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Question:

In the table shown below, match the signal type with its spectral characteristics.

GATE 2023 EC

Signal Type	Spectral Characteristics
(i) Continuous, aperiodic	(a) Continuous, aperiodic
(ii) Continuous, periodic	(b) Continuous, periodic
(iii) Discrete, aperiodic	(c) Discrete, aperiodic
(iii) Discrete, periodic	(d) Discrete, periodic

TABLE 0

- 1) (i) \rightarrow (a) , (ii) \rightarrow (b) , (iii) \rightarrow (c) , (iv) \rightarrow (d)
- 2) (i) \rightarrow (a) , (ii) \rightarrow (c) , (iii) \rightarrow (b) , (iv) \rightarrow (d)
- 3) (i) \rightarrow (d) , (ii) \rightarrow (b) , (iii) \rightarrow (c) , (iv) \rightarrow (a)
- 4) (i) \rightarrow (a) , (ii) \rightarrow (c) , (iii) \rightarrow (d) , (iv) \rightarrow (b)

Solution:

Parameter	Description
$x(t)$	Continuous Time Signal
$x(f)$	Fourier Transform of a Signal
$x[n]$	Discrete Time Signal
$X[k]$	The amplitude and phase of the k^{th} frequency component of the input signal $x[n]$
$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$	Unit step signal in continuous time
$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$	Unit step signal Discrete time
$x(t) = e^{-2t} \cdot u(t)$	Exponential Decay Continuous time Signal
$x(t) = \sin(2\pi ft)$	Sine wave Continuous time Signal
$x[n] = e^n \cdot u[n]$	Exponential Decay Discrete time Signal
$x[n] = \sin(2\pi f_0 n) \cdot u[n]$	Sinusoidal Discrete time Signal

TABLE 4

INPUT PARAMETERS

1. Continuous, aperiodic signal

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt \quad (1)$$

Let's consider the limit as f approaches a certain frequency f_0 :

$$\lim_{\epsilon \rightarrow 0} X(f_0 + \epsilon) = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi(f_0 + \epsilon)t} dt \quad (2)$$

By continuity of $x(t)$ we can interchange the limit and the integral:

$$= \int_{-\infty}^{\infty} x(t) \cdot \lim_{\epsilon \rightarrow 0} e^{-j2\pi(f_0 + \epsilon)t} dt \quad (3)$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi f_0 t} dt \quad (4)$$

$$= X(f_0) \quad (5)$$

$$\lim_{f_0 - \epsilon} X(f_0) = X(f_0) \quad (6)$$

Therefore, $X(f)$ is continuous for all frequencies f .

Let's assume $X(f)$ is periodic with period T

$$X(f + T) = X(f) \quad (7)$$

Now applying inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi ft} df \quad (8)$$

$$= \int_{-\infty}^{\infty} X(f + T) \cdot e^{j2\pi(f+T)t} df \quad (9)$$

$$= \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi(f+T)t} df \quad (10)$$

$$= e^{j2\pi Tt} \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi ft} df \quad (11)$$

$e^{j2\pi Tt}$ is a periodic function of t , which contradicts that $x(t)$ is aperiodic, Therefore, $X(f)$ cannot be periodic, hence it must be aperiodic.

For Example: Exponential Decay

From Table 4

$$x(t) = e^{-2t} \cdot u(t) \quad (12)$$

$$X(f) = \int_{-\infty}^{\infty} e^{-2ft} u(t) \cdot e^{-j2\pi ft} dt \quad (13)$$

$$= \int_0^{\infty} e^{-(2+j2\pi f)t} dt \quad (14)$$

$$= \frac{1}{2(1 + j\pi f)} \quad (15)$$

$\Rightarrow X(f)$ is continuous and aperiodic

2. Continuous, periodic signal

$$x(t) = x(t + kT) \quad (16)$$

$$x(t) \leftrightarrow X(f) \quad (17)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt \quad (18)$$

$$X(f) = \int_{-\infty}^{\infty} x(t + kT) \cdot e^{-j2\pi ft} dt \quad (19)$$

$$\text{Let } \tau = t + kT \quad (20)$$

$$X(f) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j2\pi f\tau} \cdot e^{j2\pi fkT} d\tau \quad (21)$$

