40. The prior probabilities for events $A_1, A_2,$ and A_3 are $P(A_1) = .20, P(A_2) = .50,$ and $P(A_3) = .30$. The conditional probabilities of event B given $A_1, A_2,$ and A_3 are $P(B \mid A_1) = .50, P(B \mid A_2) = .40,$ and $P(B \mid A_3) = .30$.
    a. Compute $P(B \cap A_1), P(B \cap A_2),$ and $P(B \cap A_3)$.
    b. Apply Bayes' theorem, equation (4.19), to compute the posterior probability $P(A_2 \mid B)$.
    c. Use the tabular approach to applying Bayes' theorem to compute $P(A_1 \mid B), P(A_2 \mid B),$ and $P(A_3 \mid B)$.

Applications

41. A consulting firm submitted a bid for a large research project. The firm's management initially felt they had a 50–50 chance of getting the project. However, the agency to which the bid was submitted subsequently requested additional information on the bid. Past experience indicates that for 75% of the successful bids and 40% of the unsuccessful bids the agency requested additional information.
    a. What is the prior probability of the bid being successful (that is, prior to the request for additional information)?
    b. What is the conditional probability of a request for additional information given that the bid will ultimately be successful?
    c. Compute the posterior probability that the bid will be successful given a request for additional information.

SELF test

42. A local bank reviewed its credit card policy with the intention of recalling some of its credit cards. In the past approximately 5% of cardholders defaulted, leaving the bank unable to collect the outstanding balance. Hence, management established a prior probability of .05 that any particular cardholder will default. The bank also found that the probability of missing a monthly payment is .20 for customers who do not default. Of course, the probability of missing a monthly payment for those who default is 1.
   a. Given that a customer missed one or more monthly payments, compute the posterior probability that the customer will default.
   b. The bank would like to recall its card if the probability that a customer will default is greater than .20. Should the bank recall its card if the customer misses a monthly payment? Why or why not?

43. Two Wharton professors analyzed 1,613,234 putts by golfers on the Professional Golfers Association (PGA) Tour and found that 983,764 of the putts were made and 629,470 of the putts were missed. Further analysis showed that for putts that were made, 64.0% of the time the player was attempting to make a par putt and 18.8% of the time the player was attempting to make a birdie putt. And, for putts that were missed, 20.3% of the time the player was attempting to make a par putt and 73.4% of the time the player was attempting to make a birdie putt. (D. G. Pope and M. E. Schweitzer, *Is Tiger Woods Loss Averse? Persistent Bias in the Face of Experience, Competition, and High Stakes,* June 2009, The Wharton School, University of Pennsylvania).
   a. What is the probability that a PGA Tour player makes a putt?
   b. Suppose that a PGA Tour player has a putt for par. What is the probability that the player will make the putt?
   c. Suppose that a PGA Tour player has a putt for birdie. What is the probability that the player will make the putt?
   d. Comment on the differences in the probabilities computed in parts (b) and (c).

44. According to the Open Doors 2011 Report, 9.5% of all full-time U.S. undergraduate students studied abroad during the 2009–2010 academic year (Institute of International Education, November 14, 2011). Assume that participation records show women make up 60% of the students who studied abroad during the 2009–2010 academic year, but women make up only 49% of the students who didn't participate.
   a. Let A_1 = the student studied abroad during the 2009–2010 academic year
      A_2 = the student did not study abroad during the 2009–2010 academic year
      W = the student is a female student
      Using the given information, what are the values for $P(A_1)$, $P(A_2)$, $P(W \mid A_1)$, and $P(W \mid A_2)$?
   b. What is the probability that a female student studied abroad during the 2009–2010 academic year?
   c. What is the probability that a male student studied abroad during the 2009–2010 academic year?
   d. Given the preceding results, what were the percentage of women and the percentage of men studying full-time during the 2009–2010 academic year?

