J. Biomechanics Vol. 13, pp. 49–57.

0021-9290/80/0101-0049 $02.00/0

BALLISTIC WALKING*

SIMON MOCHON and THOMAS A. MCMAHON
Division of Applied Sciences, Harvard University,
Cambridge, MA 02138, U.S.A.

Abstract – A mathematical model of the swing phase of walking is presented in this paper. The body is represented by three links, one for the stance leg and two for the thigh and shank of the swing leg. It is assumed that the muscles act only to establish an initial configuration and velocity of the limbs at the beginning of the swing phase. The swing leg and the rest of the body then moves through the remainder of the swing phase entirely under the action of gravity.

The range of possible times of swing for each step length is computed for two types of gaits; stiff-legged walking and walking with flexion at the knee. The range of times of swing found for the model with flexion at the knee are compared with published experimental results. The agreement is shown to be very close. Typical histograms of forces applied to the ground and angles of the limbs against time are also given. The computed forces and angles have the same general time course as those found experimentally in normal walking, with the exception of the vertical force, which often has a different shape. The limitations of the model apparently responsible for this discrepancy are discussed.

INTRODUCTION

When walking on the level, an individual has a range of possible step frequencies for a given step length. Nevertheless there is a preferred relationship between step frequency and walking speed.

This relationship has been reported by various investigators, among them Cotes and Meade (1960) and Grieve (1968). The most simple formula that fits the curves obtained is a power law of the form $f = \alpha v^{\beta}$, where f is the frequency and v the speed of walking. The value of β is around 0.58 and varies for different individuals (Grieve, 1968).

A more complete study has been done by Grieve and Gear (1966) in which the various relationships between time of swing, velocity and stride length are reported. It is interesting to note that the time of swing varies inversely with speed in adolescents and adults, but not in children. During the first few months of walking, children show a direct relation between time of swing and walking speed. Later, including the period up to 5 years of age, the time of swing is independent of walking speed.

No theoretical work attempting to predict the form of swing period vs speed relationship has yet been reported, but experimentally it is found that the energy consumption per unit distance is a minimum at a particular chosen frequency (Elftman, 1966). This result led Inman (1966) to describe locomotion as the translation of the center of mass through space along a path requiring the least expenditure of energy.

Various investigators used this hypothesis to model walking by minimizing some criterion which reflects the mechanical work done by the muscles of the lower extremities. Among these are the works by Beckett and Chang (1968) and Chow and Jacobson (1971).

Alexander and Goldspink (1977) have presented a series of simple and elegant mathematical models for calculating the power expended during bipedal walking. These models, which do not require a computer for their solution, pay particular attention to changes in the potential, translational and rotational kinetic energies of the body, and ascribe a metabolic cost to changes in these energies.

Another theme in theoretical models concerns the application of control theory to biped locomotion machines (Frank, 1970; Gubina *et al.*, 1974). The legs are generally considered massless in the interests of keeping the analysis manageable. Furthermore, the modelling is usually confined to the stance phase, because a particularly important question concerning robot bipeds is the stability of the trunk on the limbs.

In this paper, a purely ballistic model of walking during the swing phase is presented. The double support phase is thus specifically excluded from consideration here. We will assume that the action of the muscles during the double-support phase establishes a set of initial conditions and velocities for both the stance and swing legs. Throughout the remainder of the swing phase, the limbs move under the influence of gravity alone, and finish in a position which allows direct entry into the next step. Our model is thus essentially different from those which consider the mechanical work done by the lower extremities, since the total energy is constant during the ballistic swing. It is also quite different from those models which investigate servo control of walking, again because we assume zero muscular torques acting during the swing.

The purpose of this work is to see how well the swing phase of human gait may be described as a ballistic motion. It has always seemed plausible that the action of the swing leg is like the motion of a pure pendulum; in fact, this suggestion first appears in the literature more than a century ago (Weber and Weber. 1836).

* *Received* 15 *October* 1978; *in revised form* 3 *May* 1979

This suggestion has been often debated since, and was investigated further by Grieve and Gear (1966) who concluded that most swings are performed in times considerably less than the natural half-period of either the lower leg or the whole leg regarded as a compound pendulum.

Electromyographic traces show that the muscles of the swing leg are reasonably silent during the whole swing period, except just at the beginning and end. Basmajian (1976) stated, "very little electromyographic activity appears in any of the muscles during normal moderate-speed walking... we need only small inputs of propulsive force and balancing mechanisms to maintain forward progress". Thus, a ballistic model of walking seems plausible on electromyographic, as well as dynamic grounds.

Saunders *et al.* (1953), in their investigation with amputees, concluded that the major determinants of human gait are: (1) pelvic rotation, (2) pelvic tilt, (3) knee flexion, (4) hip flexion, (5) knee and ankle interaction, and (6) lateral pelvic displacement. They found that the loss of one of these determinants can be compensated by the other five, but the loss of two or more deranges gait so seriously that normal walking may no longer be accomplished.

As Saunders *et al.* (1953) point out, coordinated knee and ankle interaction of the stance leg is important in decreasing the vertical excursion of the center of mass of the body during a step. Even so, a "compass-gait" model of walking, where the stance leg is stiff and the whole body is represented as an inverted pendulum, has been used successfully by Cavagna *et al.* (1976) to explain the changes of kinetic and potential energies of the center of mass of the body that occur in normal walking. We take this as evidence for the argument that determinants (3) and (4) are more significant than determinants (1), (2), (5) and (6) in setting the characteristics of normal walking. Certainly, forward progression would be impossible without determinant (4).

