Approximation and Limits Modules: A4, B1

1. Consider the following Fermi estimation problem.

> "If in 2014, you had downloaded all the data on the entire internet, burned the data onto CDs, and stacked the CDs on top of each other, how tall would the stack be?"

Find upper and lower bounds for what the height of the stack might be. State any assumptions about data you use in your calculations.

In order to find the upper and lower bound, we must, first, make the following assumptions:

1. $17\, Zettabyte \;\leq\; Internet\ Data\ (Byte) \;\leq\; 18\, Zettabyte$: We will be making an estimation based on the year 2018. The assumption has been made according to the market intelligence company, "Global Datasphere", in 2018 [1].

2. $700\, MB \;\leq\; CD\ ROM\ Capacity\ (Byte) \;\leq\; 737\, MB$: The discrepancies in CD-ROM Capacity are due to the space used for error corrections data [2].

3. $1.1\, mm \;\leq\; Thickness\ of\ CD\ ROM\ (mm) \;\leq\; 1.3\, mm$: Assume that there is an uncertainty in the production of CD width of ±0.1 mm.

By Dimensional Analysis, we can obtain 2 Formulas to calculate for the Height of CD Stack for the Upper and Lower Bound:

1.) Upper Bound:

EQ1.) Max Height of CD Stack (mm) =

$$\frac{Internet\ Data\ (Byte)}{1} \times \frac{1\ (CD)}{CD\ Capacity\ (Byte)} \times \frac{1.3\ (mm)}{\#CD\ in\ 1.3mm\ (CD)}$$

2.) Lower Bound:

EQ2.) Min. Height of CD Stack (mm) =

$$\frac{Internet\ Data\ (Byte)}{1} \times \frac{1\ (CD)}{CD\ Capacity\ (Byte)} \times \frac{1.1\ (mm)}{\#CD\ in\ 1.1mm\ (CD)}$$

However, it is important to note that, to construct an Upper Bound, the value at the Numerator must be the largest value, while the value at the Denominator must be the smallest value. This means that we must use the upper bound of Internet Data of 18 Zettabyte and 1.3 mm as the largest thickness value of a CD ROM as represented by EQ1.

On the other hand, for the Lower Bound, the value in the numerator must be lowest and the value in the denominator must be greatest. Therefore, the lower bound of Internet Data of 17 Zettabyte will be used and the lowest value of thickness of a CD ROM of 1.1 mm is used, as shown in EQ2. This will lead to a resulting calculation of:

Upper Bound for CD Height = $\frac{(18E21)\ (Byte)}{1} \times \frac{1\ (CD)}{700E06\ (Byte)} \times \frac{1.3\ (mm)}{\#CD\ in\ 1.3mm\ (CD)}$ =

3.34E13 mm

Lower Bound for CD Height= $\frac{(17E21)\,(Byte)}{1} \times \frac{1\,(CD)}{737E06\,(Byte)} \times \frac{1.1\,(mm)}{\#CD\ in\ 1.1mm\,(CD)}$=

