NUMERICAL TECHNIQUES
&
PROBABILITY DISTRIBUTIONS
(A30007)

UNIT-I

Solution of algebraic and Transcendental equations and Interpolation

Solutions of Algebraic and Transcendental equations:

1) **Polynomial function:** A function $f(x)$ is said to be a polynomial function if $f(x)$ is a polynomial in x.

ie, $f(x) = a_0x^n + a_1x^{n-1} + \cdots \ldots\ldots\ldots.. + a_{n-1}x + a_n$

where $a_0 \neq 0$, the co-efficients $a_0, a_1 \ldots\ldots\ldots. a_n$ are real constants and n is a non-negative integer.

2) **Algebraic function:** A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: $f(x) = c_1e^x + c_2e^{-x} = 0$

$$f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$$

3) **Root of an equation:** A number α is called a root of an equation $f(x) = 0$ if $f(\alpha) = 0$. We also say that α is a zero of the function.

Note: The roots of an equation are the abscissae of the points where the graph $y = f(x)$ cuts the x-axis.

Methods to find the roots of *f (x)* = 0

1) **Direct method:**

We know the solution of the polynomial equations such as linear equation $ax + b = 0$, and quadratic equation $ax^2 + bx + c = 0$, using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.

2) **Bisection method:** Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzono method of successive bisection. Some times it is referred to as half-interval method. Suppose we know an equation of the form $f(x) = 0$ has exactly one real root between two real numbers x_0, x_1. The number is choosen such that $f(x_0)$ and $f(x_1)$ will have opposite sign. Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0 + x_1}{2}$. If $f(x_2) = 0$ then x_2 is a root.

If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2. The interval is taken as $[x_0, x_2]$. Otherwise the root lies in the interval $[x_2, x_1]$.

PROBLEMS

1). Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 – stages

Sol Let $f(x) = x^3 - 5x + 1$. We note that $\begin{matrix} f(0) > 0 \\ f(1) < 0 \end{matrix}$ and

$\therefore$ One root lies between 0 and 1

Consider $x_0 = 0$ and $x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$

$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \ and \ f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \dfrac{0 + 0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0$ and $f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between x_0 and x_3

The third approximation of the root is

$$x_4 = \frac{x_0 + x_3}{2} = \frac{1}{2}(0 + 0.25) = 0.125$$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0$ and $f(x_3) < 0$, the root lies between

$x_4 = 0.125$ and $x_3 = 0.25$

Considering the 4th approximation of the roots

$$x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + 0.25) = 0.1875$$

$f(x_5) = 0.06910 > 0$, since $f(x_5) > 0$ and $f(x_3) < 0$ the root must lie between

$$x_5 = 0.18758 \ and \ x_3 = 0.25$$

Here the fifth approximation of the root is

$$\begin{aligned} x_6 &= \frac{1}{2}(x_5 + x_3) \\ &= \frac{1}{2}(0.1875 + 0.25) \\ &= 0.21875 \end{aligned}$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root and it lies between 0 and 1

2) Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages

Sol Let $f(x) = x^3 - 4x - 9$

We note that $f(2) < 0$ and $f(3) > 0$

$\therefore$ One root lies between 2 and 3

Consider $x_0 = 2$ *and* $x_1 = 3$

By Bisection method $x_2 = \frac{x_0 + x_1}{2} = 2.5$

Calculating $f(x_2) = f(2.5) = -3.375 < 0$

$\therefore$ The root lies between x_2 *and* x_1

The second approximation is $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5+3}{2} = 2.75$

Now $f(x_3) = f(2.75) = 0.7969 > 0$

$\therefore$ The root lies between x_2 *and* x_3

Thus the third approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$$

Again $f(x_4) = f(2.625) = -1.421 < 0$

$\therefore$ The root lies between x_3 *and* x_4

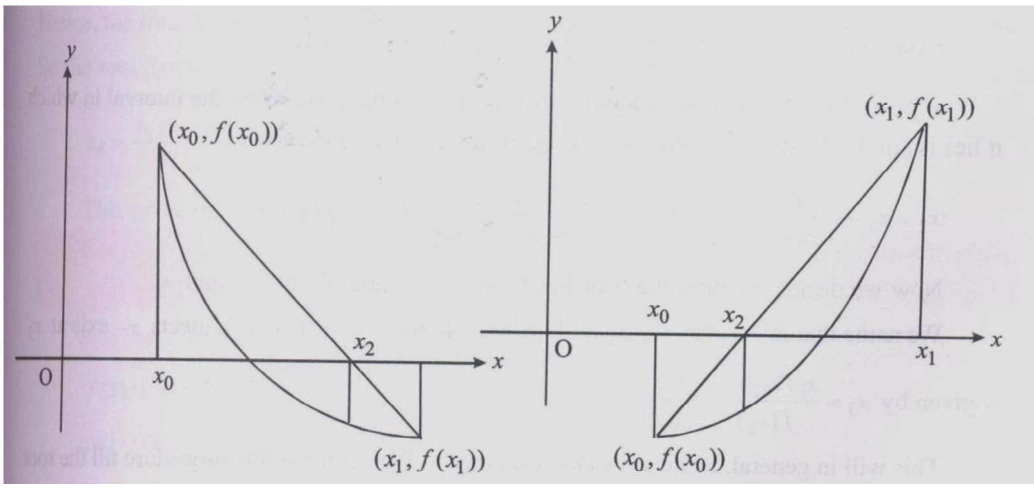
Fourth approximation is $x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

False Position Method (Regula – Falsi Method)

In the false position method we will find the root of the equation $f(x) = 0$ Consider two initial approximate values x_0 *and* x_1 near the required root so that $f(x_0)$ *and* $f(x_1)$ have different signs. This implies that a root lies between x_0 *and* x_1. The curve $f(x)$ crosses x- axis only once at the point x_2 lying between the points x_0 *and* x_1. Consider the point $A = (x_0, f(x_0))$ and $B = (x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis at x_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ *and* $f(x_2)$ are of opposite signs, then the root lies between x_0 *and* x_2 and value x_1 is replaced by x_2

Other wise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2 .

