5 Vectors

Type of Problems	Exercise	Q. Nos.
Algebra of Vectors, Position vector of a point, Unit vectors	5.1	Q.1 to 6, 9, 10
	Miscellaneous Exercise 5	Q.II (1 to 5, 7, 9, 10, 11, 13, 14)
Co-ordinates of a point in space	5.1	Q.7, 8
Collinearity and Coplanarity of Vectors	5.1	Q.11 to 14
	Miscellaneous Exercise 5	Q. II (12)
Section Formula	5.2	Q.1 to 12
	Miscellaneous Exercise 5	Q.II (15 to 20)
Scalar product of two vectors	5.3	Q.1 to 9
	Miscellaneous Exercise 5	Q.II(6, 8, 21 to 25, 28, 36(i), 48)
Direction angles, Direction ratios and Direction cosines	5.3	Q.10 to 14
	5.4	Q.16, 17, 18
	Miscellaneous Exercise 5	Q.II (26, 27, 29, 30, 31)
Vector product of two vectors	5.4	Q.1 to 15
	Miscellaneous Exercise 5	Q.II (32, 33, 35, 36(ii, iii), 37, 38, 39)
Scalar triple product	5.5	Q.1 to 7
	Miscellaneous Exercise 5	Q.II (40, 41, 42, 44, 45, 46, 47)
Vector triple product	5.5	Q.8 to 10
	Miscellaneous Exercise 5	Q.II (43)
Identify whether the given expression is a vector or a scalar	Miscellaneous Exercise 5	Q.II (34)

Syllabus

- Vectors and their types
- Section formula
- Dot Product of Vectors
- Cross Product of Vectors
- Triple Product of Vectors

Let's Study

Scalar quantity:

A quantity which can be completely described by magnitude only is called a scalar quantity.

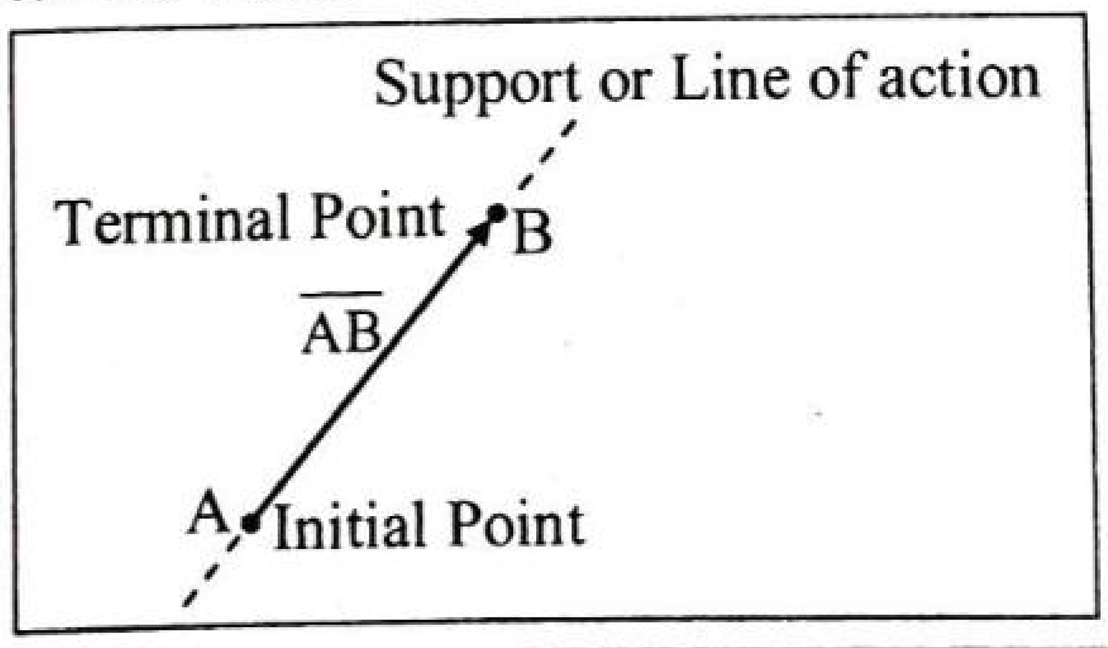
Example: mass, length, temperature, area, volume, time, distance, speed, work, money, voltage, density, resistance etc. In this book, scalars are given by real numbers.

Vector quantity:

A quantity which needs to be described using both magnitude and direction is called a vector quantity.
Example: displacement, velocity, force, electric field, acceleration, momentum etc.

Representation of Vector

Vector is represented by a directed line segment.
If AB is a segment and its direction is shown with an arrowhead as in figure, then the directed segment AB has magnitude as well as direction.
The segment AB with direction from A to B denotes the vector $\overline{AB}$ read as 'AB bar' while direction from B to A denotes the vector $\overline{BA}$.



In vector $\overline{AB}$, the point A is called the initial point and the point B is called the terminal point.
The directed line segment is a part of a line of unlimited length which is called the line of support or the line of action of the given vector.
If the initial and terminal points are not specified, then the vectors are denoted by $\bar{a}$, $\bar{b}$, $\bar{c}$ etc.

Magnitude of a Vector:
The magnitude (or size or length) of $\overline{AB}$ is denoted by $|\overline{AB}|$ and is defined as the length of segment AB. i.e.,

$|\overline{AB}| = l(AB)$

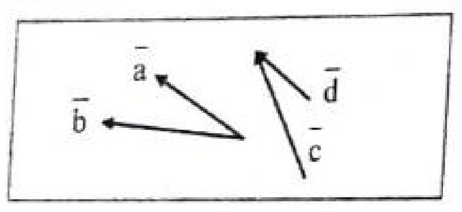
Magnitudes of vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ are $|\bar{a}|$, $|\bar{b}|$, $|\bar{c}|$ respectively.
The magnitude of a vector does not depend on its direction. Since the length is never negative, $|\bar{a}| \geq 0$.

Types of Vectors:

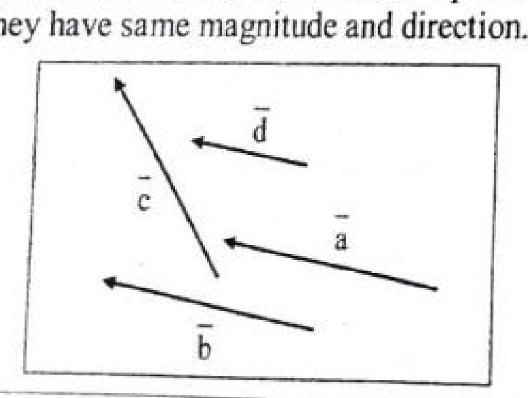
i. **Zero Vector:**
A vector whose initial and terminal points coincide, is called a zero vector (or null vector) and denoted as $\bar{0}$. Zero vector cannot be assigned a definite direction and it has zero magnitude or it may be regarded as having any suitable direction. The vectors $\overline{AA}$, $\overline{BB}$ represent the zero vector and $|\overline{AA}| = 0$

ii. **Unit Vector:**
A vector whose magnitude is unity (i.e., 1) is called a unit vector. The unit vector in the direction of a given vector $\bar{a}$ is denoted by $\hat{a}$, read as 'a-cap'.

iii. **Co-initial and Co-terminous Vectors:**
Vectors having same initial point are called co-initial vectors, whereas vectors having same terminal point are called co-terminous vectors.
Here, $\bar{a}$ and $\bar{b}$ are co-initial vectors. $\bar{c}$ and $\bar{d}$ are co-terminous vectors.

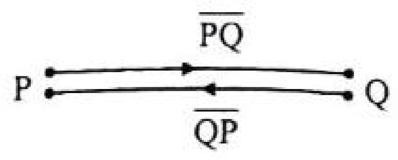


iv. **Equal Vectors:**
Two or more vectors are said to be equal vectors if they have same magnitude and direction.



- As $|\bar{a}| = |\bar{b}|$, and their directions are same regardless of initial point, we write as $\bar{a} = \bar{b}$.
- Here $|\bar{a}| = |\bar{c}|$, but directions are not same, so $\bar{a} \neq \bar{c}$.
- Here directions of $\bar{a}$ and $\bar{d}$ are same, but $|\bar{a}| \neq |\bar{d}|$, so $\bar{a} \neq \bar{d}$.

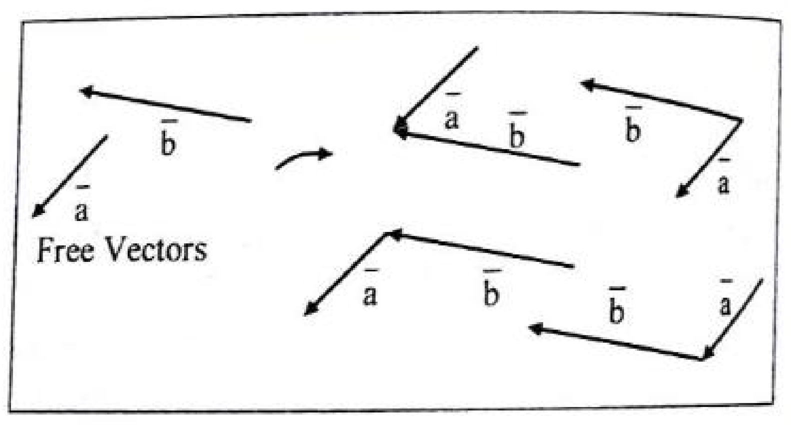
v. **Negative of a Vector:**
If $\bar{a}$ is a given vector, then the negative of $\bar{a}$ is a vector whose magnitude is same as that of $\bar{a}$ but whose direction is opposite to that of $\bar{a}$. It is denoted by $-\bar{a}$.



Thus, if $\overline{PQ} = \bar{a}$, then $\overline{QP} = -\bar{a} = -\overline{PQ}$.
Here, $|\overline{PQ}| = |-\overline{QP}|$.

vi. **Collinear Vectors:**
Vectors are said to be collinear vectors if they are parallel to same line or they are along the same line.

vii. **Free Vectors:**
If a vector can be translated anywhere in the space without changing its magnitude and direction then such a vector is called free vector. In other words, the initial point of free vector can be taken anywhere in the space keeping magnitude and direction same.



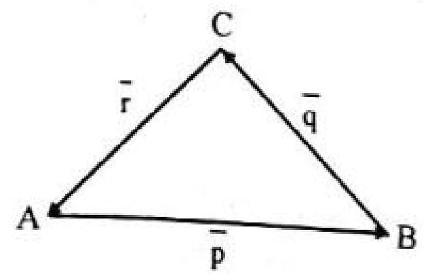
viii. **Localised Vectors:**
For a vector of given magnitude and direction, if it's initial point is fixed in space, then such a vector is called localised vector.

Example:
Force acting on a body is a localised vector.

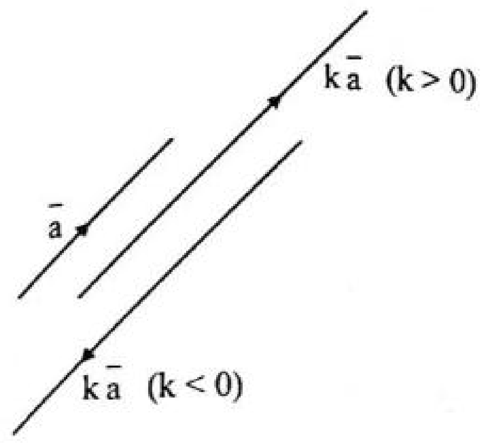
Textual Activity

Write the following vectors in terms of vectors $\bar{p}$, $\bar{q}$ and $\bar{r}$. (Textbook page no. 135)

i. $\overline{AB} = \boxed{\bar{p}}$
ii. $\overline{BA} = \boxed{-\bar{p}}$
iii. $\overline{BC} = \boxed{\bar{q}}$
iv. $\overline{CB} = \boxed{-\bar{q}}$
v. $\overline{CA} = \boxed{\bar{r}}$
vi. $\overline{AC} = \boxed{-\bar{r}}$



Algebra of Vectors:
Scalar Multiplication:
$2\bar{a}$ has the same direction as $\bar{a}$ but is twice long as $\bar{a}$.



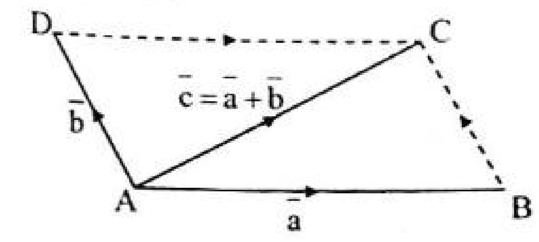
Let $\bar{a}$ be any vector and k be a scalar, then vector $k\bar{a}$, the scalar multiple vector of $\bar{a}$ is defined a vector whose magnitude is $|k\bar{a}| = |k||\bar{a}|$ and vectors $\bar{a}$ and $k\bar{a}$ have the same direction if $k > 0$ and opposite direction if $k < 0$.

Note:

i. If $k = 0$, then $k\bar{a} = \bar{0}$.
ii. $\bar{a}$ and $k\bar{a}$ are collinear or parallel vectors.
iii. Two non zero vectors $\bar{a}$ and $\bar{b}$ are collinear or parallel if $\bar{a} = m\bar{b}$, where $m \neq 0$.
iv. Let $\hat{a}$ be the unit vector along non-zero vector $\bar{a}$ then $\bar{a} = |\bar{a}|\hat{a}$ or $\dfrac{\bar{a}}{|\bar{a}|} = \hat{a}$.
v. A vector of length k in the same direction as $\bar{a}$ is $k\hat{a} = k\left(\dfrac{\bar{a}}{|\bar{a}|}\right)$.

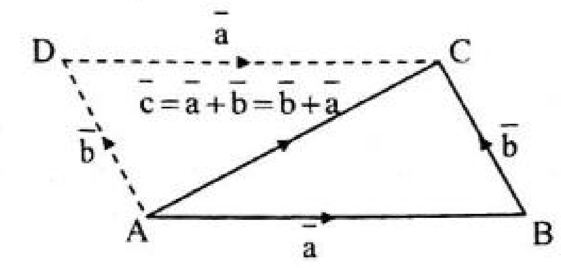
Addition of Two Vectors:
If $\bar{a}$ and $\bar{b}$ are any two vectors then their addition (or resultant) is denoted by $\bar{a} + \bar{b}$.
There are two laws of addition of two vectors.

Parallelogram Law:



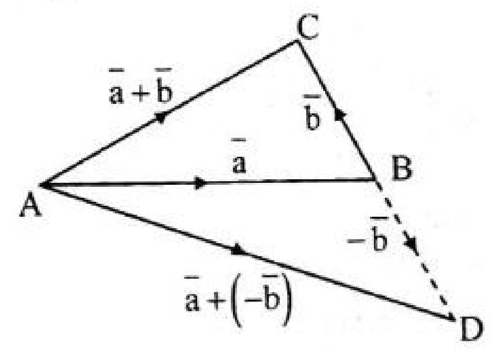
Let $\bar{a}$ and $\bar{b}$ be two vectors. Consider $\overline{AB}$ and $\overline{AD}$ along two adjacent sides of a parallelogram, such that $\overline{AB} = \bar{a}$ and $\overline{AD} = \bar{b}$ then $\bar{a} + \bar{b}$ lies along the diagonal of a parallelogram with $\bar{a}$ and $\bar{b}$ as sides.
$\overline{AC} = \overline{AB} + \overline{AD}$ i.e. $\bar{c} = \bar{a} + \bar{b}$

Triangle Law of addition of two vectors:
Let $\bar{a}$, $\bar{b}$ be any two vectors then consider triangle ABC as shown in figure such that $\overline{AB} = \bar{a}$ and $\overline{BC} = \bar{b}$ then $\bar{a} + \bar{b}$ is given by vector $\overline{AC}$ along the third side of triangle ABC.



Thus, $\bar{a} + \bar{b} = \overline{AB} + \overline{BC} = \overline{AC}$.
This is known as the triangle law of addition of two vectors $\bar{a}$ and $\bar{b}$. The triangle law can also be applied to the ΔADC.
Here, $\overline{AD} = \overline{BC} = \bar{b}$, $\overline{DC} = \overline{AB} = \bar{a}$
Hence, $\overline{AD} + \overline{DC} = \bar{b} + \bar{a} = \overline{AC}$
Thus, $\overline{AC} = \bar{b} + \bar{a} = \bar{a} + \bar{b}$

Subtraction of two vectors:
If $\bar{a}$ and $\bar{b}$ are two vectors, then $\bar{a} - \bar{b} = \bar{a} + (-\bar{b})$, where $-\bar{b}$ is the negative vector of vector $\bar{b}$. Let $\overline{AB} = \bar{a}$, $\overline{BC} = \bar{b}$, now construct a vector $\overline{BD}$ such that its magnitude is same as the vector $\overline{BC}$, but the direction is opposite to that of it.



i.e., $\overline{BD} = -\overline{BC}$
$\therefore$ $\overline{BD} = -\bar{b}$
Thus applying triangle law of addition.

We have

$\overline{AD} = \overline{AB} + \overline{BD}$

$= \overline{AB} - \overline{BC}$

$= \bar{a} - \bar{b}$

Note:

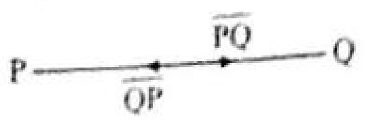
i. If (velocity) vectors $\bar{a}$ and $\bar{b}$ are acting simultaneously then we use parallelogram law of addition.

ii. If (velocity) vectors $\bar{a}$ and $\bar{b}$ are acting one after another then we use triangle law of addition.

iii. Adding vector to its opposite vector gives $\bar{0}$

As $\overline{PQ} + \overline{QP} = \overline{PP} = \bar{0}$ or

As $\overline{PQ} = -\overline{QP}$, then $\overline{PQ} + \overline{QP} = -\overline{QP} + \overline{QP} = \bar{0}$.



iv. In ΔABC, $\overline{AC} = -\overline{CA}$, so $\overline{AB} + \overline{BC} + \overline{CA} = \overline{AA} = \bar{0}$.
This means that when the vectors along the sides of a triangle are in order, their resultant is zero as initial and terminal points become same.

v. The addition law of vectors can be extended to a polygon :
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ be four vectors. Let $\overline{PQ} = \bar{a}$, $\overline{QR} = \bar{b}$, $\overline{RS} = \bar{c}$ and $\overline{ST} = \bar{d}$.

$\therefore$ $\bar{a} + \bar{b} + \bar{c} + \bar{d}$

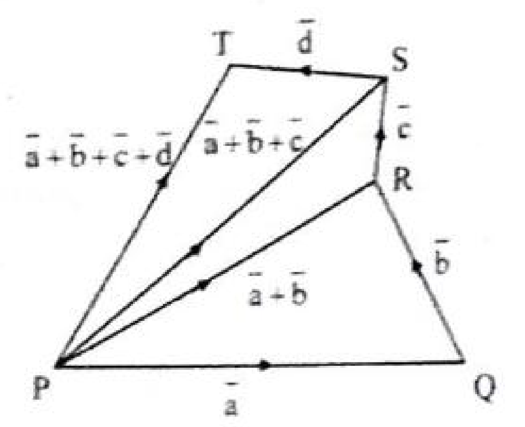
$= \overline{PQ} + \overline{QR} + \overline{RS} + \overline{ST}$

$= \left(\overline{PQ} + \overline{QR}\right) + \overline{RS} + \overline{ST}$

$= \left(\overline{PR} + \overline{RS}\right) + \overline{ST}$

$= \overline{PS} + \overline{ST}$

$= \overline{PT}$



Thus, the vector $\overline{PT}$ represents sum of all vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$.
This is also called as extended law of addition of vectors or polygonal law of addition of vectors.

vi. $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ (commutative)

vii. $\left(\bar{a} + \bar{b}\right) + \bar{c} = \bar{a} + \left(\bar{b} + \bar{c}\right)$ (associative)

viii. $\bar{a} + \bar{0} = \bar{a}$ ($\bar{0}$ is additive identity)

ix. $\bar{a} + \left(-\bar{a}\right) = \bar{0}$ ($-\bar{a}$ is additive inverse)

x. If $\bar{a}$ and $\bar{b}$ are vectors and m and n are scalars, then

a. $(m + n)\bar{a} = m\bar{a} + n\bar{a}$ (distributive)

b. $m\left(\bar{a} + \bar{b}\right) = m\bar{a} + m\bar{b}$ (distributive)

c. $m\left(n\bar{a}\right) = (mn)\bar{a} = n\left(m\bar{a}\right)$

xi. $\left|\bar{a} + \bar{b}\right| \le \left|\bar{a}\right| + \left|\bar{b}\right|$, this is known as "Triangle Inequality".
This is obtained from the triangle law, as the length of any side of triangle is less than the sum of the other two sides. i.e. in triangle ABC, AC < AB + BC,
where, AC = $\left|\bar{a} + \bar{b}\right|$, AB = $\left|\bar{a}\right|$, BC = $\left|\bar{b}\right|$

xii. Any two vectors $\bar{a}$ and $\bar{b}$ determine a plane and vectors $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ lie in the same plane.

Textual Activity

In quadrilateral PQRS, find a resultant vector.

(Textbook page no. 137)

i. $\overline{QR} + \overline{RS} = \boxed{\overline{QS}}$

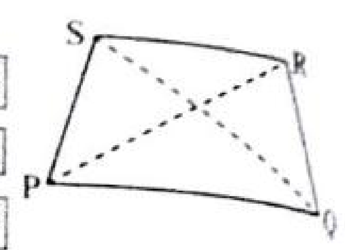
ii. $\overline{PQ} + \overline{QR} = \boxed{\overline{PR}}$

iii. $\overline{PS} + \overline{SR} + \overline{RQ} = \boxed{\overline{PQ}}$

iv. $\overline{PR} + \overline{RQ} + \overline{QS} = \boxed{\overline{PS}}$

v. $\overline{QR} - \overline{SR} - \overline{PS} = \boxed{\overline{QP}}$

vi. $\overline{QP} - \overline{RP} + \overline{RS} = \boxed{\overline{QS}}$



Theorem 1:

Two non-zero vectors $\bar{a}$ and $\bar{b}$ are collinear if and only if there exist scalars m and n, at least one of them is non-zero such that $m\bar{a} + n\bar{b} = \bar{0}$

Proof:

Only If - part:

Suppose $\bar{a}$ and $\bar{b}$ are collinear.

$\therefore$ There exists a scalar $t \neq 0$ such that $\bar{a} = t\bar{b}$

$\therefore$ $\bar{a} - t\bar{b} = 0$

i.e. $m\bar{a} + n\bar{b} = \bar{0}$, where m = 1 and n = – t.

If - part :

Conversely, suppose $m\bar{a}$ and $n\bar{b} = \bar{0}$ and $m \neq 0$.

$\therefore$ $m\bar{a} = -n\bar{b}$

$\therefore$ $\bar{a} = \left(-\dfrac{n}{m}\right)\bar{b}$, where $t = \left(-\dfrac{n}{m}\right)$ is a scalar.

i.e. $\bar{a} = t\bar{b}$.

$\therefore$ $\bar{a}$ is scalar multiple of $\bar{b}$.

$\therefore$ $\bar{a}$ and $\bar{b}$ are collinear.

Corollary 1 : If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $m\bar{a} + n\bar{b} = \bar{0}$, then m = 0, n = 0. (This can be proved by contradiction assuming $m \neq 0$ or $n \neq 0$)

Corollary 2 : If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $m\bar{a} + n\bar{b} = p\bar{a} + q\bar{b}$ then m = p, n = q.

For example, If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $3\bar{a} + y\bar{b} = x\bar{a} + 5\bar{b}$,
then 3 = x, y = 5.

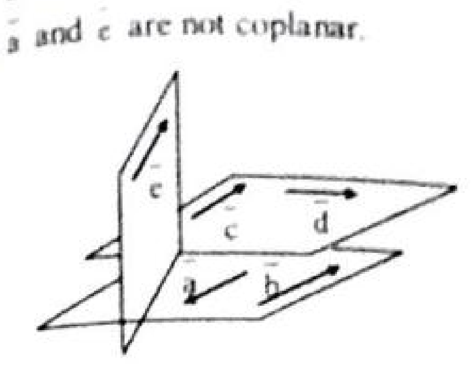
Coplanar Vectors:

Two or more vectors are coplanar, if they lie in the same plane or in parallel plane.

Vectors $\bar{a}$ and $\bar{b}$ are coplanar.

Vectors $\bar{a}$ and $\bar{c}$ are coplanar.

Vectors $\bar{a}$ and $\bar{e}$ are not coplanar.



Remark: Any two intersecting straight lines OA and OB in space determine a plane. We may choose for convenience the coordinate axes of the plane so that O is origin and axis OX is along one of OA or OB.

Theorem 2:

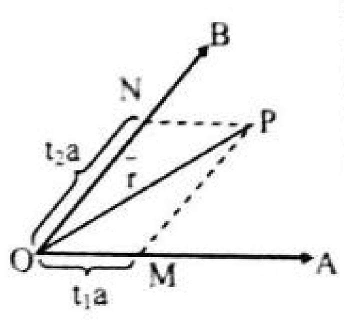
Let $\bar{a}$ and $\bar{b}$ be non-collinear vectors. A vector $\bar{r}$ is coplanar with $\bar{a}$ and $\bar{b}$ if and only if there exist unique scalars t_1 t_2, such that $\bar{r} = t_1\bar{a} + t_2\bar{b}$.

Proof:

Only If-part:

Suppose $\bar{r}$ is coplanar with $\bar{a}$ and $\bar{b}$. To show that their exist unique scalars t_1 and t_2 such that $\bar{r} = t_1\bar{a} + t_2\bar{b}$.

Let $\bar{a}$ be along OA and $\bar{b}$ be along OB. Given a vector $\bar{r}$, with initial point O, Let $\overline{OP} = \bar{r}$, draw lines parallel to OB, meeting OA in M and parallel to OA, meeting OB in N.



Then ON = $t_2\bar{b}$ and OM = $t_1\bar{a}$ for some $t_1, t_2 \in R$.
By triangle law or parallelogram law, we have $\bar{r} = t_1\bar{a} + t_2\bar{b}$

If Part:

Suppose $\bar{r} = t_1\bar{a} + t_2\bar{b}$, and we have to show that $\bar{r}$, $\bar{a}$ and $\bar{b}$ are co-planar.

As $\bar{a}$, $\bar{b}$ are coplanar, $t_1\bar{a}$, $t_2\bar{b}$ are also coplanar.

Therefore $t_1\bar{a} + t_2\bar{b}$, $\bar{a}$, $\bar{b}$ are coplanar.

Therefore $\bar{a}$, $\bar{b}$, $\bar{r}$ are coplanar.

Uniqueness:

Suppose vector $\bar{r} = t_1\bar{a} + t_2\bar{b}$ ...(1)

can also be written as $\bar{r} = s_1\bar{a} + s_2\bar{b}$ ...(2)

Subtracting (2) from (1) we get,

$\bar{0} = (t_1 - s_1)\bar{a} + (t_2 - s_2)\bar{b}$

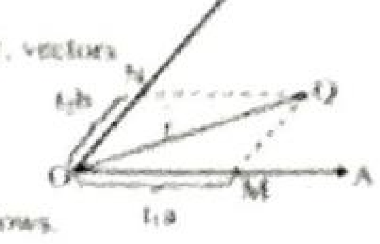
But, $\bar{a}$ and $\bar{b}$ are non-collinear, vectors

By Corollary 1 of Theorem 1,

$\therefore$ $t_1 - s_1 = 0 = t_2 - s_2$

$\therefore$ $t_1 = s_1$ and $t_2 = s_2$.

Therefore, the uniqueness follows.



Remark:

Linear combination of vectors: If $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots, \bar{a}_n$ are n vectors and $m_1, m_2, m_3, \ldots, m_n$ are n scalars, then the vector $m_1\bar{a}_1 + m_2\bar{a}_2 + m_3\bar{a}_3 + \ldots + m_n\bar{a}_n$ is called a linear combination of vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots, \bar{a}_n$. If atleast one m_i is not zero then the linear combination is non zero linear combination.

For example: Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be vectors and m, n are scalars then the vector $\bar{c} = m\bar{a} + n\bar{b}$ is called a linear combination of vector $\bar{a}$ and $\bar{b}$. Vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar vectors.

Theorem 3 :

Three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar, if and only if there exists a non-zero linear combination $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ with $(x, y, z) \neq (0, 0, 0)$.

Proof:

Only If - part:

Assume that $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar

Case - 1:

Suppose that any two of $\bar{a}$, $\bar{b}$ and $\bar{c}$ are collinear vectors, say $\bar{a}$ and $\bar{b}$.

$\therefore$ There exist scalars x, y at least one of which is non-zero such that $x\bar{a} + y\bar{b} = \bar{0}$

i.e. $x\bar{a} + y\bar{b} + 0\bar{c} = \bar{0}$ and (x, y, 0) is the required solution for $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

Case - 2:

No two vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are collinear.

As $\bar{c}$ is coplanar with $\bar{a}$ and $\bar{b}$,

$\therefore$ we have scalars x, y such that $\bar{c} = x\bar{a} + y\bar{b}$ (using Theorem 2).

$\therefore$ $x\bar{a} + y\bar{b} - \bar{c} = \bar{0}$ and (x, y, –1) is the required solution for $x\bar{a} + y\bar{b} - z\bar{c} = \bar{0}$.

If - part :

Conversely, suppose $x\bar{a} + y\bar{b} - z\bar{c} = \bar{0}$ where one of x, y, z is non-zero, say $z \neq 0$.

$\therefore$ $\bar{c} = \dfrac{-x}{z}\bar{a} - \dfrac{y}{z}\bar{b}$

$\therefore$ $\bar{c}$ is coplanar with $\bar{a}$ and $\bar{b}$.

$\therefore$ $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar vectors.

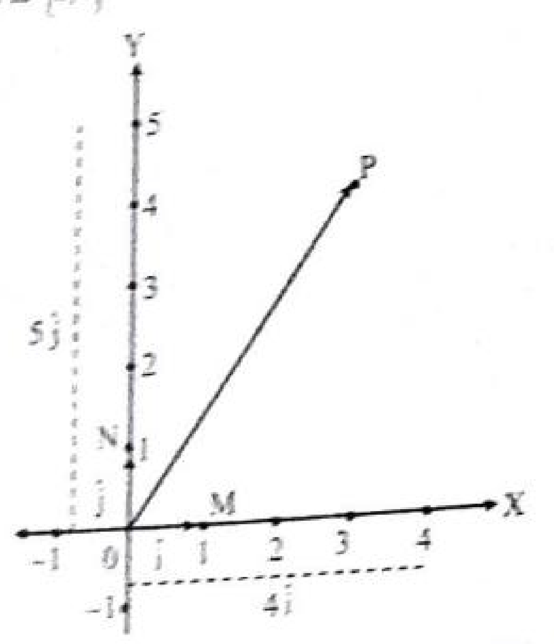
Corollary 1: If three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are not coplanar and $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$, then $x = 0$, $y = 0$ and $z = 0$ because if $(x, y, z) = (0, 0, 0)$ then $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar.

Corollary 2 : The vectors $\bar{a}$, $\bar{b}$ and $x\bar{a} + y\bar{b}$ coplanar for all values of x and y.

Vector in Two Dimensions (2-D):

The plane spanned (covered) by non collinear vectors $\bar{a}$ and $\bar{b}$ is $\{x\bar{a} + y\bar{b} \mid x, y \in R\}$, where $\bar{a}$ and $\bar{b}$ have same initial point.

This is 2-D space where generators are $\bar{a}$ and $\bar{b}$ or its basis is $\{\bar{a}, \bar{b}\}$.



For example, in XY plane, let M = (1, 0) and N = (0, 1) be two points along X and Y axis respectively.

Then, we define unit vectors $\hat{i}$ and $\hat{j}$ as $\overline{OM} = \hat{i}$, $\overline{ON} = \hat{j}$.

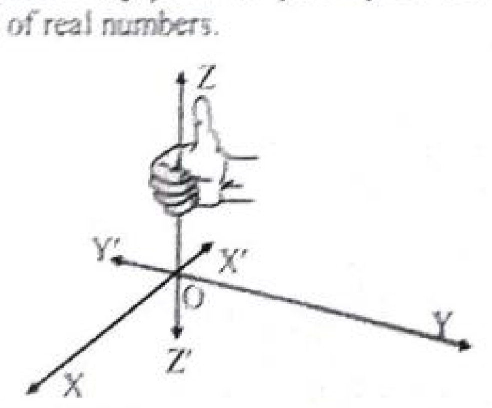
Given any other vector say $\overline{OP}$, where P = (4, 5) then $\overline{OP} = 4\hat{i} + 5\hat{j}$.

Three Dimensional (3-D) Coordinate System:

Any point in the plane is represented as an ordered pair (a, b) where a and b are distances (with suitable sign) of point (a, b) from Y-axis and X-axis respectively.

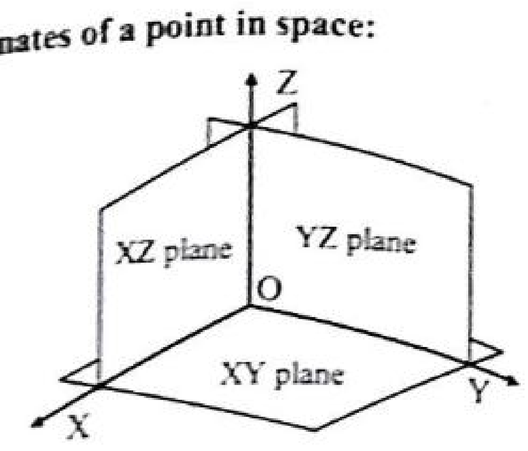
To locate a point in space, three numbers are required. Here, we need three coordinate axes OX, OY and OZ and to determine a point we need distances of it from three planes formed by these axes.

We represent any point in space by an ordered triple (a, b, c) of real numbers.



O is the origin and three directed lines through O that are perpendicular to each other are the coordinate axes. Label them as X-axis (XOX'), Y-axis (YOY') and Z-axis (ZOZ'). The direction of Z-axis is determined by right hand rule i.e. When you hold your right hand so that the fingers curl from the positive X-axis toward the positive Y-axis, your thumb points along the positive Z-axis, as shown in figure.

Co-ordinates of a point in space:



Let P be a point in the space. Draw perpendiculars PL, PM, PN through P to XY-plane, YZ-plane and XZ-plane respectively, where points L, M and N are feet of perpendicularls in XY, YZ and XZ planes respectively.

For point P(x, y, z), x, y and z are x-coordinate, y-coordinate and z-coordinate respectively.

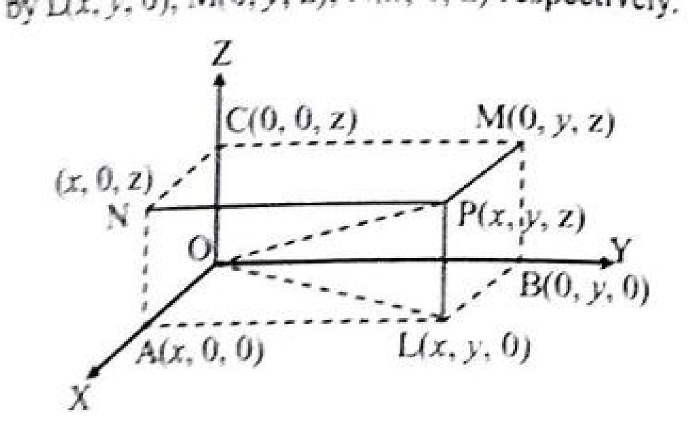
Point of intersection of all 3 planes is origin O(0, 0, 0).

Co-ordinates of points on co-ordinate axes:

Points on X-axis, Y-axis and Z-axis have coordinates given by A(x, 0, 0), B(0, y, 0) and C(0, 0, z).

Co-ordinates of points on co-ordinate planes:

Points in XY-plane, YZ-plane and ZX-plane are given by L(x, y, 0), M(0, y, z), N(x, 0, z) respectively.



Distance of P(x, y, z) from co-ordinate planes:

i. Distance of P from XY plane = |PL| = |z|.
ii. Distance of P from YZ plane = |PM| = |x|.
iii. Distance of P from XZ plane = |PN| = |y|.

Distance of any point from origin:

Distance of P (x, y, z) from the origin O(0, 0, 0) from figure we have,

$l(OP) = \sqrt{OL^2 + LP^2}$ (ΔOLP right angled triangle)

$= \sqrt{OA^2 + AL^2 + LP^2}$

$= \sqrt{OA^2 + OB^2 + OC^2}$

$= \sqrt{x^2 + y^2 + z^2}$

Distance between any two points in space

Distance between two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) in space is given by distance formula

$l(AB) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Distance of a point P(x, y, z) from coordinate axes

In given figure, PA is perpendicular to X-axis. Hence distance of P from X-axis is PA.

$\therefore$ $PA = \sqrt{(x - x)^2 + (y - 0)^2 + (z - 0)^2}$

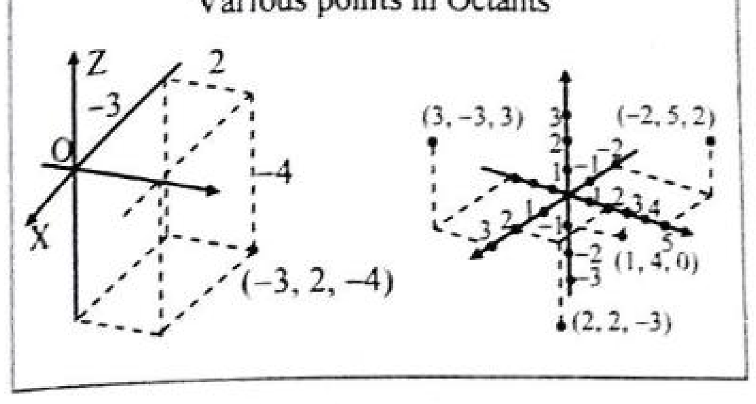
$= \sqrt{y^2 + z^2}$

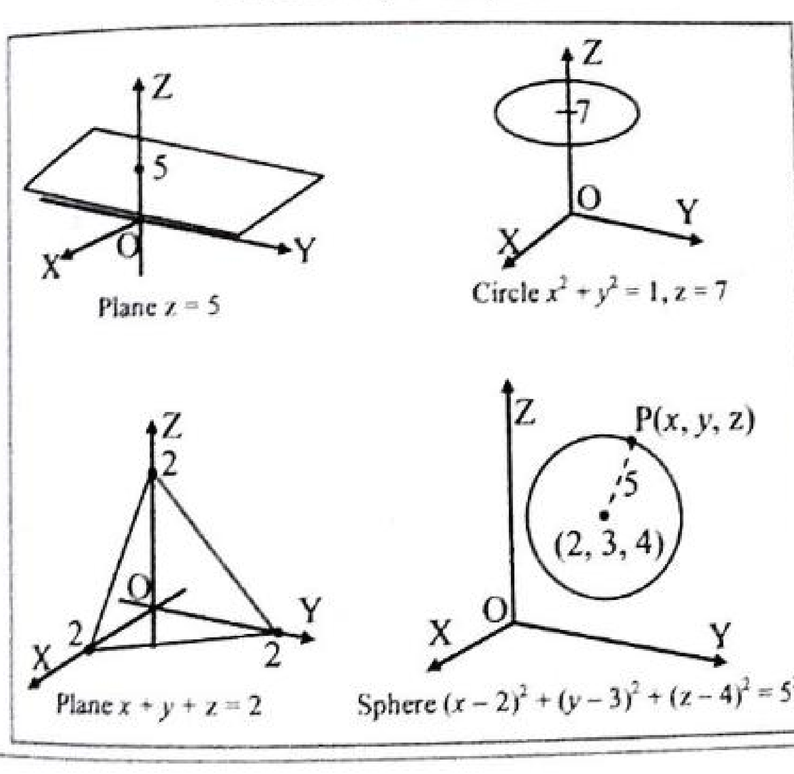
iv. In a right-handed system. Octants II, III and IV are found by rotating anti-clockwise around the positive Z-axis. Octant V is vertically below Octant I. Octants VI, VII and VIII are then found by rotating anti-clockwise around the negative Z-axis.

Signs of coordinates of a point P(x, y, z) in different octants:

Octant (x, y, z)	(I) O-XYZ (+, +, +)	(II) O-X'YZ (−, +, +)	(III) O-XY'Z (+, −, +)	(IV) O-X'Y'Z (−, −, +)
Octant (x, y, z)	(V) O-XYZ' (+, +, −)	(VI) O-X'YZ' (−, +, −)	(VII) O-XY'Z' (+, −, −)	(VIII) O-X'Y'Z' (−, −, −)

Point in Octants





Components of Vector:

In order to be more precise about the direction of a vector we can represent a vector as a linear combination of basis vectors.

Take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the X-axis, Y-axis and Z-axis, respectively.

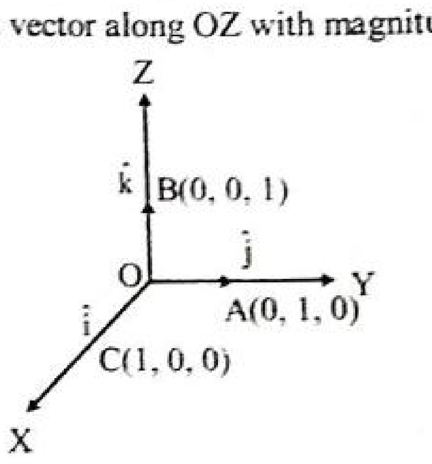
Then $|\overline{OA}| = |\overline{OB}| = |\overline{OC}|$

The vectors $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ each having magnitude 1 are called unit vectors along the axes X, Y, Z respectively. These vectors are denoted by $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively and also called as standard basis vectors or standard unit vectors.

Any vector, along X-axis is a scalar multiple of unit vector $\hat{i}$, along Y-axis is a scalar multiple of $\hat{j}$ and along Z-axis is a scalar multiple of $\hat{k}$. (Collinearity property).

Example:

i. $4\hat{i}$ is a vector along OX with magnitude 4.
ii. $6\hat{j}$ is a vector along OY with magnitude 6.
iii. $5\hat{k}$ is a vector along OZ with magnitude 5.



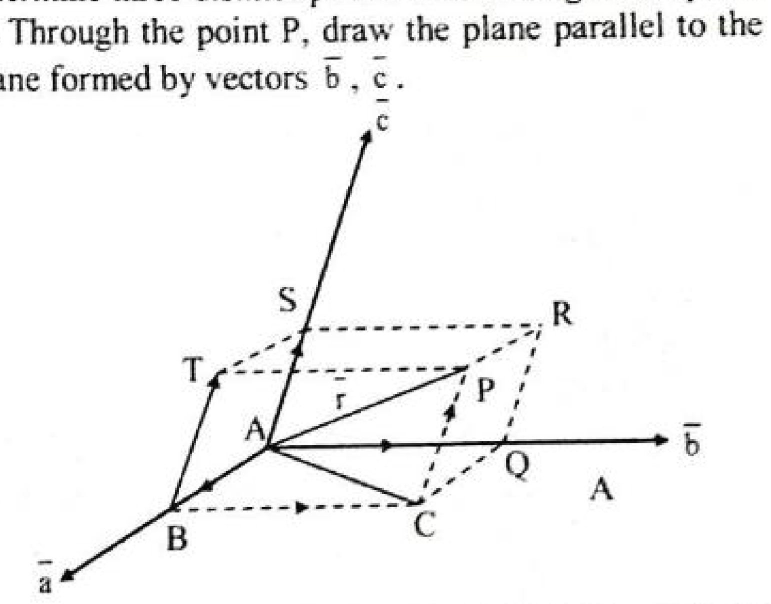
Theorem 4 :

If $\bar{a}$, $\bar{b}$, $\bar{c}$ are three non-coplanar vectors, then any vector $\bar{r}$ in the space can be uniquely expressed as a linear combination of $\bar{a}$, $\bar{b}$, $\bar{c}$.

Proof :

Let A be any point in the space, take the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{r}$, so that A becomes their initial point (See given figure).

Let $\overline{AP} = \bar{r}$. As $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors, they determine three distinct planes intersecting at the point A. Through the point P, draw the plane parallel to the plane formed by vectors $\bar{b}$, $\bar{c}$.



This plane intersects line containing $\bar{a}$ at point B. Similarly, draw the other planes and complete the parallelopiped.

Now, $\overline{AB}$ and $\bar{a}$ are collinear.

$\therefore$ $\overline{AB} = x\bar{a}$, where x is a scalar.

Similarly, we have $\overline{AQ} = y\bar{b}$ and $\overline{AS} = z\bar{c}$, where y and z are scalars.

Also, by triangle law of addition of vectors,

$\overline{AP} = \overline{AC} + \overline{CP}$ (In ΔACP)

$= \left(\overline{AB} + \overline{BC}\right) + \overline{CP}$ (In ΔABC)

$= \overline{AB} + \overline{BC} + \overline{CP}$

$\overline{AP} = \overline{AB} + \overline{BC} + \overline{CP}$

$\therefore$ $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$.

