UNIT - III

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$F(t)$

① $F(t)$ $[a,b]$

$[x_0, x_1]$ $[x_1, x_2]$ --- $[x_{n-1}, x_n]$

② Exponential

$\lim_{t \to \infty} e^{-st} f(t)\,dt = M$

Laplace transformation is a transformation applied on fun-ctions of time domain $f(t)$ to be converted to fun of frequency domain $f(s)$

Applications

In solutions of ordinary dif eqns with boundary conditions.

⇒ let $f(t)$ is a fun defined for all possible values of t then laplace transformation $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

Satisfies following conditions

(i) $F(t)$ must be piece wise continous or section-ally continous. ie $f(t)$ must be continous in each & every sub interval

(ii) Function $f(t)$ must be of exponential order

i.e $\lim_{t \to \infty} e^{-st} f(t)\,dt = M$ (fixed quantity)

(1) Find out $\mathcal{L}\{1\}$

$$L\{1\} = \int_0^\infty e^{-st}(1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\cdot\int v\,dx\right\}dx$

$$= \frac{e^{-s\infty}}{-s} - \frac{e^{-s(0)}}{-s}$$

$\boxed{\begin{matrix} e^{-\infty}=0 \\ e^{0}=1 \\ e^{\infty}=\infty \end{matrix}}$

$$= 0 + \frac{1}{s} = \frac{1}{s}$$

(2) $\mathcal{L}\{t\}$

$$= \int_0^\infty e^{-st}t\,dt$$

$$= \left[t\cdot\int e^{-st} - \int\left((t)'\int e^{-st}dt\right)dt\right]_0^\infty$$

$$= \left[t\cdot\frac{e^{-st}}{-s} - \int\frac{e^{-st}}{-s}dt\right]_0^\infty$$

$$= \left[t\,\frac{e^{-st}}{-s} + \frac{e^{-st}}{-s^2}\right]_0^\infty$$

$$= \frac{te^{-\infty}}{-s} + \frac{e^{-s\infty}}{-s^2} - 0 - \frac{e^0}{-s^2} = \frac{1}{s^2}$$

$\Rightarrow \cancel{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$$L\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{s-a}$$

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$$\Rightarrow L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{s+a}$$

$$\Rightarrow L\{\cosh at\} = L\left\{ \frac{e^{at} + e^{-at}}{2} \right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\} \right]$$

$$= \frac{1}{2} \times \frac{1}{s-a} + \frac{1}{2} \times \frac{1}{s+a}$$

$$= \frac{1}{2} \times \frac{s+a+s-a}{(s^2-a^2)}$$

$$= \frac{2s}{2(s^2-a^2)} = \frac{s}{s^2-a^2}$$

$$\Rightarrow L\{\sinh at\} = L\left\{ \frac{e^{at} - e^{-at}}{2} \right\}$$

$$= \frac{1}{2} L\{e^{at}\} - \frac{1}{2} L\{e^{-at}\}$$

$$= \frac{1}{2} \times \frac{1}{s-a} - \frac{1}{2} \times \frac{1}{s+a}$$

$$= \frac{1}{2} \times \frac{s+a-(s-a)}{s^2-a^2}$$

$$= \frac{a}{s^2-a^2}$$

$$\Rightarrow L\{\cos at\} = L\left\{ \frac{e^{iat} + e^{-iat}}{2} \right\}$$

$$= \frac{1}{2} L\{e^{iat}\} + \frac{1}{2} L\{e^{-iat}\}$$

$$= \frac{1}{2} \frac{1}{s-ia} + \frac{1}{2} \frac{1}{s+ia}$$

$$= \frac{1}{2} \frac{s+ia+s-ia}{s^2+a^2}$$

$= \frac{s}{s^2+a^2}$

$\Rightarrow L\{\sin at\} = L\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\}$

$= \frac{1}{2i} L\{e^{iat}\} - \frac{1}{2i} L\{e^{-iat}\}$

$= \frac{1}{2i}\frac{1}{s-ia} - \frac{1}{2i}\frac{1}{s+ia}$

$= \frac{1}{2i} \frac{s+ia-(s-ia)}{s^2+a^2}$

$= \frac{1}{2i} \frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$

(8) $L\{e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9\}$

By linearity property

$L\{e^{3t}\} - 2L\{e^{-2t}\} + L\{\sin 2t\} + L\{\cos 3t\} +$
$L\{\sinh 3t\} - 2L\{\cosh 4t\} + 9L\{1\}$

