Lecture 1: General Overview 08:10 AM Thu, Sep 25 2025

0.1 Introduction

Let $E \neq \emptyset$ a set.

A binary operation $\cdot$ on E is any map from $E \times E$ into E,

$$\begin{aligned} (\cdot): \quad E \times E &\longrightarrow E \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

Let $A \subset E$, we say A is a stable by $(\cdot)$if $(\cdot)$ is also a Binary Operation on A,

$$\begin{aligned} (\cdot_A): \quad A \times A &\longrightarrow A \\ (x, y) &\longmapsto x \cdot_A y = x \cdot y \end{aligned}$$

Definition 0.1.1 (Group) : *Let $G \neq \emptyset$ a set with a Binary Operation $(*)$, we say that G is a group if :*

1. *$(*)$ is associative, if :*

$$\forall x, y, z \in G: \quad (x * y) * z = x * (y * z)$$

2. *$(*)$ admits a netural elements if :*

$$\exists e \in G, \forall x \in G: \quad x * e = e * x = x$$

3.
$$\forall x \in G, \exists x' \in G: \quad x * x' = x' * x = e$$

if $(*)$ is commutative i.e.:

$$\forall x, y \in G: \quad x * y = y * x$$

then G is called an Abelian Group.

Notation: We denote $(*)$ by $(\cdot)$ if its multiplicative, and $(+)$ if its additive.

Proposition 0.1.1 : *Let $(G, \cdot)$ be a group. then:*

1. *The Neutral Element is uniuqe.*
2. *The inverse is unique*
3. $$\forall x, y \in G : \quad (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$$
4. $$\forall x, y, z \in G : \begin{cases} xy = xz \\ yx = zx \end{cases} \implies \begin{cases} y = z \\ y = z \end{cases}$$

Proof. 1. Let $e_1, e_2 \in G$ be a Neutral Element, then:

$$e_1 = e_1 \cdot e_2 = e_2$$

2. let $x \in G$ and $x_1, x_2 \in G$ be its inverses, then:

$$x_1 = x_1 \cdot e = x_1 \cdot (x \cdot x_2) = (x_1 \cdot x) \cdot x_2 = e \cdot x_2 = x_2$$

3. Let $x, y' \in G$. then:

$$\begin{aligned} (x \cdot y) \cdot (x \cdot y)^{-1} = e &\implies y \cdot (x \cdot y)^{-1} = x^{-1} \\ &\implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1} \end{aligned}$$

□

Exercise

Let $(G, \cdot)$ be a group and $x_1, x_2, \ldots, x_n \in G$. then:

- $(x_1 \cdots x_n)^{-1} = x_n^{-1} \cdots x_1^{-1}$
- $(x_1^{-1})^{-1} = x_1$

Definition 0.1.2 : *Let $(G,\cdot)$ be a group, $n \in \mathbb{Z}$ and $x \in G$, define*

$$x^n = \begin{cases} x \cdot x \cdots x \\ e \\ x^{-1} \cdot x^{-1} \cdots x^{-1} \end{cases} \implies \begin{cases} \text{if } n \geq 1 \\ \text{if } n = 0 \\ \text{if } n \leq -1 \end{cases}$$

Example:

1. $(Z,+), (\mathbb{Q}^*,\cdot), (\mathbb{R},+), (\mathbb{C}^*,\cdot)$
2. *The set $\mathcal{F}(\mathbb{R},\mathbb{R})$ with addition of maps is an Abelian Group, with the null map as Neutral Element*
3. *The set S_n of all bijection of $\{1,\ldots,n\}$ with composition of maps is a group*

Definition 0.1.3 (Sub Group) : *Let $(G,\cdot)$ be a group and $H \subset G$ we say that H is a Subgroup of G if $(H,\cdot)$ is a gorup*

Proposition 0.1.2 :
Let $(G,\cdot)$ a group and $H \subset G$. then H is a Subgroup of G if and only if:

1. $H \neq \emptyset$
2. $\forall x, y \in H : \quad x \cdot y \in H$
3. $\forall x \in H : \quad x^{-1} \in H$

Remark: The conditions (2) and (3) are equivalent to:

$$\forall x, y \in H : \quad x^{-1} \cdot y \in H$$

Proof.

$$\forall x, y \in H : \quad x^{-1} \cdot y \in H \implies \begin{cases} \forall x, y \in H : \quad x \cdot y \in H \\ \forall x \in H : \quad x^{-1} \in H \end{cases}$$

□