The model presented here includes only two of the six determinants: hip flexion (4) and knee flexion of the swing leg (3). To study the contribution of swing leg knee flexion to the dynamics of walking, the model is first analyzed without movement at the knee so that the swing leg is a stiff link. This type of gait is called stiff-legged walking.

MATHEMATICAL MODEL

The model shown schematically in Fig. 1 consists of 3 links; one representing the stance leg and two representing the thigh and shank of the swing leg. The foot of the swing leg is rigidly attached to the distal link, as explained below, and therefore does not constitute a separate link. Each link is assumed to have a distributed mass. The moment of inertia and location of the center of mass of each link is taken from the anthropometric data of Dempster (given in the Appendix). The mass of the foot is lumped into the shank. The mass of the trunk, head and arms is represented by a

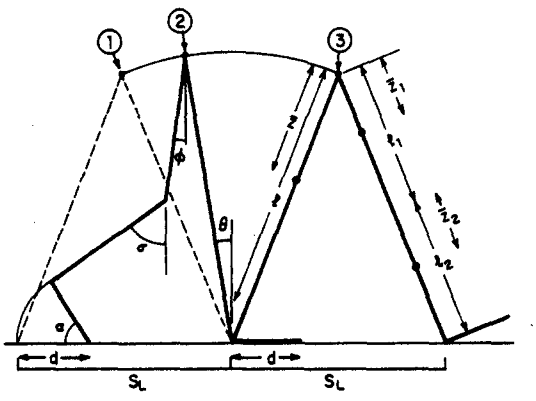

Fig. 1. Schematic representation of the model. The numbers (1), (2) and (3) give, respectively, the position of the model at heel strike, toe-off and following heel strike. The angles, lengths and positions of the centers of mass of each limb are shown in the figure. For meaning of symbols see Appendix.

point mass at the hip joint. The lengths, positions of the centers of mass and angles of each limb are shown in Fig. 1.

As explained in the Introduction, we will consider two special cases of bipedal gait; stiff-legged walking and walking with knee flexion. The equations of motion for these two cases are derived by writing expressions for the total kinetic and potential energies of the system (Fig. 1) and applying Lagrange's equations.

Stiff-legged walking

For stiff-legged walking, Lagrange's equations yield the following ballistic equations of motion:

$$J\ddot{\theta} - C\dot{\phi}^2 \sin(\theta - \phi) - C\ddot{\phi}\cos(\theta - \phi) = U \sin\theta \tag{1}$$

$$K\ddot{\phi} + C\dot{\theta}^2 \sin(\theta - \phi) - C\ddot{\theta}\cos(\theta - \phi) = -W \sin\phi, \tag{2}$$

where the meaning of the symbols is given in the Appendix and Fig. 1.

We have imposed, in this case, the following boundary conditions and constraints. Due to the geometry of the model, at the beginning and at the end of the swing (represented by points (1) and (3), respectively, in Fig. 1) both legs must make the same angle with the vertical, so that both heels are in contact with the ground. This then imposes two kinematic conditions: $\theta(0) = -\phi(0)$, and $\theta(T_s) = -\phi(T_s)$, where T_s is the time of heel strike. A third kinematic condition is that the step length should be equal at the beginning and at the end of the swing. Since a possible swing is determined by 5 parameters (2 initial angles; 2 initial angular velocities and the time of swing), with the 3 equations mentioned so far, we still have $5 - 3 = 2$ free parameters.

But there are two more conditions, which might be called dynamic constraints. One is that the vertical force at the ground must always be positive; otherwise the model will fly off the ground. The other is that the initial angular velocity of the swing leg should be positive, because if it is negative the leg will swing backwards at the beginning of the swing.

It is evident that in stiff-legged walking, the swing leg will hit the ground at some point during the swing – this is an inevitable consequence of the fact that the two legs are the same length. An individual walking with braces at the knees can avoid this by plantar flexion of the ankle of the stance leg or by tilting of the pelvis. Since these accommodations are beyond the capabilities of our model, we will ignore the fact that the foot of the swing leg drops below the surface of the ground during the swing.

Ballistic walking including knee flexion

When flexion is allowed at the knee of the swing leg, the equations become

$$K_1\ddot{\theta} - C_2\ddot{\phi}\cos(\theta-\phi) - C_2\dot{\phi}^2\sin(\theta-\phi) - C_3\ddot{\sigma}\cos(\theta-\sigma) - C_3\dot{\sigma}^2\sin(\theta-\sigma) = W_1\sin\theta \quad (3)$$

$$K_2\ddot{\phi} + C_1\ddot{\sigma}\cos(\phi-\sigma) + C_1\dot{\sigma}^2\sin(\phi-\sigma) - C_2\ddot{\theta}\cos(\theta-\phi) + C_2\dot{\theta}^2\sin(\theta-\phi) = -W_2\sin\phi \quad (4)$$

$$K_3\ddot{\sigma} - C_3\ddot{\theta}\cos(\theta-\sigma) + C_3\dot{\theta}^2\sin(\theta-\sigma) + C_1\ddot{\phi}\cos(\phi-\sigma) - C_1\dot{\phi}^2\sin(\phi-\sigma) = -W_3\sin\sigma, \quad (5)$$

where the meaning of the constants is given in the Appendix.

As was the case with stiff-legged walking, a number of initial and final conditions must be imposed to specify the motion.

