LAPLACE TRANSFORMS

(Laplace Transform is a method for solving linear differential equations arising in physics and Engineering)

The knowledge of Laplace transforms is an essential part of mathematics required by engineers and scientists. The Laplace Transform is an excellent tool for solving linear differential equations. The method of Laplace of transforms has the advantage of directly giving the solution of differential equations with given boundary values without necessity of first finding the general solution and then evaluating from it the arbitrary constants.

Definition :

Let f(t) be a function of 't' defined for all positive values of t. Then Laplace transforms of f(t) denoted by L{f(t)} is defined by

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\overline{f}(s)\rightarrow(1)$$

Provided that the integral exists. Here the parameter 'S' is a real (or) complex number.

The relation (1) can also be written as

$$f(t)=L^{-1}\{\overline{f}(s)\}$$

In such a case the function f(t) is called the inverse Laplace transform of $\overline{f}(s)$.The symbol 'L' which transform f(t) in to $\overline{f}(s)$ is called the Laplace transform operator. The symbol 'L' which transforms $\bar{f}$(s) to f(t) can be called the inverse Laplace transform operator.

Theorem : The Laplace transform operator is a Linear operator.

$i.e.,(i)\,L\{cf(t)\}=c.\mathrm{L}\{f(t)\}\ (ii)\,L\{f(t)+g(t)\}=L\{f(t)\}+L\{g(t)\}$

Where 'c' is constant

Proof : (i) By definition

$$L\{cf(t)\}=\int_0^\infty e^{-st}cf(t)\,dt=c\int_0^\infty e^{-st}f(t)\,dt=cL\{f(t)\}$$

(ii) By definition

$$L\{f(t)+g(t)\}=\int_0^\infty e^{-st}\{f(t)+g(t)\}dt$$

$$=\int_0^\infty e^{-st}f(t)dt+\int_0^\infty e^{-st}g(t)dt=L\{f(t)\}+L\{g(t)\}$$

Similarly the inverse transforms of the sum of two or more functions of 'S' is the sum of the inverse transforms of the separate functions.

Thus, $L^{-1}\left\{\overline{f}(s)+\overline{g}(s)\right\}=L^{-1}\left\{\overline{f}(s)\right\}+L^{-1}\left\{\overline{g}(s)\right\}=f(t)+g(t)$

Corolarry : $L\left\{c_1 f(t)+c_2 g(t)\right\}=c_1 L\left\{f(t)\right\}+c_2 L\left\{g(t)\right\}$, where c_1, c_2 are constants

Exponential order : A function f(t) is said to be of exponential order 'a'

if $\underset{t\to\infty}{lt}\, e^{-st} f(t)=$ a finite quantity.

1. The function t^2 is of exponential order

2. The function e^{t^3} is not of exponential order (which is not limit)

Piece – wise Continuous function : A function f(t) is said to be piece-wise continuious over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided in to a finite number of sub intervals, in each of which f(t) is continuious and has both right and left hand limits at every end point of the subinterval.

Sufficient conditions for the existence of the Laplace transform of a function:

The function f(t) must satisfy the following conditions for the existence of the L.T.

i. The function f(t) must be piece-wise continuous (or) sectionally continuous in any limited interval $0 < a \le t \le b$

ii. The function f(t) is of exponential order.

Laplace Transforms of elementary functions :

1. Prove that $L\{1\}=\frac{1}{s}$

Pf: By definition

$$L\{1\}=\int_0^\infty e^{-st}.1dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty=\frac{e^{-\infty}}{-s}-\frac{e^{0}}{-s}=0+{}^1\!/_s \; if\ s>0$$

$$L\{1\}={}^1\!/_s \left(\because e^{-\infty}=0\right)$$

2. Prove that $L\{t\}={}^1\!/_{s^2}$

Pf: By definition

$$L\{t\}=\int_0^\infty e^{-st}.tdt=\left[t.\left(\frac{e^{-st}}{-s}\right)-\int 1.\frac{e^{-st}}{-s}dt\right]_0^\infty$$

$$=\left[t.\frac{e^{-st}}{-s}-\frac{e^{-st}}{(-s)^2}\right]_0^\infty={}^1\!/_{s^2}$$

3. Prove that $L\left\{t^n\right\} = \dfrac{n!}{s^{n+1}}$ where n is a +ve integer

Prof : By definition

$$L\left\{t^n\right\} = \int_0^\infty e^{-st}.t^n dt = \left[t^n.\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n.t^{n-1}.\frac{e^{-st}}{-s}dt$$

$$= 0 - 0 + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

$$= \frac{n}{s}L\left\{t^{n-1}\right\}$$

$$Similarly\ L\left\{t^{n-1}\right\} = \frac{n-1}{s}L\left\{t^{n-2}\right\}$$

$$L\left\{t^{n-2}\right\} = \frac{n-2}{s}L\left\{t^{n-3}\right\}$$

By repeatedly applying this, we get

$$L\left\{t^n\right\} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.....\frac{2}{s}.\frac{1}{s}L\left\{t^{n-n}\right\}$$

$$= \frac{n!}{s^n}L\{1\} = \frac{n!}{s^n}.\frac{1}{s} = \frac{n!}{s^{n+1}}$$

Note : $L\left\{t^n\right\}$ can also be expressed in terms of Gamma function.

$$i.e., \mathrm{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}\left(\because \Gamma(n+1) = n!\right)$$

Def. If n>0 then Gamma function is defined by $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$

The following are some important properties of the Gamma function.

1. $\Gamma(n+1) = n\Gamma n\ if\ n > 0$

2. $\Gamma(n+1) = n!$ if n is a +ve integer

3. $\Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

We have $L\left\{t^n\right\} = \int_0^\infty e^{-st}.t^n dt$

Putting x=st on R.H.S, we get

$$L\left\{t^n\right\}t = \int_0^\infty e^{-x}.\frac{x^n}{s^n}.\frac{1}{s}dx \qquad \begin{pmatrix} x = st \\ \frac{1}{s}dx = dt \end{pmatrix}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}.x^n dx \qquad \begin{pmatrix} When \quad t = 0, x = 0 \\ When \quad t = \infty, x = \infty \end{pmatrix}$$

$L\{t^n\} = \frac{1}{s^{n+1}}.\Gamma(n+1)$

If 'n' is +ve integer then $\Gamma(n+1) = n!$

$\therefore L\{t^n\} = n!/s^{n+1}$

Note : Value of $\Gamma(n)$ in terms of factorial

$\Gamma(2) = 1 \times \Gamma(1) = 1!$

$\Gamma(3) = 2 \times \Gamma(2) = 2!$

$\Gamma(4) = 3 \times \Gamma(3) = 3!$

In general $\Gamma(n+1) = n!$ Provided 'n' is a +ve integer.

Taking n=0, it defined 0! = $\Gamma(1) = 1$

4. Prove that $L\{e^{at}\} = \frac{1}{s-a}$

Prof : $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-e^{-\infty}}{s-a} + \frac{e^0}{s-a} = \frac{1}{s-a} \; if \; s > a$$

Similarly $L\{e^{-at}\} = \frac{1}{s+a} \; if \; s > -a$

5. Prove that $L\{\sinh at\} = \frac{a}{s^2 - a^2}$

Pf. $L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right] = \frac{2a}{2(s^2-a^2)} = \frac{a}{s^2-a^2}$$

6. Prove that $L\{\cosh at\} = \frac{s}{s^2 - a^2}$

Pf. $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}\left[L\{e^{at}\}+L\{e^{-at}\}\right] = \frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{2s}{2\left(s^2-a^2\right)} = \frac{s}{s^2-a^2}$$

7. Prove that $L\{\sin at\} = \frac{a}{s^2+a^2}$

Pf. $L\{\sin at\} = \int_0^\infty e^{-st}\sin at dt$

$$= \left[\frac{e^{-st}}{s^2+a^2}\left(-s\sin at - a\cos at\right)\right]_0^\infty \left[\because \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)\right]$$

$$= \frac{a}{s^2+a^2}$$

8. Similarly $L\{\cos at\} = \frac{s}{s^2+a^2}$

Aliter : We know that

$$L\{e^{at}\} = \frac{1}{s-a}$$

Replace 'a' by 'ia' we get

$$L\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)}$$

$$i.e., \mathrm{L}\{cosat + i\sin at\} = \frac{s+ia}{s^2+a^2}$$

Equating the real and imaginary parts on both sides, we have

$$L\{\cos at\} = \frac{s}{s^2+a^2} \; and \; L\{\sin at\} = \frac{a}{s^2+a^2}$$

Problems

1. Find the Laplace transforms of $\left(t^2+1\right)^2$

Sol. Here $f(t) = \left(t^2+1\right)^2 = t^4+2t^2+1$

$$L\{(t^2+1)^2\} = L\{t^4+2t^2+1\} = L\{t^4\}+2L\{t^2\}+L\{1\}$$

$$=\frac{4!}{s^{4+1}}+2.\frac{2!}{s^3}+\frac{1}{s}=\frac{4!}{s^5}+2.\frac{2!}{s^3}+\frac{1}{s}$$

$$=\frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s}=\frac{1}{s^5}\left(24+4s^2+s^4\right)$$

2. Find the Laplace transform of $L\left\{\frac{e^{-at}-1}{a}\right\}$

Sol: $L\left\{\frac{e^{-at}-1}{a}\right\}=\frac{1}{a}L(e^{-at}-1)=\frac{1}{a}[L(e^{-at})-L(1)]$

$$=\frac{1}{a}\left[\frac{1}{s+a}-\frac{1}{s}\right]=-\frac{1}{s(s+a)}$$

1. Find the Laplace transform of Sin2tcost

Sol: *Since* $sin2t\cos t=\frac{1}{2}[2\sin 2t\cos t]=\frac{1}{2}[\sin 3t+\sin t]$

$$L(sin2t\ cost)=L\left[\frac{1}{2}(sin3t+sint)\right]$$

$$=\frac{1}{2}[L(sin3t)+L(sint)]$$

$$=\frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]=\frac{2(s^2+3)}{(s^2+1)(s^2+9)}$$

4. Find the Laplace transform of $\cosh^2 2t$

Sol: Since $\cosh^2 2t=\frac{1}{2}\left[1+\cosh 4t\right]$

$$L\{\cosh^2 2t\}=\frac{1}{2}\left[L(1)+L\{\cosh 4t\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]=\frac{s^2-8}{s\left(s^2-16\right)}$$

5. Find the Laplace transform of $\cos^3 3t$

Sol: Since cos9t=cos3(3t)

$\cos 9t=4\cos^3 3t-3\cos 3t$

(or) $\cos^3 3t=\frac{1}{4}\left[\cos 9t+3\cos 3t\right]$

$$L\{\cos^3 3t\}=\frac{1}{4}L\{\cos 9t\}+\frac{3}{4}L\{\cos 3t\}$$

$$=\frac{1}{4}.\frac{s}{s^2+81}+\frac{3}{4}.\frac{s}{s^2+9}$$

$$=\frac{s}{4}\left[\frac{1}{s^2+81}+\frac{3}{s^2+9}\right]=\frac{s\left(s^2+63\right)}{\left(s^2+9\right)\left(s^2+81\right)}$$

6. Find the Laplace transforms of $\left(\sin t+\cos t\right)^2$

Sol: Since $\left(\sin t+\cos t\right)^2=\sin^2 t+\cos^2 t+2\sin t\cos t$

$$=1+\sin 2t$$

$$L\{(\sin t+\cos t)^2\}=L\{1+\sin 2t\}$$

$$=L\{1\}+L\{\sin 2t\}$$

$$=\frac{1}{s}+\frac{2}{s^2+4}=\frac{s^2+2s+4}{s\left(s^2+4\right)}$$

7. Find the Laplace transforms of cost cos2t cos3t

Sol: $\cos t\cos 2t\cos 3t=\frac{1}{2}.\cos t\left[2.\cos 2t.\cos 3t\right]$

$$=\frac{1}{2}\cos t[\cos 5t+\cos t]=\frac{1}{2}[\cos t\cos 5t+\cos^2 t]$$

$$=\frac{1}{4}\left[2\cos t\cos 5t+2\cos^2 t\right]=\frac{1}{4}\left[\left(\cos 6t+\cos 4t\right)+\left(1+\cos 2t\right)\right]$$

$$=\frac{1}{4}[1+\cos 2t+\cos 4t+\cos 6t]$$

$$\therefore L\{\cos t\cos 2t\cos 3t\}=\frac{1}{4}L\{1+\cos 2t+\cos 4t+\cos 6t\}$$

$$=\frac{1}{4}[L\{1\}+L\{\cos 2t\}+L\{\cos 4t\}+L\{\cos 6t\}]$$

$$=\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+4}+\frac{s}{s^2+16}+\frac{s}{s^2+36}\right]$$

