Lecture 1: General Overview 08:10 AM Thu, Sep 25 2025

0.1 Introduction

Let $E \neq \emptyset$ a set.

A binary operation $\cdot$ on E is any map from $E \times E$ into E,

$$\begin{aligned} (\cdot): \quad E \times E &\longrightarrow \quad E \\ (x, y) &\longmapsto \quad x \cdot y \end{aligned}$$

Let $A \subset E$, we say A is a stable by $(\cdot)$if $(\cdot)$ is also a Binary Operation on A,

$$\begin{aligned} (\cdot_A): \quad A \times A &\longrightarrow \quad A \\ (x, y) &\longmapsto \quad x \cdot_A y = x \cdot y \end{aligned}$$

Definition 0.1.1 (Group) : *Let $G \neq \emptyset$ a set with a Binary Operation $(*)$, we say that G is a group if :*

1. *$(*)$ is associative, if :*

$$\forall x, y, z \in G : \quad (x * y) * z = x * (y * z)$$

2. *$(*)$ admits a netural elements if :*

$$\exists e \in G, \forall x \in G : \quad x * e = e * x = x$$

3. $$\forall x \in G, \exists x' \in G : \quad x * x' = x' * x = e$$

if $(*)$ is commutative i.e.:

$$\forall x, y \in G : \quad x * y = y * x$$

then G is called an Abelian Group.

Notation: We denote $(*)$ by $(\cdot)$ if its multiplicative, and $(+)$ if its additive.

Proposition 0.1.1 : *Let $(G, \cdot)$ be a group. then:*

1. *The Neutral Element is uniuqe.*
2. *The inverse is unique*
3. $$\forall x, y \in G: \quad (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$$
4. $$\forall x, y, z \in G: \begin{cases} xy = xz \\ yx = zx \end{cases} \implies \begin{cases} y = z \\ y = z \end{cases}$$

Proof. 1. Let $e_1, e_2 \in G$ be a Neutral Element, then:

$$e_1 = e_1 \cdot e_2 = e_2$$

2. let $x \in G$ and $x_1, x_2 \in G$ be its inverses, then:

$$x_1 = x_1 \cdot e = x_1 \cdot (x \cdot x_2) = (x_1 \cdot x) \cdot x_2 = e \cdot x_2 = x_2$$

3. Let $x, y' \in G$. then:

$$\begin{aligned} (x \cdot y) \cdot (x \cdot y)^{-1} = e &\implies y \cdot (x \cdot y)^{-1} = x^{-1} \\ &\implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1} \end{aligned}$$

□

Exercise

Let $(G, \cdot)$ be a group and $x_1, x_2, \dots, x_n \in G$. then:

- $(x_1 \cdots x_n)^{-1} = x_n^{-1} \cdots x_1^{-1}$
- $(x_1^{-1})^{-1} = x_1$

Definition 0.1.2 : *Let $(G,\cdot)$ be a group, $n \in \mathbb{Z}$ and $x \in G$, define*

$$x^n = \begin{cases} x \cdot x \cdots x \\ e \\ x^{-1} \cdot x^{-1} \cdots x^{-1} \end{cases} \implies \begin{cases} \text{if } n \geq 1 \\ \text{if } n = 0 \\ \text{if } n \leq -1 \end{cases}$$

Example:

1. $(Z,+), (\mathbb{Q}^*,\cdot), (\mathbb{R},+), (\mathbb{C}^*,\cdot)$
2. *The set $\mathcal{F}(\mathbb{R},\mathbb{R})$ with addition of maps is an Abelian Group, with the null map as Neutral Element*
3. *The set S_n of all bijection of $\{1,\dots,n\}$ with composition of maps is a group*

Definition 0.1.3 (Sub Group) : *Let $(G,\cdot)$ be a group and $H \subset G$ we say that H is a Subgroup of G if $(H,\cdot)$ is a gorup*

Proposition 0.1.2 :
Let $(G,\cdot)$ a group and $H \subset G$. then H is a Subgroup of G if and only if:

1. $H \neq \emptyset$
2. $\forall x, y \in H : \quad x \cdot y \in H$
3. $\forall x \in H : \quad x^{-1} \in H$

Remark: The conditions (2) and (3) are equivalent to:

$$\forall x, y \in H : \quad x^{-1} \cdot y \in H$$

Proof.

$$\forall x, y \in H : \quad x^{-1} \cdot y \in H \implies \begin{cases} \forall x, y \in H : \quad x \cdot y \in H \\ \forall x \in H : \quad x^{-1} \in H \end{cases}$$

□

Notation: if H is a Subgroup of G, we denote

$$H \leq G$$

if $H \leq G$ with $H \neq G$, we call H a proper Subgroup of G and we write $H < G$

Exercise

Let $(G, \cdot)$ be a group, then the set:

$$Z(G) = \{x \in G : \quad gx = xg, \forall g \in G\}$$

1. Prove that $Z(G) = G \iff G$ is an abelian group.

Proof. 1.

$$(\implies)$$

Suppose that G is an Abelian Group.
Let $x \in G$ and let $g \in G$, since G is an Abelian group, then $gx = xg$. then $x \in Z(G)$, then $Z(G) = G$

$$(\impliedby)$$

Suppose that $Z(G) = G$, let $x, y \in G$. then $x, y \in Z(G)$. so $\forall g \in G$:

$$\begin{cases} xg = gx \\ yg = gy \end{cases}$$

so for $g = y$, we get $xy = yx$ so G is an abelian group

2. Let $G \leq (\mathbb{Z}, +)$.

- if $G = \{0\}$. then $G = 0\mathbb{Z}$.
- if $G \neq \{0\}$, then $\exists m \in G$ with $m \neq 0$, without loss of generality. suppose $m > 0$, so $G \cap \mathbb{N} \neq \emptyset$, so $n = \min G \cap \mathbb{N}$, let $x \in n\mathbb{Z}$. then $x = kn, k \in$, so $x \in G$. hence $n\mathbb{Z} \subset G$. Let $x \in G$, so $\exists q, r \in \mathbb{Z}$, $0 \leq r \leq n-1$ such that $x = qn + r$. so $r = x - qn \in G$, if $r \neq 0$ then :

$$\begin{cases} r < n \\ r = G \cap \mathbb{N} \end{cases} \implies \begin{cases} r < n \\ n = \min G \cap \mathbb{N} \leq r, \quad \text{is a contradiction} \end{cases}$$

so $x = qn \in n\mathbb{Z}$, so $G \subset n\mathbb{Z}$

$\square$

Proposition 0.1.3 : *Let $H, K \leq G$, with G is a group. then:*

$$H \cap K \leq G$$

Proof. Since $e \in H$ and $e \in K$, then $e \in H \cap K \neq \emptyset$.

Let $x, y \in H \cap K$, then :

$$\begin{cases} x, y \in H \\ x, y \in K \end{cases} \implies \begin{cases} x^{-1}, y \in H \\ x^{-1}, y \in K \end{cases} \implies \begin{cases} x^{-1} \cdot y \in H \\ x^{-1} \cdot y \in K \end{cases} \implies x^{-1} \cdot y \in H \cap K$$

□

Proposition 0.1.4 : *Let $\{H_i\}_{i \in I}$ be a family of Subgroup of a group G, then :*

$$\bigcap_{i \in I} H_i \leq G$$

Remark: $H \cup K$ is not always a Subgroup of G.

Proposition 0.1.5 : *Let $H, K \leq G$, Then $H \cup K \leq G \iff \begin{cases} H \subset K \\ K \subset H \end{cases}$*