Complex Analysis Lecture Notes

Hand written summary from lectures

Acknowledgment

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`http://farhi.bakir.free.fr/home/index-fr.html`

Disclaimer

These notes were written in real-time during the lectures, this is not the final version, yet. so they may contain :

- Incomplete or incorrect information
- Typos, transcription mistakes, or missing content
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if you spot an error feel free to open an issue or submit a pull request, or contact me via gmail :

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Notes on Contribution :

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My Github Page

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Chapter 1

Power Series

Lecture 1

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Definition 1.0.1 (Power Series) : *A power series is a formal series of the form $\sum_{n=1}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$.*
More generally, given $z_0 \in \mathbb{C}$, a power series centered at z_0 is a formal series of the form:

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C} \quad (\forall n \in \mathbb{N})$

Remark:

The set of all complex power series (centered at 0) is denoted by $\mathbb{C}\,[[z]]$. More generally, given $z_0 \in \mathbb{C}$, the set of all complex power series at z_0 is denoted by $\mathbb{C}\,[[z - z_0]]$

Operations on Formal Power Series:

Given $z_0 \in \mathbb{C}$, we equip $\mathbb{C}\,[[z - z_0]]$, with the following operations:

1. **Additions:** For all $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{C}$:

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^n = \sum_{n=1}^{\infty} (a_n + b_n)(z - z_0)^n$$

2. **Multiplictation:**

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n \times \sum_{n=1}^{\infty} b_n (z - z_0)^n = \sum_{n=1}^{\infty} c_n (z - z_0)^n$$

where, $c_n = \sum_{k=1}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N}_0$. Also $(c_n)_{n\in\mathbb{N}}$ is called the covolution of the two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$.

3. **Scalar Multiplication:** For all $\lambda \in \mathbb{C}$, and all $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$:

$$\lambda \sum_{n=1}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} (\lambda a_n)(z - z_0)^n$$

It's straight forward to verify that $\mathbb{C}[[z - z_0]]$ equipped with these operations forms a commutative algebra over $\mathbb{C}$. The Multiplicative identity is the constant power series:

$$1 = 1 + 0 \cdot (z - z_0) + 0 \cdot (z - z_0)^2 + \ldots$$

Definition 1.0.2 (Domain of Convergence) : *The domain of convergence of a power series $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ is the set of all points $z \in \mathbb{C}$ for which the series converge the structure of this domain is very specific. Its a disk (possibly with some points in its boundary) centered at z_0.*

Proposition 1.0.1 (Abel's Lemma) : *Let $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ be a power series and let $z_1 \in \mathbb{C} \setminus \{z_0\}$. Suppose that the sequence $\{a_n (z_1 - z_0)^n\}_{n\in\mathbb{N}}$ is bounded. Then, the power series in question converges absolutely (so converges) for every $z \in \mathbb{C}$, such that:*

$$|z - z_0| < |z_1 - z_0|$$

Proof. By hypothesis, $\exists M > 0$ such that $\forall n \in \mathbb{N}_0$:

$$|a_n (z_1 - z_0)^n| \leq M$$

Then, for all $z \in \mathbb{C}$ such that $|z - z_0| < |z_1 - z_0|$ we have:

$$\begin{aligned} |a_n (z - z_0)^n| &= \underbrace{|a_n (z_1 - z_0)^n|}_{\leq M} \cdot \underbrace{\left| \frac{z - z_0}{z_1 - z_0} \right|^n}_{<1} \\ &\leq M \underbrace{\left| \frac{z - z_0}{z_1 - z_0} \right|^n}_{<1} \end{aligned}$$

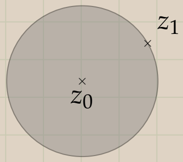
since $\left| \frac{z - z_0}{z_1 - z_0} \right| < 1$ then the geometric series

$$\sum_{n=1}^{\infty} M \left| \frac{z - z_0}{z_1 - z_0} \right|^n \text{ Converges}$$

Thus, the series $\sum_{n=1}^{\infty} |a_n (z - z_0)^n|$ also converges, that is $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ is absolutely convergent. $\square$

Corollary 1.0.2 : *Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series which converges at some $z = z_1 \in \mathbb{C}\setminus\{z_0\}$, then the power series in question converges absolutely (so converges), for every $z \in \mathbb{C}$ such that:*

$$|z-z_0| < |z-z_1|$$



Proof. $\sum_{n=1}^{\infty} a_n(z_1-z_0)^n$ converges implies that $a_n(z-z_0)^n \to 0$ as $n \to \infty$, which implies that the sequence $\{a_n(z_1-z_0)^n\}_{n\geq 0}$ is bounded. Lemma 1 (proposition 1) permits us to conclude the required result. □

Theorem 1.0.3 (Radius of Convergence) : *Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series then there exists a unique $R \in [0,\infty]$, called the radius of convergence with the following properties:*

1. *The power series converges absolutely for every $z \in \mathbb{C}$ with $|z-z_0| < R$.*

2. *The power series diverges for every $z \in \mathbb{C}$ with $|z-z_0| > R$, the disk $D(z_0,R) = \{z \in \mathbb{C} : \quad |z-z_0| < R\}$ is called the disk of convergence.*

Proof. Define the set $A \subset \mathbb{R}_{\geq 0}$ of nonegative real numbers for which the sequence $\{|a_n|\, r^n\}_{n\in\mathbb{N}_0}$ is bounded.

$$A := \left\{ r \geq 0 : \quad \sup_{n\in\mathbb{N}} |a_n|\, r^n < \infty \right\}$$

we have $A \neq \emptyset$ because $0 \in A$.

