Fast fourier Transform

$$F(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ [P.D.F of normal Dist.]

where μ = mean & σ^2 = variance, x = variable

PDF is the fourier transform of characteristic function.

Characteristic function of normal distribution:

$$\boxed{\hat{F}(t) = e^{i\mu t} \cdot e^{-\frac{1}{2}\sigma^2 t^2}}$$

characteristic function can be used to recover PDF, which can then be used to price options

$$V = \int_{R} \underbrace{e^{-rT}}_{\text{Discounting}} \cdot \underbrace{\pi(T)}_{\text{Payoff}} \cdot \underbrace{F(x)}_{\text{PDF}} \, dx$$

Inversion Lemma

$$\boxed{F(x) = \frac{1}{\pi} R\left(\int_0^\infty e^{-iux} \phi(u)\, du\right)}$$

PDF of x — Parameter — characteristic function of u. — random variable

Proof of Inversion Lemma

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \phi(u)\, du$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{0} e^{-iux} \phi(u)\, du + \int_{0}^{\infty} e^{-iux} \phi(u)\, du\right]$$

Now, taking $u = -v$, left side integral becomes

$$\int_{-\infty}^{0} e^{-iux} \phi(u)\, du = \int_{0}^{\infty} e^{ivx} \phi(-v)\, dv$$

$$\int_{0}^{\infty} e^{ivx} \phi(-v)\, dv = \overline{\int_{0}^{\infty} e^{-iux} \phi(u)\, du}$$

(Taking Conjugate of 1st Integral)

Plugging the Conjugate in first Integral

$$F(x) = \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-iux}\phi(u)\,du + \overline{\int_{0}^{\infty} e^{-iux}\phi(u)\,du}\right]$$

$$\Rightarrow \frac{1}{2\pi}\left[\int_{0}^{\infty} e^{-iux}\phi(u)\,du + \overline{\int_{0}^{\infty} e^{-iux}\phi(u)\,du}\right]$$

For any $z \in \mathbb{C}$ we have $z + \bar{z} = 2R(z)$. Therefore the equation becomes :-

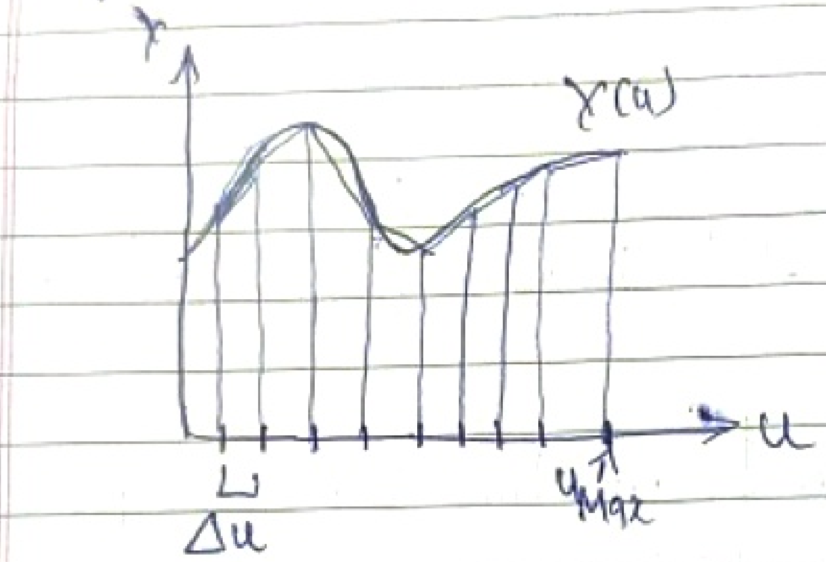
$$F(x) = \frac{1}{\pi} R\left(\int_{0}^{\infty} e^{-iux}\phi(u)\,du\right)$$

Inversion lemma.

In the Inversion lemma, replace $e^{-iux}\phi(u)$ with $\gamma(u)$.

$$\int_{0}^{\infty} e^{-iux}\phi(u)\,du = \int_{0}^{\infty}\gamma(u)\,du$$

We will ~~ean~~ evaluate this integral using Trapezoidal integration (Numerically).



$$\int_0^{u_{Max}} \gamma(u)\,du = \frac{\gamma(u_1)+\gamma(u_2)}{2}\cdot\Delta u + \frac{\gamma(u_2)+\gamma(u_3)}{2}\cdot\Delta u$$

$$\ldots\ldots \frac{\gamma(u_{n-1})+\gamma(u_N)}{2}\cdot\Delta u$$

Taking $\frac{\Delta u}{2}$ common and summing the like terms

$$\int_0^{u_{max}} \gamma(u)\,du = \frac{\Delta u}{2}\left[2\cdot\sum_{k=2}^{n-1}\gamma(u_k) + \gamma(u_1)+\gamma(u_N)\right]$$

$$= \Delta u\left[\sum_{k=2}^{N-1}\gamma(u_k) + \frac{1}{2}\left(\gamma(u_1)+\gamma(u_N)\right)\right]$$

Set the following:

$u_{max} = N \Delta_u$ → Terminal point

$u_m = (m-1) \Delta_u$ → Starting points of Trapezoids

$x_k = -b + \Delta_x (k-1)$

minimum point on x grid, $k = 1, 2, 3 \ldots N$

Replacing $\gamma(u)$ with $e^{-iux}\phi(u)$ in an discrete version of integral.

$$\int_0^{u_{max}} \gamma(u)\, du = \Delta u \sum_{m=1}^{N} e^{-i[(m-1)\Delta u][-b+\Delta_x(k-1)]} \phi(u_m)$$

$$-\frac{1}{2}\left[e^{-ixu_1}\phi(u_1) + e^{ixu_N}\phi\, u_N\right]$$

We are summing from 1 to N, hence subtraction the last term. Note the last term has no negative sign. x domain ranges from -x to +x.

We'll now rearrange the above summation so that 'fast fourier transform' in python is possible.

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$$\int_0^{u_{max}} \gamma(u)\,du \approx \Delta_u \sum_{m=1}^{N} e^{-i\Delta_x \Delta_u (m-1)(k-1)} e^{i(m-1)b\Delta_u} \phi(u_m)$$

$$-\frac{1}{2}\left[e^{-ixu_1}\phi(u_1) + e^{ixu_N}\phi(u_N)\right]$$

Setting $\Delta_x \Delta_u = \frac{2\pi}{N}$ and $u_m = (m-1)\Delta_u$

We have the following:

$$\int_0^{u_{max}} \gamma(u)\,du \approx \frac{\Delta_u}{\pi} \overbrace{\sum_{m=1}^{N} e^{-i\frac{2\pi}{N}(m-1)(k-1)}}^{\to FFT} e^{ibu_m}\phi(u_m)$$

$$-\frac{1}{2}\left[e^{-ixu_1}\phi(u_1) + e^{ixu_N}\phi(u_N)\right]$$

FFT reduce the number of multiplications required for a matrix $M \in C^{N \times N}$ the number of multiplications required are $O(N^2)$ but with FFT you only do $O(N \log_2 N)$ multiplications. This is because the matrix written in FFT form is identical about its diagonal.