HISTORY OF NAVIER - STOKES EQUATION #MillionDollarEquation

CH220:Seminar Lab

Kanishq Sahu-2022CHB1049

Indian Institute of Technology Ropar

July 10, 2024

Overview

Navier-Stokes Equations

Navier-Stokes Equations

The Navier-Stokes equations describe the motion of fluid substances:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f} \tag{1}$$

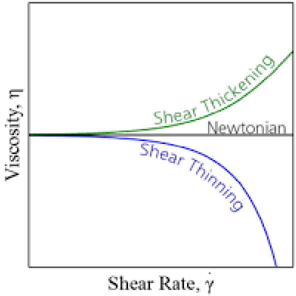
$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

where:

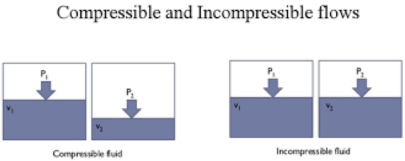
- $\mathbf{v}$ is the velocity vector field,
- ρ is the fluid density,
- $\mathbf{f}$ represents external forces per unit volume, and
- ∇ denotes the gradient operator,
- $\frac{D\mathbf{v}}{Dt}$ denotes the material derivative of velocity

Equation (1) represents the momentum equation, while Equation (3) is the continuity equation.

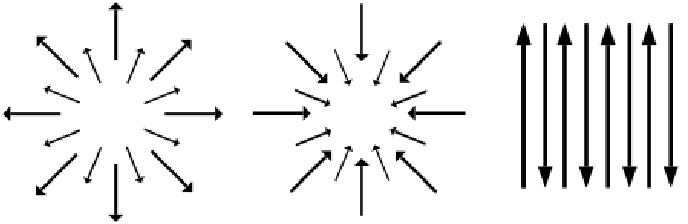
Assumptions



1. **Newtonian**
2. **Incompressible**
3. **Isothermal**



Divergence of vector

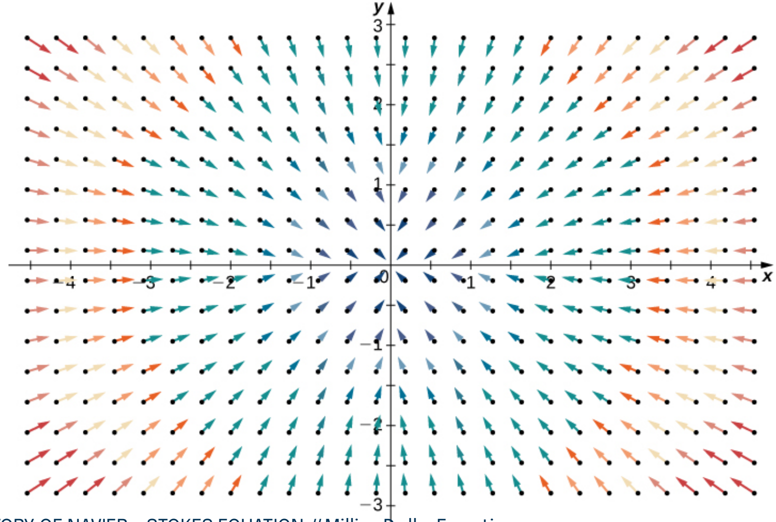


$$\partial/\partial x(\mathbf{V}_x) > 0 \qquad \partial/\partial x(\mathbf{V}_x) < 0 \qquad \partial/\partial x(\mathbf{V}_x) = 0$$

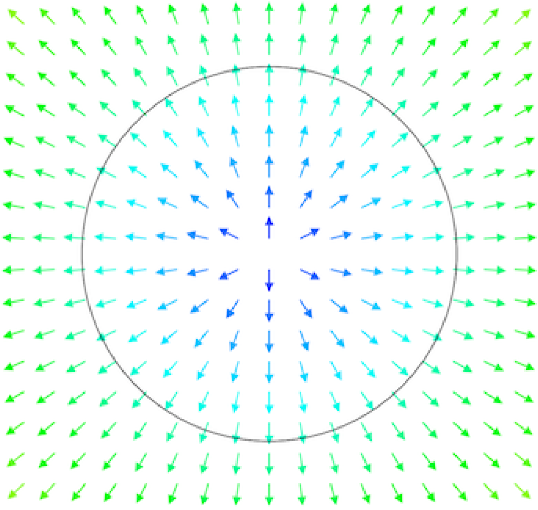
$$\partial/\partial y(\mathbf{V}_y) > 0 \qquad \partial/\partial y(\mathbf{V}_y) < 0 \qquad \partial/\partial y(\mathbf{V}_y) = 0$$

$$\nabla\bullet(\mathbf{V}) > 0 \qquad \nabla\bullet(\mathbf{V}) < 0 \qquad \nabla\bullet(\mathbf{V}) = 0$$