45. In an article about investment alternatives, *Money* magazine reported that drug stocks provide a potential for long-term growth, with over 50% of the adult population of the United States taking prescription drugs on a regular basis. For adults age 65 and older, 82% take prescription drugs regularly. For adults age 18 to 64, 49% take prescription drugs regularly. The 18–64 age group accounts for 83.5% of the adult population (*Statistical Abstract of the United States*, 2008).
   a. What is the probability that a randomly selected adult is 65 or older?
   b. Given an adult takes prescription drugs regularly, what is the probability that the adult is 65 or older?

Summary

In this chapter we introduced basic probability concepts and illustrated how probability analysis can be used to provide helpful information for decision making. We described how probability can be interpreted as a numerical measure of the likelihood that an event will occur. In addition, we saw that the probability of an event can be computed either by summing the probabilities of the experimental outcomes (sample points) comprising the event or by using the relationships established by the addition, conditional probability, and multiplication laws of probability. For cases in which additional information is available, we showed how Bayes' theorem can be used to obtain revised or posterior probabilities.

Glossary

Probability A numerical measure of the likelihood that an event will occur.

Experiment A process that generates well-defined outcomes.

Sample space The set of all experimental outcomes.

Sample point An element of the sample space. A sample point represents an experimental outcome.

Multiple-step experiment An experiment that can be described as a sequence of steps. If a multiple-step experiment has k steps with n_1 possible outcomes on the first step, n_2 possible outcomes on the second step, and so on, the total number of experimental outcomes is given by $(n_1)(n_2) \cdots (n_k)$.

Tree diagram A graphical representation that helps in visualizing a multiple-step experiment.

Combination In an experiment we may be interested in determining the number of ways n objects may be selected from among N objects without regard to the *order in which the n objects are selected*. Each selection of n objects is called a combination and the total number of combinations of N objects taken n at a time is $C_n^N = \binom{N}{n} = \dfrac{N!}{n!(N-n)!}$ for $n = 0, 1, 2, \ldots, N$.

Permutation In an experiment we may be interested in determining the number of ways n objects may be selected from among N objects when the *order in which the n objects are selected* is important. Each ordering of n objects is called a permutation and the total number of permutations of N objects taken n at a time is $P_n^N = n!\binom{N}{n} = \dfrac{N!}{(N-n)!}$ for $n = 0, 1, 2, \ldots, N$.

Basic requirements for assigning probabilities Two requirements that restrict the manner in which probability assignments can be made: (1) for each experimental outcome E_i we must have $0 \le P(E_i) \le 1$; (2) considering all experimental outcomes, we must have $P(E_1) + P(E_2) + \cdots + P(E_n) = 1.0$.

Classical method A method of assigning probabilities that is appropriate when all the experimental outcomes are equally likely.

Relative frequency method A method of assigning probabilities that is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times.

Subjective method A method of assigning probabilities on the basis of judgment.

Event A collection of sample points.

Complement of *A* The event consisting of all sample points that are not in A.

Venn diagram A graphical representation for showing symbolically the sample space and operations involving events in which the sample space is represented by a rectangle and events are represented as circles within the sample space.

Union of *A* and *B* The event containing all sample points belonging to A or B or both. The union is denoted $A \cup B$.

Intersection of *A* and *B* The event containing the sample points belonging to both *A* and *B*. The intersection is denoted $A \cap B$.

Addition law A probability law used to compute the probability of the union of two events. It is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. For mutually exclusive events, $P(A \cap B) = 0$; in this case the addition law reduces to $P(A \cup B) = P(A) + P(B)$.

Mutually exclusive events Events that have no sample points in common; that is, $A \cap B$ is empty and $P(A \cap B) = 0$.

Conditional probability The probability of an event given that another event already occurred. The conditional probability of *A* given *B* is $P(A \mid B) = P(A \cap B)/P(B)$.

Joint probability The probability of two events both occurring; that is, the probability of the intersection of two events.

Marginal probability The values in the margins of a joint probability table that provide the probabilities of each event separately.

Independent events Two events *A* and *B* where $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$; that is, the events have no influence on each other.

