HISTORY OF NAVIER - STOKES EQUATION
#MILLIONDOLLAREQUATION

A Project Report Submitted

by

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Introduction

Navier-Stokes Equations

The Navier-Stokes equations describe the motion of fluid substances:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f} \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

where:

- $\mathbf{v}$ is the velocity vector field,
- ρ is the fluid density,
- $\mathbf{f}$ represents external forces per unit volume, and
- ∇ denotes the gradient operator,
- $\frac{D\mathbf{v}}{Dt}$ denotes the material derivative of velocity

Equation (1) represents the momentum equation, while Equation (5) is the continuity equation.

Figure 1: **Motion of fluids**

In simple terms, the **Navier-Stokes equation tells you how a fluid behaves**.
A **fluid is something that you can assume to be a continuum** — i.e. not made of discrete 'lumps'. To this end, we can already see that the Navier-Stokes equations must be approximations, because fluids are made of atoms!
However, if you "zoom out" from a fluid, they look pretty much like a continuum. Can you feel the atoms in the air, or does it feel like a smooth continuum fluid?
The Navier-Stokes equations are, in essence, **just Newton's 2nd law written in a form that is applicable to continuum bodies, rather than discrete objects.** Honestly, there's nothing particularly more complex than that in there — at least from a physics perspective.

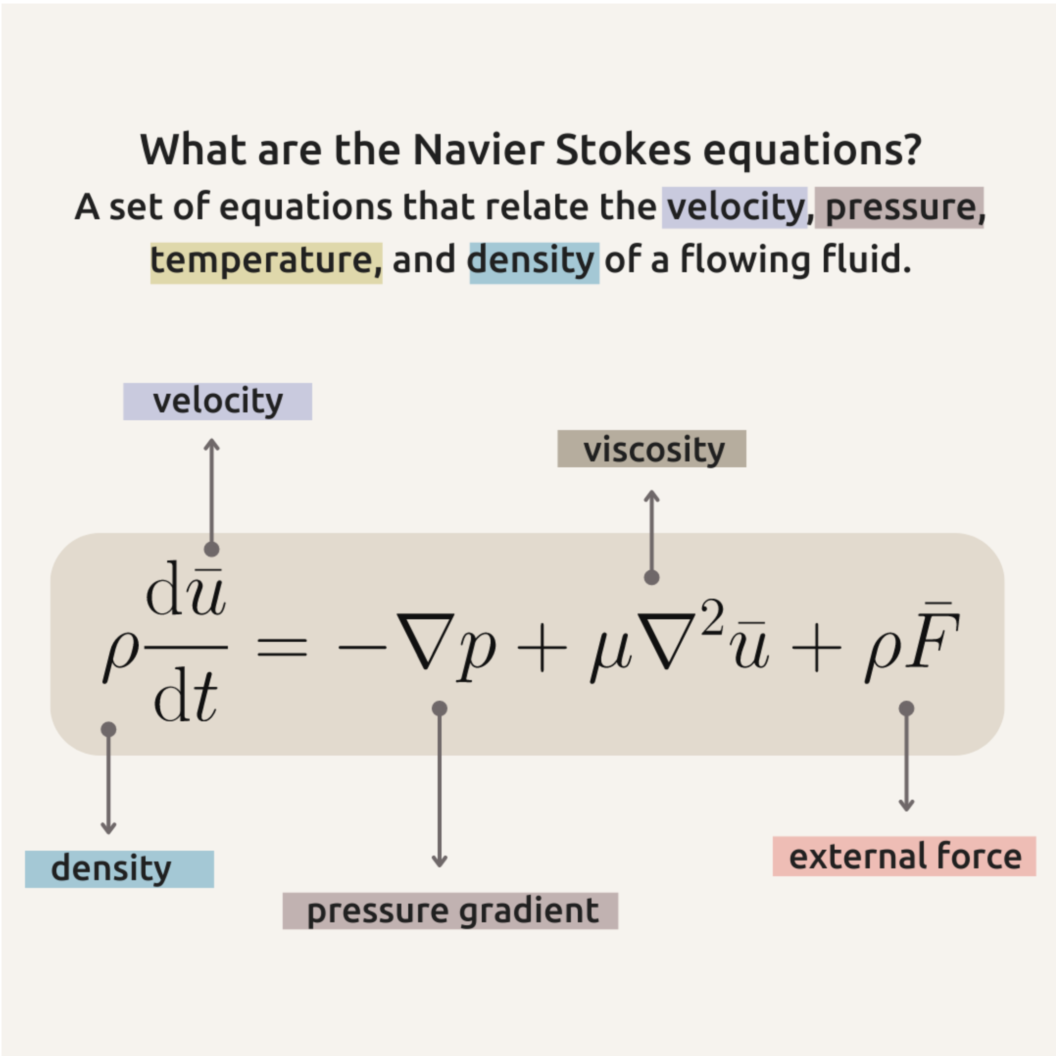


Figure 2: **Equation components**

Assumptions

There are three assumptions:

- **Newtonian Fluid**
- **Incompressible Fluid**
- **Isothermal**

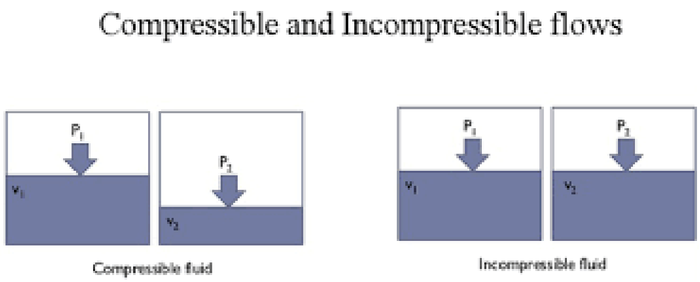


Figure 3: **Compressible and Incompressible Fluid**

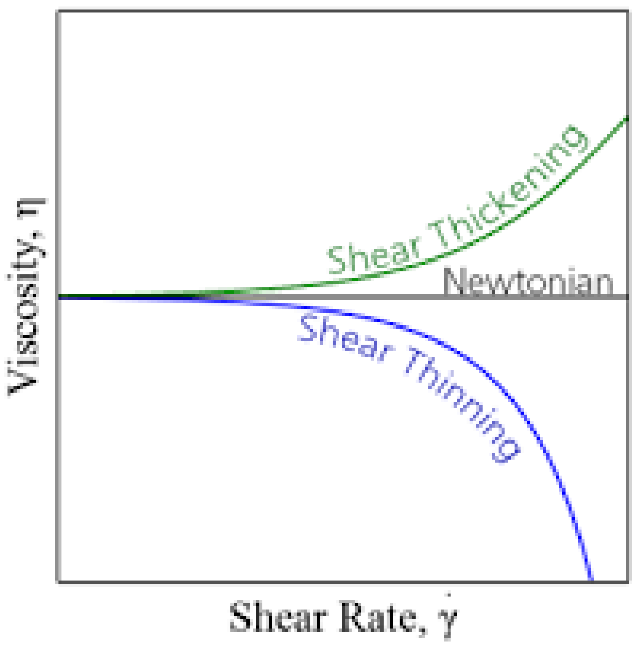


Figure 4: **Graph of Viscosity vs shear rate**

Divergence of vector

In physical terms, the divergence of a vector field is the extent to which the vector field flux behaves like a source at a given point. It is a local measure of its "outgoingness" – the extent to which there are more of the field vectors exiting from an infinitesimal region of space than entering it.