At toeing-off [(2) in Fig. 1] the angle that the foot makes with the horizontal is called α. The condition that the toe should be still in contact with the ground imposes two geometric conditions, as follows. The vertical position of the toe of the swing leg is zero (see Fig. 1). This gives the first geometric condition:

$$l\cos\theta_0 - l_1\cos\phi_0 - l_2\cos\sigma_0 - d\sin\alpha = 0. \quad (6)$$

Since the toe has not moved horizontally, it is still a distance $S_L - d$ from the ankle of the other leg. Thus the second geometric condition is:

$$l\sin\theta_0 + l_1\sin\phi_0 + l_2\sin\sigma_0 - d\cos\alpha = S_L - d \quad (7)$$

(see Appendix for meaning of constants). We arbitrarily increase the angle α linearly from 45° to 65° as step length varies from 0.5 to 1.

The above two conditions fix the position of the ankle at the beginning of the swing, but this leaves the position of the hip still undetermined. The position of the hip at toe-off depends on the movement of the body during double support. Cavagna *et al.* (1976) reported that the forward displacement of the center of mass of the body during double support is a constant for a given individual, independent of walking speed up to a speed of 7 km/hr. This constant is generally a length just smaller than the length of the foot. We have taken the constancy of the double support length as our third condition, so that

$$l\sin\phi_f - l\sin\theta_0 = 0.9d. \quad (8)$$

We also require that at heel strike [(3) in Fig. 1] the knee angle should arrive at zero and the heel of the swing leg should strike the ground simultaneously. This requirement gives two equations specifying two independent final conditions. Firstly, the condition that the knee has just locked at heel strike says $\sigma(T_s) = \phi(T_s)$. Secondly, the condition that time $t = T_S$ corresponds to heel strike requires $\phi(T_s) = -\theta(T_s)$.

Two kinematic constraints applying to the swing have been imposed. One requires that the foot must clear the ground at all times. During the swing, from the moment following toe-off, the ankle of the swing leg is assumed to be locked at 90°.

Additionally, in this case of walking including knee flexion, as was the case in stiff-legged walking, we require that the vertical force applied to the ground must always remain positive.

An extension of ballistic walking including knee flexion

We have extended this model by relaxing one of the conditions. Instead of demanding that the knee arrives at zero degrees at heel strike as we did in the strict version of the model described above, the knee is allowed to lock when it passes through zero degrees and then the swing is continued with a stiff leg until heel strike.

This locking of the knee before heel strike has to be taken as a mathematical substitute for the braking torque the muscles actually supply to the knee at the end of the swing. Instead of having a moment applied to the knee during the last part of the swing, an impulsive moment is assumed to act at the instant the knee angle reaches zero. This impulsive moment is just sufficient to bring the knee extension velocity ($\dot{\sigma} - \dot{\phi}$), to zero, locking the knee.

To represent this condition mathematically, we note that we are only applying a moment at the knee to lock it; there are no moments applied to the ankle of the stance leg or to the hip. This means that the angular momentum of the whole system around the ankle of the stance leg is the same before and after knee lock. For the same reason, the angular momentum of the swing leg around the hip is conserved through the locking collision. These two conditions specify the two angular velocities of the legs immediately after knee lock, given the angular velocities of the three links immediately before.

Conservation of angular momentum of the whole system around the ankle gives:

$$[J - C\cos(\theta-\phi)]\dot{\theta}_a + [K - C\cos(\theta-\phi)]\dot{\phi}_a = [K_1 - C\cos(\theta-\phi)]\dot{\theta}_b + [K_2 + C_1 - C_2 \times \cos(\theta-\phi)]\dot{\phi}_b + [K_3 + C_1 - C_3\cos(\theta-\phi)]\dot{\sigma}_b. \quad (9)$$

Conservation of angular momentum of the swing leg around the hip gives:

$$- C\cos(\theta-\phi)\dot{\theta}_a + K\dot{\phi}_a = - C\cos(\theta-\phi)\dot{\theta}_b + (K_2 + C_1)\dot{\phi}_b + (K_3 + C_1)\dot{\sigma}_b, \quad (10)$$

where the a and b subscripts mean after and before knee lock and θ and ϕ are the angles of the stance leg and the thigh when the knee arrives at zero degrees. These equations can also be obtained by drawing free body diagrams of each of the three links, assuming that the changes of linear and angular momenta are due to impulsive forces acting at the ground (ankle), hip joint and knee joint, and that an impulsive moment acts at the knee.

In order to keep the results as general as possible, we have changed all the equations to dimensionless form using the length of the leg l and the natural half-period of the leg $T_n = \pi(I_l/M_t gZ)^{1/2}$ as the scales for length and time. (For a leg length of 1 m, T_n is approximately 0.82 sec.) The constants appearing in the dimensionless equations are then computed from Dempster's data given in the Appendix.

RESULTS

Stiff-legged gait

The constraints imposed on the model will set limits on the possible range of the variables. This is shown for stiff-legged walking in Fig. 2. For a fixed step length there is a lower and an upper limit to the time of swing. These limits are set, respectively, by the conditions that the vertical force on the ground and the initial angular velocity of the swing leg should be positive. For times of swing to the left of the shaded area in Fig. 2, the model will fly off the ground. For times to the right of the shaded area, the swing leg will initially swing backward. The boundary $S_L = 1$ is a physiological limit to hip extension. This boundary is not precisely