$= \frac{1}{s-3} - 2\frac{1}{s+2} + \frac{2}{s^2+2^2} + \frac{s}{s^2+3^2} + \frac{3}{s^2-3^2} - 2\frac{s}{s^2-4^2} +$
$9\frac{1}{s}$

$\Rightarrow L\left\{\frac{e^{-at}-1}{a}\right\}$

$= L\left\{\frac{e^{-at}}{a}\right\} - L\left\{\frac{1}{a}\right\}$

$= \frac{1}{a} L\{e^{-at}\} - \frac{1}{a} L\{1\}$

$= \frac{1}{a} \times \frac{1}{s+a} - \frac{1}{a} \times \frac{1}{s}$

$\Rightarrow L\{(t^2+1)^2\} = L\{t^4+1+2t^2\}$

$L\{t^4\} + L\{1\} + 2L\{t^2\}$

$= \frac{4!}{s^5} + \frac{1}{s} + 2 \cdot \frac{2!}{s^3}$

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$\Rightarrow L\{(\sin t + \cos t)^2\} - L\{\sin^2 t + \cos^2 t + 2\sin t\cos t\}$

$= L\{1 + 2\sin t\cos t\}$

$= L\{1 + \sin 2t\}$

$= L\{1\} + L\{\sin 2t\}$

$= \frac{1}{s} + \frac{2}{s^2 + a^2}$

$\Rightarrow L\{(\sqrt{t} + \frac{1}{\sqrt{t}})^3\}$

$= L\{(\sqrt{t})^3 + 3(\sqrt{t})^2 \frac{1}{\sqrt{t}} + 3\sqrt{t}\left(\frac{1}{\sqrt{t}}\right)^2 + \left(\frac{1}{\sqrt{t}}\right)^3\}$

$= L\{t^{3/2} + 3t^{1/2} + 3t^{-1/2} + t^{-3/2}\}$

$= L\{t^{3/2}\} + 3L\{t^{1/2}\} + 3L\{t^{-1/2}\} + L\{t^{-3/2}\}$

$= \frac{\left(\frac{3}{2}\right)!}{s^{3/2+1}} + 3 \times \frac{\left(\frac{1}{2}\right)!}{s^{1/2+1}} + 3 \times \frac{\left(\frac{-1}{2}\right)!}{s^{-1/2+1}} + \frac{\left(\frac{-3}{2}\right)!}{s^{-3/2+1}}$

$= \frac{\Gamma\frac{5}{2}}{s^{5/2}} + 3\frac{\Gamma\frac{3}{2}}{s^{3/2}} + \frac{3\Gamma\frac{1}{2}}{s^{1/2}} + \frac{3\Gamma\frac{1}{2}}{s^{1/2}} + \frac{\Gamma\frac{-1}{2}}{s^{-1/2}}$

$\therefore \Gamma{n+1} = n!$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Gamma\frac{3}{2} = \frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$

$= \frac{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{3\frac{1}{2}\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{-2\sqrt{\pi}}{s^{-1/2}}$

$\Gamma n = n-1\Gamma{n-1}$

$\Gamma{n+1} = n\Gamma n$

$\Gamma n = \frac{\Gamma{n+1}}{n}$

$\therefore \Gamma\frac{-1}{2} = -2\sqrt{\pi}$

$\Rightarrow$ prove that the fun $f(t) = t^2$ is of exponential order.

Sol $f(t) = t^2$

$f(t) = t^2$ is of E.O

if $\lim\limits_{t\to\infty} e^{-st} t^2 =$ fixed Quantity

consider $\lim\limits_{t\to\infty} e^{-st} t^2 = \lim\limits_{t\to\infty} \frac{t^2}{e^{st}} \left(\frac{\infty}{\infty}\right)$

L' Hospitals rule

$\lim\limits_{t\to\infty} \frac{2t}{e^{st}.s} \left(\frac{\infty}{\infty}\right)$

$\lim\limits_{t\to\infty} \frac{2}{s^2.e^{st}} = 0$

$L\{t^2\} = \frac{2!}{s^3}$

⇒ prove that laplace transformation of e^{t^3}

$f(t) = e^{t^3}$

$f(t) = e^{t^3}$ is of E.O

$\lim\limits_{t\to\infty} e^{-st} e^{t^3} =$ fixed Quantity

consider $\lim\limits_{t\to\infty} e^{-st} e^{t^3} = \lim\limits_{t\to\infty} e^{(t^3-st)} = \infty$

hence $L\{e^{t^3}\}$ does not exists

(Q) prove first translation theorem or first shifting theorem.