$(\because \overline{AQ} = \overline{BC}$ and $\overline{AS} = \overline{CP})$

Therefore, any vector $\bar{r}$ in the space can be expressed as a linear combination of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Uniqueness:

Suppose $\bar{r} = x_1\bar{a} + x_2\bar{b} + x_3\bar{c}$ and also $\bar{r} = y_1\bar{a} + y_2\bar{b} + y_3\bar{c}$ for some scalars x_1, x_2, x_3 and y_1, y_2, y_3.

We need to prove $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$.

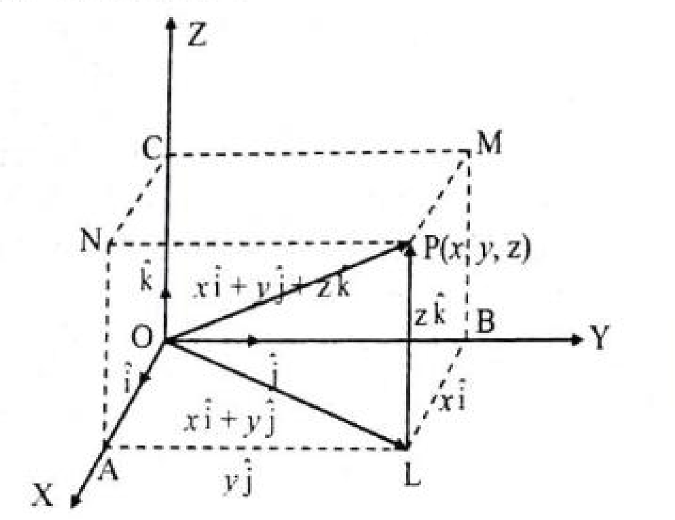
Subtracting one expression from the other we have $(x_1 - y_1)\bar{a} + (x_2 - y_2)\bar{b} + (x_3 - y_3)\bar{c} = 0$.

By Corollary 1 of Theorem 3

As $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar we must have $x_1 - y_1 = x_2 - y_2 = x_3 - y_3 = 0$ that is $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, as desired.

Position vector of a point P(x, y, z) in space:

Consider a point P in space, having coordinates (x, y, z) with respect to the origin O(0, 0, 0). Then the vector $\overline{OP}$ having O and P as its initial and terminal points, respectively is called the position vector of the point P with respect to O.



Let P (x, y, z) be a point in space.

$\therefore$ OA = PM = x, OB = PN = y, OC = PL = z.

i.e., A $\equiv$ (x, 0, 0), B $\equiv$ (0, y, 0) and C $\equiv$ (0, 0, z).

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vectors along positive directions of X-axis, Y-axis and Z-axis is respectively.

$\therefore$ $\overline{OA} = x\hat{i}$, $\overline{OB} = y\hat{j}$ and $\overline{OC} = z\hat{k}$

$\overline{OP}$ is the position vector of point P in space with respect to origin O.

Representation of $\overline{OP}$ in terms of unit vector $\hat{i}, \hat{j}, \hat{k}$.

In ΔOLP we have (see given figure)

$\overline{OP} = \overline{OL} + \overline{LP}$

$= \overline{OA} + \overline{AL} + \overline{LP}$ ...(from ΔOAL)

$= \overline{OA} + \overline{OB} + \overline{OC}$ ...$\left(\because \overline{AL} = \overline{OB}\right)$

$= x\hat{i} + y\hat{j} + z\hat{k}$ ...(i)

Magnitude of $\overline{OP}$

Now, $OP^2 = OL^2 + LP^2$ (In right angled ΔOLP)

$= OA^2 + AL^2 + LP^2$ (In right angled ΔOAL)

$= OA^2 + OB^2 + OC^2$

$= x^2 + y^2 + z^2$ ...($\because$ AL = OB)

$\therefore$ $l(OP) = \sqrt{x^2 + y^2 + z^2}$

$\therefore$ $\left(\overline{OP}\right) = \sqrt{x^2 + y^2 + z^2}$

Component form of $\bar{r}$:

If $\bar{r}$ is a position vector (p.v.) of point P w.r.t. O then $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

In this representation x, y, z are called the components of $\bar{r}$ along OX, OY and OZ.

Note that any vector in space is unique linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Note:

Some authors represent vector $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ using angle brackets as $\bar{r} = < x, y, z >$ for the point (x, y, z).

Vector joining two points:

Let O be the origin then $\overline{OA}$ and $\overline{OB}$ are the position vectors of points A and B w.r.t. origin 'O'.

In ΔAOB, we have by triangle law.

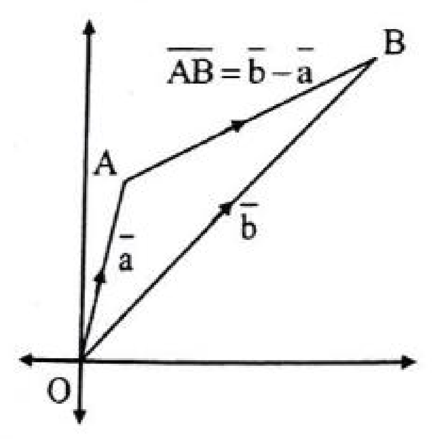
$\overline{AB} = \overline{AO} + \overline{OB}$

$= -\overline{OA} + \overline{OB}$ ...$\left(\because \overline{AO} = -\overline{OA}\right)$

$= \overline{OB} - \overline{OA}$

= position vector of B – position vector of A

That is, $\overline{AB} = \bar{b} - \bar{a}$



If A $\equiv$ (x_1, y_1, z_1) and B $\equiv$ (x_2, y_2, z_2)

Then $\overline{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and

$\overline{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore$ $\overline{AB} = \overline{OB} - \overline{AO}$

$= \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\left|\overline{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

In general, any non zero vector $\bar{a}$ in space can be expressed uniquely as the linear combination of $\hat{i}$, $\hat{j}$, $\hat{k}$ as $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where a_1, a_2, a_3 are scalars.

$\therefore$ $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Note:

i. If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a} = \bar{b}$ if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$.

ii. If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a} + \bar{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) + \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$

$= (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.

iii. If k is any scalar then $k\bar{a} = k\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$

$= ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$

Also if $\bar{b}$ and $\bar{a}$ are collinear i.e. $\bar{b} = k\bar{a}$ then $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = k$.

iv. Let $\hat{a}$ be a unit vector along a non zero vector $\bar{a}$ in space. Then $\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{a_1\hat{i} + a_2\hat{j} + a_3\hat{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$.

v. **Collinearity of 3-points:**

Three distinct points A, B and C with position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively are collinear if and only if there exist three non-zero scalars x, y and z such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and $x + y + z = 0$. (Use Theorem 1 and the fact that $\overline{AB}$ and $\overline{BC}$ are collinear.)

vi. **Coplanarity of 4-points:**

Four distinct points A, B, C and D (no three of which are collinear) with position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ respectively are coplanar if and only if there exist four scalars x, y, z and w, not all zero, such that $x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$. where $x + y + z + w = 0$. (Use Theorem 2 and the fact that $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are coplanar).

vii. **Linearly dependent vectors:**

A set of non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is said to be linearly dependent if there exist scalars x, y, z not all zero such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

Such vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar.

viii. **Linearly independent vectors:**

A set of non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is said to be linearly independent if $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$, then $x = y = z = 0$

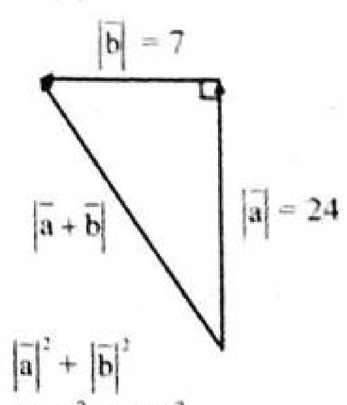
Such vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non-coplanar.

Exercise 5.1

1. The vector $\bar{a}$ is directed due north and $|\bar{a}| = 24$. The vector $\bar{b}$ is directed due west and $|\bar{b}| = 7$. Find $|\bar{a} + \bar{b}|$.

Solution:

$|\bar{a}| = 24$ and $|\bar{b}| = 7$ ...[Given]



$|\bar{a} + \bar{b}|^2 = |\bar{a}|^2 + |\bar{b}|^2$

$= (24)^2 + (7)^2$

$= 576 + 49$

$\therefore$ $|\bar{a} + \bar{b}|^2 = 625$

By taking square root of both sides, we get

$|\bar{a} + \bar{b}| = 25$

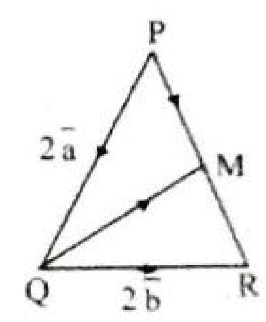
2. In the triangle PQR, $\overline{PQ} = 2\bar{a}$ and $\overline{QR} = 2\bar{b}$. The mid-point of PR is M. Find the following vectors in terms of $\bar{a}$ and $\bar{b}$.

i. $\overline{PR}$ **ii. $\overline{PM}$**

iii. $\overline{QM}$

Solution:

$\left.\begin{array}{l}\overline{PQ} = 2\bar{a} \\ \overline{QR} = 2\bar{b}\end{array}\right\}$ ...[Given]



i. In ΔPQR, by using triangle law of vector addition, we get

$\overline{PR} = \overline{PQ} + \overline{QR}$

$= 2\bar{a} + 2\bar{b}$ ...(i)

[**Note:** Answer given in the textbook is $2\bar{a} - 2\bar{b}$. However, we found that it is $2\bar{a} + 2\bar{b}$.]

ii. M is the midpoint of $\overline{PR}$. ...[Given]

$\therefore$ $\overline{PM} = \frac{1}{2}\overline{PR}$

$= \frac{1}{2}\left(2\bar{a} + 2\bar{b}\right)$ ...[From (i)]

$= \bar{a} + \bar{b}$ ...(ii)

iii. In ΔPQM,
by using triangle law of vector addition, we get

$\overline{PM} = \overline{PQ} + \overline{QM}$

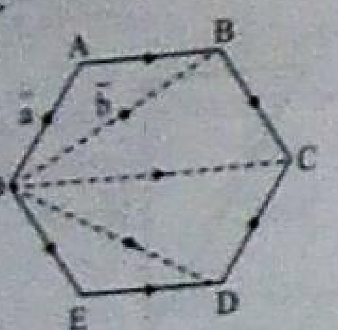
$\therefore$ $\overline{QM} = \overline{PM} - \overline{PQ}$

$= \bar{a} + \bar{b} - 2\bar{a}$ ...[From (ii)]

$= \bar{b} - \bar{a}$

3. OABCDE is a regular hexagon. The points A and B have position vectors $\bar{a}$ and $\bar{b}$ respectively, referred to the origin O. Find, in terms of $\bar{a}$ and $\bar{b}$ the position vectors of C, D and E.

Solution:



In regular hexagon OABCDE,

$\overline{OA} = \overline{DC}$ ...(i)

$\overline{BC} = \overline{OE}$ ...(ii)

In ΔOAB,
by using triangle law of vector addition, we get

$\overline{OA} + \overline{AB} = \overline{OB}$

$\therefore$ $\bar{a} + \overline{AB} = \bar{b}$

$\therefore$ $\overline{AB} = \bar{b} - \bar{a}$

$\overline{OC} = 2\overline{AB}$ ...[$\because \overline{OC} \parallel \overline{AB}$]

$= 2(\bar{b} - \bar{a})$

$\therefore$ $\overline{OC} = 2\bar{b} - 2\bar{a}$ ...(iii)

In ΔODC,
by using triangle law of vector addition, we get

$\overline{OD} + \overline{DC} = \overline{OC}$

$\therefore$ $\overline{OD} = \overline{OC} - \overline{DC}$

$\therefore$ $\overline{OD} = \overline{OC} - \overline{OA}$ ...[From (i)]

$\therefore$ $\overline{OD} = 2\bar{b} - 2\bar{a} - \bar{a}$ ...[From (iii)]

$\therefore$ $\overline{OD} = 2\bar{b} - 3\bar{a}$

In ΔOBC,
by using triangle law of vector addition, we get

$\overline{OB} + \overline{BC} = \overline{OC}$

$\therefore$ $\overline{BC} = \overline{OC} - \overline{OB}$

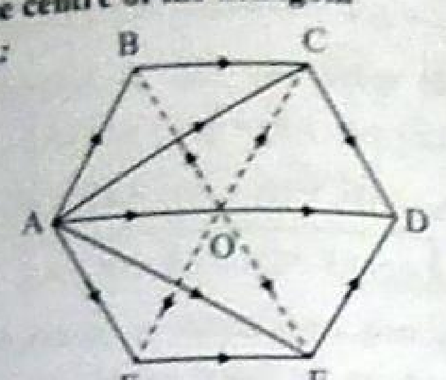
$\therefore$ $\overline{BC} = 2\bar{b} - 2\bar{a} - \bar{b}$ ...[From (iii)]

$\therefore$ $\overline{OE} = \bar{b} - 2\bar{a}$ ...[From (ii)]

[**Note:** Answer given in the textbook is $\overline{OC} = 2\bar{a} + 2\bar{b}$. However, we found that it is $2\bar{b} - 2\bar{a}$.]

4. If ABCDEF is a regular hexagon, show that $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 6\overline{AO}$, where O is the centre of the hexagon.

Solution:



By using triangle law of vector addition,

$\overline{AB} = \overline{AO} + \overline{OB}$ ...(i)

$\overline{AC} = \overline{AO} + \overline{OC}$ ...(ii)

$\overline{AE} = \overline{AO} + \overline{OE}$ ...(iii)

$\overline{AF} = \overline{AO} + \overline{OF}$ ...(iv)

Now, $\overline{AD} = \overline{AO} + \overline{OD}$

$= 2\overline{AO}$ ...(v) $[\because \overline{AO} = \overline{OD}]$

By adding (i), (ii), (iii), (iv) and (v), we get

$\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF}$

$= (\overline{AO} + \overline{OB}) + (\overline{AO} + \overline{OC}) + 2\overline{AO} + (\overline{AO} + \overline{OE}) + (\overline{AO} + \overline{OF})$

$= 6\overline{AO} + \overline{OB} + \overline{OC} + \overline{OE} + \overline{OF}$

$= 6\overline{AO} + \overline{OB} + \overline{OC} - \overline{OB} - \overline{OC}$ ...$[\because \overline{OE} = -\overline{OB}$ and $\overline{OF} = -\overline{OC}]$

$= 6\overline{AO}$

5. Check whether the vectors $2\hat{i} + 2\hat{j} + 3\hat{k}$, $-3\hat{i} + 3\hat{j} + 2\hat{k}$ and $3\hat{i} + 4\hat{k}$ form a triangle or not.

Solution:

Let $\bar{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = -3\hat{i} + 3\hat{j} + 2\hat{k}$,

$\bar{c} = 3\hat{i} + 4\hat{k}$

If $\bar{a}$, $\bar{b}$, $\bar{c}$ satisfy triangle law of vector addition, then they form a triangle.

$\bar{b} + \bar{c} = (-3\hat{i} + 3\hat{j} + 2\hat{k}) + (3\hat{i} + 4\hat{k})$

$\therefore$ $\bar{b} + \bar{c} = 0\hat{i} + 3\hat{j} + 6\hat{k}$

$\therefore$ $\bar{b} + \bar{c} \neq \bar{a}$ ...(i)

$\bar{a} + \bar{c} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + (3\hat{i} + 4\hat{k})$

$= 5\hat{i} + 2\hat{j} + 7\hat{k}$

$\therefore$ $\bar{a} + \bar{c} \neq \bar{b}$ ...(ii)

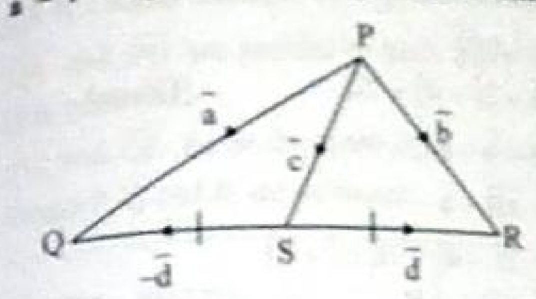
$\bar{a} + \bar{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + (-3\hat{i} + 3\hat{j} + 2\hat{k})$

$= -\hat{i} + 5\hat{j} + 5\hat{k}$

$\therefore$ $\bar{a} + \bar{b} \neq \bar{c}$ ...(iii)

$\therefore$ The given vectors do not form a triangle. ...[From (i), (ii) and (iii)]

6. In the given figure, express $\bar{c}$ and $\bar{d}$ in terms of $\bar{a}$ and $\bar{b}$. Find a vector in the direction of $\bar{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.



Solution:

$\bar{a} = \hat{i} - 2\hat{j}$

In Δ PSR and Δ PSQ,
by using triangle law of vector addition, we get

$\overline{PS} + \overline{SR} = \overline{PR}$ and $\overline{PS} + \overline{SQ} = \overline{PQ}$

$\therefore$ $\bar{c} + \bar{d} = \bar{b}$ ...(i)

$\bar{c} - \bar{d} = \bar{a}$ ...(ii)

By adding (i) and (ii), we get

$\bar{c} + \bar{d} = \bar{b}$

$\bar{c} - \bar{d} = \bar{a}$

$2\bar{c} = \bar{a} + \bar{b}$

$\therefore$ $\bar{c} = \dfrac{\bar{a} + \bar{b}}{2} = \dfrac{1}{2}\bar{a} + \dfrac{1}{2}\bar{b}$

Substituting $\bar{c} = \dfrac{\bar{a} + \bar{b}}{2}$ in (i), we get

$\dfrac{\bar{a} + \bar{b}}{2} + \bar{d} = \bar{b}$

$\therefore$ $\bar{d} = \bar{b} - \left(\dfrac{\bar{a} + \bar{b}}{2}\right)$

$= \dfrac{2\bar{b} - \bar{a} - \bar{b}}{2}$

$\therefore$ $\bar{d} = \dfrac{\bar{b} - \bar{a}}{2} = \dfrac{1}{2}\bar{b} - \dfrac{1}{2}\bar{a}$

Vector in the direction of $\bar{a}$

= (Magnitude of a vector) $\times$ $\hat{a}$

$= 7 \times \dfrac{\bar{a}}{|\bar{a}|}$

$= 7 \times \dfrac{\hat{i} - 2\hat{j}}{\sqrt{(1)^2 + (-2)^2}} = \dfrac{7}{\sqrt{5}}(\hat{i} - 2\hat{j})$

$= \dfrac{7}{\sqrt{5}}\hat{i} - \dfrac{14}{\sqrt{5}}\hat{j}$

[**Note:** The answer no. 7 n the textbook is the answer for vector in the direction of $\bar{a}$.]

7. Find the distance from (4, –2, 6) to each of the following :

i. The XY-plane **ii. The YZ-plane**

iii. The XZ-plane **iv. The X-axis**

v. The Y-axis **vi. The Z-axis**

Solution:

i. The distance from (4, –2, 6) to the XY-plane is

$|z| = \sqrt{6^2} = 6$ units

ii. The distance from (4, –2, 6) to the YZ-plane is

$|x| = \sqrt{4^2} = 4$ units

iii. The distance from (4, –2, 6) to the XZ-plane is

$|y| = \sqrt{(-2)^2} = 2$ units

iv. The distance from (4, –2, 6) to the X-axis is

$\sqrt{y^2 + z^2} = \sqrt{(-2)^2 + 6^2}$

$= \sqrt{4 + 36}$

$= \sqrt{40}$

$= 2\sqrt{10}$ units

v. The distance from (4, –2, 6) to the Y-axis is

$\sqrt{x^2 + z^2} = \sqrt{4^2 + 6^2}$

$= \sqrt{16 + 36}$

$= \sqrt{52}$

$= 2\sqrt{13}$ units

vi. The distance from (4, –2, 6) to the Z-axis is

$\sqrt{x^2 + y^2} = \sqrt{4^2 + (-2)^2}$

$= \sqrt{16 + 4}$

$= \sqrt{20}$

$= 2\sqrt{5}$ units

[**Note:** The answer no. 8 in the textbook is the answer for Q. 7.]

8. Find the coordinates of the point which is located :

i. three units behind the YZ-plane, four units to the right of the XZ-plane and five units above the XY-plane.

ii. in the YZ-plane, one unit to the right of the XZ-plane and six units above the XY-plane.

Solution:

i. Three units behind the YZ-plane i.e., $x = -3$

Four units to the right of the XZ-plane i.e., $y = 4$

Five units above the XY-plane i.e., $z = 5$

$\therefore$ The required co-ordinates are (–3, 4, 5).

ii. In the YZ-plane i.e., $x = 0$
One unit to the right of the XZ-plane i.e., $y = 1$
Six units above the XY-plane i.e., $z = 6$
$\therefore$ The required co-ordinates are (0, 1, 6).

[**Note:** The answer no. 9 in the textbook is the answer for Q. 8.]

9. Find the area of the triangle with vertices (1, 1, 0), (1, 0, 1) and (0, 1, 1).

Solution:

Let ABC be the triangle whose vertices are A(1, 1, 0), B(1, 0, 1) and C(0, 1, 1).
Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of points A, B, C respectively.

$\therefore \quad \bar{a} = \hat{i} + \hat{j},\ \bar{b} = \hat{i} + \hat{k},\ \bar{c} = \hat{j} + \hat{k}$

$\overline{AB} = \bar{b} - \bar{a} = \hat{i} + \hat{k} - \hat{i} - \hat{j} = -\hat{j} + \hat{k}$

$\overline{BC} = \bar{c} - \bar{b} = \hat{j} + \hat{k} - \hat{i} - \hat{k} = -\hat{i} + \hat{j}$

$\overline{AC} = \bar{c} - \bar{a} = \hat{j} + \hat{k} - \hat{i} - \hat{j} = -\hat{i} + \hat{k}$

$|\overline{AB}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$|\overline{BC}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$|\overline{AC}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\therefore \quad l(AB) = l(BC) = l(AC)$

$\therefore \quad \Delta ABC$ is an equilateral triangle.

$\therefore \quad A(\Delta ABC) = \dfrac{\sqrt{3}}{4}(\text{side})^2$

$= \dfrac{\sqrt{3}}{4}\left(\sqrt{2}\right)^2 = \dfrac{\sqrt{3}}{2}$ sq. units

Alternate method:

Let ABC be the triangle whose vertices are A(1, 1, 0), B(1, 0, 1) and C(0, 1, 1).
Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of points A, B, C respectively.

$\therefore \quad \bar{a} = \hat{i} + \hat{j},\ \bar{b} = \hat{i} + \hat{k},\ \bar{c} = \hat{j} + \hat{k}$

$\overline{AB} = \bar{b} - \bar{a} = \hat{i} + \hat{k} - \hat{i} - \hat{j} = -\hat{j} + \hat{k}$

$\overline{AC} = \bar{c} - \bar{a} = \hat{j} + \hat{k} - \hat{i} - \hat{j} = -\hat{i} + \hat{k}$

Area of triangle $= \dfrac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

$$\therefore \quad \overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \hat{i}(-1) - \hat{j}(1) + \hat{k}(-1)$$

$= -\hat{i} - \hat{j} - \hat{k}$

$\therefore \quad \left|\overline{AB} \times \overline{AC}\right| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2}$

$= \sqrt{1 + 1 + 1} = \sqrt{3}$

$\therefore \quad$ Area of triangle $= \dfrac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

$= \dfrac{\sqrt{3}}{2}$ sq. units

[**Note:** The answer no. 10 in the textbook is the answer for Q. 9.]

10. If $\overline{AB} = 2\hat{i} - 4\hat{j} + 7\hat{k}$ and initial point $A \equiv (1, 5, 0)$, find the terminal point B.

Solution:

Let $\bar{a}, \bar{b}$ be the position vectors of points A, B respectively.

$\therefore \quad \bar{a} = \hat{i} + 5\hat{j}$

$\overline{AB} = 2\hat{i} - 4\hat{j} + 7\hat{k}$ ...[Given]

$\overline{AB} = \bar{b} - \bar{a}$

$\therefore \quad \bar{b} = \overline{AB} + \bar{a}$

$= 2\hat{i} - 4\hat{j} + 7\hat{k} + \hat{i} + 5\hat{j}$

$\therefore \quad \bar{b} = 3\hat{i} + \hat{j} + 7\hat{k}$

$\therefore \quad$ The terminal point B is (3, 1, 7).

[**Note:** The answer no. 11 in the textbook is the answer for Q. 10.]

11. Show that the following points are collinear:
i. A (3, 2, –4), B (9, 8, –10), C (–2, –3, 1).
ii. P (4, 5, 2), Q (3, 2, 4), R (5, 8, 0).

Solution:

i. Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of points A, B, C respectively.

$\therefore \quad \bar{a} = 3\hat{i} + 2\hat{j} - 4\hat{k}$,

$\bar{b} = 9\hat{i} + 8\hat{j} - 10\hat{k}$,

$\bar{c} = -2\hat{i} - 3\hat{j} + \hat{k}$

$\therefore \quad \overline{AB} = \bar{b} - \bar{a}$

$= \left(9\hat{i} + 8\hat{j} - 10\hat{k}\right) - \left(3\hat{i} + 2\hat{j} - 4\hat{k}\right)$

$\therefore \quad \overline{AB} = 6\hat{i} + 6\hat{j} - 6\hat{k} = 6(\hat{i} + \hat{j} - \hat{k})$ ...(i)

$\overline{AC} = \bar{c} - \bar{a}$

$= \left(-2\hat{i} - 3\hat{j} + \hat{k}\right) - \left(3\hat{i} + 2\hat{j} - 4\hat{k}\right)$

$\therefore \quad \overline{AC} = -5\hat{i} - 5\hat{j} + 5\hat{k} = -5(\hat{i} + \hat{j} - \hat{k})$ ...(ii)

$\therefore \quad \overline{AC} = \left(-\dfrac{5}{6}\right)\overline{AB}$ ...[From (i) and (ii)]

$\therefore \quad \overline{AC}$ is a scalar multiple of $\overline{AB}$.

$\therefore \quad \overline{AC}$ and $\overline{AB}$ are parallel to each other with point A in common.

$\therefore \quad \overline{AB}$ and $\overline{AC}$ lie on the same line.

$\therefore \quad$ Points A, B and C are collinear.

ii. Let, $\bar{p}, \bar{q}, \bar{r}$ be the position vectors of points P, Q, R respectively.

$\therefore \quad \bar{p} = 4\hat{i} + 5\hat{j} + 2\hat{k}$,

$\bar{q} = 3\hat{i} + 2\hat{j} + 4\hat{k}$,

$\bar{r} = 5\hat{i} + 8\hat{j}$

$\therefore \quad \overline{PQ} = \bar{q} - \bar{p} = \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right) - \left(4\hat{i} + 5\hat{j} + 2\hat{k}\right)$

$\overline{PQ} = -\hat{i} - 3\hat{j} + 2\hat{k}$ ...(i)

$\overline{QR} = \bar{r} - \bar{q} = 5\hat{i} + 8\hat{j} - \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right)$

$= 2\hat{i} + 6\hat{j} - 4\hat{k}$

$\overline{QR} = (-2)\left(-\hat{i} - 3\hat{j} + 2\hat{k}\right)$ ...(ii)

$\therefore \quad \overline{QR} = (-2)\ \overline{PQ}$ ...[From (i) and (ii)]

$\therefore \quad \overline{QR}$ is a scalar multiple of $\overline{PQ}$.

$\therefore \quad \overline{QR}$ and $\overline{PQ}$ are parallel to each other with point Q in common.

$\therefore \quad \overline{PQ}$ and $\overline{QR}$ lie on the same line.

$\therefore \quad$ Points P, Q and R are collinear.

12. If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, then find the value of q.

Solution:

Let $\bar{a} = 2\hat{i} - q\hat{j} + 3\hat{k}$ and $\bar{b} = 4\hat{i} - 5\hat{j} + 6\hat{k}$.

Since $\bar{a}$ and $\bar{b}$ are collinear, there exists a scalar t such that $\bar{b} = t\bar{a}$.

$\therefore \quad 4\hat{i} - 5\hat{j} + 6\hat{k} = t(2\hat{i} - q\hat{j} + 3\hat{k})$

$= 2t\hat{i} - qt\hat{j} + 3t\hat{k}$

By equality of vectors, we get

$\therefore \quad 4 = 2t,\ -5 = -qt,\ 6 = 3t$

$4 = 2t$ and $-5 = -qt$

$\therefore \quad t = 2$ and $-5 = -q(2)$

$\therefore \quad -5 = -2q$

$\therefore \quad q = \dfrac{5}{2}$

13. Are the four points A(1, –1, 1), B(–1, 1, 1), C(1, 1, 1) and D(2, –3, 4) coplanar? Justify your answer.

Solution:

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of points A, B, C, D respectively.

$\therefore \quad \bar{a} = \hat{i} - \hat{j} + \hat{k},\ \bar{b} = -\hat{i} + \hat{j} + \hat{k},\ \bar{c} = \hat{i} + \hat{j} + \hat{k},$

$\bar{d} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$\therefore \quad \overline{AB} = \bar{b} - \bar{a} = \left(-\hat{i} + \hat{j} + \hat{k}\right) - \left(\hat{i} - \hat{j} + \hat{k}\right)$

$= -2\hat{i} + 2\hat{j}$

$\overline{AC} = \bar{c} - \bar{a} = \left(\hat{i} + \hat{j} + \hat{k}\right) - \left(\hat{i} - \hat{j} + \hat{k}\right)$

$= 2\hat{j}$

$\overline{AD} = \bar{d} - \bar{a} = \left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) - \left(\hat{i} - \hat{j} + \hat{k}\right)$

$= \hat{i} - 2\hat{j} + 3\hat{k}$

Points A, B, C, D are coplanar if $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are coplanar

Let $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ be coplanar

$\therefore \quad$ There exist unique scalars t_1 and t_2 such that

$\overline{AB} = t_1\overline{AC} + t_2\overline{AD}$

$\therefore \quad -2\hat{i} + 2\hat{j} = t_1(2\hat{j}) + t_2(\hat{i} - 2\hat{j} + 3\hat{k})$

$\therefore \quad -2\hat{i} + 2\hat{j} + 0.\hat{k} = t_2\hat{i} + (2t_1 - 2t_2)\hat{j} + 3t_2\hat{k}$

$\therefore \quad$ By equality of vectors, we get

$t_2 = -2$,

$2t_1 - 2t_2 = 2$,

$3t_2 = 0$ i.e., $t_2 = 0$

But $t_2 = -2$ and $t_2 = 0$ is not possible.

$\therefore \quad t_2$ is not unique.

$\therefore \quad \overline{AB}, \overline{AC}, \overline{AD}$ are non-coplanar.

$\therefore \quad$ Point A, B, C, D are also non-coplanar.

14. Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

Solution:

Let $\bar{a} = 2\hat{i} + \hat{j} - 4\hat{k},\ \bar{b} = 2\hat{i} - \hat{j} + 3\hat{k}$,

$\bar{c} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\bar{r} = -\hat{i} - 3\hat{j} + 4\hat{k}$

Consider, $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$ ...(i)

where x, y, z are scalars

$\therefore \quad -\hat{i} - 3\hat{j} + 4\hat{k} = x\left(2\hat{i} + \hat{j} - 4\hat{k}\right) + y\left(2\hat{i} - \hat{j} + 3\hat{k}\right) + z\left(3\hat{i} + \hat{j} - 2\hat{k}\right)$

$\therefore \quad -\hat{i} - 3\hat{j} + 4\hat{k} = (2x + 2y + 3z)\hat{i} + (x - y + z)\hat{j} + (-4x + 3y - 2z)\hat{k}$

$\therefore \quad$ By equality of vectors, we get

$2x + 2y + 3z = -1$

$x - y + z = -3$

$-4x + 3y - 2z = 4$

By Cramer's rule, we get

$$D = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 1 \\ -4 & 3 & -2 \end{vmatrix} = 2(2 - 3) - 2(-2 + 4) + 3(3 - 4)$$

$= 2(-1) - 2(2) + 3(-1)$

$= -2 - 4 - 3 = -9 \neq 0$

$$D_x = \begin{vmatrix} -1 & 2 & 3 \\ -3 & -1 & 1 \\ 4 & 3 & -2 \end{vmatrix} = -1(2 - 3) - 2(6 - 4) + 3(-9 + 4)$$

$= -1(-1) - 2(2) + 3(-5)$

$= 1 - 4 - 15 = -18$

$$D_y = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -3 & 1 \\ -4 & 4 & -2 \end{vmatrix} = 2(6 - 4) + 1(-2 + 4) + 3(4 - 12)$$

$= 2(2) + 1(2) + 3(-8)$

$= 4 + 2 - 24 = -18$

$$D_z = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & -3 \\ -4 & 3 & 4 \end{vmatrix} = 2(-4+9) - 2(4-12) - 1(3-4)$$

$$= 2(5) - 2(-8) - 1(-1)$$

$$= 10 + 16 + 1 = 27$$

$\therefore$ $x = \frac{D_x}{D} = \frac{-18}{-9} = 2,$ $\quad y = \frac{D_y}{D} = \frac{-18}{-9} = 2,$

$z = \frac{D_z}{D} = \frac{27}{-9} = -3$

$\therefore$ $\bar{r} = 2\bar{a} + 2\bar{b} - 3\bar{c}$ ...[From (i)]

Let's Study

Section Formula

Theorem 5 (Section formula for internal division) :
Let $A(\bar{a})$ and $B(\bar{b})$ be any two points in the space and $R(\bar{r})$ be a point on the line segment AB dividing it internally in the ratio m : n.

Then $\bar{r} = \frac{m\bar{b} + n\bar{a}}{m+n}$. **[Mar 98, 01, 06, 13 (old course), 15, 19; July 17; Oct 96, 97, 04]**

Proof :
R is a point on the line segment AB(A-R-B) and $\overline{AR}$ and $\overline{RB}$ are in the same direction.
Point R divides AB internally in the ratio m : n.

$\therefore$ $\frac{AR}{RB} = \frac{m}{n}$

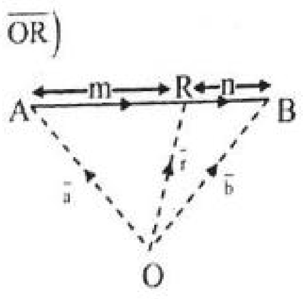
$\therefore$ n(AR) = m(RB)

As $n(\overline{AR})$ and $m(\overline{RB})$ have same direction and magnitude,

$n(\overline{AR}) = m(\overline{RB})$

$\therefore$ $n(\overline{OR} - \overline{OA}) = m(\overline{OB} - \overline{OR})$

$\therefore$ $n(\bar{r} - \bar{a}) = m(\bar{b} - \bar{r})$



$\therefore$ $n\bar{r} - n\bar{a} = m\bar{b} - m\bar{r}$

$\therefore$ $m\bar{r} + n\bar{r} = m\bar{b} + n\bar{a}$

$\therefore$ $(m+n)\bar{r} = m\bar{b} + n\bar{a}$

$\therefore$ $\bar{r} = \frac{m\bar{b} + n\bar{a}}{m+n}$

Note:
1. If A ≡ (a_1, a_2, a_3), B ≡ (b_1, b_2, b_3) and R ≡ (r_1, r_2, r_3) divides the segment AB in the ratio m : n, then $\bar{r}_i = \frac{m\bar{b}_i + n\bar{a}_i}{m+n}$, i = 1, 2, 3.

2. **Midpoint formula:**
If $R(\bar{r})$ is the mid-point of the line segment AB, then m = n.

$\therefore$ m : n = 1 : 1

$\therefore$ By section formula,

$\bar{r} = \frac{1(\bar{b}) + 1(\bar{a})}{1+1} = \frac{\bar{a} + \bar{b}}{2}$

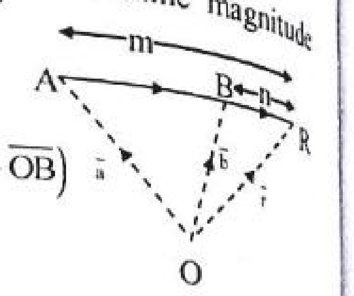
Theorem 6 (Section formula for external division):
Let $A(\bar{a})$ and $B(\bar{b})$ be any two points in the space and $R(\bar{r})$ be the third point on the line AB dividing the segment AB externally in the ratio m : n. Then $\bar{r} = \frac{m\bar{b} - n\bar{a}}{m-n}$.

Proof :
As the point R divides line segment AB externally, we have either A-B-R or R-A-B.
Assume that A-B-R and $\overline{AR} : \overline{BR}$ = m : n

$\therefore$ $\frac{AR}{BR} = \frac{m}{n}$

$\therefore$ n(AR) = m(BR)

As $n(\overline{AR})$ and $m(\overline{BR})$ have same magnitude and direction,



$n(\overline{AR}) = m(\overline{BR})$

$\therefore$ $n(\overline{OR} - \overline{OA}) = m(\overline{OR} - \overline{OB})$

$\therefore$ $n(\bar{r} - \bar{a}) = m(\bar{r} - \bar{b})$

$\therefore$ $n\bar{r} - n\bar{a} = m\bar{r} - m\bar{b}$

$\therefore$ $m\bar{b} - n\bar{a} = m\bar{r} - n\bar{r} = (m-n)\bar{r}$

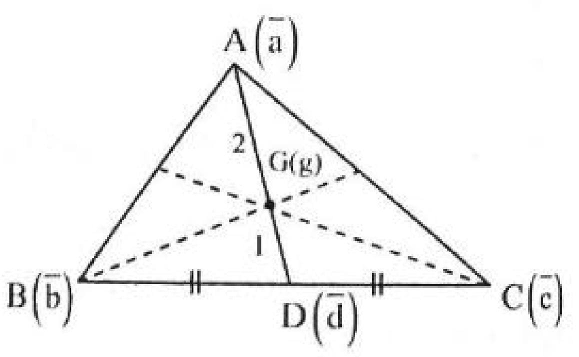
$\therefore$ $\bar{r} = \frac{m\bar{b} - n\bar{a}}{m-n}$

Note :
1. Whenever the ratio in which point R divides the join of two points A and B is required, it is convenient to take the ratio as t : 1.

Then, $\bar{r} = \frac{t\bar{b} + \bar{a}}{t+1}$, if division is internal.

$\bar{r} = \frac{t\bar{b} - \bar{a}}{t-1}$, if division is external.

2. In ΔABC, centroid G divides the medians internally in ratio 2 : 1 and is given by
$\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$.



Proof:
Consider ΔABC. Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be the position vectors of the vertices A, B and C respectively, of ΔABC.
Let D be mid-point of BC.
By mid-point formula, we get

$\therefore$ $\bar{d} = \frac{\bar{b} + \bar{c}}{2}$ ...(i)

Let G be the centroid of ΔABC and $\bar{g}$ be its position vector.

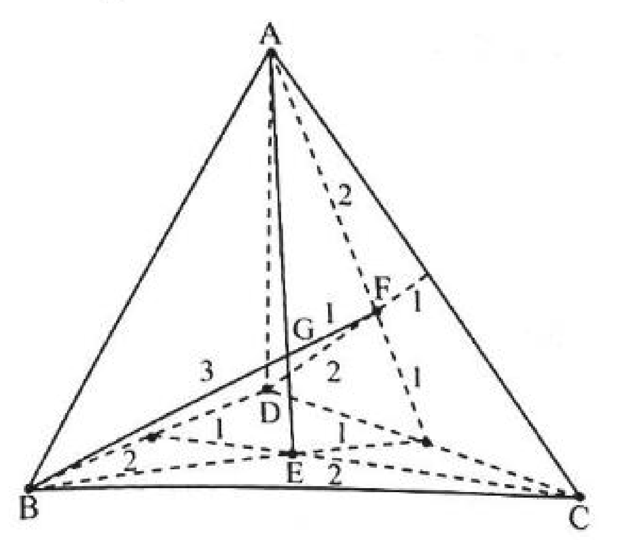
$\therefore$ G divides the median AD internally in the ratio 2 : 1

$\therefore$ By section formula, we get

$\bar{g} = \frac{2(\bar{d}) + 1(\bar{a})}{2+1} = \frac{2\left(\frac{\bar{b}+\bar{c}}{2}\right) + \bar{a}}{3}$ ...[From (i)]

$\therefore$ $\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$

3. In tetrahedron ABCD, centroid G divides the line joining the vertex of tetrahedron to centroid of opposite triangle in the ratio 3 : 1 and is given by
$\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c} + \bar{d}}{4}$.



4. In ΔABC, P.V. of Incentre is given by
$\frac{|\overline{AB}|\bar{c} + |\overline{BC}|\bar{a} + |\overline{AC}|\bar{b}}{|\overline{AB}| + |\overline{BC}| + |\overline{AC}|}$.

5. In ΔABC, P.V. of Orthocentre is given by
$\frac{\tan A\,\bar{a} + \tan B\,\bar{b} + \tan C\,\bar{c}}{\tan A + \tan B + \tan C}$.

Exercise 5.2

1. **Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $-5\hat{i} + 2\hat{j} - 5\hat{k}$ in the ratio 3 : 2**

i. **internally** ii. **externally**

Solution:

Let $\bar{p}$, $\bar{q}$ and $\bar{r}$ be the position vectors of points P, Q and R respectively.

$\therefore$ $\bar{p} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\bar{q} = -5\hat{i} + 2\hat{j} - 5\hat{k}$, m:n = 3:2

i. R divides the line PQ internally in the ratio 3: 2.

$\therefore$ By using section formula for internal division,

$\bar{r} = \frac{m\bar{q} + n\bar{p}}{m+n}$

$= \frac{3(-5\hat{i} + 2\hat{j} - 5\hat{k}) + 2(2\hat{i} - \hat{j} + 3\hat{k})}{3+2}$

$= \frac{-15\hat{i} + 6\hat{j} - 15\hat{k} + 4\hat{i} - 2\hat{j} + 6\hat{k}}{5}$

$\therefore$ $\bar{r} = \frac{-11\hat{i} + 4\hat{j} - 9\hat{k}}{5} = \frac{-11}{5}\hat{i} + \frac{4}{5}\hat{j} - \frac{9}{5}\hat{k}$

ii. R divides the line PQ externally in ratio 3: 2.

$\therefore$ By using section formula for external division,

$\bar{r} = \frac{m\bar{q} - n\bar{p}}{m-n} = \frac{3(-5\hat{i} + 2\hat{j} - 5\hat{k}) - 2(2\hat{i} - \hat{j} + 3\hat{k})}{3-2}$

$= \frac{-15\hat{i} + 6\hat{j} - 15\hat{k} - 4\hat{i} + 2\hat{j} - 6\hat{k}}{1}$

$\therefore$ $\bar{r} = -19\hat{i} + 8\hat{j} - 21\hat{k}$

2. **Find the position vector of mid-point M joining the points L (7, –6, 12) and N (5, 4, –2).**

Solution:

Let $\bar{m}$, $\bar{l}$ and $\bar{n}$ be the position vectors of points M, L and N respectively.

$\therefore$ $\bar{l} = 7\hat{i} - 6\hat{j} + 12\hat{k}$, $\bar{n} = 5\hat{i} + 4\hat{j} - 2\hat{k}$

$\therefore$ By using midpoint formula,

$\bar{m} = \frac{\bar{l} + \bar{n}}{2}$

$= \frac{7\hat{i} - 6\hat{j} + 12\hat{k} + 5\hat{i} + 4\hat{j} - 2\hat{k}}{2}$

$= \frac{12\hat{i} - 2\hat{j} + 10\hat{k}}{2}$

$\therefore$ $\bar{m} = 6\hat{i} - \hat{j} + 5\hat{k}$

3. **If the points A(3, 0, p), B (–1, q, 3) and C(–3, 3, 0) are collinear, then find**

i. **the ratio in which the point C divides the line segment AB.**

ii. **the values of p and q.**

Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of points A, B, C respectively.

$\therefore$ $\bar{a} = 3\hat{i} + 0\cdot\hat{j} + p\hat{k}$, $\bar{b} = -\hat{i} + q\hat{j} + 3\hat{k}$ and $\bar{c} = -3\hat{i} + 3\hat{j} + 0\cdot\hat{k}$

i. Let point C divides line segment AB in the ratio t : 1.