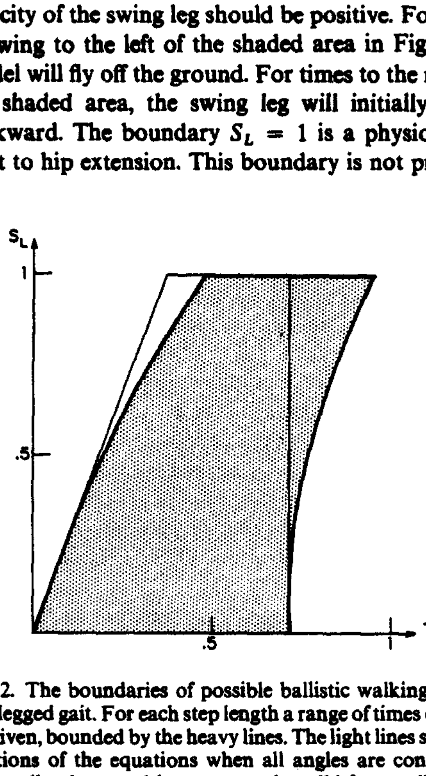

Fig. 2. The boundaries of possible ballistic walking for the stiff-legged gait. For each step length a range of times of swing are given, bounded by the heavy lines. The light lines show the solutions of the equations when all angles are confined to small values, and hence are only valid for small S_L.

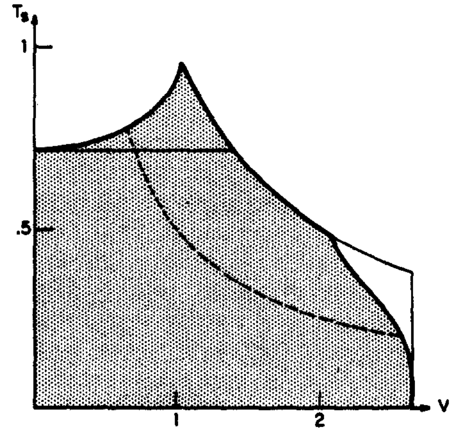

Fig. 3. The same boundaries given in Fig. 2 with the normalized time of swing T_s and velocity V now as the variables.

determined, since it varies from individual to individual, but it is included for completeness. The light lines show the analytical results of the linearized equations, and are therefore strictly valid only for small σ, ϕ and θ. Figure 3 shows the same boundaries given in Fig. 2, in a graph of time of swing against speed. The dotted line represents a constant step length ($S_L = 0.5$).

Ballistic walking including knee flexion

For the model with movement at the knee, we imposed 5 conditions, and since 7 parameters determine a solution in this case, fixing 2 parameters fixes the others automatically. The most convenient parameters to fix for computational reasons are the step length S_L and the initial angular velocity of the stance leg $\dot{\theta}_0$. For a $S_L = 0.75$ and $\dot{\theta}_0 = -0.8$, point (1) in Fig. 4 gives the initial angular velocities for the thigh $\dot{\phi}_0$ and shank $\dot{\sigma}_0$ which were found to satisfy the 5 conditions imposed. The curve on this figure shows all the possible combinations of initial velocities $\dot{\phi}_0$ and $\dot{\sigma}_0$ which ultimately bring the swing leg to the proper end condition if the knee is allowed to lock before heel strike. Each point in this curve represents a swing with a different percentage of the time of swing after knee lock. The shaded part of the curve represents those initial conditions for which the swing leg strikes the ground in mid-swing. The shaded region is therefore excluded from the range of initial conditions which result in a possible swing.

If, now, for the same step length, $\dot{\theta}_0$ is varied, a graph such as the one in Fig. 4 is obtained. Increasing the value of $\dot{\theta}_0$ (which reduces the time of swing) will bring point (1) closer to the $\dot{\phi}_0$ axis until a critical value is reached at which points (1) and (2) coincide and the toe of the swing leg just grazes the ground in mid-stride. This boundary is given by the furthest-left heavy line in Fig. 5. When the time of swing is made smaller than this limit, the toe will strike the ground in mid-stance.

Moving from left to right along any horizontal line

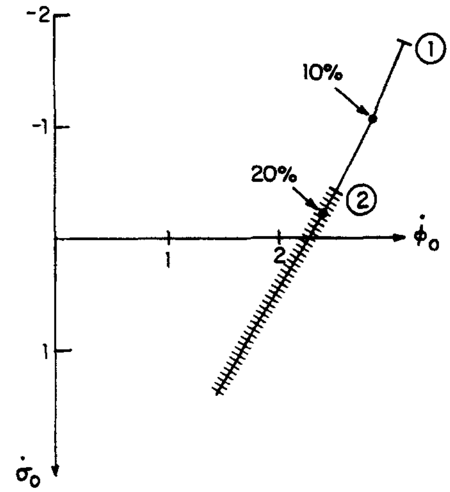

Fig. 4. Initial velocities of the thigh $\dot{\phi}_0$ and shank $\dot{\sigma}_0$ which make the model satisfy all the conditions imposed. The shaded part of the curve are those initial velocities which cause the foot of the swing leg to strike the ground in mid-swing. The 10% (20%) point represents a swing where the knee is locked when 10% (20%) of the time of swing still remains before heel-strike. Points (1) and (2) are explained in the text.

which begins at the furthest-left heavy line, higher values of the time of swing are found to correspond with higher negative initial angular velocities of the shank $\dot{\phi}_0$, which means that the knee will be flexed more during the swing. The next two heavy near-vertical lines in Fig. 5 correspond to those swings in which the knee has flexed to a maximum of 90° and 125°, respectively. This 125° limit seems to us a reasonable physiological limit for knee flexion. Not only will the knee flex more for higher times, but also the thigh will swing to higher angles at the final part of the swing, as is shown schematically in the upper diagrams of Fig. 5. Therefore times of swing higher than those given by the 125° time will be very hard to achieve from the physiological point of view.