1. Find the Laplace transforms of

$(i)t^3+5\cos t(ii)\cos^2 t(iii)\cosh at$

2. Find L.T. of $\sin^2$t

Sol. $L(sin^2 t)=L\left(\frac{1-cos2t}{2}\right)$

$$=\frac{1}{2}[L(1)-L(cos2t)]$$

$$=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

Change of Scale Property:

If $L\{f(t)\}=\bar{f}(s)$ then $L\{f(at)\}=\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Proof: By the definition we have

$L(f(at)) = \int_0^\infty e^{-st} f(at)\ dt$

Put $at = u,\qquad dt = \frac{du}{a}$

when $t = \infty$ *then* $u = \infty$

and $t = 0$ *then* $u = 0$

$$= \frac{1}{a}.\int_0^\infty e^{-\left(\frac{s}{a}\right).u} f(u)\,du = \frac{1}{a}.\overline{f}\left(s/a\right)$$

Properties of Laplace transforms:

I. Linearity Property: If a,b,c be any constants and f,g,h any functions of t, then

$L\{af(t) + bg(t) - ch(t)\} = a.L\{f(t)\} + b.L\{g(t)\} - cL\{h(t)\}$

Proof. By the definition

$$L\{af(t) + bg(t) - ch(t)\} = \int_0^\infty e^{-st}\{af(t) + bg(t) - ch(t)\}dt$$

$$= a.\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt - c\int_0^\infty e^{-st} h(t)\,dt$$

$a.L\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$

II. First shifting property: If $L\{f(t)\} = \overline{f}(s)$ then $L\{e^{at} f(t)\} = \overline{f}(s - a)$

∵Proof: By the definition

$$L\ (e^{at} f(t)) = \int_0^\infty e^{-st} e^{at} f(t)\ dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-ut} f(t)\,dt \text{ where } u = s - a$$

$$= \overline{f}(u) = \overline{f}(s - a)$$

Note : Using the above property, we have

$L\{e^{-at} f(t)\} = \overline{f}(s + a)$

-Applications of this property, we obtain the following results

1.$L\{e^{-at} t^n\} = \frac{n!}{(s-a)^{n+1}}\left\{\because L(t^n) = \frac{n!}{s^{n+1}}\right\}$

2. $L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}\left\{\because L(\sin bt) = \frac{b}{s^2 + b^2}\right\}$

3. $L\{e^{at} Cos\ bt\} = \frac{s-a}{(s-a)^2 + b^2}\left\{\because L(\mathrm{Cos}\, bt) = \frac{s}{s^2 + b^2}\right\}$

4. $L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}\left\{\because L(\sinh bt) = \frac{b}{s^2 - b^2}\right\}$

5. $L\{e^{at} Cosh\, bt\} = \frac{s-a}{(s-a)^2-b^2}\left\{\because L(\text{Cosh}\, bt) = \frac{s}{s^2-b^2}\right\}$

Problems

1. Find the Laplace Transforms of $t^3 e^{-3t}$

Sol: Since $L\{\text{t}^3\} = \frac{3!}{s^4}$

Now applying first shifting theorem, we get

$L\{\text{t}^3\, e^{-3t}\} = \frac{3!}{(s+3)^4}$

2. Find the L.T. of $e^{-t}\cos 2t$

Sol: Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

Now applying first shifting theorem we get

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Find the $L.T\ of\ e^{-2t}\cos 2t$

Ans. $\frac{s+2}{(s+2)^2+4}$

Second translation (or) second Shifting theorem:

If $\mathcal{L}\{f(t)\} = \overline{f}(s) and g(t) = \begin{cases} f(t-a), t>a \\ 0 \quad\quad t<a \end{cases} then \mathcal{L}\{g(t)\} = e^{-as}\overline{f}(s)$

Proof: By the definition

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)dt = \int_0^a e^{-st} g(t)dt + \int_a^\infty e^{-st} g(t)dt$$

$$= \int_0^\infty e^{-st}.odt + \int_a^\infty e^{-st} f(t-a)dt = \int_a^\infty e^{-st} f(t-a)dt$$

Let t-a=u so that dt=du

And also u=0 when t=a and u= ∞ when t= ∞

$$\therefore \mathcal{L}\{g(t)\} = \int_0^\infty e^{-s(u+a)} f(u)du = e^{-as}\int_0^\infty e^{-su} f(u)du = e^{-as}\int_a^\infty e^{-st} f(t)dt$$

$$= e^{-as}\mathcal{L}\{f(t)\} = e^{-as}\overline{f}(s)$$

<u>Another Form of second shifting theorem</u> :

If $\mathcal{L}\{f(t)\} = \overline{f}(s) and a > 0\ then \mathcal{L}\{F(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$ where H(t)=$\begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

and H(t) is called Heaviside unit step function.

Proof. By the definition

$$\mathcal{L}\{F(t-a)H(t-a)\} = \int_0^{\infty} e^{-st} F(t-a)H(t-a)dt \to (1)$$

Put $t-a=u\, so\, that\, dt=du\, and\, also\, when\, t=0, u=-a\, when\, t=\infty, u=\infty$

Then $\mathcal{L}\{F(t-a)H(t-a)\} = \int_a^{\infty} e^{-s(u+a)} F(u)H(u)du. [by eq(1)]$

$$= \int_{-a}^{0} e^{-s(u+a)} F(u)H(u)du + \int_{0}^{\infty} e^{-s(u+a)} F(u)H(u)du$$

$= \int_{-a}^{0} e^{-s(u+a)} F(u).0du + \int_0^{\infty} e^{-s(u+a)} F(u).1du$[By the definition of H(t)]

$$= \int_0^{\infty} e^{-s(u+a)} F(u)du = e^{-as} \int_a^{\infty} e^{-su} F(u)du$$

$= e^{-sa} \int_0^{\infty} e^{-st} F(t)dt$ by property of Definite Integrals

$$= e^{-as} \mathcal{L}\{F(t)\} = e^{-as} \overline{f}(s)$$

Note : $H(t-a)\, is\, also\, denoted\, by\, u(t-a)$

<u>Problems</u>

1. Find the L.T. of g(t) when $g(t) = \begin{cases} \cos\left(t-\pi/3\right) & if\ t > \pi/3 \\ 0 & if\ t < \pi/3 \end{cases}$

Sol. Let $f(t) = \cos t$

$$\therefore \mathcal{L}\{F(t)\} = \mathcal{L}\{cost\} = \frac{s}{s^2+1} = \overline{f}(s)$$

$$g(t) = \begin{cases} f(t-\pi/3) = cos(t-\pi/3) if t > \pi/3 \\ 0 \quad if t < \pi/3 \end{cases}$$

Now applying second shifting theorem then we get

$$\mathcal{L}\{g(t)\} = e^{\frac{-\pi s}{3}} \left(\frac{s}{s^2+1}\right) = \frac{s.e^{\frac{-\pi s}{3}}}{s^2+1}$$

2. Find the L.T. of $i.(t-2)^3 u(t-2)\ ii.e^{-3t} u(t-2)$

Sol. (i) Comparing the given function with f(t-a) u(t-a), we have a=2 and f(t)=t^3

$$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} = \overline{f}(s)$$

Now applying second shifting theorem then we get

$$\mathcal{L}\{(t-2)^3 4(t-2)\} = e^{-2s} \frac{6}{s^4} = \frac{6e^{-2s}}{s^4}$$

ii.$\mathcal{L}\{e^{-st} u(t-2)\} = \mathcal{L}\{e^{-s(t-2)}. e^{-6} u(t-2)\} = e^{-6} \mathcal{L}\{e^{-3(t-2)} u(t-2)\}$

$f(t) = e^{-3t}$ then $\overline{f}(s) = \frac{1}{s+3}$ and applying second shifting theorem then we get

$$\mathcal{L}\{e^{-3t}u(t-2)\} = e^{-6}.\, e^{-2s}\frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Practice problems:

1. Find $L\,\{cos^4 t\} and \mathcal{L}\{sin^4 t\}$

Hint :

$$\mathcal{L}\{cos^4 t\} = \frac{1}{8}[\mathcal{L}\{3 + cos4t + 4cos2t\}]$$

$$\mathcal{L}\{sin^4 t\} = \frac{1}{8}[\mathcal{L}\{3 + cos4t - 4cos2t\}]$$

2. Find $L(\sqrt{t})$

Sol) $\mathrm{L}\left\{\sqrt{t}\right\} = \mathrm{L}\left[t^{1/2}\right] = \frac{\left\lceil \frac{1}{2}+1 \right.}{s^{\frac{1}{2}+1}}$ where n is not an integer

$$= \frac{\frac{1}{2}\left\lceil \frac{1}{2} \right.}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} \because \left\lceil n+1 \right. = n\left\lceil n \right.$$

3. Find $\mathcal{L}\{sin(\omega t + \alpha)\}$,where α is a constant

Sol: $\mathcal{L}\{sin(\omega t + \alpha)\} = \mathcal{L}\{sin\omega t cos\alpha + cos\omega t sin\alpha\}$

$$= cos\alpha\mathcal{L}\{sin\omega t\} + sin\alpha\mathcal{L}\{cos\omega t\} = cos\alpha\frac{\omega}{s^2+\omega^2} + sin\alpha\frac{\omega}{s^2+\omega^2}$$

Problems in change of scale property:

1. Find $\mathcal{L}\{\sinh 3t\}$

Sol. $\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1} = \overline{f}(s)$

$\therefore \mathcal{L}\{\sinh 3t\} = \frac{1}{3}\overline{f}({}^{s}/_{3})$(Change of scale property)

$$= \frac{1}{3}\frac{1}{\left({}^{s}/_{3}\right)^2 - 1} = \frac{3}{s^2 - 9}$$

2. Find $\mathcal{L}\{\cos 7t\}$

Sol. $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1} = \overline{f}(s) - say$

$\mathcal{L}\{\cos 7t\} = \frac{1}{7}\overline{f}\left({}^{s}/_{7}\right)$ (Change of scale property)

$$\mathcal{L}\{\cos 7t\} = \frac{1}{7}\frac{{}^{s}/_{7}}{\left({}^{s}/_{7}\right)^2 + 1} = \frac{s}{s^2 + 49}$$

Theorem: If $\mathcal{L}\{f(t)\} = \overline{f}(s) then \mathcal{L}\{tf(t)\} = \frac{-d}{ds}\overline{f}(s)$

Pf: By the definition $\overline{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds}\{\overline{f}(s)\} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

By Leibnitz's rule for differentiating under the integral sign,

$$\frac{d}{ds}\overline{f}(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt = -\int_0^\infty e^{-st}\{tf(t)\}dt = -\mathcal{L}\{tf(t)\}$$

Thus $\mathcal{L}\{tf(t)\} = \frac{-d}{ds}\overline{f}(s)$

$$\therefore \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} = \overline{f}(s)$$

Note : Leibnitz's Rule

If $f(x,\alpha)$ and $\frac{\partial}{\partial\alpha} f(x,\alpha)$ be continuous functions of x and α then

$$\frac{d}{d\alpha}\left\{\int_a^b f(x,\alpha)dx\right\} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx$$

Where a,b are constants independent of α

Problems:

1. Find L.T of tcos at

Sol. Since $\mathcal{L}\{tcosat\} = \frac{s}{s^2+a^2}$

$$\mathcal{L}\{tcosat\} = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$=\frac{-s^2+a^2-s.2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$

2. Find t^2sin at

Sol. Since $\mathcal{L}\{sinat\} = \frac{a}{s^2+a^2}$

$$\mathcal{L}\{t^2.sinat\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$=\frac{d}{ds}\left(\frac{-2as}{\left(s^2+a^2\right)^2}\right)=\frac{2a\left(3s^2-a^2\right)}{\left(s^2+a^2\right)^3}$$