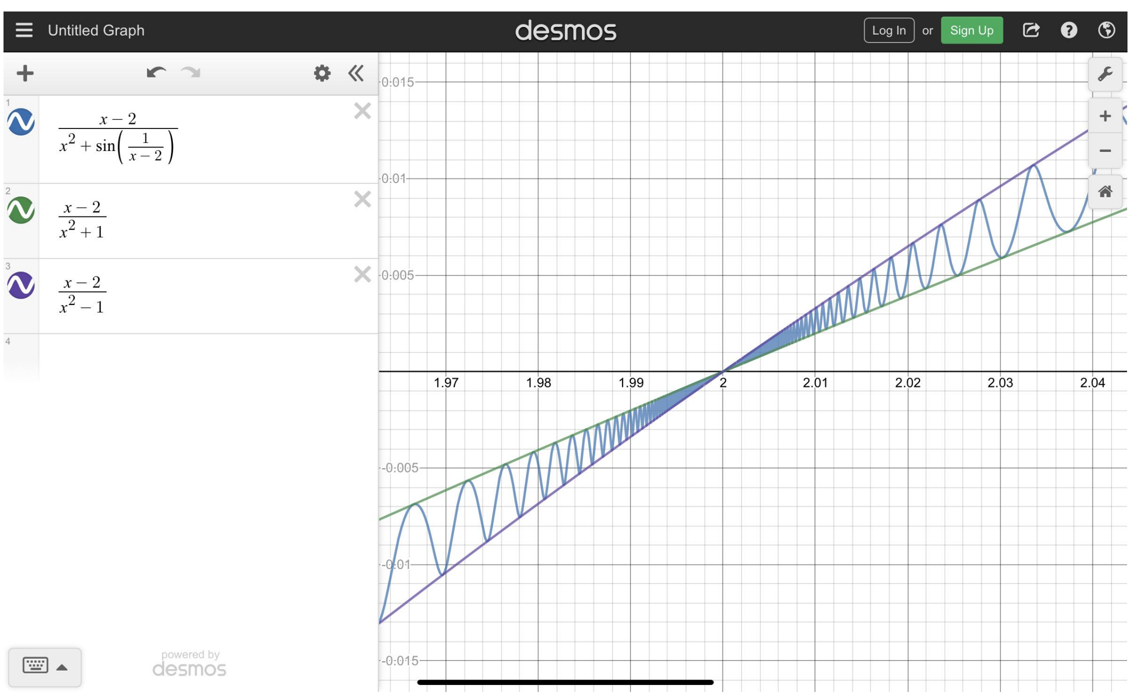
2.537E13 mm

Hence; $2.537E13\ mm \ \leq\ Actual\ Value\ \leq\ 3.34E13\ mm$

2. Find the following limit, fully justifying your reasoning using mathematical principles we have covered. Remember to use complete sentences as part of your response.

$$\lim_{x \to 2}\left(\frac{x-2}{x^2+\sin\left(\frac{1}{x-2}\right)}\right)$$

Since $sin(\frac{1}{x-2})$is a periodic function that is continuous on the range [-1,1], the function does converge to a specific point. Therefore, to compute $\lim_{x \to 2}\ (\frac{x-2}{x^2+sin(\frac{1}{x-2})})$, we need to apply the “Squeeze Theorem” to $sin(\frac{1}{x-2})$. By definition of the squeeze theorem, if $f(x)\ \leq\ g(x)\ \leq\ h(x)$for all value of x and at some point where x = a, where $\lim_{x \to a}\ f(a)\ =\ \lim_{x \to a}\ h(a)\ =\ L$, then we can prove that g(x) converges to a value “L” that as x approaches x=a [3]. The function g(x) in this case is $(\frac{x-2}{x^2+sin(\frac{1}{x-2})})$, which we know for certain that it is periodic due to the presence of the sine function. Since the $sin(\frac{1}{x-2})$is a periodic function over the range [-1,1] and is on the denominator of the function, we can determine the upper and lower limit of the function. The upper limit of the function would be $\lim_{x \to 2}\ (\frac{x-2}{x^2-1})$and the lower limit of the function would be $\lim_{x \to 2}\ (\frac{x-2}{x^2+1})$, since a smaller value in denominator leads to larger number and larger value in denominator leads to smaller number. Hence, we obtain that:



$\lim_{x \to 2} \left(\frac{x-2}{x^2+1}\right) \leq \lim_{x \to 2} \left(\frac{x-2}{x^2+sin(\frac{1}{x-2})}\right) \leq \lim_{x \to 2} \left(\frac{x-2}{x^2-1}\right)$. By computing the upper and lower limit, we find that: $\lim_{x \to 2} \left(\frac{x-2}{x^2+1}\right) = \lim_{x \to 2} \left(\frac{x-2}{x^2-1}\right) = 0$. Since, both the lower and upper limit as x approaches 2 is "zero", by Squeeze Theorem, we find that $\lim_{x \to 2} \left(\frac{x-2}{x^2+sin(\frac{1}{x-2})}\right) = 0$.

Figure 1: Graph showing the Lower and Upper Limit Function converging to 0 as x approaches x=2.

3. In three sentences or less, reflect on the similarities and differences in the math used to solve these problems.

There are several similarities in the math used to solve these 2 questions. First, both problems rely on the use of maximum and minimum boundary as a mathematical approach to solve the problem. Second, both math used will provide only an approximated value of the answer, but will NOT give the real, exact answer. On the other hand, the difference is that the first problem uses the construction of upper and lower bound to construct a range of possible real answers. However, the second problem uses the upper and lower bound of the periodic function (sine) with the idea of limit through the method of Squeeze theorem to find a limit as x approaches x=2. The second problem provides a much closer estimate to the answer, which is the real value at x=2.

Reference Lists:

[1] "How Much Data Is There In the World?," *Bernard Marr*. [Online]. Available: https://www.bernardmarr.com/default.asp?contentID=1846.

[2] "CD-ROM," *Wikipedia*, 06-Oct-2020. [Online]. Available: https://en.wikipedia.org/wiki/CD-ROM

[3] " The Squeeze Theorem," *World Web Math: The Squeeze Theorem*. [Online]. Available: http://web.mit.edu/wwmath/calculus/limits/squeeze.html.