UNIT-II

Numerical Method-II.

07/10/2021.

1. Numerical Integration
2. Solution to IVP

Numerical Integration :-

1) Trapezoidal Rule
2) Simpson's Rule (or) Simpson's $\frac{1}{3}$ Rule.
3) Simpson's $\frac{3}{8}$ Rule.

Consider the set of $(n+1)$ points $(x_0, y_0), (x_1, y_1) \cdots (x_n, y_n)$ of the function $y = f(x)$, where $f(x)$ is not known explesively to evaute integral x_0 to x_n $f(x)dx$ we use Newtons forward interpolation formula & for different values on 'n' we can derive trapezoidal & Simpson's Rule.

$$I = \int_{x_0}^{x_n} f(x)\,dx$$

$$= \int_{x_0}^{x_n} \left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \cdots \right) dx.$$

$$p = \frac{x - x_0}{h} \Rightarrow x = x_0 + ph$$

$$dx = h\,dp$$

$$= h \int_0^n \left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \cdots \right) dp$$

$$= h \left[n y_0 + \frac{n^2}{2} \Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2} \right) \Delta^2 y_0 + \cdots \right]$$

↳ Quadrature formula

Newton's cote's

① Trapezoidal Rule.

put $n=1$, in the Quadrature formula.

$$I=\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots y_n)\right]$$

Pascal Δle

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$=\frac{h}{2}\Big[\text{(sum of the first \& last ordinates)} + 2\,\text{(Remaing ordinates)}\Big]$$

② Simpson's Rule.

When n & h are not given we should take multiples of 3 (6, 9, 12)... in any formula

put $n=2$, in the Quadrature formula

$$I=\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+y_5+\cdots y_{n-1}) + 2(y_2+y_4+y_6+\cdots y_{n-2})\Big]$$

$$=\frac{h}{3}\Big[\text{(sum of the first \& last ordinates)} + 4\,\text{(odd)} + 2\,\text{(even)}\Big]$$

Note:- n = sub intervals. (no. of Sub intervals)
& 'n' should be even.

③ Simpson's $\frac{3}{8}$th Rule.

put $n=3$, in the Quadrature formula.

$$I=\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n)+3(\underbrace{y_1+y_2+y_4+y_5+y_7\cdots y_{n-2}\,y_{n-1}}_{\text{Remaing}}) + 2(\underbrace{y_3+y_6+y_9\cdots y_{n-3}}_{\text{multiple of 3}})\Big]$$

Note: The no. of sub intervals n should be a multiple of 3.

1) Evaluate intergal $I=\int_0^1 \frac{1}{(1+x)}\,dx$ using Trapezoidal Rule. $h=0.125$

$$I=\int_0^1 \frac{1}{1+x}\,dx$$

$x_0=0,\quad x_n=1,\quad h=0.125$

$$n=\frac{x_n-x_0}{h}=\frac{1-0}{0.125}=8$$

x	$f(x)=\frac{1}{1+x}$
0	1 (y_0)
0.125	0.8888 (y_1)
0.25	0.8 (y_2)
0.375	0.7272 (y_3)
0.5	0.6666 (y_4)
0.625	0.6153 (y_5)
0.75	0.5714 (y_6)
0.875	0.5333 (y_7)
1	0.5 (y_8)

$$I=\frac{h}{2}\Big[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\Big]$$

$$I=\frac{0.125}{2}\Big[(1+0.5)+2(0.8888+0.8+0.7272+0.6666+0.6153+0.5714+0.5333)\Big]$$

$$I=0.694075$$

08/010/2021

Evaluate $\int_0^{\pi/2}\sqrt{\sin x}\,dx$ using
1) Trapezoidal Rule taking $h=\pi/12$
2) Simpsons Rule

$x_0=0,\ x_n=\frac{\pi}{2}\quad h=\frac{\pi}{12}$

$$n=\frac{x_n-x_0}{h}=\frac{\frac{\pi}{2}-0}{\pi/12}=6\ \text{(sub intervals)}$$

$f(x)=\sqrt{\sin x}$

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12}$
$y=f(x)$	0	~~0.0955~~ 0.5087	~~0.1470~~ 0.7071	0.8408	0.9306	0.9828	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Trapezoidal Rule.

$$I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 \cdots y_{n-1})\right]$$

$$= \frac{\pi/12}{2}\left[(0+1) + 2(0.5087 + 0.7071 + 0.8408 + 0.9306 + 0.9828)\right]$$

$$= \cancel{0.1514}\ 1.1702$$

Simpsons Rule (more accurate).