By using section formula,

$\bar{c} = \frac{t\bar{b} + 1\bar{a}}{t+1}$

$\therefore \quad -3\hat{i} + 3\hat{j} + 0\hat{k} = \frac{t\left(-\hat{i} + q\hat{j} + 3\hat{k}\right) + \left(3\hat{i} + 0\hat{j} - p\hat{k}\right)}{t+1}$

$\therefore \quad (t+1)\left(-3\hat{i} + 3\hat{j} + 0\hat{k}\right) = -t\hat{i} + tq\hat{j} + 3t\hat{k} + 3\hat{i} + 0\hat{j} - p\hat{k}$

$\therefore \quad -3(t+1)\hat{i} + 3(t+1)\hat{j} + 0\hat{k} = (-t+3)\hat{i} + tq\hat{j} + (3t - p)\hat{k}$

$\therefore$ By equality of vectors, we get

$-3(t+1) = -t + 3 \quad \ldots(i)$

$3(t+1) = tq \quad \ldots(ii)$

$0 = 3t - p \quad \ldots(iii)$

From (i), we get

$-3t - 3 = -t + 3$

$\therefore \quad -2t = 6$

$\therefore \quad t = -3$

$\therefore$ C divides segment AB externally, (since t is negative) in the ratio 3 : 1.

iii. Putting $t = -3$ in (ii), we get

$3(-3+1) = -3q$

$\therefore \quad -6 = -3q$

$\therefore \quad q = 2$

Putting $t = -3$ in (iii), we get

$0 = -9 + p$

$\therefore \quad p = 9$

$\therefore \quad p = 9$ and $q = 2$

4. The position vectors of points A and B are $6\bar{a} + 2\bar{b}$ and $\bar{a} - 3\bar{b}$ respectively. If the point C divides AB in the ratio 3 : 2, then show that the position vector of C is $3\bar{a} - \bar{b}$.

Solution:

Let $\bar{c}$ be the position vector of point C.

The position vector of point A is $6\bar{a} + 2\bar{b}$ and the position vector of point B is $\bar{a} - 3\bar{b}$.

Point C divides AB in the ratio 3 : 2.

$\therefore$ By using section formula,

$\bar{c} = \frac{3\left(\bar{a} - 3\bar{b}\right) + 2\left(6\bar{a} + 2\bar{b}\right)}{3+2}$

$= \frac{3\bar{a} - 9\bar{b} + 12\bar{a} + 4\bar{b}}{5}$

$= \frac{15\bar{a} - 5\bar{b}}{5}$

$\therefore \quad \bar{c} = 3\bar{a} - \bar{b}$

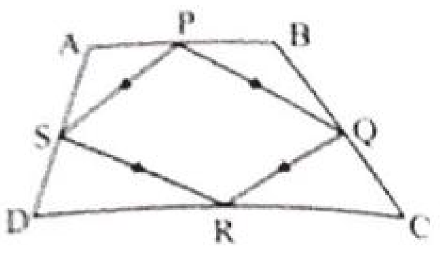
The position vector of C is $3\bar{a} - \bar{b}$.

5. Prove that the line segments joining mid-points of adjacent sides of a quadrilateral form a parallelogram.

Solution:

Let ABCD be a quadrilateral. Let P, Q, R, S be the midpoints of sides AB, BC, CD, DA respectively.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}, \bar{q}, \bar{r}, \bar{s}$ be the position vectors of A, B, C, D, P, Q, R, S respectively.



$\therefore$ By using midpoint formula,

$\bar{p} = \frac{\bar{a} + \bar{b}}{2}, \bar{q} = \frac{\bar{b} + \bar{c}}{2}, \bar{r} = \frac{\bar{c} + \bar{d}}{2}$ and $\bar{s} = \frac{\bar{d} + \bar{a}}{2} \quad \ldots(i)$

Prove that □PQRS is a parallelogram

i.e., to prove that

$\left|\overline{PQ}\right| = \left|\overline{SR}\right|$ and $\overline{PQ} \parallel \overline{SR}$,

$\left|\overline{QR}\right| = \left|\overline{PS}\right|$ and $\overline{QR} \parallel \overline{PS}$

Consider, $\overline{PQ} = \bar{q} - \bar{p}$

$= \left(\frac{\bar{b} + \bar{c}}{2}\right) - \left(\frac{\bar{a} + \bar{b}}{2}\right) \quad \ldots[\text{From (i)}]$

$= \frac{1}{2}\left(\bar{b} + \bar{c} - \bar{a} - \bar{b}\right)$

$\therefore \quad \overline{PQ} = \frac{1}{2}\left(\bar{c} - \bar{a}\right) \quad \ldots(ii)$

$\overline{SR} = \bar{r} - \bar{s}$

$= \left(\frac{\bar{c} + \bar{d}}{2}\right) - \left(\frac{\bar{d} + \bar{a}}{2}\right)$

$= \frac{1}{2}\left(\bar{c} + \bar{d} - \bar{d} - \bar{a}\right)$

$\therefore \quad \overline{SR} = \frac{1}{2}\left(\bar{c} - \bar{a}\right) \quad \ldots(iii)$

$\therefore \quad \overline{PQ} = \overline{SR} \quad \ldots[\text{From (ii) and (iii)}]$

$\therefore \quad \left|\overline{PQ}\right| = \left|\overline{SR}\right|$ and $\overline{PQ} \parallel \overline{SR}$

Consider, $\overline{QR} = \bar{r} - \bar{q}$

$= \frac{\bar{c} + \bar{d}}{2} - \frac{\bar{b} + \bar{c}}{2} \quad \ldots[\text{From (i)}]$

$= \frac{1}{2}\left(\bar{c} + \bar{d} - \bar{b} - \bar{c}\right)$

$\therefore \quad \overline{QR} = \frac{1}{2}\left(\bar{d} - \bar{b}\right) \quad \ldots(iv)$

$\overline{PS} = \bar{s} - \bar{p}$

$= \frac{\bar{d} + \bar{a}}{2} - \frac{\bar{a} + \bar{b}}{2} \quad \ldots[\text{From (i)}]$

$= \frac{1}{2}\left(\bar{d} + \bar{a} - \bar{a} - \bar{b}\right)$

$\overline{PS} = \frac{1}{2}\left(\bar{d} - \bar{b}\right) \quad \ldots(v)$

$\therefore \quad \overline{QR} = \overline{PS} \quad \ldots[\text{From (iv) and (v)}]$

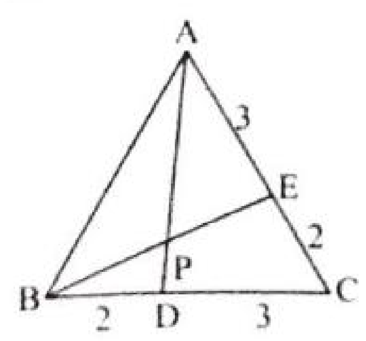
$\therefore \quad \left|\overline{QR}\right| = \left|\overline{PS}\right|$ and $\overline{QR} \parallel \overline{PS}$

$\therefore$ Opposite sides of □PQRS are congruent and parallel.

$\therefore$ The line segments joining the midpoints of adjacent sides of a quadrilateral form a parallelogram.

6. D and E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.

Solution:



Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}$ be the position vectors of points A, B, C, D, E respectively.

D divides BC in the ratio 2 : 3.

$\therefore \quad \bar{d} = \frac{2\bar{c} + 3\bar{b}}{2+3} = \frac{2\bar{c} + 3\bar{b}}{5}$

E divides CA in the ratio 2 : 3.

$\therefore \quad \bar{e} = \frac{2\bar{a} + 3\bar{c}}{2+3} = \frac{2\bar{a} + 3\bar{c}}{5}$

Let the point of intersection $P(\bar{p})$ of AD and BE divides BE in the ratio k : 1 and AD in the ratio m : 1.

Then, the position vectors of P in these cases are

$\bar{p} = \frac{k\bar{e} + \bar{b}}{k+1}$ and $\bar{p} = \frac{m\bar{d} + \bar{a}}{m+1}$ respectively.

Substituting the values of $\bar{d}$ and $\bar{e}$, we get

$\bar{p} = \frac{k\left(\frac{2\bar{a} + 3\bar{c}}{5}\right) + \bar{b}}{k+1}$ and $\bar{p} = \frac{m\left(\frac{2\bar{c} + 3\bar{b}}{5}\right) + \bar{a}}{m+1}$

$\therefore \quad \bar{p} = \frac{2k}{5(k+1)}\bar{a} + \frac{1}{(k+1)}\bar{b} + \frac{3k}{5(k+1)}\bar{c} \quad \ldots(i)$

and $\bar{p} = \frac{1}{(m+1)}\bar{a} + \frac{3m}{5(m+1)}\bar{b} + \frac{2m}{5(m+1)}\bar{c}$

$\therefore \quad \frac{2k}{5(k+1)}\bar{a} + \frac{1}{(k+1)}\bar{b} + \frac{3k}{5(k+1)}\bar{c}$

$= \frac{1}{(m+1)}\bar{a} + \frac{3m}{5(m+1)}\bar{b} + \frac{2m}{5(m+1)}\bar{c}$

Equating the position vectors of P, we get

$\frac{2k}{5(k+1)} = \frac{1}{m+1} \quad \ldots(ii)$

$\frac{1}{(k+1)} = \frac{3m}{5(m+1)} \quad \ldots(iii)$

$\frac{3k}{5(k+1)} = \frac{2m}{5(m+1)} \quad \ldots(iv)$

Dividing (iv) by (iii), we get

$\frac{3k}{5(k+1)} \times \frac{k+1}{1} = \frac{2m}{5(m+1)} \times \frac{5(m+1)}{3m}$

$\therefore \quad \frac{3k}{5} = \frac{2}{3}$

$\therefore \quad k = \frac{10}{9} \quad \ldots(v)$

Dividing (iii) by (ii), we get

$\frac{1}{k+1} \times \frac{5(k+1)}{2k} = \frac{3m}{5(m+1)} \times \frac{m+1}{1}$

$\therefore \quad \frac{5}{2k} = \frac{3m}{5}$

$\therefore \quad \frac{5}{2 \times \frac{10}{9}} = \frac{3m}{5} \quad \ldots[\text{From (v)}]$

$\therefore \quad m = \frac{15}{4}$

Substituting $k = \frac{10}{9}$ in (i), we get

$\bar{p} = \frac{2\left(\frac{10}{9}\right)}{5\left(\frac{10}{9} + 1\right)}\bar{a} + \frac{1}{\left(\frac{10}{9} + 1\right)}\bar{b} + \frac{3\left(\frac{10}{9}\right)}{5\left(\frac{10}{9} + 1\right)}\bar{c}$

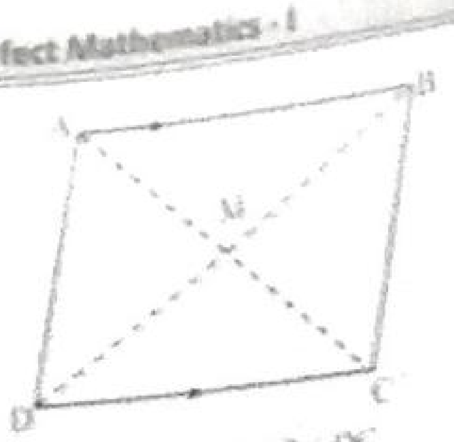
$= \frac{4}{19}\bar{a} + \frac{9}{19}\bar{b} + \frac{6}{19}\bar{c}$

$\therefore$ The position vector of the point of intersection of AD and BE is $\frac{4}{19}\bar{a} + \frac{9}{19}\bar{b} + \frac{6}{19}\bar{c}$ and the ratio in which this point divides AD and BE are 15: 4 and 10 : 9 respectively.

7. Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Solution:

Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ be the position vectors of the vertices A, B, C and D respectively of the parallelogram ABCD.



Then, $l(AB) = l(DC)$ and $AB \parallel DC$

...[Opposite sides of a parallelogram]

$\therefore \overline{AB} = \overline{DC}$

$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d}$

$\therefore \bar{a} + \bar{c} = \bar{b} + \bar{d}$

$\therefore \dfrac{\bar{a}+\bar{c}}{2} = \dfrac{\bar{b}+\bar{d}}{2}$

The position vectors of the midpoints of the diagonals AC and BD are $\dfrac{\bar{a}+\bar{c}}{2}$ and $\dfrac{\bar{b}+\bar{d}}{2}$ and they are equal.

The midpoints of the diagonals AC and BD are the same.

This shows that the diagonals AC and BD bisect each other.

Conversely, suppose that the diagonals AC and BD of □ABCD bisect each other.

They have the same midpoint.

The position vectors of the midpoints of AC and BD are equal.

$\therefore \dfrac{\bar{a}+\bar{c}}{2} = \dfrac{\bar{b}+\bar{d}}{2}$

$\therefore \bar{a} + \bar{c} = \bar{b} + \bar{d}$

$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d}$

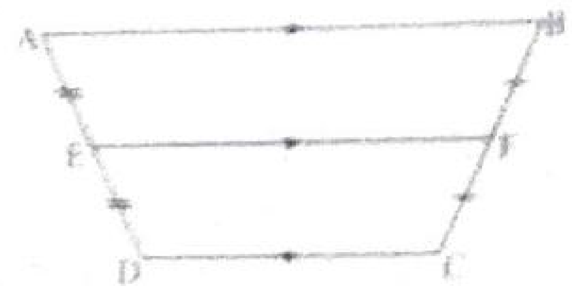
$\therefore \overline{AB} = \overline{DC}$

$\therefore l(AB) = l(DC)$ and $AB \parallel DC$

$\therefore$ □ABCD is a parallelogram.

8. Prove that the median of a trapezium is parallel to the parallel sides of the trapezium and its length is half the sum of parallel sides.

Solution:



Let ABCD be a trapezium with side AB parallel to side DC.

Let E and F be the midpoints of sides AD and BC respectively.

$\therefore$ EF is the median of trapezium ABCD.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be the position vectors of points A, B, C, D, E, F respectively.

Prove that EF is parallel to AB and DC.

Since side AB is parallel to side DC,

$\overline{AB}$ is parallel to $\overline{DC}$

There exists a scalar t such that

$\overline{AB} = t\,\overline{DC}$ ...(i)

E $(\bar{e})$ and F $(\bar{f})$ are the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively.

$\bar{e} = \dfrac{\bar{a}+\bar{d}}{2}$ and $\bar{f} = \dfrac{\bar{b}+\bar{c}}{2}$

Now, consider $\overline{EF} = \bar{f} - \bar{e}$

$= \dfrac{\bar{b}+\bar{c}}{2} - \dfrac{\bar{a}+\bar{d}}{2}$

$= \dfrac{\bar{b}-\bar{a}}{2} + \dfrac{\bar{c}-\bar{d}}{2}$

$= \dfrac{\overline{AB}}{2} + \dfrac{\overline{DC}}{2}$ ...(ii)

$= \dfrac{t\,\overline{DC}}{2} + \dfrac{\overline{DC}}{2}$ ...[From (i)]

$\overline{EF} = \left(\dfrac{t+1}{2}\right)\overline{DC}$, where $\left(\dfrac{t+1}{2}\right)$ is a scalar.

$\therefore \overline{EF}$ is a scalar multiple of $\overline{DC}$

$\therefore \overline{EF}$ is parallel to $\overline{DC}$

$\therefore \overline{EF}$ is parallel to $\overline{AB}$ as $\overline{AB}$ and $\overline{DC}$ are parallel to each other.

Also from (ii) we get,

$\overline{EF} = \dfrac{1}{2}\left(\overline{AB} + \overline{DC}\right)$

$\therefore \left|\overline{EF}\right| = \dfrac{1}{2}\left|\overline{AB} + \overline{DC}\right|$

$\therefore l(EF) = \dfrac{1}{2}\left|\overline{AB} + \overline{DC}\right|$

9. If two of the vertices of the triangle are A(3, 1, 4) and B(−4, 5, −3) and the centroid of a triangle is G(−1, 2, 1), then find the coordinates of the third vertex C of the triangle.

Solution:

Let $\bar{a}, \bar{b}, \bar{c}, \bar{g}$ be the position vectors of points A, B, C, G respectively.

$\therefore \bar{a} = 3\hat{i} + \hat{j} + 4\hat{k}$, $\bar{b} = -4\hat{i} + 5\hat{j} - 3\hat{k}$,

$\bar{g} = -\hat{i} + 2\hat{j} + \hat{k}$.

Given, G is the centroid of ΔABC.

By using centroid formula,

$\bar{g} = \dfrac{\bar{a}+\bar{b}+\bar{c}}{3}$

$\therefore 3\bar{g} = \bar{a} + \bar{b} + \bar{c}$

$\therefore \bar{c} = 3\bar{g} - \bar{a} - \bar{b}$

$\bar{c} = 3\left(-\hat{i}+2\hat{j}+\hat{k}\right) - \left(3\hat{i}+\hat{j}+4\hat{k}\right) - \left(-4\hat{i}+5\hat{j}-3\hat{k}\right)$

$= -3\hat{i} + 6\hat{j} + 3\hat{k} - 3\hat{i} - \hat{j} - 4\hat{k} + 4\hat{i} - 5\hat{j} + 3\hat{k}$

$\therefore \bar{c} = -2\hat{i} + 0\hat{j} + 2\hat{k}$

$\therefore$ The co-ordinates of the third vertex C are (−2, 0, 2).

Smart Check

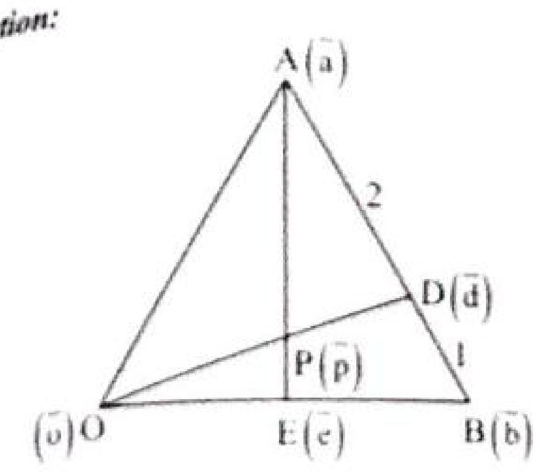
Centroid $= \left(\dfrac{3-4-2}{3}, \dfrac{1+5+0}{3}, \dfrac{4-3+2}{3}\right)$

$= (-1, 2, 1)$

Thus, our answer is correct.

10. In ΔOAB, E is the mid-point of OB and D is the point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.

Solution:



Let $\bar{o}, \bar{a}, \bar{b}, \bar{d}, \bar{e}, \bar{p}$ be the position vectors of points O, A, B, D, E, P respectively.

E is the midpoint of OB.

$\therefore \bar{e} = \dfrac{\bar{o}+\bar{b}}{2}$

D divides AB in the ratio 2:1.

$\therefore \bar{d} = \dfrac{2\bar{b}+\bar{a}}{2+1} = \dfrac{2\bar{b}+\bar{a}}{3}$

Let the point of intersection $P(\bar{p})$ of AE and OD divides OD in the ratio k : 1 and AE in the ratio m : 1.

Then, the position vectors of P in these cases are

$\bar{p} = \dfrac{k\bar{d}+\bar{o}}{k+1}$ and $\bar{p} = \dfrac{m\bar{e}+\bar{a}}{m+1}$ respectively.

Substituting the values of $\bar{d}$ and $\bar{e}$, we get

$\bar{p} = \dfrac{k\left(\dfrac{2\bar{b}+\bar{a}}{3}\right)+\bar{o}}{k+1}$ and $\bar{p} = \dfrac{m\left(\dfrac{\bar{o}+\bar{b}}{2}\right)+\bar{a}}{m+1}$

$\therefore \bar{p} = \dfrac{k}{3(k+1)}\bar{a} + \dfrac{2k}{3(k+1)}\bar{b} + \dfrac{1}{(k+1)}\bar{o}$

and $\bar{p} = \dfrac{1}{(m+1)}\bar{a} + \dfrac{m}{2(m+1)}\bar{b} + \dfrac{m}{2(m+1)}\bar{o}$

$\therefore \dfrac{k}{3(k+1)}\bar{a} + \dfrac{2k}{3(k+1)}\bar{b} + \dfrac{1}{(k+1)}\bar{o}$

$= \dfrac{1}{(m+1)}\bar{a} + \dfrac{m}{2(m+1)}\bar{b} + \dfrac{m}{2(m+1)}\bar{o}$

Equating the position vectors of P, we get

$\dfrac{k}{3(k+1)} = \dfrac{1}{(m+1)}$ ...(i)

$\dfrac{2k}{3(k+1)} = \dfrac{m}{2(m+1)}$ ...(ii)

$\dfrac{1}{(k+1)} = \dfrac{m}{2(m+1)}$ ...(iii)

Dividing (ii) by (iii), we get

$\dfrac{2k}{3(k+1)} \times \dfrac{(k+1)}{1} = \dfrac{m}{2(m+1)} \times \dfrac{2(m+1)}{m}$

$\therefore \dfrac{2k}{3} = 1$

$\therefore k = \dfrac{3}{2}$

$\therefore$ OP : PD = 3 : 2

11. If the centroid of a tetrahedron OABC is (1, 2, −1), where A = (a, 2, 3), B = (1, b, 2), C = (2, 1, c) respectively, find the distance of P (a, b, c) from the origin.

Solution:

Let G be the centroid of the tetrahedron OABC.

Let $\bar{o}, \bar{a}, \bar{b}, \bar{c}$ and $\bar{g}$ be the position vectors of points O, A, B, C and G respectively.

$\therefore \bar{o} = 0\hat{i} + 0\hat{j} + 0\hat{k}$, $\bar{a} = a\hat{i} + 2\hat{j} + 3\hat{k}$,

$\bar{b} = \hat{i} + b\hat{j} + 2\hat{k}$, $\bar{c} = 2\hat{i} + \hat{j} + c\hat{k}$, $\bar{g} = \hat{i} + 2\hat{j} - \hat{k}$

$\therefore$ By using centroid formula,

$\bar{g} = \dfrac{\bar{o}+\bar{a}+\bar{b}+\bar{c}}{4}$

$\therefore 4\bar{g} = \bar{o} + \bar{a} + \bar{b} + \bar{c}$

$\therefore 4\left(\hat{i}+2\hat{j}-\hat{k}\right) = \left(0\hat{i}+0\hat{j}+0\hat{k}\right) + \left(a\hat{i}+2\hat{j}+3\hat{k}\right) + \left(\hat{i}+b\hat{j}+2\hat{k}\right) + \left(2\hat{i}+\hat{j}+c\hat{k}\right)$

$\therefore 4\hat{i} + 8\hat{j} - 4\hat{k} = (3+a)\hat{i} + (3+b)\hat{j} + (5+c)\hat{k}$

$\therefore$ By equality of vectors, we get

$4 = 3 + a$ $\quad\therefore a = 1$

$8 = 3 + b$ $\quad\therefore b = 5$

$-4 = 5 + c$ $\quad\therefore c = -9$

$\therefore$ The distance of P (1, 5, −9) from the origin is

$OP = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{1^2 + 5^2 + (-9)^2}$

$= \sqrt{1 + 25 + 81}$

$= \sqrt{107}$ units

12. Find the centroid of tetrahedron with vertices K(5, –7, 0), L(1, 5, 3), M(4, – 6, 3), N(6, –4, 2).

Solution:

Let G be the centroid of the tetrahedron KLMN.

Let $\bar{k}$, $\bar{l}$, $\bar{m}$, $\bar{n}$ and $\bar{g}$ be the position vectors of points K, L, M, N and G respectively.

$\therefore$ $\bar{k} = 5\hat{i} - 7\hat{j} + 0\hat{k}$, $\bar{l} = \hat{i} + 5\hat{j} + 3\hat{k}$,

$\bar{m} = 4\hat{i} - 6\hat{j} + 3\hat{k}$, $\bar{n} = 6\hat{i} - 4\hat{j} + 2\hat{k}$

$\therefore$ By using centroid formula,

$$\bar{g} = \frac{\bar{k} + \bar{l} + \bar{m} + \bar{n}}{4}$$

$$= \frac{1}{4}\left[\left(5\hat{i} - 7\hat{j} + 0\hat{k}\right) + \left(\hat{i} + 5\hat{j} + 3\hat{k}\right) + \left(4\hat{i} - 6\hat{j} + 3\hat{k}\right) + \left(6\hat{i} - 4\hat{j} + 2\hat{k}\right)\right]$$

$$= \frac{16\hat{i} - 12\hat{j} + 8\hat{k}}{4}$$

$\therefore$ $\bar{g} = 4\hat{i} - 3\hat{j} + 2\hat{k}$

$\therefore$ The centroid of the tetrahedron is (4, –3, 2).

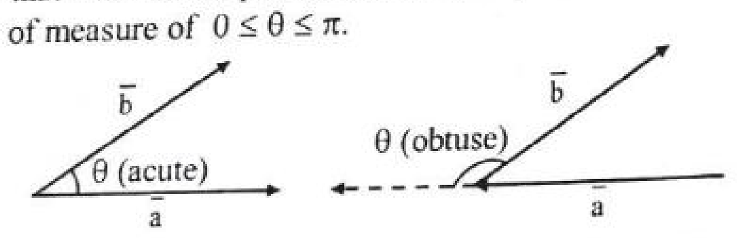
Let's Study

Product of vectors

The product of two vectors is defined in two different ways. One form of product results in a scalar quantity while other form gives a vector quantity. Let us study these products and interpret them geometrically.

Angle between two vectors:

When two non- zero vectors $\bar{a}$ and $\bar{b}$ are placed such that their initial points coincide, they form an angle θ of measure of $0 \le \theta \le \pi$.



The angle between the collinear vectors is 0 if they point in the same direction ant π if they are in opposite directions.

Scalar product of two vectors:

The scalar product of two non-zero vectors $\bar{a}$ and $\bar{b}$ is denoted by $\bar{a} \cdot \bar{b}$, and is defined as $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos \theta$, where θ is the angle between $\bar{a}$ and $\bar{b}$.

$\bar{a} \cdot \bar{b}$ is a real number, that is a scalar.

Thus, the scalar product is also called the dot product.

Note:

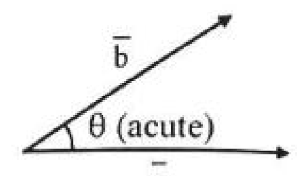
1. If either $\bar{a} = 0$ or $\bar{b} = 0$ then θ is not defined and in this case, we define $\bar{a} \cdot \bar{b} = 0$.
2. If $\theta = 0$, then $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos 0 = |\bar{a}||\bar{b}|$.
   In particular, $\bar{a} \cdot \bar{a} = |\bar{a}|^2$ as $\theta = 0$.
3. If $\theta = \pi$, then $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos \pi = -|\bar{a}||\bar{b}|$.
4. If $\bar{a}$ and $\bar{b}$ are perpendicular or orthogonal, then $\theta = \pi/2$.
   Conversely if $\bar{a} \cdot \bar{b} = 0$ then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ or $\theta = \pi/2$.
   Also, $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos \theta = |\bar{b}||\bar{a}| \cos \theta = \bar{b} \cdot \bar{a}$.
5. Dot product is distributive over vector addition. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are any three vectors, then $\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$.
6. If $\bar{a}$ and $\bar{b}$ are vectors and m, n are scalars, then
   i. $(m\bar{a}) \cdot (n\bar{b}) = mn(\bar{a} \cdot \bar{b})$
   ii. $(m\bar{a}) \cdot \bar{b} = m(\bar{a} \cdot \bar{b}) = \bar{a} \cdot (m\bar{b})$
7. $|\bar{a} \cdot \bar{b}| \le |\bar{a}||\bar{b}|$ (This is known as Cauchy Schwartz Inequality.)

Finding angle between two vectors:

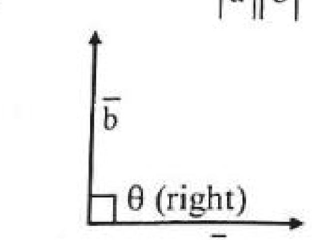
Angle θ, $(0 \le \theta \le \pi)$ between two non-zero vectors $\bar{a}$ and $\bar{b}$ is given by $\cos \theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|}$, that is $\theta = \cos^{-1}\left(\frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|}\right)$.

Note:

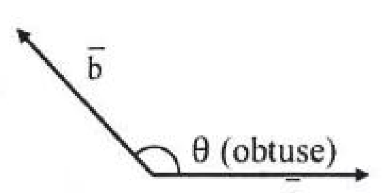
1. If $0 \le \theta < \frac{\pi}{2}$, then $\cos \theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} > 0$, that is $\bar{a} \cdot \bar{b} > 0$.



2. If $\theta = \frac{\pi}{2}$, then $\cos \theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} = 0$, that is $\bar{a} \cdot \bar{b} = 0$.



3. If $\frac{\pi}{2} < \theta \le \pi$, then $\cos \theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} < 0$, that is $\bar{a} \cdot \bar{b} < 0$.



8. In particular scalar product of $\hat{i}$, $\hat{j}$, $\hat{k}$ vectors are
   i. $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and
   ii. $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.
9. The scalar product of vectors $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\bar{a} \cdot \bar{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \cdot \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$
$$\cos \theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Projections:

$\overline{PQ}$ and $\overline{PR}$ represent the vectors $\bar{a}$ and $\bar{b}$ with same initial point P. If M is the foot of perpendicular from R to the line containing $\overline{PQ}$, then $\overline{PM}$ is called the scalar projection of $\bar{b}$ on $\bar{a}$. We can think of it as a shadow of $\bar{b}$ on $\bar{a}$, when sun is overhead.

Scalar Projection of $\bar{b}$ on $\bar{a} = \overline{PM} = |\bar{b}| \cos \theta$

$$= \frac{|\bar{a}||\bar{b}| \cos \theta}{|\bar{a}|}$$

Scalar Projection of $\bar{b}$ on $\bar{a} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|}$

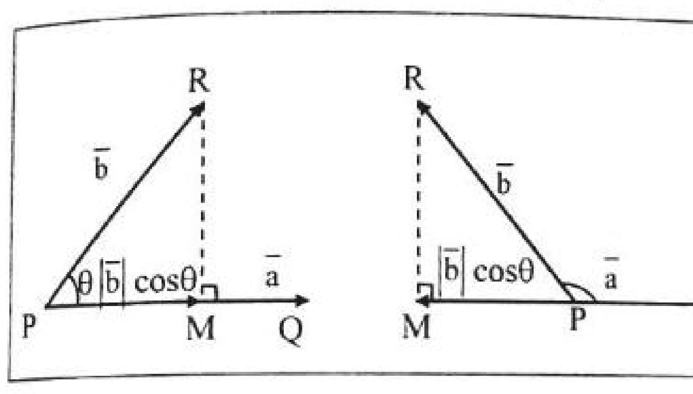
Vector Projection of $\bar{b}$ on $\bar{a} = \overline{PM}$

$= \overline{PM}\,\hat{a}$, where $\hat{a}$ is unit vector along $\bar{a}$

$$= \left(\frac{\bar{a} \cdot \bar{b}}{|\bar{a}|}\right)\hat{a}$$

$$= \left(\frac{\bar{a} \cdot \bar{b}}{|\bar{a}|}\right)\frac{\bar{a}}{|\bar{a}|}$$

Vector Projection of $\bar{b}$ on $\bar{a} = (\bar{a} \cdot \bar{b})\frac{\bar{a}}{|\bar{a}|^2}$



Direction Angles and Direction Cosines:

The direction angles of a non-zero vector $\bar{a}$ are angles α, β and γ $(\in [0, \pi])$ that $\bar{a}$ makes with the positive X, Y and Z axes respectively. These angles completely determine the direction of the vector $\bar{a}$.

The cosines of these direction angles, that is $\cos \alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines (d.c.s) of vector $\bar{a}$.

If α is the angle between $\hat{i}$ (unit vector) along X-axis and $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then

$$\cos \alpha = \frac{\bar{a} \cdot \hat{i}}{|\bar{a}||\hat{i}|} = \frac{\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \cdot \hat{i}}{|\bar{a}|(1)} = \frac{a_1}{|\bar{a}|},$$

Similarly $\cos \beta = \frac{a_2}{|\bar{a}|}$ and $\cos \gamma = \frac{a_3}{|\bar{a}|}$,

where $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

By squaring and adding, we get $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{a_1^2}{|\bar{a}|^2} + \frac{a_2^2}{|\bar{a}|^2} + \frac{a_3^2}{|\bar{a}|^2}.$$

As $|\bar{a}|^2 = a_1^2 + a_2^2 + a_3^2$

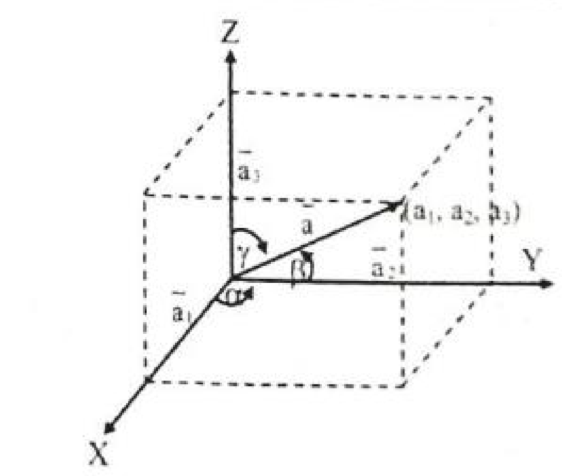
$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Also, $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$= |\bar{a}| \cos \alpha\,\hat{i} + |\bar{a}| \cos \beta\,\hat{j} + |\bar{a}| \cos \gamma\,\hat{k}$$
$$= |\bar{a}|(\cos \alpha\,\hat{i} + \cos \beta\,\hat{j} + \cos \gamma\,\hat{k})$$

$\therefore$ $\frac{\bar{a}}{|\bar{a}|} = \cos \alpha\,\hat{i} + \cos \beta\,\hat{j} + \cos \gamma\,\hat{k} = \hat{a}$,

which means that the direction cosines of $\bar{a}$, are components of the unit vector in the direction of $\bar{a}$.



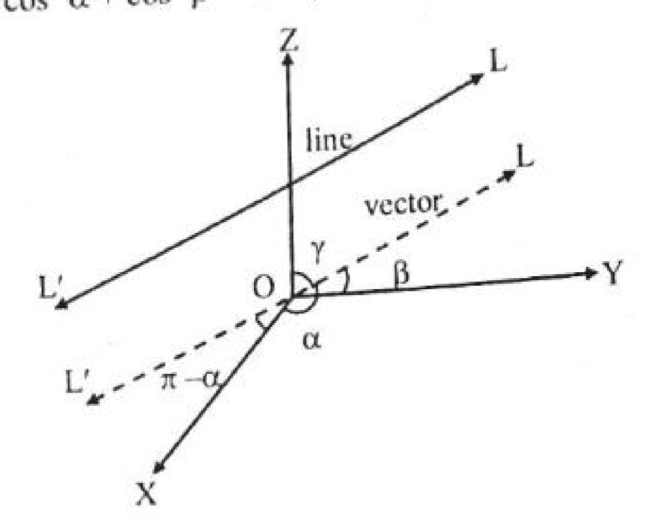
Direction cosines (d.c.s) of any line along a vector $\bar{a}$ has same direction cosines as that of $\bar{a}$.

Direction cosines are generally denoted by l, m, n, where $l = \cos \alpha$, m = $\cos \beta$, n = $\cos \gamma$.

As the unit vectors along X, Y and Z axes are $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively. Then $\hat{i}$ makes the direction angles 0, $\frac{\pi}{2}$, $\frac{\pi}{2}$. So its direction cosines are cos 0, $\cos \frac{\pi}{2}$, $\cos \frac{\pi}{2}$ i.e. 1, 0, 0.

Similarly direction cosines of Y- and Z-axes are 0, 1, 0 and 0, 0, 1 respectively.

Let $\overline{OL}$ and $\overline{OL'}$ be the vectors in the direction of line LL′. If α, β and γ are direction angles of $\overline{OL}$, then the direction angles of $\overline{OL'}$ are $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$. Therefore, direction cosines of $\overline{OL}$ are cos α, cos β, cos γ i.e. l, m, n whereas direction cosines of $\overline{OL'}$ are $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$ and $\cos(\pi - \gamma)$.
i.e. $-\cos\alpha$, $-\cos\beta$ and $-\cos\gamma$. i.e. $-l$, $-m$, $-n$.
Therefore direction cosines of line LL′ are same as that of vectors $\overline{OL}$ or $\overline{OL'}$ in the direction of line LL′.
i.e. either l, m, n or $-l$, $-m$, $-n$.
As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, so $l^2 + m^2 + n^2 = 1$.



Direction ratios:
Any 3 real numbers which are proportional to direction cosines of the line are called the direction ratios (d.r.s) of the line. Generally the direction ratios are denoted by a, b, c.
If a, b, c are real numbers and l, m, n are the direction cosines of a line such that

$\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \lambda$ (a constant),

then the direction ratios are $a = \lambda l$, $b = \lambda m$, $c = \lambda n$, for some $\lambda \in R$.

Example:
If a line makes angles 45°, 60°, 60° with positive directions of X-axis, Y-axis, Z-axis respectively, then direction cosines of the line will be

$l = \cos 45° = \frac{1}{\sqrt{2}}$

$m = \cos 60° = \frac{1}{2}$

and $n = \cos 60° = \frac{1}{2}$

And direction ratios would be any real numbers proportional to the above direction cosines.

$\therefore$ 1, $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$ or $\sqrt{2}$, 1, 1 etc. will be direction ratios of the given line.

Note:
A line has infinitely many direction ratios but unique direction cosines.

Relation between direction ratios and direction cosines:
Let a, b, c be direction ratios and l, m, n be direction cosines of a line.
By definition of d.r.s.,

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$

i.e., $l = \lambda a$, $m = \lambda b$, $n = \lambda c$ …(i)

But $l^2 + m^2 + n^2 = 1$

$\therefore$ $(\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2 = 1$

$\therefore$ $\lambda^2(a^2 + b^2 + c^2) = 1$

$\therefore$ $\lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

From (i), we get

$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$ and

$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$

Note: The direction cosines are similar to the definition of unit vectors, that is if $\bar{x} = a\hat{i} + b\hat{j} + c\hat{k}$ be any vector (d.r.s), then $\hat{x} = \pm\frac{\bar{x}}{|\bar{x}|} = \pm\frac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2 + b^2 + c^2}}$ is unit vector (d.c.s) along $\bar{x}$.

Exercise 5.3

1. Find two unit vectors each of which is perpendicular to both $\bar{u}$ and $\bar{v}$, where $\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution:

Let $\bar{w} = a\hat{i} + b\hat{j} + c\hat{k}$ be perpendicular to both $\bar{u}$ and $\bar{v}$.

Then, $\bar{w} \cdot \bar{u} = 0$

$\therefore$ $(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 0$

$\therefore$ $2a + b - 2c = 0$ …(i)

and $\bar{w} \cdot \bar{v} = 0$

$\therefore$ $(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 0$

$\therefore$ $a + 2b - 2c = 0$ …(ii)

Solving (i) and (ii),

$$\frac{a}{\begin{vmatrix} 1 & -2 \\ 2 & -2 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 2 & -2 \\ 1 & -2 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$\therefore$ $\frac{a}{2} = \frac{b}{2} = \frac{c}{3}$

Let $\frac{a}{2} = \frac{b}{2} = \frac{c}{3} = \lambda$

$\therefore$ $a = 2\lambda$, $b = 2\lambda$, $c = 3\lambda$

Since $\bar{w}$ is a unit vector

$|\bar{w}| = 1$

$\therefore$ $a^2 + b^2 + c^2 = 1$

$\therefore$ $4\lambda^2 + 4\lambda^2 + 9\lambda^2 = 1$

$\therefore$ $17\lambda^2 = 1$

$\therefore$ $\lambda^2 = \pm\frac{1}{\sqrt{17}}$

$\therefore$ $a = 2\lambda = \pm\frac{2}{\sqrt{17}}$, $b = 2\lambda = \pm\frac{2}{\sqrt{17}}$,

$\therefore$ $c = 3\lambda = \pm\frac{3}{\sqrt{17}}$

Hence, the component of $\bar{w}$ are

$\pm\frac{2i}{\sqrt{17}} \pm \frac{2j}{\sqrt{17}} \pm \frac{3k}{\sqrt{17}}$ i.e., $\pm\frac{1}{\sqrt{17}}(2\hat{i} + 2\hat{j} + 3\hat{k})$

Alternate method:

$\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$

Let $\bar{w}$ be unit vector perpendicular to two given vectors i.e., $\bar{u}$ and $\bar{v}$

$\bar{w} = \bar{u} \times \bar{v}$

$\therefore$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

$= \hat{i}(-2 + 4) - \hat{j}(-4 + 2) + \hat{k}(4 - 1)$

$\bar{w} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

Required vector is $2\hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore$ Unit vector $\hat{w} = \pm\frac{\text{Vector}}{\text{Magnitude}}$

$= \pm\frac{2i + 2j + 3k}{\sqrt{2^2 + 2^2 + 3^2}}$

$= \pm\frac{2\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{4 + 4 + 9}}$

$= \pm\frac{2\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{17}}$

Thus, $\pm\frac{2\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{17}}$ are unit vectors perpendicular to two given vectors.

2. If $\bar{a}$ and $\bar{b}$ are two vectors perpendicular to each other, prove that $(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$.

Solution:

$\bar{a}$ is perpendicular to $\bar{b}$.

$\therefore$ $\bar{a} \cdot \bar{b} = 0$

$(\bar{a} + \bar{b})^2 = (\bar{a})^2 + 2\bar{a} \cdot \bar{b} + (\bar{b})^2$

$= (\bar{a})^2 + 2(0) + (\bar{b})^2$

$= (\bar{a})^2 + (\bar{b})^2$ …(i)

$(\bar{a} - \bar{b})^2 = (\bar{a})^2 - 2\bar{a} \cdot \bar{b} + (\bar{b})^2$

$= (\bar{a})^2 - 2(0) + (\bar{b})^2$

$= (\bar{a})^2 + (\bar{b})^2$ …(ii)

From (i) and (ii), we get

$(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$

3. Find the values of c so that for all real x the vectors $xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle.

Solution:

Let $\bar{a} = xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $\bar{b} = x\hat{i} + 2\hat{j} + 2cx\hat{k}$

Vector $\bar{a}$ and $\bar{b}$ make an obtuse angle.

i.e., $\cos\theta < 0$

But $\cos\theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}| \cdot |\bar{b}|}$

$\therefore$ $\frac{\bar{a} \cdot \bar{b}}{|\bar{a}| \cdot |\bar{b}|} < 0$

$\bar{a} \cdot \bar{b} < 0$

$\therefore$ $x^2c + 6xc - 12 < 0$

$\because$ It is quadratic equation.

$ax^2 + bx + c < 0$ For all x

$a < 0$ and $\Delta < 0$ …(i)

$\therefore$ $a = c$, $\Delta = b^2 - 4ac$

$= (6c)^2 - 4(c)(-12)$

$= 36c^2 + 48c$

$\therefore$ From (i)

$c < 0$ and $36c^2 + 48c < 0$

$c < 0$ and $c(3c + 4) < 0$

$\therefore$ $\frac{-4}{3} < c < 0$

4. Show that the sum of the length of projections of $p\hat{i} + q\hat{j} + r\hat{k}$ on the coordinate axes, where p = 2, q = 3 and r = 4, is 9.

Solution:

Let $\bar{a} = p\hat{i} + q\hat{j} + r\hat{k}$

$\therefore$ $p = 2$, $q = 3$, $r = 4$

$\therefore$ $\bar{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

Projection of $\bar{a}$ on X-axis is

$= \frac{\bar{a} \cdot \hat{i}}{|\hat{i}|} = \frac{(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot \hat{i}}{1} = 2$

Projection of $\bar{a}$ on Y-axis is

$= \frac{\bar{a} \cdot \hat{j}}{|\hat{j}|} = \frac{(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot \hat{j}}{1} = 3$

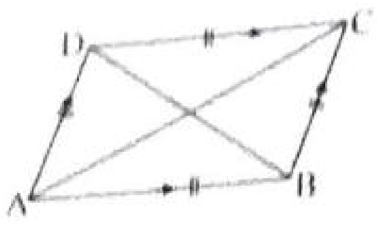
Projection of $\bar{a}$ on Z axis is

$$\frac{\bar{a}\cdot\hat{k}}{|\hat{k}|} = \frac{(2\hat{i}+3\hat{j}+4\hat{k})\cdot\hat{k}}{1} = 4$$

∴ Sum of all projections on coordinate axes is $2 + 3 + 4 = 9$

5. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

Solution:



Let ABCD be quadrilateral having equal length and opposite sides parallel.

$|\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{AD}|$ ...[Given]

$AB \parallel DC$ and $AD \parallel BC$

AC and BD are diagonals of quadrilateral ABCD.

∴ $\overline{AC}\cdot\overline{BD} = (\overline{AB}+\overline{BC})\cdot(\overline{BC}+\overline{CD})$

...(Triangle Law of Vector addition)

$= (\overline{AB}+\overline{BC})\cdot(\overline{BC}-\overline{AB})$ ...$(\overline{CD} = -\overline{AB})$

$= \overline{AB}\cdot\overline{BC} - \overline{AB}\cdot\overline{AB} + \overline{BC}\cdot\overline{BC} - \overline{BC}\cdot\overline{AB}$

$= \overline{AB}\cdot\overline{AB} - 0 + 0 - \overline{BC}\cdot\overline{BC}$ ...$(\overline{AB} = \overline{BC})$

$= |\overline{AB}|^2 - |\overline{BC}|^2$

$\overline{AC}\cdot\overline{BD} = 0$ ...$(\overline{AB} = \overline{BC})$

∴ $\overline{AC}$ is perpendicular to $\overline{BD}$.