$e^{j2\pi fkT}$ is a Periodic function in frequency domain with period $\frac{1}{T}$ Hz

$$X(f) = e^{j2\pi fkT} \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j2\pi f\tau} d\tau \quad (22)$$

$$\Rightarrow X(f) \text{ is aperiodic} \quad (23)$$

$X(f)$ will be a sum of scaled copies of the Fourier transform of $x(t)$, each copy shifted by multiples of $\frac{1}{T}$ Hz. This results in a discrete frequency domain representation.

For example: Sine wave function

From Table 4

$$x(t) = \sin(2\pi ft) \quad (24)$$

$$\sin(f_0 t) = \frac{1}{2j} [e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}] \quad (25)$$

$$X(f) = \frac{1}{2j} [2\pi\delta(2\pi f - 2\pi f_0) - 2\pi\delta(2\pi f + 2\pi f_0)] \quad (26)$$

$$= j\pi [\delta(2\pi f + 2\pi f_0) - \delta(2\pi f - 2\pi f_0)] \quad (27)$$

$\Rightarrow X(f)$ is Discrete and Aperiodic Signal

3. Discrete, aperiodic signal

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi fn} \quad (28)$$

Let's consider the Discrete time Fourier Transform (DTFT) of $x[n]$, denoted by $X[k]$ which is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \quad (29)$$

The DFT is inherently periodic in frequency domain with period $\frac{1}{N}$.

As N approaches infinity, the spacing between frequency samples in the frequency domain $\frac{1}{N}$ approaches zero. Therefore, the DFT $X[k]$ approaches the continuous Fourier Transform $X(f)$.

Since the DFT $X(f)$ is periodic in the frequency domain, its limiting form $X(f)$ is also periodic, continuous in frequency domain.

For example: Exponential Decay

From Table 4

$$x[n] = e^n \cdot u[n] \quad (30)$$

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} e^n \cdot u[n] \cdot e^{-j2\pi f n} \quad (31)$$

$$= \sum_{n=0}^{\infty} e^{(1-j2\pi f)n} \quad (32)$$

$$= \frac{1}{1 - e^{(1-j2\pi f)}} \quad (33)$$

\Rightarrow Continuous and Periodic signal

4. Discrete,periodic signal

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f n} \quad (34)$$

The Discrete Fourier Transform(DFT) is periodic in frequency domain with period $\frac{1}{N}$

As N is finite , Therefore DFT has finite period $\frac{1}{N}$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} kn} \quad (35)$$

The DFT computation involves summing over a finite number of samples . Therefore, the resulting spectrum $X[k]$ is inherently discrete.

For example: Sinusoidal Signal

From Table 4

$$x[n] = \sin(2\pi f_0 n) \cdot u[n] \quad (36)$$

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} \sin(2\pi f_0 n) \cdot u[n] \cdot e^{-j2\pi f n} \quad (37)$$

$$= \sum_{n=0}^{\infty} \frac{e^{j2\pi f_0 n} - e^{-j2\pi f_0 n}}{2j} \cdot e^{-j2\pi f n} \quad (38)$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j(2\pi f_0 n - 2\pi f n)} - e^{-j(2\pi f_0 n + 2\pi f n)}) \quad (39)$$

$$= \frac{1}{2j} \left(\frac{1}{1 - e^{j(2\pi f_0 - 2\pi f)}} - \frac{1}{1 - e^{-j(2\pi f_0 + 2\pi f)}} \right) \quad (40)$$

\Rightarrow Discrete and Periodic

(i) \rightarrow (a) , (ii) \rightarrow (c) , (iii) \rightarrow (b) , (iv) \rightarrow (d)