Another line is drawn by connecting the newly obtained pair of values. Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points $x_2, x_3, x_4, \ldots$ obtained converge to the expected root of the equation $y = f(x)$



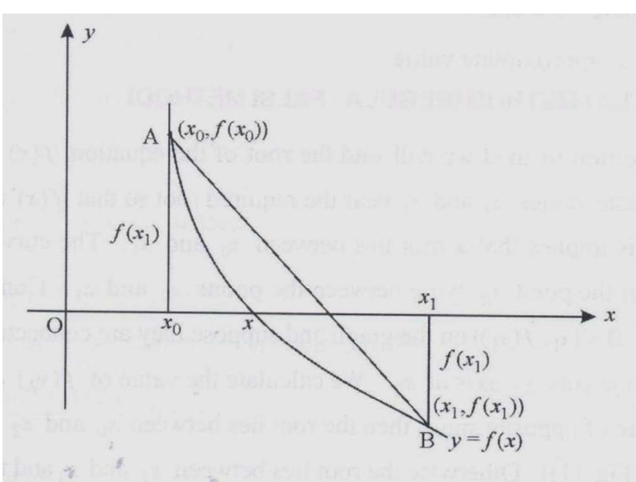
<u>To Obtain the equation to find the next approximation to the root</u>

Let $A = (x_0, f(x_0))$ *and* $B = (x_1, f(x_1))$ be the points on the curve $y = f(x)$ Then the equation to the chord AB is $\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} ------(1)$

At the point C where the line AB crosses the x – axis, where $f(x) = 0$ $ie,\ y = 0$

From (1), we get $x = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$ --------------(2)

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes



$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \text{ - } \quad \text{-------------(3)}$$

Now we decide whether the root lies between

x_0 *and* x_2 (or) x_2 *and* x_1

We name that interval as (x_1, x_2) The line joining $(x_1, y_1), (x_2, y_2)$ meets x – axis at x_3 is given by $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.

Also $x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

PROBLEMS:

1. **By using Regula - Falsi method, find an approximate root of the equation** $x^4 - x - 10 = 0$ **that lies between 1.8 and 2. Carry out three approximations**

Sol.Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs,the equation $f(x) = 0$ has a root between x_0 *and* x_1

The first order approximation of this root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$

$$= 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between x_2 *and* x_1 and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right].f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

We find that $f(x_3) = f(1.8548)$

$$= -0.019$$

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3.But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the third order approximate value of the root is $x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$$= 1.8557$$

This gives the approximate value of x.

2. **Find out the roots of the equation $x^3 - x - 4 = 0$ using False position method**

Sol. Let $f(x) = x^3 - x - 4 = 0$

Then $f(0) = -4, f(1) = -4, f(2) = 2$

Since $f(1)$ *and* $f(2)$ have opposite signs the root lies between 1 and 2

By False position method $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{(1 \times 2) - 2(-4)}{2 - (-4)}$$

$$= \frac{2 + 8}{6} = \frac{10}{6} = 1.666$$

$$f(1.666) = (1.666)^3 - 1.666 - 4$$

$$= -1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$$

$$f(1.780) = (1.780)^3 - 1.780 - 4$$

$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$$

$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$$

$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796$$ The root is 1.796

ITERATION METHOD:-

Consider an equation $f(x) = 0$ which can taken in the form $x = \phi(x)$ --------------(1)

where $\phi(x)$ satisfies the following conditions

(i) For two real numbers a and b, $a \le x \le b$ implies $a \le \phi(x) \le b$ and

(ii) For all x^1, x^{11} lying in the interval(a, b), we have $\left|\phi(x^1) - \phi(x^{11})\right| \le m\left|x^1 - x^{11}\right|$

where m is a constant such that $0 \le m \le 1$

Then, it can be proved that the equation (1) has a unique root 'α' in the interval (a,b). To find the approximate value of this root, we start with an initial approximation x_0 of the root'α' and find $\phi(x_0)$

We put $x_1 = \phi(x_0)$ and take x_1 as the first approximation of 'α'

Next, we put $x_2 = \phi(x_1)$ and take x_2 as the second approximation of α .Continuing the process, we get the third approximation x_3, the forth approximation x_4, and so on.

The n^th approximation is given by $x_n = \phi(x_n - 1), n \ge 1 \to (2)$

In this process of finding successive approximation of the root α, an approximation of α is obtained by substituting the preceding approximation in the function $\phi(x)$ which is known. Such a process is called on iteration process. The successive approximations $x_1, x_2 \ldots$ obtained by iteration are called the iterates. The n^th approximation x_n is called the n^th iterate.

A formula $x_n = \phi(x_n - 1), n \ge 1$ is called an iterative formula

Convergence of An Iteration:-

Since $x_1, x_2, x_3........x_n$ are taken as successive approximations of the root α, each approximation is nearest to α than the preceding approximation, so that for large n, x_n may be taken to be almost equal to α. In other words, if the sequence $\{x_{II}\}$ converges to α, we can say that the iteration process is convergent. We state below a theorem with out proof giving a sufficient condition for the convergence of the iteration given by $x_n = \phi(x_n - 1), n \geq 1$

Note:- Let I be an interval containing a root α of the equation (1). If $[\phi^1(x)] < 1$ for all x in I, then for any value of x_0 in I, the iteration given by (2) is convergent

PROBLEMS:

1) . Explain the iterative method approach in solving the problems.

Sol. In Latin the word iterate means to repeat. Iterative methods use a process of obtaining better and better estimates of solution with each iteration (or) repetitive computation. This process continues until an acceptable solution is found

The steps involved in an iterative method are

1. Develop an algorithm to solve a problem step-by-step
2. Make an initial guess or estimate for the variables (or) variables of the solution. The initial estimates should be reasonable. Success in the solution is dependent of the selection of proper initial values of variables
3. Better and better estimates are obtained in the progressive iterations by using the algorithm developed.
4. Stop the iteration process after reaching an acceptable solution, based on a reasonable criteria being met.