6. Determine whether $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neither.

i. $\bar{a} = -9\hat{i}+6\hat{j}+15\hat{k},\ \bar{b} = 6\hat{i}-4\hat{j}-10\hat{k}$

ii. $\bar{a} = 2\hat{i}+3\hat{j}-\hat{k},\ \bar{b} = 5\hat{i}-2\hat{j}+4\hat{k}$

iii. $\bar{a} = -\frac{3}{5}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{3}\hat{k},\ \bar{b} = 5\hat{i}+4\hat{j}+3\hat{k}$

iv. $\bar{a} = 4\hat{i}-\hat{j}+6\hat{k},\ \bar{b} = 5\hat{i}-2\hat{j}+4\hat{k}$

Solution:

i. $\bar{a} = -9\hat{i}+6\hat{j}+15\hat{k},\ \bar{b} = 6\hat{i}-4\hat{j}-10\hat{k}$

$\bar{a}\cdot\bar{b} = (-9\hat{i}+6\hat{j}+15\hat{k})\cdot(6\hat{i}-4\hat{j}-10\hat{k})$

$= (-9)(6) + 6(-4) + 15(-10)$

$= -54 - 24 - 150$

$= -228$

Since $\bar{a}\cdot\bar{b} \neq 0$,

$\bar{a}$ and $\bar{b}$ are not orthogonal.

Now, $\frac{a_1}{b_1} = \frac{-9}{6} = -\frac{3}{2}, \frac{a_2}{b_2} = \frac{6}{-4} = -\frac{3}{2}, \frac{a_3}{b_3} = \frac{15}{-10} = -\frac{3}{2}$

Since $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$,

$\bar{a}$ and $\bar{b}$ are parallel.

ii. $\bar{a} = 2\hat{i}+3\hat{j}-\hat{k},\ \bar{b} = 5\hat{i}-2\hat{j}+4\hat{k}$

$\bar{a}\cdot\bar{b} = (2\hat{i}+3\hat{j}-\hat{k})\cdot(5\hat{i}-2\hat{j}+4\hat{k})$

$= 10 - 6 - 4 = 0$

∴ $\bar{a}\cdot\bar{b} = 0$

∴ $\bar{a}$ and $\bar{b}$ are orthogonal.

iii. $\bar{a} = -\frac{3}{5}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{3}\hat{k},\ \bar{b} = 5\hat{i}+4\hat{j}+3\hat{k}$

$\bar{a}\cdot\bar{b} = \left(\frac{-3}{5}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{3}\hat{k}\right)\cdot(5\hat{i}+4\hat{j}+3\hat{k})$

$= \frac{-3}{5}(5)+\frac{1}{2}(4)+\frac{1}{3}(3)$

$= -3 + 2 + 1$

∴ $\bar{a}\cdot\bar{b} = 0$

∴ $\bar{a}$ and $\bar{b}$ are orthogonal.

iv. $\bar{a} = 4\hat{i}-\hat{j}+6\hat{k},\ \bar{b} = 5\hat{i}-2\hat{j}+4\hat{k}$

$\bar{a}\cdot\bar{b} = (4\hat{i}-\hat{j}+6\hat{k})\cdot(5\hat{i}-2\hat{j}+4\hat{k})$

$= (4)(5) + (-1)(-2) + 6(4)$

$= 20 + 2 + 24$

$= 46$

∴ $\bar{a}\cdot\bar{b} \neq 0$

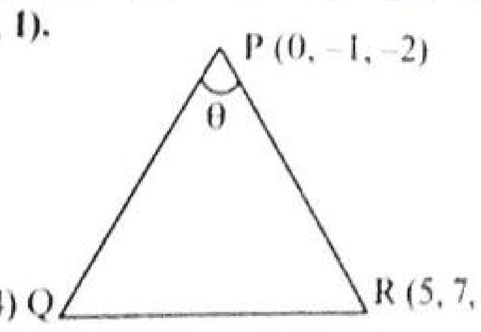
Now, $\frac{a_1}{b_1} = \frac{4}{5}, \frac{a_2}{b_2} = \frac{-1}{-2} = \frac{1}{2}, \frac{a_3}{b_3} = \frac{6}{4} = \frac{3}{2}$

Since $\bar{a}\cdot\bar{b} \neq 0$ and $\frac{a_1}{b_1} \neq \frac{a_2}{b_2} \neq \frac{a_3}{b_3}$,

$\bar{a}$ and $\bar{b}$ is neither orthogonal nor parallel.

7. Find the angle P of the triangle whose vertices are P(0, −1, −2), Q(3, 1, 4) and R(5, 7, 1).

Solution:



Let $\bar{p}$, $\bar{q}$ and $\bar{r}$ be position Vectors of P, Q and R respectively.

$\bar{p} = 0\hat{i} - \hat{j} - 2\hat{k}$

$\bar{q} = 3\hat{i} + \hat{j} + 4\hat{k}$

$\bar{r} = 5\hat{i} + 7\hat{j} + \hat{k}$

$\overline{PR} = \bar{r} - \bar{p} = 5\hat{i} + 8\hat{j} + 3\hat{k}$

$\overline{PQ} = \bar{q} - \bar{p} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$$\cos P = \frac{\overline{PR}\cdot\overline{PQ}}{|\overline{PR}||\overline{PQ}|} = \frac{(5\hat{i}+8\hat{j}+3\hat{k})\cdot(3\hat{i}+2\hat{j}+6\hat{k})}{\sqrt{5^2+8^2+3^2}\sqrt{3^2+2^2+6^2}}$$

$$= \frac{15+16+18}{\sqrt{98}\sqrt{49}} = \frac{49}{\sqrt{2\times 49}\times\sqrt{49}}$$

$$= \frac{49}{\sqrt{2}\times 49}$$

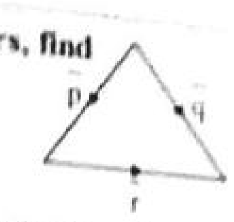
$\cos P = \frac{1}{\sqrt{2}} = \cos 45°$

$P = \frac{\pi}{4}$ or $45°$

8. If $\hat{p}$, $\hat{q}$ and $\hat{r}$ are unit vectors, find

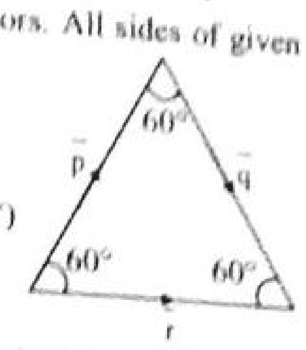
i. $\hat{p}\cdot\hat{q}$

ii. $\hat{p}\cdot\hat{r}$



Solution:

$\hat{p}$, $\hat{q}$ and $\hat{r}$ are unit vectors. All sides of given triangle are same i.e., 1.



i. $\hat{p}\cdot\hat{q} = |\hat{p}|\,|\hat{q}|\cdot\cos\theta$

$= (1)(1)\cos 60$ ...($\because \theta = 60°$)

$\hat{p}\cdot\hat{q} = \frac{1}{2}$

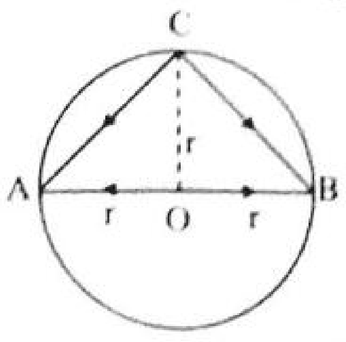
ii. $\hat{p}\cdot\hat{r} = |\hat{p}|\cdot|(\hat{r})|\cdot\cos\theta$

$= (1)(1)\cos 60$ ...($\because \theta = 60°$)

$\hat{p}\cdot\hat{r} = \frac{1}{2}$

9. Prove by vector method that the angle subtended on semicircle is a right angle.

Solution:



Let, r be the radius and O be the centre of the circle. A, B and C are three points on the circle such that, AB is the diameter.

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be the position vectors of points A, B and C respectively.

∴ C is on the circle, $|\bar{c}| = r$

Also, $|\bar{a}| = |\bar{b}| = r$ and $\bar{b} = -\bar{a}$

Consider $\overline{CA}\cdot\overline{CB} = (\bar{a}-\bar{c})\cdot(\bar{b}-\bar{c})$

$= (\bar{a}-\bar{c})\cdot(-\bar{a}-\bar{c})$

$= -(\bar{a}-\bar{c})\cdot(\bar{a}+\bar{c})$

$= -(\bar{a}-\bar{c})\cdot(\bar{a}+\bar{c})$

$= -\left(|\bar{a}|^2 - |\bar{c}|^2\right)$

$= r^2 - r^2$

$= 0$

$\overline{CA}\cdot\overline{CB} = 0$

$\overline{CA}$ is perpendicular to $\overline{CB}$

The angle between $\overline{CA}$ and $\overline{CB}$ is a right angle

$m\angle ACB = 90°$

The angle subtended on a semicircle is a right angle

10. If a vector has direction angles 45° and 60°, find the third direction angle.

Solution:

Let α, β, γ be the angles made by the vector with positive directions of X, Y, Z axes respectively

$\alpha = 45°, \beta = 60°$

We know that,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

∴ $\cos^2 45° + \cos^2 60° + \cos^2\gamma = 1$

∴ $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$

∴ $\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$

∴ $\cos^2\gamma = 1 - \frac{3}{4}$

∴ $\cos^2\gamma = \frac{1}{4}$

∴ $\cos\gamma = \pm\frac{1}{2}$

∴ $\gamma = \frac{\pi^c}{3}$ or $\frac{2\pi^c}{3}$

∴ Third direction angle is $\frac{\pi^c}{3}$ or $\frac{2\pi^c}{3}$

[**Note:** Answer given in the textbook is $\frac{\pi^c}{3}$. However as per our calculation it is $\frac{\pi^c}{3}$ or $\frac{2\pi^c}{3}$.]

11. If a line makes angles 90°, 135°, 45° with the X, Y and Z axes respectively, then find its direction cosines.

Solution:

Let α, β, γ be the angles made by the line with positive directions of X, Y, Z axes respectively

∴ $\alpha = 90°, \beta = 135°, \gamma = 45°$

Let the direction cosines of the line be l, m, n.

∴ $l = \cos 90°$, $m = \cos 135°$, $n = \cos 45°$

Now, $m = \cos 135^\circ = \cos(180^\circ - 45^\circ)$

$= -\cos 45^\circ = -\frac{1}{\sqrt{2}}$

$\therefore \quad l = 0, m = -\frac{1}{\sqrt{2}}, n = \frac{1}{\sqrt{2}}$

$\therefore$ Direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

12. If a line has the direction ratios 4, –12, 18, then find its direction cosines.

Solution:

Direction ratios of the line are 4, –12, 18.

$\therefore$ Direction cosines of the line are

$$\frac{4}{\sqrt{(4)^2+(-12)^2+(18)^2}}, \frac{-12}{\sqrt{(4)^2+(-12)^2+(18)^2}}, \frac{18}{\sqrt{(4)^2+(-12)^2+(18)^2}}$$

i.e., $\frac{4}{\sqrt{16+144+324}}, \frac{-12}{\sqrt{16+144+324}}, \frac{18}{\sqrt{16+144+324}}$

i.e., $\frac{4}{22}, -\frac{12}{22}, \frac{18}{22}$

i.e., $\frac{2}{11}, -\frac{6}{11}, \frac{9}{11}$

$\therefore$ Direction cosines of the line are $\frac{2}{11}, -\frac{6}{11}, \frac{9}{11}$.

13. The direction ratios of $\overline{AB}$ are –2, 2, 1. If A = (4, 1, 5) and l(AB) = 6 units, find B.

Solution:

The direction ratios of $\overline{AB}$ are –2, 2, 1

Let l, m, n be the direction cosines of $\overline{AB}$.

$$\therefore \quad l = \pm\frac{(-2)}{\sqrt{(-2)^2+2^2+1^2}} = \pm\left(-\frac{2}{3}\right)$$

$$m = \pm\frac{2}{\sqrt{(-2)^2+2^2+1^2}} = \pm\frac{2}{3}$$

$$n = \pm\frac{1}{\sqrt{(-2)^2+2^2+1^2}} = \pm\frac{1}{3}$$

Now, A ≡ (4, 1, 5) and $|\overline{AB}| = 6$ ...[Given]

$\therefore$ If B ≡ (x, y, z), then

$x - 4 = \pm\left(-\frac{2}{3}\right)|\overline{AB}|$

$y - 1 = \pm\frac{2}{3}|\overline{AB}|$

$z - 5 = \pm\frac{1}{3}|\overline{AB}|$

$\therefore \quad x = 4 \pm\left(-\frac{2}{3}\right)(6)$,

$\therefore \quad x = 0$ or $x = 8$

$y = 1 \pm \frac{2}{3}(6)$,

$\therefore \quad y = 5$ or $y = -3$

$z = 5 \pm \frac{1}{3}(6)$,

$\therefore \quad z = 7$ or $z = 3$

$\therefore \quad$ B ≡ (0, 5, 7) or B ≡ (8, –3, 3)

14. Find the angle between the lines whose direction cosines l, m, n satisfy the equations $5l + m + 3n = 0$ and $5mn - 2nl + 6lm = 0$.

Solution:

$5l + m + 3n = 0$ ...[Given]

$\therefore \quad m = -(5l + 3n)$ ...(i)

$5mn - 2nl + 6lm = 0$ ...[Given]

$\therefore \quad -5(5l + 3n)n - 2nl - 6l(5l + 3n) = 0$ ...[From (i)]

$\therefore \quad -25ln - 15n^2 - 2nl - 30l^2 - 18nl = 0$

$\therefore \quad -30l^2 - 45ln - 15n^2 = 0$

$\therefore \quad 2l^2 + 3ln + n^2 = 0$

$\therefore \quad (2l + n)(l + n) = 0$

$\therefore \quad n = -2l$ or $n = -l$

If $n = -2l$, then from (i), we get

$m = -[5l + 3(-2l)]$

$\therefore \quad m = l$

$\therefore \quad m = l, n = -2l$

$\therefore$ Direction ratios of the first line are proportional to $l, l, -2l$ i.e., 1, 1, –2

If $n = -l$, then from (i), we get

$m = -[5l + 3(-l)]$

$\therefore \quad m = -2l$

$\therefore \quad m = -2l, n = -l$

$\therefore$ Direction ratios of the second line are proportional to $l, -2l, -l$ i.e., 1, –2, –1

Let θ be the angle between the two lines.

$$\therefore \quad \cos\theta = \left|\frac{1(1)+1(-2)+(-2)(-1)}{\sqrt{1^2+1^2+(-2)^2}\cdot\sqrt{1^2+(-2)^2+(-1)^2}}\right|$$

$$= \left|\frac{1-2+2}{\sqrt{1+1+4}\cdot\sqrt{1+4+1}}\right|$$

$$= \left|\frac{1}{\sqrt{6}\cdot\sqrt{6}}\right|$$

$$= \frac{1}{6}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

[**Note:** Answer given in the textbook is – 1, 1, 2 or 1, 2, 3. However, as per our calculation it is $\cos^{-1}\left(\frac{1}{6}\right)$.]

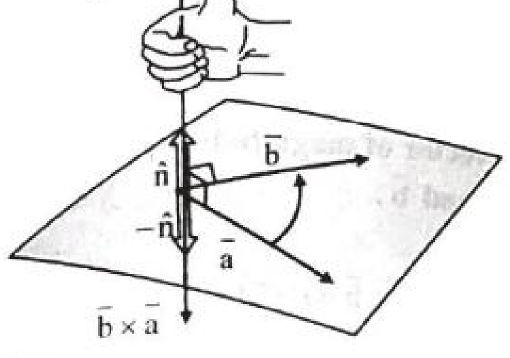
Let's Study

Vector Product of two vectors

In a plane, to describe how a line is tilting we used the notions of slope and angle of inclination. In space, we need to know how plane is tilting. We get this by multiplying two vectors in the plane together to get the third vector perpendicular to the plane. Third vector tell us inclination of the plane. The product we use for finding the third vector is called vector product.

Let $\bar{a}$ and $\bar{b}$ be two nonzero vectors in space. If $\bar{a}$ and $\bar{b}$ are not collinear, they determine a plane. We choose a unit vector $\hat{n}$ perpendicular to the plane by the right-hand rule. Which means $\hat{n}$ points in the way, right thumb points when our fingers curl through the angle from (See given figure). Then we define a new vector $\bar{a} \times \bar{b}$ as $\bar{a} \times \bar{b} = |\bar{a}||\bar{b}| \sin\theta\, \hat{n}$

The vector product is also called as the cross product of two vectors.



Remarks:

i. $|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}|\sin\theta$ as $|\hat{n}| = 1$

ii. $\bar{a} \times \bar{b}$ is perpendicular vector to the plane of $\bar{a}$ and $\bar{b}$.

iii. The unit vector $\hat{n}$ along $\bar{a} \times \bar{b}$ is given by $\hat{n} = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$

iv. If $\bar{a}$ and $\bar{b}$ are any two coplanar (but non collinear) vectors, then any vector $\bar{c}$ in the space can be given by $\bar{c} = x\bar{a} + y\bar{b} + z(\bar{a} \times \bar{b})$. This is because $\bar{a} \times \bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$ and thus $\bar{a}$, $\bar{b}$ and $\bar{a} \times \bar{b}$ span the whole space.

v. If $\bar{a}$, $\bar{b}$, $\hat{n}$ form a right handed triplet, then $\bar{b}$, $\bar{a}$, $-\hat{n}$ also form a right handed triplet and $\bar{b} \times \bar{a} = |\bar{a}||\bar{b}|\sin\theta(-\hat{n}) = -|\bar{a}||\bar{b}|\sin\theta(\hat{n}) = -\bar{a} \times \bar{b}$. Thus vector product is anticommutative.

vi. If $\bar{a}$ and $\bar{b}$ are non zero vectors such that $\bar{a}$ is parallel to $\bar{b}$, then
$\theta = 0$ i.e. $\sin\theta = 0$ i.e $\bar{a} \times \bar{b} = \bar{0}$

Conversely if $\bar{a} \times \bar{b} = \bar{0}$, then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ or $\sin\theta = 0$ that is $\theta = 0$.

Thus the cross product of two non zero vectors is zero only when $\bar{a}$ and $\bar{b}$ are collinear.

In particular if $\bar{a} = k\bar{b}$, then $\bar{a} \times \bar{b} = k\bar{b} \times \bar{b}$
$= k(\bar{0}) = \bar{0}$

vii. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are any three vectors, then

a. $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$ ...[Left distributive law]

b. $(\bar{b} + \bar{c}) \times \bar{a} = \bar{b} \times \bar{a} + \bar{c} \times \bar{a}$ ...[Right distributive law]

viii. If $\bar{a}$ and $\bar{b}$ are any two vectors and m, n are two scalars, then

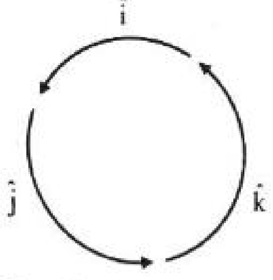
a. $m(\bar{a} \times \bar{b}) = (m\bar{a}) \times \bar{b} = \bar{a} \times (m\bar{b})$

b. $m\bar{a} \times n\bar{b} = mn(\bar{a} \times \bar{b})$

ix. If $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors along the co-ordinate axes, then

a. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \bar{0}$
$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

b. $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$



x. If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors in space, then

$\bar{a} \times \bar{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

This is given using determinant by $\bar{a} \times \bar{b}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Angle between two vectors:

Let θ be the angle between $\bar{a}$ and $\bar{b}$ $(0 \le \theta \le \pi)$,

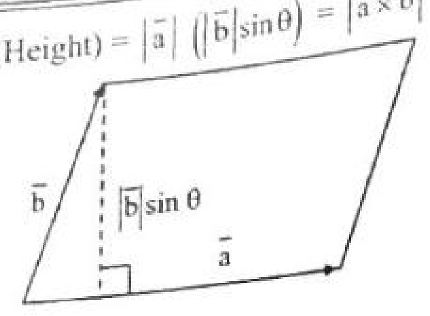
Then, $|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}|\sin\theta$

$$\therefore \quad \sin\theta = \frac{|\bar{a} \times \bar{b}|}{|\bar{a}||\bar{b}|}$$

Geometrical meaning of vector product of $\bar{a}$ and $\bar{b}$:

If $\bar{a}$ and $\bar{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\bar{a}|$, height $|\bar{b}|\sin\theta$ and area of parallelogram,

$A = (\text{Base})(\text{Height}) = |\bar{a}| \left(|\bar{b}|\sin\theta\right) = |\bar{a} \times \bar{b}|$



Exercise 5.4

1. **If $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$, find $(\bar{a}+\bar{b}) \times (\bar{a}-\bar{b})$.**

Solution:

$\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$

$$(\bar{a}+\bar{b}) = (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 4\hat{j} + 2\hat{k})$$
$$= 3\hat{i} - \hat{j} + \hat{k}$$
$$(\bar{a}-\bar{b}) = (2\hat{i} + 3\hat{j} - \hat{k}) - (\hat{i} - 4\hat{j} + 2\hat{k})$$
$$= \hat{i} + 7\hat{j} - 3\hat{k}$$

$\therefore$ $$(\bar{a}+\bar{b}) \times (\bar{a}-\bar{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 7 & -3 \end{vmatrix}$$
$$= \hat{i}(3-7) - \hat{j}(-9-1) + \hat{k}(21+1)$$
$$= -4\hat{i} + 10\hat{j} + 22\hat{k}$$

2. **Find a unit vector perpendicular to the vectors $\hat{j} + 2\hat{k}$ and $\hat{i} + \hat{j}$.**

Solution:

Let $\bar{a} = \hat{j} + 2\hat{k}$ and $\bar{b} = \hat{i} + \hat{j}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i}(0-2) - \hat{j}(0-2) + \hat{k}(0-1)$$
$$= -2\hat{i} + 2\hat{j} - \hat{k}$$

$\therefore$ $$|\bar{a} \times \bar{b}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2}$$
$$= \sqrt{4+4+1} = \sqrt{9} = 3$$

$\therefore$ Unit vectors perpendicular to $\bar{a}$ and $\bar{b}$

$$= \pm \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|} = \pm\left(\frac{-2\hat{i} + 2\hat{j} - \hat{k}}{3}\right)$$
$$= \pm\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$$

3. **If $\bar{a} \cdot \bar{b} = \sqrt{3}$ and $\bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, find the angle between $\bar{a}$ and $\bar{b}$.**

Solution:

$\bar{a} \cdot \bar{b} = \sqrt{3}$ and $\bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

Let θ be the angle between $\bar{a}$ and $\bar{b}$.

We know that,

$|\bar{a}| \cdot |\bar{b}| \sin\theta = |\bar{a} \times \bar{b}|$ and $|\bar{a}| \cdot |\bar{b}| \cos\theta = \bar{a} \cdot \bar{b}$

$\therefore$ $$\frac{|\bar{a}| \cdot |\bar{b}| \sin\theta}{|\bar{a}| \cdot |\bar{b}| \cos\theta} = \frac{|\bar{a} \times \bar{b}|}{\bar{a} \cdot \bar{b}}$$

$\therefore$ $$\tan\theta = \frac{|\bar{a} \times \bar{b}|}{\bar{a} \cdot \bar{b}}$$

$$|\bar{a} \times \bar{b}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$\therefore$ $\tan\theta = \frac{3}{\sqrt{3}}$

$\therefore$ $\tan\theta = \sqrt{3}$

$\therefore$ $\theta = 60^\circ$

4. **If $\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$.**

Solution:

$\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$, $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$
$$= \hat{i}(1-6) - \hat{j}(2+3) + \hat{k}(-4-1)$$
$$= -5\hat{i} - 5\hat{j} - 5\hat{k}$$
$$= -5(\hat{i} + \hat{j} + \hat{k})$$

$\therefore$ $|\bar{a} \times \bar{b}| = 5\sqrt{1^2 + 1^2 + 1^2} = 5\sqrt{3}$

$\therefore$ Vectors of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$ $= 5 \times \pm \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$

$$= \pm \frac{5 \times [-5(\hat{i} + \hat{j} + \hat{k})]}{5\sqrt{3}}$$
$$= \pm \frac{5}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

[**Note:** Answer given in the textbook is $\pm \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$. However, as per our calculation it is $\pm \frac{5}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$.]

5. **Find**

i. **$\bar{u} \cdot \bar{v}$ if $|\bar{u}| = 2$, $|\bar{v}| = 5$, $|\bar{u} \times \bar{v}| = 8$**

ii. **$|\bar{u} \times \bar{v}|$ if $|\bar{u}| = 10$, $|\bar{v}| = 2$, $\bar{u} \cdot \bar{v} = 12$**

Solution:

i. $|\bar{u}| = 2$, $|\bar{v}| = 5$, $|\bar{u} \times \bar{v}| = 8$

We know that,

$|\bar{u}| \cdot |\bar{v}| \sin\theta = |\bar{u} \times \bar{v}|$

$2 \times 5 \sin\theta = 8$

$\therefore$ $\sin\theta = \frac{4}{5}$

$\therefore$ $$\cos\theta = \sqrt{1 - \sin^2\theta}$$

$\therefore$ $$= \sqrt{1 - \frac{16}{25}}$$
$$= \sqrt{\frac{9}{25}}$$
$$= \pm \frac{3}{5} \qquad \ldots[\because 0 < \theta < \pi]$$

$\therefore$ $\bar{u} \cdot \bar{v} = |\bar{u}| \cdot |\bar{v}| \cos\theta = 2 \times 5 \times \left(\pm\frac{3}{5}\right)$

$\therefore$ $\bar{u} \cdot \bar{v} = \pm 6$

$\therefore$ $\bar{u} \cdot \bar{v} = 6$, if $0 < \theta < \frac{\pi}{2}$

$\bar{u} \cdot \bar{v} = -6$, if $\frac{\pi}{2} < \theta < \pi$

ii. $|\bar{u}| = 10$, $|\bar{v}| = 2$, $\bar{u} \cdot \bar{v} = 12$

We know that,

$\bar{u} \cdot \bar{v} = |\bar{u}| \cdot |\bar{v}| \cos\theta$

$\therefore$ $12 = 10 \times 2 \cos\theta$

$\therefore$ $\cos\theta = \frac{3}{5}$

$\therefore$ $$\sin\theta = \sqrt{1 - \cos^2\theta}$$
$$= \sqrt{1 - \frac{9}{25}}$$
$$= \sqrt{\frac{16}{25}}$$
$$= \frac{4}{5} \qquad \ldots[\because 0 < \theta < \pi]$$

$\therefore$ $|\bar{u}| \cdot |\bar{v}| \sin\theta = |\bar{u} \times \bar{v}|$

$\therefore$ $10 \times 2 \times \left(\frac{4}{5}\right) = |\bar{u} \times \bar{v}|$

$\therefore$ $|\bar{u} \times \bar{v}| = 16$

[**Note:** Answer given in the textbook is 1.6. However, as per our calculation it is 16.]

6. **Prove that $2(\bar{a} - \bar{b}) \times (\bar{a} + \bar{b}) = 4(\bar{a} \times \bar{b})$**

Solution:

$$\text{L. H. S.} = 2(\bar{a} - \bar{b}) \times (\bar{a} + \bar{b})$$
$$= 2(\bar{a} \times \bar{a} + \bar{a} \times \bar{b} - \bar{b} \times \bar{a} - \bar{b} \times \bar{b})$$
$$= 2(\bar{0} + \bar{a} \times \bar{b} + \bar{a} \times \bar{b} - \bar{0})$$
$$= 2[2(\bar{a} \times \bar{b})]$$
$$= 4(\bar{a} \times \bar{b}) = \text{R. H. S.}$$

[**Note:** *The question has been modified.*]

7. **If $\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{b} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\bar{c} = \hat{i} - \hat{j} + 2\hat{k}$, verify that $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$.**

Solution:

$\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\bar{c} = \hat{i} - \hat{j} + 2\hat{k}$

$\bar{b} + \bar{c} = (4\hat{i} - 3\hat{j} + \hat{k}) + (\hat{i} - \hat{j} + 2\hat{k}) = 5\hat{i} - 4\hat{j} + 3\hat{k}$

$\therefore$ $$\bar{a} \times (\bar{b} + \bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 5 & -4 & 3 \end{vmatrix}$$
$$= \hat{i}(-6+12) - \hat{j}(3-15) + \hat{k}(-4+10)$$
$$= 6\hat{i} + 12\hat{j} + 6\hat{k} \qquad \ldots\text{(i)}$$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 4 & -3 & 1 \end{vmatrix}$$
$$= \hat{i}(-2+9) - \hat{j}(1-12) + \hat{k}(-3+8)$$
$$= 7\hat{i} + 11\hat{j} + 5\hat{k}$$

$$\bar{a} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$$
$$= \hat{i}(-4+3) - \hat{j}(2-3) + \hat{k}(-1+2)$$
$$= -\hat{i} + \hat{j} + \hat{k}$$

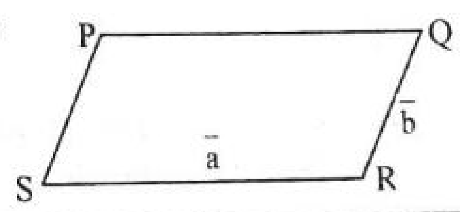
$\therefore$ $$(\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c}) = (7\hat{i} + 11\hat{j} + 5\hat{k}) + (-\hat{i} + \hat{j} + \hat{k})$$
$$= 6\hat{i} + 12\hat{j} + 6\hat{k} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$\bar{a} \times (\bar{b} + \bar{c}) = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})$

8. **Find the area of the parallelogram whose adjacent sides are the vectors $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} - 3\hat{k}$.**

Solution:



Let PQRS be the parallelogram whose adjacent sides are $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} - 3\hat{k}$.

$\therefore$ Area of parallelogram $= |\bar{a} \times \bar{b}|$

$$\therefore \quad \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -3 & -3 \end{vmatrix}$$

$$= \hat{i}(6+3) - \hat{j}(-6-1) + \hat{k}(-6+2)$$

$$= 9\hat{i} + 7\hat{j} - 4\hat{k}$$

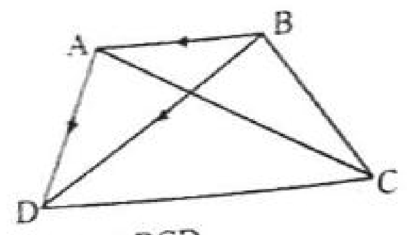
$$\therefore \quad |\bar{a} \times \bar{b}| = \sqrt{(9)^2 + (7)^2 + (-4)^2} = \sqrt{81+49+16}$$

$$= \sqrt{146}$$

$\therefore$ The area of the parallelogram is $\sqrt{146}$ sq. units.

9. Show that vector area of a quadrilateral ABCD is $\frac{1}{2}\left(\overline{AC} \times \overline{BD}\right)$, where AC and BD are its diagonals.

Solution:



Vector area of $\square$ABCD

= vector area of ΔABC + vector area of ΔADC

$$= \frac{1}{2}\,\overline{AB} \times \overline{AC} + \frac{1}{2}\,\overline{AC} \times \overline{AD}$$

$$= \frac{1}{2}\,\overline{AB} \times \overline{AC} + \overline{AC} \times \overline{AD}$$

$$= \frac{1}{2}\,-\overline{AC} \times \overline{AB} + \overline{AC} \times \overline{AD}$$

...[$\because \bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$]

$$= \frac{1}{2}\,\overline{AC} \times \overline{BA} + \overline{AC} \times \overline{AD} \quad ...[\because \overline{AB} = -\overline{BA}]$$

$$= \frac{1}{2}\,\overline{AC} \times (\overline{BA} + \overline{AD})$$

$\therefore$ Area of $\square$ABCD $= \frac{1}{2}\left(\overline{AC} \times \overline{BD}\right)$

...[$\because \overline{BA} + \overline{AD} = \overline{BD}$]

$\therefore$ Vector area of quadrilateral ABCD is $\frac{1}{2}\left(\overline{AC} \times \overline{BD}\right)$, where $\overline{AC}$ and $\overline{BD}$ are its diagonals.

10. Find the area of parallelogram whose diagonals are determined by the vectors $\bar{a} = 3\hat{i} - \hat{j} - 2\hat{k}$ and $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$.

Solution:

Given, diagonals of parallelogram are $\bar{a} = 3\hat{i} - \hat{j} - 2\hat{k}$ and $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$.

$\therefore$ Area of parallelogram $= \frac{1}{2}|\bar{a} \times \bar{b}|$

$$\therefore \quad \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -2 \\ -1 & 3 & -3 \end{vmatrix}$$

$$= \hat{i}(3+6) - \hat{j}(-9-2) + \hat{k}(9-1)$$

$$= 9\hat{i} + 11\hat{j} + 8\hat{k}$$

$$\therefore \quad |\bar{a} \times \bar{b}| = \sqrt{9^2 + 11^2 + 8^2} = \sqrt{81+121+64}$$

$$= \sqrt{266}$$

$\therefore$ Area of parallelogram $= \frac{1}{2}|\bar{a} \times \bar{b}|$

$$= \frac{1}{2}\sqrt{266}$$

$$= \sqrt{\frac{266}{4}}$$

$$= \sqrt{\frac{133}{2}} \text{ sq. units}$$

[**Note:** Answer given in the textbook is $\sqrt{42}$ sq. units. However, as per our calculation it is $\sqrt{\frac{133}{2}}$ sq. units.]

11. If $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ are four distinct vectors such that $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$, prove that $\bar{a} - \bar{d}$ is parallel to $\bar{b} - \bar{c}$.

Solution:

$\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$ ...[Given]

Two non-zero vectors are parallel if and only if their cross – product is zero.

$$\therefore \quad (\bar{a} - \bar{d}) \times (\bar{b} - \bar{c}) = \bar{a} \times \bar{b} - \bar{a} \times \bar{c} - \bar{d} \times \bar{b} + \bar{d} \times \bar{c}$$

$$= \bar{c} \times \bar{d} - \bar{b} \times \bar{d} + \bar{b} \times \bar{d} - \bar{c} \times \bar{d}$$

...[$\because -\bar{d} \times \bar{b} = \bar{b} \times \bar{d},\ \bar{d} \times \bar{c} = -\bar{c} \times \bar{d}$]

$$\therefore \quad (\bar{a} - \bar{d}) \times (\bar{b} - \bar{c}) = 0$$

$\therefore$ $(\bar{a} - \bar{d})$ is parallel to $(\bar{b} - \bar{c})$.

12. If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{j} - \hat{k}$, find a vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$.

Solution:

Let $\bar{b} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{c} = \hat{j} - \hat{k}$

$\bar{a} \times \bar{b} = \bar{c}$

$$\therefore \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$$

$$\therefore \quad \hat{j} - \hat{k} = \hat{i}(z-y) - \hat{j}(z-x) + \hat{k}(y-x)$$

By equality of vectors, we get

$z - y = 0$, $-(z-x) = 1$ and $y - x = -1$

$\therefore$ $z = y$, $z = x - 1$ and $y = x - 1$

Now,

$\bar{a} \cdot \bar{b} = 3$

$\therefore$ $(\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$\therefore$ $x + y + z = 3$

$\therefore$ $x + x - 1 + x - 1 = 3$

$\therefore$ $3x = 5$

$\therefore$ $x = \frac{5}{3}$

$\therefore$ $y = x - 1 = \frac{5}{3} - 1 = \frac{2}{3}$

$\therefore$ $z = y = \frac{2}{3}$

$$\therefore \quad \bar{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} = \frac{1}{3}\left(5\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

13. Find $\bar{a}$, if $\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = 0$.

Solution:

Let $\bar{a} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\bar{a} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-z) + \hat{k}(0-y)$$

$$= z\hat{j} - y\hat{k}$$

Now, $\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = 0$

$\therefore$ $\left(z\hat{j} - y\hat{k}\right) + 2\left(x\hat{i} + y\hat{j} + z\hat{k}\right) - 5\hat{j} = 0$

$\therefore$ $2x\hat{i} + (z + 2y - 5)\hat{j} + (-y + 2z)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\therefore$ By equality of vectors, we get

$2x = 0$, $z + 2y - 5 = 0$, $-y + 2z = 0$

$\therefore$ $x = 0$

$z + 2y = 5$ ...(i)

$2z - y = 0$ ...(ii)

Solving (i) and (ii), we get

$z = 1, y = 2$

$\therefore$ $\bar{a} = 2\hat{j} + \hat{k}$

14. If $|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ and $\bar{a} \cdot \bar{b} < 0$, then find the angle between $\bar{a}$ and $\bar{b}$.

Solution:

We know that,

$\bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cos\theta$

$\therefore$ $|\bar{a} \cdot \bar{b}| = \left||\bar{a}| \cdot |\bar{b}| \cos\theta\right|$

$\therefore$ $|\bar{a} \cdot \bar{b}| = -|\bar{a}| \cdot |\bar{b}| \cos\theta$ ...(i) [$\bar{a} \cdot \bar{b} < 0$]

Also, $|\bar{a} \times \bar{b}| = |\bar{a}| \cdot |\bar{b}| \sin\theta$ ...(ii)

$|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ ...[Given]

$\therefore$ $-|\bar{a}| \cdot |\bar{b}| \cos\theta = |\bar{a}| \cdot |\bar{b}| \sin\theta$ ...[From (i) and (ii)]

$\therefore$ $-1 = \tan\theta$

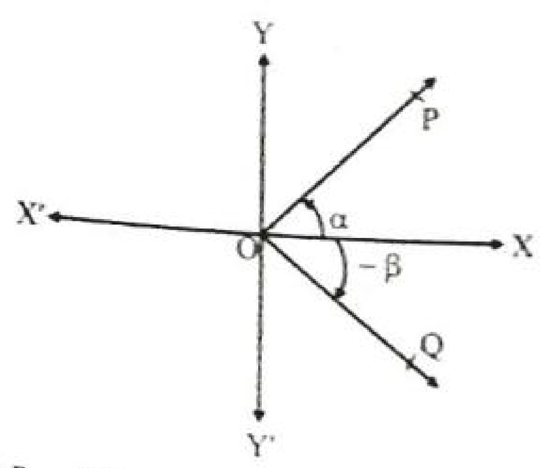
$\therefore$ $\tan\theta = -1$

$\therefore$ $\theta = \tan^{-1}(-1) = 135^\circ$ or $\frac{3\pi}{4}$

$\therefore$ The angle between $\bar{a}$ and $\bar{b}$ is $\frac{3\pi}{4}$.

15. Prove by vector method that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$.

Solution:



Let P and Q be two points in XY plane such that OP makes an angle α and OQ makes an angle $(-\beta)$ with positive direction of X-axis.

Let $|\overline{OP}| = |\overline{OQ}| = 1$

$\therefore$ P $\equiv (\cos\alpha, \sin\alpha)$ and

Q $\equiv (\cos(-\beta), \sin(-\beta)) \equiv (\cos\beta, -\sin\beta)$

If $\bar{p}$ and $\bar{q}$ are the position vectors of points P and Q respectively, then $|\bar{p}| = |\bar{q}| = 1$ and $\overline{OP} = \bar{p}$ and $\overline{OQ} = \bar{q}$

$\therefore$ $\bar{p} = (\cos\alpha)\hat{i} + (\sin\alpha)\hat{j}$

and $\bar{q} = (\cos\beta)\hat{i} - (\sin\beta)\hat{j}$

Consider, $\overline{OP} \times \overline{OQ} = \bar{p} \times \bar{q}$

Now, $\bar{p} \times \bar{q} = |\bar{p}|\,|\bar{q}| \sin\theta$, where θ is the angle between $\bar{p}$ and $\bar{q}$

$$\therefore \quad \bar{p} \times \bar{q} = 1 \cdot 1 \cdot \sin(-\alpha - \beta)\,\hat{k}$$

$$= \sin[-(\alpha + \beta)]\,\hat{k}$$

$$\therefore \quad \bar{p} \times \bar{q} = -\sin(\alpha + \beta)\,\hat{k} \quad ...(i)$$

$$\text{Also, } \bar{p} \times \bar{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & -\sin\beta & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(-\cos\alpha \sin\beta - \sin\alpha \cos\beta)$$

$$\therefore \quad \bar{p} \times \bar{q} = -(\cos\alpha \sin\beta + \sin\alpha \cos\beta)\,\hat{k} \quad ...(ii)$$

$\therefore$ From (i) and (ii), we get
$-\sin(\alpha+\beta)\hat{k} = -(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\hat{k}$

$\therefore$ By equality of vectors, we get
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

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16. Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are

i. $-2, 1, -1$ and $-3, -4, 1$

ii. $1, 3, 2$ and $-1, 1, 2$

Solution:

i. Let L_1 and L_2 be the two lines with direction ratios $-2, 1, -1$ and $-3, -4, 1$ respectively.
Let the direction ratios of the vector perpendicular to L_1 and L_2 be a, b, c.

$\therefore$ $-2a + b - c = 0$

and $-3a - 4b + c = 0$

$$\therefore \quad \frac{a}{\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} -2 & 1 \\ -3 & -4 \end{vmatrix}}$$

$$\therefore \quad \frac{a}{1-4} = \frac{-b}{-2-3} = \frac{c}{8+3}$$

$$\therefore \quad \frac{a}{-3} = \frac{b}{5} = \frac{c}{11}$$

$\therefore$ The direction ratios of the vector are $-3, 5, 11$.

Alternate method:
The two lines are perpendicular to the vector whose direction ratios are required.

$$\therefore \quad \bar{L}_1 \times \bar{L}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix}$$

$$= \hat{i}(-3) - \hat{j}(-5) + \hat{k}(11)$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$\therefore$ The direction ratios of the vector are $-3, 5, 11$.

ii. Let L_1 and L_2 be the two lines with direction ratios $1, 3, 2$ and $-1, 1, 2$ respectively.
Let the direction ratios of the vector perpendicular to L_1 and L_2 be a, b, c.

$\therefore$ $a + 3b + 2c = 0$

and $-a + b + 2c = 0$

$$\therefore \quad \frac{a}{\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}}$$

$$\therefore \quad \frac{a}{6-2} = \frac{-b}{2+2} = \frac{c}{1+3}$$

$$\therefore \quad \frac{a}{4} = \frac{-b}{4} = \frac{c}{4}$$

$\therefore$ The direction ratios of the vector are $4, -4, 4$.

17. Prove that two vectors whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$.

Solution:

$al + bm + cn = 0$ ...(i)[Given]

$fmn + gnl + hlm = 0$ ...(ii)[Given]

From (i), we get

$$m = -\left(\frac{al + cn}{b}\right) \quad \text{...(iii)}$$

Substituting (iii) in (ii), we get

$$(fn + hl)\left[-\frac{(al + cn)}{b}\right] + gnl = 0$$

$\therefore$ $-(afnl + hal^2 + fcn^2 + hcnl) + gbnl = 0$

$\therefore$ $ahl^2 + (af + hc - gb)\,nl + fcn^2 = 0$ ...(iv)

Here, l and n cannot be zero at a time, since if $l = n = 0$ then from (iii) we get, $m = 0$ which is not possible as $l^2 + m^2 + n^2 = 1$

$\therefore$ $l \neq 0$ or $n \neq 0$

Let $n \neq 0$

Dividing (iv) throughout by n^2, we get

$$ah\left(\frac{l}{n}\right)^2 + (af + ch - gb)\left(\frac{l}{n}\right) + fc = 0$$

This is a quadratic equation in $\left(\frac{l}{n}\right)$.