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 \cdots y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2})\right]$$

$$= \frac{\pi/12}{3}\left[(0+1) + 4(0.5087 + 0.8408 + 0.9828) + 2(0.7071 + 0.9306)\right]$$

$$= 1.1872$$

Simpsons $\frac{3}{8}$ Rule

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(\text{Remaining}) + 2(\text{multiple of } 3)\right]$$

$$= \frac{3 \times \frac{\pi}{12}}{8}\left[(0+1) + 3(0.5087 + 0.7071 + 0.9306 + 0.9828) + 2(0.8408)\right]$$

$$= 1.1848$$

② Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ by using.

Trapezoidal, simpsons Rule, simpsons $\frac{3}{8}$ Rule.

$x_0 = 0 \quad x_n = 1$

$n = 6$

$\frac{x_0 - x_n}{h} = 6 \qquad h = 0.1666$

$\frac{0-1}{h} = 6 \qquad h = 1/6$

$\frac{0-1}{6} = h$

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	1	0.9729	0.9	0.8	0.692	0.590	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Trapezoidal Rule.

$$I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 \cdots y_{n-1})\right]$$

$$= \frac{1/6}{2}\left[(1 + 0.5) + 2(0.9729 + 0.9 + 0.8 + 0.692 + 0.590)\right]$$

$$= 0.7843$$

Simpsons Rule

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(\text{odd}) + 2(\text{even})\right]$$

$$= \frac{1/6}{3}\left[(1 + 0.5) + 4(0.9729 + 0.8 + 0.590) + 2(0.9 + 0.692)\right]$$

$$= 0.7853$$

Simpons $\frac{3}{8}$ Rule

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(\text{Remaining}) + 2(\text{multiple of } 3)\right]$$

$$= \frac{3 \times \frac{1}{6}}{8}\left[(1 + 0.5) + 3(0.9729 + 0.9 + 0.692 + 0.590) + 2(0.8)\right]$$

$$= 0.7852$$

Actual value $\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right] = \pi/4 = 0.785$

1) $\int_0^2 e^{x^2} dx$ taking $n=10$ using (i) Trapezoidal Rule (ii) Simpson's Rule

2) $(0, 2.3)$ $(0.5, .19)$ $(1.0, 14)$ $(1.5, 11)$ $(2.0, 13)$
$(2.5, 16)$ $(3.0, 19)$ $(3.5, 20)$ $(4.0, 20)$

$\int_0^4 f(x)\,dx = ?$ not given (no need to find)

Trapezoidal Rule.

3) The velocity of a bike (km/min) which is starts from rest is given at fixed intervals of time (min) as follows.

Time	0	2	4	6	8	10	12	14	16	18	20
Velocities	0	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 min using simpsons Rule.

20/10/2021

Numerical Solutions of ODE

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \quad \text{(Initial value problem)}$$

1) Taylors Series Method } Direct method
2) picard's Method or Successive Approximate method.
3) Euler's method
4) Modified Euler's Method
5) Range kutta Method

Taylors Series Method.

Consider the inital value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Consider $y' = f(x,y)$ with the inital condition $y(x_0) = y_0$.

The Taylors Series expansion of f $y(x)$ around the point $x = x_0$ is given by $y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0''$

$$\boxed{y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \dots + \frac{(x-x_0)^n}{n!}y_0^{(n)} + \dots}$$

Find y_0', y_0'' ..... by successive differention & evaluating x_0, we can find $y(x)$.