Statement : If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$

$(s-a) > 0$

proof : Given $L\{f(t)\} = F(s) \Rightarrow L\{f(t)\}$

$= \int_0^\infty e^{-st} f(t)dt = F(s)$

consider $L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt$

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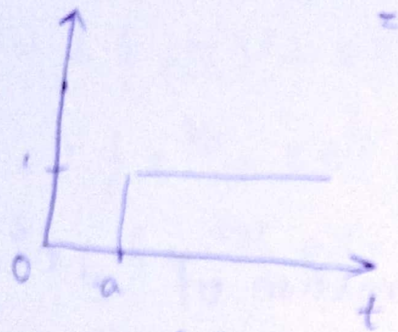
$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt.$$

$$= F(s-a)$$

Unit Step function :- [Heaviside]

The unit step function / Heavisides unit step function is defined as

$H(t-a)$ or $u(t-a) = 0 \quad t<a$
$\qquad\qquad\qquad\quad = 1 \quad t>a$



$L\{H(t-a)\} / L\{u(t-a)\}$

Q) If $L\{f(t)\} = F(s)$ and $g(t)$ is defined as $f(t-a)$
$g(t) = H(t-a) \quad t>a$ then $L\{g(t)\} = e^{-as}F(s)$
$\qquad = 0 \quad t<a$ [second translation theorem]

proof: Given $L\{f(t)\} = F(s)$ and $g(t) = f(t-a) \quad t>a$
$\qquad = 0 \quad t<a$

consider $L\{f(t)\} = \int_0^\infty e^{-st} g(t)\,.dt$

$$= \int_0^a e^{-st} g(t)dt + \int_a^\infty e^{-st}. g(t)dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

Let $t-a = u$ $\qquad$ LL : $u = a-a = 0$
$dt = du$ $\qquad$ UL : $u = \infty - a = \infty$

$$L\{f(t)\} = \int_0^\infty e^{-(s)(u+a)} f(u).du$$

$$= \int_0^\infty e^{-as}. e^{-su}. f(u)du$$

$= e^{-as} F(s)$

Change of scale property: If $L\{f(t)\} = F(s)$

then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

proof: Given $L\{f(t)\} = F(s)$

consider $L\{f(at)\} = \int_0^\infty e^{-st} \cdot f(at)\,dt$

let $at = u$

$a\,dt = du \quad \cdot\ dt = \frac{du}{a}$

$a(0) = u \qquad u = 0$

$a(\infty) = u \qquad u = \infty$

$= \int_0^\infty e^{-s(u/a)} f(u) \frac{du}{a}$

$= \frac{1}{a} \int_0^\infty e^{(-s/a)u} f(u)\,du = \frac{1}{a} f(s/a)$

$\Rightarrow$ 1) $L\{e^{-3t}(2\cos 5t - \sin 5t\}$

2) $L\{e^{-at} (\cosh bt\}$

1) $L\{e^{-3t}(2\cos 5t - \sin 5t\}$

Let $f(t) = 2\cos 5t - \sin 5t$

$L\{f(t)\} = L\{2\cos 5t\} - L\{\sin 5t\}$

$= \frac{2s}{s^2+25} - \frac{5}{s^2+25}$

By first translation theorem

$L\{e^{at} f(t)\} = F(s-a)$

$L\{e^{-3t}(2\cos 5t - \sin 5t)\} = \frac{2(s-(-3))}{(s-(-3))^2+25} - \frac{5}{(s-(-3))^2+25}$

$= \frac{2(s+3)-5}{(s+9)^2+25}$

$= \frac{2s+1}{(s+9)^2+25}$

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2) $L\{e^{-at}\cosh bt\}$

let $f(t) = \cosh bt$

$$L\{f(t)\} = L\{\cosh bt\} = \frac{s}{s^2-b^2}$$

By first translation theorem $L\{e^{at}f(t)\} = F(s-a)$

$$L\{e^{-at}\cosh bt\} = F(s+a)$$

$$= \frac{(s+a)}{(s+a)^2-b^2}$$

Q) $L\{e^{3t}\sin^2 t\}$

$$f(t) = \sin^2 t = \frac{1-\cos 2t}{2}$$

$$L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

By first translation theorem

$$L\{e^{3t}\sin^2 t\} = F(s-(-3)) = \frac{1}{2}\left[\frac{1}{s-3} - \frac{(s-3)}{(s-3)^2+4}\right]$$