Let $\frac{l_1}{n_1}$ and $\frac{l_2}{n_2}$ be the roots of this equation. Here l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the given vectors.

$$\therefore \quad \left(\frac{l_1}{n_1}\right)\left(\frac{l_2}{n_2}\right) = \frac{fc}{ah}$$

$$\therefore \quad \frac{l_1 l_2}{n_1 n_2} = \frac{\left(\frac{f}{a}\right)}{\left(\frac{h}{c}\right)}$$

$$\therefore \quad \frac{l_1 l_2}{\left(\frac{f}{a}\right)} = \frac{n_1 n_2}{\left(\frac{h}{c}\right)}$$

Similarly we prove that,

$$\frac{l_1 l_2}{\left(\frac{f}{a}\right)} = \frac{m_1 m_2}{\left(\frac{g}{b}\right)}$$

Let $\frac{l_1 l_2}{\left(\frac{f}{a}\right)} = \frac{m_1 m_2}{\left(\frac{g}{b}\right)} = \frac{n_1 n_2}{\left(\frac{h}{c}\right)} = k$ ...(v)

The given vectors are perpendicular iff
$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

i.e., if $k\left(\frac{f}{a}\right) + k\left(\frac{g}{b}\right) + k\left(\frac{h}{c}\right) = 0$ ...[From (v)]

i.e., if $k\left[\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right] = 0$

i.e., if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ ...($\because k \neq 0$)

$\therefore$ $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

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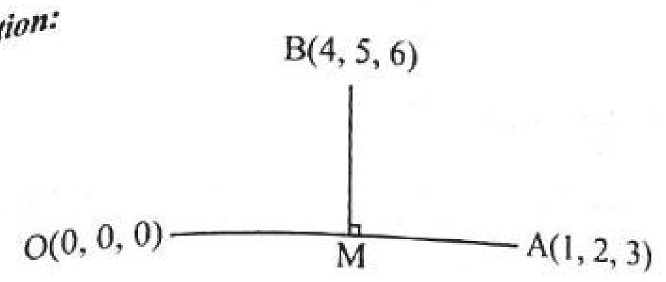
18. If A(1, 2, 3) and B(4, 5, 6) are two points, then find the foot of the perpendicular from the point B to the line joining the origin and point A.

Solution:



Let M be the foot of perpendicular from point B to line OA.

Let M divide line OA internally in the ratio $\lambda : 1$.

$$\therefore \quad M \equiv \left(\frac{\lambda + 0}{\lambda + 1}, \frac{2\lambda + 0}{\lambda + 1}, \frac{3\lambda + 0}{\lambda + 1}\right) \quad \text{...(i)}$$

$\therefore$ Direction ratios of BM are

$$\frac{\lambda}{\lambda+1} - 4, \frac{2\lambda}{\lambda+1} - 5, \frac{3\lambda}{\lambda+1} - 6$$

i.e., $\frac{-3\lambda - 4}{\lambda + 1}, \frac{-3\lambda - 5}{\lambda + 1}, \frac{-3\lambda - 6}{\lambda + 1}$

Direction ratios of OA are

$1 - 0, 2 - 0, 3 - 0$ i.e., $1, 2, 3$

Since BM is perpendicular to OA,

$$1\left(\frac{-3\lambda - 4}{\lambda + 1}\right) + 2\left(\frac{-3\lambda - 5}{\lambda + 1}\right) + 3\left(\frac{-3\lambda - 6}{\lambda + 1}\right) = 0$$

$$\therefore \quad \frac{-3\lambda - 4 - 6\lambda - 10 - 9\lambda - 18}{\lambda + 1} = 0$$

$\therefore$ $-18\lambda - 32 = 0$

$$\therefore \quad \lambda = -\frac{32}{18} = -\frac{16}{9}$$

Substituting the value of λ in (i), we get

$$M \equiv \left(\frac{\frac{-16}{9}}{\frac{-16}{9} + 1}, \frac{2\left(\frac{-16}{9}\right)}{\frac{-16}{9} + 1}, \frac{3\left(\frac{-16}{9}\right)}{\frac{-16}{9} + 1}\right)$$

$$\therefore \quad M \equiv \left(\frac{16}{7}, \frac{32}{7}, \frac{48}{7}\right)$$

Let's Study

Scalar Triple Product

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be three vectors, then the scalar product $\bar{a}\cdot(\bar{b}\times\bar{c})$ is called the scalar triple product of $\bar{a}$, $\bar{b}$ and $\bar{c}$ (appearing in the same order). It is also called box product and is denoted by $[\bar{a}\,\bar{b}\,\bar{c}]$

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
$\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Consider,
$[\bar{a}\,\bar{b}\,\bar{c}] = \bar{a}\cdot(\bar{b}\times\bar{c})$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \left[\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)\right]$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of scalar triple product:

Using the properties of determinant, we get the following properties of scalar triple product.

1. A cyclic change of vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ in a scalar triple product does not change its value
i.e., $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{c}\ \bar{a}\ \bar{b}] = [\bar{b}\ \bar{c}\ \bar{a}]$
This follows as a cyclic change is equivalent to interchanging a pair of rows in the determinant two times.

2. A single interchange of vectors in a scalar triple product changes the sign of its value
i.e., $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{b}\ \bar{a}\ \bar{c}] = -[\bar{c}\ \bar{b}\ \bar{a}]$
$= -[\bar{a}\ \bar{c}\ \bar{b}]$
This follows as interchange of any 2 rows changes the value of determinant by sign only.

3. If a row of determinant can be expressed as a linear combination of other rows, then the determinant is zero. Using this fact we get the following properties.
The scalar triple product of vectors is zero if any one of the following is true.

i. One of the vectors is a zero vector.

ii. Any two vectors are collinear.

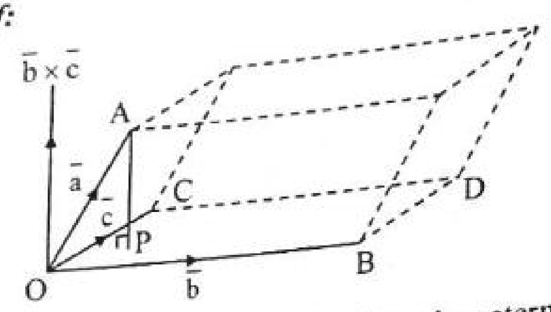
iii. The three vectors are coplanar.

iv. Any two vectors are repeated.

4. An interchange of 'dot' and 'cross' in a scalar triple product does not change its value
i.e., $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\times\bar{b}\right)\cdot\bar{c}$
This is followed by property (1) and the commutativity of dot product.

Theorem 9:
The volume of a parallelopiped with coterminous edges as $\bar{a}$, $\bar{b}$ and $\bar{c}$ is $\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$.
[Mar 05; Oct 96, 00, 02, 04, 08, 14, 15; July 16]

Proof:



Let $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$ be the coterminous edges of parallelopiped.
Let AP be the height of the parallelopiped.
Volume of parallelopiped
= (Area of parallelogram OBDC) × (Height AP)
But AP = Scalar Projection of $\bar{a}$ on $(\bar{b}\times\bar{c})$

$$= \frac{\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)}{\left|\bar{b}\times\bar{c}\right|} \quad \ldots\left[\because \text{scalar projection of } \bar{p} \text{ on } \bar{q} \text{ is } \frac{\bar{p}\cdot\bar{q}}{\left|\bar{q}\right|}\right]$$

and area of parallelogram OBDC = $\left|\bar{b}\times\bar{c}\right|$

$$\therefore \quad \text{Volume of parallelopiped} = \frac{\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)}{\left|\bar{b}\times\bar{c}\right|}\left|\bar{b}\times\bar{c}\right|$$
$$= \bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$$
$$= \left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

Theorem 10:
The volume of a tetrahedron with coterminous edges $\bar{a}$, $\bar{b}$ and $\bar{c}$ is $\frac{1}{6}\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$.

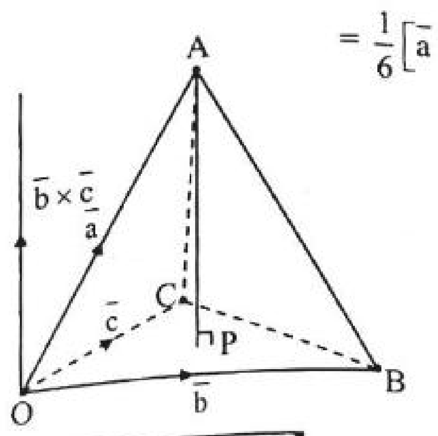
Proof:
Let $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$ be the coterminous edges of tetrahedron OABC.
Let AP be the height of the tetrahedron.
Volume of tetrahedron
$= \frac{1}{3}$ (area of ΔOCB) × (Height AP)

But AP = Scalar projection of $\bar{a}$ on $(\bar{b}\times\bar{c})$

$$= \frac{\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)}{\left|\bar{b}\times\bar{c}\right|} \quad \ldots\left[\because \text{scalar projection of } \bar{p} \text{ on } \bar{q} \text{ is } \frac{\bar{p}\cdot\bar{q}}{\left|\bar{q}\right|}\right]$$

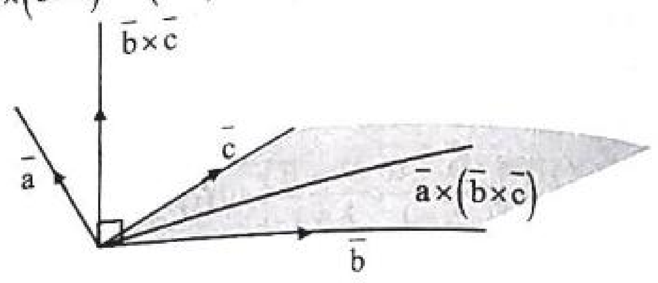
Area of ΔOCB $= \frac{1}{2}\left|\bar{b}\times\bar{c}\right|$

$$\therefore \quad \text{Volume of tetrahedron} = \frac{1}{3}\times\frac{1}{2}\left|\bar{b}\times\bar{c}\right|\frac{\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)}{\left|\bar{b}\times\bar{c}\right|}$$
$$= \frac{1}{6}\left[\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)\right]$$
$$= \frac{1}{6}\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$



Vector Triple Product

For vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ in the space, we define the vector triple product without proof as
$\bar{a}\times\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}$



Properties of vector triple product:
1. $\bar{a}\times\left(\bar{b}\times\bar{c}\right) = -\left(\bar{b}\times\bar{c}\right)\times\bar{a}$ $\quad\ldots\left[\because \bar{p}\times\bar{q} = -\bar{q}\times\bar{p}\right]$
2. $\left(\bar{a}\times\bar{b}\right)\times\bar{c} = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{b}\cdot\bar{c}\right)\bar{a}$
3. $\bar{a}\times\left(\bar{b}\times\bar{c}\right) \neq \left(\bar{a}\times\bar{b}\right)\times\bar{c}$
4. $\hat{i}\times\left(\hat{j}\times\hat{k}\right) = \bar{0}$
5. $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ is a linear combination of $\bar{b}$ and $\bar{c}$, hence it is coplanar with $\bar{b}$ and $\bar{c}$.

Exercise 5.5

1. Find $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$, if $\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $c = -5\hat{i} + 2\hat{j} + 3\hat{k}$.

Solution:
$\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}$,
$c = -5\hat{i} + 2\hat{j} + 3\hat{k}$

$$\bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 3 & -1 \\ -5 & 2 & 3 \end{vmatrix}$$
$$= 3(9+2) + 1(6-5) + 4(4+15)$$
$$= 3(11) + 1\,(1) + 4(19)$$
$$= 33 + 1 + 76$$
$$\therefore \quad \bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = 110$$

2. If the vectors $3\hat{i} + 5\hat{k}$, $4\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} + 4\hat{k}$ are coterminous edges of the parallelopiped, then find the volume of the parallelopiped.

Solution:
Let $\bar{a} = 3\hat{i} + 5\hat{k}$, $\bar{b} = 4\hat{i} + 2\hat{j} - 3\hat{k}$ and $\bar{c} = 3\hat{i} + \hat{j} + 4\hat{k}$
Volume of parallelopiped
$\therefore \quad = \left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$

$$= \begin{vmatrix} 3 & 0 & 5 \\ 4 & 2 & -3 \\ 3 & 1 & 4 \end{vmatrix}$$
$$= 3(8+3) - 0(16+9) + 5(4-6)$$
$$= 3(11) - 0 + 5(-2)$$
$$= 33 - 10$$
$$= 23$$

$\therefore$ The volume of the parallelopiped is 23 cubic units.

3. If the vectors $-3\hat{i} + 4\hat{j} - 2\hat{k}$, $\hat{i} + 2\hat{k}$ and $\hat{i} - p\hat{j}$ are coplanar, then find the value of p.

Solution:
Let $\bar{a} = -3\hat{i} + 4\hat{j} - 2\hat{k}$, $\bar{b} = \hat{i} + 2\hat{k}$, $\bar{c} = \hat{i} - p\hat{j}$
Since $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar,
$\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] = 0$

$$\therefore \quad \begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -p & 0 \end{vmatrix} = 0$$
$$\therefore \quad -3(0+2p) - 4(0-2) - 2(-p-0) = 0$$
$$\therefore \quad -6p + 8 + 2p = 0$$
$$\therefore \quad -4p = -8$$
$$\therefore \quad p = \frac{-8}{-4} = 2$$

Smart Check

If $\begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{vmatrix} = 0$, then our answer is correct.

$$\begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{vmatrix} = -3(0+4) - 4(0-2) - 2(-2-0)$$
$$= -12 + 8 + 4 = 0$$

Thus, our answer is correct.

4. Prove that:
i. $\left[\bar{a}\ \ \bar{b}+\bar{c}\ \ \bar{a}+\bar{b}+\bar{c}\right] = 0$
ii. $\left(\bar{a} + 2\bar{b} - \bar{c}\right)\cdot\left[\left(\bar{a}-\bar{b}\right)\times\left(\bar{a}-\bar{b}-\bar{c}\right)\right] = 3\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$

Solution:
i. $\left[\bar{a}\ \ \bar{b}+\bar{c}\ \ \bar{a}+\bar{b}+\bar{c}\right]$
$$= \bar{a}\cdot\left[\left(\bar{b}+\bar{c}\right)\times\left(\bar{a}+\bar{b}+\bar{c}\right)\right]$$
$$= \bar{a}\cdot\left[\bar{b}\times\bar{a} + \bar{b}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{c}\times\bar{b} + \bar{c}\times\bar{c}\right]$$
$$= \bar{a}\cdot\left[\bar{b}\times\bar{a} + \bar{0} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a} - \bar{b}\times\bar{c} + \bar{0}\right]$$
$$= \bar{a}\cdot\left[\bar{b}\times\bar{a} + \bar{c}\times\bar{a}\right] = \bar{a}\cdot\left(\bar{b}\times\bar{a}\right) + \bar{a}\cdot\left(\bar{c}\times\bar{a}\right)$$
$$= \left[\bar{a}\,\bar{b}\,\bar{a}\right] + \left[\bar{a}\,\bar{c}\,\bar{a}\right]$$
$$= 0 + 0 = 0$$
$$\therefore \quad \left[\bar{a}\ \ \bar{b}+\bar{c}\ \ \bar{a}+\bar{b}+\bar{c}\right] = 0$$

ii. $\left(\bar{a}+2\bar{b}-\bar{c}\right)\cdot\left[\left(\bar{a}-\bar{b}\right)\times\left(\bar{a}-\bar{b}-\bar{c}\right)\right]$
$$= \left(\bar{a}+2\bar{b}-\bar{c}\right)\cdot\left[\bar{a}\times\bar{a} - \bar{a}\times\bar{b} - \bar{a}\times\bar{c} - \bar{b}\times\bar{a} + \bar{b}\times\bar{b} + \bar{b}\times\bar{c}\right]$$
$$= \left(\bar{a}+2\bar{b}-\bar{c}\right)\cdot\left[\bar{0} - \bar{a}\times\bar{b} - \bar{a}\times\bar{c} + \bar{a}\times\bar{b} + \bar{0} + \bar{b}\times\bar{c}\right]$$
$$= \left(\bar{a}+2\bar{b}-\bar{c}\right)\cdot\left[-\bar{a}\times\bar{c} + \bar{b}\times\bar{c}\right]$$
$$= -\bar{a}\cdot\left(\bar{a}\times\bar{c}\right) + \bar{a}\cdot\left(\bar{b}\times\bar{c}\right) - 2\bar{b}\cdot\left(\bar{a}\times\bar{c}\right) + 2\bar{b}\cdot\left(\bar{b}\times\bar{c}\right) + \bar{c}\cdot\left(\bar{a}\times\bar{c}\right) - \bar{c}\cdot\left(\bar{b}\times\bar{c}\right)$$
$$= -\left[\bar{a}\,\bar{a}\,\bar{c}\right] + \left[\bar{a}\,\bar{b}\,\bar{c}\right] - 2\left[\bar{b}\,\bar{a}\,\bar{c}\right] + 2\left[\bar{b}\,\bar{b}\,\bar{c}\right] + \left[\bar{c}\,\bar{a}\,\bar{c}\right] - \left[\bar{c}\,\bar{b}\,\bar{c}\right]$$
$$= -0 + \left[\bar{a}\,\bar{b}\,\bar{c}\right] + 2\left[\bar{a}\,\bar{b}\,\bar{c}\right] + 0 + 0 - 0$$
$$= 3\left[\bar{a}\,\bar{b}\,\bar{c}\right]$$
$$\therefore \quad \left(\bar{a}+2\bar{b}-\bar{c}\right)\cdot\left[\left(\bar{a}-\bar{b}\right)\times\left(\bar{a}-\bar{b}-\bar{c}\right)\right] = 3\left[\bar{a}\,\bar{b}\,\bar{c}\right]$$

[**Note:** *The question has been modified.*]

5. If $\bar{c} = 3\bar{a} - 2\bar{b}$, then prove that $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = 0$. **[Oct 14]**

Solution:
$\bar{c} = 3\bar{a} - 2\bar{b}$ $\quad\ldots$[Given]
$$\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] = \bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$$
$$= \bar{a}\cdot\left[\bar{b}\times\left(3\bar{a}-2\bar{b}\right)\right]$$
$$= \bar{a}\cdot\left[\bar{b}\times 3\bar{a} - \bar{b}\times 2\bar{b}\right]$$
$$= \bar{a}\cdot\left[\bar{b}\times 3\bar{a} - \bar{0}\right] \quad\ldots\left[\because \bar{b}\times\bar{b} = \bar{0}\right]$$
$$= 3\bar{a}\cdot\left[\bar{b}\times\bar{a}\right]$$
$$= 3\left[\bar{a}\ \ \bar{b}\ \ \bar{a}\right] = 3(0)$$
$$\therefore \quad \left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] = 0$$

6. If $\bar{u} = \hat{i} - 2\hat{j} + \hat{k}$, $\bar{r} = 3\hat{i} + \hat{k}$ and $\bar{w} = \hat{j} - \hat{k}$ are given vectors, then find $\left[\bar{u}+\bar{w}\right]\cdot\left[\left(\bar{u}\times\bar{r}\right)\times\left(\bar{r}\times\bar{w}\right)\right]$.

Solution:

$$\bar{u}\times\bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \hat{i}(-2-0) - \hat{j}(1-3) + \hat{k}(0+6)$$

$$= -2\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\bar{r}\times\bar{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(-3-0) + \hat{k}(3-0)$$

$$= -\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\bar{u}+\bar{w} = \left(\hat{i}-2\hat{j}+\hat{k}\right) + \left(\hat{j}-\hat{k}\right) = \hat{i} - \hat{j}$$

$$\left[\bar{u}+\bar{w}\right]\cdot\left[\left(\bar{u}\times\bar{r}\right)\times\left(\bar{r}\times\bar{w}\right)\right]$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -2 & 2 & 6 \\ -1 & 3 & 3 \end{vmatrix}$$

$$= 1(6-18) + 1(-6+6) + 0 = -12 + 0 = -12$$

$\therefore \quad \left[\bar{u}+\bar{w}\right]\cdot\left[\left(\bar{u}\times\bar{r}\right)\times\left(\bar{r}\times\bar{w}\right)\right] = -12$

[**Note:** *The question has been modified.*]

7. Find the volume of a tetrahedron whose vertices are A(–1, 2, 3) B(3, –2, 1), C(2, 1, 3) and D(–1, –2, 4).

Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ be the position vectors of points A, B, C and D respectively.

$\therefore \quad \bar{a} = -\hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = 3\hat{i} - 2\hat{j} + \hat{k}$,

$\bar{c} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\bar{d} = -\hat{i} - 2\hat{j} + 4\hat{k}$

$$\overline{AB} = \bar{b} - \bar{a} = \left(3\hat{i}-2\hat{j}+\hat{k}\right) - \left(-\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$= 4\hat{i} - 4\hat{j} - 2\hat{k}$$

$$\overline{AC} = \bar{c} - \bar{a} = \left(2\hat{i}+\hat{j}+3\hat{k}\right) - \left(-\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$= 3\hat{i} - \hat{j} + 0\hat{k}$$

$$\overline{AD} = \bar{d} - \bar{a} = \left(-\hat{i}-2\hat{j}+4\hat{k}\right) - \left(-\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$= 0\hat{i} - 4\hat{j} + \hat{k}$$

$$\text{Volume of tetrahedron} = \frac{1}{6}\left[\overline{AB} \quad \overline{AC} \quad \overline{AD}\right]$$

$$= \frac{1}{6}\begin{vmatrix} 4 & -4 & -2 \\ 3 & -1 & 0 \\ 0 & -4 & 1 \end{vmatrix}$$

$$= \frac{1}{6}[4(-1+0) + 4(3-0) - 2(-12+0)]$$

$$= \frac{1}{6}(-4+12+24)$$

$$= \frac{1}{6}(32)$$

$$= \frac{16}{3} \text{ cubic units}$$

8. If $\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = 3\hat{i} + 2\hat{j}$ and $\bar{c} = 2\hat{i} + \hat{j} + 3\hat{k}$, then verify that $\bar{a}\times\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}$.

Solution:

$\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = 3\hat{i} + 2\hat{j}$,

$\bar{c} = 2\hat{i} + \hat{j} + 3\hat{k}$

$$\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= \hat{i}(6-0) - \hat{j}(9-0) + \hat{k}(3-4)$$

$$= 6\hat{i} - 9\hat{j} - \hat{k}$$

$$\therefore \quad \bar{a}\times\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 6 & -9 & -1 \end{vmatrix}$$

$$= \hat{i}(-2+27) - \hat{j}(-1-18) + \hat{k}(-9-12)$$

$\therefore \quad \bar{a}\times\left(\bar{b}\times\bar{c}\right) = 25\hat{i} + 19\hat{j} - 21\hat{k} \quad \ldots\text{(i)}$

$$\bar{a}\cdot\bar{c} = (\hat{i}+2\hat{j}+3\hat{k})\cdot(2\hat{i}+\hat{j}+3\hat{k})$$

$$= 2 + 2 + 9 = 13$$

$$\bar{a}\cdot\bar{b} = (\hat{i}+2\hat{j}+3\hat{k})\cdot(3\hat{i}+2\hat{j})$$

$$= 3 + 4 + 0 = 7$$

$$\left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}$$

$$= 13\left(3\hat{i}+2\hat{j}\right) - 7\left(2\hat{i}+\hat{j}+3\hat{k}\right)$$

$$= 39\hat{i} + 26\hat{j} - 14\hat{i} - 7\hat{j} - 21\hat{k}$$

$= 25\hat{i} + 19\hat{j} - 21\hat{k} \quad \ldots\text{(ii)}$

From (i) and (ii), we get

$$\bar{a}\times\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}$$

[**Note:** *The question has been modified.*]

9. If $\bar{a} = \hat{i} - 2\hat{j}$, $\bar{b} = \hat{i} + 2\hat{j}$ and $\bar{c} = 2\hat{i} + \hat{j} - 2\hat{k}$, then find

i. $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$

ii. $\left(\bar{a}\times\bar{b}\right)\times\bar{c}$ Are the results same? Justify.

Solution:

i. $\bar{a}\times\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}$

$$\bar{a}\cdot\bar{c} = (\hat{i}-2\hat{j})\cdot(2\hat{i}+\hat{j}-2\hat{k}) = 2 - 2 - 0 = 0$$

$\bar{a}\cdot\bar{b} = (\hat{i}-2\hat{j})$ … -3

$\therefore \quad \bar{a}\times\left(\bar{b}\times\bar{c}\right) = 0(\hat{i}+2\hat{j}) - (-3)(2\hat{i}+\hat{j}-2\hat{k})$

$$= 0 + 6\hat{i} + 3\hat{j} - 6\hat{k}$$

$$= 6\hat{i} + 3\hat{j} - 6\hat{k}$$

Alternate method:

$$\bar{b}\times\bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \hat{i}(-4-0) - \hat{j}(-2-0) + \hat{k}(1-4)$$

$$= -4\hat{i} + 2\hat{j} - 3\hat{k}$$

$\therefore \quad \bar{a}\times\left(\bar{b}\times\bar{c}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ -4 & 2 & -3 \end{vmatrix}$$

$$= \hat{i}(6-0) - \hat{j}(-3+0) + \hat{k}(2-8)$$

$$= 6\hat{i} + 3\hat{j} - 6\hat{k}$$

ii. $\left(\bar{a}\times\bar{b}\right)\times\bar{c} = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{b}\cdot\bar{c}\right)\bar{a}$

$$\bar{a}\cdot\bar{c} = (\hat{i}-2\hat{j})\cdot(2\hat{i}+\hat{j}-2\hat{k}) = 2 - 2 - 0 = 0$$

$$\bar{b}\cdot\bar{c} = (\hat{i}+2\hat{j})\cdot(2\hat{i}+\hat{j}-2\hat{k}) = 2 + 2 - 0 = 4$$

$\therefore \quad \left(\bar{a}\times\bar{b}\right)\times\bar{c} = 0(\hat{i}+2\hat{j}) - 4(\hat{i}-2\hat{j})$

$$= 0 - 4\hat{i} + 8\hat{j}$$

$$= -4\hat{i} + 8\hat{j}$$

No the results are not same as $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ lies in the plane of $\bar{b}$ and $\bar{c}$ whereas $\left(\bar{a}\times\bar{b}\right)\times\bar{c}$ lies in the plane of $\bar{a}$ and $\bar{b}$.

[**Note:** *The question has been modified.*
Answer given in the textbook is $-2\hat{i} + 4\hat{j}$.
However, as per our calculation it is $-4\hat{i} + 8\hat{j}$.]

10. Show that $\bar{a}\times\left(\bar{b}\times\bar{c}\right) + \bar{b}\times\left(\bar{c}\times\bar{a}\right) + \bar{c}\times\left(\bar{a}\times\bar{b}\right) = 0$.

Solution:

$$\text{L.H.S.} = \bar{a}\times\left(\bar{b}\times\bar{c}\right) + \bar{b}\times\left(\bar{c}\times\bar{a}\right) + \bar{c}\times\left(\bar{a}\times\bar{b}\right)$$

$$= \left[\left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}\right] + \left[\left(\bar{b}\cdot\bar{a}\right)\bar{c} - \left(\bar{b}\cdot\bar{c}\right)\bar{a}\right] + \left[\left(\bar{c}\cdot\bar{b}\right)\bar{a} - \left(\bar{c}\cdot\bar{a}\right)\bar{b}\right]$$

$$= \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c} + \left(\bar{a}\cdot\bar{b}\right)\bar{c} - \left(\bar{b}\cdot\bar{c}\right)\bar{a} + \left(\bar{b}\cdot\bar{c}\right)\bar{a} - \left(\bar{a}\cdot\bar{c}\right)\bar{b}$$

$$= 0 = \text{R. H. S.}$$

Miscellaneous Exercise – 5

I. Select the correct option from the given alternatives :

1. If $|\bar{a}| = 2$, $|\bar{b}| = 3$, $|\bar{c}| = 4$, then $\left[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}-\bar{a}\right]$ is equal to
   (A) 24 (B) –24
   (C) 0 (D) 48

2. If $|\bar{a}| = 3$, $|\bar{b}| = 4$, then the value of λ for which $\bar{a} + \lambda\bar{b}$ is perpendicular to $\bar{a} - \lambda\bar{b}$, is
   (A) $\frac{9}{16}$ (B) $\frac{3}{4}$
   (C) $\frac{3}{2}$ (D) $\frac{4}{3}$

3. If sum of two unit vectors is itself a unit vector, then the magnitude of their difference is
   (A) $\sqrt{2}$ (B) $\sqrt{3}$
   (C) 1 (D) 2

4. If $|\bar{a}| = 3$, $|\bar{b}| = 5$, $|\bar{c}| = 7$ and $\bar{a} + \bar{b} + \bar{c} = 0$, then the angle between $\bar{a}$ and $\bar{b}$ is
   (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$
   (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

[**Note:** *Option (B) has been modified.*]

5. The volume of tetrahedron whose vertices are (1, –6, 10), (–1, –3, 7), (5, – 1, λ) and (7, –4, 7) is 11 cu. units, then the value of λ is
   (A) 7 (B) $\frac{\pi}{3}$
   (C) 1 (D) 5

6. If α, β, γ are direction angles of a line and $\alpha = 60°$, $\beta = 45°$, then $\gamma =$
   (A) 30° or 90° (B) 45° or 60°
   (C) 90° or 30° (D) 60° or 120°

7. The distance of the point (3, 4, 5) from Y- axis is
   (A) 3 (B) 5
   (C) $\sqrt{34}$ (D) $\sqrt{41}$

8. The line joining the points (–2, 1, –8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2, 3. The values of a, b, c are
   (A) 4, 3, –5 (B) 1, 2, $\frac{-13}{2}$
   (C) 10, 5, –2 (D) 3, 5, 11

9. If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of a line, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is
(A) 1 (B) 2
(C) 3 (D) 4

10. If l, m, n are direction cosines of a line, then $l\hat{i} + m\hat{j} + n\hat{k}$ is
(A) null vector
(B) the unit vector along the line
(C) any vector along the line
(D) a vector perpendicular to the line

11. If $|\bar{a}| = 3$ and $-1 \le k \le 2$, then $|k\bar{a}|$ lies in the interval
(A) [0, 6] (B) [−3, 6]
(C) [3, 6] (D) [1, 2]

12. Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$, $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$
(A) are collinear
(B) form an equilateral triangle
(C) form a scalene triangle
(D) form a right angled triangle

13. Let $\bar{p}$ and $\bar{q}$ be the position vectors of P and Q respectively, with respect to O and $|\bar{p}| = p$, $|\bar{q}| = q$. The points R and S divide PQ internally and externally in the ratio 2 : 3 respectively. If OR and OS are perpendicular, then
(A) $9p^2 = 4q^2$ (B) $4p^2 = 9q^2$
(C) $9p = 4q$ (D) $4p = 9q$

14. The 2 vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is
(A) $\frac{\sqrt{34}}{2}$ (B) $\frac{\sqrt{48}}{2}$
(C) $\sqrt{18}$ (D) None of these

15. If $\bar{a}$ and $\bar{b}$ are unit vectors, then what is the angle between $\bar{a}$ and $\bar{b}$ for $\sqrt{3}\bar{a} - \bar{b}$ to be a unit vector?
(A) 30° (B) 45°
(C) 60° (D) 90°

16. If θ be the angle between any two vectors $\bar{a}$ and $\bar{b}$, then $|\bar{a}\cdot\bar{b}| = |\bar{a}\times\bar{b}|$, when θ is equal to
(A) 0 (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$ (D) π

17. The value of $\hat{i}\cdot(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$ is
(A) 0 (B) −1
(C) 1 (D) 3

18. Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then c is
(A) the arithmetic mean of a and b
(B) the geometric mean of a and b
(C) the harmonic mean of a and b
(D) 0

19. Let $\bar{a} = \hat{i} - \hat{j}$, $\bar{b} = \hat{j} - \hat{k}$, $\bar{c} = \hat{k} - \hat{i}$. If $\bar{d}$ is a unit vector such that $\bar{a}\cdot\bar{d} = 0 = [\bar{b}\ \bar{c}\ \bar{d}]$, then $\bar{d}$ equals
(A) $\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ (B) $\pm\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$
(C) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ (D) $\pm\hat{k}$

[Note: *Option (B) has been modified.*]

20. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are non coplanar unit vectors such that $\bar{a}\times(\bar{b}\times\bar{c}) = \frac{(\bar{b}+\bar{c})}{\sqrt{2}}$, then the angle between $\bar{a}$ and $\bar{b}$ is
(A) $\frac{3\pi}{4}$ (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$ (D) π

Answers:

1.	(C)	2.	(B)	3.	(B)	4.	(B)
5.	(A)	6.	(D)	7.	(C)	8.	(A)
9.	(B)	10.	(B)	11.	(A)	12.	(B)
13.	(A)	14.	(A)	15.	(A)	16.	(B)
17.	(C)	18.	(B)	19.	(A)	20.	(A)

Hints:

1. $(\bar{a}+\bar{b})\cdot[(\bar{b}+\bar{c})\times(\bar{c}-\bar{a})]$
$= (\bar{a}+\bar{b})\cdot[\bar{b}\times\bar{c} - \bar{b}\times\bar{a} + \bar{c}\times\bar{c} - \bar{c}\times\bar{a}]$
$= (\bar{a}+\bar{b})\cdot[\bar{b}\times\bar{c} - \bar{b}\times\bar{a} - \bar{c}\times\bar{a}]$
$= \bar{a}\cdot(\bar{b}\times\bar{c}) - 0 - 0 + 0 - 0 - \bar{b}\cdot(\bar{c}\times\bar{a})$
$= [\bar{a}\ \bar{b}\ \bar{c}] - [\bar{b}\ \bar{c}\ \bar{a}]$
$= [\bar{a}\ \bar{b}\ \bar{c}] - [\bar{a}\ \bar{b}\ \bar{c}]$
$= 0$

2. $\bar{a}+\lambda\bar{b}$ is perpendicular to $\bar{a}-\lambda\bar{b}$.
∴ $(\bar{a}+\lambda\bar{b})\cdot(\bar{a}-\lambda\bar{b}) = 0$
∴ $\bar{a}\cdot\bar{a} - \lambda\bar{a}\cdot\bar{b} + \lambda\bar{b}\cdot\bar{a} - \lambda^2\bar{b}\cdot\bar{b} = 0$
∴ $|\bar{a}|^2 - \lambda^2|\bar{b}|^2 = 0$
∴ $3^2 - \lambda^2(4)^2 = 0$
∴ $\lambda = \frac{3}{4}$

3. Let $\bar{a}$ and $\bar{b}$ be two unit vectors.
$|a+b|^2 = 1$
∴ $|\bar{a}|^2 + |\bar{b}|^2 + 2\bar{a}\cdot\bar{b} = 1$
∴ $1 + 1 + 2|\bar{a}||\bar{b}|\cos\theta = 1$
∴ $2 + 2\cos\theta = 1$
∴ $\cos\theta = \frac{-1}{2}$
Now,
$|a-b|^2 = |\bar{a}|^2 + |\bar{b}|^2 - 2\bar{a}\cdot\bar{b}$
$= 1 + 1 - 2|\bar{a}||\bar{b}|\cos\theta$
$= 2 - 2\left(\frac{-1}{2}\right)$
$= 2 - (-1)$
∴ $|a-b|^2 = 3$
∴ $|a-b| = \sqrt{3}$

4. $\bar{a}+\bar{b}+\bar{c} = 0$
$\bar{a}+\bar{b} = -\bar{c}$
∴ $|\bar{a}+\bar{b}|^2 = |\bar{c}|^2$
∴ $|\bar{a}|^2 + |\bar{b}|^2 + 2\bar{a}\cdot\bar{b} = |\bar{c}|^2$
∴ $3^2 + 5^2 + 2|\bar{a}||\bar{b}|\cos\theta = 7^2$
∴ $2\times3\times5\cos\theta = 15$
∴ $30\cos\theta = 15$
∴ $\cos\theta = \frac{1}{2}$
∴ $\theta = \frac{\pi}{3}$

5. Let A(1, −6, 10), B(−1, −3, 7), C(5, −1, λ) and D(7, −4, 7) be the vertices of tetrahedron.
∴ $\overline{AB} = \bar{b}-\bar{a} = -2\hat{i}+3\hat{j}-3\hat{k}$
$\overline{AC} = \bar{c}-\bar{a} = 4\hat{i}+5\hat{j}+(\lambda-10)\hat{k}$
$\overline{AD} = \bar{d}-\bar{a} = 6\hat{i}+2\hat{j}-3\hat{k}$
Volume of tetrahedron $= \frac{1}{6}[\overline{AB}\ \overline{AC}\ \overline{AD}]$
∴ $11 = \frac{1}{6}\begin{vmatrix} -2 & 3 & -3 \\ 4 & 5 & \lambda-10 \\ 6 & 2 & -3 \end{vmatrix}$
∴ $66 = -2(-15 - 2\lambda + 20) - 3(-12 - 6\lambda + 60) - 3(8 - 30)$

∴ $66 = -10 + 4\lambda - 144 + 18\lambda + 66$
∴ $22\lambda = 154$
∴ $\lambda = 7$

6. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
∴ $\cos^2 60° + \cos^2 45° + \cos^2\gamma = 1$
∴ $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$
∴ $\frac{1}{4} + \frac{1}{2} + \cos^2\gamma = 1$
∴ $\cos^2\gamma = 1 - \frac{3}{4}$
∴ $\cos^2\gamma = \frac{1}{4}$
∴ $\cos\gamma = \pm\frac{1}{2}$
∴ $\gamma = 60°$ or $120°$

7. Distance of the point (3, 4, 5) from Y-axis
$= \sqrt{(0-3)^2 + (4-4)^2 + (0-5)^2}$
$= \sqrt{9+25}$
$= \sqrt{34}$

8. Direction ratios of a line joining the points (−2, 1, −8) and (a, b, c) are a + 2, b − 1, c + 8, is parallel to the line whose direction ratios are 6, 2, 3.
∴ a + 2 = 6 implies a = 4
∴ b − 1 = 2 implies b = 3
∴ c + 8 = 3 implies c = −5

9. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$
$= 1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma$
$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$
$= 3 - 1$
$= 2$

12. Let $\bar{a} = \alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$, $\bar{b} = \beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}$,
$\bar{c} = \gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$
$|\bar{a}| = |\bar{b}| = |\bar{c}| = \sqrt{\alpha^2+\beta^2+\gamma^2}$
∴ It forms an equilateral triangle.

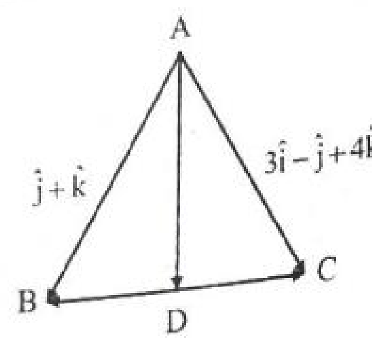
13. R divides PQ internally in the ratio 2:3.
∴ $\bar{r} = \frac{2\bar{q}+3\bar{p}}{5}$
S divides PQ externally in the ratio 2:3.
∴ $\bar{s} = \frac{2\bar{q}-3\bar{p}}{2-3}$
∴ $\bar{s} = 3\bar{p} - 2\bar{q}$
OR and OS are perpendicular.
∴ $\bar{r}\cdot\bar{s} = 0$

$\therefore \quad \frac{2}{5}(-2)|\bar{q}|^2 + \frac{3}{5}(3)|\bar{p}|^2 = 0$

$\therefore \quad \frac{9}{5}p^2 = \frac{4}{5}q^2$

$\therefore \quad 9p^2 = 4q^2$

14.



In ΔABC,
by using triangle law of vector addition, we get

$\overline{BC} = \overline{AC} - \overline{AB}$

$= (3\hat{i}-\hat{j}+4\hat{k}) - (\hat{j}+\hat{k})$

$= 3\hat{i}-2\hat{j}+3\hat{k}$

$\overline{BD} = \frac{1}{2}\overline{BC} = \frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}$

Now, In ΔABD,
by using triangle law of vector addition, we get

$\overline{AD} = \overline{AB} + \overline{BD}$

$= (\hat{j}+\hat{k}) + \left(\frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}\right)$

$= \frac{3}{2}\hat{i}+\frac{5}{2}\hat{k}$

$\therefore$ Length of median $= |\overline{AD}|$

$= \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{5}{2}\right)^2}$

$= \frac{\sqrt{34}}{2}$

15. Let $|\sqrt{3}\,\bar{a}-\bar{b}| = 1$

Squaring on both sides, we get

$(\sqrt{3})^2|\bar{a}|^2 - 2\sqrt{3}\,\bar{a}\cdot\bar{b} + |\bar{b}|^2 = 1$

$\therefore \quad 3 - 2\sqrt{3}|\bar{a}||\bar{b}|\cos\theta + 1 = 1$

$\therefore \quad 3 - 2\sqrt{3}\cos\theta = 0$

$\therefore \quad \cos\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad \theta = 30^\circ$

16. $|\bar{a}\cdot\bar{b}| = |\bar{a}\times\bar{b}|$

$\therefore \quad |\bar{a}||\bar{b}|\cos\theta = |\bar{a}||\bar{b}|\sin\theta$

$\therefore \quad \tan\theta = 1$

$\therefore \quad \theta = \frac{\pi^c}{4}$

17. $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j})$

$= \hat{i}\cdot(\hat{i})+\hat{j}\cdot(-\hat{j})+\hat{k}\cdot(\hat{k})$

$= 1 - 1 + 1$

$= 1$

18. $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$, $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane.

$\therefore \quad \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$\therefore \quad a(-c) - a(b-c) + c(c) = 0$

$\therefore \quad -ac - ab + ac + c^2 = 0$

$\therefore \quad c^2 = ab$

$\therefore$ c is the geometric mean of a and b.

19. Let $\bar{d} = x\hat{i}+y\hat{j}+z\hat{k}$

$\bar{a}\cdot\bar{d} = 0$

$\therefore \quad x - y = 0$

$\therefore \quad x = y$

$\therefore \quad \bar{d} = x\hat{i}+x\hat{j}+z\hat{k}$

$[\bar{b}\ \ \bar{c}\ \ \bar{d}] = 0$

$\therefore \quad \bar{b}\cdot(\bar{c}\times\bar{d}) = 0$

$\therefore \quad \bar{c}\times\bar{d} = (\hat{k}-\hat{i})\times(x\hat{i}+x\hat{j}+z\hat{k})$

$= -x\hat{i}+(z+x)\hat{j}-x\hat{k}$

Now $\bar{b}\cdot(\bar{c}\times\bar{d}) = (\hat{j}-\hat{k})\cdot(-x\hat{i}+(z+x)\hat{j}-x\hat{k})$

$\therefore \quad 0 = z + x + x$

$\therefore \quad z = -2x$

$|\bar{d}| = 1$

$\therefore \quad \sqrt{x^2+x^2+(-2x)^2} = 1$

$\therefore \quad \sqrt{6x^2} = 1$

$\therefore \quad x = \pm\frac{1}{\sqrt{6}}$

$\therefore \quad \bar{d} = \pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$

20. $\bar{a}\times(\bar{b}\times\bar{c}) = \frac{(\bar{b}+\bar{c})}{\sqrt{2}}$

$\therefore \quad (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c} = \frac{\bar{b}+\bar{c}}{\sqrt{2}}$

$\therefore \quad (\bar{a}\cdot\bar{c})\bar{b} - \frac{\bar{b}}{\sqrt{2}} - (\bar{a}\cdot\bar{b})\bar{c} - \frac{\bar{c}}{\sqrt{2}} = 0$

$\therefore \quad \left(\bar{a}\cdot\bar{c}-\frac{1}{\sqrt{2}}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}+\frac{1}{\sqrt{2}}\right)\bar{c} = 0$

$\bar{a}$, $\bar{b}$, $\bar{c}$ are non coplanar linearly independent.

$\therefore \quad \bar{a}\cdot\bar{b} = \frac{-1}{\sqrt{2}}$

$\therefore \quad |\bar{a}||\bar{b}|\cos\theta = \frac{-1}{\sqrt{2}}$

$\therefore \quad \cos\theta = \frac{-1}{\sqrt{2}}$

$\therefore \quad \theta = \frac{3\pi}{4}$

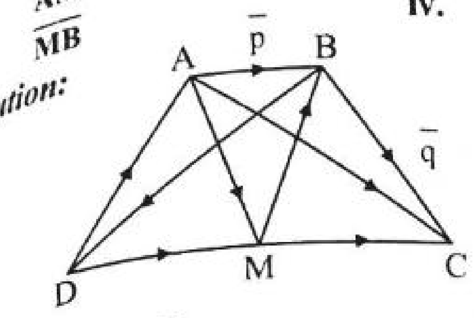
II. Answer the following:

1. ABCD is a trapezium with AB parallel to DC and DC = 3AB. M is the mid-point of DC, $\overline{AB} = \bar{p}$ and $\overline{BC} = \bar{q}$. Find in terms of $\bar{p}$ and $\bar{q}$.

i. $\overline{AM}$ ii. $\overline{BD}$

iii. $\overline{MB}$ iv. $\overline{DA}$

Solution:



DC = 3AB ...[Given]

$\therefore \quad \overline{DC} = 3\,\overline{AB}$

$\therefore \quad \overline{DC} = 3\,\bar{p}$

M is the midpoint of DC. ...[Given]

$\therefore \quad \overline{MC} = \frac{1}{2}\overline{DC}$

$\therefore \quad \overline{MC} = \frac{3}{2}\bar{p}$

i. In ΔAMC and ΔABC,
by using triangle law of vector addition, we get

$\overline{AC} = \overline{AM} + \overline{MC}$, $\overline{AC} = \overline{AB} + \overline{BC} = \bar{p} + \bar{q}$

$\therefore \quad \overline{AM} = \overline{AC} - \overline{MC}$

$= \bar{p} + \bar{q} - \frac{3}{2}\bar{p}$

$\therefore \quad \overline{AM} = \frac{2\bar{p}+2\bar{q}-3\bar{p}}{2}$

$\therefore \quad \overline{AM} = \frac{2\bar{q}-\bar{p}}{2} = \bar{q} - \frac{1}{2}\bar{p}$

ii. In ΔBDC,
by using triangle law of vector addition, we get

$\overline{BC} = \overline{BD} + \overline{DC}$

$\therefore \quad \overline{BD} = \overline{BC} - \overline{DC}$

$\therefore \quad \overline{BD} = \bar{q} - 3\,\bar{p}$

iii. In ΔMBC,
by using triangle law of vector addition, we get

$\overline{MC} = \overline{MB} + \overline{BC}$

$\therefore \quad \overline{MB} = \overline{MC} - \overline{BC}$

$= \frac{3}{2}\bar{p} - \bar{q}$

iv. In ΔADB,
by using triangle law of vector addition, we get

$\overline{DB} = \overline{DA} + \overline{AB}$

$\therefore \quad \overline{DA} = \overline{DB} - \overline{AB}$

$= -\overline{BD} - \overline{AB}$

$= 3\bar{p} - \bar{q} - \bar{p}$

$= 2\bar{p} - \bar{q}$

[**Note:** Answer given in the textbook are in terms of $\bar{a}$ and $\bar{b}$.]

