# Pair of straight line

1. Slope - point form

$$\boxed{y - y_1 = m(x - x_1)}$$

2. Two - point form

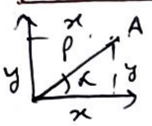
$$\boxed{\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}}$$

3. Slope - intercept form

$$\boxed{y = mx + c}$$

4. Normal form

$$\boxed{x \cos\alpha + y \sin\alpha = p}$$



5. Double - intercept form

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

**Note:**
1. $h^2 - ab > 0 \Rightarrow$ lines are distinct or represent pair of lines
2. $h^2 - ab = 0 \Rightarrow$ for coincident
3. $h^2 - ab < 0 \Rightarrow$ X represent pairs of Imaginary lines.

| | |
|---|---|
| $m_1 + m_2 = \frac{-2h}{b}$ | $m_1 \cdot m_2 = \frac{a}{b}$ |
| $m_1 = \frac{-h - \sqrt{h^2 - ab}}{b}$ | $m_2 = \frac{-h + \sqrt{h^2 - ab}}{b}$ |

auxillary eq$^n$ : $bm^2 + 2hm + a = 0$

# Combined eq$^n$ of lines $u = 0$ & $v = 0$ is $u \cdot v = 0$

# Trick :- Eq$^n$ of req$^{rd}$ line passing through origin & perpendicular to lines represented by $ax^2 + 2hxy + by^2 = 0$ is

$$bx^2 \mp 2hxy + ay^2 = 0$$

## Slopes

1. $(x_1, y_1)$ —— $(x_2, y_2)$

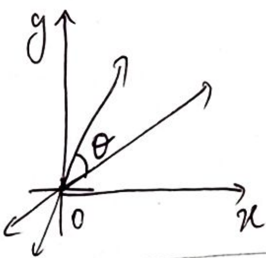
$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

2. $ax + by + c = 0$

$$\boxed{m = -\frac{a}{b}}$$

3. $m = \tan\theta$

# Angle between lines $ax^2 + 2hxy + by^2 = 0$



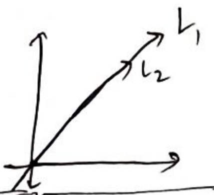
$$\boxed{\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|}$$

**Note**

X-intercept $= \frac{-c}{a}$

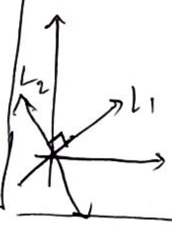
Y-intercept $= \frac{-c}{b}$

Note: for coincident cond$^n$:



$$\boxed{h^2 = ab}$$

for Perpendicular



$$\boxed{a + b = 0}$$

Homogenous eq$^n$ of degree two in $x$ & $y$ has form

$$\boxed{ax^2 + 2hxy + by^2 = 0}$$

$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$

$h^2 - ab \geq 0$ for real & coincident.

separate eq$^n$ → factors

→ $\frac{ax^2}{x^2} + \frac{2hxy}{x^2} + \frac{by^2}{x^2} = \frac{0}{x^2}$ → $b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$

$$\boxed{\frac{y}{x} = \frac{-2h \pm \sqrt{4h^2 - 4ba}}{2b}}$$

## General second Degree Equation in $x$ & $y$.

equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

- The necessary conditions for a general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of lines are

i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ → $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

ii) $h^2 - ab \geq 0$

Remark :- ① If eq$^n$ $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then these lines are parallel to the line represented by $ax^2 + 2hxy + by^2 = 0$

② Acute angle Btw them is

$$\boxed{\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|}$$

③ $a + b = 0$ lines are $\perp^{rd}$

④ $h^2 - ab = 0$ lines are $\parallel$

⑤ $h^2 - ab \geq 0$ distinct lines

⑥ $h^2 - ab < 0$ lines does not exist.

⑦ Condition for lines to intersect each other is $h^2 - ab > 0$

Co-ordinates of their point of intersection are

$$\left[\left(\frac{hf - bg}{ab - h^2}\right), \frac{gh - af}{ab - h^2}\right]$$

Joint eqⁿ of angle Bisector of $ax^2 + 2hxy + by^2 = 0$ is

$$\boxed{hx^2 - hy^2 - (a-b)xy = 0}$$ or $$\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$

# Pair of straight line

**Theorem**: Homogeneous eq$^n$ of degree two in $x$ & $y$ $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through origin if $h^2 - ab \geq 0$

**proof**:- Consider the homogeneous eq$^n$ of degree two in $x$ & $y$, $ax^2 + 2hxy + by^2 = 0$ . . . (i)

Consider two cases $b = 0$ & $b \neq 0$. These two cases are exhaustive

**Case I**: If $b = 0$ then eq$^n$ (1) becomes $ax^2 + 2hxy = 0$

$\therefore$ $x(ax + 2hy) = 0$, which is combined eq$^n$ of lines

$x = 0$ & $ax + 2hy = 0$

we observe that these lines pass through the origin.

**Case 2**: If $b \neq 0$ then we multiply eq$^n$ (1) by $b$

$\therefore$ $abx^2 + 2bhxy + b^2y^2 = 0$

$2bhxy + b^2y^2 = -abx^2$

To make L.H.S. complete square we add $h^2x^2$ to both sides.

