Complex Analysis Lecture Notes

Hand written summary from lectures

Acknowledgment

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`http://farhi.bakir.free.fr/home/index-fr.html`

Disclaimer

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- Interpretations or notations that reflect my own understanding at the moment

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Contents

Chapter 1

Power Series

Lecture 1 08:06 AM Mon, Sep 29 2025

Definition 1.0.1 (Power Series) : *A power series is a formal series of the form* $\sum_{n=1}^{\infty} a_n z^n$, *where* $a_n \in \mathbb{C}$ *for all* $n \in \mathbb{N}_0$.

More generally, given $z_0 \in \mathbb{C}$, *a power series centered at* z_0 *is a formal series of the form:*

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C} \quad (\forall n \in \mathbb{N})$

Remark:

The set of all complex power series (centered at 0) is denoted by $\mathbb{C}[[z]]$. More generally, given $z_0 \in \mathbb{C}$, the set of all complex power series at z_0 is denoted by $\mathbb{C}[[z - z_0]]$

Operations on Formal Power Series:

Given $z_0 \in \mathbb{C}$, we equip $\mathbb{C}[[z - z_0]]$, with the following operations:

1. **Additions:** For all $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{C}$:

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^n = \sum_{n=1}^{\infty} (a_n + b_n)(z - z_0)^n$$

2. **Multiplictation:**

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n \times \sum_{n=1}^{\infty} b_n (z - z_0)^n = \sum_{n=1}^{\infty} c_n (z - z_0)^n$$

where, $c_n = \sum_{k=1}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N}_0$. Also $(c_n)_{n\in\mathbb{N}}$ is called the covolution of the two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$.

3. **Scalar Multiplication:** For all $\lambda \in \mathbb{C}$, and all $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$:

$$\lambda \sum_{n=1}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} (\lambda a_n)(z - z_0)^n$$

It's straight forward to verify that $\mathbb{C}[[z - z_0]]$ equipped with these operations forms a commutative algebra over $\mathbb{C}$. The Multiplicative identity is the constant power series:

$$1 = 1 + 0 \cdot (z - z_0) + 0 \cdot (z - z_0)^2 + \dots$$

Definition 1.0.2 (Domain of Convergence) : *The domain of convergence of a power series $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ is the set of all points $z \in \mathbb{C}$ for which the series converge the structure of this domain is very specific. Its a disk (possibly with some points in its boundary) centered at z_0.*

Proposition 1.0.1 (Abel's Lemma) : *Let $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ be a power series and let $z_1 \in \mathbb{C} \setminus \{z_0\}$. Suppose that the sequence $\{a_n (z_1 - z_0)^n\}_{n\in\mathbb{N}}$ is bounded. Then, the power series in question converges absolutely (so converges) for every $z \in \mathbb{C}$, such that:*

$$|z - z_0| < |z_1 - z_0|$$

Proof. By hypothesis, $\exists M > 0$ such that $\forall n \in \mathbb{N}_0$:

$$|a_n (z_1 - z_0)^n| \leq M$$

Then, for all $z \in \mathbb{C}$ such that $|z - z_0| < |z_1 - z_0|$ we have:

$$\begin{aligned} |a_n (z - z_0)^n| &= \underbrace{|a_n (z_1 - z_0)^n|}_{\leq M} \cdot \underbrace{\left| \frac{z - z_0}{z_1 - z_0} \right|^n}_{<1} \\ &\leq M \underbrace{\left| \frac{z - z_0}{z_1 - z_0} \right|^n}_{<1} \end{aligned}$$

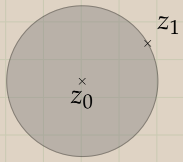
since $\left| \frac{z - z_0}{z_1 - z_0} \right| < 1$ then the geometric series

$$\sum_{n=1}^{\infty} M \left| \frac{z - z_0}{z_1 - z_0} \right|^n \text{ Converges}$$

Thus, the series $\sum_{n=1}^{\infty} |a_n (z - z_0)^n|$ also converges, that is $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ is absolutely convergent. □

Corollary 1.0.2 : *Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series which converges at some $z = z_1 \in \mathbb{C}\setminus\{z_0\}$, then the power series in question converges absolutely (so converges), for every $z \in \mathbb{C}$ such that:*

$$|z-z_0| < |z-z_1|$$



Proof. $\sum_{n=1}^{\infty} a_n(z_1-z_0)^n$ converges implies that $a_n(z-z_0)^n \to 0$ as $n \to \infty$, which implies that the sequence $\{a_n(z_1-z_0)^n\}_{n\geq 0}$ is bounded. Lemma 1 (proposition 1) permits us to conclude the required result. □