2. The points A, B and C have position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively. The point P is midpoint of AB. Find in terms of $\bar{a}$, $\bar{b}$ and $\bar{c}$ the vector $\overline{PC}$.

***Solution*:**

Point P$(\bar{p})$ is the midpoint of AB.

By mid-point formula, we get

$\bar{p} = \frac{\bar{a}+\bar{b}}{2}$ ...(i)

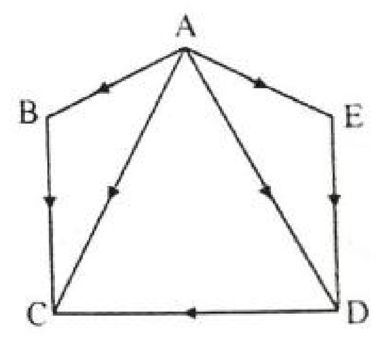
Consider $\overline{PC} = \bar{c} - \bar{p}$

$= \bar{c} - \left(\frac{\bar{a}+\bar{b}}{2}\right)$ ...[From (i)]

$\therefore \quad \overline{PC} = -\frac{1}{2}\bar{a} - \frac{1}{2}\bar{b} + \bar{c}$

3. In a pentagon ABCDE, show that $\overline{AB}+\overline{AE}+\overline{BC}+\overline{DC}+\overline{ED} = 2\overline{AC}$.

***Solution*:**



Consider L.H.S.

$= \overline{AB}+\overline{AE}+\overline{BC}+\overline{DC}+\overline{ED}$

$= (\overline{AB}+\overline{BC})+(\overline{AE}+\overline{ED})+\overline{DC}$

$= \overline{AC}+\overline{AD}+\overline{DC}$ ...[by triangle law of vector addition]

$= \overline{AC}+(\overline{AD}+\overline{DC})$

$= \overline{AC}+\overline{AC}$ ...[by triangle law of vector addition]

$= 2\overline{AC}$ = R.H.S.

$\therefore \quad \overline{AB}+\overline{AE}+\overline{BC}+\overline{DC}+\overline{ED} = 2\overline{AC}$

Alternate Method:

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}$ be the position vectors of points A, B, C, D, E respectively w. r. t. fixed point.

Consider L.H.S. $= \overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED}$

$= \bar{b} - \bar{a} + \bar{e} - \bar{a} + \bar{c} - \bar{b} + \bar{c} - \bar{d} + \bar{d} - \bar{e}$

$= 2\bar{c} - 2\bar{a}$

$= 2(\bar{c} - \bar{a})$

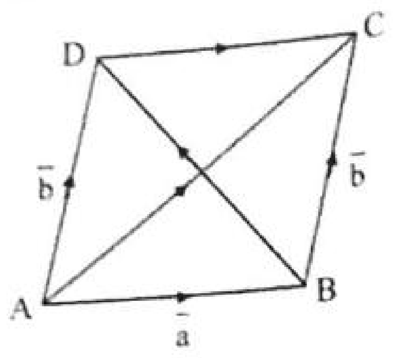
$= 2\,\overline{AC}$

= R.H.S.

$\therefore \quad \overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} = 2\,\overline{AC}$

4. If in parallelogram ABCD, diagonal vectors are $\overline{AC} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overline{BD} = -6\hat{i} + 7\hat{j} - 2\hat{k}$, then find the adjacent side vectors $\overline{AB}$ and $\overline{AD}$.

Solution:



Let $\overline{AB} = \bar{a}$ and $\overline{AD} = \bar{b}$

In ΔABC and ΔABD,

by using triangle law of vector addition, we get

$\overline{AC} = \overline{AB} + \overline{BC}$ and $\overline{BD} = \overline{BA} + \overline{AD}$

$\therefore \quad \overline{AC} = \bar{a} + \bar{b}$ ...(i)

and $\overline{BD} = -\overline{AB} + \overline{AD}$

$\therefore \quad \overline{BD} = -\bar{a} + \bar{b}$ ...(ii)

Subtracting (ii) from (i), we get

$2\bar{a} = \overline{AC} - \overline{BD}$

$\therefore \quad \bar{a} = \dfrac{\overline{AC} - \overline{BD}}{2}$

$\therefore \quad \overline{AB} = \dfrac{(2\hat{i} + 3\hat{j} + 4\hat{k}) - (-6\hat{i} + 7\hat{j} - 2\hat{k})}{2}$

$\therefore \quad \overline{AB} = \dfrac{8\hat{i} - 4\hat{j} + 6\hat{k}}{2}$

$\therefore \quad \overline{AB} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

Adding (i) and (ii), we get

$2\bar{b} = \overline{AC} + \overline{BD}$

$\therefore \quad \bar{b} = \dfrac{\overline{AC} + \overline{BD}}{2}$

$\therefore \quad \overline{AD} = \dfrac{(2\hat{i} + 3\hat{j} + 4\hat{k}) + (-6\hat{i} + 7\hat{j} - 2\hat{k})}{2}$

$\therefore \quad \overline{AD} = \dfrac{-4\hat{i} + 10\hat{j} + 2\hat{k}}{2}$

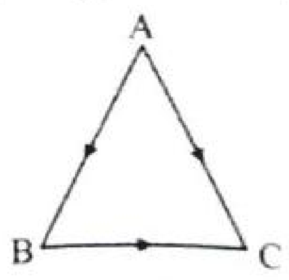
$\therefore \quad \overline{AD} = -2\hat{i} + 5\hat{j} + \hat{k}$

[Note: Answers given in the textbook $\overline{AB} = -2\hat{i} + 5\hat{j} + \hat{k}$ and $\overline{AD} = 4\hat{i} - 2\hat{j} + 3\hat{k}$. However, as per our calculation $\overline{AB} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\overline{AD} = -2\hat{i} + 5\hat{j} + \hat{k}$]

5. If two sides of a triangle are $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{k}$, then find the length of the third side.

Solution:

Let $\overline{AB} = \hat{i} + 2\hat{j}$, $\overline{BC} = \hat{i} + \hat{k}$



In ΔABC,

by using triangle law of vector addition, we get

$\overline{AC} = \overline{AB} + \overline{BC}$

$\therefore \quad \overline{AC} = \hat{i} + 2\hat{j} + \hat{i} + \hat{k}$

$\therefore \quad \overline{AC} = 2\hat{i} + 2\hat{j} + \hat{k}$

$\therefore \quad$ Length of the third side $= |\overline{AC}|$

$= \sqrt{(2)^2 + (2)^2 + 1^2}$

$= \sqrt{9}$

$= 3$ units

6. If $|\bar{a}| = |\bar{b}| = 1$, $\bar{a} \cdot \bar{b} = 0$ and $\bar{a} + \bar{b} + \bar{c} = 0$, then find $|\bar{c}|$.

Solution:

$|\bar{a}| = |\bar{b}| = 1$, $\bar{a} \cdot \bar{b} = 0$ ...[Given]

Consider $\bar{a} + \bar{b} + \bar{c} = 0$

$\therefore \quad \bar{a} + \bar{b} = -\bar{c}$

Squaring on both sides, we get

$|\bar{a} + \bar{b}|^2 = |\bar{c}|^2$

$\therefore \quad |\bar{a}|^2 + 2\bar{a} \cdot \bar{b} + |\bar{b}|^2 = |\bar{c}|^2$

$\therefore \quad 1^2 + 2(0) + 1^2 = |\bar{c}|^2$

$\therefore \quad |\bar{c}|^2 = 2$

$\therefore \quad |\bar{c}| = \sqrt{2}$

7. Find the lengths of the sides of the triangle and also determine the type of a triangle.

i. A(2, –1, 0), B(4, 1, 1), C(4, –5, 4)

ii. L(3, –2, –3), M(7, 0,1), N (1, 2, 1)

Solution:

i. Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of points A, B, C respectively.

$\bar{a} = 2\hat{i} - \hat{j} + 0\hat{k}$, $\bar{b} = 4\hat{i} + \hat{j} + \hat{k}$,

$\bar{c} = 4\hat{i} - 5\hat{j} + 4\hat{k}$

Consider

$\overline{AB} = \bar{b} - \bar{a}$

$= (4\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 0\hat{k})$

$= 2\hat{i} + 2\hat{j} + \hat{k}$

$\therefore \quad |\overline{AB}| = \sqrt{2^2 + 2^2 + 1^2}$

$= \sqrt{9}$

$\therefore \quad |\overline{AB}| = 3$

$\therefore \quad l(AB) = 3$ units ...(i)

Now, $\overline{BC} = \bar{c} - \bar{b}$

$= (4\hat{i} - 5\hat{j} + 4\hat{k}) - (4\hat{i} + \hat{j} + \hat{k})$

$= 0\hat{i} - 6\hat{j} + 3\hat{k}$

$\therefore \quad |\overline{BC}| = \sqrt{0^2 + (-6)^2 + 3^2}$

$= \sqrt{45}$

$\therefore \quad |\overline{BC}| = 3\sqrt{5}$

$\therefore \quad l(BC) = 3\sqrt{5}$ units ...(ii)

Also, $\overline{AC} = \bar{c} - \bar{a}$

$= (4\hat{i} - 5\hat{j} + 4\hat{k}) - (2\hat{i} - \hat{j} + 0\hat{k})$

$\overline{AC} = 2\hat{i} - 4\hat{j} + 4\hat{k}$

$\therefore \quad |\overline{AC}| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{36}$

$\therefore \quad |\overline{AC}| = 6$

$\therefore \quad l(AC) = 6$ units ...(iii)

Consider $AB^2 + AC^2 = 3^2 + 6^2$ ...[From (i) and (iii)]

$= 9 + 36$

$\therefore \quad AB^2 + AC^2 = 45$ ...(iv)

From (ii), we get

$BC^2 = (3\sqrt{5})^2 = 45$ ...(v)

$\therefore \quad AB^2 + AC^2 = BC^2$ ...[From (iv) and (v)]

$\therefore \quad \Delta$ABC is a right angled triangle.

...[By converse of Pythagoras theorem]

ii. Let $\bar{l}, \bar{m}, \bar{n}$ be the position vectors of points L, M, N respectively.

$\therefore \quad \bar{l} = 3\hat{i} - 2\hat{j} - 3\hat{k}$, $\bar{m} = 7\hat{i} + 0\hat{j} + \hat{k}$, $\bar{n} = \hat{i} + 2\hat{j} + \hat{k}$

Consider $\overline{LM} = \bar{m} - \bar{l}$

$= (7\hat{i} + 0\hat{j} + \hat{k}) - (3\hat{i} - 2\hat{j} - 3\hat{k})$

$\therefore \quad \overline{LM} = 4\hat{i} + 2\hat{j} + 4\hat{k}$

$\therefore \quad |\overline{LM}| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36}$

$\therefore \quad |\overline{LM}| = 6$

$\therefore \quad l(LM) = 6$ units ...(i)

Now, $\overline{MN} = \bar{n} - \bar{m}$

$= (\hat{i} + 2\hat{j} + \hat{k}) - (7\hat{i} + 0\hat{j} + \hat{k})$

$\therefore \quad \overline{MN} = -6\hat{i} + 2\hat{j} + 0\hat{k}$

$\therefore \quad |\overline{MN}| = \sqrt{(-6)^2 + 2^2 + 0^2}$

$\therefore \quad |\overline{MN}| = \sqrt{40} = 2\sqrt{10}$

$\therefore \quad l(MN) = 2\sqrt{10}$ units ...(ii)

Also, $\overline{LN} = \bar{n} - \bar{l}$

$= (\hat{i} + 2\hat{j} + \hat{k}) - (3\hat{i} - 2\hat{j} - 3\hat{k})$

$\therefore \quad \overline{LN} = (-2\hat{i} + 4\hat{j} + 4\hat{k})$

$\therefore \quad |\overline{LN}| = \sqrt{(-2)^2 + 4^2 + 4^2} = \sqrt{36}$

$\therefore \quad |\overline{LN}| = 6$

$\therefore \quad l(LN) = 6$ units ...(iii)

$\therefore \quad l(LM) = l(LN)$ ...[From (i) and (iii)]

$\therefore \quad \Delta$LMN is an isosceles triangle.

8. Find the component form of $\bar{a}$, if

i. It lies in YZ plane and makes 60° with positive Y-axis and $|\bar{a}| = 4$

ii. It lies in XZ plane and makes 45° with positive Z-axis and $|\bar{a}| = 10$

Solution:

i. Let $\bar{a} = 0\hat{i} + y\hat{j} + z\hat{k}$ ...(YZ plane)

$\bar{b} = 0\hat{i} + y\hat{j} + 0\hat{k}$ ...(Y-axis)

$|\bar{a}| = 4$, $|\bar{b}| = y$

$\cos\theta = \dfrac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|}$

$\therefore \quad \cos 60^\circ = \dfrac{(0\hat{i} + y\hat{j} + z\hat{k}) \cdot (0\hat{i} + y\hat{j} + 0\hat{k})}{4y}$

$\therefore \quad \dfrac{1}{2} = \dfrac{(0)(0) + (y)(y) + (z)(0)}{4y}$

$\therefore \quad \dfrac{1}{2} = \dfrac{y^2}{4y}$

$\therefore \quad y = 2$ ...(i)

$|\bar{a}| = 4$ ...[Given]

$\therefore \quad \sqrt{y^2 + z^2} = 4$

$\therefore \quad \sqrt{2^2 + z^2} = 4$ ...[From (i)]

$\therefore \quad 4 + z^2 = 16$

$\therefore \quad z^2 = 12$

$\therefore \quad z = \pm 2\sqrt{3}$ ...(ii)

$\bar{a} = y\hat{j} + z\hat{k}$

$\therefore \quad \bar{a} = 2\hat{j} \pm 2\sqrt{3}\hat{k}$ ...[From (i) and (ii)]

ii. Let $\bar{a} = x\hat{i} + 0\hat{j} + z\hat{k}$ ...(XZ plane)

$\bar{b} = 0\hat{i} + 0\hat{j} + z\hat{k}$ ...(Z-axis)

$|\bar{a}| = 10,\ |\bar{b}| = z$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$$

$\therefore \quad \cos 45° = \dfrac{(x\hat{i}+0\hat{j}+z\hat{k})\cdot(0\hat{i}+0\hat{j}+z\hat{k})}{10z}$

$\therefore \quad \dfrac{1}{\sqrt{2}} = \dfrac{(x)(0)+(0)(0)+(z)(z)}{10z}$

$\therefore \quad \dfrac{1}{\sqrt{2}} = \dfrac{z^2}{10z}$

$\therefore \quad z = \dfrac{10}{\sqrt{2}}$

$\therefore \quad z = 5\sqrt{2}$ ...(i)

$|\bar{a}| = 10$ ...[Given]

$\therefore \quad \sqrt{x^2+z^2} = 10$

$\therefore \quad \sqrt{x^2+(5\sqrt{2})^2} = 10$ ...[From (i)]

$\therefore \quad x^2 + 50 = 100$

$\therefore \quad x^2 = 50$

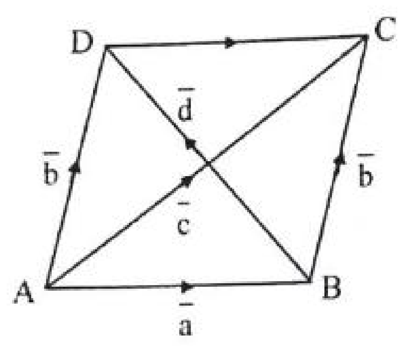
$\therefore \quad x = \pm 5\sqrt{2}$ ...(ii)

$\bar{a} = x\hat{i} + z\hat{k}$

$\therefore \quad \bar{a} = \pm 5\sqrt{2}\,\hat{i} + 5\sqrt{2}\,\hat{k}$ ...[From (i) and (ii)]

9. Two sides of a parallelogram are $3\hat{i}+4\hat{j}-5\hat{k}$ and $-2\hat{j}+7\hat{k}$. Find the unit vectors parallel to the diagonals.

Solution:



Let $\bar{a} = 3\hat{i}+4\hat{j}-5\hat{k}$ and $\bar{b} = -2\hat{j}+7\hat{k}$

Let $\bar{c}$, $\bar{d}$ be the diagonals of a parallelogram.

By using triangle law of vector addition, we get

$\bar{c} = \bar{a}+\bar{b}$ and $\bar{d} = \bar{b}-\bar{a}$

$\therefore \quad \bar{c} = (3\hat{i}+4\hat{j}-5\hat{k})+(-2\hat{j}+7\hat{k})$ and

$\bar{d} = (-2\hat{j}+7\hat{k})-(3\hat{i}+4\hat{j}-5\hat{k})$

$\therefore \quad \bar{c} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\bar{d} = -3\hat{i}-6\hat{j}+12\hat{k}$

$\therefore \quad |\bar{c}| = \sqrt{3^2+2^2+2^2}$ and

$|\bar{d}| = \sqrt{(-3)^2+(-6)^2+(12)^2}$

$\therefore \quad |\bar{c}| = \sqrt{9+4+4}$ and $|\bar{d}| = \sqrt{9+36+144}$

$\therefore \quad |\bar{c}| = \sqrt{17}$ and $|\bar{d}| = 3\sqrt{21}$

Unit vectors parallel to the diagonals $\bar{c}$ and $\bar{d}$ are given by

$\hat{c} = \dfrac{\bar{c}}{|\bar{c}|}$ and $\hat{d} = \dfrac{\bar{d}}{|\bar{d}|}$

$\therefore \quad \hat{c} = \dfrac{3\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{17}}$ and $\hat{d} = \dfrac{-3\hat{i}-6\hat{j}+12\hat{k}}{3\sqrt{21}}$

$\therefore \quad \hat{c} = \dfrac{1}{\sqrt{17}}(3\hat{i}+2\hat{j}+2\hat{k})$ and $\hat{d} = \dfrac{1}{\sqrt{21}}(-\hat{i}-2\hat{j}+4\hat{k})$

10. If D, E, F are the mid-points of the sides BC, CA, AB of a triangle ABC, prove that $\overline{AD}+\overline{BE}+\overline{CF} = \bar{0}$.

Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ be the position vectors of points A, B, C, D, E and F respectively with respect to a fixed point.

Given, D, E and F are the midpoints of sides BC, CA and AB respectively.

$\therefore \quad \bar{d} = \dfrac{\bar{b}+\bar{c}}{2},\ \bar{e} = \dfrac{\bar{c}+\bar{a}}{2},\ \bar{f} = \dfrac{\bar{a}+\bar{b}}{2}$

Consider, L.H.S. $= \overline{AD}+\overline{BE}+\overline{CF}$

$= (\bar{d}-\bar{a})+(\bar{e}-\bar{b})+(\bar{f}-\bar{c})$

$= (\bar{d}+\bar{e}+\bar{f})-(\bar{a}+\bar{b}+\bar{c})$

$= \left[\dfrac{\bar{b}+\bar{c}}{2}+\dfrac{\bar{c}+\bar{a}}{2}+\dfrac{\bar{a}+\bar{b}}{2}\right]-(\bar{a}+\bar{b}+\bar{c})$

$= \dfrac{2(\bar{a}+\bar{b}+\bar{c})}{2}-(\bar{a}+\bar{b}+\bar{c})$

$= \bar{0}$ = R.H.S.

$\therefore \quad \overline{AD}+\overline{BE}+\overline{CF} = \bar{0}$.

11. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point (2, 4).

Solution:

Comparing $y = x^2$ with $x^2 = 4ay$, we get

$a = \dfrac{1}{4}$

Equation of a tangent to the parabola is given by

$xx_1 = 2a(y+y_1)$

$\therefore \quad 2x = 2\left(\dfrac{1}{4}\right)(y+4)$ ...[$x_1 = 2, y_1 = 4$]

$\therefore \quad 4x = y+4$

$\therefore \quad 4x - y = 4$

Let $\bar{b}$ be the vector parallel to the tangent of the parabola $y = x^2$.

$\therefore \quad \bar{b} = 4\hat{i}-\hat{j}$

$|\bar{b}| = \sqrt{4^2+(-1)^2+0^2}$

$|\bar{b}| = \sqrt{17}$

Required unit vectors that are parallel to the tangent of the parabola are

$\pm\dfrac{\bar{b}}{|\bar{b}|} = \pm\dfrac{1}{\sqrt{17}}(4\hat{i}-\hat{j})$

[**Note:** Answer given in the textbook is $\pm\dfrac{1}{\sqrt{17}}(\hat{i}+4\hat{j})$.

However, as per our calculation it is $\pm\dfrac{1}{\sqrt{17}}(4\hat{i}-\hat{j})$.]

12. Express the vector $\hat{i}+4\hat{j}-4\hat{k}$ as a linear combination of the vectors $2\hat{i}-\hat{j}+3\hat{k}$, $\hat{i}-2\hat{j}+4\hat{k}$ and $-\hat{i}+3\hat{j}-5\hat{k}$.

Solution:

Let $\bar{a} = 2\hat{i}-\hat{j}+3\hat{k}$, $\bar{b} = \hat{i}-2\hat{j}+4\hat{k}$,

$\bar{c} = -\hat{i}+3\hat{j}-5\hat{k}$ and $\bar{r} = \hat{i}+4\hat{j}-4\hat{k}$

Consider $\bar{r} = x\bar{a}+y\bar{b}+z\bar{c}$, where x, y, z are scalars.

$\therefore \quad \hat{i}+4\hat{j}-4\hat{k} = x(2\hat{i}-\hat{j}+3\hat{k})+y(\hat{i}-2\hat{j}+4\hat{k})$
$\qquad + z(-\hat{i}+3\hat{j}-5\hat{k})$ ...(i)

$\therefore \quad \hat{i}+4\hat{j}-4\hat{k} = (2x+y-z)\hat{i}+(-x-2y+3z)\hat{j}$
$\qquad +(3x+4y-5z)\hat{k}$

By equality of vectors, we get

$\therefore \quad 2x+y-z = 1,$

$-x-2y+3z = 4$ and

$3x+4y-5z = -4$

$$D = \begin{vmatrix} 2 & 1 & -1 \\ -1 & -2 & 3 \\ 3 & 4 & -5 \end{vmatrix}$$

$= 2(10-12)-1(5-9)-1(-4+6)$

$= 2(-2)-1(-4)-1(2)$

$= -4+4-2$

$= -2 \neq 0$

$$D_x = \begin{vmatrix} 1 & 1 & -1 \\ 4 & -2 & 3 \\ -4 & 4 & -5 \end{vmatrix}$$

$= 1(10-12)-1(-20+12)-1(16-8)$

$= 1(-2)-1(-8)-1(8)$

$= -2+8-8 = -2$

$$D_y = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 4 & 3 \\ 3 & -4 & -5 \end{vmatrix}$$

$= 2(-20+12)-1(5-9)-1(4-12)$

$= 2(-8)-1(-4)-1(-8)$

$= -16+4+8 = -4$

$$D_z = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -2 & 4 \\ 3 & 4 & -4 \end{vmatrix}$$

$= 2(8-16)-1(4-12)+1(-4+6)$

$= 2(-8)-1(-8)+1(2)$

$= -16+8+2$

$= -6$

By Cramer's rule, we get

$x = \dfrac{D_x}{D} = \dfrac{-2}{-2} = 1$

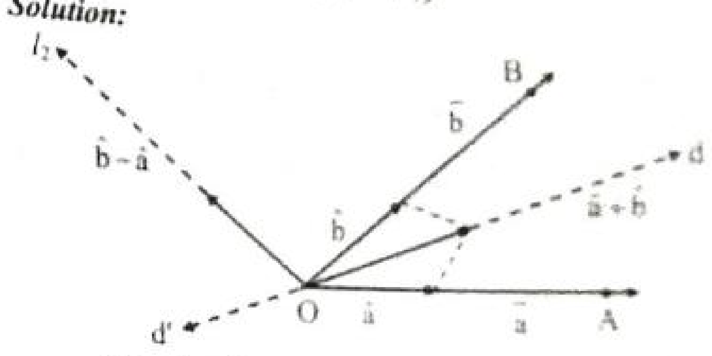
$y = \dfrac{D_y}{D} = \dfrac{-4}{-2} = 2$

$z = \dfrac{D_z}{D} = \dfrac{-6}{-2} = 3$

$\therefore \quad \hat{i}+4\hat{j}-4\hat{k} = 1(2\hat{i}-\hat{j}+3\hat{k})+2(\hat{i}-2\hat{j}+4\hat{k})$
$\qquad +3(-\hat{i}+3\hat{j}-5\hat{k})$ ...[From (i)]

13. If $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$, then show that the vector along the angle bisector of angle AOB is given by $\bar{d} = \lambda\left(\dfrac{\bar{a}}{|\bar{a}|}+\dfrac{\bar{b}}{|\bar{b}|}\right)$.

Solution:



From the diagram,

$\hat{a}$, $\hat{b}$ are the unit vectors along with A and B.

d, d' be the angle bisectors of AOB

$l_1 = \hat{a}+\hat{b}$

$l_2 = \hat{b}-\hat{a}$

Direction of d $= \hat{a}+\hat{b}$

$= \dfrac{\bar{a}}{|\bar{a}|}+\dfrac{\bar{b}}{|\bar{b}|}$

Now, vector of magnitude (λ) along d is $\lambda\left(\dfrac{\bar{a}}{|\bar{a}|}+\dfrac{\bar{b}}{|\bar{b}|}\right)$.

Vector of magnitude (λ) along d' is $-\lambda\left(\dfrac{\bar{a}}{|\bar{a}|}+\dfrac{\bar{b}}{|\bar{b}|}\right)$

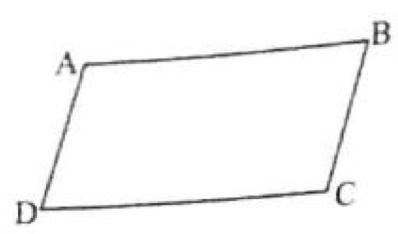
$\therefore$ Angle bisector of angle AOB is given by

$\bar{d} = \lambda\left(\dfrac{\bar{a}}{|\bar{a}|}+\dfrac{\bar{b}}{|\bar{b}|}\right)$

[**Note:** *The question has been modified.*]

14. **The position vectors of three consecutive vertices of a parallelogram are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+3\hat{j}+5\hat{k}$ and $7\hat{i}+9\hat{j}+11\hat{k}$. Find the position vector of the fourth vertex.**

Solution:



Let □ABCD be a parallelogram with

$\bar{a} = \hat{i}+\hat{j}+\hat{k}$, $\bar{b} = \hat{i}+3\hat{j}+5\hat{k}$ and

$\bar{c} = 7\hat{i}+9\hat{j}+11\hat{k}$

$\overline{AD} = \overline{BC}$ ...[Sides of a parallelogram]

$\therefore \quad \bar{d} - \bar{a} = \bar{c} - \bar{b}$

$\therefore \quad \bar{d} = \bar{c} - \bar{b} + \bar{a}$

$\therefore \quad \bar{d} = \left(7\hat{i}+9\hat{j}+11\hat{k}\right) - \left(\hat{i}+3\hat{j}+5\hat{k}\right) + \left(\hat{i}+\hat{j}+\hat{k}\right)$

$\therefore \quad \bar{d} = \left(8\hat{i}+10\hat{j}+12\hat{k}\right) - \left(\hat{i}+3\hat{j}+5\hat{k}\right)$

$\therefore \quad \bar{d} = 7\hat{i}+7\hat{j}+7\hat{k}$

$\therefore \quad \bar{d} = 7\left(\hat{i}+\hat{j}+\hat{k}\right)$

15. **A point P with p·v· $\dfrac{-14\hat{i}+39\hat{j}+28\hat{k}}{5}$ divides the line joining A(–1, 6, 5) and B in the ratio 3:2, then find the point B.**

Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{p}$ be the position vectors of points A, B, P respectively.

Then, $\bar{p} = \dfrac{-14\hat{i}+39\hat{j}+28\hat{k}}{5}$, $\bar{a} = -\hat{i}+6\hat{j}+5\hat{k}$

P divides the segment AB in the ratio 3:2.

$\therefore$ By using section formula, we get

$\bar{p} = \dfrac{3\bar{b}+2\bar{a}}{3+2}$

$\therefore \quad \dfrac{-14\hat{i}+39\hat{j}+28\hat{k}}{5} = \dfrac{3\bar{b}+2\left(-\hat{i}+6\hat{j}+5\hat{k}\right)}{5}$

$\therefore \quad -14\hat{i}+39\hat{j}+28\hat{k} = 3\bar{b} + \left(-2\hat{i}+12\hat{j}+10\hat{k}\right)$

$\therefore \quad 3\bar{b} = -12\hat{i}+27\hat{j}+18\hat{k}$

$\therefore \quad \bar{b} = -4\hat{i}+9\hat{j}+6\hat{k}$

$\therefore \quad B \equiv (-4, 9, 6)$

16. **Prove that the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.**

Solution:

Let points D, E and F be the midpoints of sides BC, CA and AB respectively.

Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ be the position vectors of the points A, B, C, D, E and F respectively.

$\therefore \quad \bar{d} = \dfrac{\bar{b}+\bar{c}}{2}$, $\bar{e} = \dfrac{\bar{c}+\bar{a}}{2}$, $\bar{f} = \dfrac{\bar{a}+\bar{b}}{2}$ ...(i)

Consider, $\overline{AD} + \overline{BE} + \overline{CF}$

$= \left(\bar{d}-\bar{a}\right)+\left(\bar{e}-\bar{b}\right)+\left(\bar{f}-\bar{c}\right)$

$= \left(\bar{d}+\bar{e}+\bar{f}\right)-\left(\bar{a}+\bar{b}+\bar{c}\right)$

$= \left(\dfrac{\bar{b}+\bar{c}}{2}+\dfrac{\bar{c}+\bar{a}}{2}+\dfrac{\bar{a}+\bar{b}}{2}\right)-\left(\bar{a}+\bar{b}+\bar{c}\right)$ ...[From (i)]

$= \dfrac{2\left(\bar{a}+\bar{b}+\bar{c}\right)}{2}-\left(\bar{a}+\bar{b}+\bar{c}\right)$

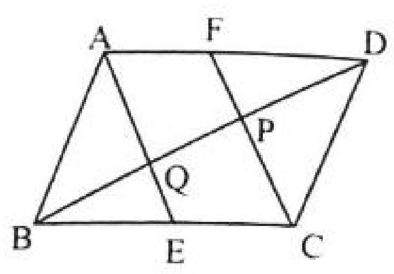
$= \left(\bar{a}+\bar{b}+\bar{c}\right) - \left(\bar{a}+\bar{b}+\bar{c}\right)$

$= \bar{0}$

$\therefore$ the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.

17. **ABCD is a parallelogram. E, F are the mid points of BC and AD respectively. AE, CF meet the diagonal BD at Q and P respectively. Show that P and Q trisect DB.**

Solution:



Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ be the position vectors of points A, B, C, D, E and F respectively.

Prove that P and Q trisect DB.

$\because$ E is the midpoint of side BC of parallelogram ABCD

$\therefore \quad \bar{e} = \dfrac{\bar{b}+\bar{c}}{2}$

$\therefore \quad 2\bar{e} = \bar{b} + \bar{c}$ ...(i)

$\because$ F is the midpoint of side AD of parallelogram ABCD

$\therefore \quad \bar{f} = \dfrac{\bar{a}+\bar{d}}{2}$

$\therefore \quad 2\bar{f} = \bar{a} + \bar{d}$ ...(ii)

Also, $\overline{BC} = \overline{AD}$ as BC and AD are the opposite sides of the parallelogram.

$\therefore \quad \bar{c} - \bar{b} = \bar{d} - \bar{a}$

$\therefore \quad -\bar{b} + \bar{c} = \bar{d} - \bar{a}$ ...(iii)

Subtract (iii) from (i),

$\bar{b} + \bar{c} = 2\bar{e}$

$-\bar{b} + \bar{c} = \bar{d} - \bar{a}$

$+ \quad - \qquad - \quad +$

$2\bar{b} = 2\bar{e} - \bar{d} + \bar{a}$

$2\bar{b} + \bar{d} = 2\bar{e} + \bar{a}$

$\therefore \quad \dfrac{2\bar{b}+\bar{d}}{2+1} = \dfrac{2\bar{e}+\bar{a}}{2+1} = \bar{q}$ (say)

$\therefore \quad Q(\bar{q})$ is a point which divides $\overline{DB}$ and $\overline{AE}$ internally in the ratio 2:1.

$\therefore \quad \overline{DB}$ and $\overline{AE}$ trisect each other at Q. ...(iv)

Now, add (iii) and (ii),

$2\bar{d} = 2\bar{f} - \bar{b} + \bar{c}$

$2\bar{d} + \bar{b} = 2\bar{f} + \bar{c}$

$\therefore \quad \dfrac{2\bar{d}+\bar{b}}{2+1} = \dfrac{2\bar{f}+\bar{c}}{2+1} = \bar{p}$ (say)

$\therefore \quad P(\bar{p})$ is a point which divides $\overline{BD}$ and $\overline{CF}$ internally in the ratio 2:1.

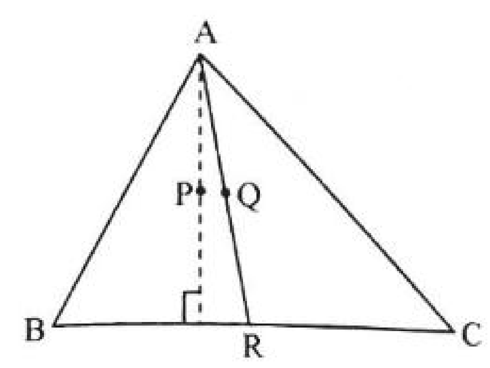
$\therefore \quad \overline{BD}$ and $\overline{CF}$ trisect each other at P. ...(v)

$\therefore$ From (iv) and (v) we get, P and Q trisect DB.

[**Note:** *The question has been modified.*]

18. **If ABC is a triangle whose orthocentre is P and the circumcentre is Q, then prove that $\overline{PA}+\overline{PC}+\overline{PB} = 2\overline{PQ}$.**

Solution:



Let $\bar{a}, \bar{b}, \bar{c}, \bar{q}$ be the position vectors of points A, B, C, Q respectively w.r.t. orthocentre P.

$\therefore \quad \overline{PA} = \bar{a}$, $\overline{PB} = \bar{b}$, $\overline{PC} = \bar{c}$ and $\overline{PQ} = \bar{q}$

Let R be the midpoint of side BC with position vector $\bar{r}$ w.r.t. P i.e. $\overline{PR} = \bar{r}$

$\therefore \quad \bar{r} = \dfrac{\bar{b}+\bar{c}}{2}$

$\therefore \quad 2\bar{r} = \bar{b}+\bar{c}$ ...(i)

$\because$ AP ∥ QR and AP = 2QR

$\therefore \quad \overline{PA} = 2\overline{RQ}$

$\therefore \quad \bar{a} = 2\left(\bar{q}-\bar{r}\right)$

$= 2\bar{q} - 2\bar{r}$

$\therefore \quad \bar{a} + 2\bar{r} = 2\bar{q}$

$\therefore \quad \bar{a} + \bar{b} + \bar{c} = 2\bar{q}$ ...[From (i)]

$\therefore \quad \overline{PA} + \overline{PB} + \overline{PC} = 2\overline{PQ}$

19. **If P is orthocentre, Q is circumcentre and G is centroid of a triangle ABC, then prove that $\overline{QP} = 3\overline{QG}$.**

Solution:

Let $\bar{p}$, $\bar{q}$ and $\bar{g}$ be the position vectors of points P, Q and G respectively.

Since P is orthocentre, Q is circumcentre and G is the centroid of ΔABC.

$\therefore$ P, Q and G are collinear.

Also, centroid G of triangle divides segment QP internally in the ratio 1 : 2

$\therefore$ By using section formula,

$\bar{g} = \dfrac{2\bar{q}+\bar{p}}{2+1}$

$\therefore \quad 3\bar{g} = 2\bar{q} + \bar{p}$

$\therefore \quad 3\bar{g} - 3\bar{q} = 2\bar{q} + \bar{p} - 3\bar{q}$

$\therefore \quad 3\left(\bar{g}-\bar{q}\right) = \bar{p} - \bar{q}$

$\therefore \quad \overline{QP} = 3\overline{QG}$

20. **In a triangle OAB, E is the midpoint of BO and D is a point on AB such that AD: DB = 2:1. If OD and AE intersect at P, determine the ratio OP:PD using vector methods.**

Solution:

Refer Exercise 5.2 Q. 10

21. **Dot-product of a vector with vectors $3\hat{i} - 5\hat{k}$, $2\hat{i} + 7\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively –1, 6 and 5. Find the vector.**

Solution:

Let $\bar{a} = 3\hat{i}-5\hat{k}$, $\bar{b} = 2\hat{i}+7\hat{j}$,

$\bar{c} = \hat{i}+\hat{j}+\hat{k}$ and $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ be the required vector.

Consider

$\bar{r}\cdot\bar{a} = -1$ ...[Given]

$\therefore \quad \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(3\hat{i}-5\hat{k}\right) = -1$

$\therefore \quad 3x - 5z = -1$ ...(i)

Now, $\bar{r}\cdot\bar{b} = 6$ ...[Given]

$\therefore \quad \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(2\hat{i}+7\hat{j}\right) = 6$

$\therefore \quad 2x + 7y = 6$ ...(ii)

Also, $\bar{r}\cdot\bar{c} = 5$

$\therefore \quad \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) = 5$

$\therefore \quad x + y + z = 5$ ...(iii)

Consider

$$D = \begin{vmatrix} 3 & 0 & -5 \\ 2 & 7 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 3(7) - 5(2-7)$$

$= 21 - 5(-5)$
$= 21 + 25$
$= 46 \neq 0$

$$D_x = \begin{vmatrix} -1 & 0 & -5 \\ 6 & 7 & 0 \\ 5 & 1 & 1 \end{vmatrix} = -1(7) - 5(6-35)$$

$= -7 - 5(-29)$
$= -7 + 145$
$= 138$

$$D_y = \begin{vmatrix} 3 & -1 & -5 \\ 2 & 6 & 0 \\ 1 & 5 & 1 \end{vmatrix} = 3(6) + 1(2) - 5(10-6)$$

$= 18 + 2 - 5(4)$
$= 20 - 20$
$= 0$

$$D_z = \begin{vmatrix} 3 & 0 & -1 \\ 2 & 7 & 6 \\ 1 & 1 & 5 \end{vmatrix} = 3(35-6) - 1(2-7)$$

$= 3(29) - 1(-5)$
$= 87 + 5$
$= 92$

By Cramer's rule, we get

$x = \dfrac{D_x}{D} = \dfrac{138}{46} = 3$

$y = \dfrac{D_y}{D} = 0$

$z = \dfrac{D_z}{D} = \dfrac{92}{46} = 2$

$\therefore \quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = 3\hat{i} + 2\hat{k}$

22. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are unit vectors such that $\bar{a} + \bar{b} + \bar{c} = 0$, then find the value of $\bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{c}\cdot\bar{a}$.

Solution:

$\bar{a} + \bar{b} + \bar{c} = 0$ …[Given]

Squaring on both sides, we get

$(\bar{a} + \bar{b} + \bar{c})^2 = 0$

$\therefore \quad |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 + 2\bar{a}\cdot\bar{b} + 2\bar{b}\cdot\bar{c} + 2\bar{c}\cdot\bar{a} = 0$

$\therefore \quad 1^2 + 1^2 + 1^2 + 2\bar{a}\cdot\bar{b} + 2\bar{b}\cdot\bar{c} + 2\bar{c}\cdot\bar{a} = 0$

…[$\bar{a}$, $\bar{b}$, $\bar{c}$ are unit vectors.]

$\therefore \quad 3 + 2\bar{a}\cdot\bar{b} + 2\bar{b}\cdot\bar{c} + 2\bar{c}\cdot\bar{a} = 0$

$\therefore \quad 2(\bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{c}\cdot\bar{a}) = -3$

$\therefore \quad \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{c}\cdot\bar{a} = \dfrac{-3}{2}$

23. If a parallelogram is constructed on the vectors $\bar{a} = 3\bar{p} - \bar{q}$, $\bar{b} = \bar{p} + 3\bar{q}$ and $|\bar{p}| = |\bar{q}| = 2$ and angle between $\bar{p}$ and $\bar{q}$ is $\dfrac{\pi}{3}$, show that the ratio of the lengths of the sides is $\sqrt{7} : \sqrt{13}$.

Solution:

$\bar{a} = 3\bar{p} - \bar{q}$ …[Given]

Squaring on both sides, we get

$|\bar{a}|^2 = |3\bar{p} - \bar{q}|^2$

$\therefore \quad |\bar{a}|^2 = 9|\bar{p}|^2 + |\bar{q}|^2 - 6|\bar{p}|\cdot|\bar{q}|\cos\theta$

$\therefore \quad |\bar{a}|^2 = 9(2)^2 + (2)^2 - 6(2)(2)\cos 60°$

$\therefore \quad |\bar{a}|^2 = 36 + 4 - 24\left(\dfrac{1}{2}\right)$

$\therefore \quad |\bar{a}|^2 = 28$

$\therefore \quad |\bar{a}| = \sqrt{28} = 2\sqrt{7}$

Now, $\bar{b} = \bar{p} + 3\bar{q}$ …[Given]

Squaring on both sides, we get

$|\bar{b}|^2 = |\bar{p} + 3\bar{q}|^2$

$\therefore \quad |\bar{b}|^2 = |\bar{p}|^2 + 9|\bar{q}|^2 + 6|\bar{p}|\cdot|\bar{q}|\cos\theta$

$\therefore \quad |\bar{b}|^2 = (2)^2 + 9(2)^2 + 6(2)(2)\cos 60°$

$\therefore \quad |\bar{b}|^2 = 4 + 36 + 24\left(\dfrac{1}{2}\right)$

$\therefore \quad |\bar{b}|^2 = 52$

$\therefore \quad |\bar{b}| = \sqrt{52} = 2\sqrt{13}$

Ratio of the lengths of the sides $= \dfrac{|\bar{a}|}{|\bar{b}|}$

$= \dfrac{2\sqrt{7}}{2\sqrt{13}}$

$= \dfrac{\sqrt{7}}{\sqrt{13}}$

24. Express the vector $\bar{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as a sum of two vectors such that one is parallel to the vector $\bar{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\bar{b}$.

Solution:

Let $\bar{a}_1$ and $\bar{a}_2$ be the two vectors which are parallel to the vector $\bar{b}$ and perpendicular to the vector $\bar{b}$.