Q) $g(t) = \cos(t-\pi/3),\ t > \pi/3$

$= 0,\ t < \pi/3$

If $g(t) = f(t-a),\ t > a$

$0,\ t < 0$

$f(t) = \cos t$

$L\{f(t)\} = \frac{s}{s^2+1}$

$$L\{g(t)\} = e^{-as}F(s)$$

$g(t) = \cos(t-\frac{\pi}{3})\ t > \pi/3$

$= 0,\ t < \pi/3$

$$L\{g(t)\} = e^{-\pi/3\, s} \cdot \frac{s}{s^2+1}$$

Laplace transformation of derivatives

$f(t)$ is continous and is of exponential order and $f'(t)$ is piece wise continous then laplace transformation of $L\{f'(t)\} = sF(s) - f(0)$

$$= sL\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - sf'(0)$$

$$L\{f^n(t)\} = s^n F(s) - s^{n-1}f(0) --- s^{n-n} f^{n-1} f(0)$$

Laplace transformation of integrals

If $L\{f(t)\} = F(s)$ then $L\{\int_0^t f(t)\,dt\} = \frac{1}{s}F(s)$

let $g(t) = \int_0^t f(t)\,dt$

$$g'(t) = \frac{d}{dt}\int_0^t f(t)\,dt = f(t) - f(0)$$

$$g'(t) = f(t)$$

$$L\{g'(t)\} = sL'\{g(t)\} = g(0)$$

$$\Rightarrow L\{g'(t)\} = sL\{g(t)\}$$

$$L\{g(t)\} = \frac{1}{s}L\{g(t)\}$$

$$L\{\int_0^t f(t)\,dt\} = \frac{1}{s}L\{f(t)\} = \frac{1}{s}F(s)$$

Laplace transformation of $t\{f(t)\}$

$$L\{tf(t)\} = \frac{-d}{ds}F(s)$$

consider $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{tf(t)\} = \int_0^\infty e^{-st}\, t\, f(t)\,dt$$

$$= \left[t f(t)\int e^{-st}dt - \int [tf(t)]^{+1}\int e^{-st}dt)dt\right]_0^\infty$$

$$= t f(t)\frac{e^{-st}}{-s} - \int [f(t) + tf'(t)]\frac{e^{-st}}{-s}dt\Big]_0^\infty$$

$$= 0 + \left\{\int \frac{f(t)e^{-st}}{s}dt + \int t f'(t)\frac{e^{-st}}{s}dt\right]_0^\infty$$

The integral is recursive. Hence consider $\frac{d}{ds}F(s)$

$\Rightarrow \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt$

$\int_0^\infty \frac{d}{ds} e^{-st} f(t)dt$

$\int_0^\infty e^{-st}(-t) f(t) dt$

$-\int_0^\infty e^{-st} t f(t) dt$

$= -L\{t f(t)\}$

$\Rightarrow$ If $L\{f(t)\} = F(s)$ then $L\{\frac{f(t)}{t}\} = \int_s^\infty F(s)ds$ provide the integral.

Sol $\int_s^\infty F(s)ds = \int_s^\infty\left[\int_0^\infty e^{-st} f(t)dt\right]ds$

$= \int_0^\infty f(t)dt \int_s^\infty e^{-st} ds$

$= \int_0^\infty f(t)dt \left[\frac{e^{-st}}{-t}\right]_s^\infty$

$= \int_0^\infty e^{-st}\frac{f(t)}{t} dt$

$= L\{\frac{f(t)}{t}\}$

$\Rightarrow L\{\int_0^t e^{-t} \cos t dt\}$

let $f(t) = \cos t$

$L\{f(t)\} = \frac{s}{s^2+1}$

$L\{e^{-t}\cos t\}$ By first translation theorem

$L\{e^{at}f(t)\} = F(s-a)$

$L\{e^{-t}\cos t\} = \frac{s-(-1)}{(s-(-1))^2+1}$

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

$$L\left\{\int_0^t e^{-t}\cos t\,dt\right\} = \frac{1}{s}\left(\frac{s+1}{(s+1)^2+1}\right)$$

$\Rightarrow L\left\{\int_0^t \int_0^t \cosh at\,dt\right\}$

$f(t) = \cosh at$

$$L\{f(t)\} = \frac{s}{s^2-a^2}$$

$$L\left\{\int_0^t \cosh at\,dt\right\} = \frac{1}{s}\left(\frac{s}{s^2-a^2}\right)$$

$$L\left\{\int_0^t \int_0^t \cosh at\,dt\right\} = \frac{1}{s}\left(\frac{s}{s(s^2-a^2)}\right) = \frac{1}{s^2}\left(\frac{s}{s^2-a^2}\right)$$