Q. Find $y(0.1)$ correct to 4 decimal places using Taylors series if $y(x)$ satisfies, $y' = x - y^2$, $y(0) = 1$

$y' = x - y^2$, $y(0) = 1$ compare $y' = f(x,y)$, $y(x_0) = y_0$

$x_0 = 0$; $y_0 = 1$

$y' = f(x,y) = x - y^2$, $x_0 = 0$, $y_0 = 1$

$\frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u$

$u = y \quad v = y'$

$y\frac{d}{dx}y' + y'\frac{d}{dx}y$

$yy'' + y'y'$

$y' = x - y^2$, $y_0' = x_0 - y_0^2 = -1$

$y'' = 1 - 2\underbrace{yy'}_{uv}$, $y_0'' = 1 - 2y_0y_0' = 1 - 2(1)(-1) = 3$

$y''' = -2\underbrace{yy''}_{uv} - 2(y')^2$, $y_0''' = -2y_0y_0'' - 2(y_0')^2 = -2(1)(3) - 2(-1)^2 = -8$

$y^{iv} = -2yy''' - 2y'y'' - 4y'y''$, $y_0^{iv} = -2(1)(-8) - 6(-1)(3) = 34$

$\therefore y(x) = 1 + (x-0)(-1) + \frac{(x-0)^2}{2!}(3) + \frac{(x-0)^3}{3!}(-8) + \frac{(x-0)^4}{4!}(34)$

At $x = 0.1$

$y(0.1) = 1 + (0.1-0)(-1) + \frac{(0.1-0)^2}{2!}(3) + \frac{(0.1-0)^3}{3!}(-8) + \frac{(0.1-0)^4}{4!}(34)$

$= 0.9138$

$\frac{dy}{dx} = 1 + xy$, $y(0) = 1$ calculate $y(0.1)$

use Taylors series Method.

$y' = f(x,y) = 1 + xy$, $y(0) = 1$ $\quad xy$

$x_0 = 0, y_0 = 1$ $\quad u = x \quad v = y$

$y' = 1 + xy$, $y_0' = xy' - y$ $\quad \frac{d}{dx}uv = u\frac{d}{dx}v + v\frac{d}{dx}u$

$= x_0 y_0' - y_0$ $\quad x\frac{d}{dx}y + y\frac{d}{dx}x$

$= 0\cdot(1)\cdot$ $\quad xy' + y$

$y' = 1 + xy$, $y_0' = 1 + x_0 y_0 \Rightarrow 1 + (0(1)) = 1$

$y'' = xy' + y = y_0'' = x_0 y_0' + y_0 \Rightarrow 0$ $\quad xy'$

$y''' = xy'' + y' + y' \Rightarrow y_0''' = x_0 y_0'' + y_0' + y_0'$

$y^{iv} = xy''' + y'' + y'' + y'' = y_0^{iv} = x_0 y_0''' +$

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$\frac{dy}{dx} = \frac{1}{x^2+y}$, $y(4) = 4$ find $y(4.1)$ & $y(4.2)$.

$y' = f(x,y) \quad x_0 = 4 \quad y_0 = 4$

$y_0' = \frac{1}{x^2+y} = \frac{1}{4^2+4} = \frac{1}{20} \Rightarrow 0.05$

$y_0'' = \frac{-1}{(x^2+y)^2}(2x+y') \Rightarrow y_0'' = \frac{-[2(4)+0.05]}{(16+4)^2} = -0.020125$

$y''' = -\left[\frac{(x^2+y)^2(2+y'') - (2x+y')\,2(x^2+y)(2x+y')}{(x^2+y)^4}\right] \quad \frac{u}{v} = \frac{vu' - uv'}{v^2}$

$= \left[\frac{(2+y'')(x^2+y) - 2(2x+y')^2}{(x^2+y)^3}\right]$

$= -\left[\frac{(2+(-0.020125))(4^2+4) - 2(2(4)+0.05)^2}{(4^2+4)^3}\right]$

$= +0.01125$

By Taylors Method

$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \dots$

$= 4 + (x-4)0.05 + \frac{(x-4)^2}{2!}(-0.020125) + \frac{(x-4)^3}{3!}(0.01125)$

$y(4.1)$

$= 4 + (4.1-4)0.05 + \frac{(4.1-4)^2}{2!}(-0.020125) + \frac{(4.1-4)^3}{3!}(0.01125)$

$= 4.0049$

$y(4.2)$

$= 4 + (4.2-4)0.05 + \frac{(4.2-4)^2}{2!}(-0.020125) + \frac{(4.2-4)^3}{3!}(0.01125)$

$= 4.0096$

Picard's Method.