$\therefore \quad \bar{a}_1 = 6\hat{i} + 2\hat{k}$

Now $\bar{a} = \bar{a}_1 + \bar{a}_2$

$\bar{a}_2 = \bar{a} - \bar{a}_1$

$\bar{a}_2 = (5\hat{i} - 2\hat{j} + 5\hat{k}) - (6\hat{i} + 2\hat{k})$

$\bar{a}_2 = -\hat{i} - 2\hat{j} + 3\hat{k}$

Consider $\bar{a}_2 \cdot \bar{b} = (-\hat{i} - 2\hat{j} + 3\hat{k})\cdot(3\hat{i} + \hat{k})$

$= (-1)(3) + (-2)(0) + (3)(1)$
$= -3 + 0 + 3$
$= 0$

$\bar{a}_2$ is perpendicular to $\bar{b}$

$\bar{a}_2 = -\hat{i} - 2\hat{j} + 3\hat{k}$ and $\bar{a}_1 = 6\hat{i} + 2\hat{k}$

$\bar{a} = \bar{a}_1 + \bar{a}_2$

$(5\hat{i} - 2\hat{j} + 5\hat{k}) = (6\hat{i} + 2\hat{k}) + (-\hat{i} - 2\hat{j} + 3\hat{k})$

25. Find two unit vectors each of which makes equal angles with $\bar{u}$, $\bar{v}$ and $\bar{w}$.
$\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\bar{w} = 2\hat{i} - 2\hat{j} + \hat{k}$

Solution:

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Angle between $\bar{r}$ and $\bar{u}$ is given by,

$\cos\theta = \dfrac{\bar{r}\cdot\bar{u}}{|\bar{r}||\bar{u}|} = \dfrac{2x + y - 2z}{\sqrt{x^2 + y^2 + z^2}\sqrt{2^2 + 1^2 + (-2)^2}}$

$= \dfrac{2x + y - 2z}{3\sqrt{x^2 + y^2 + z^2}}$ …(i)

Angle between $\bar{r}$ and $\bar{v}$ is given by,

$\cos\theta = \dfrac{\bar{r}\cdot\bar{v}}{|\bar{r}||\bar{v}|} = \dfrac{x + 2y - 2z}{\sqrt{x^2 + y^2 + z^2}\sqrt{1^2 + 2^2 + (-2)^2}}$

$= \dfrac{x + 2y - 2z}{3\sqrt{x^2 + y^2 + z^2}}$ …(ii)

Angle between $\bar{r}$ and $\bar{w}$ is given by,

$\cos\theta = \dfrac{\bar{r}\cdot\bar{w}}{|\bar{r}||\bar{w}|} = \dfrac{2x - 2y + z}{\sqrt{x^2 + y^2 + z^2}\sqrt{2^2 + (-2)^2 + 1^2}}$

$= \dfrac{2x - 2y + z}{3\sqrt{x^2 + y^2 + z^2}}$ …(iii)

Since angle between $\bar{r}$ and $\bar{u}$, $\bar{r}$ and $\bar{v}$, $\bar{r}$ and $\bar{w}$ is equal, we get

$\dfrac{2x + y - 2z}{3\sqrt{x^2 + y^2 + z^2}} = \dfrac{x + 2y - 2z}{3\sqrt{x^2 + y^2 + z^2}} = \dfrac{2x - 2y + z}{3\sqrt{x^2 + y^2 + z^2}}$

$\therefore \quad 2x + y - 2z = x + 2y - 2z$ or
$x + 2y - 2z = 2x - 2y + z$ or
$2x + y - 2z = 2x - 2y + z$

$\therefore \quad x - y = 0$ or $x - 4y + 3z = 0$ or $3y - 3z = 0$

$\therefore \quad x - y = 0$ i.e., $x = y$ …(iv)

Also, $3y - 3z = 0$ i.e., $y = z$ …(v)

From (iv) and (v), we get

$x = y = z$

Let $\bar{r} = p\hat{i} + p\hat{j} + p\hat{k}$

$\hat{r} = \dfrac{p\hat{i} + p\hat{j} + p\hat{k}}{\sqrt{p^2 + p^2 + p^2}}$

$\therefore \quad \hat{r} = \dfrac{p(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3p^2}}$

$\therefore \quad \hat{r} = \dfrac{p(\hat{i} + \hat{j} + \hat{k})}{p\sqrt{3}}$

$\therefore \quad \hat{r} = \dfrac{\pm(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}}$

$\therefore \quad \hat{r} = \pm\left(\dfrac{1}{\sqrt{3}}\hat{i} + \dfrac{1}{\sqrt{3}}\hat{j} + \dfrac{1}{\sqrt{3}}\hat{k}\right)$

26. Find the acute angles between the curves at their points of intersection. $y = x^2$, $y = x^3$.

Solution:

$y = x^2$, $y = x^3$ …[Given]

$\therefore \quad x^2 = x^3$

$\therefore \quad x^3 - x^2 = 0$

$\therefore \quad x^2(x - 1) = 0$

$\therefore \quad x = 0$ or $x = 1$

When $x = 0$, $y = 0$

When $x = 1$, $y = 1$

Let B and C be the points of intersection.

$B \equiv (0, 0)$, $C \equiv (0, 0)$

Equation of a tangent to the curve $y = x^2$

$y = x^2$

$\dfrac{dy}{dx} = 2x$

At $(0, 0)$, slope $= 0$

$\therefore$ Equation of a tangent to the curve $y = x^2$ at $(0, 0)$ is

$y = 0$, $\hat{j} = 0$

At $(1, 1)$, slope $= 2$

Equation of a tangent to the curve $y = x^2$ at $(1, 1)$ is

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$\therefore \quad 2x - y = 1$

$\therefore \quad 2\hat{i} - \hat{j} = 1$

Equation of a tangent to the curve $y = x^3$

$y = x^3$

$\dfrac{dy}{dx} = 3x^2$

At $(0, 0)$, slope $= 0$

$\therefore$ Equation of a tangent to the curve $y = x^3$ at $(0, 0)$ is

$y = 0$, $\hat{j} = 0$

At $(1, 1)$, slope $= 3$

$\therefore$ Equation of a tangent to the curve $y = x^3$ at $(1, 1)$ is

$y - 1 = 3(x - 1)$

$y - 1 = 3x - 3$

$3x - y = 2$

$3\hat{i} - \hat{j} = 2$

Angle between $y = x^2$ and $y = x^3$ at (0, 0) will be

$\cos\theta = \frac{\hat{j}\cdot\hat{j}}{|\hat{j}||\hat{j}|} = 1.$

$\therefore \quad \theta = 0^\circ$

Angle between $y = x^2$ and $y = x^3$ at (1, 1) will be

$$\cos\theta = \frac{(2)(3)+(-1)(-1)}{\sqrt{2^2+(-1)^2}\sqrt{(3)^2+(-1)^2}}$$

$$\therefore \quad \cos\theta = \frac{7}{\sqrt{5}\sqrt{10}}$$

$$\therefore \quad \cos\theta = \frac{7}{5\sqrt{2}}$$

27. Find the direction cosines and direction angles of the vector.

i. $2\hat{i} + \hat{j} + 2\hat{k}$ **ii.** $\left(\frac{1}{2}\right)\hat{i} + \hat{j} + \hat{k}$

Solution:

i. $2\hat{i}+\hat{j}+2\hat{k}$

Direction ratio of a given vector is

a = 2, b = 1, c = 2

$\therefore$ Direction cosines are given by

$$l = \cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{2^2+1^2+2^2}} = \frac{2}{3}$$

$$m = \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{2^2+1^2+2^2}} = \frac{1}{3}$$

$$n = \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{2^2+1^2+2^2}} = \frac{2}{3}$$

ii. $\left(\frac{1}{2}\right)\hat{i}+\hat{j}+\hat{k}$

Direction ratio of a given vector is

$a = \frac{1}{2}$, b = 1, c = 1

$\therefore$ Direction cosines are given by

$$l = \cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{\frac{1}{2}}{\sqrt{\left(\frac{1}{2}\right)^2+1^2+1^2}} = \frac{\frac{1}{2}}{\sqrt{\frac{1}{4}+2}} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)} = \frac{1}{3}$$

$$m = \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2+1^2+1^2}} = \frac{1}{\sqrt{\frac{1}{4}+2}} = \frac{1}{\frac{3}{2}}$$

$$\therefore \quad m = \frac{2}{3}$$

$$n = \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2+1^2+1^2}} = \frac{1}{\sqrt{\frac{1}{4}+2}} = \frac{1}{\frac{3}{2}}$$

$$\therefore \quad n = \cos\gamma = \frac{2}{3}$$

[**Note:** Answer given in the textbook is $\cos\alpha = \frac{1}{4}$.

However, as per our calculation it is $\cos\alpha = \frac{1}{3}$.]

28. Let $\bar{b} = 4\hat{i} + 3\hat{j}$ and $\bar{c}$ be two vectors perpendicular to each other in the XY-plane. Find vectors in the same plane having projection 1 and 2 along $\bar{b}$ and $\bar{c}$ respectively.

Solution:

Let $\bar{c} = x\hat{i}+y\hat{j}$

Since $\bar{b}$ and $\bar{c}$ are perpendicular to each other, then $\bar{b}\cdot\bar{c} = 0$

$\therefore \quad \left(4\hat{i}+3\hat{j}\right)\cdot\left(x\hat{i}+y\hat{j}\right) = 0$

$\therefore \quad 4x + 3y = 0$

$\therefore \quad \frac{x}{y} = \frac{-3}{4}$

$\therefore \quad \bar{c} = \lambda\left(3\hat{i}-4\hat{j}\right)$ where λ is constant of ratio.

Let the required vector be $\bar{d} = p\hat{i}+q\hat{j}$

Projection of $\bar{d}$ on $\bar{b} = \frac{\bar{d}\cdot\bar{b}}{|\bar{b}|}$

$$\therefore \quad 1 = \frac{\left(p\hat{i}+q\hat{j}\right)\cdot\left(4\hat{i}+3\hat{j}\right)}{\sqrt{4^2+3^2}}$$

$$\therefore \quad 1 = \frac{4p+3q}{5}$$

$\therefore \quad 4p + 3q = 5 \quad \ldots$(i)

Also, projection of $\bar{d}$ on $\bar{c} = \frac{\bar{d}\cdot\bar{c}}{|\bar{c}|}$

$$\therefore \quad 2 = \frac{\left(p\hat{i}+q\hat{j}\right)\cdot\lambda\left(3\hat{i}-4\hat{j}\right)}{\sqrt{(3\lambda)^2+(-4\lambda)^2}}$$

$$\therefore \quad 2 = \frac{3\lambda p-4\lambda q}{5\lambda}$$

$$\therefore \quad 2 = \frac{3p-4q}{5}$$

$\therefore \quad 3p - 4q = 10 \quad \ldots$(ii)

Multiplying equation (i) by 3, we get

$12p + 9q = 15 \quad \ldots$(iii)

Multiplying equation (ii) by 4, we get

$12p - 16q = 40 \quad \ldots$(iv)

Subtracting equation (iii) from (iv), we get

$$\begin{array}{r} 12p - 16q = 40 \\ 12p + 9q = 15 \\ \underline{+ \quad - \qquad - \quad +} \\ -25q = 25 \end{array}$$

$\therefore \quad q = -1$

Substituting q = −1 in equation (ii), we get

$3p - 4(-1) = 10$

$\therefore \quad 3p = 10 - 4$

$\therefore \quad 3p = 6$

$\therefore \quad p = 2$

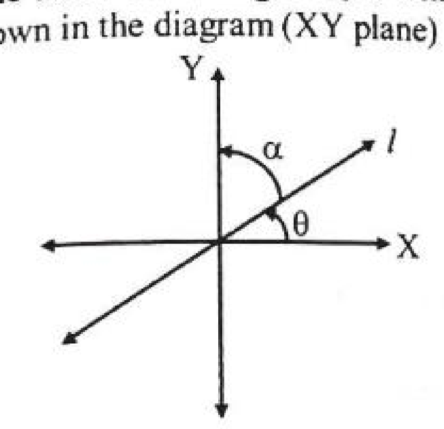
$\therefore$ Required vector $\bar{d} = p\hat{i}+q\hat{j} = 2\hat{i}-\hat{j}$

[**Note:** *The question has been modified.*]

29. Show that no line in space can make angle $\frac{\pi}{6}$ and $\frac{\pi}{4}$ with X- axis and Y-axis.

Solution:

Let the line l make angles θ, α with X, Y axes as shown in the diagram (XY plane)



From the diagram,

$\theta + \alpha = \frac{\pi}{2} \quad \ldots$(i)

$\theta = \frac{\pi}{6}, \alpha = \frac{\pi}{4} \quad \ldots$[Given]

Let $\theta + \alpha = \frac{\pi}{6} + \frac{\pi}{4}$

$= \frac{4\pi+6\pi}{24}$

$= \frac{10\pi}{24}$

$\therefore \quad \theta + \alpha = \frac{5\pi}{12} \quad \ldots$(ii)

From (i) and (ii), we get

$\theta + \alpha \neq \frac{\pi}{2}$

$\therefore$ No line in space can make angle $\frac{\pi}{6}$ and $\frac{\pi}{4}$ with X-axis and Y-axis.

30. Find the angle between the lines whose direction cosines are given by the equations $6mn - 2nl + 5lm = 0$, $3l + m + 5n = 0$.

Solution:

$3l + m + 5n = 0 \quad \ldots$[Given]

$\therefore \quad m = -3l - 5n \quad \ldots$(i)

$6mn - 2nl + 5lm = 0 \quad \ldots$[Given]

$\therefore \quad 6n(-3l - 5n) - 2nl + 5l(-3l - 5n) = 0 \quad \ldots$[From (i)]

$\therefore \quad -18nl - 30n^2 - 2nl - 15l^2 - 25nl = 0$

$\therefore \quad -30n^2 - 45nl - 15l^2 = 0$

$\therefore \quad 2n^2 + 3nl + l^2 = 0$

$\therefore \quad 2n^2 + 2nl + nl + l^2 = 0$

$\therefore \quad (2n + l)(n + l) = 0$

$\therefore \quad l = -2n \quad$ or $\quad l = -n$

Substituting $l = -2n$ in (i), we get

$m = -3(-2n) - 5n$

$\therefore \quad m = 6n - 5n$

$\therefore \quad m = n$

$\therefore \quad (l, m, n) = (-2n, n, n) = (2, -1, -1)$

$\therefore \quad (a_1, b_1, c_1) = (2, -1, -1)$

Also, substituting $l = -n$ in (i), we get

$m = -3(-n) - 5n$

$\therefore \quad m = 3n - 5n$

$\therefore \quad m = -2n$

$\therefore \quad (l, m, n) = (-n, -2n, n) = (1, 2, -1)$

$\therefore \quad (a_2, b_2, c_2) = (1, 2, -1)$

Angle between two lines is given by

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$\therefore \quad \cos\theta = \left|\frac{(2)(1)+(-1)(2)+(-1)(-1)}{\sqrt{2^2+(-1)^2+(-1)^2}\sqrt{(1)^2+(2)^2+(-1)^2}}\right|$$

$$\therefore \quad \cos\theta = \left|\frac{2-2+1}{\sqrt{6}\sqrt{6}}\right|$$

$$\therefore \quad \cos\theta = \frac{1}{6}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

31. If Q is the foot of the perpendicular from P(2, 4, 3) on the line joining the points A(1, 2, 4) and B(3, 4, 5), find coordinates of Q.

Solution:

Let PQ be the perpendicular drawn from point P(2, 4, 3) to the line joining the points A(1, 2, 4) and B (3, 4, 5).

Let Q divides AB internally in the ratio $\lambda : 1$.

$\therefore \quad Q \equiv \left(\frac{3\lambda+1}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{5\lambda+4}{\lambda+1}\right)$ ...(i)

Direction ratios of PQ are

$\frac{3\lambda+1}{\lambda+1} - 2, \frac{4\lambda+2}{\lambda+1} - 4, \frac{5\lambda+4}{\lambda+1} - 3$

i.e., $\frac{\lambda-1}{\lambda+1}, \frac{-2}{\lambda+1}, \frac{2\lambda+1}{\lambda+1}$

Now, direction ratios of AB are, 3 – 1, 4 – 2, 5 – 4 i.e., 2, 2, 1.

Since PQ is perpendicular to AB,

$2\left(\frac{\lambda-1}{\lambda+1}\right) + \frac{2(-2)}{\lambda+1} + 1\left(\frac{2\lambda+1}{\lambda+1}\right) = 0$

$\therefore \quad \frac{2\lambda-2-4+2\lambda+1}{\lambda+1} = 0$

$\therefore \quad 4\lambda - 5 = 0$

$\therefore \quad 4\lambda = 5$

$\therefore \quad \lambda = \frac{5}{4}$

Putting $\lambda = \frac{5}{4}$ in (i),

Coordinates of Q are,

$\frac{3\left(\frac{5}{4}\right)+1}{\left(\frac{5}{4}\right)+1} = \frac{19}{9}, \frac{4\left(\frac{5}{4}\right)+2}{\left(\frac{5}{4}\right)+1} = \frac{28}{9}$

and $\frac{5\left(\frac{5}{4}\right)+4}{\left(\frac{5}{4}\right)+1} = \frac{41}{9}$

$\therefore \quad Q \equiv \left(\frac{19}{9}, \frac{28}{9}, \frac{41}{9}\right)$

32. Show that the area of a triangle ABC, the position vectors of whose vertices are $\bar{a}$, $\bar{b}$ and $\bar{c}$ is $\frac{1}{2}\left[\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a}\right]$.

Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of points A, B, C respectively.

In ΔABC,

$\left.\begin{array}{l}\overline{AB} = \overline{OB} - \overline{OA} = \bar{b} - \bar{a} \\ \overline{AC} = \overline{OC} - \overline{OA} = \bar{c} - \bar{a}\end{array}\right\}$ where, O is the origin

Vector area of triangle ABC

$= \frac{1}{2}\left[\overline{AB}\times\overline{AC}\right]$

$= \frac{1}{2}\left[\left(\bar{b}-\bar{a}\right)\times\left(\bar{c}-\bar{a}\right)\right]$

$= \frac{1}{2}\left[\bar{b}\times\bar{c} - \bar{b}\times\bar{a} - \bar{a}\times\bar{c} + \bar{a}\times\bar{a}\right]$

$= \frac{1}{2}\left[\bar{b}\times\bar{c} + \bar{a}\times\bar{b} + \bar{c}\times\bar{a} + 0\right]$

$= \frac{1}{2}\left[\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a}\right]$

33. Find a unit vector perpendicular to the plane containing the points (a, 0, 0), (0, b, 0), and (0, 0, c). What is the area of the triangle with these vertices?

Solution:

Let A ≡ (a, 0, 0), B ≡ (0, b, 0) and C ≡ (0, 0, c)

$\therefore \quad \bar{a} = a\hat{i}, \ \bar{b} = b\hat{j}$ and $\bar{c} = c\hat{k}$

Now, $\overline{AB} = \bar{b} - \bar{a}$

$= b\hat{j} - a\hat{i}$

$\therefore \quad \overline{AB} = -a\hat{i} + b\hat{j}$

Also, $\overline{AC} = \bar{c} - \bar{a}$

$= c\hat{k} - a\hat{i}$

$\therefore \quad \overline{AC} = -a\hat{i} + c\hat{k}$

Consider $\overline{AB}\times\overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$

$= \hat{i}\,(bc) - \hat{j}\,(-ac) + \hat{k}\,(ab)$

$\therefore \quad \overline{AB}\times\overline{AC} = (bc)\,\hat{i} + (ac)\,\hat{j} + (ab)\,\hat{k}$

Also, $\left|\overline{AB}\times\overline{AC}\right| = \sqrt{(bc)^2 + (ac)^2 + (ab)^2}$

$= \sqrt{b^2c^2 + a^2c^2 + a^2b^2}$

Unit vector perpendicular to the plane containing the points A, B and C is given by,

$\frac{\overline{AB}\times\overline{AC}}{\left|\overline{AB}\times\overline{AC}\right|} = \frac{bc\hat{i} + ac\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$

Required area $= \frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|$

$\therefore \quad$ Required area $= \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$

34. State whether each expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar.

i. $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$ **ii.** $\bar{a}\times\left(\bar{b}\cdot\bar{c}\right)$

iii. $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ **iv.** $\bar{a}\cdot\left(\bar{b}\cdot\bar{c}\right)$

v. $\left(\bar{a}\cdot\bar{b}\right)\times\left(\bar{c}\cdot\bar{d}\right)$ **vi.** $\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{c}\times\bar{d}\right)$

vii. $\left(\bar{a}\cdot\bar{b}\right)\cdot\bar{c}$ **viii.** $\left(\bar{a}\cdot\bar{b}\right)\bar{c}$

ix. $\left(\left|\bar{a}\right|\right)\left(\bar{b}\cdot\bar{c}\right)$ **x.** $\bar{a}\cdot\left(\bar{b}+\bar{c}\right)$

xi. $\bar{a}\cdot\bar{b}+\bar{c}$ **xii.** $\left|\bar{a}\right|\cdot\left(\bar{b}+\bar{c}\right)$

Solution:

i. $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$ is meaningful.

It is a scalar triple product.

$\therefore$ It will give a scalar.

ii. $\bar{a}\times\left(\bar{b}\cdot\bar{c}\right)$ is meaningless.

Here, $\bar{a}$ is a vector and $\bar{b}\cdot\bar{c}$ is a scalar.

Cross product of vector and scalar is not possible.

iii. $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ is meaningful.

It is a vector triple product.

$\therefore$ It will give a vector.

iv. $\bar{a}\cdot\left(\bar{b}\cdot\bar{c}\right)$ is meaningless.

Here, $\bar{a}$ is a vector and $\bar{b}\cdot\bar{c}$ is a scalar.

Dot product of vector and scalar is not possible.

v. $\left(\bar{a}\cdot\bar{b}\right)\times\left(\bar{c}\cdot\bar{d}\right)$ is meaningless.

Here, $\bar{a}\cdot\bar{b}$ and $\bar{c}\cdot\bar{d}$ both are scalars.

Cross product of two scalars is not possible.

vi. $\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{c}\times\bar{d}\right)$ is meaningful.

Here, $\bar{a}\times\bar{b}$ and $\bar{c}\times\bar{d}$ both are vectors.

Dot product of two vectors is possible.

$\therefore$ It will give a scalar.

vii. $\left(\bar{a}\cdot\bar{b}\right)\cdot\bar{c}$ is meaningless.

Here, $\bar{a}\cdot\bar{b}$ is a scalar and $\bar{c}$ is a vector.

Dot product of a scalar and a vector is not possible.

viii. $\left(\bar{a}\cdot\bar{b}\right)\bar{c}$ is meaningful.

Here, $\bar{a}\cdot\bar{b}$ is a scalar and $\bar{c}$ is a vector.

Multiplication of a scalar and a vector is possible.

$\therefore$ It will give a vector.

ix. $\left(\left|\bar{a}\right|\right)\left(\bar{b}\cdot\bar{c}\right)$ is meaningful.

Here, $\left(\left|\bar{a}\right|\right)$ and $\left(\bar{b}\cdot\bar{c}\right)$ both are scalars.

Multiplication of these two scalars is possible.

$\therefore$ It will give a scalar.

x. $\bar{a}\cdot\left(\bar{b}+\bar{c}\right)$ is meaningful.

Here, $\bar{a}$ and $\left(\bar{b}+\bar{c}\right)$ both are vectors.

Dot product of two vectors is possible.

$\therefore$ It will give a scalar.

xi. $\bar{a}\cdot\bar{b}+\bar{c}$ is meaningless.

Here, $\bar{a}\cdot\bar{b}$ is a scalar and $\bar{c}$ is a vector.

Addition of a scalar and a vector is not possible.

xii. $\left|\bar{a}\right|\cdot\left(\bar{b}+\bar{c}\right)$ is meaningless.

Here, $\left|\bar{a}\right|$ is a scalar and $\left(\bar{b}+\bar{c}\right)$ is a vector.

Dot product of a scalar and a vector is not possible.

35. Show that, for any vectors

$$\left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{c} + \left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{b} + \left(\bar{b}-\bar{c}\right)\times\bar{a} = 2\bar{a}\times\bar{c}$$

Solution:

$\left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{c} + \left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{b} + \left(\bar{b}-\bar{c}\right)\times\bar{a}$

$= \bar{a}\times\bar{c} + \bar{b}\times\bar{c} + \bar{c}\times\bar{c} + \bar{a}\times\bar{b} + \bar{b}\times\bar{b} + \bar{c}\times\bar{b} + \bar{b}\times\bar{a} - \bar{c}\times\bar{a}$

$= \bar{a}\times\bar{c} + \bar{b}\times\bar{c} + \bar{0} + \bar{a}\times\bar{b} + \bar{0} - \bar{b}\times\bar{c} - \bar{a}\times\bar{b} + \bar{a}\times\bar{c}$

...$[\because \bar{c}\times\bar{b} = -\bar{b}\times\bar{c}$ and $\bar{b}\times\bar{a} = -\bar{a}\times\bar{b}]$

$= 2\bar{a}\times\bar{c}$

[**Note:** *The question has been modified.*]

36. Suppose that $\bar{a} = 0$.

i. If $\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c}$, then is $\bar{b} = \bar{c}$?

ii. If $\bar{a}\times\bar{b} = \bar{a}\times\bar{c}$, then is $\bar{b} = \bar{c}$?

iii. If $\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c}$ and $\bar{a}\times\bar{b} = \bar{a}\times\bar{c}$, then is $\bar{b} = \bar{c}$?

Solution:

i. $\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c}$ ...[Given]

$\therefore \quad \left|\bar{a}\right|\left|\bar{b}\right|\cos X = \left|\bar{a}\right|\left|\bar{c}\right|\cos Y$

$\therefore \quad \left|\bar{b}\right|\cos X = \left|\bar{c}\right|\cos Y$

$\therefore \quad \left|\bar{b}\right| \neq \left|\bar{c}\right|$

$\therefore \quad \bar{b} \neq \bar{c}$

ii. $\bar{a}\times\bar{b} = \bar{a}\times\bar{c}$ ...[Given]

$\therefore \quad \left|\bar{a}\right|\left|\bar{b}\right|\sin X = \left|\bar{a}\right|\left|\bar{c}\right|\sin Y$

$\therefore \quad \left|\bar{b}\right|\sin X = \left|\bar{c}\right|\sin Y$

$\therefore \quad \left|\bar{b}\right| \neq \left|\bar{c}\right|$

$\therefore \quad \bar{b} \neq \bar{c}$

iii. $\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c}$ ...[Given]

$\therefore$ $\bar{a}\cdot(\bar{b}-\bar{c}) = 0$ ...(i)

Now, $\bar{a}\times\bar{b} = \bar{a}\times\bar{c}$ ...[Given]

$\therefore$ $\bar{a}\times\bar{b} - \bar{a}\times\bar{c} = 0$

$\therefore$ $\bar{a}\times(\bar{b}-\bar{c}) = 0$ ...(ii)

Let $\bar{b} - \bar{c} = \bar{d}$

$\therefore$ $\bar{a}\cdot\bar{d} = 0$ ...(iii) [From (i)]

and $\bar{a}\times\bar{d} = 0$ ...(iv) [From (ii)]

$\therefore$ $\bar{a}\cdot\bar{d} = \bar{a}\times\bar{d} = 0$ ...[From (iii) and (iv)]

$\therefore$ It is possible only when $\bar{d} = 0$

$\therefore$ $\bar{b} - \bar{c} = 0$

$\therefore$ $\bar{b} = \bar{c}$

37. If A(3, 2, –1), B(–2, 2, –3), C(3, 5, –2), D(–2, 5, –4), then

i. verify that the points are the vertices of a parallelogram and

ii. find its area.

Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ be the position vectors of points A, B, C, D respectively.

$\therefore$ $\bar{a} = 3\hat{i}+2\hat{j}-\hat{k}$, $\bar{b} = -2\hat{i}+2\hat{j}-3\hat{k}$,

$\bar{c} = 3\hat{i}+5\hat{j}-2\hat{k}$, $\bar{d} = -2\hat{i}+5\hat{j}-4\hat{k}$

Consider $\overline{AB} = \bar{b}-\bar{a}$

$= (-2\hat{i}+2\hat{j}-3\hat{k}) - (3\hat{i}+2\hat{j}-\hat{k})$

$= -5\hat{i}-2\hat{k}$

Now, $\overline{CD} = \bar{d}-\bar{c}$

$= (-2\hat{i}+5\hat{j}-4\hat{k}) - (3\hat{i}+5\hat{j}-2\hat{k})$

$= -5\hat{i}-2\hat{k}$

$\therefore$ $\overline{AB} = \overline{CD}$ ...(i)

$\overline{AC} = \bar{c}-\bar{a}$

$= (3\hat{i}+5\hat{j}-2\hat{k}) - (3\hat{i}+2\hat{j}-\hat{k})$

$= 3\hat{j}-\hat{k}$

$\overline{BD} = \bar{d}-\bar{b}$

$= (-2\hat{i}+5\hat{j}-4\hat{k}) - (-2\hat{i}+2\hat{j}-3\hat{k})$

$= 3\hat{j}-\hat{k}$

$\therefore$ $\overline{AC} = \overline{BD}$ ...(ii)

$\therefore$ □ABDC is a parallelogram. ...[From (i) and (ii)]

$\therefore$ The given points A, B, C, D are the vertices of a parallelogram.

Area of parallelogram $= |\overline{AB}\times\overline{BD}|$

$\therefore$ $\overline{AB}\times\overline{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 0 & -2 \\ 0 & 3 & -1 \end{vmatrix}$

$= \hat{i}(6) - \hat{j}(5) + \hat{k}(-15)$

$= 6\hat{i}-5\hat{j}-15\hat{k}$

$\therefore$ $|\overline{AB}\times\overline{BD}| = \sqrt{6^2+(-5)^2+(-15)^2}$

$= \sqrt{36+25+225}$

$= \sqrt{286}$

$\therefore$ Area of parallelogram $= \sqrt{286}$ sq. units

38. Let A, B, C, D be any four points in space. Prove that $|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}|$ = 4(area of ΔABC).

Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{0}$ be the position vectors of points A, B, C, D respectively.

Consider

$|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}|$

$= |(\bar{b}-\bar{a})\times(\bar{0}-\bar{c})+(\bar{c}-\bar{b})\times(\bar{0}-\bar{a})+(\bar{a}-\bar{c})\times(\bar{0}-\bar{b})|$

$= |(\bar{b}-\bar{a})\times(-\bar{c})+(\bar{c}-\bar{b})\times(-\bar{a})+(\bar{a}-\bar{c})\times(-\bar{b})|$

$= |-\bar{b}\times\bar{c}+\bar{a}\times\bar{c}-\bar{c}\times\bar{a}+\bar{b}\times\bar{a}-\bar{a}\times\bar{b}+\bar{c}\times\bar{b}|$

$= |-\bar{b}\times\bar{c}-\bar{c}\times\bar{a}-\bar{c}\times\bar{a}-\bar{a}\times\bar{b}-\bar{a}\times\bar{b}-\bar{b}\times\bar{c}|$

$= |-2(\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a})|$

$= 2|\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}|$ ...(i)

Area of ΔABC $= \frac{1}{2}|\overline{AB}\times\overline{AC}|$

$= \frac{1}{2}|(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})|$

$= \frac{1}{2}|\bar{b}\times\bar{c}-\bar{b}\times\bar{a}-\bar{a}\times\bar{c}+\bar{a}\times\bar{a}|$

$\therefore$ Area of ΔABC $= \frac{1}{2}|\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a}+0|$

$\therefore$ 2 (Area of ΔABC) $= |\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}|$ ...(ii)

$\therefore$ $|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}|$

= 2×2 (Area of ΔABC) ...[From (i) and (ii)]

= 4(Area of ΔABC)

39. Let $\hat{a}, \hat{b}, \hat{c}$ be unit vectors such that $\hat{a}\cdot\hat{b} = \hat{a}\cdot\hat{c} = 0$ and the angle between $\hat{b}$ and $\hat{c}$ be $\frac{\pi}{6}$. Prove that $\hat{a} = \pm 2(\hat{b}\times\hat{c})$.

Solution:

$\hat{a}\cdot\hat{b} = \hat{a}\cdot\hat{c} = 0$

$\therefore$ $\hat{a}$ is perpendicular to $\hat{b}$ and $\hat{a}$ is also perpendicular to $\hat{c}$.

$\therefore$ $\hat{a}$ is parallel to $(\hat{b}\times\hat{c})$.

$\therefore$ $\hat{a} = \lambda(\hat{b}\times\hat{c})$ ...(i)

$\therefore$ $|\hat{a}| = |\lambda||\hat{b}||\hat{c}| \sin\theta$

$\therefore$ $1 = |\lambda| \sin\frac{\pi}{6}$

$\therefore$ $1 = |\lambda|\left(\frac{1}{2}\right)$

$\therefore$ $|\lambda| = 2$

$\therefore$ $\lambda = \pm 2$

$\therefore$ $\hat{a} = \pm 2(\hat{b}\times\hat{c})$ ...[From (i)]

40. Find the value of 'a' so that the volume of parallelopiped formed by $\hat{i} + a\hat{j} + \hat{k}$, $\hat{j}+a\hat{k}$ and $a\hat{i} + \hat{k}$ becomes minimum.

Solution:

Let V be the volume of parallelopiped.

$\therefore$ $V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$

$= 1(1-0) - a(0-a^2) + 1(0-a)$

$V = 1 + a^3 - a$

$\therefore$ Differentiating w.r.t. a, we get

$f'(V) = 3a^2 - 1$ ...(i)

Now, $f'(V) = 0$

$\therefore$ $3a^2 - 1 = 0$

$\therefore$ $3a^2 = 1$

$\therefore$ $a^2 = \frac{1}{3}$

$\therefore$ $a = \pm\frac{1}{\sqrt{3}}$

Differentiating (i) w.r.t. a, we get

$f''(V) = 6a$

When $a = \frac{1}{\sqrt{3}}$,

$f''\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} > 0$

$\therefore$ Volume is minimum when $a = \frac{1}{\sqrt{3}}$.

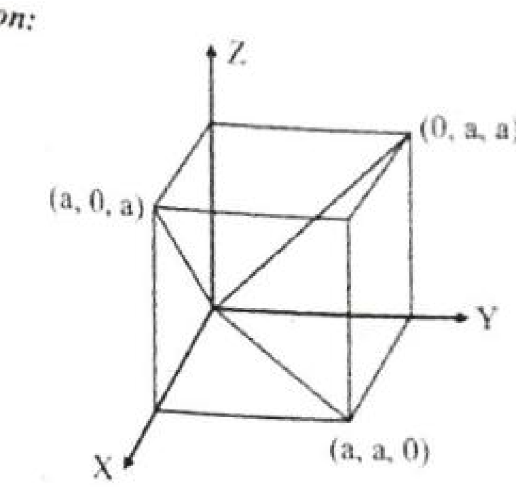
[**Note:** *The question has been modified.*

Answer given in the textbook is $a = \pm\frac{1}{\sqrt{3}}$.

However, as per our calculation it is $a = \frac{1}{\sqrt{3}}$.]

41. Find the volume of the parallelopiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.

Solution:



From the diagram,

the three faces of a cube will be $(a\hat{i}+a\hat{j}+0\hat{k})$, $(0\hat{i}+a\hat{j}+a\hat{k})$ and $(a\hat{i}+0\hat{j}+a\hat{k})$.

Volume of parallelopiped $= \begin{vmatrix} a & a & 0 \\ 0 & a & a \\ a & 0 & a \end{vmatrix}$

$= a(a^2-0) - a(0-a^2) + 0$

$= a^3 + a^3$

$= 2a^3$ cu. units

42. If $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar vectors, then show that $\frac{\bar{a}\cdot(\bar{b}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}} + \frac{\bar{b}\cdot(\bar{a}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}} = 0$.

Solution:

Consider L.H.S. $= \frac{\bar{a}\cdot(\bar{b}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}} + \frac{\bar{b}\cdot(\bar{a}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}}$

$= \frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]} + \frac{[\bar{b}\ \bar{a}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]}$

$= \frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{a}\ \bar{b}\ \bar{c}]} + \frac{[\bar{b}\ \bar{a}\ \bar{c}]}{-[\bar{b}\ \bar{a}\ \bar{c}]}$

$= 1 - \frac{[\bar{b}\ \bar{a}\ \bar{c}]}{[\bar{b}\ \bar{a}\ \bar{c}]}$

$= 1 - 1$

$= 0 =$ R.H.S.

$\therefore$ $\frac{\bar{a}\cdot(\bar{b}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}} + \frac{\bar{b}\cdot(\bar{a}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}} = 0$

43. Prove that $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$.

Solution:

Let $(\bar{a}\times\bar{b}) = \bar{p}$

$$\begin{aligned}
\text{L.H.S.} &= (\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})\\
&= \bar{p}\cdot(\bar{c}\times\bar{d})\\
&= (\bar{p}\times\bar{c})\cdot\bar{d}\\
&= \left[(\bar{a}\times\bar{b})\times\bar{c}\right]\cdot\bar{d}\\
&= \left[\bar{b}(\bar{a}\cdot\bar{c})-\bar{a}(\bar{b}\cdot\bar{c})\right]\cdot\bar{d}\\
&= (\bar{b}\cdot\bar{d})(\bar{a}\cdot\bar{c})-(\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c}) \quad \text{...(i)}
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S.} &= \begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}\\
&= (\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d})-(\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c})\\
&= (\bar{b}\cdot\bar{d})(\bar{a}\cdot\bar{c})-(\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c}) \quad \text{...(ii)}
\end{aligned}$$

$\therefore$ L.H.S. = R.H.S. ...[From (i) and (ii)]

$\therefore$ $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$

44. Find the volume of a parallelopiped whose coterminous edges are represented by the vectors $\hat{j}+\hat{k}$, $\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}$. Also find volume of tetrahedron having these coterminous edges.

Solution:

Let $\bar{a} = \hat{j}+\hat{k}$, $\bar{b} = \hat{i}+\hat{k}$, $\bar{c} = \hat{i}+\hat{j}$ be the coterminous edges of the parallelopiped and also of the tetrahedron.

$$\begin{aligned}
\left[\bar{a}\ \bar{b}\ \bar{c}\right] &= \begin{vmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{vmatrix}\\
&= 0-1(0-1)+1(1-0)\\
&= 1+1\\
&= 2 \quad \text{...(i)}
\end{aligned}$$

$\therefore$ Volume of parallelopiped $= \left[\bar{a}\ \bar{b}\ \bar{c}\right] = 2$

$$\begin{aligned}
\text{Volume of tetrahedron} &= \frac{1}{6}\left[\bar{a}\ \bar{b}\ \bar{c}\right]\\
&= \frac{1}{6}(2) \quad \text{...[From (i)]}\\
&= \frac{1}{3}\ \text{cubic units.}
\end{aligned}$$

$\therefore$ Volume of parallelopiped is 2 cubic units and that of a tetrahedron is $\frac{1}{3}$ cubic units.

45. Using properties of scalar triple product, prove that $\left[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}\right] = 2\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

Solution:

$$\begin{aligned}
&\left[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}\right]\\
&= (\bar{a}+\bar{b})\cdot\left[(\bar{b}+\bar{c})\times(\bar{c}+\bar{a})\right]\\
&= (\bar{a}+\bar{b})\cdot\left[\bar{b}\times\bar{c}+\bar{b}\times\bar{a}+\bar{c}\times\bar{c}+\bar{c}\times\bar{a}\right]\\
&= (\bar{a}+\bar{b})\cdot\left[\bar{b}\times\bar{c}+\bar{b}\times\bar{a}+\bar{0}+\bar{c}\times\bar{a}\right]\\
&= \bar{a}\cdot(\bar{b}\times\bar{c})+\bar{a}\cdot(\bar{b}\times\bar{a})+\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{b}\cdot(\bar{b}\times\bar{c})\\
&\qquad +\bar{b}\cdot(\bar{b}\times\bar{a})+\bar{b}\cdot(\bar{c}\times\bar{a})\\
&= \left[\bar{a}\,\bar{b}\,\bar{c}\right]+\left[\bar{a}\,\bar{b}\,\bar{a}\right]+\left[\bar{a}\,\bar{c}\,\bar{a}\right]+\left[\bar{b}\,\bar{b}\,\bar{c}\right]\\
&\qquad +\left[\bar{b}\,\bar{b}\,\bar{a}\right]+\left[\bar{b}\,\bar{c}\,\bar{a}\right]\\
&= \left[\bar{a}\,\bar{b}\,\bar{c}\right]+0+0+0+0+\left[\bar{a}\,\bar{b}\,\bar{c}\right]
\end{aligned}$$

$\therefore$ $\left[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}\right] = 2\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

46. If four points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ and $D(\bar{d})$ are coplanar, then show that $\left[\bar{a}\ \bar{b}\ \bar{d}\right]+\left[\bar{b}\ \bar{c}\ \bar{d}\right]+\left[\bar{c}\ \bar{a}\ \bar{d}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

Solution:

If points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$, $D(\bar{d})$ are coplanar, then $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ are also coplanar.

$\therefore$ $\overline{AB}\cdot(\overline{AC}\times\overline{AD}) = 0$ ...(i)

Here, $\overline{AB} = \bar{b}-\bar{a}$
$\overline{AC} = \bar{c}-\bar{a}$
$\overline{AD} = \bar{d}-\bar{a}$

From (i), we get

$(\bar{b}-\bar{a})\cdot\left[(\bar{c}-\bar{a})\times(\bar{d}-\bar{a})\right] = 0$

$\therefore$ $(\bar{b}-\bar{a})\cdot\left[\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d}+\bar{a}\times\bar{a}\right] = 0$

$\therefore$ $(\bar{b}-\bar{a})\cdot\left[\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d}+\bar{0}\right] = 0$

$\therefore$ $(\bar{b}-\bar{a})\cdot\left[\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d}\right] = 0$

$\therefore$ $\bar{b}\cdot(\bar{c}\times\bar{d})-\bar{b}\cdot(\bar{c}\times\bar{a})-\bar{b}\cdot(\bar{a}\times\bar{d})-\bar{a}\cdot(\bar{c}\times\bar{d})+\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{a}\cdot(\bar{a}\times\bar{d})=0$

$\therefore$ $\left[\bar{b}\,\bar{c}\,\bar{d}\right]-\left[\bar{b}\,\bar{c}\,\bar{a}\right]-\left[\bar{b}\,\bar{a}\,\bar{d}\right]-\left[\bar{a}\,\bar{c}\,\bar{d}\right]+\left[\bar{a}\,\bar{c}\,\bar{a}\right]+\left[\bar{a}\,\bar{a}\,\bar{d}\right]=0$

$\therefore$ $\left[\bar{b}\,\bar{c}\,\bar{d}\right]-\left[\bar{a}\,\bar{b}\,\bar{c}\right]+\left[\bar{a}\,\bar{b}\,\bar{d}\right]+\left[\bar{c}\,\bar{a}\,\bar{d}\right]+0+0=0$

$\therefore$ $\left[\bar{a}\,\bar{b}\,\bar{d}\right]+\left[\bar{b}\,\bar{c}\,\bar{d}\right]+\left[\bar{c}\,\bar{a}\,\bar{d}\right]=\left[\bar{a}\,\bar{b}\,\bar{c}\right]$

47. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are three non coplanar vectors, then $(\bar{a}+\bar{b}+\bar{c})\cdot\left[(\bar{a}+\bar{b})\times(\bar{a}+\bar{c})\right]=-\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

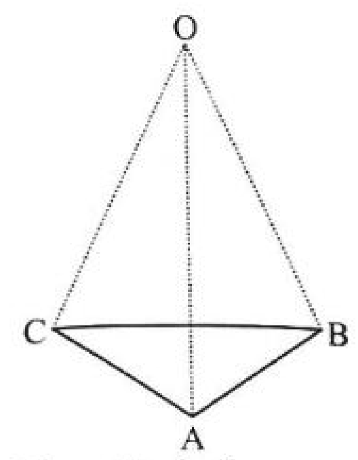
Solution:

$$\begin{aligned}
&(\bar{a}+\bar{b}+\bar{c})\cdot\left[(\bar{a}+\bar{b})\times(\bar{a}+\bar{c})\right]\\
&= (\bar{a}+\bar{b}+\bar{c})\cdot\left[(\bar{a}\times\bar{a})+(\bar{a}\times\bar{c})+(\bar{b}\times\bar{a})+(\bar{b}\times\bar{c})\right]\\
&= (\bar{a}+\bar{b}+\bar{c})\cdot\left[0+(\bar{a}\times\bar{c})+(\bar{b}\times\bar{a})+(\bar{b}\times\bar{c})\right]\\
&= \bar{a}\cdot(\bar{a}\times\bar{c})+\bar{a}\cdot(\bar{b}\times\bar{a})+\bar{a}\cdot(\bar{b}\times\bar{c})\\
&\quad +\bar{b}\cdot(\bar{a}\times\bar{c})+\bar{b}\cdot(\bar{b}\times\bar{a})+\bar{b}\cdot(\bar{b}\times\bar{c})\\
&\quad +\bar{c}\cdot(\bar{a}\times\bar{c})+\bar{c}\cdot(\bar{b}\times\bar{a})+\bar{c}\cdot(\bar{b}\times\bar{c})\\
&= \left[\bar{a}\ \bar{a}\ \bar{c}\right]+\left[\bar{a}\ \bar{b}\ \bar{a}\right]+\left[\bar{a}\ \bar{b}\ \bar{c}\right]\\
&\quad +\left[\bar{b}\ \bar{a}\ \bar{c}\right]+\left[\bar{b}\ \bar{b}\ \bar{a}\right]+\left[\bar{b}\ \bar{b}\ \bar{c}\right]\\
&\quad +\left[\bar{c}\ \bar{a}\ \bar{c}\right]+\left[\bar{c}\ \bar{b}\ \bar{a}\right]+\left[\bar{c}\ \bar{b}\ \bar{c}\right]\\
&= 0+0+\left[\bar{a}\ \bar{b}\ \bar{c}\right]+\left[\bar{b}\ \bar{a}\ \bar{c}\right]+0+0+0\\
&\quad +\left[\bar{c}\ \bar{b}\ \bar{a}\right]+0\\
&= \left[\bar{a}\ \bar{b}\ \bar{c}\right]-\left[\bar{a}\ \bar{b}\ \bar{c}\right]-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\\
&= \left[\bar{a}\ \bar{b}\ \bar{c}\right]-2\left[\bar{a}\ \bar{b}\ \bar{c}\right]\\
&= -\left[\bar{a}\ \bar{b}\ \bar{c}\right]
\end{aligned}$$

$\therefore$ $(\bar{a}+\bar{b}+\bar{c})\cdot\left[(\bar{a}+\bar{b})\times(\bar{a}+\bar{c})\right]=-\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

48. If in a tetrahedron, edges in each of the two pairs of opposite edges are perpendicular, then show that the edges in the third pair are also perpendicular.