$\Rightarrow$ ~~$L\{t f(t)\}$~~ -

$L\{t\sin 3t\cos 2t\}$

$L\{t f(t)\} \Rightarrow f(t) = \sin 3t\cos 2t$

$$= \frac{1}{2}\, 2\sin 3t\cos 2t$$

$$= \frac{1}{2}\left[\sin 5t + \sin t\right]$$

$$L\{f(t)\} = \frac{1}{2} L\{\sin 5t\} + \frac{1}{2} L\{\sin t\}$$

$$= \frac{1}{2}\cdot\frac{5}{s^2+25} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$L\{t f(t)\} = -\frac{d}{ds}F(s)$$

$$= -\left[\frac{1}{2}\frac{d}{ds}\left(\frac{5}{s^2+25}\right) + \frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\,5\,\frac{\left(\frac{d}{ds}(1)(s^2+25) - 1\frac{d}{ds}(s^2+25)\right)}{(s^2+25)^2} -$$

$$\frac{1}{2}\,\frac{\frac{d}{ds}(1)(s^2+25)}{(s^2+25)^2} - \frac{1}{2}\,\frac{\frac{d}{ds}(1)(s^2+1) - \frac{d}{ds}(s^2+1)}{(s^2+1)^2}$$

$$= -\frac{5}{2}\frac{(-25)}{(s^2+25)^2} - \frac{25}{2(s^2+25)^2}$$

$\Rightarrow L\{te^{t}\cosh t\}$

$$f(t) = \cosh t$$

$$L\{f(t)\} = L\{\cosh t\}$$

$$= \frac{s}{s^2-1}$$

$L\{e^{-t}\cosh t\}$ By F.T.T

$$L\{e^{-t}\cosh t\} = \frac{(s-(-1))}{(s-(-1))^2-1}$$

$$= \frac{s+1}{(s+1)^2-1}$$

$$L\{t(e^{-t}\cosh t)\} = \frac{-d}{ds}\left(\frac{s+1}{s^2+25}\right)$$

$$= \frac{\frac{d}{ds}(s+1)(s^2+25) - (s+1)\frac{d}{ds}(s^2+25)}{(s^2+25)^2}$$

$$= \frac{-s^2+25-(s+1)(2s+2)}{(s^2+25)^2}$$

$$= \frac{s^2+2s+2}{(s^2+25)^2}$$

$\Rightarrow L\{t^2e^{-2t}\}$

$$f(t) = e^{-2t}$$

$$L\{f(t)\} = \frac{1}{s+2}$$

$$L\{t^2f(t)\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{1}{s+2}\right)\right)$$

$$= \frac{d}{ds}(-1)(s+2)^{-1-1}$$

$$= -\frac{d}{ds}(s+2)^{-2}$$

$= (-1)(-2)(s+2)^{-3}$

$= \frac{2}{(s+2)^3}$

$\Rightarrow L\{ \int_0^t t e^{-t} \sin 4t \, dt \}$

$f(t) = \sin 4t$

$L\{f(t)\} = \frac{4}{s^2+4^2}$

$L\{ e^{-t} \sin 4t \} = \frac{4}{(s-(-1))^2+4^2)}$

By F.T.T

$L\{ t e^{-t} \sin 4t \} = -\frac{d}{ds} \left(\frac{4}{s^2+2s+17} \right)$

$= \frac{+4(2s+2)}{(s^2+2s+1)^2}$

$L\{ \int_0^t t e^{-t} \sin 4t \, dt \}$

$= \frac{1}{s} \frac{8s+8}{(s^2+2s+1)^2}$

$\Rightarrow L\{ \frac{\sin t}{t} \}$

$f(t) = \sin t$

$L\{f(t)\} = \frac{1}{s^2+1}$

$L\{ \frac{f(t)}{t} \} = \int_s^\infty \frac{1}{s^2+1} ds$

$= [\tan^{-1} s]_s^\infty$

$\tan^{-1}\infty - \tan^{-1} s$

$\frac{\pi}{2} - \tan^{-1} s$

$\Rightarrow L\{ \frac{e^{-at} - e^{-bt}}{t} \}$

$f(t) = e^{-at} - e^{-bt}$

$L\{f(t)\} = \frac{1}{s+a} - \frac{1}{s+b}$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$\left[\log(s+a) - \log(s+b)\right]_s^{\infty}$$