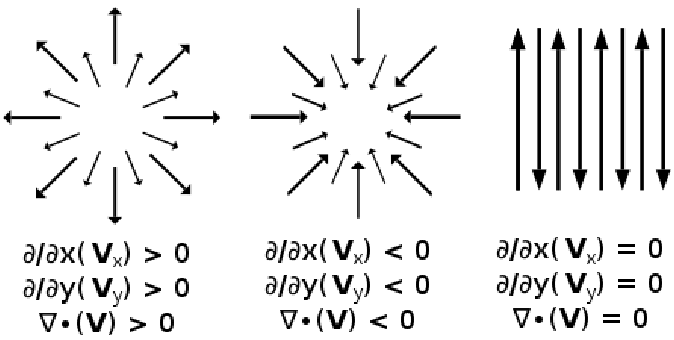


Figure 5: **Divergence of vector**

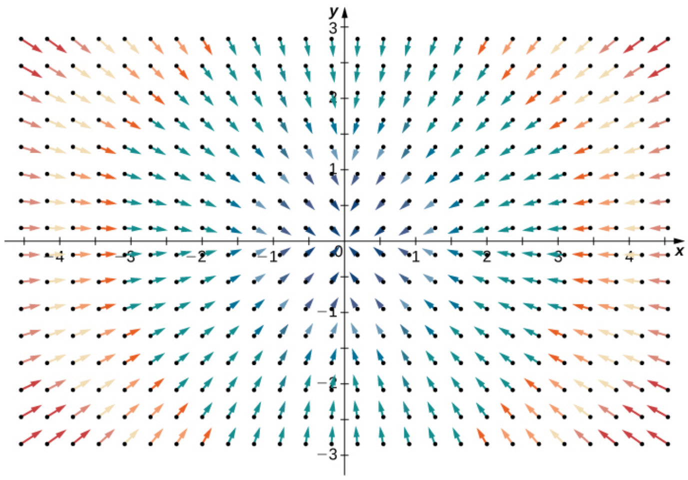


Figure 6: **Inflow**

As in the above case we can see that the **divergence of the vector is < 0**. This tell us that all the flow is towards a particular source.

$$\nabla \cdot \mathbf{v} < 0 \tag{3}$$

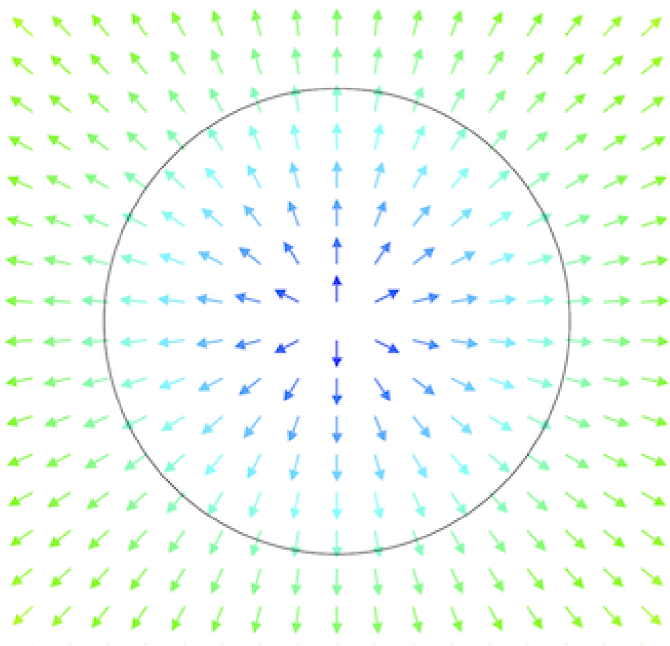


Figure 7: **Outflow**

As in the above case we can see that the **divergence of the vector is > 0**. This tell us that the flow is generating from a source.

By this we can make a point that, when fluid flows it cannot generate or cannot disappear at any source just as we saw above, therefore we can say that fluids always have divergence = 0.

$$\nabla \cdot \mathbf{v} = 0 \tag{4}$$

1st Equation

From here comes our 1st equation of Navier-Stokes

$$\nabla \cdot \mathbf{v} = 0 \tag{5}$$

which is nothing but Conservation of mass

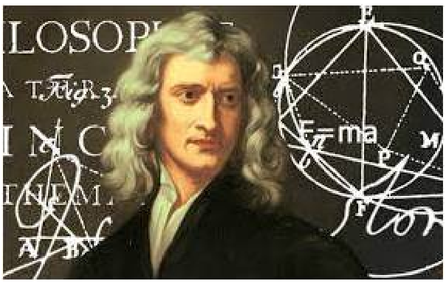
History

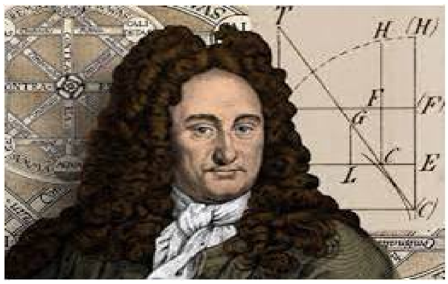
The Beginning:

: **Issac Newton(1642-1726)**

- Considered to have begun on 17th century.
- In 1687, Issac Newton with **“Mathematical Principles of Natural philosophy”**
- A **huge impact on physics** and most particularly mechanics
- This work stablished the foundations of the machanic as we know today, giving three laws.

Then with the help of Gottfried Leibniz, new invention came in!