Theorem 1.0.3 (Radius of Convergence) : *Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series then there exists a unique $R \in [0,\infty]$, called the radius of convergence with the following properties:*

1. *The power series converges absolutely for every $z \in \mathbb{C}$ with $|z-z_0| < R$.*
2. *The power series diverges for every $z \in \mathbb{C}$ with $|z-z_0| > R$, the disk $D(z_0,R) = \{z \in \mathbb{C} : \ |z-z_0| < R\}$ is called the disk of convergence.*

Proof. Define the set $A \subset \mathbb{R}_{\geq 0}$ of nonegative real numbers for which the sequence $\{|a_n|\,r^n\}_{n\in\mathbb{N}_0}$ is bounded.

$$A := \left\{ r \geq 0 : \quad \sup_{n\in\mathbb{N}} |a_n|\,r^n < \infty \right\}$$

we have $A \neq \emptyset$ because $0 \in A$.

Define $R := \sup A \in [0,\infty]$, we now show that R has the stated properties.

1. Let $z \in D(z_0,R)$. By definition of the supremum, there exists $r \in A$, (i.e., $|a_n|\,r^n$ is bounded) such that $|z-z_0| < r \leq R$, since $|z-z_0| < r$ and $\{|a_n|\,r^n\}_{n\geq 0}$ is bounded. then by Abel's lemma, we deduce that the series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converges absolutely.
2. Let $z \in \mathbb{C}$ such that $|z-z_0| > R$, suppose for contradictions that the power series converges at z. Then by the corollary 2, it would converge absolutely for any ω with $|\omega - z_0| < |z-z_0|$, In particular, for any r such that:

$$R < r < |z-z_0|$$

The series would converge at points on the circle $C(z_0,r)$ implying $r \in A$. This contradicts the fact that $R = \sup A$, Therefore. the power series diverges.

The Uniqueness of R:

If another $R \in [0, \infty]$ satisfies the same properties, a point z such that $|z - z_0|$ lies between R and R' would lead to a contradiction regarding the converges or divergence of the power series. □

1.1 Formulas for Calculating the Radius of Convergence

Proposition 1.1.1 (Hadamard's Formula) : *Let $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ be a power series centered at $z_0 \in \mathbb{C}$. Denote by R its radius of convergence. Then:*

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$

Proof. Let $L := \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} \in [0, \infty]$, we must show that $R = \frac{1}{L}$. Let $z \in \mathbb{C} \setminus \{z_0\}$, we distinguish three cases:

1. if $(L = 0)$. In this case, we have:

$$0 \leq \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$$

   Thus, $\lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$ This implies that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and equals to 0, so for all n sufficiently large, we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{z\,|z - z_0|}$$

   That is

$$|a_n (z - z_0)^n| < \frac{1}{z^n}$$

   for all n sufficiently large, since the geometric series $\sum_{n=1}^{\infty} \frac{1}{z^n}$ converges then the series $\sum_{n=1}^{\infty} |a_n (z - z_0)^n|$ converges $\forall z \in \mathbb{C}$, thus $R = +\infty = \frac{1}{L}$

2. $(L = +\infty)$, we have $L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = +\infty$ is equivallent to the fact the sequence $\left\{ |a_n|^{\frac{1}{n}} \right\}_{n \in \mathbb{N}}$ is bounded. Therefore, the sequence:

$$|a_n (z - z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}} |z - z_0|$$

   is also unbounded. This implies that $|a_n (z - z_0)^n|$ is unbounded, thus $|a_n (z - z_0)^n|$ does not converge to 0 as $n \to \infty$. Hence $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ diverges. Hence $R = 0$

3. $(L \in (0, \infty))$, Let $z \in \mathbb{C}$. We consider two subcases:

   (a) if $|z - z_0| < r < \frac{1}{L}$, choose r such that $|z - z_0| < r < \frac{1}{L}$, thus $L < \frac{1}{r}$. By defintion of a $\lim_{n\to\infty} \sup$, for all n sufficiently large we have:

   $$|a_n|^{\frac{1}{n}} < \frac{1}{r}$$

   which implies that:

   $$|a_n(z - z_0)^n| < \underbrace{\left(\frac{|z - z_0|}{r}\right)^n}_{<1}$$

   since $\left|\frac{z-z_0}{r}\right| < 1$, the geometric series $\sum_{n=1}^{\infty} \left|\frac{z-z_0}{r}\right|^n$ converges, by comparison, the power series $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ converges absolutely

   (b) if $(|z - z_0| > \frac{1}{L})$, in this case, we have:

   $$\begin{aligned}\lim_{n\to\infty} \sup |a_n(z - z_0)^n|^{\frac{1}{n}} &= \lim_{n\to\infty} \sup \left(|a_n|^{\frac{1}{n}} |z - z_0|\right) \\ &= L\,|z - z_0| > 1\end{aligned}$$

   Thus, $\{a_n(z - z_0)^n\}_{n\in\mathbb{N}}$ is unbounded, hence $|a_n(z - z_0)^n|$ does not converge to zero as $n \to \infty$, implying that $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ diverges therefore:

   $$R = \frac{1}{L}$$

□

Lecture 2

08:00 AM Mon, Oct 06 2025

Proposition 1.1.2 (Ratio Test Formula) : *Let $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ be a power series. suppose that the limit*

$$\alpha = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists (i.e., $\in [0, \infty]$). Then the radius of convergence R of the power series in question is $R = \alpha$.

Proof. We use the d'Allembert rule for the series

$$\sum_{n=1}^{\infty} a_n(z - z_0)^n \qquad (z \in \mathbb{C} \setminus \{z_0\})$$

let $z \in \mathbb{C} \setminus \{z_0\}$. we have:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \cdot |z-z_0| \\ &= |z-z_0| \cdot \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \frac{|z-z_0|}{\alpha}\end{aligned}$$

By the d'Allembert rule, we have:

- The series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converges if

$$\frac{|z-z_0|}{\alpha} < 1 \quad \text{i.e.} \quad |z-z_0| < \alpha$$

- The series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ diverges if

$$\frac{|z-z_0|}{\alpha} > 1 \quad \text{i.e.} \quad |z-z_0| > \alpha$$

Hence $R = \alpha$.

$\square$

Example: *Determine the radius of convergence of the power series* $\sum_{n=1}^{\infty} \frac{z_n}{n!}$ *where* $z_0 = 0$.

1 st method: (By Hadamard formula)

We must compute $\lim_{n\to\infty}\sup\left(\frac{1}{n!}\right)^{\frac{1}{n}}$*, by the stirling formula, we have that:*

$$n! \overset{+\infty}{\sim} n^n e^{-n}\sqrt{2\pi n}$$

Thus we get:

$$(n!)^{\frac{1}{n}} \overset{+\infty}{\sim} ne^{-1}(2\pi n)^{\frac{1}{2n}}$$

Thus

$$\left(\frac{1}{n!}\right)^{\frac{1}{n}} \overset{+\infty}{\sim} \frac{e}{n}(2\pi n)^{-\frac{1}{2n}} \to 0 \text{ as } n \to \infty$$

Thus $R = \frac{1}{0} = +\infty$.

This means that the power series $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ *converges for all* $z \in \mathbb{C}$.

2 nd method:

We use proposition 2. we have:

$$\lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty}(n+1) = +\infty$$

Thus $R = +\infty$

1.2 Analytic Functions

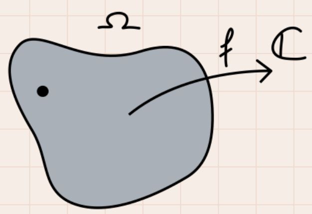
Definition 1.2.1 : *Let* Ω *be a non empty open subset of* $\mathbb{C}$ *and let* $z_0 \in \Omega$.