$$= -\log(s+a) + \log(s+b)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

$$\Rightarrow L\left\{\frac{1-e^{t}}{t}\right\}$$

$$\Rightarrow f(t) = 1-e^{t}$$

$$L\{f(t)\} = \frac{1}{s} - \frac{1}{s-1}$$

$$L\left\{\frac{1-e^{t}}{t}\right\} = \int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s-1}\right)ds$$

$$= \left[\log s - \log(s-1)\right]_s^{\infty}$$

$$= \log\frac{s-1}{s}$$

$$\Rightarrow L\left\{\int_0^t \frac{e^{t}\sin t}{t}\,dt\right\}$$

$$f(t) = \sin t$$

$$\rightarrow L\left\{\frac{1-\cos at}{t}\right\}$$

$$t = s^2+a^2$$

$$dt = 2s\,ds$$

$$s\,ds = \frac{dt}{2}$$

$$f(t) = 1-\cos at$$

$$L\{f(t)\} = L\{1\} - L\{\cos at\}$$

$$= \frac{1}{s} - \frac{s}{s^2+a^2}$$

$$= \int_0^{\infty} \frac{1}{s} - \frac{s}{s^2+a^2}\,ds$$

$$= \log s - \frac{1}{2}\log(s^2+a^2)$$

$$= \log s - \frac{1}{s}\log(s^2+a^2)$$

$$\Rightarrow L\left\{\frac{1-\cos t}{t^2}\right\}$$

$$f(t) = 1-\cos t$$

$$L\{f(t)\} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty \left(\frac{1}{s} - \frac{1}{2}\frac{2s}{s^2+1}\right)ds$$

$$= \log s - \frac{1}{2}\log(s^2+1)\Big]_0^\infty$$

$$= \frac{1}{2}\log\frac{s^2}{s^2+1}\Big]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1}{1+\frac{1}{s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1}{1} - \log\frac{1}{1+1/s^2}\right]$$

$$= \frac{1}{2}\log\left(1+\frac{1}{s^2}\right)$$

$$L\left\{\frac{f(t)}{t^2}\right\} = L\left\{\frac{1}{t}\cdot\frac{f(t)}{t}\right\}$$

$$= \int_s^\infty \frac{1}{2}\log\left(1+\frac{1}{s}\right)ds$$

$$= \frac{1}{2}\left\{\log\left(1+\frac{1}{s^2}\right)\int 1\,ds - \int_s \frac{d}{ds}\log\left(1+\frac{1}{s^2}\right)\right\}.$$

$$\int 1\,ds)\,ds$$

$$= \frac{1}{2}\left\{s\log\left(1+\frac{1}{s}\right) - \int_s^\infty s\cdot\frac{1}{1+\frac{1}{s}}\cdot\frac{-2}{s^2}\,ds\right\}$$

$$= \frac{1}{2}\left\{s.\log\left(1+\frac{1}{s}\right) + 2\int_s^\infty \frac{1}{1+s^2}\,ds\right\}$$

$$= \frac{1}{2}\left\{s\log\left(1+\frac{1}{s}\right) + 2.\tan^{-1}(s)\Big]_s^\infty\right\}_s^\infty$$

$$= \frac{1}{2}\,s\log\left(1+\frac{1}{s^2}\right) + \left(\frac{\pi}{2} - \tan^{-1}(s)\right\}$$

$$= -\frac{1}{2}\,s\log\left(1+\frac{1}{s^2}\right) + \cot^{-1}(s)$$

** $L\left\{ \cancel{\frac{e^{-3t}\sin 3t}{t}} \right\}$ $L\left\{ e^{-3t} \int_0^t \frac{\sin t}{t} dt \right\}$

$f(t) = \sin t$

$L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{1}{s^2+1} ds$

$= [\tan^{-1} s]_s^\infty = \frac{\pi}{2} - \tan^{-1} s$

$L\left\{ \int_0^t \frac{\sin t}{t} dt \right\} = \frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1} s\right)$

$L\left\{ e^{-3t} \int_0^t \frac{\sin t}{t} dt \right\}$

$L\left\{ e^{-3t} \left[\int_0^t \frac{\sin t}{t} dt \right] \right\} = \frac{1}{(s-(-3))} \left[\frac{\pi}{2} - \tan^{-1}(s-(-3)) \right]$