2) **Explain the classification of iterative method based on the number of guesses**

Sol. Iterative methods can be classified into two categories based on the number of guesses

1. **Interpolation methods – also called as bracketing methods**
2. **Extrapolation methods – also called as open end methods**

Two estimates are made for the root in the interpolation methods. One is positive value for the function $f(x)$ and the other gives a negative value for the function $f(x)$. This means that the root is bracketed between these two values

By proper selection, the gap between the two estimates can be reduced further to arrive at a very small gap. Two popular methods of this type are

a) Bi-section method b) False position method

A single value, which is called as initial estimate is chosen in the extrapolation methods. The new value of the root is successively computed in each step, depending on the algorithm. This process is continued untill the difference between the values of two successive iterations is small enough to stop the iteration process. Some methods of this type are Newton- Raphson method

3. **By the fixed point iteration process, find the root correct to 3-decimal places, of the equation** $x = \cos x \ \ near \ \ x = \pi / 4$

Sol:- The given equation is of the form $x = \phi(x)$

Where $\phi(x) = \cos x$

$\left|\phi^1(x)\right| = \left|\sin x\right| < 1 \ for \ all \ x$

Hence, the iteration process $x_n = \phi(x_{n-1})$ is convergent in every interval. Since the root is near $\pi / 4$, we take the initial approximation of the root as $x_0 = \pi / 4 = 0.785398$

Then, by iteration formula $x_n = \phi(x_{n-1})$

$$x_1 = \phi(x_0) = \cos(\pi / 4) = 0.7071068$$
$$x_2 = \phi(x_1) = \cos x_1 = 0.7602446$$
$$x_3 = \phi(x_2) = \cos x_2 = 0.7246675$$

Similarly we get,

$$x_4 = 0.7487199$$
$$x_5 = 0.7325608,$$
$$x_6 = 0.7434642.......$$

By observing these iterations, we conclude the approximation as 0.739 for the required root correct to three decimal places

4. Solve $x = 1 + \tan^{-1} x$ **by iteration method**

Sol Here $\phi(x) = 1 + tan^{-1}x, \phi^1(x) = \frac{1}{1+x^2} < 1$

Hence the process converges and we take $x_{i+1} = 1 + \tan^{-1} x_i$

Where $x_0 = 1, x_1 = 1.7854, x_2 = 2.0602$

$$x_3 = 2.1189, x_4 = 2.1318$$

$x_5 = 2.1322, x_6 = 2.1323 \quad \therefore x_6 = x_7$

Hence the root is 2.1323............

Newton- Raphson Method:-

The Newton- Raphson method is a powerful and eligant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1) = 0$. We use Taylor's theorem and expand $f(x_1) = f(x_0 + h) = 0$

$$\Rightarrow f(x_0) + hf^1(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 ,we get

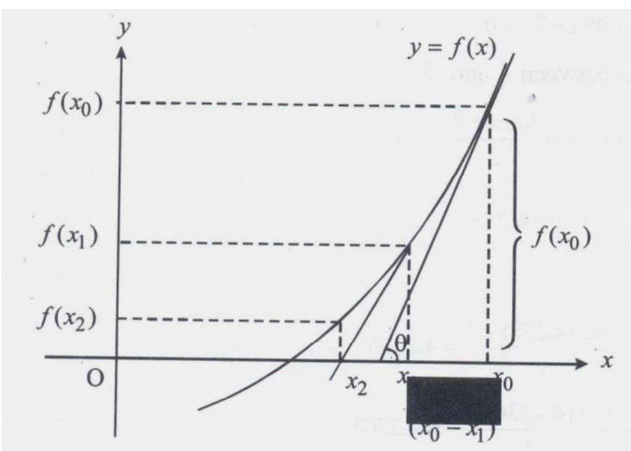
$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$x_2, x_3 \dots\dots\dots.. x_{n+1}$ where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$



Problems:

1. Apply Newton – Raphson method to find an approximate root, correct to three decimal places, of the equation $x^3 - 3x - 5 = 0,$ which lies near $x = 2$

<u>Sol:-</u> Here $f(x) = x^3 - 3x - 5 = 0 \;\; and \;\; f^1(x) = 3(x^2 - 1)$

$\therefore$ The Newton – Raphson iterative formula

$$x_{i+1} = x_i - \frac{x_i^3 - 3x_i - 5}{3(x_i^2 - 1)} = \frac{2x_i^3 + 5}{3(x_i^2 - 1)}, i = 0,1,2....(1)$$

To find the root near $x = 2$, we take $x_0 = 2$ then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3\left(x_0^2 - 1\right)} = \frac{16+5}{3(4-1)} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3\left(x_1^2 - 1\right)} = \frac{2\times(2.3333)^3 + 5}{3\left[(2.3333)^2 - 1\right]} = 2.2806$$

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^3 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since x_3 and x_4 are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal

2. Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

Sol. a) **<u>Square root:-</u>**

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found. The solution to $f(x)$ is then $x = \sqrt{N}$

Here $f'(x) = 2x$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Using the above iteration formula the square root of any number N can be found to any desired accuracy. For example, we will find the square root of $N = 24$.

Let the initial approximation be $x_0 = 4.8$

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{23.04 + 24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01 + 24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904 + 24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3$, there fore the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That means, the square root of 24 is 4.898

b) Reciprocal:-

Let $f(x)=\frac{1}{x}-N=0$ where N is the number whose reciprocal is to be found

The solution to $f(x)$ is then $x=\frac{1}{N}$. Also, $f^1(x)=\frac{-1}{x^2}$

To find the solution for $f(x)=0$, apply Newton – Raphson method

$$x_{i+1}=x_i-\frac{\left(\frac{1}{x_i}-N\right)}{-1/x_i^2}=x_i(2-x_iN)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0=0.045$

$\therefore x_1=0.045(2-0.045\times 22)$

$=0.045(2-0.99)$

$=0.0454(1.01)=0.0454$

$x_2=0.0454(2-0.0454\times 22)$

$=0.0454(2-0.9988)$

$=0.0454(1.0012)=0.04545$

$x_3=0.04545(2-0.04545\times 22)$

$=0.04545(1.0001)=0.04545$

$$x_4=0.04545(2-0.04545\times 22)$$
$$=0.04545(2-0.99998)$$
$$=0.04545(1.00002)$$

$=0.0454509$

$\therefore$ The reciprocal of 22 is 0.04545

3. **Find by Newton's method, the real root of the equation $xe^x-2=0$ correct to three decimal places.**

Sol. Let $f(x)=xe^x-2\rightarrow(1)$

Then $f(0)=-2$ and $f(1)=e-2=0.7183$

So root of $f(x)$ lies between 0 and 1

It is near to 1. So we take $x_0=1$ and $f^1(x)=xe^x+e^x$ *and* $f^1(1)=e+e=5.4366$

$\therefore$ By Newton's Rule

First approximation $x_1=x_0-\frac{f(x_0)}{f^1(x_0)}$

$$= 1 - \frac{0.7183}{5.4366} = 0.8679$$

$\therefore f(x_1) = 0.0672 \qquad f^1(x_1) = 4.4491$

The second approximation $x_2 = x_1 - \dfrac{f(x_1)}{f^1(x_1)}$

$$= 0.8679 - \frac{0.0672}{4.4491}$$
$$= 0.8528$$

$\therefore$ Required root is 0.853 correct to 3 decimal places.