Multiplication law A probability law used to compute the probability of the intersection of two events. It is $P(A \cap B) = P(B)P(A \mid B)$ or $P(A \cap B) = P(A)P(B \mid A)$. For independent events it reduces to $P(A \cap B) = P(A)P(B)$.

Prior probabilities Initial estimates of the probabilities of events.

Posterior probabilities Revised probabilities of events based on additional information.

Bayes' theorem A method used to compute posterior probabilities.

Key Formulas

Counting Rule for Combinations

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{4.1}$$

Counting Rule for Permutations

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!} \tag{4.2}$$

Computing Probability Using the Complement

$$P(A) = 1 - P(A^c) \tag{4.5}$$

Addition Law

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{4.6}$$

Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{4.7}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \tag{4.8}$$

Multiplication Law

$$P(A \cap B) = P(B)P(A \mid B) \tag{4.11}$$

$$P(A \cap B) = P(A)P(B \mid A) \tag{4.12}$$

Multiplication Law for Independent Events

$$P(A \cap B) = P(A)P(B) \qquad \text{(4.13)}$$

Bayes' Theorem

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)} \qquad \text{(4.19)}$$

Supplementary Exercises

46. *The Wall Street Journal*/Harris Personal Finance poll asked 2082 adults if they owned a home (All Business website, January 23, 2008). A total of 1249 survey respondents answered Yes. Of the 450 respondents in the 18–34 age group, 117 responded Yes.
    a. What is the probability that a respondent to the poll owned a home?
    b. What is the probability that a respondent in the 18–34 age group owned a home?
    c. What is the probability that a respondent to the poll did not own a home?
    d. What is the probability that a respondent in the 18–34 age group did not own a home?

47. A financial manager made two new investments—one in the oil industry and one in municipal bonds. After a one-year period, each of the investments will be classified as either successful or unsuccessful. Consider the making of the two investments as an experiment.
    a. How many sample points exist for this experiment?
    b. Show a tree diagram and list the sample points.
    c. Let O = the event that the oil industry investment is successful and M = the event that the municipal bond investment is successful. List the sample points in O and in M.
    d. List the sample points in the union of the events ($O \cup M$).
    e. List the sample points in the intersection of the events ($O \cap M$).
    f. Are events O and M mutually exclusive? Explain.

48. Forty-three percent of Americans use social media and other websites to voice their opinions about television programs (*The Huffington Post,* November 23, 2011). Below are the results of a survey of 1400 individuals who were asked if they use social media and other websites to voice their opinions about television programs.

	Uses Social Media and Other Websites to Voice Opinions About Television Programs	Doesn't Use Social Media and Other Websites to Voice Opinions About Television Programs
Female	395	291
Male	323	355

    a. What is the probability a respondent is female?
    b. What is the conditional probability a respondent uses social media and other websites to voice opinions about television programs given the respondent is female?
    c. Let F denote the event that the respondent is female and A denote the event that the respondent uses social media and other websites to voice opinions about television programs. Are events F and A independent?

49. A study of 31,000 hospital admissions in New York State found that 4% of the admissions led to treatment-caused injuries. One-seventh of these treatment-caused injuries resulted in death, and one-fourth were caused by negligence. Malpractice claims were filed in one out of 7.5 cases involving negligence, and payments were made in one out of every two claims.
    a. What is the probability a person admitted to the hospital will suffer a treatment-caused injury due to negligence?

b. What is the probability a person admitted to the hospital will die from a treatment-caused injury?

c. In the case of a negligent treatment-caused injury, what is the probability a malpractice claim will be paid?

50. A telephone survey to determine viewer response to a new television show obtained the following data.

Rating	Frequency
Poor	4
Below average	8
Average	11
Above average	14
Excellent	13

a. What is the probability that a randomly selected viewer will rate the new show as average or better?

b. What is the probability that a randomly selected viewer will rate the new show below average or worse?

51. The following crosstabulation shows household income by educational level of the head of household (*Statistical Abstract of the United States*, 2008).