Let $f : \Omega \longrightarrow \mathbb{C}$ *be a map. then:*

1. f *is said to be analytic at* z_0 *if there exists* $r > 0$ *and a complex sequence* $(a_n)_{n\in\mathbb{N}_0}$ *such that* $D(z_0, r) \subset \Omega$ *and:*

$$f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n \quad * \,\forall z \in D(z_0, r)$$

2. f *is said to be analytic on* Ω *if its analytic at every point of* Ω



Example:

1. *Every complex polynomial is analytic on* $\mathbb{C}$. *Indeed, let* $P \in \mathbb{C}[\mathbb{Z}]$, *and* $z_0 \in \mathbb{C}$. *since* $P(z + z_0) \in \mathbb{C}[\mathbb{Z}]$, *we can write:*

$$P(z + z_0) = \sum_{n=0}^{d} a_n z^n \quad (d \in \mathbb{N}_0)$$

substituting z by $(z - z_0)$*. we get:*

$$P(z) = \sum_{n=0}^{d} a_n (z - z_0)^n$$

which is a power series centered at z_0 *with infinite randius of convergence. Thus,* P *is analytic at* z_0*, since* z_0 *was arbitrary* P *is analytic on* $\mathbb{C}$*.*

2. *The function* $z \longrightarrow \frac{1}{z}$ *is analytic on* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$*. Indeed, let* $z_0 \in \mathbb{C}^*$ *arbitrary. For* $z \in D(z_0, |z_0|)$*, we have:*

$$\left| \frac{z - z_0}{z_0} \right| < 1$$

we can write

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{z_0 + (z - z_0)} \\
&= \frac{1}{z_0} \cdot \frac{1}{1 + \frac{z - z_0}{z_0}} \\
&= \frac{1}{z_0} \cdot \sum_{n=1}^{\infty} (-1)^n \left(\frac{z - z_0}{z_0} \right)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z - z_0)^n
\end{aligned}$$

which is a power series centered at z_0*, valid on* $D(z_0, |z_0|)$*. Hence* $z \longrightarrow \frac{1}{z}$*, is analytic at* z_0*. since* $z_0 \in \mathbb{C}^*$ *was arbitrary, then* $z \longrightarrow \frac{1}{z}$ *is analytic on* $\mathbb{C}^*$

1.2.1 Properties of Analytic Functions

Proposition 1.2.1 : *Let* Ω *be a non empty open subset of* $\mathbb{C}$ *and let* $z_0 \in \Omega$ *if,* $f, g : \Omega \longrightarrow \mathbb{C}$ *are analytic at* z_0*, then the same for* $(f + g)$ *and* $(f \cdot g)$*. Moreover, if* f *and* g *are represented by power series with radii of convergence* R_f *and* R_g *respectively then* $(f + g)$ *and* $(f \cdot g)$ *represented by power series with radii of convergence* $\geq \min(R_f, R_g)$

Proof. Exercise □

Corollary 1.2.2 : *Let* Ω *be a non empty open subset of* $\mathbb{C}$ *and let* $f, g : \Omega \longrightarrow \mathbb{C}$*. If* f *and* g *are both analytic on* Ω*, then the same is for* $(f + g)$ *and* $(f \cdot g)$*.*

Proposition 1.2.3 (Analyticity $\implies$ Continuity) : *Let Ω be a non empty open subset of $\mathbb{C}$ and let $z_0 \in \Omega$, Let also $f : \Omega \longrightarrow \mathbb{C}$ be a map.*
if f is analytic at z_0 then f is continuous at z_0

Proof. Suppose that f is analytic at z_0 then there exists $R > 0$ and a complex sequence $(a_n)_{n\in\mathbb{N}_0}$ such that $D(z_0, R) \subset \Omega$ and:

$$f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n \quad (\forall z \in D(z_0, R))$$

in particular, $f(z_0) = a_0$. Thus for all $z \in D(z_0, R)$ we have:

$$\begin{aligned} f(z) - f(z_0) &= \sum_{n=1}^{\infty} a_n(z - z_0)^n \\ &= (z - z_0) \sum_{n=1}^{\infty} a_n(z - z_0)^{n-1} \\ &= (z - z_0) \underbrace{\sum_{n=1}^{\infty} a_{n+1}(z - z_0)^n}_{(1)} \end{aligned}$$

By the Hadamard formula, we see that the power series $\sum_{n=1}^{\infty} a_{n+!}(z - z_0)^n$ has the same radius of convergence as the original power series $\sum_{n=1}^{\infty} a_n(z - z_0)^n$

$$\left(R' = \frac{1}{\lim_{n\to\infty} \sup |a_{n+1}|^{\frac{1}{n}}} = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n-1}}} \right)$$