Interpolation

Introduction:-

If we consider the statement $y = f(x) x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

$$x : x_0 \quad x_1 \quad x_2........x_n$$
$$y : y_0 \quad y_1 \quad y_2........y_n$$

Satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, then it is possible to find a simple function say $f(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

Errors in Polynomial Interpolation:-

Suppose the function $y(x)$ which is defined at the points $(x_i, y_i) i = 0,1,2,3----n$ is continuous and differentiable $(n+1)$ times let $\phi_n(x)$ be polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i, i = 1,2---n \to (1)$ be the approximation of $y(x)$ using this $\phi_n(x_i)$ for other value of x, not defined by (1) the error is to be determined since $y(x)-\phi_n(x)=0 \;\; for \;\; x-x_0,x_1,......x_n$ we put

$y(x)-\phi_n(x)=L\pi_{n+1}(x)$

Where $\pi_{n+1}(x)=(x-x_0).........(x-x_n)\rightarrow(3)$ and L to be determined such that the equation (2) holds for any intermediate value of x such as $x=x^1, x_0<x^1<x_n$

Clearly $L=\dfrac{y(x^1)-\phi_n(x^1)}{\pi_{n+1}(x^1)}\rightarrow(4)$

We construct a function $F(x)$ such that $F(x)=F(x_n)=F(x^1)$. Then $F(x)$ vanishes $(n+2)$ times in the interval $[x_0,x_n]$. Then by repeated application of Rolle's theorem. $F^1(x)$ must be zero $(n+1)$ times, $F^{11}(x)$ must be zero n times…….. in the interval $[x_0,x_n]$. Also $F^{n+1}(x)=0$ once in this interval. suppose this point is $x=\varepsilon$, $x_0<\varepsilon<x_n$ differentiate (5) $(n+1)$ times with respect to x and putting $x=\varepsilon$, we get

$y^{n+1}(\varepsilon)-L(n+1)!=0$ which implies that $L=\dfrac{y^{n+1}(\varepsilon)}{(n+1)!}$

Comparing (4) and (6) , we get

$$y(x^1)-\phi_n(x^1)=\frac{y^{n+1}(\varepsilon)}{(n+1)!}\pi_{n+1}(x^1)$$

Which can be written as $y(x)-\phi_n(x)=\dfrac{\pi_{n+1}(x)}{(n+1)!}y^{n+1}(\varepsilon)$

This given the required expression $x_0<\varepsilon<x_n$ for error

Finite Differences:-

1.**Introduction:-**

In this chapter, we introduce what are called the forward, backward and central differences of a function $y=f(x)$. These differences and three standard examples of finite differences and play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics

2.**Forward Differences:-**

Consider a function $y=f(x)$ of an independent variable x. let $y_0,y_1,y_2,....y_r$ be the values of y corresponding to the values $x_0,x_1,x_2....x_r$ of x respectively. Then the differences

$y_1 - y_0, y_2 - y_1 -----$ are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, ...$ that is

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2 ....$$

In general $\Delta y_r = y_{r+1} - y_r \therefore r = 0,1,2-----$

Here, the symbol Δ is called the forward difference operator

The first forward differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1......$ that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2.......$ similarly, the n^th^ forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2.......$$

While using this formula for $n = 1$, use the notation $\Delta^0 y_r = y_r$ and we have $\Delta^n y_r = 0 \forall n = 1,2......$ *and* $r = 0,2,.........$ the symbol Δ^n is referred as the n^th^ forward difference operator.

3.Forward Difference Table:-

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
	y_4	$= y_4 - y_3$			

Example finite forward difference table for $y = x^3$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

4. Backward Differences:-

As mentioned earlier, let $y_0, y_1......y_r......$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2............x_r......$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2,....$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1},\ r = 1,2,3......... \to (1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0,1,2....... \to (2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 --- \nabla^2{}_r ----$i.e.,..

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2 \,\ldots\ldots\ldots$$

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3..... \to (3)$ similarly, the nth backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1..... \to (4)$ While using this formula, for n = 1 we employ the notation $\nabla^0 y_r = y_r$

If $y = f(x)$ is a constant function , then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the nth backward difference operator

5. Backward Difference Table:-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\Delta^2 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

6. Central Differences:-

With $y_0, y_1, y_2 .... y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2 ..... x_r .... of\ x$, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \to (1)$$

The symbol δ is called the central differences operator. This operator is a linear operator

Comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2 .....$$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0,1,2...... \to (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2 ...$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{\ 3/2} ....$

$$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \to (3)$$

Higher order central differences are similarly defined. In general the n[th] central differences are given by

i) for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1,2.... \to (4)$

ii) for even $n : \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1,2.... \to (5)$

while employing for formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$

If y is a constant function, that is if $y = c$ a constant, then $\delta^n y_r = 0\ \ for\ \ all\ \ n \geq 1$

7. Central Difference Table

x_0	y_0	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Example: Given $f(-2)=12, f(-1)=16, f(0)=15, f(1)=18, f(2)=20$ from the central difference table and write down the values of $\delta y_{3/2}, \delta^2 y_0$ *and* $\delta^3 y_{7/2}$ by taking $x_0 = 0$

Sol. The central difference table is

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		
		2			
2	20				

Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by symbolic methods

Definition:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right]$

Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Relationship Between Δ *and* E

We have

$$\begin{aligned}\Delta y_0 &= y_1 - y_0 \\ &= Ey_0 - y_0 = (E-1)y_0 \\ \Rightarrow \Delta &= E - y(or)E = 1 + \Delta\end{aligned}$$