	Household Income ($1000s)					
Education Level	Under 25	25.0– 49.9	50.0– 74.9	75.0– 99.9	100 or more	Total
Not H.S. Graduate	4207	3459	1389	539	367	9961
H.S. Graduate	4917	6850	5027	2637	2668	22,099
Some College	2807	5258	4678	3250	4074	20,067
Bachelor's Degree	885	2094	2848	2581	5379	13,787
Beyond Bach. Deg.	290	829	1274	1241	4188	7822
Total	13,106	18,490	15,216	10,248	16,676	73,736

a. Develop a joint probability table.

b. What is the probability of a head of household not being a high school graduate?

c. What is the probability of a head of household having a bachelor's degree or more education?

d. What is the probability of a household headed by someone with a bachelor's degree earning $100,000 or more?

e. What is the probability of a household having income below $25,000?

f. What is the probability of a household headed by someone with a bachelor's degree earning less than $25,000?

g. Is household income independent of educational level?

52. An MBA new-matriculants survey provided the following data for 2018 students.

		Applied to More Than One School	
		Yes	No
Age Group	23 and under	207	201
	24–26	299	379
	27–30	185	268
	31–35	66	193
	36 and over	51	169

   a.   For a randomly selected MBA student, prepare a joint probability table for the experi-
        ment consisting of observing the student's age and whether the student applied to one
        or more schools.
   b.   What is the probability that a randomly selected applicant is 23 or under?
   c.   What is the probability that a randomly selected applicant is older than 26?
   d.   What is the probability that a randomly selected applicant applied to more than one
        school?

53. Refer again to the data from the MBA new-matriculants survey in exercise 52.
   a.   Given that a person applied to more than one school, what is the probability that the
        person is 24–26 years old?
   b.   Given that a person is in the 36-and-over age group, what is the probability that the
        person applied to more than one school?
   c.   What is the probability that a person is 24–26 years old or applied to more than one
        school?
   d.   Suppose a person is known to have applied to only one school. What is the probability
        that the person is 31 or more years old?
   e.   Is the number of schools applied to independent of age? Explain.

54. In February 2012, the Pew Internet & American Life project conducted a survey that included
     several questions about how Internet users feel about search engines and other websites col-
     lecting information about them and using this information either to shape search results or tar-
     get advertising to them (Pew Research Center, March 9, 2012). In one question, participants
     were asked, "If a search engine kept track of what you search for, and then used that infor-
     mation to personalize your future search results, how would you feel about that?" Respon-
     dents could indicate either "Would *not* be okay with it because you feel it is an invasion of
     your privacy" or "Would be *okay* with it, even if it means they are gathering information about
     you." Frequencies of responses by age group are summarized in the following table.

Age	Not Okay	Okay
18–29	0.1485	0.0604
30–49	0.2273	0.0907
50+	0.4008	0.0723

   a.   What is the probability a survey respondent will say she or he is *not okay* with this
        practice?
   b.   Given a respondent is 30–49 years old, what is the probability the respondent will say
        she or he is *okay* with this practice?
   c.   Given a respondent says she or he is *not okay* with this practice, what is the probabil-
        ity the respondent is 50+ years old?
   d.   Is the attitude about this practice independent of the age of the respondent? Why or
        why not?
   e.   Do attitudes toward this practice for respondents who are 18–29 years old and
        respondents who are 50+ years old differ?

55. A large consumer goods company ran a television advertisement for one of its soap prod-
     ucts. On the basis of a survey that was conducted, probabilities were assigned to the fol-
     lowing events.

     B = individual purchased the product
     S = individual recalls seeing the advertisement
     $B \cap S$ = individual purchased the product and recalls seeing the advertisement

     The probabilities assigned were $P(B) = .20$, $P(S) = .40$, and $P(B \cap S) = .12$.
   a.   What is the probability of an individual's purchasing the product given that the in-
        dividual recalls seeing the advertisement? Does seeing the advertisement increase

the probability that the individual will purchase the product? As a decision maker, would you recommend continuing the advertisement (assuming that the cost is reasonable)?

b. Assume that individuals who do not purchase the company's soap product buy from its competitors. What would be your estimate of the company's market share? Would you expect that continuing the advertisement will increase the company's market share? Why or why not?

c. The company also tested another advertisement and assigned it values of $P(S) = .30$ and $P(B \cap S) = .10$. What is $P(B \mid S)$ for this other advertisement? Which advertisement seems to have had the bigger effect on customer purchases?

