Infectious Disease Modeling — Modules: D2, D3

In this problem set, we examine a simple model for the spread of a contagious disease. Our model makes the following assumptions:

- the disease can only be transmitted by human-to-human contact in a single specific population;
- the size of the population remains constant in time (that is, no births or deaths occur and the population is cut off from the outside world);
- the disease confers no immunity after someone has recovered from infection.

Of course, these assumptions limit the applicability of our model to real-life epidemics, but they provide a good foundation upon which more complex and accurate models can be based. Let time be denoted by $t \in [0, \infty)$, measured in days. From the above assumptions, the population can be divided into two groups:

- $S(t)$ = fraction of the population that is susceptible to infection but not actually infected at time t, and
- $I(t)$ = fraction of the population that is infected at time t.

Our model requires two parameters with values in $(0, 1)$: the infection rate a and the recovery rate b, both with units of $(\text{days})^{-1}$ (in practice, these numbers would need to be estimated using patient data). In words, we can think of $b\Delta t$ as the proportion of infected individuals recovering and returning to the susceptible group over a small time interval of length Δt. Similarly, we can think of $a\Delta t$ in the following way: out of all those susceptibles who came into contact with an infected individual during the small time interval, $a\Delta t$ is the proportion who actually end up getting infected as a result of that contact.

1. Explain why the following identity holds for all t:

$$S(t) + I(t) = 1 \tag{1}$$

   Your answer must be contained in one or two sentences.

2. In this problem, we derive an ordinary differential equation that can be solved (or simulated) to find $I(t)$. For now, suppose that at every instant in time everyone in the population is in contact with one and only one other person. Every Δt days, the contacts change up. So the proportion of people whose contact is an infected person during a time interval is approximately $I(t)$ and the proportion of people whose contact is a susceptible person during a time interval is approximately $S(t)$.

   (a) Let $\Delta I_{\text{recovered}}$ denote the change in $I(t)$ due only to recoveries occurring in the time interval $[t, t + \Delta t]$. Argue why we have

$$\Delta I_{\text{recovered}} \approx -b\Delta t I(t) \tag{2}$$

   (b) Let $\Delta I_{\text{infected}}$ denote the change in $I(t)$ due only to infections occurring in the time interval $[t, t + \Delta t]$. Argue why we have

$$\Delta I_{\text{infected}} \approx a\Delta t I(t) S(t) \tag{3}$$

Adding (2) to (3) and dividing both sides by Δt,

$$\frac{\Delta I}{\Delta t} = \frac{\Delta I_{\text{infected}}}{\Delta t} + \frac{\Delta I_{\text{recovered}}}{\Delta t} \approx aIS - bI \tag{4}$$

and by taking $\lim\limits_{\Delta t \to 0}$ of both sides of (4),

$$\frac{dI}{dt} = aIS - bI. \tag{5}$$

Using (1) to eliminate S, we can convert (5) to the following differential equation for $I(t)$ alone.

$$\frac{dI}{dt} = (a - b)I - aI^2 \tag{6}$$

3. Recall that an equilibrium solution of (6) is a solution $I_0(t)$ of (6) that is constant in time. Give a condition on the ratio $R_0 \doteq a/b$ that is necessary for a nonzero equilibrium solution to exist. Write down the equilibrium solution in terms of R_0.

4. How do you think solutions of (6) will behave as $t \longrightarrow \infty$ when a and b are chosen so that $R_0 \in (0, 1)$? Answer in four sentences or less.

	Clarity and conciseness of written exposition; consideration of audience	**Good movement between English & Mathematics**	**Mathematical Thinking**
Below expectations (0)	Writing is confusing, or shows signs of carelessness through high level of error; does not demonstrate application of engineering communication; only understandable to the writer.	Math is presented without context; it is difficult to reconstruct the meaning of equations.	No demonstration of any significant level of mathematical analysis at an appropriate level. No serious attempt to apply mathematics.
Shows potential (1)	Writing requires some effort to understand, or has some errors rarely affecting understanding; some evidence of communication principles but may be awkward or simplistic; offers sufficient information and demonstrates mostly logical development of ideas; generally directed to the instructor or TA.	While the writing is clear, the math is kept completely separate from it; requires some effort to understand how the math and writing are connected.	A serious attempt is made to apply appropriate math and explain mathematical reasoning, regardless of completeness or correctness.
Meets expectations (2)	Writing shows appropriate selection and balance of text and image (if necessary); demonstrates clarity at the paragraph, sentence, and word choice levels, with minimal error; writing applies principles in ways that enhance the reading experience; generally directed towards a fellow student who understands the math but not this particular instance.	The interaction of the text with the math is adequate; any transitions between math and text are appropriately placed and make sense; any equations and variables are introduced and explained in an organized manner; any solutions are interpreted in a real-world context where appropriate; no guesswork is needed by the reader.	The response shows a good ability to analyze a situation using math; relevant topics are applied with proper notation; any solutions are critically analyzed for sensibility; any obviously incorrect solutions are remarked upon and followed up with a sketch of an alternative idea or plan.
Above expectations; bonus (3)	The writing demonstrates professional polish through word choice, sentence structure choices, clarity, concision, proofreading, etc. This solution is presentable as a textbook example or exposition.	The math is merged very well with the writing; any equations and variables are embedded, clear, and concise; any transitions between text and mathematics aid understanding; any solutions are contextualized and explained in a professional manner.	The analysis is excellent, creating a large amount of insight into the prompt and demonstrates mastery of the topic at or above the course level.