Divergence of vector



Divergence of vector



1st Equation

From here comes our 1st equation of Navier-Stokes

$$\nabla \cdot \mathbf{v} = 0 \qquad (3)$$

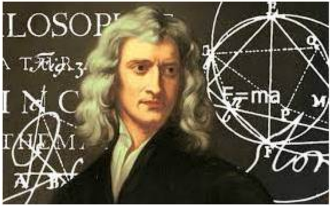
which is nothing but Conservation of mass

History - The Beginning

Issac Newton(1642-1726)

- Considered to have begun on 17th century.
- In 1687, Issac Newton with **“Mathematical Principles of Natural philosophy”**
- A **huge impact on physics** and most particularly mechanics
- This work stablished the foundations of the machanic as we know today, giving three laws.

History - The Beginning



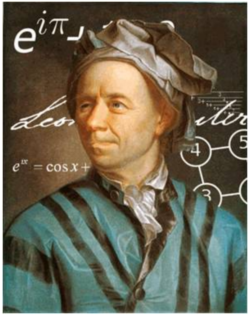
Issac Newton
(1642-1726)

Gottfried Leibniz
(1646-1719)

- They set off a mathematical tools - **infinitesimal calculus**
- With the development of calculus many problems were solve in the frame of ideal fluid or inviscid fluid (fluid without viscosity)

History - In 1738

Daniel Bernoulli (1700-1782)



Leonhard Euler (1707-1783)

- Bernoulli proved the **gradient of pressure is proportional to the acceleration of a fluid.**
- Euler derived **"Euler's equations"** which closely resemble the Navier-Stokes equations.
- In particular, it correspond to the **Navier-Stokes equations with zero viscosity and zero thermal conductivity.**
- The action of viscosity was not considered in these

History – In 1758

Jean le Rond d’Alembert (1717–1783)

- Proved that the **drag on a body of any shape moving through a fluid with no viscosity zero: “D’Alembert paradox”**
- Hence a mathematical fluid mechanics and engineering hydrodynamics were developed into separate branches.

History - In 19th century

Many research tried to add a friction term in Euler's equation in order to obtain realistic results

Claude Louis Navier (1785-1836)

George Stokes (1819-1903)

- Navier made the first known derivation of the Navier-stokes equations in order to represent friction based on a molecular mechanism.
- Stokes made the first mathematically rigorous derivation of the Navier-stokes equations

History - In 1934

figureLudwig Prandtl (1875-1953)

- From the Euler's equation up today, different scientists wrote the Navier-Stoke equations in many form
- Ludwig Prandtl, in 1934 who wrote the **Navier-stoke equations in the form that is most widely used today.**

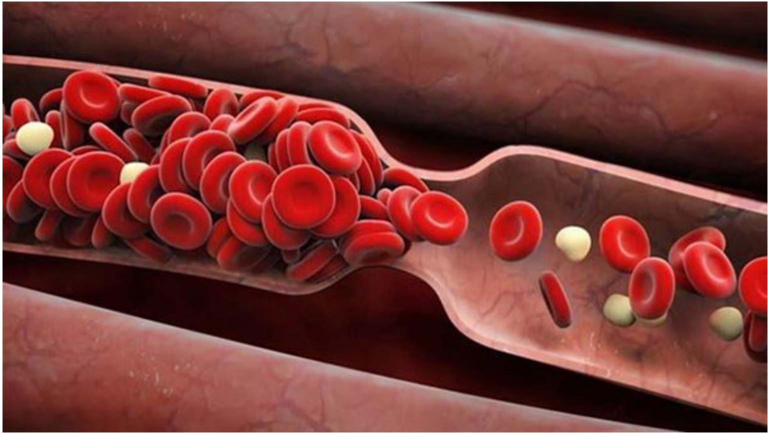
Prandtl form of Navier-Stokes Equation

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Applications



Blood flow

Applications

Aeroplane Design

Applications

Applications

Pollution track

2nd equation

- The **Navier-Stokes equations is just Newton's 2nd law** written in a form that is applicable to continuum bodies, rather than discrete objects.
- We know that Newton's second law as **F=ma**, but it is more correct to formulate it as:

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$$

- If we are dealing with a **fluid**, **we don't care about mass, we care about density** – which for now we assume is constant – i.e., the **fluid is incompressible.**
- taking the **continuum form of Newton's Second Law**, we get:

$$\mathbf{f} = \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

2nd equation

- **f is the "body force"** – the infinitesimal force on an infinitesimal chunk of the fluid.
- Since the velocity of the fluid is (presumably) both a function of space and of time – we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\mathbf{x},t) = \frac{\partial \mathbf{v}}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial \mathbf{v}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \mathbf{v}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \mathbf{v}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\mathbf{x},t) = \frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}$$

2nd equation

- Using

$$\mathbf{f} = \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

- Gives us:

$$\frac{1}{\rho}\mathbf{f} = \frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$$

- So now we need to think about what these forces could be.

2nd equation

- There are two types of forces that can act on a fluid:
  - **Internal forces**, due to interactions between the components
  - **External forces** such as gravity
- There are two major internal forces that we need to consider: **pressure forces, and the viscosity.**

$$\mathbf{f} = \mathbf{f}_{viscous} + \mathbf{f}_{ext} - \nabla P$$

- We plug this in:

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{\rho}(\mathbf{f}_{viscous} + \mathbf{f}_{ext} - \nabla P)$$

2nd equation

- The Navier-Stokes equation has one final step which is to write down the form of the viscous force
- The **viscous force** is the internal friction of all the particles rubbing up against each other, takes the form:

$$\frac{1}{\rho}\mathbf{f}_{viscous} = \nu\nabla^2\mathbf{v}$$

- Just by applying Newton's 2nd law to a small chunk of a fluid, we have derived:

$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{v} + \frac{1}{\rho}\mathbf{f}_{ext}$$

2nd equation

- This really is Navier-stokes equation!!

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{v} + \frac{1}{\rho}\mathbf{f}_{ext}$$

- It is just **"force is mass times acceleration"** souped up a bit, with pressure and viscosity added.
- The **mathematics behind** this, however, is where things get interesting. That's because, basically: this is an **awful equation to solve.**
- We can solve it exactly for certain specific setups, and we can make approximations to make it easier to study, but fundamentally....this is just awful.
- It's so awful that **it hasn't even been proved that solutions always exist.**

Millenium Prize problems

1. Yang-Mills and Mass Gap
2. Riemann Hypothesis
3. P vs NP Problem
4. **Existence and smoothness of the Navier-Stokes equation.**
5. Hodge Conjecture
6. Poincare Conjecture
7. Birch and Swinnerton-Dyer Conjecture

- If you can **show that equation up there has smooth solutions** that exist for any given setup, you will win $1,000,000.
- They don't even ask you to find the solutions – it's simply showing that they exist that is the problem!

Conclusion

- So the **Navier-Stokes equations are, physically, very simple**.
- All you are doing is **applying classical physics to little chunks of a fluid**, and asking how they respond when acted on with a force.
- The **difficulty is in finding valid mathematical solutions** to this.
- At this point, the vast majority of interest is in the actual Navier-Stokes equations for mathematical reasons, rather than physical ones.
- Whilst **physicists make use of them (mostly approximations in the correct regimes)**, it is mathematicians who will do the heavy lifting.
- Physicists are much, much more interested in some of the weird and wonderful behaviour that emerges from these very simple rules – a complex and fascinating subject in its own right.

Acknowledgement

- I want to thank **Dr. Chandi Sasmal** for letting me present this interesting topic. I learned a lot about this under-researched topic. Thanks a lot.....

References

- Academia.edu
- Quora
- Wikipedia
- Google photos
- Youtube

The End

Million Dollars

- The exact statement is :
- Prove or give a counter-example of the following statement:
- In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, then solve the Navier-Stokes equations.