Some more relations

$$\begin{aligned}\Delta^3 y_0 = (E-1)^3 y_0 &= (E^3 - 3E^2 + 3E - 1)y_0 \\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

Definition

Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$

In general $E^{-n} y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$

ii) $\delta \equiv E^{1/2} - E^{-1/2}$

iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Definition The operator D is defined as $Dy(x) = \frac{\partial}{\partial x}\left[y(x)\right]$

Relation Between The Operators D And E

Using Taylor's series we have, $y(x+h) = y(x) + hy^1(x) + \frac{h^2}{2!}y^{11}(x) + \frac{h^3}{3!}y^{111}(x) + ----$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + ----\right]y_x \quad = e^{hD}.y_x$$

We obtain in the relation $E = e^{hD} \to (3)$

❖ If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Proof:

Let $f(x) = a_0 x^n + a_1 x^{n-1} + - - - - - - + a_{n-1} x + a_n$ where $a_0, a_1, a_2 .... a_n$ are constants and $a_0 \neq 0$. If h is the step- length, we know the formula for the first forward difference

$$\Delta f(x) = f(x+h) - f(x) = \left[a_0 (x+h)^n + a_1 (x+h)^{n-1} + - - - - + a_{n-1}(x+h) + a_n \right]$$
$$- \left[a_0 x^n + a_1 x^{n-1} + - - - - + a_{n-1} x + a_n \right]$$
$$= a_0 \left[\left\{ x^n + n.x^{n-1} h + \frac{n(n-1)}{2!} x^{n-2}.h^2 + - - - \right\} - x^n \right] +$$
$$a_1 \left[\left\{ x^{n-1} + (n-1) x^{n-2}.h + \frac{(n-1)(n-2)}{2!} x^{n-3}.h^2 + - - - \right\} - x^{n-1} \right] +$$
$$- - - - - - + a_{n-1} h$$
$$= a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + - - - - + b_{n-3} x + b_{n-2}$$

Where $b_2, b_3, ... b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$, thus, the first difference of a polynomial of n^th degree is a polynomial of degree $(n-1)$

Now

$$\Delta^2 f(x) = \Delta\left[\Delta f(x) \right]$$
$$= \Delta\left[a_0 n h . x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + - - - - + b_{n-1} x + b_{n-2} \right]$$
$$= a_0 n h \left[(x+h)^{n-1} - x^{n-1} \right] + b_2 \left[(x+h)^{n-2} - x^{n-2} \right] + - - - + b_{n-1} \left[(x+h) - x \right]$$
$$= a_0 n^{(n-1)} h^2 x^{n-2} + c_3 x^{n-3} + - - - - - - + c_{n-4} x + c_{n-3}$$

Where $c_3 .... c_{n-3}$ are constants. This polynomial is of degree $(n-2)$

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$ continuing like this we get $\Delta^n f(x) = a_0 n(n-1)(n-2) - - - - - 2.1.h^n = a_0 h^n (n!)$

$\therefore$ which is constant

Note:-

1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0, \Delta^{n+2} f(x) = 0, ...$
2. The converse of above result is also true that is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n

Example:-

1. Form the forward difference table and write down the values of $\Delta f(10)$, $\Delta^2 f(10), \Delta^3 f(15)$ *and* $\Delta^4 y(15)$

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	19.97					
		1.54				
15	21.51		- 0.58			
		0.96		0.67		
20	22.47		0.09		- 0.68	
		1.05		- 0.01		0.72
25	23.52		0.08		0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

We note that the values of x are equally spaced with step- length h = 5

Note: - $\therefore x_0 = 10, x_1 = 15 ---- x_5 = 35$ and

$$y_0 = f(x_0) = 19.97$$
$$y_1 = f(x_1) = 21.51$$

$$y_5 = f(x_5) = 25.89$$

$$y_5 = f(x_5) = 25.89$$

From table

$$\Delta f(10) = \Delta y_0 = 1.54$$
$$\Delta^2 f(10) = \Delta^2 y_0 = -0.58$$
$$\Delta^3 f(15) = \Delta^3 y_1 = -0.01$$
$$\Delta^4 f(15) = \Delta^3 y_1 = 0.04$$

2. **Evaluate**

$(i)\Delta \cos x$

$(ii)\Delta^2 \sin(px+q)$

$(iii)\Delta^n e^{ax+b}$

Sol. Let h be the interval of differencing

$$(i)\Delta \cos x = \cos(x+h) - \cos x$$

$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{\text{h}}{2}$$

$$(ii)\Delta \sin(px+q) = \sin\left[p(x+h)+q\right] - \sin(px+q)$$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin(px+q)+\frac{1}{2}(\pi+ph)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

$$(iii)\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$$

$$= e^{(ax+b)}\left(e^{ah-1}\right)$$

$$\Delta^2 e^{ax+b} = \Delta\left[\Delta\left(e^{ax+b}\right)\right] - \Delta\left[\left(e^{ah}-1\right)\left(e^{ax+b}\right)\right]$$

$$= \left(e^{ah}-1\right)^2 \Delta\left(e^{ax+h}\right)$$

$$= \left(e^{ah}-1\right)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$

3. **Using the method of separation of symbols show that**

$$\Delta^n \mu_{x-n} = \mu_{x-n} - n\mu_{x-1} + \frac{n(n-1)}{2}\mu_{x-2} + ----+(-1)^n \mu_{x-n}$$

Sol. To prove this result, we start with the right hand side. Thus

$$\mu x - n\mu x - 1 + \frac{n(n-1)}{2}\mu x - 2 + ------ + (-1)^n \mu x - n$$

$$= \mu x - nE^{-1}\mu x + \frac{n(n-1)}{2}E^{-2}\mu x + ------ + (-1)^n E^{-n}\mu x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + ------ + (-1)^n E^{-n}\right]\mu x = \left(1 - E^{-1}\right)^n \mu x$$

$$= \left(1 - \frac{1}{E}\right)^n \mu n = \frac{(E-1)^n}{E}\mu n$$

$$= \frac{\Delta^n}{E^n}\mu x = \Delta^n E^{-n}\mu x$$

$= \Delta^n \mu_{x-n}$ which is left hand side

4. **Find the missing term in the following data**

x	0	1	2	3	4
y	1	3	9	-	81

Why this value is not equal to 3^3. Explain

Sol. Consider $\Delta^4 y_0 = 0$

$\Rightarrow 4y_0 - 4y_3 + 5y_2 - 4y_1 + y_0 = 0$

Substitute given values we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$

Newton's Forward Interpolation Formula:-

Let $y = f(x)$ be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + ---$$
$$+ b_n(x - x_0)(x - x_1) ---- (x - x_{n-1}) \rightarrow (1)$$