56. Cooper Realty is a small real estate company located in Albany, New York, specializing primarily in residential listings. They recently became interested in determining the likelihood of one of their listings being sold within a certain number of days. An analysis of company sales of 800 homes in previous years produced the following data.

		Days Listed Until Sold			
		Under 30	**31–90**	**Over 90**	**Total**
Initial Asking Price	**Under $150,000**	50	40	10	100
	$150,000–$199,999	20	150	80	250
	$200,000–$250,000	20	280	100	400
	Over $250,000	10	30	10	50
	Total	100	500	200	800

a. If A is defined as the event that a home is listed for more than 90 days before being sold, estimate the probability of A.
b. If B is defined as the event that the initial asking price is under $150,000, estimate the probability of B.
c. What is the probability of $A \cap B$?
d. Assuming that a contract was just signed to list a home with an initial asking price of less than $150,000, what is the probability that the home will take Cooper Realty more than 90 days to sell?
e. Are events A and B independent?

57. A company studied the number of lost-time accidents occurring at its Brownsville, Texas, plant. Historical records show that 6% of the employees suffered lost-time accidents last year. Management believes that a special safety program will reduce such accidents to 5% during the current year. In addition, it estimates that 15% of employees who had lost-time accidents last year will experience a lost-time accident during the current year.

a. What percentage of the employees will experience lost-time accidents in both years?
b. What percentage of the employees will suffer at least one lost-time accident over the two-year period?

58. A survey showed that 8% of Internet users age 18 and older report keeping a blog. Referring to the 18–29 age group as young adults, the survey showed that for bloggers 54% are young adults and for nonbloggers 24% are young adults (Pew Internet & American Life Project, July 19, 2006).

a. Develop a joint probability table for these data with two rows (bloggers vs. nonbloggers) and two columns (young adults vs. older adults).
b. What is the probability that an Internet user is a young adult?
c. What is the probability that an Internet user keeps a blog and is a young adult?
d. Suppose that in a follow-up phone survey we contact someone who is 24 years old. What is the probability that this person keeps a blog?

59. An oil company purchased an option on land in Alaska. Preliminary geologic studies assigned the following prior probabilities.

$$P(\text{high-quality oil}) = .50$$
$$P(\text{medium-quality oil}) = .20$$
$$P(\text{no oil}) = .30$$

a. What is the probability of finding oil?
b. After 200 feet of drilling on the first well, a soil test is taken. The probabilities of finding the particular type of soil identified by the test follow.

$$P(\text{soil} \mid \text{high-quality oil}) = .20$$
$$P(\text{soil} \mid \text{medium-quality oil}) = .80$$
$$P(\text{soil} \mid \text{no oil}) = .20$$

How should the firm interpret the soil test? What are the revised probabilities, and what is the new probability of finding oil?

60. The five most common words appearing in spam emails are *shipping!, today!, here!, available,* and *fingertips!* (Andy Greenberg, "The Most Common Words In Spam Email," *Forbes* website, March 17, 2010). Many spam filters separate spam from ham (email not considered to be spam) through application of Bayes' theorem. Suppose that for one email account, 1 in every 10 messages is spam and the proportions of spam messages that have the five most common words in spam email are given below.

shipping!	.051
today!	.045
here!	.034
available	.014
fingertips!	.014

Also suppose that the proportions of ham messages that have these words are

shipping!	.0015
today!	.0022
here!	.0022
available	.0041
fingertips!	.0011

a. If a message includes the word *shipping!,* what is the probability the message is spam? If a message includes the word *shipping!,* what is the probability the message is ham? Should messages that include the word *shipping!* be flagged as spam?
b. If a message includes the word *today!,* what is the probability the message is spam? If a message includes the word *here!,* what is the probability the message is spam? Which of these two words is a stronger indicator that a message is spam? Why?
c. If a message includes the word *available*, what is the probability the message is spam? If a message includes the word *fingertips!*, what is the probability the message is spam? Which of these two words is a stronger indicator that a message is spam? Why?
d. What insights do the results of parts (b) and (c) yield about what enables a spam filter that uses Bayes' theorem to work effectively?