Consider the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

integrating on b/s

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x,y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

The 1st approx is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\,dx, \quad y^{(0)} = y_0$$

2nd approx :

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

nth approx

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx.$$

Find $y(0.1)$ using picards method given that

$$y' = x + y^2 \qquad y(0) = 1$$

$$y' = f(x,y) = x + y^2, \quad x_0 = 0, \quad y_0 = 1 = y^{(0)}$$

1st approx $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\,dx$

$$= 1 + \int_0^x (x+1^2)\,dx.$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + x + \frac{x^2}{2}$$

2nd approx $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$

$$y^{(2)} = 1 + \int_0^x \left[x + \underbrace{\left(1 + x + \frac{x^2}{2}\right)^2}_{(a+b+c)^2}\right] dx$$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$

$$y^{(2)} = 1 + \int_0^x \left(x + 1 + x^2 + \frac{x^4}{4} + 2x + x^3 + x^2\right) dx$$

$$= 1 + \frac{x^2}{2} + x + \frac{x^3}{3} + \frac{x^5}{20} + x^2 + \frac{x^4}{4} + \frac{x^3}{3}$$

$$= 1 + x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{20}$$

$$y(0.1) = 1 + 0.1 + \frac{3(0.1)^2}{2} + \frac{2(0.1)^3}{3} + \frac{(0.1)^4}{4} + \frac{(0.1)^5}{20}$$

$$= 1.1156$$

$\frac{dy}{dx} = x^2+y^2 \quad y(0)=0$, find $y(0.4)$

$y' = x^2+y^2$, $x_0 = 0$ $\quad y_0 = 0 = y^{(0)}$

1st approx $\quad y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$

$= 0 + \int_{0}^{x} (x^2+0^2)\, dx$

$= \frac{x^3}{3}$

2nd approx $\quad y^{(2)} = y_0 + \int_{x_0}^{x} (f(x, y^{(1)}))\, dx$

$= 0 + \int_{0}^{x} \left(x^2 + \left(\frac{x^3}{3}\right)^2\right) dx$

$= \cancel{\frac{x^3}{3}} + \cancel{\frac{x^4}{12}} \int_0^x x^2 + \frac{x^6}{9}$

3rd approx $\quad y^{(3)} = y_0 + \int_{x_0}^{x} f(\quad = \frac{x^3}{3} + \frac{x^7}{63}$

$y(0.4)$

$= \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63}$

$= 0.02135$

$y' = 1+xy$, $y(0) = 1$, compute $y(0.1)$ & $y(0.2)$

$y' = 1+xy \quad x_0 = 0 \quad y_0 = 1 = y^{(0)}$

1st approx $\quad y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$

$= 1 + \int_0^x (1+x)\, dx$

$= 1 + x + \frac{x^2}{2}$

2nd approx $\quad y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$

$= 1 + \int_0^x 1 + x\left(1+x+\frac{x^2}{2}\right)$

$= 1 + \int_0^x 1 + x + x^2 + \frac{x^3}{2}$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$

$y(0.1)$

$= 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{3} + \frac{0.1^4}{8}$

$= 1.1053$

$y(0.2)$

$= 1 + 0.2 + \frac{0.2^2}{2} + \frac{0.2^3}{3} + \frac{0.2^4}{8}$

$= 1.2228$

22/10/2021

Euler's Method :-

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

$$y(x_1) = y_1 = y_0 + h f(x_0, y_0) \qquad \boxed{x_1 = x_0 + h}$$

$$y(x_2) = y(x_1 + h) = y_2 = y_1 + h f(x_1, y_1)$$

$$\vdots$$

$$y(x_{n+1}) = y(x_n + h) = y_{n+1} = y_n + h f(x_n, y_n)$$

Taking the step size h the value of y at $x = x_1$

Q:- using Euler's Method find $y(2)$ in 4 steps, given that $\frac{dy}{dx} = \frac{2y}{x}$, $y(1) = 2$

$\frac{y(2)}{y(1)}$ $2-1 = \frac{1}{4}$

$h = 0.25$

Here $h = 0.25$

$y(1) = 2$

$x_0 = 1 ; y_0 = 2$

Step-1

$x_1 = x_0 + h$
$= 1 + 0.25$
$= 1.25$

$$y_1 = y(1.25) = y_0 + h f\left(\frac{2y}{x}\right)$$
$$= 2 + 0.25\left(\frac{2(2)}{1}\right)$$
$$= 3$$

