Contents

The Uniqueness of R:

If another $R \in [0, \infty]$ satisfies the same properties, a point z such that $|z - z_0|$ lies between R and R' would lead to a contradiction regarding the converges or divergence of the power series. □

1.1 Formulas for Calculating the Radius of Convergence

Proposition 1.1.1 (Hadamard's Formula) : *Let $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ be a power series centered at $z_0 \in \mathbb{C}$. Denote by R its radius of convergence. Then:*

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$$

with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$

Proof. Let $L := \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} \in [0, \infty]$, we must show that $R = \frac{1}{L}$. Let $z \in \mathbb{C} \setminus \{z_0\}$, we distinguish three cases:

1. if $(L = 0)$. In this case, we have:

$$0 \leq \lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} \leq \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$$

   Thus, $\lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$ This implies that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and equals to 0, so for all n sufficiently large, we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{z\,|z - z_0|}$$

   That is

$$|a_n(z - z_0)^n| < \frac{1}{z^n}$$

   for all n sufficiently large, since the geometric series $\sum_{n=1}^{\infty} \frac{1}{z^n}$ converges then the series $\sum_{n=1}^{\infty} |a_n(z - z_0)^n|$ converges $\forall z \in \mathbb{C}$, thus $R = +\infty = \frac{1}{L}$

2. $(L = +\infty)$, we have $L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = +\infty$ is equivallent to the fact the sequence $\left\{|a_n|^{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$ is bounded. Therefore, the sequence:

$$|a_n(z - z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}\,|z - z_0|$$

   is also unbounded. This implies that $|a_n(z - z_0)^n|$ is unbounded, thus $|a_n(z - z_0)^n|$ does not converge to 0 as $n \to \infty$. Hence $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ diverges. Hence $R = 0$

3. $(L \in (0, \infty))$, Let $z \in \mathbb{C}$. We consider two subcases:

   [i]if $|z - z_0| < r < \frac{1}{L}$, choose r such that $|z - z_0| < r < \frac{1}{L}$, thus $L < \frac{1}{r}$. By defintion of a $\lim_{n\to\infty} \sup$, for all n sufficiently large we have:

   $$|a_n|^{\frac{1}{n}} < \frac{1}{r}$$

   which implies that:

   $$|a_n(z - z_0)^n| < \underbrace{\left(\frac{|z - z_0|}{r}\right)^n}_{<1}$$

   since $\left|\frac{z-z_0}{r}\right| < 1$, the geometric series $\sum_{n=1}^{\infty} \left|\frac{z-z_0}{r}\right|^n$ converges, by comparison, the power series $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ converges absolutely if $(|z - z_0| > \frac{1}{L})$, in this case, we have:

   $$\begin{aligned}\lim_{n\to\infty} \sup |a_n(z - z_0)^n|^{\frac{1}{n}} &= \lim_{n\to\infty} \sup \left(|a_n|^{\frac{1}{n}} |z - z_0|\right) \\ &= L\,|z - z_0| > 1\end{aligned}$$

   Thus, $\{a_n(z - z_0)^n\}_{n\in\mathbb{N}}$ is unbounded, hence $|a_n(z - z_0)^n|$ does not converge to zero as $n \to \infty$, implying that $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ diverges therefore:

   $$R = \frac{1}{L}$$

□