ec23...

EE23BTECH11029 - Kanishk

Question:
In the table shown below, match the signal type with its spectral characteristics. GATE 2023 EC

Signal Type	Spectral Characterstics
(i) Continuous, aperiodic	(a) Continuous, aperiodic
(ii) Continuous, periodic	(b) Continuous, periodic
(iii) Discrete, aperiodic	(c) Discrete, aperiodic
(iii) Discrete, periodic	(d) Discrete, periodic

TABLE 0

1) $(i)\ \rightarrow (a)$, $(ii)\ \rightarrow (b)$, $(iii)\ \rightarrow (c)$, $(iv)\ \rightarrow (d)$
2) $(i)\ \rightarrow (a)$, $(ii)\ \rightarrow (c)$, $(iii)\ \rightarrow (b)$, $(iv)\ \rightarrow (d)$
3) $(i)\ \rightarrow (d)$, $(ii)\ \rightarrow (b)$, $(iii)\ \rightarrow (c)$, $(iv)\ \rightarrow (a)$
4) $(i)\ \rightarrow (a)$, $(ii)\ \rightarrow (c)$, $(iii)\ \rightarrow (d)$, $(iv)\ \rightarrow (b)$

Solution:

Parameter	Description
$x(t)$	Continuous Time Signal
$x(f)$	Fourier Transform of a Signal
$x[n]$	Discrete Time Signal
$X[k]$	Tthe amplitude and phase of the k^{th}frequency component of the input signal $x[n]$

TABLE 4
INPUT PARAMETERS

1. Continuous, aperiodic signal

$$X(f) = \int_{-\infty}^{\infty} x(t) . e^{-j2\pi f t}\, dt \tag{1}$$

Let's consider the limit as f approaches a certain frequency f_0:

$$lim_{\epsilon\to 0}X(f_0+\epsilon) = lim_{\epsilon\to 0}\int_{-\infty}^{\infty} x(t).e^{-j2\pi ft}\,dt \tag{2}$$

By continuity of $x(t)$ we can interchange the limit and the integral:

$$= \int_{-\infty}^{\infty} x(t).lim_{\epsilon\to 0}e^{-j2\pi(f_0+\epsilon)t}\,dt \tag{3}$$

$$= \int_{-\infty}^{\infty} x(t).e^{-j2\pi f_0 t}\,dt \tag{4}$$

$$= X(f_0) \tag{5}$$

$$lim_{f_0-\epsilon} = X(f_0) \tag{6}$$

Therefore, X(f) is continuous for all frequencies f.
Let's assume $X(f)$ is periodic with period T

$$X(f+T) = X(f) \tag{7}$$

Now applying inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f).e^{j2\pi ft}\,df \tag{8}$$

$$= \int_{-\infty}^{\infty} X(f+T).e^{j2\pi(f+T)t}\,df \tag{9}$$

$$= \int_{-\infty}^{\infty} X(f).e^{j2\pi(f+T)t}\,df \tag{10}$$

$$= e^{j2\pi Tt}\int_{-\infty}^{\infty} .X(f).e^{2\pi ft}\,df \tag{11}$$

$e^{j2\pi Tt}$ is a periodic function of t, which contradicts that $x(t)$ is aperiodic,
Therefore, $X(f)$ cannot be periodic, hence it must be aperiodic.

For Example: Exponential Decay

$$x(t) = e^{-\alpha t} \tag{12}$$

$$X(f) = \int_{-\infty}^{\infty} e^{-\alpha t}.e^{-j2\pi ft}\,dt \tag{13}$$

2. Continuous, periodic signal

$$x(t) = x(t+kT) \tag{14}$$

$$x(t) \leftrightarrow X(f) \tag{15}$$

$$X(f) = \int_{-\infty}^{\infty} x(t).e^{-j2\pi ft}\,dt \tag{16}$$

$$X(f) = \int_{-\infty}^{\infty} x(t+kT).e^{-j2\pi ft}\,dt \tag{17}$$

$$\text{Let } \tau = t+kT \tag{18}$$

$$X(f) = \int_{-\infty}^{\infty} x(\tau).e^{-j2\pi f\tau}.e^{j2\pi fkT}\,d\tau \tag{19}$$

$e^{j2\pi fkT}$ is a Periodic function in frequency domain with period $\frac{1}{T}$ Hz

$$X(f) = e^{j2\pi fkT} \int_{-\infty}^{\infty} x(\tau).e^{-j2\pi f\tau}\, d\tau \tag{20}$$
$$\implies X(f) \text{ is aperiodic} \tag{21}$$

$X(f)$ will be a sum of scaled copies of the Fourier transform of $x(t)$, each copy shifted by multiples of $\frac{1}{T}$Hz This results in a discrete frequency domain representation.

For example: Square Wave Equation

$$x(t) = sgn(sin(2\pi ft)) \tag{22}$$
$$X(f) = \int_{-\infty}^{\infty} sgn(sin(2\pi ft)).e^{-j2\pi ft}\, dt \tag{23}$$

3. Discrete, aperiodic signal

$$X(f) = \sum_{n=-\infty}^{\infty} x[n].e^{-j2\pi fn} \tag{24}$$

Let's consider the Discrete time Fourier Transform (DFT) of $x[n]$, denoted by $X[k]$ which is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \tag{25}$$

The DFT is inherently periodic in frequency domain with period $\frac{1}{N}$.

As N approaches infinity, the spacing between frequency samples in the frequency domain $\frac{1}{N}$ approaches zero. Therefore, the DFT $X[k]$ approaches the continuous Fourier Transform $X(f)$.

Since the DFT $X(f)$ is periodic in the frequency domain, its limiting form $X(f)$ is also periodic, continuous in frequency domain.

For example: Exponential Decay

$$x[t] = e^{-2n} \tag{26}$$
$$X(f) = \sum_{n=-\infty}^{\infty} e^{-2n}.e^{-j2\pi fn} \tag{27}$$

4. Discrete,periodic signal

$$X(f) = \sum_{n=-\infty}^{\infty} x[n].e^{-j2\pi fn} \tag{28}$$

The Discrete Fourier Transform(DFT) is periodic in frequency domain with period $\frac{1}{N}$

As N is finite , Therefore DFT has finite period $\frac{1}{N}$

$$X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi}{N}kn} \tag{29}$$

The DFT computation involves summing over a finite number of samples . Therefore, the resulting spectrum $X[k]$ is inherently discrete.

For example: Sinusoidal Signal

$$x[n] = 2sin(2\pi fn) \tag{30}$$

$$X(f) = \sum_{n=-\infty}^{\infty} 2sin(2\pi fn) \, .e^{-j2\pi fn} \tag{31}$$

(i) $\rightarrow(a)$, (ii) $\rightarrow(c)$, (iii) $\rightarrow(b)$, (iv) $\rightarrow(d)$