$h^2x^2 + 2bhxy + b^2y^2 = -abx^2 + h^2x^2$

$(hx + by)^2 = x^2(h^2 - ab)$

$(hx + by)^2 = \left(\sqrt{h^2 - ab}\right)^2 x^2$ , as $h^2 - ab \geq 0$

$(by + hx)^2 - \left(\sqrt{h^2 - ab}\right)^2 x^2 = 0$

$(by + hx)^2 - \left(\sqrt{h^2 - ab} \cdot x\right)^2 = 0$

$\left(by + hx + \sqrt{h^2 - ab} \cdot x\right)\left(by + hx - \sqrt{h^2 - ab} \cdot x\right) = 0$

$\left[by + x\left(h + \sqrt{h^2 - ab}\right)\right]\left[by + x\left(h - \sqrt{h^2 - ab}\right)\right] = 0$

which is the combined eq$^n$ of lines $\left(h + \sqrt{h^2 - ab}\right)x + by = 0$

& $\left(h - \sqrt{h^2 - ab}\right)x + by = 0$

## Short trick : Pair of straight line

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines, then the distance between them is given by

$$\boxed{2\sqrt{\frac{g^2 - ac}{a(a+b)}} \quad \text{or} \quad 2\sqrt{\frac{f^2 - bc}{b(a+b)}}}$$

e.g Find distance between the parallel straight lines $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$ is :

(a) $\frac{2}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{10}}$ (c) $\frac{3}{\sqrt{10}}$ (d) $\frac{4}{\sqrt{10}}$

$\rightarrow$ $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$

$\boxed{a=9}$ $\boxed{b=1}$ $\boxed{2g=18}$ $\boxed{2f=-6}$ $\boxed{c=8}$

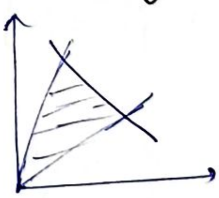
Distance between ∥ lines $= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$

$$= 2\sqrt{\frac{(9)^2 - 9(8)}{9(9+1)}} = 2\sqrt{\frac{81-72}{90}}$$

$$= 2\sqrt{\frac{9}{90}} = \frac{2}{\sqrt{10}}$$

### * Area of triangle (for equilateral only)

The area of triangle formed by $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$



$$\boxed{\text{Area} = \left|\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2mlh + bl^2}\right|}$$

Q) The area of the triangle formed by the lines $x^2 + 8xy - 9y^2 = 0$ and line $3x + 2y - 5 = 0$

$\rightarrow$ $\boxed{a=1}$ $2h = 8$ $\boxed{h=4}$ $\boxed{b=-9}$ | $\boxed{l=3}$ $\boxed{m=2}$ $\boxed{n=-5}$

$\rightarrow$ Area $= \left| \dfrac{n^2\sqrt{h^2-ab}}{am^2-2mlh+bl^2} \right|$

$= \left| \dfrac{25\sqrt{16-1\times-9}}{1(2)^2-2(2)(3)(4)+(-9)(3)^2} \right|$

$= \left| \dfrac{25\sqrt{16+9}}{4-48-81} \right| = \left| \dfrac{\cancel{25}\times\cancel{5}}{-\cancel{125}} \right|$

$A = 1$ sq unit

<u>Short cut trick</u> (3)

$\rightarrow$ The co-ordinates $(x,y)$ of the orthocenter of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is given by

$$\boxed{\frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2-2hlm+bl^2}}$$

$\Rightarrow$ Centroid of the triangle formed by lines $ax^2+2hxy+by^2=0$ & $lx+my=1$ is given by $\boxed{\dfrac{x}{bl-hm} = \dfrac{y}{am-hl} = \dfrac{2}{3(am^2-2hlm+bl^2)}}$

<u>Q)</u> The co-ordinates of the orthocentre of the triangle formed by $5x^2-2xy-3y^2=0$ and $x-2y=1$ is

$\rightarrow$ $\boxed{a=5}$  $2h=-2$ $\boxed{h=-1}$  $\boxed{b=-3}$  $\boxed{l=1}$  $\boxed{m=-2}$

$\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{a+b}{am^2-2hlm+bl^2}$

$\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{2}{5(-2)^2-2(-1)(1)(-2)+(-3)(1)^2}$

$x = \dfrac{y}{-2} = \dfrac{2}{20-4-3}$

$\boxed{x = \dfrac{2}{13}}$

$\dfrac{y}{-2} = \dfrac{2}{13}$

$\boxed{y = -\dfrac{4}{13}}$

$\left(\dfrac{2}{13}, \dfrac{-4}{13}\right)$

## Shortcut (4)

→ If the slopes of the lines $ax^2 + 2hxy + by^2 = 0$ differ by $p$, then $\boxed{4(h^2 - ab) = p^2 b^2}$

→ If the slope of one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $p$ times the slope of the other then

$$\boxed{4ph^2 = ab(1+p)^2}$$

Q) If the slope of the line $kx^2 - 4xy + 5y^2 = 0$ differ by 2, then k?

→ $\boxed{a = K}$ $2h = -4$ $\boxed{h = -2}$ $\boxed{b = 5}$, $\boxed{p = 2}$

$4((-2)^2 - k(5)) = (2)^2(5)^2$

$4(4 - 5k) = 100$

$16 - 20k = 100$

$16 - 100 = 20k$

$-\frac{84}{20} = k$

$k = -\frac{21}{5}$

Shortcut

⇒ Isosceles triangle

The lines $ax^2 + 2hxy + by^2 = 0$ & $lx + my + n = 0$ form an isosceles triangle, then

$$\boxed{\frac{l^2 - m^2}{lm} = \frac{a-b}{h}}$$

Eg. If the lines $2x^2 + 2kxy - 3y^2 = 0$ and $3x + 2y = 5$ form an isosceles triangle then $k = ?$

a) 4  b) 5  c) 6  d) 7

$$\frac{l^2 - m^2}{lm} = \frac{a-b}{h} \Rightarrow \frac{9-4}{3\times 2} = \frac{2+3}{k} \Rightarrow \boxed{k=6}$$

## Shortcut

To find the point of intersection.

# If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of intersecting lines, then the point of intersection of lines is

$$\boxed{\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)}$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

<u>Important Note</u> :- gradient of one line - means slope of one line.