$= \frac{1}{s+3} \left[\frac{\pi}{2} - \tan^{-1}(s+3) \right]$

Evaluation of integrals

$\Rightarrow \int_0^\infty t e^{-3t} dt$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$f(t) = t \quad s = 3$

$\int_0^\infty e^{-st} \cdot t\, dt = L\{f(t)\}$

$= \frac{1}{s^2}$

$= \frac{1}{9}$

** $\Rightarrow \int_0^\infty \frac{\sin 2t}{t} dt$

** $\Rightarrow \int_0^\infty t^{\circ} e^{-4t} \sin 2t\, dt = \frac{11}{500}$

$\Rightarrow \int_0^\infty \frac{\sin 2t}{t}\, dt$

$= \int_0^\infty e^{-0t} \frac{\sin 2t}{t}\, dt$

let $s = 0$

$\int_0^\infty e^{-st} \frac{\sin 2t}{t}\, dt$

$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty \frac{2}{s^2+4}\, ds$

$= \left[\frac{2}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}(s/2)$

but $s = 0$

$= \pi/2$

$\Rightarrow \int_0^\infty t^2 e^{-4t} \sin 2t\, dt = \frac{11}{500}$

let $s = 4$

$\int_0^\infty e^{-st}(t^2 \sin 2t)\, dt$

$L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$

$= 2 \cdot \frac{d}{ds}\left(\frac{-2s}{(s^2+4)^2}\right)$

$= -\frac{4(s^2+4)^2 - 4s^2(s^2+4)}{(s^2+4)^4}$

let $s = 4$

$= \frac{11}{500}$

$\Rightarrow \int_0^\infty t e^{-t} \sin t\, dt$

$= \int_0^\infty e^{-st}(t \sin t)\, dt$

by $s = 1$

$L\{t \sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$$= -\frac{(-2s)}{(s^2+1)^2}$$

$$= \frac{2s}{(s^2+1)^2} \quad \text{by } s=1$$

$$= \frac{1}{2}$$

$$\Rightarrow \int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt$$

$$\int_0^\infty e^{-st}\left[\frac{e^{-t}-e^{-2t}}{t}\right]dt$$

by $s = 0$

$$f(t) = \frac{e^{-t}-e^{-2t}}{t}$$

$$L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+2}\right)ds$$

$$= \left[\log(s+1) - \log(s+2)\right]_s^\infty$$

$$= -\log(s+1) + \log(s+2)$$

$$= \log 2.$$

unit delta
direct impulse
periodic

unit impulse function or direct delta function

The function is defined ~~to~~ represent when large forces act for a very shorter time

$\Rightarrow$ unit impulse fun $f_\varepsilon(t-a)$ is defined as zero for values $< a$.

$$f_\varepsilon(t-a) = \begin{cases} 0 & a<t \\ \frac{1}{\varepsilon} & a<t<a+\varepsilon \\ 0 & a+\varepsilon<t \end{cases}$$

laplace transformation for unit impulse fun is defined as $L\{f_\varepsilon(t-a)\} = \int_0^\infty e^{-st} f_\varepsilon(t-a)$

$$= \int_0^a e^{-st} f_\varepsilon(t-a)\,dt + \int_a^{a+\varepsilon} e^{-st} f_\varepsilon(t-a)\,dt + \int_{a+\varepsilon}^{\infty} e^{-st} f_\varepsilon(t-a)\,dt$$

$$= 0 + \int_a^{a+\varepsilon} e^{-st} \frac{1}{\varepsilon}\,dt + 0$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon}$$

$$= -\frac{1}{\varepsilon s}\left(e^{-s(a+\varepsilon)} - e^{-sa}\right)$$

$$= \frac{-e^{-as}}{\varepsilon s}\left(e^{-s\varepsilon} - 1\right)$$

periodic fun

A fun $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for some value of T. T is called periodicity. we also observe $f(t+2T) = f(t)$ $f(t+3T) = f(t)$ --- $f(T+nT) = f(t)$

If $f(t)$ = periodic fun then $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

**

proof: Given $f(t)$ is a periodic fun when periodicity T.

$$f(t) = T$$

$$f(t) = f(t+T) = \dots + f(t+nT)$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt +$$

$$= \int_0^T e^{-st} f(t)\,dt$$

$t = U + T$

$dt = dU$

$2T = U+T$ $\quad$ $T = U+T$

$U = T$ $\quad$ $U = 0$

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$t = U+2T$ $\qquad$ $t = U+3T$