3. Find $\mathcal{L}\{e^{-2t}\cos 2t\}$ Ans: $\frac{s+2}{(s+2)^2+4}$

4. Find $\mathcal{L}\{tsinat\}$Ans : $\frac{2as}{\left(s^2+a^2\right)^2}$

5. Find $L.T\ of\ te^{-t}\sin 3t$

Sol. Since $\mathcal{L}\{sin\ 3t\}=\frac{3}{s^2+3^2}$

$$\therefore\ \mathcal{L}\{tsin\ 3t\}=\frac{-d}{ds}\left[\frac{3}{s^2+3^2}\right]=\frac{6s}{(s^2+9)^2}$$

Now using the shifting property, we get

$$\mathcal{L}\{te^{-t}sin\ 3t\}=\frac{6(s+1)}{((s+1)^2+9)^2}=\frac{6(s+1)}{(s^2+2s+10)^2}$$

6. Find $\mathcal{L}\{te^{2t}sin\ 3t\}$

Sol. Since $\mathcal{L}\{sin\ 3t\}=\frac{3}{s^2+9}$

$$\therefore \mathrm{L}\left\{e^{2t}\sin 3t\right\}=\frac{3}{\left(s-2\right)^2+9}=\frac{3}{s^2-4s+13}$$

$$\mathcal{L}\{te^{2t}sin\ 3t\}=(-1)\frac{d}{ds}\left[\frac{3}{s^2-4s+13}\right]=(-1)\left[\frac{0-3(2s-4)}{(s^2-4s+13)^2}\right]$$

$$=\frac{3\left(2s-4\right)}{\left(s^2-4s+13\right)^2}=\frac{6\left(s-2\right)}{\left(s^2-4s+13\right)^2}$$

7. Find the L.T. of $\left(1+te^{-t}\right)^2$

Sol. Since $(1+te^{-t})^2=1+2te^{-t}+t^2e^{-2t}$

$$\therefore \mathrm{L}\left(1+te^{-t}\right)^2=\mathrm{L}\left\{1\right\}+2\mathrm{L}\left\{te^{-t}\right\}+\mathrm{L}\left\{t^2e^{-2t}\right\}$$

$$=\frac{1}{s}+2\left(-1\right)\frac{d}{ds}\left(\frac{1}{s+1}\right)+\left(-1\right)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$=\frac{1}{s}+\frac{2}{\left(s+1\right)^2}+\frac{d}{ds}\left(\frac{-1}{\left(s+2\right)^2}\right)$$

$$=\frac{1}{s}+\frac{2}{\left(s+1\right)^2}+\frac{2}{\left(s+2\right)^3}$$

8. Find the L.T of t^3e^{-3t} (already we have solved by another method)

Sol. $\mathcal{L}\{t^3e^{-3t}\} = (-1)^3 \frac{d^3}{ds^3}\mathcal{L}\{e^{-3t}\} = -\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = \frac{-3!(-1)^3}{(s+3)^4}$

$= \frac{3!}{(s+3)^4}$

9. Find the L.T of $f(t)=te^{-t}\sin t$

10. Find the L.T of $t^2\sin at$

11. Find $\mathcal{L}(\cosh at \sin at)$

Sol. $\mathcal{L}(\cosh at \sin at) = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}.\sin at\right\}$

$=\frac{1}{2}[\mathcal{L}\{e^{at}\sin at\} + \mathcal{L}\{e^{-at}\sin at\}]$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$

12. Find the L.T of the function $f(t) = (t-1)^2, \quad t > 1$
$= 0 \quad 0 < t < 1$

Sol. By the definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \int_0^1 e^{-st}f(t)dt + \int_1^\infty e^{-st}f(t)dt$$

$= \int_0^1 e^{-st}0dt + \int_1^\infty e^{-st}(t-1)^2 dt$

$$= \int_1^\infty e^{-st}(t-2)^2dt = \left[(t-1)^2\frac{e^{-st}}{-s}\right]_1^\infty - \int_1^\infty 2(t-1)\frac{e^{-st}}{-s}dt$$

$= 0 + \frac{2}{s}\int_1^\infty e^{-st}(t-1)dt$

$= \frac{2}{s}\left[\left\{(t-1)\left(\frac{e^{-st}}{-s}\right)\right\}_1^\infty - \int_1^\infty \frac{e^{-st}}{-s}dt\right]$

$= \frac{2}{s}\left[0 + \frac{1}{s}\int_1^\infty e^{-st}dt\right] = \frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_1^\infty = \frac{-2}{s^3}\left(e^{-st}\right)_1^\infty$

$= \frac{-2}{s^3}\left(0 - e^{-s}\right) = \frac{2}{s^3}e^{-s}$

13. Find the L.T of f(t) defined as $f(t) = (t-1)^2$, t >1

$= 0$, 0<t<1

Sol. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$

$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$

$$= \int_0^2 e^{-st}.0dt + \int_2^\infty e^{-st}\, 3dt$$

$= 0 + \int_2^\infty e^{-st} 3dt = \frac{-3}{s}\left(e^{-st}\right)_2^\infty = \frac{-3}{s}\left(0 - e^{-2s}\right)$

$= \frac{3}{s} e^{-2s}$

14. $\mathcal{L}\{t.e^{-4t} Sin3t\}$ Ans : $\frac{6(s+4)}{\left(s^2+8s+25\right)^2}$

15. $\mathcal{L}\{t.sin^2 3t\}$ Ans : $\frac{54\left(s^2+12\right)}{s^2\left(s^2+36\right)^2}$

16. S.T $\mathcal{L}\{t.\sin 3t.cos\, 2t\} = s\left[\frac{5}{(s^2+25)^2} + \frac{1}{(s^2+1)^2}\right]$

17.. Find the L.T of $te^{-3t}\cos 2t$ Ans : $\frac{s^2+6s+5}{(s^2+6s+13)^2}$

18. Find the L.T of $t\cos(at+b)$ Ans : $\frac{1}{\left(s^2+a^2\right)^2}\left[\left(s^2-a^2\right)\cos b - 2as\sin b\right]$

19. Find the L.T of $te^{-t}\cosh t$

Theorem: If $\mathcal{L}\{f(t)\} = \overline{f}(s) then \mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \overline{f}(s)ds$

Proof: We have $\overline{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Now integrating both sides w.r.t s from s to ∞, we have

$\int_0^\infty \overline{f}(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right] ds$

$= \int_0^\infty \int_s^\infty f(t) e^{-st} dsdt$ (change the order of integration)

$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st} ds\right] dt$ ($\because$ t is independent of 's')

$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt$

$= \int_0^\infty e^{-st}\frac{f(t)}{t} dt (or) \mathcal{L}\left\{\frac{1}{t}f(t)\right\}$

Problems:

1. Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

Sol. Since $\mathcal{L}\{sint\} = \frac{1}{s^2+1} = \overline{f}(s)$

Division by 't', we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{1}{s^2+1}ds$$

$$= [Tan^{-1}s]_s^\infty = Tam^{-1}\infty - Tan^{-1}s$$

$$= \pi/2 - Tan^{-1}s = \cot^{-1}s$$

2. Find the L.T of $\frac{\sin at}{t}$

Sol. Since $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Division by t, we have

$$\mathcal{L}\left\{\frac{\text{sina}\, t}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{a}{s^2+a^2}ds$$

$$= a.\frac{1}{a}\left[Tan^{-1}\, s/a\right]_s^\infty = Tan^{-1}\infty - Tan^{-1}\, s/a$$

$$= \pi/2 - Tan^{-1}\left(s/a\right) = \cot^{-1} s/a$$

3. Find $\mathcal{L}\left\{\frac{\sin 2t}{t}\right\}$ Ans : $\cot^{-1}\left(s/2\right)$

4. Evaluate $\mathcal{L}\left\{\frac{1-\cos \text{a}\, t}{t}\right\}$

Sol. Since $\mathcal{L}\{1-\cos \text{a}\, t\} = \mathcal{L}\{1\} - \mathcal{L}\{cosat\} = \frac{1}{s} - \frac{s}{s^2+a^2}$

$$\mathcal{L}\left\{\frac{1-\cos \text{a}\, t}{t}\right\} = \int_s^\infty\left(\frac{1}{s} - \frac{s}{s^2+a^2}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log s - \log\left(s^2+a^2\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+a^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[l\,og\left(\frac{1}{1+a^2/s^2}\right)\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log\frac{s^2}{s^2+a^2}\right]$$

$$= -\frac{1}{2}l\,og\left(\frac{s^2}{s^2+a^2}\right) = \log\left(\frac{s^2}{s^2+a^2}\right)^{\frac{-1}{2}} = \log\sqrt{\frac{s^2+a^2}{s^2}}$$

5. Evaluate $\mathcal{L}\left\{\frac{1-\cos t}{t}\right\}$Ans: $\log\sqrt{\frac{s^2+1}{s}}$

(Putting a=1 in the above problem)

6. Find $\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol. $\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty=\left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$=\underset{s\to\infty}{lt}\left\{\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right\}-\log\left(\frac{s+a}{s+b}\right)$$

$$=\log 1-\log(s+a)+\log(s+b)=\log\left(\frac{s+b}{s+a}\right)$$

7. Find $\mathcal{L}\{tcos(at+b)\}$

Sol. $\mathcal{L}\{cos(at+b)\}=\mathcal{L}\{\cos at\cos b-\sin at\sin b\}$

$=\cos b.\mathcal{L}\{\cos at\}-\sin b\,\mathcal{L}\{\sin at\}$

$=\cos b.\frac{s}{s^2+a^2}-\sin b.\frac{a}{s^2+a^2}$

$\mathcal{L}\{t.cos(at+b)\}=\frac{-d}{ds}\left[\cos b.\frac{s}{s^2+a^2}-\sin b.\frac{a}{s^2+a^2}\right]$

$$=-\cos b.\left(\frac{s^2+a^2.1-s.2s}{\left(s^2+a^2\right)^2}\right)+\sin b\left(\frac{\left(s^2+a^2\right).0-a.2s}{\left(s^2+a^2\right)^2}\right)$$

$$=\frac{1}{\left(s^2+a^2\right)^2}\left[\left(s^2-a^2\right)^2\cos b-2as\sin b\right]$$

Laplace transforms of Derivatives:

If $f^1(t)$ be continuous and $\mathcal{L}\{f(t)\}=\overline{f}(s) then \mathcal{L}\{f^1(t)\}=s\overline{f}(s)-f(0)$

Proof : By the definition

$$\mathcal{L}\{f^1(t)\}=\int_0^\infty e^{-st}f^1(t)dt$$

$=\left[e^{-st}f(t)\right]_0^\infty-\int_0^\infty(-s)e^{-st}f(t)dt$ (integrating by parts)

$=\left[e^{-st}f(t)\right]_0^\infty+s\int_0^\infty e^{-st}f(t)dt$

$=\underset{t\to\infty}{lt}\,e^{-st}f(t)-f(0)+s.\mathcal{L}\{f(t)\}$

Since f(t) is exponential order

$\therefore \underset{t\to\infty}{lt}\, e^{-st} f(t) = 0$

$$\therefore \mathcal{L}\{f^1(t)\} = 0 - f(0) + s\mathcal{L}\{f(t)\}$$

$$= s\overline{f}(s) - f(0)$$

The Laplace Transform of the second derivative $f^{11}(t)$ is similary obtained.

$$\therefore \mathcal{L}\{f^{11}(t)\} = s.\mathcal{L}\{f^1(t)\} - f^1(0)$$

$= s.\left[s\overline{f}(s) - f(0)\right] - f^1(0)$

$= s^2\overline{f}(s) - sf(0) - f^1(0)$

$$\therefore \mathcal{L}\{f^{111}(t)\} = s.\mathcal{L}\{f^{11}(t)\} - f^{11}(0)$$

$$= s[s^2\mathcal{L}\{f(t)\} - sf(0) - f^1(0)] - f^{11}(0)$$

$$= s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf^1(0) - f^{11}(0)$$

Proceeding similarly, we have

$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^1(0) \dots \dots f^{n-1}(0)$

Note 1 : $\mathcal{L}\{f^n(t)\} = s^n\overline{f}(s)\ if\ f(0) = 0\ and\ f^1(0) = 0, f^{11}(0) = 0 \dots\ f^{n-1}(0) = 0$