Step-2

$$y_2 = y(1.5) = y_1 + h f\left(\frac{2(y_1)}{x_1}\right)$$
$$= 3 + 0.25\left(\frac{2(3)}{1.25}\right)$$
$$= 4.2$$

Step-3

$$y_3 = y(1.75) = y_2 + h f\left(\frac{2(y_2)}{x_2}\right)$$
$$= 4.2 + 0.25\left(\frac{2(4.2)}{1.75}\right)$$
$$= 5.6$$

Step-4

$$y_4 = y(2) = y_3 + h f\left(\frac{2(y_3)}{x_3}\right)$$
$$= 5.6 + 0.25\left(\frac{2(5.6)}{1.75}\right)$$
$$= 7.2.$$

Q: $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$ Solve for $x = 2$ taking step size 0.5

$x_0 = 1 \quad y_0 = 2.$

$x_1 = x_0 + h$
$= 1 + 0.5$
$= 1.5$

Step-1

$$y_1 = y(\underset{x_1}{1.5}) = 2 + h f(3x_0^2 + 1)$$
$$= 2 + 0.5(3(1)^2 + 1)$$
$$= \cancel{5.875}\ 4$$

$x_2 = (x_1 + h$
$= 1.5 + 0.5$
$= 2.$

$$y_2 = y(\underset{x_2}{2}) = y_1 + h f(3x_1^2 + 1)$$
$$= \cancel{5.875}\ 4 + 0.5\{3(1.5)^2 + 1)$$
$$= \cancel{12.375} = 7.875$$

Modified Euler's Method 10 M

Consider the differential equation.

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Step-1:-

Initial approx to y_1

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$\vdots$$

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(k)})\right]$$

Step-2:-

Starting with x_1, y_1

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y) + f(x_2, y_2^{(1)})\right]$$

$$\vdots$$

$$y_2^{(k+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(k)})\right]$$

Given that $\frac{dy}{dx} = x+y$, $y(0) = 1$ compute $y(0.3)$ taking $h = 0.1$; also compare the numerical Sol^n with analatical sol^n.

$$f(x,y) = x+y, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.1$$

Step-1 To find $y_1 = y(0.1)$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$.

$$y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + 0.1(0+1) = 1.1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{(0)})\right] = 1 + \frac{0.1}{2}\left[(0+1) + (0.1 + 1.1)\right] = 1.11$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{(1)})\right] = 1 + \frac{0.1}{2}\left[(0+1) + (0.1 + 1.11)\right] = 1.1105$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{(2)})\right] = 1 + \frac{0.1}{2}\left[(0+1) + (0.1 + 1.1105)\right] = 1.1105$$

$$y_1 = y(0.1) = 1.1105$$

step-2 $x_1 = 0.1$, $y_1 = 1.1105$; $h = 0.1$, $x_2 = 0.2$

$$y_2^{(0)} = y_1 + h f(x_1, y_1) = 1.1105 + 0.1(0.1 + 1.1105) = 1.23155$$

$y_2^{(1)} = y_1 + \frac{h}{2} f\left((x_1, y_1) + (x_2\ y_2^{(0)})\right)$

$= 1.1105 + \frac{0.1}{2}\left((0.1 + 1.1105) + (0.2 + 1.23155)\right)$

$= 1.2426025.$

$y_2^{(2)} = y_1 + \frac{h}{2} f\left((x_1, y_1) + (x_2\ y_2^{(1)})\right)$

$= 1.1105 + \frac{0.1}{2}\left[(0.1 + 1.1105) + (0.2 + 1.2426025)\right]$

$= 1.243155125.$

$y_2^{(3)} = y_1 + \frac{h}{2} f\left((x_1, y_1) + (x_2\ y_2^{(2)})\right)$

$= 1.1105 + \frac{0.1}{2}\left[(0.1 + 1.1105) + (0.2 + 1.243155125)\right]$

$= 1.24318\cancel{2}.$

$y_2 = 1.2431 \quad x_2 = 0.2, \quad y_0 \quad x_3 = 0.3$

Step-3.