When knee lock and heel strike are not synchronous

The results given up until now are for the strict version of the model, where the knee reaches full extension at the moment the heel strikes the ground. If we now look at the results of the extended model, where heel strike occurs some time after knee lock, the limit curves will be shifted to higher values of the time of swing. In Fig. 5, the broken lines show how the 125° maximum knee flexion line is shifted when 10 and 20% of the swing time still remains after knee lock.

A light line is also shown in Fig. 5. This line divides those swings in which the model will fly off the ground (to the left of this line) from those which will not (to the right). Note that the shaded area is well to the right of this line so that the condition of positive vertical force at the ground is not a determining factor in setting a lower limit to the time of swing.

No results are shown in Fig. 5 for step lengths below 0.5. This is because the computer analysis has only been carried out for step lengths between 0.5 and 1. This range is the one most commonly used in normal walking (Grieve, 1968). Figure 6 shows the same results given in Fig. 5, but now with the velocity of walking as a variable. The walking velocity v is computed:

$$v = \frac{S_L - S_{DS}}{T_s},$$

where $S_{DS} = 0.9d$ is the distance the body moves forward during the double support phase. It can be seen from these graphs that low (high) speeds of walking correspond to small (large) step lengths and to large (small) times of swing.

Typical histograms of force and angles against time during the swing phase are given in Figs. 8 and 9. For comparison, experimental traces of the vertical and forward forces at the ground reproduced from Cavagna and Margaria (1966) are presented in Fig. 7. The arrows added to Fig. 7 indicate the moment of toeing-off (full arrow) and heel strike (broken arrow). The portion between the arrows is then the swing phase. It is this portion of the complete step cycle which our model attempts to simulate.

Figure 8 shows the calculated angles and ground reactions during the swing for the limit case where the foot will just clear the ground ($S_L = 0.75$, $T_s = 0.53$). The predicted angles and ground reactions look very much like those found experimentally for normal walking during the swing phase, except for the vertical

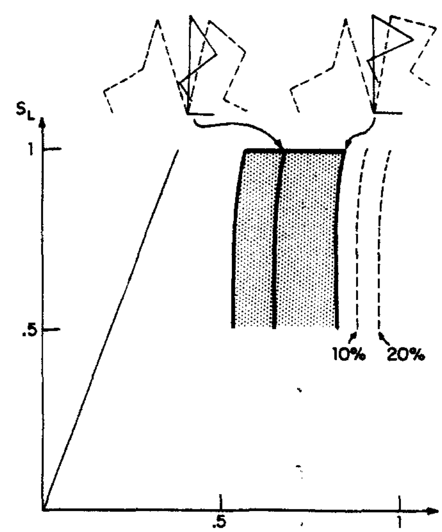

Fig. 5. The range of possible times of swing for each step length is given by the shaded area for the model including swing leg knee flexion. The broken lines show how the furthest-right boundary (representing the case where knee flexion reaches a maximum of 125°) is shifted if knee lock occurs 10% (20%) of the time of swing before heel strike. The upper diagrams show the moment of toe-off (left broken configuration), maximum knee flexion (continuous configuration) and maximum hip flexion (right broken configuration) for a step length of 1.0 and a maximum knee flexion of 90° (left diagram) and 125° (right diagram).

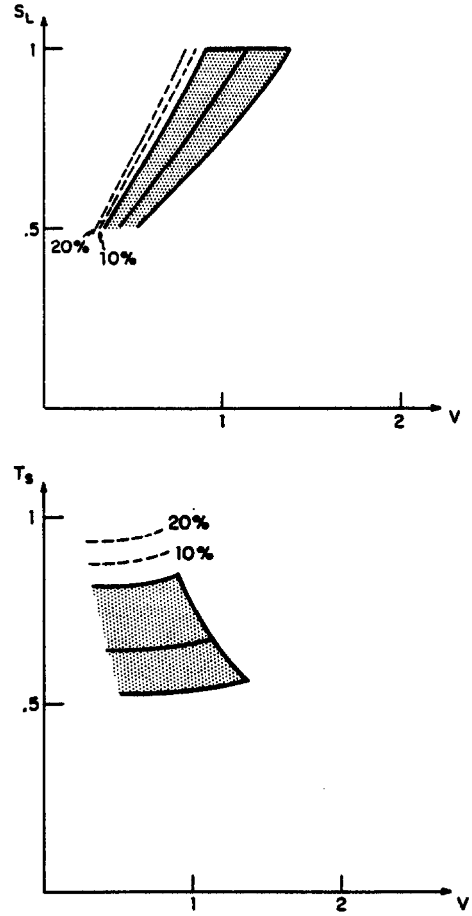

Fig. 6. The same boundaries given in Fig. 5 are given here in graphs of (a) step length S_L against velocity of walking V; and (b) time of swing T_s against velocity of walking V.