Case Problem   Hamilton County Judges

Hamilton County judges try thousands of cases per year. In an overwhelming majority of the cases disposed, the verdict stands as rendered. However, some cases are appealed, and of those appealed, some of the cases are reversed. Kristen DelGuzzi of *The Cincinnati Enquirer* conducted a study of cases handled by Hamilton County judges over a three-year period. Shown in Table 4.8 are the results for 182,908 cases handled (disposed) by

TABLE 4.8   TOTAL CASES DISPOSED, APPEALED, AND REVERSED IN HAMILTON
COUNTY COURTS

WEB file

Judge

Common Pleas Court

Judge	Total Cases Disposed	Appealed Cases	Reversed Cases
Fred Cartolano	3037	137	12
Thomas Crush	3372	119	10
Patrick Dinkelacker	1258	44	8
Timothy Hogan	1954	60	7
Robert Kraft	3138	127	7
William Mathews	2264	91	18
William Morrissey	3032	121	22
Norbert Nadel	2959	131	20
Arthur Ney, Jr.	3219	125	14
Richard Niehaus	3353	137	16
Thomas Nurre	3000	121	6
John O'Connor	2969	129	12
Robert Ruehlman	3205	145	18
J. Howard Sundermann	955	60	10
Ann Marie Tracey	3141	127	13
Ralph Winkler	3089	88	6
Total	43,945	1762	199

Domestic Relations Court

Judge	Total Cases Disposed	Appealed Cases	Reversed Cases
Penelope Cunningham	2729	7	1
Patrick Dinkelacker	6001	19	4
Deborah Gaines	8799	48	9
Ronald Panioto	12,970	32	3
Total	30,499	106	17

Municipal Court

Judge	Total Cases Disposed	Appealed Cases	Reversed Cases
Mike Allen	6149	43	4
Nadine Allen	7812	34	6
Timothy Black	7954	41	6
David Davis	7736	43	5
Leslie Isaiah Gaines	5282	35	13
Karla Grady	5253	6	0
Deidra Hair	2532	5	0
Dennis Helmick	7900	29	5
Timothy Hogan	2308	13	2
James Patrick Kenney	2798	6	1
Joseph Luebbers	4698	25	8
William Mallory	8277	38	9
Melba Marsh	8219	34	7
Beth Mattingly	2971	13	1
Albert Mestemaker	4975	28	9
Mark Painter	2239	7	3
Jack Rosen	7790	41	13
Mark Schweikert	5403	33	6
David Stockdale	5371	22	4
John A. West	2797	4	2
Total	108,464	500	104

38 judges in Common Pleas Court, Domestic Relations Court, and Municipal Court. Two of the judges (Dinkelacker and Hogan) did not serve in the same court for the entire three-year period.

The purpose of the newspaper's study was to evaluate the performance of the judges. Appeals are often the result of mistakes made by judges, and the newspaper wanted to know which judges were doing a good job and which were making too many mistakes. You are called in to assist in the data analysis. Use your knowledge of probability and conditional probability to help with the ranking of the judges. You also may be able to analyze the likelihood of appeal and reversal for cases handled by different courts.

Managerial Report

Prepare a report with your rankings of the judges. Also, include an analysis of the likelihood of appeal and case reversal in the three courts. At a minimum, your report should include the following:

1. The probability of cases being appealed and reversed in the three different courts.
2. The probability of a case being appealed for each judge.
3. The probability of a case being reversed for each judge.
4. The probability of reversal given an appeal for each judge.
5. Rank the judges within each court. State the criteria you used and provide a rationale for your choice.