Issac Newton

(1642-1726)

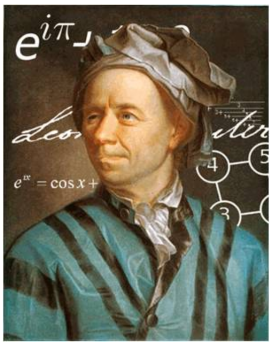
Gottfried Leibniz

(1646-1719)

- They set off a mathematical tools – **infinitesimal calculus**
- With the development of calculus many problems were solve in the frame of ideal fluid or inviscid fluid (fluid without viscosity)

History - In 1738

: **Daniel Bernoulli(1700-1782)**



: **Leonhard Euler(1707-1783)**

History - In 1758

- Bernoulli proved the **gradient of pressure is proportional to the acceleration of a fluid.**
- Euler derived **"Euler's equations"** which closely resemble the Navier-Stokes equations.
- In particular, it correspond to the **Navier–Stokes equations with zero viscosity and zero thermal conductivity.**
- The action of viscosity was not considered in these equation, providing unrealistic results.

: **Jean le Rond d'Alembert(1717-1783)**

History - In 19th century

- Proved that the **drag on a body of any shape moving through a fluid with no viscosity zero: "D'Alembert paradox"**
- Hence a mathematical fluid mechanics and engineering hydrodynamics were developed into separate branches.

Many researchers tried to add a friction term in Euler's equation in order to obtain realistic results

: **Claude Louis Navier(1785-1836)**

: **George Stokes(1819-1903)**

- Navier made the first known derivation of the Navier-stokes equations in order to represent friction based on a molecular mechanism.
- Stokes made the first mathematically rigorous derivation of the Navier-stokes equations

History - In 1934

Figure 8: **Ludwig Prandtl (1875-1953)**

- From the Euler's equation up today, different scientists wrote the Navier-Stoke equations in many form
- Ludwig Prandtl, in 1934 who wrote the **Navier-stoke equations in the form that is most widely used today.**

Prandtl form of Navier-Stokes Equation

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Applications

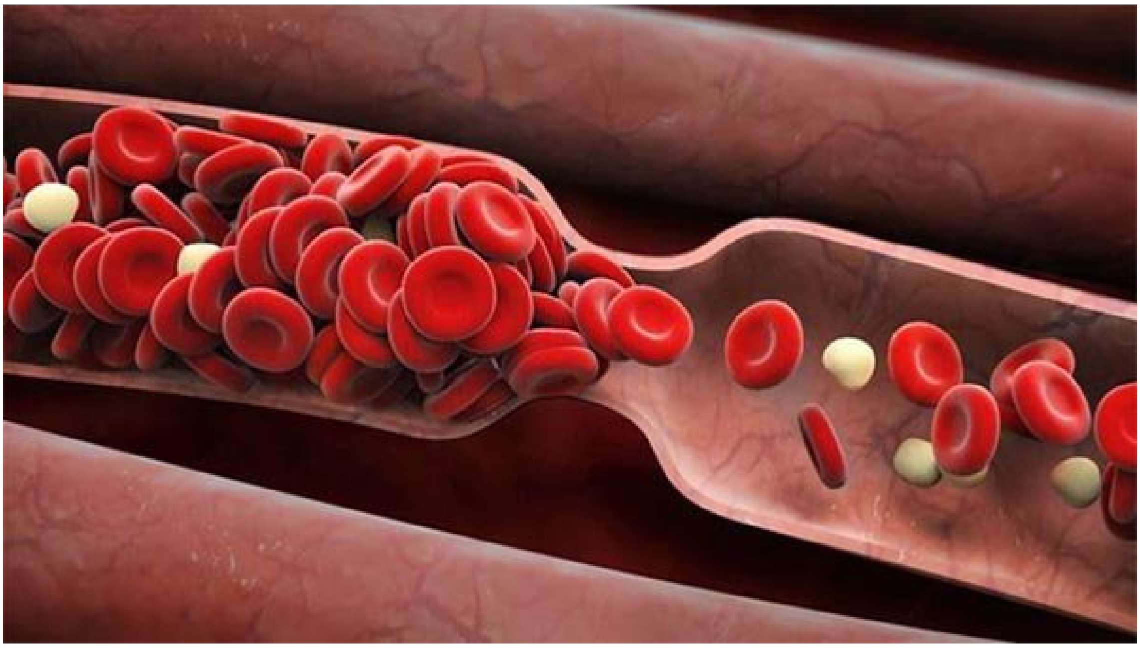


Figure 9: **Blood Flow**

Figure 10: **Aeroplane Design**

By **Reynolds averaging**, we can average out the velocity per square unit area, and then using Navier Stokes equation, it's easy to predict the weather or the notion of air, tornadoes, etc.