Note 2: Now $|f(t)| \le M.e^{at} forallt \ge 0\ andforsomeconstants\ a\ and\ M.$

We have $|e^{-st}f(t)| = e^{-st}|f(t)| \le e^{at}.Me^{at}$

=M.$e^{-(s-a)t} \to 0\ as\ t\ \to \infty$ if s>a

$$\therefore \underset{t\to\infty}{lt}\, e^{-st}f(t) = 0\ for\ \ s > a$$

Problems:

Using the theorem on transforms of derivatives, find the Laplace Transform of the following functions.

i. e^{at} ii. Cosat iii. t sin at

i. $Let\ f(t) = e^{at}\ Then\ f^1(t) = a.e^{at}\ and\ f(0) = 1$

$$Now\ \mathcal{L}\{f^1(t)\} = s.\mathcal{L}\{\ f(t)\} - f(0)$$

$$i.e., \mathcal{L}\{ae^{at}\} = s.\mathcal{L}\{e^{at}\} - 1$$

$$i.e., \mathcal{L}\{e^{at}\} - s.\mathcal{L}\{e^{at}\} = -1$$

$i.e., (a - s)\mathcal{L}\{e^{at}\} = -1$

$\therefore \mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

ii. $Let\ f(t) = cosat\ then\ f^1(t) = -asinat\ and\ f^{11}(t) = -a^2cosat$

$$\therefore \mathcal{L}\{f^{11}(t)\} = s^2\mathcal{L}\{\ f(t)\} - s.f(0) - f^1(0)$$

$Now\ f(0) = \cos 0 = 1\ and\ f^1(0) = -a\sin 0 = 0$

$Then\ \mathcal{L}\{-a^2\cos at\} = s^2\mathcal{L}\{\cos at\} - s.1 - 0$

$$\Rightarrow -a^2\mathcal{L}\{\cos at\} - s^2\mathcal{L}\{\cos at\} = -s$$

$$\Rightarrow -(s^2 + a^2)\mathcal{L}\{\cos at\} = -s \Rightarrow \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

iii. $Let\ f(t) = t\sin at\ then\ f^1(t) = \sin at + at\cos at$

$f^{11}(t) = a\cos at + a[\cos at - at\sin at] = 2a\cos at - a^2 t\sin at$

Also $f(0) = 0$ and $f^1(0) = 0$

$Now\ \mathcal{L}\{f^{11}(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f^1(0)$

$$i.e.,\mathcal{L}\{2a\cos at - a^2 t\sin at\} = s^2\mathcal{L}\{t\sin at\} - 0 - 0$$

$$i.e.,2a\ \mathcal{L}\{\cos at\} - a^2\mathcal{L}\{t\sin at\} - s^2\mathcal{L}\{t\sin at\} = 0$$

$$i.e., -(s^2 + a^2)\mathcal{L}\{t\sin at\} = \frac{-2as}{s^2 + a^2} \Rightarrow \mathcal{L}\{t\sin at\} = \frac{2as}{(s^2 + a^2)^2}$$

Laplace Transform of Integrals:

If $\mathcal{L}\{f(t)\} = \overline{f}(s)\ then\ \mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Pf. $Let\ g(t) = \int_0^t f(x)dx$

Then $g^1(t) = \frac{d}{dt}\left[\int_0^t f(x)\,dx\right] = f(t)\ and\ g(0) = 0$

Taking Laplace Transform on both sides

$$\mathcal{L}\{g^1(t)\} = \mathcal{L}\{f(t)\}$$

But $\mathcal{L}\{g^1(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\} - 0 \quad \therefore g(0) = 0$

$$\therefore \mathcal{L}\{g^1(t)\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow s\mathcal{L}\{g(t)\} = \mathcal{L}\{f(t)\} \Rightarrow \mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

$$But\ g(t) = \int_0^t f(x)\,dx$$

$$\therefore \mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$$

Problems:

1. Find the L.T of $\int_0^t \sin at dt$

Sol. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

$\therefore \mathcal{L}\left\{\int_0^t \sin at\right\} = \frac{a}{s(s^2+a^2)}$

2. Find the L.T of $\int_0^t \frac{\sin t}{t} dt$

Sol. $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ $also$ $\underset{t\to 0}{lt} \frac{\sin t}{t} = 1$ $exists$

$\therefore \mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \mathcal{L}\{\sin t\} ds = \int_s^\infty \frac{1}{s^2+1} ds$

$= \left[Tan^{-1} s\right]_s^\infty = Tan^{-1}\infty - Tan^{-1} s = \pi/2 - Tan^{-1} s = \cot^{-1} s\,(or)\, Tan^{-1}\left(1/s\right)$

$i.e., \mathcal{L}\left\{\frac{\sin t}{t}\right\} = Tan^{-1}(1/s)(or)cot^{-1}s$

$\therefore\ \mathcal{L}\left\{\int_0^t \frac{\sin t}{t} dt\right\} = \frac{1}{s} Tan^{-1}(1/s)(or)\frac{1}{s}cot^{-1}s$

Laplace Transform of Some special functions:

1. The Unit step function or Heaviside's Unit functions:

It is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Laplace Transform of unit step function:

To prove that $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

Pf. Unit step function is defined as

$$u(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$$

Then $\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$

$= \int_0^a e^{-st} u(t-a) dt + \int_a^\infty e^{-st} u(t-a) dt$

$= \int_0^a e^{-st}.0 dt + \int_a^\infty e^{-st}.1 dt$

$= \int_a^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}.\left[e^{-\infty} - e^{-as}\right] = \frac{e^{-as}}{s}$

$$\therefore \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

2. The dirac delta function or Unit impulse function:

Consider the function.

$$f_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \le t \le \epsilon \\ 0 & t > \epsilon \end{cases}$$

Laplace Transforms of Dirac Delta Function:

Prove that $\mathcal{L}\{f_\epsilon(t)\} = \frac{1-e^{-s\epsilon}}{s\epsilon}$ hence show that $\mathcal{L}\{\delta(t)\} = 1$

Pf: By the definition $f_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \le t \le \epsilon \\ 0 & t > \epsilon \end{cases}$

And Hence $\mathcal{L}\{f_\epsilon(t)\} = \int_0^\infty e^{-st} f_\epsilon(t)\,dt$

$$= \int_0^\epsilon e^{-st} f_\epsilon(t)\,dt + \int_\epsilon^\infty e^{-st} f_\epsilon(t)\,dt$$

$$= \int_0^\epsilon e^{-st}\frac{1}{\epsilon}dt + \int_\epsilon^\infty e^{-st}.0\,dt$$

$= \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_0^\epsilon = -{}^{1}/_{\epsilon\, s}\,[e^{-s\epsilon} - e^0] = \frac{1-e^{-s\epsilon}}{s\epsilon}$

$$\therefore \mathcal{L}\{f_\epsilon(t)\} = \frac{1-e^{-s\epsilon}}{s\,\epsilon}$$

Now $\mathcal{L}\{\delta(t)\} = \underset{\epsilon\to 0}{lt}\, \mathcal{L}\{f_\epsilon(t)\}$

$$= \underset{\epsilon\to 0}{lt}\, \frac{1-e^{-s\epsilon}}{s\,\epsilon}$$

$\therefore \mathcal{L}\{\delta(t)\} = 1\ using$ L' hospital rule.

Properties of Dirac Delta Function:

1. $\int_0^\infty \delta(t)\,dt = 0$
2. $\int_0^\infty \delta(t)G(t)\,dt = G(0)$Where G(t) is some continuous function.
3. $\int_0^\infty \delta(t-a)G(t)\,dt = G(a)$Where G(t) is some continuous function.
4. $\int_0^\infty G(t)\delta^1(t-a) = -G^1(a)$

Problems

1. Prove that $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

Sol. By Translation theorem

$\mathcal{L}\{\delta(t-a)\} = e^{-as}\mathcal{L}\{\delta(t)\}$

$= e^{-as}.\sin ce\, \mathcal{L}\{\delta(t)\} = 1$

2. Evaluate $\int_0^\infty \cos 2t\, \delta\left(t - {}^{\pi}/_{3}\right) dt$

Sol. By using property (3) then we get

$$\int_0^\infty \delta(t-a)G(t)dt = G(a)$$

Here $a = {}^{\pi}/_{3}, G(t) = \cos 2t$

$\therefore G(a) = G\left({}^{\pi}/_{3}\right) = \cos 2\,{}^{\pi}/_{3} = -{}^{1}/_{2}$

$$\therefore \int_0^\infty \cos 2at\, \delta\left(t - {}^{\pi}/_{3}\right) dt = \cos 2\,{}^{\pi}/_{3} = {}^{-\pi}/_{2}$$

3. Evaluate $\int_0^\infty e^{-4t}\delta^1(t-2)\,dt$

By the IVth Property then we get

$$\int_0^\infty \delta^1(t-a)G(t)\,dt = -G^1(a)$$

$$G(t) = e^{-4t} \; and \; a = 2$$

$$G^1(t) = -4.e^{-4t}$$

$$\therefore G^1(a) = G^1(2) = -4.e^{-8}$$

$$\therefore \int_0^\infty e^{-4t}\delta^1(t-2)\,dt = -G^1(a) = 4.e^{-8}$$

Inverse Transforms:

If $\overline{f}(s)$ is the Laplace transforms of a function of f(t) i.e. $\mathcal{L}\{f(t)\} = \overline{f}(s)$ then f(t) is called the inverse Laplace transform of $\overline{f}(s)$ and is written as $f(t) = \mathcal{L}^{-1}\{\overline{f}(s)\}$

$\therefore$ $\mathcal{L}^{-1}$ is called the inverse L.T operator.

Table of Inverse Laplace Transforms

S.No.	$\mathcal{L}\{f(t)\} = \overline{f}(s)$	$\mathcal{L}^{-1}\{\overline{f}(s)\} = f(t)$
1.	$\mathcal{L}\{1\} = {}^{1}/_{s}$	$\mathcal{L}^{-1}\{{}^{1}/_{s}\} = 1$
2.	$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$	$\mathcal{L}^{-1}\{{}^{1}/_{s-a}\} = e^{at}$
3.	$\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$	$\mathcal{L}^{-1}\{{}^{1}/_{s+a}\} = e^{at}$
4.	$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $n\ is\ a + ve\ integer$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{n!}$
5.	$\mathcal{L}\{t^{n-1}\} = \dfrac{(n-1)!}{s^n}$	$\mathcal{L}^{-1}\{{}^{1}/_{s^n}\} = \dfrac{t^{n-1}}{(n-1)!}, n = 1,2,3 \ldots$
6.	$\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{a}.\sin at$
7.	$\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$
8.	$\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-a^2}\right\} = \dfrac{1}{a}\sinh at$
9.	$\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$
10.	$\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^2+b^2}\right\} = \dfrac{1}{b}.e^{at}\sin bt$
11.	$\mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}^{-1}\left\{\dfrac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$
12.	$\mathcal{L}\{e^{at}\sinh bt\} = \dfrac{b}{(s-a)^2-b^2}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^2-b^2}\right\} = \dfrac{1}{b}.e^{at}\sinh bt$
13.	$\mathcal{L}\{e^{at}\cosh bt\} = \dfrac{s-a}{(s-a)^2-b^2}$	$\mathcal{L}^{-1}\left\{\dfrac{(s-a)}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$
14.	$\mathcal{L}\{e^{-at}\sin bt\} = \dfrac{b}{(s+a)^2+b^2}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{(s+a)^2+b^2}\right\} = \dfrac{1}{b}.e^{-at}\sin bt$
15.	$\mathcal{L}\{e^{-at}\cos bt\} = \dfrac{s+a}{(s+a)^2+b^2}$	$\mathcal{L}^{-1}\left\{\dfrac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$
16.	$\mathcal{L}\{e^{at}f(t)\} = \overline{f}(s-a)$	$\mathcal{L}^{-1}\{\overline{f}(s-a)\} = e^{at}\mathcal{L}^{-1}\{\overline{f}(s)\}$
17.	$\mathcal{L}\{e^{-at}f(t)\} = \overline{f}(s+a)$	$\mathcal{L}^{-1}\{\overline{f}(s+a)\} = e^{-at}f(t) = e^{-at}\mathcal{L}^{-1}\{\overline{f}(s)\}$