$y_3^{(0)} = y_2 + hf\left((x_2\ y_2)\right)$

$= 1.2431 + 0.1\ (0.2 + 1.2431)$

$= 1.38741$

$y_3^{(1)} = y_2 + \frac{h}{2} f\left[(x_2\ y_2) + (x_3\ y_3^{(0)})\right]$

$= 1.2431 + \frac{0.1}{2}\left((0.2 + 1.2431) + (0.3 + 1.38741)\right]$

$= 1.3996255$

$y_3^{(2)} = y_2 + \frac{h}{2} f\left[(x_2 y_2) + (x_3\ y_3^{(1)})\right]$

$= 1.2431 + \frac{0.1}{2}\left[(0.2 + 1.2431) + (0.3 + 1.3996255)\right]$

$\doteq 1.400236275.$

$y_3^{(2)} = y_2 + \frac{h}{2} f\left[(x_2 y_2) + (x_3\ y_3^{(2)})\right]$

$= 1.2431 + \frac{0.1}{2}\left[(0.2 + 1.2431) + (0.3 + 1.400236275)\right]$

$= 1.400266814$

$y_3 = 1.4002$

Analytical sol^n.

$\frac{dy}{dx} = x + y, \quad y(0) = 1$

$\frac{dy}{dx} - y = x \Rightarrow$ linear in y.

$\frac{dy}{dx} + P(x)\, y = Q(x)$

$IF = e^{\int P dx} = e^{\int -dx} = e^{-x}$

G.S $y\,(IF) = \int Q\,(IF)\, dx + c$

$y e^{-x} = \int x e^{-x} dx + c$

$y e^{-x} = x(-e^{-x}) - 1 \cdot e^{-x} + c$

$= -e^{-x}(x+1) + c \longrightarrow (1)$

put $x = 0, y = 1$

$1 = -1 + c$

$c = 2$ in eq (1)

$y e^{-x} = -e^{-x}(x+1) + 2$

or $y = -(x+1) + 2e^{x}.$

put $x = 0.3$

$y(0.3) = 1.3997$

compute $y(0.2)$ & $y(0.4)$ given that

$$\frac{dy}{dx} = -xy^2 \quad y(0) = 2$$

$h = 0.2$, $x_0 = 0$, $x_1 = 0.2$, $y_0 = 2$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

step-1

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 2 + 0.2(-0.2^2)$

$= 2$

$y_1^{(1)} = y_0 + \frac{h}{2} f((x_0, y_0) + (x_1, y_1^{(0)}))$

$= 2 + \frac{0.2}{2}((-0 \times 2^2) + (-0.2 \times 2^2))$

$= 1.92$

$y_1^{(2)} = y_0 + \frac{h}{2} f((x_0, y_0) + (x_1, y_1^{(1)}))$

$= 2 + \frac{0.2}{2}((0) + (-0.2 \times 1.92^2))$

$= 1.926272.$

$y_1^{(3)} = y_0 + \frac{h}{2} f((x_0, y_0) + (x_1, y_1^{(2)}))$

$= 2 + \frac{0.2}{2}((0) + (-0.2 \times 1.926272^2))$

$= 1.925789524$

$y_1 = 1.926$

!

$y_1^{(5)} = 1.9258.$

step-2

$x_1 = 0.2$, $y_1 = 1.9258$, $y_0 = 2$, $h = 0.2$

$y_2^{(0)} = y_0 + hf(x_1, y_1)$

$= 2 + 0.2(-0.2 \times 1.926^2)$

$= 1.851620965177\mathrm{f}$

$y_2^{(1)} = y_0 + \frac{h}{2} f[(x_1, y_1) + (x_2\ y_2^{(0)})]$

$= 2 + \frac{0.2}{2}[(-0.2 \times 1.9258^2) + (-0.4 \times 1.851651777\mathrm{f}^2)]$