Solution:



Let O-ABC be a tetrahedron.

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of points A, B, C respectively.

OA,BC; OB,AC; OC,AB are the pairs of opposite edges of the tetrahedron.

$\overline{OA}\perp\overline{BC}$, $\overline{OB}\perp\overline{AC}$ ...[Given]

Prove that $\overline{OC}\perp\overline{AB}$

Consider, $\overline{OA}\perp\overline{BC}$

$\therefore$ $\overline{OA}\cdot\overline{BC} = 0$

$\therefore$ $\bar{a}\cdot(\bar{c}-\bar{b}) = 0$

$\therefore$ $\bar{a}\cdot\bar{c}-\bar{a}\cdot\bar{b} = 0$

$\bar{a}\cdot\bar{c} = \bar{a}\cdot\bar{b}$ ...(i)

Also, $\overline{OB}\perp\overline{AC}$

$\therefore$ $\overline{OB}\cdot\overline{AC} = 0$

$\therefore$ $\bar{b}\cdot(\bar{c}-\bar{a}) = 0$

$\therefore$ $\bar{b}\cdot\bar{c}-\bar{b}\cdot\bar{a} = 0$

$\therefore$ $\bar{b}\cdot\bar{c}-\bar{a}\cdot\bar{b} = 0$

$\therefore$ $\bar{b}\cdot\bar{c} = \bar{a}\cdot\bar{b}$ ...(ii)

$\therefore$ $\bar{a}\cdot\bar{c} = \bar{b}\cdot\bar{c}$ ...[From (i) and (ii)]

$\therefore$ $\bar{b}\cdot\bar{c}-\bar{a}\cdot\bar{c} = 0$

$\therefore$ $(\bar{b}-\bar{a})\cdot\bar{c} = 0$

$\therefore$ $\overline{AB}\cdot\overline{OC} = 0$

$\therefore$ $\overline{AB}\perp\overline{OC}$

$\therefore$ In a tetrahedron, if two pairs of opposite edges are perpendicular, then the third pair of opposite edges is also perpendicular.

Activities for Practice

1. If $\bar{a} = 4\hat{i}+3\hat{j}+\hat{k}$ and $\bar{b} = \hat{i}-2\hat{k}$, then find $|2\bar{b}\times\bar{a}|$ by filling the boxes.

   $\bar{b}\times\bar{a} = \square\,\hat{i}-\square\,\hat{j}+3\hat{k}$

   $\therefore$ $2\bar{b}\times\bar{a} = \square(2\hat{i}-3\hat{j}+\hat{k})$

   $\therefore$ $|2\bar{b}\times\bar{a}| = \square$

2. Find the cosine of the angle between the vectors $\bar{a}$ and $\bar{b}$, if $\bar{a} = \hat{i}-2\hat{j}+\hat{k}$ and $\bar{b} = 2\hat{i}-2\hat{j}+2\hat{k}$ by filling the boxes.

   Let θ be the angle between $\bar{a} = \hat{i}-2\hat{j}+\hat{k}$ and $\bar{b} = 2\hat{i}-2\hat{j}+2\hat{k}$.

   $\therefore$ $\cos\theta = \dfrac{\square}{|\bar{a}|.|\bar{b}|}$

   $\bar{a}\cdot\bar{b} = \square$

   $|\bar{a}| = \sqrt{6}$ and $|\bar{b}| = \square$

   $\therefore$ $\cos\theta = \dfrac{\square}{\square}$

3. If A(2, −2, 3), B(x, 4, −1), C(3, x, −5) are the vertices and G(2, 1, −1) is the centroid of the triangle ABC, then by vector method find the value of x by filling the boxes.

   Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of vertices A, B, C respectively and $\bar{g}$ be the position vector of centroid G of triangle ABC.

   $\therefore$ $\bar{a} = 2\hat{i}-2\hat{j}+3\hat{k}$, $\bar{b} = x\hat{i}+4\hat{j}-\hat{k}$, $\bar{c} = 3\hat{i}+x\hat{j}-5\hat{k}$, $\bar{g} = 2\hat{i}+\hat{j}-\hat{k}$

   Since G is the centroid of ΔABC, by centroid formula,

   $\bar{g} = \dfrac{\square}{3}$

   $\therefore$ $6\hat{i}+3\hat{j}-3\hat{k} = \square\,\hat{i}+\square\,\hat{j}-3\hat{k}$

   By equality of vectors, we get

   $6 = \square$ and $3 = \square$

   $\therefore$ $x = \square$

4. Find two vectors each of magnitude 3 and perpendicular to both of the vectors $2\hat{i}+2\hat{j}+\hat{k}$ and $3\hat{i}-2\hat{j}-\hat{k}$ by filling the boxes.

Let $\bar{a} = 2\hat{i}+2\hat{j}+\hat{k}$, $\bar{b} = 3\hat{i}-2\hat{j}-\hat{k}$

$\bar{a}\times\bar{b} = 5(\hat{j}-\square)$

∴ Two vectors each of magnitude 3 and perpendicular to $\bar{a}$ and $\bar{b}$

$= \pm\dfrac{3(\bar{a}\times\bar{b})}{\square}$

$= \pm\dfrac{3\times5(\hat{j}-\square)}{5\sqrt{1+\square}}$

$= \pm\dfrac{\square}{\square}(\hat{j}-\square)$

5. Show that the four points A(−6, 3, 2), B(3, −2, 4), C(5, 7, 3) and D(−13, 17, −1) are coplanar by filling the boxes.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of points A, B, C, D respectively.

∴ $\bar{a} = -6\hat{i}+3\hat{j}+2\hat{k}$, $\bar{b} = 3\hat{i}-2\hat{j}+4\hat{k}$,

$\bar{c} = 5\hat{i}+7\hat{j}+3\hat{k}$, $\bar{d} = -13\hat{i}+17\hat{j}-\hat{k}$

Points A, B, C, D are coplanar if vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ are $\square$.

Now, $\overline{AB} = 9\hat{i}-5\hat{j}+2\hat{k}$

$\overline{AC} = 11\hat{i}+\square\hat{j}+\hat{k}$

$\overline{AD} = \square\hat{i}+\square\hat{j}-3\hat{k}$

$[\overline{AB}\ \overline{AC}\ \overline{AD}] = \square$

∴ Vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ are $\square$ with point A in common.

∴ Points A, B, C, D are coplanar.

One Mark Questions

1. Find the angle between two vectors $\bar{a}$ and $\bar{b}$ having the same magnitude $\sqrt{2}$ and their scalar product is 1.

2. Find the distance of the point (4, 3, 5) from the Y- axis.

3. Find the value of λ, if vectors $\bar{a} = \lambda\hat{i}+2\hat{j}+\hat{k}$ and $\bar{b} = 4\hat{i}-9\hat{j}+2\hat{k}$ are perpendicular to each other.

4. If $\bar{a} = \hat{i}-\hat{j}$ and $\bar{b} = -\hat{j}+2\hat{k}$, find

5. If $\bar{a} = 2\hat{i}+3\hat{j}+\hat{k}$, $\bar{b} = \hat{i}-2\hat{j}+\hat{k}$ and $\bar{c} = -3\hat{i}+\hat{j}+2\hat{k}$, find $\bar{a}\cdot(\bar{b}\times\bar{c})$.

Multiple Choice Questions

1. The position vector of the midpoint of segment PQ where position vectors of P and Q are given by $3\hat{i}+2\hat{j}+8\hat{k}$ and $\hat{i}-8\hat{j}-2\hat{k}$ is
   (A) $4\hat{i}-6\hat{j}+6\hat{k}$ (B) $2\hat{i}+3\hat{j}-\hat{k}$
   (C) $\hat{i}-3\hat{j}+2\hat{k}$ (D) $2\hat{i}-3\hat{j}+3\hat{k}$

2. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are unit vectors such that $\bar{a}+\bar{b}+\bar{c} = \bar{0}$, then the value of $\bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{c}\cdot\bar{a} =$
   (A) 1 (B) 2
   (C) $-\dfrac{3}{2}$ (D) $\dfrac{2}{3}$

3. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are unit vectors then $|\bar{a}-\bar{b}|^2+|\bar{b}-\bar{c}|^2+|\bar{c}-\bar{a}|^2$ does not exceed
   (A) 4 (B) 9
   (C) 8 (D) 6

4. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are vectors such that $\bar{c} = \bar{a}+\bar{b}$ and $\bar{a}\cdot\bar{b} = 0$, then
   (A) $a^2+b^2+c^2 = 0$ (B) $a^2-b^2 = 0$
   (C) $a^2+b^2 = c^2$ (D) $\bar{c} = \bar{a}\times\bar{b}$

5. Given $\bar{a} = \hat{i}+\hat{j}-\hat{k}$, $\bar{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\bar{c} = -\hat{i}+2\hat{j}-\hat{k}$, a unit vector perpendicular to both $\bar{a}+\bar{b}$ and $\bar{b}+\bar{c}$ is,
   (A) $\hat{i}$ (B) $\hat{j}$
   (C) $\hat{k}$ (D) $(\hat{i}+\hat{j}+\hat{k})\sqrt{3}$

6. If $\bar{a} = 2\hat{i}-3\hat{j}+\hat{k}$, $\bar{b} = -\hat{i}+\hat{k}$ and $\bar{c} = 2\hat{j}-\hat{k}$, then area of parallelogram having diagonals $(\bar{a}+\bar{b})$ and $(\bar{b}+\bar{c})$ is
   (A) $\sqrt{21}$ (B) $\dfrac{1}{2}\sqrt{21}$
   (C) $\sqrt{23}$ (D) $\dfrac{1}{2}\sqrt{23}$

7. If $\bar{a}\times\bar{b} = \bar{a}\times\bar{c}$, $\bar{a} \neq 0$ and $\bar{b} \neq \bar{c}$, then
   (A) $\bar{b} = \bar{c}+\lambda\bar{a}$ (B) $\bar{c} = \bar{a}+\lambda\bar{b}$
   (C) $\bar{a} = \bar{b}+\lambda\bar{c}$ (D) $\bar{c} = 2\bar{a}+\lambda\bar{b}$

8. $\bar{i}\times(\bar{x}\times\bar{i})+\bar{j}\times(\bar{x}\times\bar{j})+\bar{k}\times(\bar{x}\times\bar{k})$ is
   (A) D (B) $\bar{0}$

9. Let a, b, c be the three vectors such that $a\cdot(b+c) = b\cdot(c+a) = c\cdot(a+b) = 0$ and $|a| = 1$, $|b| = 4$, $|c| = 8$, then $|a+b+c| =$
   (A) 13 (B) 81
   (C) 9 (D) 5

10. The vector $(i-j)\times(j-k)\times(i+5k)$ is equal to
   (A) $5i-4j-k$ (B) $3i-2j+5k$
   (C) $4i-5j-k$ (D) $5i+4j-k$

11. Let $\bar{a} = \hat{i}+\hat{j}+p\hat{k}$ and $\bar{b} = \hat{i}+\hat{j}+\hat{k}$, then $|\bar{a}+\bar{b}| = |\bar{a}|+|\bar{b}|$, holds for
   (A) all real p (B) no real p
   (C) $p = -1$ (D) $p = 1$

12. If $|\bar{\alpha}+\bar{\beta}| = |\bar{\alpha}-\bar{\beta}|$, $\bar{\alpha}, \bar{\beta} \neq 0$, then
   (A) $\bar{\alpha} \parallel \bar{\beta}$ (B) $\bar{\alpha} \perp \bar{\beta}$
   (C) $|\bar{\alpha}| = |\bar{\beta}|$ (D) $|\bar{\alpha}| \neq |\bar{\beta}|$

13. The position vectors of P and Q are $5\hat{i}+4\hat{j}+a\hat{k}$ and $-\hat{i}+2\hat{j}-2\hat{k}$ respectively. If the distance between them is 7, then the value of a =
   (A) −5, 1 (B) 5, 1
   (C) 0, 5 (D) 1, 0

14. If the vectors $3\hat{i}+2\hat{j}-\hat{k}$ and $6\hat{i}-4x\hat{j}+y\hat{k}$ are parallel, then the value of x and y will be
   (A) −1, −2 (B) 1, −2
   (C) −1, 2 (D) 1, 2

15. If $3\hat{i}+4\hat{j}$ and $-5\hat{i}+7\hat{j}$ are the vector sides of any triangle, then its area is given by
   (A) 41 (B) 47 (C) $\dfrac{41}{2}$ (D) $\dfrac{47}{2}$

16. The points with position vectors $10\hat{i}+3\hat{j}$, $12\hat{i}-5\hat{j}$ and $a\hat{i}+11\hat{j}$ are collinear if the value of a is
   (A) −8 (B) 4 (C) 8 (D) 12

17. If $\bar{a}$ and $\bar{b}$ are two vectors then value of $(\bar{a}\cdot\bar{b})^2+(\bar{a}\times\bar{b})^2$ is
   (A) a^2b^2 (B) 1
   (C) $2a^2b^2$ (D) ab^2

18. If the position vectors of three points are $\bar{a}-2\bar{b}+3\bar{c}$, $2\bar{a}+3\bar{b}-4\bar{c}$, $-7\bar{b}+10\bar{c}$ then the points are
   (A) collinear (B) coplanar
   (C) non-collinear (D) non-coplanar

19. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors, then $\dfrac{\bar{a}\cdot\bar{b}\times\bar{c}}{\bar{c}\times\bar{a}\cdot\bar{b}}+\dfrac{\bar{b}\cdot\bar{a}\times\bar{c}}{\bar{c}\cdot\bar{a}\times\bar{b}} =$
   (C) 1 (D) ∞

20. The area of the Δ whose vertices are A(1, −1, 2), B(2, 1, −1), C(3, −1, 2) is
   (A) $\sqrt{13}$ (B) 8
   (C) 14 (D) 16

21. If $\bar{a}\times\bar{r} = \bar{b}+\lambda\bar{a}$ and $\bar{a}\cdot\bar{r} = 3$ where $\bar{a} = 2\hat{i}+\hat{j}-\hat{k}$ and $\bar{b} = -\hat{i}-2\hat{j}+\hat{k}$ then r and λ are equal to
   (A) $\bar{r} = \dfrac{7}{6}\hat{i}+\dfrac{2}{3}\hat{j}, \lambda = \dfrac{6}{5}$
   (B) $\bar{r} = \dfrac{7}{6}\hat{i}+\dfrac{2}{3}\hat{j}, \lambda = \dfrac{5}{6}$
   (C) $\bar{r} = \dfrac{6}{7}\hat{i}+\dfrac{2}{3}\hat{j}, \lambda = \dfrac{6}{7}$
   (D) $\bar{r} = \dfrac{6}{7}\hat{i}+\dfrac{2}{3}\hat{j}, \lambda = \dfrac{5}{6}$

22. If P(1, 2, 3) Q(1, 0, 3) and R(4, 1 − 3) form a triangle, with centroid G, then length RG =
   (A) $\sqrt{15}$ (B) $\sqrt{20}$
   (C) 4 (D) $\sqrt{6}$

23. If in □ABCD, L and M are the midpoints of the diagonals AC and BD respectively, then $\overline{AB}+\overline{CB}+\overline{AD}+\overline{CD} =$
   (A) $\overline{LM}$ (B) $2\overline{LM}$
   (C) $4\overline{LM}$ (D) $3\overline{LM}$

24. In ΔABC, D is the midpoint of AB. Then $\overline{AB}+\overline{BC}+\overline{AC}$ equals
   (A) $2(\overline{AD}-\overline{BD})$ (B) $2(\overline{AD}-\overline{CD})$
   (C) $2(\overline{BD}-\overline{CA})$ (D) $2(\overline{BD}-\overline{AC})$

25. The position vectors of A and B are $\bar{a}$ and $\bar{b}$ respectively. Then position vector of a point P dividing AB in the ratio 2 : 3 externally is
   (A) $\dfrac{1}{5}(2\bar{a}+3\bar{b})$ (B) $\dfrac{1}{5}(2\bar{b}+3\bar{a})$
   (C) $\dfrac{1}{6}(\bar{a}+\bar{b})$ (D) $3\bar{a}-2\bar{b}$

26. $A \equiv (\bar{a}+2\bar{b})$, P is $\bar{a}$ and P divides AB in the ratio 2 : 3. The position vector of B is
   (A) $2\bar{a}-\bar{b}$ (B) $\bar{b}-2\bar{a}$
   (C) $\bar{a}-3\bar{b}$ (D) $\bar{b}$

27. P, Q, R, S have position vectors p, q, r, s, such that $(p-q) = 2(s-r)$, then
   (A) PQ and RS bisect each other.
   (B) PQ and RS trisect each other.
   (C) QS and PR bisect each other.
   (D) QS and PR trisect each other.

28. Volume of parallelopiped whose coterminus edges are $2\bar{i}-3\bar{j}+4\bar{k}$, $\bar{i}+2\bar{j}-2\bar{k}$, $3\bar{i}-\bar{j}+\bar{k}$, is
(A) 5 cu. units (B) 6 cu. units
(C) 7 cu. units (D) 8 cu. Units

29. Value of k, for which the vectors $2\hat{i}+2\hat{j}-3\hat{k}$, $3\hat{i}+k\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ are coplanar, is
(A) 49 (B) $\frac{40}{9}$
(C) $\frac{30}{9}$ (D) $\frac{9}{40}$

30. $\bar{a}=\hat{i}+\hat{j}+\hat{k}$, $\bar{b}=2\hat{i}-4\hat{k}$, $\bar{c}=\hat{i}+\lambda\hat{j}+3\hat{k}$ are coplanar, then the value of λ is
(A) $\frac{5}{2}$ (B) $\frac{3}{5}$ (C) $\frac{7}{3}$ (D) $\frac{5}{3}$

31. $(3\bar{a}\times 2\bar{b})\cdot\bar{c}+(3\bar{b}\times 2\bar{c})\cdot\bar{a}+(4\bar{c}\times 3\bar{b})\cdot\bar{a}=$
(A) 0 (B) $24[\bar{a}\ \bar{b}\ \bar{c}]$
(C) $24[\bar{b}\ \bar{a}\ \bar{c}]$ (D) 24

32. $\bar{a}\cdot[(\bar{b}+\bar{c})\times(\bar{a}+\bar{b}+\bar{c})]$ is equal to :
(A) $[\bar{a}\ \bar{b}\ \bar{c}]$ (B) $2[\bar{a}\ \bar{b}\ \bar{c}]$
(C) $3[\bar{a}\ \bar{b}\ \bar{c}]$ (D) 0

33. $\bar{p}$, $\bar{q}$, $\bar{r}$ are three vectors. Then the scalar triple product $(\bar{p}-\bar{q})\cdot[(\bar{q}-\bar{r})\times(\bar{r}-\bar{p})]=$
(A) 0 (B) $2\bar{p}\cdot(\bar{q}\times\bar{r})$
(C) $\bar{p}\cdot(\bar{q}\times\bar{r})$ (D) $3\bar{p}\cdot\bar{q}\times\bar{r}$

34. Volume of the parallelopiped whose sides are given by $\overline{OA}=2\hat{i}-3\hat{j}$, $\overline{OB}=\hat{i}+\hat{j}-\hat{k}$ and $\overline{OC}=3\hat{i}-\hat{k}$, is equal to
(A) $\frac{4}{13}$ (B) 4
(C) $\frac{2}{7}$ (D) 6

35. Let $\bar{a}=\hat{i}-\hat{k}$, $\bar{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$ and $\bar{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$. Then $[\bar{a}\ \bar{b}\ \bar{c}]$ depends on
(A) only x (B) only y
(C) neither x nor y (D) both x and y

36. If $\bar{a}$, $\bar{b}$, $\bar{c}$ be three non-coplanar vector, then $[\bar{a}\times\bar{b},\ \bar{b}\times\bar{c},\ \bar{c}\times\bar{a}]=$
(A) $[\bar{a}\ \bar{b}\ \bar{c}]$ (B) $2[\bar{a}\ \bar{b}\ \bar{c}]$
(C) 0 (D) $[\bar{a}\ \bar{b}\ \bar{c}]^2$

37. If the vectors ai + j + k, i − bj + k and i + j − ck are co-planar, then : abc + 2 =
(A) a + b − c (B) a − b − c
(C) a + b + c (D) a − b + c

38. The volume of a tetrahedron PQRS where P is (3, 4, 7), Q ≡ (−2, 1, 5), R ≡ (1, 3, −1), S ≡ (−3, 6, 8) is
(A) $-\frac{205}{6}$ (B) $\frac{205}{6}$
(C) $\frac{215}{6}$ (C) $-\frac{215}{6}$

39. If the vectors 4i + 11j + mk, 7i + 2j + 6k and i + 5j + 4k are coplanar, then m is
(A) 38 (B) 0 (C) 10 (D) −10

40. If a, b, c, d be the position vectors of the points A, B, C and D respectively referred to same origin O such that no three of these points are collinear and a + c = b + d, then quadrilateral ABCD is a
(A) Square (B) Rhombus
(C) Rectangle (D) Parallelogram

41. The direction cosines of the line drawn from P(−5, 3, 1) to Q(1, 5, −2) is
(A) (6, 2, −3) (B) (2, − 4, 1)
(C) (− 4, 8, −1) (D) $\left(\frac{6}{7},\frac{2}{7},-\frac{3}{7}\right)$

42. If a straight line makes an angle of $\frac{\pi}{4}$ with X-axis and Y-axis, then the angle made by the line with Z-axis is
(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{4}$ (D) $\frac{2\pi}{3}$

43. If a line makes 120° and 60° angles with Y-axis and Z-axis respectively, then it will make with X-axis an angle of
(A) 60° or 120° (B) 45° or 135°
(C) 30° or 150° (D) 30° or 60°

44. If the co-ordinates of the points P, Q, R, S be (1, 2, 3), (4, 5, 7), (− 4, 3, − 6) and (2, 0, 2) respectively, then
(A) PQ ∥ RS (B) PQ ⊥ RS
(C) PQ = RS (D) PQ ≠ RS

45. The angle between the lines whose direction cosines are proportional to (1, 2, 1) and (2, −3, 6) is
(A) $\cos^{-1}\left(\frac{2}{7\sqrt{6}}\right)$ (B) $\cos^{-1}\left(\frac{1}{7\sqrt{6}}\right)$
(C) $\cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$ (D) $\cos^{-1}\left(\frac{5}{7\sqrt{6}}\right)$

46. If the direction ratios of a line are 1, −3, 2, then the direction cosines of the line are
(A) $\frac{1}{\sqrt{14}},\frac{-3}{\sqrt{14}},\frac{2}{\sqrt{14}}$ (B) $\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}$
(C) $\frac{-1}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{-2}{\sqrt{14}}$ (D) $\frac{-1}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{-3}{\sqrt{14}}$

47. The direction cosines of the line joining the points (4, 3, − 5) and (− 2, 1, − 8) are
(A) $\left(\frac{6}{7},\frac{2}{7},\frac{1}{7}\right)$ (B) $\left(\frac{2}{7},\frac{3}{7},\frac{6}{7}\right)$
(C) $\left(\frac{6}{7},\frac{3}{7},\frac{2}{7}\right)$ (D) $\left(\frac{6}{7},\frac{2}{7},\frac{3}{7}\right)$

48. If A (−1, 3, 2), B(2, 3, 5) and C (3, 5, −2), then ∠A =
(A) 90° (B) 60°
(C) 45° (D) 0°

49. If a line makes 45° and 60° angles with X-axis and Y-axis respectively, then with Z-axis it will make an angle of
(A) 30° or 150° (B) 60° or 120°
(C) 45° or 135° (D) 90° or 180°

50. If A(2, 3, −1), B(3, 5, −3), C(1, 2, 3) and D(3, 5, 7) are four points, then angle between AB and CD is
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

51. If direction ratios of two lines are 5, −12, 13 and −3, 4, 5, then the angle between them is
(A) $\cos^{-1}\left(\frac{1}{65}\right)$ (B) $\cos^{-1}\left(\frac{2}{65}\right)$
(C) $\cos^{-1}\left(\frac{3}{65}\right)$ (D) $\frac{\pi}{2}$

52. If direction cosines of two lines are proportional to (2, 3, − 6) and (3, − 4, 5), then the acute angle between them is
(A) $\cos^{-1}\left(\frac{49}{36}\right)$ (B) $\cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$
(C) $\cos^{-1}\left(\frac{18}{35}\right)$ (D) $\cos^{-1}\left(\frac{18}{35}\right)$

53. Direction ratios of a line are 1, 0, 1. The angle made by it with Y-axis is
(A) 0° (B) 60°
(C) 90° (D) 30°

54. If the co-ordinates of the points P and Q be (1, −2, 1) and (2, 3, 4) and O be the origin, then
(A) OP = OQ (B) OP ⊥ OQ
(C) OP ∥ OQ (D) OP ≠ OQ

55. If the direction cosines of a straight line are k, k, k, then
(A) k > 0 (B) 0 < k < 1
(C) k = 1 (D) $k=\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

56. The angle between a line with direction ratios 2 : 2 : 1 and a line joining (3, 1, 4) to (7, 2, 12) is
(A) $\cos^{-1}\left(\frac{2}{3}\right)$ (B) $\cos^{-1}\left(\frac{3}{2}\right)$
(C) $\tan^{-1}\left(-\frac{2}{3}\right)$ (D) $\tan^{-1}\left(\frac{3}{2}\right)$

57. The direction cosines of the ray from (0, 0, 0) to (2, −3, 6) are
(A) $\frac{2}{7},\frac{-3}{7},\frac{6}{7}$ (B) $\frac{-2}{7},\frac{3}{7},\frac{-6}{7}$
(C) 2, −3, 6 (D) −2, 3, − 6

58. If $\left(\frac{1}{\sqrt{2}},\frac{1}{2},k\right)$ represent direction cosines of a line, then k =
(A) $\pm\frac{1}{2}$
(B) $\frac{1}{2}$
(C) $-\frac{1}{2}$
(D) k can take any value.

59. If projections of a line on X, Y, and Z-axes are 6, 2 and 3 respectively, then direction cosines of the line are
(A) $\frac{6}{7},\frac{2}{7},\frac{3}{7}$ (B) $\frac{3}{7},\frac{5}{7},\frac{6}{7}$
(C) $\frac{1}{7},\frac{2}{7},\frac{3}{7}$ (D) None of these

60. The ratio in which the line joining (1, 2, 3) and (−3, 4, −5) is divided by XY-plane, is
(A) 5 : 3 internally (B) 3 : 5 internally
(C) 2 : 3 internally (D) 1 : 2 internally

61. If a line OP makes an angle of measure 120° with X-axis and an angle of measure 60° with Y-axis, then the angle made by the line with Z-axis is
(A) 45° (B) 120°
(C) 60° (D) 0°

62. The acute angle between the lines whose direction cosines are proportional to (2, 3, − 6) and (3, − 4, 5), is
(A) $\cos^{-1}\left(\frac{36}{7\sqrt{48}}\right)$ (B) $\cos^{-1}\left(\frac{36}{7\sqrt{50}}\right)$
(C) $\cos^{-1}\left(\frac{18}{7\sqrt{50}}\right)$ (D) $\cos^{-1}\left(\frac{18}{7\sqrt{48}}\right)$

63. In ΔABC the mid points of the sides AB, BC, and CA are $(l, 0, 0)$, $(0, m, 0)$ and $(0, 0, n)$ respectively. Then, $\frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$ is equal to
(A) 2 (B) 4
(C) 8 (D) 16

64. The angle between the lines whose direction cosines are $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{-\sqrt{3}}{2}\right)$, is
(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

65. The coordinates of the foot of the perpendicular drawn from the point A(1, 0, 3) to the join of the points B(4, 7, 1) and C(3, 5, 3) are
(A) $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$ (B) (5, 7, 17)
(C) $\left(\frac{5}{3}, -\frac{7}{3}, \frac{17}{3}\right)$ (D) $\left(-\frac{5}{3}, \frac{7}{3}, -\frac{17}{3}\right)$

66. If the vectors $-3\hat{i} + 4\hat{j} - 2\hat{k}$, $\hat{i} + 2\hat{k}$, $\hat{i} - p\hat{j}$ are coplanar, then the value of p is [Mar 13]
(A) -2 (B) 1
(C) -1 (D) 2

67. If a line is inclined at $60°$ and $30°$ with the X and Y-axes respectively, then the angle which it makes with Z-axis is [Oct 13]
(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

68. Which of the following represents direction cosines of the line? [Mar 14]
(A) $0, \frac{1}{\sqrt{2}}, \frac{1}{2}$ (B) $0, \frac{-\sqrt{3}}{2}, \frac{1}{\sqrt{2}}$
(C) $0, \frac{\sqrt{3}}{2}, \frac{1}{2}$ (D) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

69. If $\left[\bar{a}\ \bar{b}\ \bar{c}\right] \neq 0$ and $\bar{p} = \frac{\bar{b} \times \bar{c}}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$, $\bar{q} = \frac{\bar{c} \times \bar{a}}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$, $\bar{r} = \frac{\bar{a} \times \bar{b}}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$, then $\bar{a}.\bar{p} + \bar{b}.\bar{q} + \bar{c}.\bar{r}$ is equal to [Oct 14]
(A) 0 (B) 1 (C) 2 (D) 3

70. Direction cosines of the line passing through the points A(− 4, 2, 3) and B(1, 3, −2) are [Oct 14]
(A) $\pm\frac{1}{\sqrt{51}}, \pm\frac{5}{\sqrt{51}}, \pm\frac{1}{\sqrt{51}}$
(B) $\pm\frac{5}{\sqrt{51}}, \pm\frac{1}{\sqrt{51}}, \pm\frac{-5}{\sqrt{51}}$
(C) $\pm 5, \pm 1, \pm 5$
(D) $\pm\sqrt{51}, \pm\sqrt{51}, \pm\sqrt{51}$

71. If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, then value of q is [July 16]
(A) 5 (B) 10
(C) $\frac{5}{2}$ (D) $\frac{5}{4}$

72. If the points A(2, 1, 1), B(0, −1, 4) and C(k, 3, −2) are collinear, then k = ______. [Mar 17]
(A) 0 (B) 1
(C) 4 (D) − 4

73. If $\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{c} = -5\hat{i} + 2\hat{j} + 3\hat{k}$, then $\bar{a} \cdot \left(\bar{b} \times \bar{c}\right) =$ ______. [July 17]
(A) 100 (B) 101
(C) 110 (D) 109

74. If a line makes angles $90°$, $135°$, $45°$ with the X, Y, and Z axes respectively, then its direction cosines are ______. [July 17]
(A) $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ (B) $0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$
(C) $1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (D) $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

75. The measure of acute angle between the lines whose direction ratios are 3, 2, 6 and −2, 1, 2 is ______. [Mar 18]
(A) $\cos^{-1}\left(\frac{1}{7}\right)$ (B) $\cos^{-1}\left(\frac{8}{15}\right)$
(C) $\cos^{-1}\left(\frac{1}{3}\right)$ (D) $\cos^{-1}\left(\frac{8}{21}\right)$

76. If the vectors $\hat{i} - 2\hat{j} + \hat{k}$, $a\hat{i} - 5\hat{j} + 3\hat{k}$ and $5\hat{i} - 9\hat{j} + 4\hat{k}$ are coplanar, then the value of a is ______. [July 18]
(A) 3 (B) −3
(C) 2 (D) −2

77. The direction ratios of the line which is perpendicular to the lines with direction ratios −1, 2, 2 and 0, 2, 1 are ______. [Mar 19]
(A) −2, −1, −2 (B) 2, 1, 2
(C) 2, −1, − 2 (D) −2, 1, −2

78. If a line makes angles α, β, γ with co-ordinate axes, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$ ______. [July 19]
(A) 1 (B) − 1
(C) 2 (D) − 2

79. If $\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\bar{c} = -5\hat{i} + 2\hat{j} + 3\hat{k}$, then $\bar{a} \cdot \left(\bar{b} \times \bar{c}\right)$ is ______. [Feb 20]
(A) 100 (B) 110
(C) 109 (D) 10

Time: 1 Hour

TOPIC TEST

Total Marks: 20

SECTION A

Q.1. Select and write the correct answer. [4]

i. If $|\bar{a}| = 2$, $|\bar{b}| = 3$ $|\bar{c}| = 4$, then $\left[\bar{a} + \bar{b}\ \ \bar{b} + \bar{c}\ \ \bar{c} - \bar{a}\right]$ is equal to
(A) 24 (B) −24 (C) 0 (D) 48

ii. If $|\bar{a}| = 3$, $|\bar{b}| = 5$, $|\bar{c}| = 7$ and $\bar{a} + \bar{b} + \bar{c} = 0$, then the angle between $\bar{a}$ and $\bar{b}$ is
(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Q.2. Answer the following. [2]

i. Find the angle between two vectors $\bar{a}$ and $\bar{b}$ having the same magnitude $\sqrt{2}$ and their scalar product is 1.

ii. Find the distance from (4, −2, 6) to the Z-axis.

SECTION B

Attempt any two of the following: [4]

Q.3. Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $-5\hat{i} + 2\hat{j} - 5\hat{k}$ in the ratio 3 : 2 internally

Q.4. The vector $\bar{a}$ is directed due north and $|\bar{a}| = 24$. The vector $\bar{b}$ is directed due west and $|\bar{b}| = 7$. Find $|\bar{a} + \bar{b}|$.

Q.5. If $|\bar{a}| = |\bar{b}| = 1$, $\bar{a} \cdot \bar{b} = 0$ and $\bar{a} + \bar{b} + \bar{c} = 0$, then find $|\bar{c}|$.

SECTION C

Attempt any two of the following: [6]

Q.6. Find the angle between the lines whose direction cosines are given by the equation $6mn - 2nl + 5lm = 0$, $3l + m + 5n = 0$.

Q.7. If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{j} - \hat{k}$, find a vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$.

Q.8. Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

SECTION D

Attempt any one of the following: [4]

Q.9. If four points $A\left(\bar{a}\right)$, $B\left(\bar{b}\right)$, $C\left(\bar{c}\right)$ and $D\left(\bar{d}\right)$ are coplanar, then show that
$$\left[\bar{a}\ \bar{b}\ \bar{d}\right] + \left[\bar{b}\ \bar{c}\ \bar{d}\right] + \left[\bar{c}\ \bar{a}\ \bar{d}\right] = \left[\bar{a}\ \bar{b}\ \bar{c}\right].$$

Q.10. If in a tetrahedron, edges in each of the two pairs of opposite edges are perpendicular, then show that the edges in the third pair are also perpendicular.

Answers

Activities for Practice

1. i. 6 ii. 9
   iii. 6 iv. $6\sqrt{14}$
2. i. $\bar{a}\cdot\bar{b}$ ii. 8
   iii. $\sqrt{12}$ iv. $\frac{4}{3\sqrt{2}}$
3. i. $\bar{a}+\bar{b}+\bar{c}$ ii. $x+5$
   iii. $x+2$ iv. $x+5$
   v. $x+2$ vi. 1
4. i. $2\hat{k}$ ii. $|\bar{a}\times\bar{b}|$
   iii. $2\hat{k}$ iv. 4
   v. $\frac{3}{\sqrt{5}}$ vi. $2\hat{k}$
5. i. coplanar ii. 4
   iii. -7 iv. 14
   v. 0 vi. coplanar

One Mark Questions

1. $\frac{\pi}{3}$
2. $\sqrt{41}$ units
3. 4
4. -9
5. -30

Multiple Choice Questions

1.	(D)	2.	(C)	3.	(B)	4.	(C)
5.	(C)	6.	(B)	7.	(A)	8.	(C)
9.	(C)	10.	(A)	11.	(D)	12.	(B)
13.	(A)	14.	(A)	15.	(C)	16.	(C)
17.	(A)	18.	(A)	19.	(A)	20.	(A)
21.	(B)	22.	(B)	23.	(C)	24.	(B)
25.	(D)	26.	(C)	27.	(D)	28.	(C)
29.	(B)	30.	(D)	31.	(A)	32.	(D)
33.	(A)	34.	(B)	35.	(C)	36.	(D)
37.	(B)	38.	(B)	39.	(C)	40.	(D)
41.	(D)	42.	(A)	43.	(B)	44.	(D)
45.	(A)	46.	(A)	47.	(D)	48.	(A)
49.	(B)	50.	(C)	51.	(A)	52.	(B)
53.	(C)	54.	(B)	55.	(D)	56.	(A)
57.	(A)	58.	(A)	59.	(A)	60.	(B)
61.	(A)	62.	(B)	63.	(C)	64.	(C)
65.	(A)	66.	(D)	67.	(C)	68.	(C)
69.	(D)	70.	(B)	71.	(C)	72.	(C)
73.	(C)	74.	(D)	75.	(D)	76.	(C)
77.	(D)	78.	(C)	79.	(B)		

Topic Test

1. i. (C) ii. (B)
2. i. $\frac{\pi}{3}$ ii. $2\sqrt{5}$ units
3. $\frac{-11}{5}\hat{i}+\frac{4}{5}\hat{j}-\frac{9}{5}\hat{k}$
4. 25
5. $\sqrt{2}$
6. $\cos^{-1}\left(\frac{1}{6}\right)$
7. $\frac{1}{3}\left(5\hat{i}+2\hat{j}+2\hat{k}\right)$

Competitive Corner

1. Let the volume of a parallelopiped whose coterminous edges are given by $\bar{u}=\hat{i}+\hat{j}+\lambda\hat{k}$, $\bar{v}=\hat{i}+\hat{j}+3\hat{k}$ and $\bar{w}=2\hat{i}+\hat{j}+\hat{k}$ be 1 cu. unit. If θ be the angle between the edges $\bar{u}$ and $\bar{w}$, then $\cos\theta$ can be **[JEE (Main) 2020]**
   (A) $\frac{7}{6\sqrt{3}}$ (B) $\frac{7}{6\sqrt{6}}$
   (C) $\frac{5}{3\sqrt{3}}$ (D) $\frac{5}{7}$

2. The projection of the line segment joining the points $(1, -1, 3)$ and $(2, -4, 11)$ on the line joining the points $(-1, 2, 3)$ and $(3, -2, 10)$ is ________. **[JEE (Main) 2020]**

3. Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be three vectors such that $|\bar{a}|=\sqrt{3}$, $|\bar{b}|=5$, $\bar{b}\cdot\bar{c}=10$ and the angle between $\bar{b}$ and $\bar{c}$ is $\frac{\pi}{3}$. If $\bar{a}$ is perpendicular to the vector $\bar{b}\times\bar{c}$, then $\left|\bar{a}\times\left(\bar{b}\times\bar{c}\right)\right|$ is equal to ________. **[JEE (Main) 2020]**

4. If $A \equiv (3, x, 1)$, $B \equiv (y, -2, 2)$, $C \equiv (2x, 2y, -3)$ are the vertices of ΔABC and $G \equiv (2, 1, 0)$ is its centroid, then **[MHT CET 2019]**
   (A) $x=\frac{-1}{3}, y=\frac{-7}{3}$ (B) $x=\frac{7}{3}, y=\frac{1}{3}$
   (C) $x=\frac{1}{3}, y=\frac{7}{3}$ (D) $x=\frac{5}{3}, y=\frac{4}{3}$

5. If $\bar{a}, \bar{b}, \bar{c}$ are unit coplanar vectors, then $\left[2\bar{a}-\bar{b}\quad 2\bar{b}-\bar{c}\quad 2\bar{c}-\bar{a}\right]=$ **[MHT CET 2019]**
   (A) 0 (B) 1
   (C) $\sqrt{3}$ (D) $-\sqrt{3}$

Answers:
1. (A) 2. 8 3. 30 4. (C)
5. (A)

Hints:

1. Volume of parallelopiped = 1
$\left[\bar{u}\quad\bar{v}\quad\bar{w}\right]=1$
$\therefore \begin{vmatrix}1 & 1 & \lambda\\ 1 & 1 & 3\\ 2 & 1 & 1\end{vmatrix}=1$
$\therefore |1(1-3)-1(1-6)+\lambda(1-2)|=1$
$\therefore |-2+5-\lambda|=1$
$\therefore \lambda=4$ or $\lambda=2$
When $\lambda=2$,
$\bar{u}\cdot\bar{w}=|\bar{u}||\bar{w}|\cos\theta$
$\left(\hat{i}+\hat{j}+2\hat{k}\right)\cdot\left(2\hat{i}+\hat{j}+\hat{k}\right)=\sqrt{6}\cdot\sqrt{6}\cos\theta$
$\therefore 2+1+2=6\cos\theta$
$\therefore \cos\theta=\frac{5}{6}$
When $\lambda=4$,
$\bar{u}\cdot\bar{w}=|\bar{u}||\bar{w}|\cos\theta$
$\left(\hat{i}+\hat{j}+4\hat{k}\right)\cdot\left(2\hat{i}+\hat{j}+\hat{k}\right)=\sqrt{18}\sqrt{6}\cos\theta$
$\therefore 2+1+4=\sqrt{18}\sqrt{6}\cos\theta$
$\therefore \cos\theta=\frac{7}{\sqrt{18}\sqrt{6}}$
$=\frac{7}{6\sqrt{3}}$

2. Let P(1, −1, 3), Q(2, −4, 11), R(−1, 2, 3) and S(3, −2, 10) be the given points.
$\overline{PQ}=\bar{q}-\bar{p}=\hat{i}-3\hat{j}+8\hat{k}$
$\overline{RS}=\bar{s}-\bar{r}=4\hat{i}-4\hat{j}+7\hat{k}$
Projection of $\overline{PQ}$ on $\overline{RS}$
$=\frac{\overline{PQ}\cdot\overline{RS}}{|\overline{RS}|}$
$=\frac{(1)(4)+(-3)(-4)+(8)(7)}{\sqrt{4^2+(-4)^2+7^2}}$
$=\frac{72}{\sqrt{81}}=8$

3. $\left|\bar{a}\times\left(\bar{b}\times\bar{c}\right)\right|=|\bar{a}|\left|\bar{b}\times\bar{c}\right|\sin\frac{\pi}{2}$
$=|\bar{a}|\left|\bar{b}\times\bar{c}\right|$
$=|\bar{a}|\left(|\bar{b}||\bar{c}|\sin\frac{\pi}{3}\right)$
$=\left(\sqrt{3}\right)(5)|\bar{c}|\left(\frac{\sqrt{3}}{2}\right)$
$\therefore \left|\bar{a}\times\left(\bar{b}\times\bar{c}\right)\right|=\frac{15}{2}|\bar{c}|$ ...(i)
$\cos\theta=\frac{\bar{b}\cdot\bar{c}}{|\bar{b}||\bar{c}|}$
$\therefore \cos\frac{\pi}{3}=\frac{10}{5|\bar{c}|}$
$\therefore \frac{1}{2}=\frac{2}{|\bar{c}|}$
$\therefore |\bar{c}|=4$ ...(ii)
$\therefore \left|\bar{a}\times\left(\bar{b}\times\bar{c}\right)\right|=30$ ...[From (i) and (ii)]

4. Since, G is the centroid of ΔABC.
$\therefore 2=\frac{3+y+2x}{3}$ and $1=\frac{x-2+2y}{3}$
$\Rightarrow 6=3+y+2x$ and $3=x-2+2y$
$\Rightarrow 2x+y=3$ and $x+2y=5$
Solving these equations, we get
$x=\frac{1}{3}$ and $y=\frac{7}{3}$

5. $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors.
$\Rightarrow \left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$
Let $\bar{\alpha}=2\bar{a}-\bar{b}$, $\bar{\beta}=2\bar{b}-\bar{c}$ and $\gamma=2\bar{c}-\bar{a}$. Then,
$\left[\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\right]=\begin{vmatrix}2 & -1 & 0\\ 0 & 2 & -1\\ -1 & 0 & 2\end{vmatrix}\left[\bar{a}\ \bar{b}\ \bar{c}\right]$
$\Rightarrow \left[\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\right]=7\left[\bar{a}\ \bar{b}\ \bar{c}\right]=7(0)=0$

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