Problems

1. Find the inverse transform of $\frac{s^2-3s+4}{s^3}$

Sol. $\mathcal{L}^{-1}\left\{\frac{s^3-3s+4}{s^3}\right\} = \mathcal{L}^{-1}\left\{1/s - 3.1/s^2 + 4/s^3\right\}$

$= \mathrm{L}^{-1}\left\{1/s\right\} - 3\mathrm{L}^{-1}\left\{1/s^2\right\} + \mathrm{L}^{-1}\left\{4/s^3\right\}$

$= 1 - 3t + 4.\frac{t^2}{2!} = 1 - 3t + 2t^2$

2. Find the inverse transform of $\frac{s+2}{s^2-4s+13}$

Sol. $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2+4}{(s-2)^2+3^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + 4.\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+3^2}\right\}$

$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$

3. Find the inverse Laplace transform of $\frac{2s-5}{s^2-4}$

Sol. $\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2-4} - \frac{5}{s^2-4}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$

$= 2.\cosh 2t - 5.\frac{1}{2}\sinh 2t$

4. Find $\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

Sol. $\mathcal{L}^{-1}\left\{\frac{s+s+1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{1}{s}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = e^{-t} + 1$

5. Find $\mathcal{L}^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{3s}{4s^2+25}\right\} - 8\mathcal{L}^{-1}\left\{\frac{1}{4s^2+25}\right\}$

$= \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(5/2\right)^2}\right\} - \frac{8}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\left(5/2\right)^2}\right\}$

$= \frac{3}{4}.\cos\frac{5}{2}t - \frac{8}{4}.\frac{2}{5}\sin\frac{5}{2}t$

$= \frac{3}{4}\cos\frac{5}{2}t - \frac{4}{5}\sin\frac{5}{2}t$

6. Find the Laplace inverse transform of $\frac{s}{(s+a)^2}$

Sol. $\mathcal{L}^{-1}\left\{\frac{s}{(s+a)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+a-a}{(s+a)^2}\right\} = e^{-at}\mathcal{L}^{-1}\left\{\frac{s-a}{s^2}\right\}$

$= e^{-at}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{a}{s^2}\right\}$

$= e^{-at}\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - a.\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right]$

$= e^{-at}\left[1-at\right]$

7. Find $\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Sol. $Let \frac{3s+7}{s^2-2s-3} = \frac{A}{s+1} + \frac{B}{s-3}$

$$A(s-3) + B(s+1) = 3s+7$$

$$put\ s = 3, 4B = 16 \Rightarrow B = 4$$

$$put\ s = -1, -4A = 4 \Rightarrow A = -1$$

$\therefore \frac{3s+7}{s^2-2s-3} = \frac{-1}{s+1} + \frac{4}{s-3}$

$\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1} + \frac{4}{s-3}\right\} = -1\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$= -e^{-t} + 4.e^{3t}$

8. Find $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}$

Sol. $\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$

$A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 = s$

Equating Co-efficients of s^3, A+C=0……..(1)

Equating Co-efficients of s^2, A+B+2C+D=0…….(2)

Equating Co-efficients of s, A+C+2D=1…….(3)

$put\ s = -1, 2B = -1 \Rightarrow B = -\frac{1}{2}$

Substituting (1) in (3) $2D = 1 \Rightarrow D = \frac{1}{2}$

Substituting the values of B and D in (2)

i.e., $A - \frac{1}{2} + 2C + \frac{1}{2} = 0 \Rightarrow A + 2C = 0, also\ A + C = 0 \Rightarrow A = 0, C = 0$

$$\therefore \frac{s}{(s+1)^2(s^2+1)} = \frac{\frac{-1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = \frac{1}{2}\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - te^{-t}\right]$$

9. Find $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Sol. Since $s^4 + 4a^4 = \left(s^2 + 2a^2\right)^2 - (2as)^2$

$$= (s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$$

$$\therefore Let\ \frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

$$(As + B)(s^2 - 2as + 2a^2) + (Cs + D)(s^2 + 2as + 2a^2) = s$$

Solving we get $A = 0, C = 0, B = \frac{-1}{4a}, D = \frac{1}{4a}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4 + 4a^4}\right\} = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4a}}{s^2 + 2as + 2a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{4a}}{s^2 - 2as + 2a^2}\right\}$$

$$= \frac{-1}{4}a.\mathrm{L}^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a}..\mathrm{L}^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$= \frac{-1}{4a}.\frac{1}{a}.e^{-at}\sin at + \frac{1}{4a}.\frac{1}{a}e^{at}\sin at$$

$$= \frac{1}{4a^2}\sin at\left(e^{at} - e^{-at}\right)$$

$$= \frac{1}{4a^2}.\sin at.2\sinh at = \frac{1}{2a^2}\sin at \sinh at$$

10. Find i. $\mathcal{L}^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$ ii. $\mathcal{L}^{-1}\left\{\frac{3\left(s^2-2\right)^2}{2s^5}\right\}$

Sol. i. $\mathcal{L}^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3t + 4\frac{t^2}{2!} = 1 - 3t + 2t^2$$

ii. $\mathcal{L}^{-1}\left\{\frac{3\left(s^2-2\right)^2}{2s^5}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{\left(s^2-2\right)^2}{s^5}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s^4-4s^2+4}{s^5}\right\}$

$$=\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{4}{s^3}+\frac{4}{s^5}\right\}+\frac{3}{2}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-4\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}+4\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}\right\}$$

$$=\frac{3}{2}\left[1-4\frac{t^2}{2!}+\frac{4t^4}{4!}\right]=\frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]=\frac{1}{4}\left[t^4-6t^2+6\right]$$

(11) Find $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right]$

Sol:

$$\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right]=\mathcal{L}^{-1}\left[\frac{2s}{2(s^2-a^2)}\right]=\frac{1}{2}\mathcal{L}^{-1}\left[\frac{2s}{(s-a)(s+a)}\right]=\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$=\frac{1}{2}\left[e^{at}+e^{-at}\right]=\cosh at$$

(12) Find $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]$

Sol: $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]=4L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=4L^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]=4[e^{-t}-e^{-2t}]$

(13) Find $L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]$

Hint : $\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$

$A=1/3, B=4/15, C=2/5$

Ans : $\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$

(14) Find $L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]$

Hint : $\frac{s^2+2s-4}{(s^2+9)(s-5)}=\frac{A}{s-5}+\frac{Bs+C}{s^2+9}$

$A=31/34, B=3/34, C=83/34$

Ans : $\frac{31}{34}e^{5t}+1/34\left[3\cos 3t+\frac{83}{3}\sin 3t\right]$

<u>First Shifting Theorem :</u>

If $L^{-1}\left\{\overline{f}(s)\right\}=f(t)$, then $L^{-1}\left\{\overline{f}(s-a)\right\}=e^{at}f(t)$

P.f. : We have seen that $L\left\{e^{at} f(t)\right\} = \overline{f}(s-a)$

$\therefore L^{-1}\left\{\overline{f}(s-a)\right\} = e^{at} f(t) = e^{at} L^{-1}\left\{\overline{f}(s)\right\}$

Problems

1. Find (i) $L^{-1}\left\{\dfrac{1}{(s+2)^2+16}\right\} = L^{-1}\left\{\overline{f}(s+2)\right\}$

$= e^{-2t} L^{-1}\left\{\dfrac{1}{s^2+16}\right\} = e^{-2t}.\dfrac{1}{4}\sin 4t = \dfrac{e^{-2t}\sin 4t}{4}$

(ii) $L^{-1}\left\{\dfrac{3s-2}{s^2-4s+20}\right\} = L^{-1}\left\{\dfrac{3s-2}{(s-2)^2+16}\right\} = L^{-1}\left\{\dfrac{3(s-2)+4}{(s-2)^2+4^2}\right\}$

$= 3L^{-1}\left\{\dfrac{s-2}{(s-2)^2+4^2}\right\} + 4L^{-1}\left\{\dfrac{1}{(s-2)^2+4^2}\right\}$

$= 3e^{2t} L^{-1}\left\{\dfrac{s}{s^2+4^2}\right\} + 4e^{2t} L^{-1}\left\{\dfrac{1}{s^2+4^2}\right\}$

$= 3e^{2t}\cos 4t + 4e^{2t}\dfrac{1}{4}\sin 4t$

(iii) $\dfrac{1}{(s+1)^2(s^2+4)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{Cs+D}{s^2+4}$

$A = \dfrac{2}{25}, B = \dfrac{1}{5}, C = \dfrac{-2}{25}, D = \dfrac{-3}{25}$

$\therefore L^{-1}\left\{\dfrac{1}{(s+1)^2(s^2+4)}\right\} = \dfrac{2}{25} L^{-1}\left\{\dfrac{1}{s+1}\right\} + \dfrac{1}{5} L^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} - \dfrac{2}{25} L^{-1}\left\{\dfrac{s}{s^2+4}\right\} - \dfrac{3}{25} L^{-1}\left\{\dfrac{1}{s^2+4}\right\}$

$= \dfrac{2}{25} e^{-t}\ L^{-1}\left\{\dfrac{1}{s}\right\} + \dfrac{1}{5} e^{-t} L^{-1}\left\{\dfrac{1}{s^2}\right\} - \dfrac{2}{25}\cos 2t - \dfrac{3}{25}.\dfrac{1}{2}\sin 2t$

$= \dfrac{2}{25} e^{-t} + \dfrac{1}{5} e^{-t}.t - \dfrac{2}{25}\cos 2t - \dfrac{3}{50}\sin 2t$

(iv) Find $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\}$

Sol : $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\} = L^{-1}\left\{\dfrac{s+3}{(s-5)^2+2^2}\right\} = L^{-1}\left\{\dfrac{s-5+8}{(s-5)^2+2^2}\right\}$

$= e^{5t} L^{-1}\left\{\dfrac{s+8}{s^2+2^2}\right\} = e^{5t}\left\{\cos 2t + 8.\dfrac{1}{2}\sin 2t\right\}$

(V). Find $L^{-1}\left\{\log\dfrac{s+1}{s-1}\right\}$

Sol. Let $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}=f(t)$

$$L\{f(t)\}=\log\frac{s+1}{s-1}$$

$$L\{tf(t)\}=\frac{-d}{ds}\left\{\log\frac{s+1}{s-1}\right\}$$

$$L\{tf(t)\}=\frac{-1}{s+1}+\frac{1}{s-1}$$

$$tf(t)=L^{-1}\left\{\frac{-1}{s+1}+\frac{1}{s-1}\right\}$$

$$tf(t)=-1.L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$=-e^{-t}+e^{t}$$

$$t\,f(t)=2\sinh t\Rightarrow f(t)=\frac{2\sinh t}{t}$$

$$\therefore L^{-1}\left\{\log\frac{s+1}{s-1}\right\}=\frac{2\sinh t}{t}$$

(VI). $L^{-1}\left\{\log\frac{1+s}{s}\right\}$ Ans : $\frac{1-e^{-t}}{t}$

(VII). Find $L^{-1}\left\{\cot^{-1}(s)\right\}$

Sol. Let $L^{-1}\left\{\cot^{-1}(s)\right\}=f(t)$

$$L\{f(t)\}=\cot^{-1}(s)$$

$$L\{tf(t)\}=\frac{-d}{ds}[\cot^{-1}(s)=-\left[\frac{-1}{1+s^2}\right]=\frac{1}{1+s^2}$$

$$tf(t)=L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$$

$$f(t)=\frac{\sin t}{t}$$

$$\therefore L^{-1}\left\{\cot^{-1}(s)\right\}=\frac{1}{t}\sin t$$

Convolution theorem :

If $L\{f(t)\}=\overline{f}(s)$ and $L\{g(t)\}=\overline{g}(s)$ then $L\left\{\int_0^t f(u)\,g(t-u)\,du\right\}=\overline{f}(s).\overline{g}(s)$

Convolution Theorem :

Definition : If f(t) and g(t) are two functions defined for $t \geq 0$ then the convolution of f(t) and g(t) is defined as

$$f(t) * g(t) = \int_0^t f(u) g(t-u)\, du$$

$f(t) * g(t)$ *can also be written as* $(f * g)(t)$

Properties :