$= 1.788681316$

$y_2^{(2)} = y_0 + \frac{h}{2} f[(x_1, y_1) + (x_2\ y_2^{(1)})]$

$= 2 + \frac{0.2}{2}[(-0.2 \times 1.9258^2) + (-0.4 \times 1.788681316^2)]$

$= \cancel{1.85488}\ 1.797850653$

$y_2^{(3)} = y_0 + \frac{h}{2} f[(x_1, y_1) + (x_2\ y_2^{(2)})]$

$= 2 + \frac{0.2}{2}[(-0.2 \times 1.9258^2) + (-0.4 \times 1.797850653^2)]$

1.7317

$= 1.796535208$

$y_2^{(4)} = y_0 + \frac{h}{2} f[(x_1, y_1) + (x_2\ y_2^{(3)})]$

$= 2 + \frac{0.2}{2}[(-0.2 \times 1.9258^2) + (-0.4 \times 1.796535208^2)]$

$= 1.796275775$

$y_2^{(5)} = y_0 + \frac{h}{2} f[(x_1, y_1) + (x_2\ y_2^{(4)})]$

$= 1.796761621$

$y_2 = 1.796$

23/10/2024

$\frac{dy}{dx} = x^2 y \quad y(0)=1$ find $y(0.1)$

taking step size $h = 0.05$

$\frac{dy}{dx} = \log(x+y) \quad y(0)=2$ compute $y(0.2)$ & $y(0.4)$

$h = 0.2$

Runge kutta Method of 4^{th} order. 10M

$\frac{dy}{dx} = f(x,y), \; y(x_0) = y_0$

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

the process is continued starting with next x_1, y_1

$y' = 3x + \frac{y}{2}, \; y(0)=1$ calculate $y(0.2)$, $h=0.1$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.1$

$x_0 = 0$

$x_1 = x_0 + h$

$x_1 = 0 + 0.1$

$x_1 = 0.1 \quad x_2 = 0.2$

$f(x,y) = 3x + \frac{y}{2}$

Step-1

$k_1 = hf(x_0, y_0)$

$= 0.1\left(3(0) + \frac{1}{2}\right)$

$= 0.05$

$k_2 = hf\left(\underbrace{x_0 + \frac{h}{2}}_{3x}, \underbrace{y_0 + \frac{k_1}{2}}_{\frac{y}{2}}\right)$

$k_2 = 0.1\left(0 + \frac{0.1}{2} + 3\left(0 + \frac{0.1}{2}\right) + \frac{\left(1 + \frac{0.05}{2}\right)}{2}\right)$

$k_2 = 0.06625$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \quad 0.1\left(3(0) + \frac{0.7}{2}\right)$

$= 0.1\left(3\left(0 + \frac{0.1}{2}\right) + \frac{\left(1 + \frac{0.06625}{2}\right)}{2}\right)$

$= 0.06665625$

$k_4 = hf\left(x_0 + \frac{h}{2}, \frac{y_0 + k_3}{y}\right)$

$= 0.1\left(3\left(0 + \cancel{\frac{0.1}{2}}\right) + \frac{1 + 0.06665625}{2}\right) \quad = \cancel{0.13333}$

$= 0.08333$

$y_1 = y(0.1) = 1 + \frac{1}{6}\left[0.05 + 2(0.06625) + 2(0.06665) + 0.08333\right] = 1.0665 \quad \cancel{1.0708}$

step-2

$f(x,y) = 3x + \frac{y}{2}, \; x_1 = 0.1, \; y_1 = 1.0665, \; h = 0.1$

$k_1 = hf(x_1, y_1) = (0.1)\left[3(0.1)\right.$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1004$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1008$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1183$

$y_2 = y(x_2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.1672$

26/10/2021

(2) Find $y(0.2)$ & $y(0.4)$ using 4th order R.K method given that $y' = y(1+x^2)$ $y(0) = 1$

$y_0 = 1$ $x_0 = 0$

$x_1 = x_0 + h$ $\quad x_2 = x_1 + h$

$x_1 = 0 + 0.2$ $\quad = 0.2 + 0.2$

$x_1 = 0.2$ $\quad = 0.4$

$f(x,y) = y(1+x^2)$

Step-1

$\boxed{k_1 = hf(x_0, y_0)}$

$k_1 = 0.2(0.1(1+0^2))$

$= 0.2$

$\boxed{k_2 = h.f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})}$

$= 0.2.(\cancel{0+\frac{0.2}{2}}\ 1 + \frac{0.2}{2}(1+(0+\frac{0.2}{2})^2))$

$= 0.2202$

$\boxed{k_3 = h.f(x_0+\frac{h}{2}, y_0 + \frac{k_2}{2})}$

$= 0.2(1+\frac{0.2202}{2}(1+(0+\frac{0.2}{2})^2))$

$= 0.22224$

$\boxed{k_4 = h.f(x_0+\frac{h}{2}, y_0+\frac{k_3}{2})}$

$= 0.2(1+\frac{0.22224}{2}(1+(0+\frac{0.2}{2})^2))$

$= 0.22244$ $= \cancel{0.244892}$ 0.25422

$y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(0.2) = 1 + \frac{1}{6}(0.2 + 2(0.2202) + 2(0.22224) + \cancel{0.22244})$ 0.25422