It's interesting to know that with such **basic concepts of conservation of momentum and mass**, we can solve complex world problems.

Figure 11: **Weather Forecast**

Figure 12: **Pollution track**

2nd Equation

- The **Navier-Stokes equations is just Newton's 2nd law** written in a form that is applicable to continuum bodies, rather than discrete objects.
- We know that Newton's second law as **F=ma**, but it is more correct to formulate it as:

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$$

- If we are dealing with a **fluid**, **we don't care about mass, we care about density** — which for now we assume is constant — i.e., the **fluid is incompressible.**
- taking the **continuum form of Newton's Second Law**, we get:

$$\mathbf{f} = \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

- **f is the "body force"** — the infinitesimal force on an infinitesimal chunk of the fluid.
- Since the velocity of the fluid is (presumably) both a function of space and of time — we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\mathbf{x},t) = \frac{\partial\mathbf{v}}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial\mathbf{v}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\mathbf{v}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\\ + \frac{\partial\mathbf{v}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\mathbf{x},t) = \frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}$$

- Using

$$\mathbf{f} = \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

- Gives us:

$$\frac{1}{\rho}\mathbf{f} = \frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}$$

- So now we need to think about what these forces could be.

- There are two types of forces that can act on a fluid:
  - **Internal forces**, due to interactions between the components
  - **External forces** such as gravity
- There are two major internal forces that we need to consider: **pressure forces, and the viscosity.**

$$\mathbf{f} = \mathbf{f}_{viscous} + \mathbf{f}_{ext} - \nabla P$$

- We plug this in:

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{\rho}(\mathbf{f}_{viscous} + \mathbf{f}_{ext} - \nabla P)$$

- The Navier-Stokes equation has one final step which is to write down the form of the viscous force
- The **viscous force** is the internal friction of all the particles rubbing up against each other, takes the form:

$$\frac{1}{\rho}\mathbf{f}_{viscous} = \nu \nabla^2 \mathbf{v}$$

- Just by applying Newton's 2nd law to a small chunk of a fluid, we have derived:

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{v} + \frac{1}{\rho}\mathbf{f}_{ext}$$

- This really is Navier-stokes equation!!
- It is just **"force is mass times acceleration"** souped up a bit, with pressure and viscosity added.
- The **mathematics behind** this, however, is where things get interesting. That's because, basically: this is an **awful equation to solve.**
- We can solve it exactly for certain specific setups, and we can make approximations to make it easier to study, but fundamentally. . . .this is just awful.
- It's so awful that **it hasn't even been proved that solutions always exist.**

Millenium Prize problems

1. Yang-Mills and Mass Gap
2. Riemann Hypothesis
3. P vs NP Problem
4. **Existence and smoothness of the Navier-Stokes equation.**
5. Hodge Conjecture
6. Poincare Conjecture
7. Birch and Swinnerton-Dyer Conjecture

- If you can **show that equation up there has smooth solutions** that exist for any given setup, you will win $1,000,000.
- They don't even ask you to find the solutions — it's simply showing that they exist that is the problem!

To **Grab the Million Dollar:**

- Prove or give a counter-example of the following statement:
  In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, then solve the Navier–Stokes equations.

Conclusion

- So the **Navier-Stokes equations are, physically, very simple**.
- All you are doing is **applying classical physics to little chunks of a fluid**, and asking how they respond when acted on with a force.
- The **difficulty is in finding valid mathematical solutions** to this.
- At this point, the vast majority of interest is in the actual Navier-Stokes equations for mathematical reasons, rather than physical ones.
- Whilst **physicists make use of them (mostly approximations in the correct regimes)**, it is mathematicians who will do the heavy lifting.
- Physicists are much, much more interested in some of the weird and wonderful behaviour that emerges from these very simple rules — a complex and fascinating subject in its own right.

Acknowledgement

- I want to thank **Dr. Chandi Sasmal** and **Mayank Chopra sir** for letting me work on this interesting topic. I learned a lot about this under-researched topic. Thanks a lot.....

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The End