Consequently, the power series $\sum_{n=1}^{\infty} a_{n+1}(z - z_0)^n$ converges absolutely for $|z - z_0| < R$, let $r \in \mathbb{R}$ such that $0 < r < R$. then for all $z \in D(z_0, r)$, we have from (1) the estimate:

$$\begin{aligned} |f(z) - f(z_0)| &= |z - z_0| \cdot \left| \sum_{n=1}^{\infty} a_{n+1}(z - z_0)^n \right| \\ &\le |z - z_0| \sum_{n=1}^{\infty} |a_{n+1}|\, |z - z_0|^n \\ &\le |z - z_0| \underbrace{\sum_{n=1}^{\infty} |a_{n+1}| \cdot r^n}_{(\text{since } r<R)} \end{aligned}$$

Taking the limit as $z \to z_0$, we conclude that $\lim_{z\to z_0} f(z) = f(z_0)$, so f is continuous at z_0. □

Corollary 1.2.4 (Immediate) : *Let Ω be a non empty open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$. If f is analytic on Ω, then f is continuous on Ω.*

Proposition 1.2.5 (Composition of Analytic :unctions) : *Let Ω_1 and Ω_2 be two non empty open subsets of $\mathbb{C}$ and let $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \mathbb{C}$ be two maps. Let also $z_0 \in \Omega_1$. If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)$ is analytic at z_0.*

Proof. Exercise □

Corollary 1.2.6 (Immediate) : *Let Ω_1 and Ω_2 be two non empty open subsets of $\mathbb{C}$ and let $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \mathbb{C}$ be two maps. If f is analytic on Ω_1 and g is analytic on Ω_2 then $(g \circ f)$ is analytic on Ω_1.*

Proposition 1.2.7 (Quotient of Analytic Functions) : *Let Ω be a non empty open subsets of $\mathbb{C}$ and let $z_0 \in \Omega$. let also $f, g : \Omega \longrightarrow \mathbb{C}$ be two functions which are both analytic at z_0 and such that $g(z_0) \neq 0$. Then the function $\frac{f}{g}$ is analytic at z_0*

Proof. since $g(z_0) \neq 0$ then the function $h : w \longrightarrow \frac{1}{w}$ is analytic at $g(z_0)$ (as seen in previous examples), therefore. by proposition 5, the function $\frac{1}{g} = h \circ g$ is anayltic at z_0.
It then follows from proposition 1 that the product $f \cdot \left(\frac{1}{g}\right)$ is analytic at z_0 □

Corollary 1.2.8 (Immediate) : *Let Ω be a non empty open subset of $\mathbb{C}$ and let $f, g : \Omega \longrightarrow \mathbb{C}$ be two analytic functions on Ω such that $g(z) \neq 0$. for every $z \in \Omega$. Then the function $\frac{f}{g}$ is analytic on Ω .*

Example: *Every rational function is analytic on its domain of definition. this is because a rational function is a quotient of two polynomials, and polynomials are analytic on $\mathbb{C}$*

1.3 Power series define Analytic functions

Theorem 1.3.1 : *A power series with a positive radius of converges defines an analytic function on its disk of convergence*

Proof. Let $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ be a power series ($z_0 \in \mathbb{C}$, $(a_n)_{n \in \mathbb{N}}) \subset \mathbb{C}$) with radius of convergence

$R > 0$. define the function f on the disk $D(z_0, R)$ by:

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

we must show that f is analytic on $D(z_0, R)$. arbitrary we will show that f is analytic at z_1 for $z \in D(z_1, R - |z_1 - z_0|)$, we have

$$|z - z_0| \overset{I.I}{\leq} \underbrace{|z - z_1|}_{<R-|z_1-z_0|} + |z_1 - z_0|$$

Thus $D(z_1, R - |z_1 - z_0|) \subset D(z_0, R)$, so the power series $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ converges absolutely. so:

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} a_n (z - z_0)^n \\
&= \sum_{n=1}^{\infty} a_n ((z - z_1) + (z_1 - z_0))^n \\
&= \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \\
&= \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} a_k \binom{n}{k} (z_1 - z_0)^{n-k} \right) (z - z_1)^k
\end{aligned}$$

The interchange of summation is justified by the absolute convergence of the double series for $z \in D(z_1, R - |z_1 - z_0|)$.

This express $f(z)$ as a power series in $(z - z_1)$ in the disk $D(z_1, R - |z_1 - z_0|)$ proving that f is analytic at z_1. since z_1 was arbitrary in $D(z_0, R)$ then f is analytic on $D(z_0, R)$. □