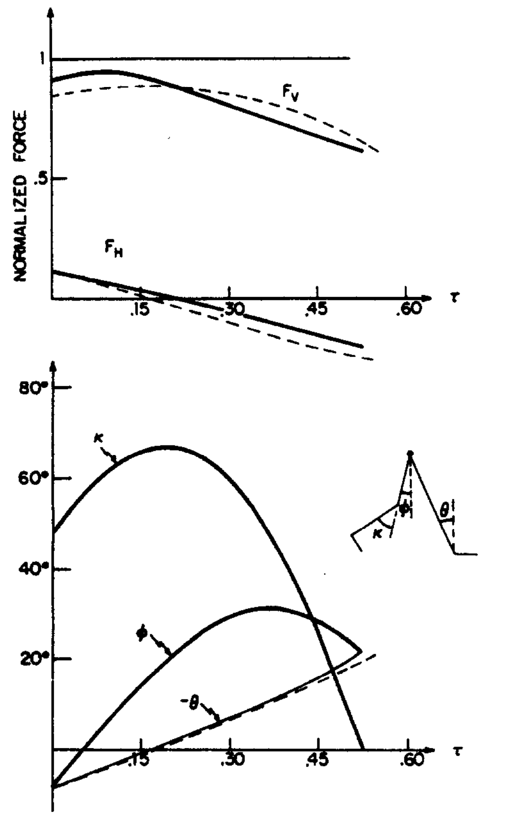

Fig. 8. The solid lines show calculated angles and ground reactions as a function of time during the swing phase for a limit case where the foot just clears the ground. The broken lines show F_v, F_H and θ for a simple inverted pendulum which starts with the same initial angle and velocity.

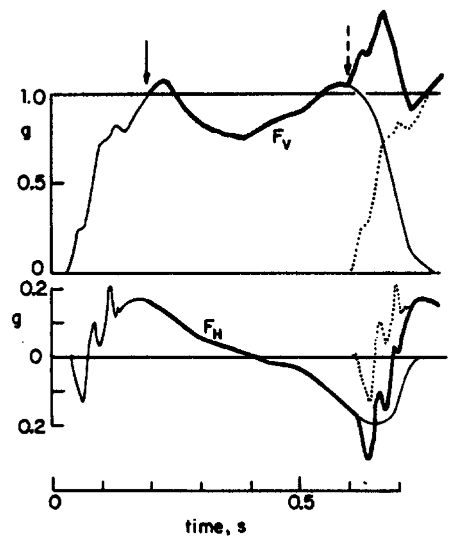

Fig. 7. Experimental traces of the vertical and forward forces on the ground in normal walking reproduced from Cavagna and Margaria (1966). The solid arrow shows the time of toeing-off; the broken arrow, the moment of heel strike. The dotted lines show the force contributed by the swing leg after it strikes the ground.

force, which looks like the experimental trace until mid-stance, after which the experimental trace diverges upward.

Figure 9 gives an example of a swing in the case where the knee locks before heel strike (25% of the total swing period occurs after knee lock in this figure). The first thing to note is that a dip followed by a rising phase appears in the vertical force calculated for the model, just as it does in the experimental traces shown in Fig. 7. This is due to the pulling down of the center of mass of the body produced by the centripetal acceleration of the swing leg at the middle of the swing. After knee lock, the vertical force begins falling again.

The effect of the impulsive moment at the knee is to stop the swing of the shank and lock it with the thigh. This, in turn, slows down and therefore reduces the backward movement of the thigh because if no moment had been applied to the knee, the thigh would have continued to swing, as shown by the broken line for ϕ in Fig. 9. The locking of the knee and consequent reduction of negative θ acts to keep the thigh flexed for heel strike.

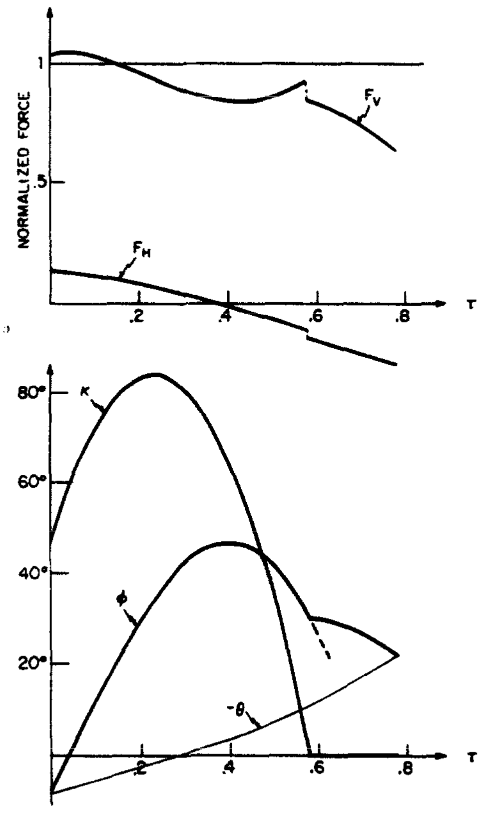

Fig. 9. Angles and ground reactions during the swing phase obtained for a case in which knee lock occurred when 25% of the time of swing remained before heel strike.

DISCUSSION

Limitations of the analysis

The model described in this paper does not take into account some of the determinants of normal walking discussed in the introduction. General pelvic movement, as well as knee and ankle interactions of the support leg, are absent in the model. It is important to note that the absent determinants, when taken into account, will change many details of the geometry of Fig. 1, and hence change some of the equations of constraint deduced from geometric considerations. The introduction of stance leg knee flexion, for example, will introduce new terms into equations (6), (7) and (8). Most of the basic assumptions of our model, however, such as the constancy of step length and the prohibition of the toe from striking the ground, are characteristics of walking which will not change when the absent determinants are introduced. Another strong example here is our assumption of a double support displacement S_{DS} which was taken to be independent of walking speed and step length – this assumption is backed by experimental evidence, and would not be changed by making the model more complex.

The constraint of positive initial angular velocity of the swing leg in stiff-legged walking, although reasonable, is otherwise arbitrary. We could not find physiological evidence for or against its realism.