$dt = du$ $\qquad$ $dt = dU$

$2T = U+2T$ $\qquad$ $3T = U+3T$

$U = 0$ $\qquad$ $U = 0$

$3T = U+2T$ $\qquad$ $4T = U+3T$

$U = T$ $\qquad$ $U = T$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(U+T)} f(U+T)\,du + \int_0^T e^{-s(U+2T)} f(U+2T)\,du$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^T e^{-sU} f(u)\,du + e^{-s2T} \int_0^T e^{-sU} f(U)\,du$$

$$= \int_0^T e^{-sU} f(U)\,du \left(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \dots\right)$$

$$(1 + x + x^2 + x^3 + \dots)$$

$$= (1-x)^{-1}$$

$$= \left(1 + e^{-sT} + (e^{-sT})^2 + (e^{-sT})^3 + \dots\right) \int_0^T e^{-sU} f(u)\,du$$

$$= (1-e^{-sT})^{-1} \int_0^T e^{-sU} f(u)\,du$$

$$= \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$\Rightarrow$ find $L\{f(t)\}$ where $f(t)$ is a periodic fun

$$f(t) = \sin t \qquad 0 < t < \pi$$
$$= 0 \qquad \pi < t < 2\pi$$

Sol $\quad T = 2\pi$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\,dt$$

$$= \frac{1}{1-e^{-s2\pi}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} (0)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} (-s\sin t - \cos t) \right]_0^{\pi} + 0$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1} (-(-1)) + \frac{1}{s^2+1} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} \right]$$

$$\boxed{\int e^{ax} \sin x\, dx = \frac{e^{ax}}{a^2+b^2} (a\sin bx - b\cos bx)}$$

⇒ find $L\{f(t)\}$ where $f(t)$ is given by

$$f(t) = t \quad 0 < t < b$$
$$\quad 2b - t \quad b < t < 2b \qquad T = 2b$$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} t\, dt + \int_b^{2b} e^{-st} (2b-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\left[t \int e^{-st} dt - \int \frac{dt}{dt} \int e^{-st} dt \right) dt \right]_0^b$$
$$+ \left[(2b-t) \int e^{-st} - \int \left(\frac{d}{dt} (2b-t) \int e^{-st} dt \right) dt \right]_b^{2b}$$

$$= \frac{1}{1-e^{-2bs}} \left[\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^b + \left[(2b-t) \frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_b^{2s} \right.$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{te^{-bs}}{-s} - \frac{e^{-bs}}{s^2} - \left[-\frac{1}{s^2} \right] + \left[(2b-2s) \frac{e^{-2s^2}}{-s} \right. \right.$$
$$\left. + \frac{e^{-2s^2}}{s^2} - \left[(2b-b) \frac{e^{-bs}}{-s} + \frac{e^{-bs}}{s^2} \right] \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{1}{s^2} + \frac{e^{-2bs}}{s^2} - \frac{2e^{-sb}}{s^2} \right]$$

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Questions

→ State F.S.T and find $L^{-1}\{e^{-3t}(2\cos 5t - 3\sin$

→ Find L.T of $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

→ $L\{\sinh^3 2t\}$

→ $L\{e^{-t}\cos^2 t\}$

→ $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$

③ Sol $\sinh^3 2t = \left(\frac{e^{2t}-e^{-2t}}{2}\right)^3$

$$= \frac{1}{8}\left[(e^{2t})^3 - 3(e^{2t})^2 e^{-2t} + 3e^{2t}(e^{-2t})^2 - (e^{-2t})^3\right]$$

$$= \frac{1}{8}\left[e^{6t} - 3e^{2t} + 3e^{-2t} - e^{-6t}\right]$$

→ Obtain the first two approximations of picard's method for $y' = 1 + xy$, $y(0) = 1$ hence compute $y(0.1)$

Formula → write fourth order R.K formula. Find y at $x = x_1$ given that $y' = f(x,y)$ $y = y_0$ at $x = x_0$

→ Evaluate $\int_0^{\pi} x\sin x\,dx$ by 1) T.R 2) S 1/3 rule
upto 10 radians

→ By modified Euler $y(0.2)$ and $y(0.4)$.

$\frac{dy}{dx} = x - y^2$, $y(0) = 1$

→ State T.R → write Euler formula

→ use R.K's 4th order to find y when $x = 0.2$

$x_0 = 0$, $y_0 = 1$ and $h = 0.1$

$$y' = x - y^2$$