MAT186 Writing Task 5:

Cooling Fins Modules: E1-E4

For a body with total surface area A_s and uniform surface temperature T_s , the instantaneous rate of convective heat transfer out of the body is modeled by Newton's law of cooling:

$$Q_{conv} = hA_s (T_s - T_\varnothing)$$

where h is a constant and $T_\varnothing$ is the ambient temperature (temperature of the room). The units are as follows:

Q_{conv} : Watts
h : Watts/(meters squared degrees Kelvin)
A_s : meters squared
$(T_s - T_\varnothing)$: degrees Kelvin

What if we extended this model a bit? Say we constructed a cooling fin shaped as a solid of revolution. Let our fin be the solid created when the area between a function r(x) and the x axis between x = 0 to x = L is revolved around the x-axis. All distances are in meters. As a simplifying assumption, suppose the surface temperature of the fin $T_s = T_s(x)$ depends only on the horizontal distance from the origin (note this is no longer a uniform surface temperature). We want to find an expression for the instantaneous rate of convective heat transfer out of our entire cooling fin, Q_{conv}.

1. Explain a plan to break this problem up into parts, providing the notation you will use.

= The problem involves 4 main variables and constants: Surface Area(A_s), Surface Temperature(T_s), Ambient Temperature($T_\varnothing$), and Constant "h". In addition to this, we can introduce more variables that constitute the Surface Area and Surface Temperature. Let:

f(x) = Fin's surface function representing the height of fin for each horizontal distance "x".

Arc Length $(\Delta s) = \sqrt{1 + (f'(x_{n-1}))^2}\Delta x$,where $\Delta x = Horizontal\ Width\ of\ ribbon\ strip.$

$T_s(x)$ = Surface Temperature function that depends on the horizontal distance "x".

We can break this problem into parts by taking the half of the total vertical height of the fin f(x) on the interval from x=0 to x=L at which the fin is rotated about the x-axis. Then, make a perpendicular cut to the horizontal x-axis. We will divide the fin into "n" equal slices $[x_0, x_1],\ [x_1, x_2],\ .\ .\ .\ , [x_{n-1}, x_n]$ of surface horizontal width equivalent to the arc length$(\Delta s) = \sqrt{1 + (f'(x_{n-1}))^2}\Delta x$ of the fin from $[x_{k-1}, x_k]$.

2. Explain how you are approximating Q_{conv} on one of your individual parts. Be specific in explaining what is being approximated by what, and any choices you are making. Units are an important part of this explanation.

= Approximating Q_{conv} on one individual part is equivalent to estimating only the surface area of the fin since the component "h" and ambient temperature($T_Ø$) are both constants, and the surface temperature(T_s) is a function which gives the surface temp. according to varying horizontal distance. All these 3 elements do not have to be approximated. On the other hand, the surface area of the fin function must be approximated, since the fin is a 3D object that is subjected to rotation about the x-axis. To obtain an accurate expression for the surface area of a fin, an approximation has to be made through riemann sum.

We can approximate the Surface Area of the fin between horizontal width x_{k-1} and width x_k by selecting a representative width x_k^* from within the interval $[x_{k-1}, x_k]$. Assume that the slice is a "ribbon" with width(meter) Δs and a uniform circular circumference $2\pi r$(meter). The width is approximated by the "Arc Length" of the ribbon and the circumference is approximated by radius of the ribbon. Since the fin is oriented horizontally and revolves about the x-axis, the radius is equal to the vertical distance from y = 0 to y = f(x). Therefore, the circumference can be represented by $2\pi f(x_i)$. By making this simplification, we obtain that:

$Surface\ Area\ of\ a\ Ribbon\ (\Delta A_s)\ =\ Circumference\ \times\ Width$, which is equivalent to:

$Surface\ Area\ of\ a\ Ribbon\ (\Delta A_s) = 2\pi f(x_i)\Delta s = 2\pi f(x_i) \times \sqrt{1 + (f'(x_i))^2}\Delta x.$

We can perform a check through the units:

Since;$Circumference\ = 2\pi f(x_i)$ & $Width\ =\ \Delta s\ = \sqrt{1 + (f'(x_i))^2}\Delta x\ are\ both\ measured\ in\ meter,$

Surface Area = $Circumference\ \times\ Width$ is measured in "meter squared(m^2)".

Since Newton's Law of Cooling is given by $Q_{conv} = hA_s(T_s - T_Ø)$, we can obtain an expression for Q_{conv} on one of the individual parts as:

$\Delta Q_{conv} = h\Delta A_s(T_s - T_Ø)\ =\ h \times 2\pi f(x_i)\Delta s \times (T_s(x) - T_Ø)$, which can be further derived as:

$\Delta Q_{conv} = h \times 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}\Delta x \times (T_s(x) - T_Ø)$

3. Add up your approximations to come up with an expression for the approximate value of Q_{conv} for the whole fin.

= By adding our approximations of the surface area(A_s) of all ribbon slices comprising the full horizontal width of the fin, we can obtain an approximation of the full Surface Area of the fin. Therefore:

Total Surface Area (As) = $\sum_{i=0}^{n-1} 2\pi f(x_i)\Delta s$ = $\sum_{i=0}^{n-1} 2\pi f(x_i) \times \sqrt{1 + (f'(x_i))^2}\Delta x$.

Since Newton's Law of Cooling is given by: $Q_{conv} = hA_s (T_s - T_Ø)$, we have that:

$$Q_{conv} = \sum_{i=0}^{n-1} h \times 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}\Delta x \times (T_s(x) - T_Ø)$$

4. 4. Use your answer to the previous part to explain how to obtain a definite integral formula for the Q_{conv} for the whole fin; use appropriate mathematical notation to relate the two.

= By Definition of Definite integral, the definite integral for Q_{conv} can be obtained by taking the limit of the Riemann Sum as "n" approaches infinity. By doing so, the approximation becomes more accurate as the number of subintervals representing the width gets increasingly large. Hence, we obtain that the expression for the definite integral is:

$$Q_{conv} = \lim_{n \to \infty} [\sum_{i=0}^{n-1} h \times 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}\Delta x \times (T_s(x) - T_Ø)]$$

$$= \int_0^L h \times 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}dx \times (T_s(x) - T_Ø)$$. Rearranging, we get that:

$$Q_{conv} = \int_0^L h \times (T_s(x) - T_Ø) \times 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}dx$$

Your answer to this problem will involve the functions r(x) and T_s (x) as well as the constants h, TØ, and L. You should not assume r(x) looks as pictured below.