We have taken the ratio of foot length to leg length to be 0.25. (The appropriate measure of d for our model is the length from the toe to the ankle joint and not the whole length of the foot.) Although this is approximately correct, the value 0.25 is by no means precise, and a different value will shift the furthest-left curve of Fig. 5. A lower value, for example 0.2, will move this curve to the left, to a time of swing near 0.4. The time of swing will also change if movement of the ankle of the swing leg is taken into account, instead of locking it at 90° as was done in this paper.

The most important assumption of all, that the muscles do not act during the ballistic swing, is only justified in "normal" walking, not in walking at very low or high speeds. Alexander (1977) discusses how, in walking at normal speeds, the potential energy of the body is lowest and the kinetic energy is highest during the double support phase, and how fluctuations in the two act directly opposite each other during the step, keeping the total energy about constant. The situation for walking at normal speeds is thus energetically somewhat similar to that of a simple inverted pendulum which starts with sufficient initial velocity (at heel strike) to carry it "over the top" of its arcing motion. As Alexander notes, in fast walking, the fluctuations in potential energy grow smaller and the fluctuations in kinetic energy grow larger. This means that the total energy of the center of mass goes through significant fluctuations during a step, but these fluctuations can only occur if the muscles add or subtract energy, violating the conditions of the ballistic model.

The analog between normal walking and the simple inverted pendulum has been proposed so often that we decided to compare the inverted pendulum to our ballistic walking model including flexion at the knee of the swing leg. In Fig. 8, the broken lines show the vertical force F_V, the horizontal force F_H and the angle with the horizontal θ made by an inverted pendulum starting from the same θ and $\dot{\theta}$ taken for this particular run of the ballistic walking model. The entire mass of the body, including the legs, is assumed concentrated at the top of the inverted pendulum. As might have been expected, the horizontal and vertical forces and the angle θ all execute trajectories which are quite close to those of the ballistic walking model: the only notable difference is that the swing is prolonged for the pendulum, to a dimensionless time of $\tau = 0.55$, as opposed to 0.53 for the walking model.

At this point, the reader might be wondering why, if the inverted pendulum gives force trajectories so similar to those of the ballistic walking model, we have bothered to construct a ballistic walking model at all. The answer is that the inverted pendulum does not have a time of swing built in. Changing the initial angular velocity of the inverted pendulum changes the time of swing without restrictions. We believe that the ballistic walking model, and the coupling it provides between an inverted pendulum representing the stance leg and a compound pendulum representing the swing leg, is the simplest representation of walking which still possesses a natural period while getting the ground reaction forces about right. Neither the inverted pendulum nor the compound pendulum alone can accomplish both of these objectives.

CONCLUSIONS

As discussed earlier, Fig. 5 gives the possible range of times of swing for a given step length. In this figure, T_s is the dimensionless swing time, i.e. the actual swing time scaled by the parameter $T_n = \pi\,(I_l/M_l g\bar{Z})^{1/2}$. Using the values of these constants presented in the Appendix, we can write this parameter in the form: $T_n = 2.58\,(l/g)^{1/2}$; or, taking the ratio of length leg l to stature S to be $\frac{1}{2}$ (from Williams and Lissner, 1974), we can write $T_n = 1.82\,(S/g)^{1/2}$. This means that the range of times of swing for each individual depends on his height; a small person will have relatively short times of swing. To compare our results with experiment, then, the range of times of swing should be given as a function of stature. Grieve and Gear (1966) made precisely these observations; their results are reproduced in Fig. 10. In this figure each vertical line represents the range of possible times of swing for a subject. The broken line gives the half-period of the whole leg regarded as a passive compound pendulum, calculated by Grieve and Gear. Their expression for this half-period is 1.81 $(S/g)^{1/2}$, which is essentially identical to the one obtained above. We have added to this figure two ruled lines. These represent the same limits given in Fig. 5. The lower line corresponds to the limit where the toe just clears the ground and the upper line to our 125° maximum knee flexion line. (The shaded area then corresponds to the shaded area of Fig. 5.) We can see from Fig. 10 that the range of times of swing predicted in this paper encompasses most of the times of swing found experimentally. The discrepancy for small statures suggests that young children have a different mechanism of walking, as was discussed in the introduction.

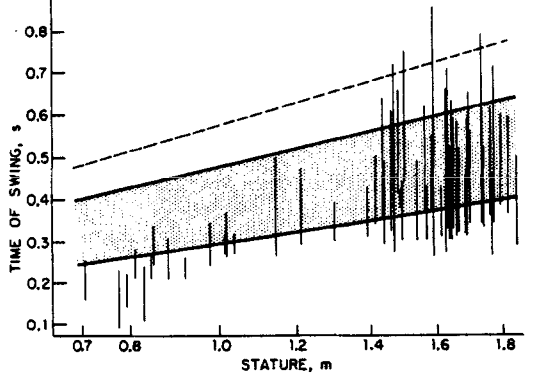

Fig. 10. The range of times of swing observed in each subject is represented by a vertical line plotted against stature, reproduced from Grieve and Gear (1966). Superimposed are the range of times predicted by the "strict" ballistic walking model including swing knee flexion described in this paper, represented by the shaded area.

As a general conclusion, we may observe that for each step length, the limiting factor in going to higher speeds is the condition that the swing leg must clear the ground. If muscles are allowed to act during the swing, as we expect them to at the higher walking speeds, this conclusion must be modified. Nevertheless, even when muscles act, the inertial dynamics of the leg will continue to play a role, so that the parameter d/l (the ratio of foot length to leg length, whose influence on the ballistic model was found to be considerable) could be important in determining the maximum speed of walking.