This polynomial passes through all the points $[xi; yi]$ for i = 0 to n. there fore, we can obtain the $y_i's$ by substituting the corresponding $x_i's$ as

$$at\ \ x = x_0, y_0 = b_0$$
$$at\ \ x = x_1, y_1 = b_0 + b_1(x_1 - x_0)$$
$$at\ \ x = x_2, y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \rightarrow (1)$$

Let 'h' be the length of interval such that $x_i's$ represent

$$x_0, x_0 + h, x_0 + 2h, x_0 + 3h ---- x_0 + xh$$

This implies $x_1 - x_0 = h, x_2 - x_0 - 2h, x_3 - x_0 = 3h ----x_n - x_0 = nh \rightarrow (2)$

From (1) and (2), we get

$$y_0 = b_0$$
$$y_1 = b_0 + b_1 h$$
$$y_2 = b_0 + b_1 2h + b_2 (2h) h$$
$$y_3 = b_0 + b_1 3h + b_2 (3h)(2h) + b_3 (3h)(2h) h$$

....

....

$$y_n = b_0 + b_1 (nh) + b_2 (nh)(n-1)h + ---- + b_n (nh)\left[(n-1)h\right]\left[(n-2)h\right] \rightarrow (3)$$

Solving the above equations for $b_0, b_{11}, b_2 .....b_n$, we get $b_0 = y_0$

$$b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = y_2 - y_0 - \frac{(y_1 - y_0)}{h} 2h$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore b_2 = \frac{\Delta^2 y_0}{2! h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3! h^3}, b_4 = \frac{\Delta^4 y_0}{4! h^4} ------ b_n = \frac{\Delta^n y_0}{n! h^n}$$

$$\therefore y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2! h^2}(x - x_0)(x - x_1)$$

$$+ \frac{\Delta^3 y_0}{3! h^3}(x - x_0)(x - x_1)(x - x_2) + ---- +$$

$$+ \frac{\Delta^n y_0}{n! h^n}(x - x_0)(x - x_1) --- (x - x_{n-1}) \rightarrow (3)$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph, \ where \ p = 0,1,2,.....n$

Then

$$x-x_1=x-(x_0+h)=(x-x_0)-h$$
$$=ph-h=(p-1)h$$
$$x-x_2=x-(x_1+h)=(x-x_1)-h$$
$$=(p-1)h-h=(p-2)h$$
....
$$x-x_i=(p-i)h$$
....
$$x-x_{n-1}=\left[p-(n-1)\right]h$$

Equation (3) becomes

$$y=f(x)=f(x_0+ph)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+----+ \frac{p(p-1)(p-2)----(p-(n-1))}{n!}\Delta^n y_0 \to (4)$$

Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x)=a_0+a_1(x-x_n)+a_2(x-x_n)(x-x_{n-1})+a_3(x-x_n)(x-x_{n-1})(x-x_{n-2})+----(x-x_i)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated points

$x_n, x_n-1, ...x_2, x_1, x_0$

We obtain

$$y_n(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2i}\nabla^2 y_n+---$$
$$\frac{p(p+1)----\left[p+(n-1)\right]}{n!}\nabla^n y_n+----\to(6)$$

Where $p=\dfrac{x-x_n}{h}$

This uses tabular values of the left of y_n. Thus this formula is useful formula is useful for interpolation near the end of the tabule values

Formula for Error in Polynomial Interpolation:-

If $y=f(x)$ is the exact curve and $y=\phi_n(x)$ is the interpolating curve, then the error in polynomial interpolation is given by

$$Error=f(x)-\phi_n(x)=\frac{(x-x_0)(x-x_1)----(x-x_n)}{(n+1)!}f^{n+1}(\varepsilon)\to(7)$$

for any x, where $x_0<x<x_n$ *and* $x_0<\varepsilon<x_n$

The error in Newton's forward interpolation formula is given by

$$f(x)-\phi_n(x)=\frac{p(p-1)(p-2).......(p-n)}{(n+1)!}\Delta^{n+1}f(\varepsilon)$$

Where $p=\frac{x-x_0}{h}$

The error in Newton's backward interpolation formula is given by

$$f(x)-\phi_n(x)=\frac{p(p+1)(p+2).......(p+n)}{(n+1)!}h^{n+1}y^{n+1}f(\varepsilon) \text{ Where } p=\frac{x-x_n}{h}$$

Examples:-

1. find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q°c)$	205	225	248	274

Sol. The difference table is

x	y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

$$x_0+ph=24, x_0=50, h=10$$
$$50+p(10)=54\ (or)\ p=0.4$$

By Newton's forward interpolation formula

$$f(x_0+ph)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{n!}\Delta^3 y_0+---$$

$$f(54)=205+0.4(20)+\frac{0.4(0.4-1)}{2!}(3)+\frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$
$$=205+8-0.36$$
$$=212.64$$

Melting point = 212.64

2. Using Newton's forward interpolation formula, and the given table of values

x	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)$ $when\ x = 1.4$

Sol.