The convolution operation * has the following properties

1. commutative i.e., $(f * g)(t) = (g * f)(t)$
2. Associative $[f * (g * h)](t) = [(f * g) * h](t)$
3. Distributive $[f * (g + h)](t) = (f * g)(t) + (f * h)(t)$ *for* $t \geq 0$

Theorem : If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$ then

$$\mathrm{L}\{f(t) * g(t)\} = \mathrm{L}\{f(t)\}.\mathrm{L}\{g(t)\} = \overline{f}(s).\overline{g}(s)$$

i.e., The L.T of convolution of f(t) and g(t) is equal to the product of the L.T of f(t) and g(t)

Proof : Let $\phi(t) = \int_0^t f(u) g(t-u)\, du$

$$\mathrm{L}\{\phi(t)\} = \int_0^\infty e^{-st} \left\{\int_0^t f(u)\, g(t-u)\, du\right\} dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt$$

The double integral is considered within the region enclosed by the line u=0 and u=t

On changing the order of integration, we get

$$\mathrm{L}\ \{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, dt\, du$$

$$= \int_0^\infty e^{-su} f(u) \left\{\int_u^\infty e^{-s(t-u)} g(t-u)\, dt\right\} du$$

$$= \int_0^\infty e^{-su} f(u) \left\{\int_0^\infty e^{-sv} g(v)\, dv\right\} du \qquad put\ t-u=v$$

$$= \int_0^\infty e^{-su} f(u) \left\{\overline{g}(s)\right\} du = \overline{g}(s) \int_0^\infty e^{-su} f(u)\, du = \overline{g}(s).\overline{f}(s)$$

$$\therefore \mathrm{L}\{\phi(t)\} = \overline{f}(s).\overline{g}(s)$$

Problems:

1. Using the convolution theorem find $\mathrm{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

Sol. $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right\}$

$Let\ \overline{f}(s) = \frac{s}{s^2+a^2}\ and\ \overline{g}(s) = \frac{1}{s^2+a^2}$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t) - say$

$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{1}{a}\sin at = g(t) - say$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au.\frac{1}{a}.\sin a(t-u)du$$

$$= \frac{1}{2a}\int_0^t \left[\sin(\mathrm{au}+at-au) - \sin(au-at+au)\right]du$$

$$= \frac{1}{2a}\int_0^t \left[\sin at - \sin(2au-at)\right]du$$

$$= \frac{1}{2a}\left[\sin at.u + \frac{1}{2a}.\cos(2au-at)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos(2at-at) - \frac{1}{2a}\cos(-at)\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos at\right]$$

$$= \frac{t}{2a}\sin at$$

2. Use convolution theorem to evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol. $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\right\}$

$Let\ \overline{f}(s) = \frac{s}{s^2+a^2}\ and\ \overline{g}(s) = \frac{s}{s^2+b^2}$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t) - say$

$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{s}{(s^2+b^2)}\right\} = \cos bt = g(t) - say$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}=\int_0^t \cos au.\cos b(t-u)du$$

$$=\frac{1}{2}\int_0^t\left[\cos(au-bu+bt)+\cos(au+bu-bt)\right]du$$

$$=\frac{1}{2}\left[\frac{\sin(au-bu+bt)}{a-b}+\frac{\sin(au+bu-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right]=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

3. Use convolution theorem to evaluate $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

Sol. $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}=L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}$

Let $\overline{f}(s)=\frac{1}{s^2}$ *and* $\overline{g}(s)=\frac{s}{\left(s^2+4\right)^2}$

So that $L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s^2}\right\}=t=g(t)-say$

$$L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}=\frac{t.\sin 2t}{4}=f(t)-say\left[\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{t s\text{in} 2t}{2a}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}=\int_0^t\frac{u}{4}\sin 2u(t-u)du$$

$$=\frac{t}{4}\int_0^t u\sin 2u du-\frac{1}{4}\int_0^t u^2\sin 2u du=\frac{t}{4}\left(-\frac{u}{2}\cos 2u+\frac{1}{4}\sin 2u\right)_0^t$$

$$=-\frac{1}{4}\left[\frac{-u^2}{2}\cos 2u+\frac{u}{2}\sin 2u+\frac{1}{4}\cos 2u\right]_0^t$$

$$=\frac{1}{16}\left[1-t\sin 2t-\cos 2t\right]$$

4. Find $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]$

Sol. $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]=L^{-1}\left[\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right]$

Let $\overline{f}(s)=\frac{1}{s-2}$ *and* $\overline{g}(s)=\frac{1}{s^2+1}$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} = f(t) - say$

$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = g(t) - say$

$\therefore L^{-1}\left\{\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right\} = \int_0^t f(u).\mathrm{g}(\mathrm{t}-u)\,\mathrm{du}$ (By Convolution theorem)

$= \int_0^t e^{2u}\sin(\mathrm{t}-\mathrm{u})\,\mathrm{du}$ (or) $\int_0^t \mathrm{sinu}.e^{2(t-u)}\,\mathrm{du}$

$= e^{2t}\int_0^t \sin u e^{-2u}\,du$

$= e^{2t}\left[\frac{e^{-2u}}{2^2+1}\left[-2\sin u - \cos u\right]\right]_0^t$

$= e^{2t}\left[\frac{1}{5}e^{-2t}\left(-2\sin t - \cos t\right) - \frac{1}{5}(-1)\right]$

$= \frac{1}{5}\left(-2\sin t - \cos t\right) + \frac{1}{5}e^{2t}$

$= \frac{1}{5}\left(e^{2t} - 2\sin t - \cos t\right)$

5. Find $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$

Sol. $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s-2}\right\}$

$Let\,\overline{f}(s) = \frac{1}{s+1}\ and\ \overline{g}(s) = \frac{1}{s-2}$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} = f(t) - say$

$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} = g(t) - say$

$\therefore$ By using convolution theorem, we have

$L^{-1}\left\{\frac{1}{(s+1)(\mathrm{s}-2)}\right\} = \int_0^t e^{-u}e^{2(t-u)}\,du$

$= \int_0^t e^{2t}e^{-3u}\,du = e^{2t}\int_0^t e^{-3u}\,du = e^{2t}\left[\frac{e^{-3u}}{-3}\right]_0^t = \frac{1}{3}\left[e^{2t} - e^{-t}\right]$

6. Find $L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Sol. $L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = L^{-1}\left\{\frac{1}{s^2}.\frac{1}{s^2-a^2}\right\}$

$Let\,\overline{f}(s) = \frac{1}{s^2}\, and\, \overline{g}(s) = \frac{1}{s^2-a^2}$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t) - say$

$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at = g(t) - say$

By using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \int_0^t u.\frac{1}{a}\sinh a(t-u)du$$

$$= \frac{1}{a}\int_0^t u\sinh(at-au)du$$

$$= \frac{1}{a}\left[\frac{-u}{a}\cosh(at-au) - \frac{\sin(at-au)}{a^2}\right]_0^t$$

$$\frac{1}{a}\left[\frac{-t}{a}\cosh(at-at) - 0 - \frac{1}{a^2}[0-\sinh at]\right]$$

$$= \frac{1}{a}\left[\frac{-t}{a} + \frac{1}{a^2}\sinh at\right]$$

$$= \frac{1}{a^3}\left[-at + \sinh at\right]$$

Application of L.T – Solutions of Ordinary Differential Equations

1. Solve $y^{111} + 2y^{11} - y^{1} - 2y = 0$ using Laplace Transformation given that $y(0) = y^{1}(0) = 0$ and $y^{11}(0) = 6$

Sol. Given that $y^{111} + 2y^{11} - y^{1} - 2y = 0$

Taking the Laplace transform on both sides, we get

$$L\left\{y^{111}(t)\right\} + 2L\left\{y^{11}(t)\right\} - L\left\{y^{1}\right\} - 2L\left\{y\right\} = 0$$

$$\Rightarrow s^3L\left\{y(t)\right\} - s^2y(0) - sy^{1}(0) - y^{11}(0) + 2\left\{s^2L\left\{y(t)\right\} - sy(0) - y^{1}(0)\right\} -$$

$$\left\{sL\left\{y(t)\right\} - y(0)\right\} - 2L\left\{y(t)\right\} = 0$$

$$\Rightarrow \left\{s^3 + 2s^2 - s - 2\right\}L\left\{y(t)\right\} = s^2y(0) + sy^{1}(0) + y^{11}(0) + 2sy(0) + 2y^{1}(0) - y(0)$$

$= 0+0+6+2.0+2.0-0$

$\Rightarrow \left\{s^3+2s^2-s-2\right\} \mathrm{L}\left\{y(t)\right\} = 6$

$$\mathrm{L}\left\{y(t)\right\} = \frac{6}{s^3+2s^2-s-2} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$= \frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}$$

$\Rightarrow A(s+1)(s+2)+B(s-1)(s+2)+C(s-1)(s+1) = 6$

$\Rightarrow A(s^2+3s+2)+B(s^2-s-2)+C(s^2-1) = 6$

$\Rightarrow A+B+C=0, 3A-B=0, 2A-2B-C=6$

$\mathrm{A}+\mathrm{B}+\mathrm{C}=0$

$2\mathrm{A}-2\mathrm{B}-\mathrm{C}=6$

$3A-B=6$

$3A+B=0$

$6A=6 \Rightarrow A=1$

$3A+B=0 \Rightarrow B=-3A \Rightarrow B=-3$

$\therefore A+B+C=0 \Rightarrow C=-A-B=-1+3=2$

$$\therefore \mathrm{L}\left\{y(t)\right\} = \frac{1}{s-1}-\frac{3}{s+1}+\frac{2}{s+2}$$

$$y(t) = \mathrm{L}^{-1}\left\{\frac{1}{s-1}\right\} - 3.\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2.\mathrm{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$= e^t - 3e^{-t} + 2.e^{-2t}$

Which is the desired result

2. Solve $y^{11}-3y^1+2y = 4t+e^{3t}$ using Laplace Transformation given that $y(0)=1 \, and \, y^1(0)=-1$

Sol. Given that $y^{11}-3y^1+2y = 4t+e^{3t}$

Taking the Laplace transform on both sides, we get

$\mathrm{L}\left\{y^{11}(t)\right\} - 3\mathrm{L}\left\{y^1(t)\right\} + 2\mathrm{L}\left\{y(t)\right\} = 4\mathrm{L}\left\{t\right\} + \mathrm{L}\left\{e^{3t}\right\}$

$$\Rightarrow s^2\mathrm{L}\left\{y(t)\right\} - sy(0) - y^1(0) - 3\left[s\mathrm{L}\left\{y(t)\right\} - y(0)\right] + 2\mathrm{L}\left\{y(t)\right\} = \frac{4}{s^2}+\frac{1}{s-3}$$

$$\Rightarrow (s^2-3s+2)\mathrm{L}\left\{y(t)\right\} = \frac{4}{s^2}+\frac{1}{s-3}+s-4$$

$$\Rightarrow (s^2-3s+2)\mathrm{L}\{y(t)\} = \frac{4s-12+s^4+s^2-3s^3-4s^3+12s^2}{s^2(s-3)}$$

$$\Rightarrow \mathrm{L}\{y(t)\} = \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s^2-3s+2)}$$

$$\Rightarrow \mathrm{L}\{y(t)\} = \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)} = \frac{As+B}{s^2}+\frac{C}{s-3}+\frac{D}{s-1}+\frac{E}{s-2}$$

$$= \frac{(As+B)(s-1)(s-2)(s-3)+C(s^2)(s-1)(s-2)+D(s^2)(s-2)(s-3)+E(s^2)(s-1)(s-3)}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow s^4-7s^3+13s^2+4s-12 = (As+B)(s^3-6s^2+11s-6)+$$
$$C(s^2)(s^2-3s+2)+D(s^2)(s^2-5s+6)+E.s^2(s^2-4s+3)$$

$A+C+D+E=1$......................(1)

$-6A+B-3C-5D-4E=-7$......................(2)

$put\ s=1, 2D=-1 \Rightarrow D=\frac{-1}{2}$

$put\ s=2, -4E=8 \Rightarrow E=-2$

$put\ s=3, 18C=9 \Rightarrow C=\frac{1}{2}$

$from\ eq.(1)\ A=1-\frac{1}{2}+\frac{1}{2}+2$

$\Rightarrow A=3$

from equation (2) B= -7+18+$\frac{3}{2}-\frac{5}{2}-8=3-1=2$

$$y(t)=\mathrm{L}^{-1}\left\{\frac{3}{s}+\frac{2}{s^2}+\frac{1}{2(s-3)}-\frac{1}{2(s-1)}-\frac{2}{s-2}\right\}$$

$$y(t)=3+2t+\frac{1}{2}e^{3t}-\frac{1}{2}e^{t}-2.e^{2t}$$

3. Using Laplace Transform Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t$, given that $y=\frac{dy}{dt}=0$ when t=0