In summary, it is reasonable to claim that many aspects of walking at normal speed, from a prediction of the foot forces to an understanding of the relationship between walking cadence and body stature, are well represented by a model which completely disregards the action of muscles, except for setting the initial positions and velocities of the limbs at the beginning of the swing phase. We offer this model not in the hope of saying the last word on walking, which is, after all, a very complicated and subtle matter, but only in the hope of saying something simple about it.

Acknowledgements – This work was supported in part by grant number AM19638-01 from the National Institute of Arthritis, Metabolic and Digestive Diseases and by Rehab Engr. Ctr. Grant 23-P-55854 from the Rehabilitation Services Administration and by the Division of Applied Sciences, Harvard University. Grateful acknowledgment is also made of helpful criticism from Professor R. E. Kronauer and Dr. Sheldon Simon.

REFERENCES

Alexander, R. McN. and Goldspink, G. (1977) *Mechanics and Energetics of Animal Locomotion*. Halsted Press, New York (Division of Wiley, New York).

Basmajian, J. V. (1976) The human bicycle. *Biomechanics*, Vol. 5-A. (Edited by Komi, P. V.). University Park Press, Baltimore.

Beckett, R. and Chang, K. (1968) An evaluation of the kinematics of gait by minimum energy. *J. Biomechanics* **1**, 147–159.

Cavagna, G. A. and Margaria, R. (1966) Mechanics of walking. *J. appl. Physiol.* **21**, 271–278.

Cavagna, G. A., Thys., H. and Zamboni, A. (1976) The sources of external work in level walking and running. *J. Physiol.* **262**, 639–657.

Chow, C. K. and Jacobson, D. H. (1971) Studies of human locomotion via optimal programming. *Math. Biosciences* **10**, 239.

Cotes, J. E. and Meade, F. (1960) The energy expenditure and mechanical energy demand in walking. *Egonomics* **3**, 97–119.

Elftman, H. (1966) Biomechanics of muscle. *J. Bone Jt Surg.* (A) **48**, 363–377.

Frank, A. A. (1970) An approach to the dynamic analysis and synthesis of biped locomotion machines. *Med. biol. Engng* **8**, 465–476.

Grieve, D. W. and Gear, R. J. (1966) The relationship between the length of stride, step frequency, time of swing and speed of walking for children and adults. *Ergonomics* **9**, 379–399.

Grieve, D. W. (1968) Gait patterns and the speed of walking. *Bio-med. Engng* **3**, 119–122.

Gubina, F., Hemami, H. and McGhee, R. B. (1974) On the dynamic stability of biped locomotion. *IEEE Trans. biomed. Engng* **BME-21**, 102–108.

Inman, V. T. (1966) Human locomotion. *Can. med. Ass. J.* **94**, 1047–1054.

Saunders, J. B., Inman, V. T. and Eberhart, H. D. (1953) The major determinants in normal and pathological gait. *J. Bone Jt Surg.* (A) **35**, 543–558.

Weber, W. and Weber, E. (1836) *Mechanik der Menschlichen Gehwerkzauge* (Mechanics of human locomotion). Göttinger, Gottingen.

Williams, M. and Lissner, H. R. (1974) *Biomechanics of Human Motion.* Saunders, Philadelphia.

APPENDIX

List of Symbols

M_T, M_l, M_1, M_2	mass of body, leg, thigh and shank (+foot)
l, l_1, l_2	length of the leg, thigh and shank
$\bar{Z}, \bar{Z}_1, \bar{Z}_2$	distance of the center of mass of the leg, thigh and shank (+foot) to proximal end
η_1, η_2	radius of gyration of the thigh and shank (+foot) around proximal end
I_l, I_1, I_2	moments of inertia of the leg, thigh and shank (+foot) around proximal end
g	gravitational constant
S_L	step length
τ	t/T_n = normalized time
T_s	normalized time of swing
$T_n =$	$\pi(I_l/M_l g\bar{Z})^{1/2}$
V	velocity of walking = forward distance travelled $\|T_s$
θ, ϕ, σ	angle that the leg, thigh and shank make with the vertical (see Fig. 1)
$\dot{\theta}, \dot{\phi}, \dot{\sigma}$	velocities of the leg, thigh and shank
κ	knee angle = $\phi - \sigma$

(a zero and f subscript mean the initial and final state of the variable).

Dempster's data
(from Williams and Lissner, 1974)

$M_1|M_T = 0.097$ $\quad$ $M_2|M_T = 0.06$

$\bar{Z}_1|l_1 = 0.433$ $\quad$ $\bar{Z}_2|l_2 = 0.437$

$\eta_1 = 0.54$ $\quad$ $\eta_2 = 0.735$

$l_1 = l_2$

Any other constant appearing in the paper can be evaluated from this data.

Constants Appearing in Equations

$$J = I_l + M_T l^2 - 2M_l l\bar{Z}$$
$$K = I_l$$
$$C = M_l l\bar{Z}$$
$$U = (M_T l - M_l\bar{Z})g$$
$$W = M_l\bar{Z}g$$
$$K_1 = I_l + M_T l^2 - 2M_l l\bar{Z}$$
$$K_2 = I_1 + M_2 l_1^2$$
$$K_3 = I_2$$
$$C_1 = M_2 l_1 \bar{Z}_2$$
$$C_2 = (M_2 l_1 + M_1\bar{Z}_1)l$$
$$C_3 = M_2 l\bar{Z}_2$$
$$W_1 = (M_T l - M_1\bar{Z})g$$
$$W_2 = (M_2 l_1 + M_1\bar{Z}_1)g$$
$$W_3 = M_2\bar{Z}_2 g.$$