X	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.1	0.21				
		0.48			
1.3	0.69		0.08		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take $x_0 = 1.3\ then\ y_0 = 0.69$,

$$\Delta y_0 = 0.56, \Delta^2 y_0 = 0.08, \Delta^3 y_0 = 0, L = 0.2, x = 1.3$$

$$x_0 + ph = 1.4 (or) 1.3 + p(0.2) = 1.4, p = \frac{1}{2}$$

Using Newton's interpolation formula

$$f(1.4) = 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} \times 0.08$$

$$= 0.69 + 0.28 - 0.01 = 0.96$$

3. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

Sol. Putting $L = 10, x_0 = 1891, x = 1895$ in the formula $x = x_0 + ph$ we obtain $p = 2/5 = 0.4$

x	y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
192	93		-4		
		8			
1931	101				

$$y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{6} - (-5)$$
$$+ \frac{(0.4-1)0.4(0.4-2)}{6}(2)$$
$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}$$
$$= 54.45 \ thousands$$

Gauss's Interpolation Formula:- We take x_0 as one of the specified of x that lies around the middle of the difference table and denote $x_0 - rh$ by $x - r$ and the corresponding value of y by $y - r$. Then the middle part of the forward difference table will appear as shown in the next page

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
x_{-3}	y_{-3}	Δy_{-4}				
x_{-2}	y_{-2}	Δy_{-3}	$\Delta^2 y_{-4}$			
x_{-1}	y_{-1}	Δy_{-2}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$		
x_0	y_0	Δy_{-1}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-4}$	
x_1	y_1	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-4}$
x_2	y_2	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$
x_3	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$
x_4	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	$\Delta^5 y_{-1}$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} -----(1) \quad and$$
$$\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$$
$$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$$
$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} -----(2)$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's forward interpolation formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + ----] \quad \text{-----------------3}$$

Here y_p is the value of y at $x = x_p = x_0 + ph$

Gauss Forward Interpolation Formula:-

Substituting for $\Delta^2 y_0, \Delta^3 y_0, ...$ from (1)in the formula (3), we get

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1} + \Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + ----]$$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1} + \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ----]$$

Substituting $\Delta^4 y_{-1}$ *from* (2), this becomes

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \quad \text{-----------------4}$$

Note:- we observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula (4) can be written in the notation of central differences as given below

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$
$$+\frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + ----] \quad \text{-----------------5}$$

2. **Gauss's Backward Interpolation formula:-**

Let us substitute for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ ----- from (1) in the formula (3), thus we obtain

$$y_p = [y_0 + p\left(\Delta y_{-1} + \Delta^2 y_{-1}\right) + \frac{p(p-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{(p-1)p(p-2)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) +$$
$$\frac{(p-1)(p-2)p(p-3)}{4!}\left(\Delta^4 y_{-1} + \Delta^5 y_{-1}\right) + ----]$$

$$= [y_0 + p\left(\Delta y_{-1}\right) + \frac{(p+1)}{2!}p\left(\Delta^2 y_{-1}\right) + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\Delta^4 y_{-1}\right) + ----]$$

Substituting for $\Delta^3 y_{-1}$ and $\Delta^4 y_{-1}$ from (2) this becomes

$$y_p = [y_0 + p\left(\Delta y_{-1}\right) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-2}\right)$$
$$+\frac{(p+1)p(p-1)(p-2)}{4!}\left(\Delta^4 y_{-2} + \Delta^5 y_{-2}\right) + ----]$$

Lagrange's Interpolation Formula:-

Let $x_0, x_1, x_2, ...\ x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2........y_n$ be the corresponding values of $y = f(x)$ let the polynomial of degree n for the function $y = f(x)$ passing through the $(n+1)$ points $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right) ---- \left(x_n, f(x_n)\right)$ be in the following form

$$y = f(x) = a_0(x-x_1)(x-x_2)......(x-x_n) + a_1(x-x_0)(x-x_2).........(x-x_n) +$$
$$a_2(x-x_0)(x-x_1).......(x-x_n) + ... + a_n(x-x_0)(x-x_1)......(x-x_{n-1}) \rightarrow (1)$$

Where $a_0, a_1, a_2....\ a^n$ are constants

Since the polynomial passes through $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right)......\left(x_n, f(x_n)\right)$. The constants can be determined by substituting one of the values of $x_0, x_1, ...x_n$ *for* x in the above equation

Putting $x = x_0$ in (1) we get, $f(x_0) = a_0(x-x_1)(x_0-x_2)(x_0-x_n)$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x-x_1)(x_0-x_2)....(x_0-x_n)}$$

Putting $x = x_1$ in (1) we get, $f(x_1) = a_1(x-x_0)(x_1-x_2) ---- (x_1-x_n)$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}$$

Similarly substituting $x = x_2$ in (1), we get

$$\Rightarrow a_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)......(x_2-x_n)}$$

Continuing in this manner and putting $x = x_n$ in (1) we get

$$a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)----(x_n-x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2....a_n$, we get

$$f(x) = \frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2).........(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}$$

$$f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_2).....(x-x_n)}{(x_2-x_0)(x_2-x_1)......(x_2-x_n)} + ...f(x_2) + \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_1)(x_n-x_2).....(x_n-x_{n-1})} f(x_n)$$

Examples:-

1. Using lagrange formula calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_5 = 6, x_4 = 5$

$$f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$$

From langrange's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2)$$

$$-------------------$$

$$-------------------$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times 1+$$

$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times 14+$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times 15+$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19$$

$$=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$

$$=0.05-4.2+11.25+3.75-1.8+0.95$$

$$=10$$

$$f(x_3)=10$$

1) Find $f(3.5)$ using lagrange method of 2^{nd} and 3^{rd} order degree polynomials.

x	1	2	3	4
$f(x)$	1	2	9	28

Sol: By lagrange's interpolation formula

$$f(x)=\sum_{k=0}^{n} f(x_k)\frac{(x-x_0).......(x-x_{k-1})(x-x_{k+1})(x-x_n)}{(x_k-x_0).......(x_k-x_{k-1}).......(x_k-x_n)}$$

For $n=4$,we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)+$$

$$\therefore f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\quad\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)+$$

$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+$$

- - - -

$$\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)+$$

=0.0625+(-0.625)+8.4375+8.75

=16.625

$$f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$

$$+\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$=\frac{(x^2-5x+6)(x-4)}{-6}+(x^2-4x+3)(x-4)+\frac{(x^2-3x+2)}{-2}(x-4)(9)+\frac{(x^2-3x+2)}{6}(x-3)(28)$$

$$=\frac{x^3-9x^2+26x-24}{-6}+x^3-8x^2+9x-12+\frac{x^3-7x^2+14x-8}{-2}(9)+\frac{x^3-6x^2+11x-6}{6}(28)$$

$$=\frac{\left[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x+216+308x+28x^3-168x^2-168\right]}{6}$$

$$=\frac{6x^3-18x^2+18x}{6}\Rightarrow f(x)=x^3-3x^2+3x$$

$$\therefore f(3.5)=(3.5)^3-3(3.5)^2+3(3.5)=16.625$$

Example:

Find y(25), given that $y_{20}=24$, $y_{24}=32$, $y_{28}=35$ $y_{32}=40$ using Guass forward difference Formula :

Solution: Given

x	20	24	28	32
y	24	32	35	40

By Gauss Forward difference formula

$$y_p=[y_0+p(\Delta y_0)+\frac{p(p-1)}{2!}\left(\Delta^2y_{-1}+\Delta^3y_{-1}\right)+\frac{p(p-1)(p-2)}{3!}\Delta^3y_{-1}$$

$$+\Delta^4y_{-1}+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4y_{-1}+\Delta^5y_{-1}+---]$$

We take x= 24 as origin.