Sol. Given equation is $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t.$

$$\mathrm{L}\{y^{11}(t)\}+2\mathrm{L}\{y^{1}(t)\}-3\mathrm{L}\{y(t)\}=\mathrm{L}\{\sin t\}$$

$$s^2\mathrm{L}\{y(t)\}-sy(0)-y^1(0)+2\left[s\mathrm{L}\{y(t)\}-y(0)\right]-3.\mathrm{L}\{y(t)\}=\frac{1}{s^2+1}$$

$$\Rightarrow \left(s^2+2s-3\right)\mathrm{L}\left\{y(t)\right\} = \frac{1}{s^2+1}$$

$$\Rightarrow \mathrm{L}\left\{y(t)\right\} = \left(\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}\right)$$

$$\Rightarrow y(t) = \mathrm{L}^{-1}\left(\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right)$$

Now consider

$$\frac{1}{(s-1)(s+3)\left(s^2+1\right)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$A(s+3)\left(s^2+1\right) + B(s-1)\left(s^2+1\right) + (Cs+D)(s-1)(s+3) = 1$$

$$put\, s = 1, 8A = 1 \Rightarrow A = \frac{1}{8}$$

$$put\, s = -3, -40B = 1 \Rightarrow B = \frac{-1}{40}$$

$$A + B + C = 0 \Rightarrow C = 0 - \frac{1}{8} + \frac{1}{40}$$

$$C = \frac{-5+1}{40} = \frac{-4}{40} = \frac{-1}{10}$$

$$3A - B + 2C + D = 0 \Rightarrow D = -\frac{3}{8} - \frac{1}{40} + \frac{1}{5}$$

$$D = \frac{-15-1+8}{40} = \frac{-8}{40} = \frac{-1}{5}$$

$$\therefore y(t) = \mathrm{L}^{-1}\left\{\frac{\frac{1}{8}}{s-1} + \frac{\frac{-1}{40}}{s+3} + \frac{\frac{-1}{10}s - \frac{1}{5}}{s^2+1}\right\}$$

$$= \frac{1}{8}\mathrm{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{40}\mathrm{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{10}\mathrm{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{5}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\therefore y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

4. Solve $\frac{dx}{dt} + x = \sin\omega\, \mathrm{t}, x(0) = 2$

Sol. Given equation is $\frac{dx}{dt} + x = \sin\omega\, \mathrm{t}$

$$L\left\{x^{1}(t)\right\}+L\left\{x(t)\right\}=L\left\{\sin\omega t\right\}$$

$$\Rightarrow s.L\left\{x(t)\right\}-x(0)+L\left\{x(t)\right\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow s.L\left\{x(t)\right\}-2+L\left\{x(t)\right\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow (s+1)L\left\{x(t)\right\}=\frac{\omega}{s^2+\omega^2}+2$$

$$\Rightarrow x(t)=L^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}+\frac{2}{s+1}\right\}$$

$$=2L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}\right\}$$

$$=2e^{-t}+L^{-1}\left\{\frac{\frac{\omega}{\omega^2+1}}{s+1}-\frac{\frac{s\omega}{1+\omega^2}}{s^2+\omega^2}+\frac{\frac{\omega}{1+\omega^2}}{s^2+\omega^2}\right\}$$

$$=2e^{-t}+\frac{\omega}{\omega^2+1}e^{-t}-\frac{\omega}{1+\omega^2}\cos\omega t+\frac{\omega}{1+\omega^2}.\frac{1}{\omega}\sin\omega t$$

5. Solve $\left(D^2+n^2\right)x=a\sin(nt+\alpha)$ given that x=Dx=0, when t=0

Sol. Given equation is $\left(D^2+n^2\right)x=a\sin(nt+\alpha)$

$$x^{11}(t)+n^2x(t)=a\sin(nt+\alpha)$$

$$L\left\{x^{11}(t)\right\}+n^2L\left\{x(t)\right\}=L\left\{a\sin nt\cos\alpha+a\cos nt\sin\alpha\right\}$$

$$\Rightarrow s^2L\left\{x(t)\right\}-sx(0)-x^{1}(0)+n^2L\left\{x(t)\right\}=a\cos\alpha L\left\{\sin nt\right\}+a\sin\alpha L\left\{\cos nt\right\}$$

$$\Rightarrow\left(s^2+n^2\right)L\left\{x(t)\right\}=a\cos\alpha\frac{n}{s^2+n^2}+a\sin\alpha.\frac{s}{s^2+n^2}$$

$$\Rightarrow L\left\{x(t)\right\}=a\cos\alpha\frac{n}{\left(s^2+n^2\right)^2}+a\sin\alpha\frac{s}{\left(s^2+n^2\right)^2}$$

$$x(t)=a\cos\alpha L^{-1}\left\{\frac{n}{\left(s^2+n^2\right)^2}\right\}+a\sin\alpha L^{-1}\left\{\frac{s}{\left(s^2+n^2\right)^2}\right\}\rightarrow(1)$$

$$=na\cos\alpha\int_0^t\frac{1}{n}.\sin nx.\frac{1}{n}\sin n(t-x)\,dx-\frac{a\sin\alpha}{2}L^{-1}\left\{\frac{d}{ds}\frac{1}{\left(s^2+n^2\right)}\right\}$$

$$=\frac{a\cos\alpha}{2n}\int_0^t\left\{\cos(nt-2nx)-\cos nt\right\}dx+\frac{a\sin\alpha}{2}t\frac{1}{n}\sin nt$$

$$=\frac{a\cos\alpha}{2n}\left[\int_0^t\{\cos n(t-2x)-\cos nt\}\,dx+\frac{a}{2n}\sin\alpha t\sin nt\right]$$

$$=\frac{a\cos\alpha}{2n}\left[\frac{-1}{2n}.\sin n(t-2x)-x\cos nt\right]_0^t+\frac{at\sin\alpha}{2n}\sin nt$$

$$=\frac{a\cos\alpha}{2n}\left[\frac{\sin nt}{2n}-t\cos nt\right]+\frac{at\sin\alpha}{2n}\sin nt$$

$$=\frac{a\cos\alpha\sin nt}{2n^2}-\frac{at}{2n}[\cos\alpha\cos nt-\sin\alpha\sin nt]$$

$$=\frac{a\cos\alpha\sin nt}{2n^2}-\frac{at}{2n}\cos(\alpha+nt)$$

6. Solve $y^{11}+4y^1+3y=0, y(0)=3, y^1(0)=1$

7. Solve $\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=e^{-t}, x(0)=0, x^1(0)=1$

8. Solve $\frac{dx}{dt}+x=\sin\omega t, x(0)=2$

Ans. $2e^{-t}+\frac{\omega e^{-t}}{\omega^2+1}-\frac{\omega}{\omega^2+1}\cos\omega t+\frac{1}{\omega^2+1}\sin\omega t$

Laplace transform of Periodic functions :

If f(t) is a periodic function with period 'a'.

i.e, $f(t+a)=f(t)$ then

$$\text{L}\{f(t)\}=\frac{1}{1-e^{-sa}}\int_0^a e^{-st}f(t)\,dt$$

$$\left(f(t)=f(a+t)=f(2a+t)=..........=f(na+t)\right)$$

Ex : sin x is a periodic function with period 2π

$i.e.,\ \sin x=\sin(2\pi+x)=\sin(4\pi+x)$............

Problems

1. A function f(t) is periodic in (0,2b) and is defined as $f(t) = 1\ if\ \ 0<t<b$
$\qquad = -1\ if\ \ b<t<2b$

Find its Laplace Transform.

Sol. $\text{L}\{f(t)\}=\frac{1}{1-e^{-2bs}}\int_0^{2b}e^{-st}f(t)\,dt$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}f(t)\,dt+\int_b^{2b}e^{-st}f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}dt - \int_b^{2b} e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s\left(1-e^{-2bs}\right)}\left[-\left(e^{-sb}-1\right)+\left(e^{-2bs}-e^{-sb}\right)\right]$$

$$\mathrm{L}\left\{f(t)\right\} = \frac{1}{s\left(1-e^{-2bs}\right)}\left[1-2e^{-sb}+e^{-2bs}\right]$$

2. Find the L.T of the function $f(t) = \sin\omega t \; if \; 0 < t < \frac{\pi}{\omega}$

$$= 0 \; if \; \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \; where \; f(t) \; has \; period \; \frac{2\pi}{\omega}$$

Sol : Since f(t) is a periodic function with period $\frac{2\pi}{\omega}$

$$\mathrm{L}\left\{f(t)\right\} = \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t)dt$$

$$\mathrm{L}\left\{f(t)\right\} = \frac{1}{1-e^{-s2\pi/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s\pi/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0dt\right]$$

$$= \frac{1}{1-e^{-2s\pi/\omega}}\left[\frac{e^{-st}\left(-s\sin\omega t-\omega\cos\omega t\right)}{s^2+\omega^2}\right]_0^{\pi/\omega} \therefore \int e^{at}\sin bt = \frac{e^{at}}{a^2+b^2}\left(a\sin bt - b\cos bt\right)$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{1}{s^2+\omega^2}\left(e^{-s\pi/\omega}.\omega+\omega\right)\right]$$

Problem: Find $\mathrm{L}\left\{\frac{1-\cos t}{t^2}\right\}$

Sol. $\mathrm{L}\left\{\frac{1-\cos t}{t^2}\right\} = \mathrm{L}\left\{\frac{1}{t}.\frac{1-\cos t}{t}\right\}.....(1)$

Now $\mathrm{L}\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds = \left[\log s - \frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty = \frac{-1}{2}\left[\log\frac{s^2}{s^2+1}\right] = \frac{1}{2}\log\frac{s^2+1}{s^2}$$

$$\therefore \mathrm{L}\left[\frac{1-\cos t}{t^2}\right]=\int_s^\infty \frac{1}{2}\log\frac{s^2+1}{s^2}ds$$

$$=\frac{1}{2}\left[\left\{\log\left(\frac{s^2+1}{s^2}\right)\right\}.s\right]_s^\infty-\int_s^\infty \frac{s^2}{s^2+1}\left(\frac{-2}{s^3}\right).sds$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s.\log\left(1+\frac{1}{s^2}\right)\right\}-s\log\left(\frac{s^2+1}{s^2}\right)+2\int_s^\infty \frac{ds}{s^2+1}\right]$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s\left(\frac{1}{s^2}-\frac{1}{2s^4}+\frac{1}{3s^6}+....\right)-s\log\frac{s^2+1}{s^2}\right\}+2Tan^{-1}s\right]_s^\infty$$

$$=\frac{1}{2}\left[\left\{0-s\log\left(1+\frac{1}{s^2}\right)+2\left(\frac{\pi}{2}-Tan^{-1}s\right)\right\}\right]\quad \therefore\left(\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....\right)$$

$$=\cot^{-1}s-\frac{1}{2}s\log\left(1+\frac{1}{s^2}\right)$$

Short Answer Questions

1. Define Laplace transform.
2. Define exponential order.
3. Find $L\{t^n\}$ where n is a +ve integer
4. Find $L\{\sin at\}$
5. Find $L\{\cos^3 2t\}$
6. Find $L\{e^{2t}+4t^3-2\sin 3t+3\cos 3t\}$
7. Find the Laplace transform of $f(t)=\begin{cases}\sin t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$
8. State first shifting theorem
9. If $L\{f(t)\}=\dfrac{9s^2-12s+15}{(s-1)^3}$,find $L\{f(3t)\}$ using change of scale property
10. To prove $L\{f^n(t)\}=s^n\overline{f}(s)-s^{n-1}f(0)-s^{n-2}f^1(0)....-f^{n-1}(0)$
11. Evaluate $L\{t\sin 3t\cos 2t\}$
12. Find $L\left\{\dfrac{\sin 3t\cos t}{t}\right\}$
13. Define Heaviside's unit function
14. Find Laplace transform of Dirac delta function.