Define $R := \sup A \in [0,\infty]$, we now show that R has the stated properties.

1. Let $z \in D(z_0,R)$. By definition of the supremum, there exists $r \in A$, (i.e., $|a_n|\, r^n$ is bounded) such that $|z-z_0| < r \leq R$, since $|z-z_0| < r$ and $\{|a_n|\, r^n\}_{n\geq 0}$ is bounded. then by Abel's lemma, we deduce that the series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converges absolutely.

2. Let $z \in \mathbb{C}$ such that $|z-z_0| > R$, suppose for contradictions that the power series converges at z. Then by the corollary 2, it would converge absolutely for any ω with $|\omega - z_0| < |z-z_0|$, In particular, for any r such that:

$$R < r < |z-z_0|$$

The series would converge at points on the circle $C(z_0,r)$ implying $r \in A$. This contradicts the fact that $R = \sup A$, Therefore. the power series diverges.

The Uniqueness of R:

If another $R \in [0, \infty]$ satisfies the same properties, a point z such that $|z - z_0|$ lies between R and R' would lead to a contradiction regarding the converges or divergence of the power series. $\square$

1.1 Formulas for Calculating the Radius of Convergence

Proposition 1.1.1 (Hadamard's Formula) : *Let $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ be a power series centered at $z_0 \in \mathbb{C}$. Denote by R its radius of convergence. Then:*

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$$

with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$

Proof. Let $L := \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} \in [0, \infty]$, we must show that $R = \frac{1}{L}$. Let $z \in \mathbb{C} \setminus \{z_0\}$, we distinguish three cases:

1. if $(L = 0)$. In this case, we have:

$$0 \leq \lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} \leq \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$$

   Thus, $\lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$ This implies that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and equals to 0, so for all n sufficiently large, we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{z\,|z - z_0|}$$

   That is

$$|a_n(z - z_0)^n| < \frac{1}{z^n}$$

   for all n sufficiently large, since the geometric series $\sum_{n=1}^{\infty} \frac{1}{z^n}$ converges then the series $\sum_{n=1}^{\infty} |a_n(z - z_0)^n|$ converges $\forall z \in \mathbb{C}$, thus $R = +\infty = \frac{1}{L}$

2. $(L = +\infty)$, we have $L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = +\infty$ is equivallent to the fact the sequence $\left\{|a_n|^{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$ is bounded. Therefore, the sequence:

$$|a_n(z - z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}} |z - z_0|$$

   is also unbounded. This implies that $|a_n(z - z_0)^n|$ is unbounded, thus $|a_n(z - z_0)^n|$ does not converge to 0 as $n \to \infty$. Hence $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ diverges. Hence $R = 0$

3. $(L \in (0, \infty))$, Let $z \in \mathbb{C}$. We consider two subcases:

   (a) if $|z - z_0| < r < \frac{1}{L}$, choose r such that $|z - z_0| < r < \frac{1}{L}$, thus $L < \frac{1}{r}$. By defintion of a $\lim_{n\to\infty} \sup$, for all n sufficiently large we have:

   $$|a_n|^{\frac{1}{n}} < \frac{1}{r}$$

   which implies that:

   $$|a_n(z - z_0)^n| < \underbrace{\left(\frac{|z - z_0|}{r}\right)^n}_{<1}$$

   since $\left|\frac{z-z_0}{r}\right| < 1$, the geometric series $\sum_{n=1}^{\infty} \left|\frac{z-z_0}{r}\right|^n$ converges, by comparison, the power series $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ converges absolutely

   (b) if $(|z - z_0| > \frac{1}{L})$, in this case, we have:

   $$\begin{aligned}\lim_{n\to\infty} \sup |a_n(z - z_0)^n|^{\frac{1}{n}} &= \lim_{n\to\infty} \sup \left(|a_n|^{\frac{1}{n}} |z - z_0|\right) \\ &= L\,|z - z_0| > 1\end{aligned}$$

   Thus, $\{a_n(z - z_0)^n\}_{n\in\mathbb{N}}$ is unbounded, hence $|a_n(z - z_0)^n|$ does not converge to zero as $n \to \infty$, implying that $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ diverges therefore:

   $$R = \frac{1}{L}$$

□