$= \cancel{1.21788}$ $= 1.2232$

Step-2 $\quad x_1 = 0.2 \quad y_1 = \cancel{0}.\ 1.21788$

$k_1 = hf(x_1, y_1)$

$= \cancel{h.f}\ (y_1(1+x_1^2))$

$= 0.2(1.21788(1+0.2^2))$

$= 0.253319$

$k_2 = hf(x_1 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= h.f(y_0 + \frac{k_1}{2}(1+(x_1+\frac{h}{2})^2))$

$= 0.2(1.21788 + \frac{0.253319}{2}(1+(0.2+\frac{0.2}{2})^2))$

$= 0.271187$

$k_3 = hf(x_1+\frac{h}{2}, y_1 + \frac{k_2}{2})$

$= 0.\ hf(y_1 + \frac{k_2}{2}(1+(x_1+\frac{h}{2})^2))$

$= 0.2(1.21788 + \frac{0.271187}{2}(1+(0.2+\frac{0.2}{2})^2))$

$= 0.273135$

$k_4 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_3}{2})$

$= hf(y_1 + \frac{k_3}{2}(1+(x_1+\frac{h}{2})^2))$

$= 0.2(1.21788 + \frac{0.273135}{2}(1+(0.2+\frac{0.2}{2})^2))$

$= 0.27334$

$y_2 = y(x_2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.21788 + \frac{1}{6}(0.253319 + 2(0.271189) + 2(0.273135) + 0.27334)$

$= 1.487097$

H/w $y' = x^2 + y^2$, $y(1) = 1.5$, compute $y(1.2)$ with $h = 0.1$

$y_0 = 1.5$ $x_0 = 1$

$x_1 = x_0 + h$ $h = 0.1$

$= 1 + 0.1$

$= 1.1$

$\left.\begin{array}{l} y_1 = 1.8955 \\ y_2 = 2.5043 \end{array}\right\}$ A

$k_1 = h \cdot f(x_0; y_0)$

$= h \cdot f(x_0^2 + y_0^2)$

$= 0.1 (1^2 + 1.5^2)$

$= 0.325$

$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1 \left(\left(1 + \frac{0.1}{2}\right)^2 + \left(1.5 + \frac{0.325}{2}\right)^2\right)$

$= 0.386640$

$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1 \left(\left(1 + \frac{0.1}{2}\right)^2 + \left(1.5 + \frac{0.386640}{2}\right)^2\right)$

$= 0.3969832$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$

$= 0.1 \left((1 + 0.1)^2 + (1.5 + 0.3969832)^2\right)$

$= 0.480797$

$y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.5 + \frac{1}{6}(0.325 + 2(0.386640) + 2(0.3969832) + 0.480797)$

$= 1.8955$

$x_1 = 1.1$, $y_1 = 1.8955$ $h = 0.1$

$k_1 = h f(x_1, y_1)$

$= 0.1 ((1.1)^2 + 1.8955^2)$

$= 0.480292$

$k_2 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$= 0.1 \left[\left(1.1 + \frac{0.1}{2}\right)^2 + \left(1.8955 + \frac{0.480292}{2}\right)\right]$

$= 0.588348$

$k_3 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= 0.1 \left[\left(1.1 + \frac{0.1}{2}\right)^2 + \left(1.8955 + \frac{0.588348}{2}\right)^2\right]$

$= 0.6117303$

$k_4 = h \cdot f(x_1 + h, y_1 + k_3)$

$= 0.1 \left((1.1 + 0.1)^2 + (1.8955 + 0.6117303)^2\right)$

$= 0.7726203$

$y_2 = y(x_2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.8955 + \frac{1}{6}(0.480292 + 2(0.588348) + 2(0.6117303) + 0.7726203)$

$= 2.5043$