15. Find $L^{-1}\left\{\dfrac{3s-8}{4s^2+25}\right\}$

16. Find $L^{-1}\left\{\dfrac{3\left(s^2-2\right)^2}{2s^5}\right\}$

17. Find $L^{-1}\left\{\dfrac{4}{(s+1)(s+2)}\right\}$

18. Find $L^{-1}\left\{\dfrac{2s+3}{s^2+2s+2}\right\}$

19. Evaluate $L^{-1}\left\{\dfrac{1+e^{-\pi s}}{s^2+1}\right\}$

20. Find $L^{-1}\left\{\dfrac{e^{-2s}}{s^2+4s+5}\right\}$

21. Evaluate $L^{-1}\left\{\cot^{-1} s\right\}$

22. Find $L^{-1}\left\{\dfrac{s}{(s-4)^5}\right\}$

23. State Convolution theorem

24. Find $L^{-1}\left\{\dfrac{2s^2-4s+5}{s^3}\right\}$

25. Find $\int_0^\infty te^{-3t}\sin t\,dt$

26. Find $L\left\{\dfrac{\sin t}{t}\right\}$

27. Define an inverse Laplace transform of a function

28. Find $L^{-1}\left\{\log\left(\dfrac{s+3}{s+2}\right)\right\}$

29. Define convolution function.

30. Find $L^{-1}\left\{\dfrac{1}{(s-a)^2+b^2}\right\}$

Essay Questions

1. State and prove first shifting theorem

2. Find Laplace transform of $e^{-3t}\left(2\cos 5t-3\sin 5t\right)$

3. Find $L\left\{e^{3t}\sin^2 t\right\}$

4. State and prove second shifting theorem

5. Find the Laplace transform of g(t), where g(t) $=\begin{cases} \cos\left(t-\dfrac{2\pi}{3}\right), & if\ t>{2\pi}/{3} \\ 0, & if\ t<{2\pi}/{3} \end{cases}$

6. If $L\left\{f(t)\right\}=\overline{f}(s)$, then $L\left\{f(at)\right\}=\dfrac{1}{a}\overline{f}\left({s}/{a}\right)$

7. If $L\left\{f(t)\right\}=\overline{f}(s)$, then $L\left\{\int_0^t f(u)\,du\right\}=\dfrac{1}{s}\overline{f}(s)$

8. If $L\left\{f(t)\right\}=\overline{f}(s)$, then $L\left\{t^n f(t)\right\}=(-1)^n\dfrac{d^n}{ds^n}\left[\overline{f}(s)\right]$, where $n=1,2,3.....$

9. If $L\left\{f(t)\right\}=\overline{f}(s)$, then $L\left\{\dfrac{f(t)}{t}\right\}=\int_s^\infty \overline{f}(s)\,ds$

10. Find $L\left\{\dfrac{\cos 4t\sin 2t}{t}\right\}$

11. Using Laplace transform, evaluate $\int_0^\infty \dfrac{e^{-t}-e^{-2t}}{t}dt$

12. Using Laplace transform, evaluate $\int_0^\infty t^2e^{-4t}\sin 2t\,dt$

13. Find $L\left\{f(t)\right\}$ where f(t) is given by $f(t)=t, 0<t<b. f(t)=2b-t, b<t<2b, 2b$ being the period of f(t)

14. Find $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+9)(s^2+25)}\right]$

15. Find $L^{-1}\left\{\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$

16. Find $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\}$

17. Find $L^{-1}\left\{\dfrac{1}{s(s^2-1)(s^2+1)}\right\}$

18. Find $L^{-1}\left\{\log\left(\dfrac{s+1}{s-1}\right)\right\}$

19. Find $L^{-1}\left\{\dfrac{s+3}{\left(s^2+6s+13\right)^2}\right\}$

20. Using the convolution theorem find $L^{-1}\left\{\dfrac{s}{\left(s^2+a^2\right)^2}\right\}$

21. Using the convolution theorem find $L^{-1}\left\{\dfrac{1}{s\left(s^2+2s+2\right)}\right\}$

22. Using Laplace transform, solve $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+5y=e^{-t}\sin t$, given that y(0)=0, $y^1(0)=1$

23. Using Laplace transform, solve

$\left(D^3-D^2+4D-4\right)y=68e^x\sin 2x, y=1, Dy=-19, D^2y=-37$ at x=0

Objective type Questions:

1. $L(e^{3t})$
   (a) 1/s + 1 (b) 1/s - 3 (c) 1/s + 3 (d) 1/s
2. $L(t^2)$
   (a) $1/s^2$ (b) $1/s^3$ (c) $2/s^3$ (d) $2/s^4$
3. $L(t^5)$
   (a) $1/s^5$ (b) $1/s^6$ (c) $24/s^5$ (d) $120/s^6$
4. $L(\sin^2 2t)$
   (a) $s^2/s(s^2+16)$ (b) $s^2+2/s(s^2+16)$ (c) $s^2+4/s(s^2+16)$ (d) $s^2+8/s(s^2+16)$
5. L(cosh2t)
   (a) $2/s^2-4$ (b) s/s^2-4 (c) s/s^2+4 (d) 2/s2 + 4
6. $L(te^{2t})$
   (a) $1/(s-2)^2$ (b) $s/(s-2)^2$ (c) $s+2/(s-2)^2$ (d) $s+2/s^2+4$
7. L(tsinht)=
   (a) $2/(s^2-1)^2$ (b) $s/(s^2-1)^2$ (c) $2s/(s^2-1)^2$ (d) $s-2/(s^2-1)^2$
8. L^{-1}[1/s - 5]
   (a) e^{5t} (b) e^{-5t} (c) sint (d) te^{5t}
9. L^{-1}[2/s - 9)
   (a) $2e^{-9t}$ (b) 2sint (c) e^{9t} (d) e^{5t}
10. $L^{-1}[6/s^4]$
    (a) t^2 (b) t^3 (c) t^4 (d) $t^3/6$
11. L^{-1}[1/(s + 2)(s - 4)]
    (a) $e^{-2t}-e^{-4t}$ (b) $e^{4t}-e^{-2t}$ (c) $1/6[e^{-4t}-e^{-2t}]$ (d) ½$[e^{4t}-e^{-2t}]$
12. L(sintcost)
    (a) $2/s^2+4$ (b) $1/s^2+4$ (c) $2/s^2-4$ (d) s/s^2+4
13. $L^{-1}[s+5/s^2-4s+5]$
    (a) $e^{-2t}\cos 2t$ (b) $e^{-2t}\sin t$ (c) $e^{-2t}\cos t+5e^{-2t}\sin t$ (d) $e^{-2t}(\sin t+\cos t)$
14. $L[e^{2t}-e^{3t}/t]$

(a) log[(s - 3)/s -2] (b) log [s - 2/s - 3] (c) log[s + 4/s – 9] (d) log[s + 2/s + 3]

15. L($\cos^2 t$)

(a) $s/s^2 + 4$ (b) $s/2(s^2 + 4)$ (c) $1/2s + ½(s^2 + 4)$ (d) $1/2s + s/2(s^2 + 4)$

16. L^{-1}[log s + 6/s – 2]

(a) $e^{2t} – e^{-6t}/t$ (b) $e^{-6t} – e^{2t}/t$ (c) $e^{2t} + e^{6t}/t$ (d) $e^{2t} + e^{-6t}/t$

17. $L^{-1}[5/s^5]$

(a) $5t^4$ (b) $t^4/24$ (c) $(5/24) t^4$ (d) t^5

18. L^{-1} $[3s/s^2 + 16]$

(a) cos4t/3 (b) 3co4t (c) cos4t (d) 3sin4t

19.If [F(t-a)]=0 , 0<t<a then L[F(t-a)]=

(a) e^{-as} (b) s e^{-as} (c) $\frac{e^{-as}}{s}$ (d) $\frac{e^{-as}}{s^2}$

20. L(sinh4t) =

(a) $\frac{4}{s^2-16}$ (b) $\frac{s}{s^2-16}$ (c) $\frac{s}{s^2+16}$ (d) $\frac{4}{s^2+16}$

21. If L(f(t))=$\frac{6}{s^2+4}$ then L$\left[\int_0^1 f(t)dt\right]$ =

(a) $\frac{6s}{s^2+4}$ (b) $\frac{s}{s^2+4}$ (c) $\frac{6}{s^2+4}$ (d) $\frac{6}{s(s^2+4)}$

22. $L^{-1}\left[1/s^n\right]$ is possible only when n is

(a) Positive integer (b) zero (c) negative Integer (d) negative rational

23. $L^{-1}\left[\frac{e^{-\pi s}}{(s^2+1)}\right]$ =

(a) cost u(t-π) (b) sint u(t-π) (c) -sint u(t-π)(d) -cost u(t-π)

24. $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

(a) $\frac{1}{(b-a)}\left(e^{-at} - e^{-bt}\right)$ (b) $\frac{1}{(a-b)}\left(e^{-bt} - e^{at}\right)$

(c) $\frac{1}{(a+b)}\left(e^{-at} + e^{-bt}\right)$ (d) $\frac{1}{(a-b)}\left(e^{at} + e^{bt}\right)$

25. $L^{-1}[1]$

(a) 0 (b)1 (c)δt (d) δ(t-1)

26. $L^{-1}[2^t]$

(a) $\frac{1}{s-log2}$ (b) $\frac{1}{s}$ (c) $\frac{1}{s+log2}$ (d) $\frac{1}{log2}$

27. $L^{-1}[3!/s^4]$

(a) t^3 (b) t^5 (c) $\frac{t^3}{logt}$ (d) t^8

28. Laplace transform of f(t) is defined as

(a) $\int_0^{\propto} e^{-st} f(t)dt$ (b) $\int_0^{\propto} f(t)dt$ (c) $\int_0^{\propto} e^{st} f(t)dt$ (d) $\int_0^{\propto} e^{st} dt$

29. $\Gamma(n)$ =

(a) $\int_0^{\propto} e^{-x} x^{n-1} dx$ (b) $\int_0^{\propto} e^{x} x^{n+1} dx$ (c) $\int_0^{\propto} e^{x} dx$ (d) none

30. $\Gamma\left(\frac{1}{2}\right)$

(a) $\sqrt{\pi}$ (b) π (c) π^2 (d) 0

31. When s>a L($e^{at}t^{n}$)

(a) $\frac{n!}{(s-a)^n}$ (b) $\frac{n}{(s-a)^n}$ (c) $\frac{n!}{(s+a)^n}$ (d) $\frac{n}{s+a}$

32. $\int_0^{\propto}\frac{sint}{t}\,dt$

(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) ∞

33. If Lt(t-a) is a unit step function,L{H(t-a)}

(a) $\frac{e^{-as}}{s}$ (b) $\frac{1}{s}$ (c) e^{a} (d) $\frac{e^{as}}{s}$

34. L($\sqrt{t}$)

(a) $\frac{\sqrt{t}}{2s^{\frac{3}{2}}}$ (b) $\frac{\pi}{2s}$ (c) $\frac{1}{s^{\frac{3}{2}}}$ (d) $\frac{\pi}{2s^{\frac{3}{2}}}$

35. The value of $\int_0^{\propto} e^{-2t}cos3tdt$ ____________________

36. $L^{-1}\left[\frac{1}{(4s^2-25)}\right]$= ________________

37. $L^{-1}\left[\frac{1}{\sqrt{s+4}}\right]$ = __________________

38. L $\left[\frac{1-e^{t}}{t}\right]$= ______________________

39. $L^{-1}\left[\frac{1}{s(s^2+w^2)}\right]$= ________________

40. $L^{-1}\left[\frac{1}{\sqrt{s}}\right]$= ________________

41. $L^{-1}\left[\frac{1}{s^{\frac{5}{2}}}\right]$= ________________

42. L$[t^2e^t]$= ________________

43. If L[f(t)]=$\frac{1}{s}e^{\frac{-1}{s}}$ then L[f(2t)]=________________

44. If f(0)=0 then L(f^1 $(t)) =$ ________________

45. $L^{-1}\left[\frac{1}{(2s-5)}\right]$= ________________

46. $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$= ________________

47. $L\,[f^1\,(t)]$= ________________

48. $L^{-1}\left[\frac{1}{(s-a)(s-b)}\right]$= ________________