$X_0=24$, h = 4, x = 25 p = x-x_0/ h, p = 25-24/4 = 2.5

Gauss Forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	24			
24	32	$\Delta y_{-1} = 8$		
28	35	$\Delta y_0 = 3$	$\Delta^2 y_{-1} = -5$	
32	40	$\Delta y_1 = 5$	$\Delta^2 y_0 = 2$	$\Delta^3 y_{-1} = 7$

By gauss Forward interpolation Formula

We y(25) = 32 +.25(3) + ($\frac{.25)(.25-1)}{2}(-5) + \frac{(.25+1)(.25)(.25-1)}{6}(7)$ = 32 + .75

+ .46875 - .2734 = 32.945

Y(25) = 32.945.

Example:

Use Gauss Backward interpolation formula to find f(32) given that f(25) = .2707, f(30) = .3027, f(35) = .3386 f(40) = .3794.

Solution: let x_0 = 35 and difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	.2707			
30	.3027	.032		
35	.3386	.0359	.0039	
40	.3794	.0408	.0049	.0010

From the table y_0 = 0.3386

Δy_{-1} = 0.0359 , $\Delta^2 y_{-1}$ = 0.0049, $\Delta^3 y_{-2}$ = 0.0010, x_p = 32 p = x_p- x_0/h = 32-35/5 = -.6

By Gauss Backward difference formula

f(32) = .3386 + (-.6)(.0359) + (-.6)(-.6+1)(.0049)/2 + (-.6)(.36-1)(0.00010)/6 = .3165

PRACTICE PROBLEMS ON INTERPOLATION

1.Find $y(42)$ from the following data. Using Newton's interpolation formula

X	20	25	30	35	40	45
Y	354	332	291	260	231	204

2.Construct difference table for the following data

X	0.1	0.3	0.5	0.7	0.9	1.1	1.3
Y	0.003	0.67	0.148	0.248	0.37	0.518	0.697

And find f(x) that fits at x = 0.3, 0.5, 0.7 and 0.9 using newton's forward formula

3.Find f(22) from the following table using Gauss forward forumula

X	20	25	30	35	40	45
Y	354	332	291	260	231	204

4. Using gauss back ward difference formula find y(8) from the following table

X	0	5	10	15	20	25
Y	7	11	14	18	24	32

5. Find the interpolating polynomial f(x) from the table

X	0	1	4	5
Y	4	3	24	39

6. Using lagrange's formula to fit a polynomial to the data and hence find y(1).

X	-1	0	2	3
Y	-8	3	1	12

Objective bits on Interpolations

1.

x	0	1	2
f(x)	7	10	13

By Newton's forward formula f(2.5)=

a) 15.25 b) 16.75 c) 16.25 d) 16.10

2.

x	1	2	3	4
f(x)	1	4	27	64

If x=2.5 then p=

a) 1.5 b) 1 c) 2.5 d) 2

3.

X	0.1	0.2	0.3	0.4
f(x)	1.005	1.02	1.045	1.081

When p=0.6, x=

a) 0.16 b) 0.26 c) 0.1 d) 3.0

4.

x	2	3	4
f(x)	2.6	3.4	4.7

f(1.5) by Gauss formula

a) 1.5 b) 2.4 c) 2.1 d) 2.2

5.

x	2	3	4
f(x)	2.626	3.454	4.784

f(3.5) by Gauss forward difference formula is

a) 4.0653 b) 3.9512 c) 3.7523 d) 2.75

6. $y_{20} = 2854, y_{24} = 3162, y_{28} = 3544, y_{32} = 3992\, then\, y_{25}$ using Gauss forward formula is

a) 3050 b) 3250.3 c) 3518.5 d) 3725.2

7.

x	16	18	20	22
y	43	89	K	15

$\Rightarrow K =$

a) 80 b) 95 c) 100 d) 105

8. In Gauss forward formula involves differences below the central line and even differences on the line in (Δ) then formula is useful if

a) 0<p<1 b) -1<p<0 c) $-\infty$<p<0 d) 0<p<∞

9.

x	20	30	40	50	60
f(x)	0.34	0.5	0.64	0.76	0.87

f(45) by Gauss backward formula

a)0.6517 b) 0.6925 c) 0.7175 d) 0.7624

9. A linear version of the largrange's interpolation formula for f(x) is

a) $\frac{x-x_1}{(x_0-x_1)}f(x_0)+\frac{(x-x_0)}{(x_1-x_0)}f(x_1)$ b) $\frac{x-x_1}{(x_0-x_1)}f(x_0)-\frac{(x-x_0)}{(x_1-x_0)}f(x_1)$

c) $\left(\frac{x-x_1}{x_0-x_1}\right)f(x_0)+\frac{(x-x_0)(x_2-x_0)}{(x_1-x_0)}f(x_1)$ d) $\frac{(x-x_0)}{(x_0-x_1)}f(x_0)+\frac{(x_1-x_0)}{(x-x_0)}f(x_1)$

10. The following is used for unequal interval of x values

a) Lagrange's formula b) Newton's forward

c) Newton's backward interpolation formula d) Gauss forward interpolation formula

11. If $x_0=5, y_0=12, x_1=6, y_1=13, x_2=9, y_2=14$ then y_{10} by lagrange's interpolation formula is

a) 13.667 b) 15.333 c) 16.333 d) 17

12.

x	1	3	4	7
f(x)	0	6	60	42

use Lagrange's formula to find the polynomial

a) x^2+1 b) x^3-2x c) x^2-x d) x^2-x+2