Complex Analysis Lecture Notes

Hand written summary from lectures

Acknowledgment

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`http://farhi.bakir.free.fr/home/index-fr.html`

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- Incomplete or incorrect information.
- Typos, transcription mistakes, or missing content.
- Interpretations or notations that reflect my own understanding. at the moment

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Chapter 1

Power Series

Lecture 1 `08:06 AM Mon, Sep 29 2025`

Definition 1.0.1 (Power Series) : *A power series is a formal series of the form $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$.*
More generally, given $z_0 \in \mathbb{C}$, a power series centered at z_0 is a formal series of the form:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n \in \mathbb{C} \quad (\forall n \in \mathbb{N}_0)$

Remark ✎

The set of all complex power series (centered at 0) is denoted by $\mathbb{C}[[z]]$. More generally, given $z_0 \in \mathbb{C}$, the set of all complex power series centered at z_0 is denoted by $\mathbb{C}[[z - z_0]]$.

☞ Operations on Formal Power Series:

Given $z_0 \in \mathbb{C}$, we equip $\mathbb{C}[[z - z_0]]$. with the following operations:

① **Additions:** For all $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(z - z_0)^n.$$

② **Multiplication**

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \times \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

where, $c_n := \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N}_0$. Also $(c_n)_{n\in\mathbb{N}}$ is called the covolution of the two sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$.

③ **Scalar Multiplication:** For all $\lambda \in \mathbb{C}$, and all $(a_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$:

$$\lambda \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} (\lambda a_n)(z-z_0)^n.$$

It's straightforward to verify that $\mathbb{C}\,[[z-z_0]]$ equipped with these operations forms a commutative algebra over $\mathbb{C}$. The Multiplicative identity is the constant power series:

$$1 = 1 + 0 \cdot (z-z_0) + 0 \cdot (z-z_0)^2 + \dots$$

Definition 1.0.2 (Domain of Convergence) : *The domain of convergence of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is the set of all points $z \in \mathbb{C}$ for which the series converge. The structure of this domain is very specific. Its a disk (possibly with some points in its boundary) centered at z_0.*

Proposition 1.0.1 (Abel's Lemma) : *Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series and let $z_1 \in \mathbb{C}\setminus\{z_0\}$. Suppose that the sequence $\{a_n(z_1-z_0)^n\}_{n\in\mathbb{N}_0}$ is bounded. Then, the power series in question converges absolutely (so converges) for every $z \in \mathbb{C}$, such that:*

$$|z-z_0| < |z_1-z_0|$$

Proof. By hypothesis, $\exists M > 0$ such that $\forall n \in \mathbb{N}_0$:

$$|a_n(z_1-z_0)^n| \leq M$$

Then, for all $z \in \mathbb{C}$ such that $|z-z_0| < |z_1-z_0|$ we have:

$$\begin{aligned} |a_n(z-z_0)^n| &= \underbrace{|a_n(z_1-z_0)^n|}_{\leq M} \cdot \underbrace{\left|\frac{z-z_0}{z_1-z_0}\right|^n}_{<1} \\ &\leq M \underbrace{\left|\frac{z-z_0}{z_1-z_0}\right|^n}_{<1}. \end{aligned}$$

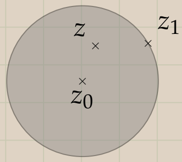
Since $\left|\frac{z-z_0}{z_1-z_0}\right| < 1$ then the geometric series

$$\sum_{n=0}^{\infty} M \left|\frac{z-z_0}{z_1-z_0}\right|^n \text{ Converges .}$$

Thus, the series $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ also converges, that is $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent. ☐

Corollary 1.0.2 : *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series which converges at some $z = z_1 \in \mathbb{C}\setminus\{z_0\}$. Then the power series in question converges absolutely (so converges), for every $z \in \mathbb{C}$ such that:*

$$|z-z_0| < |z-z_1|$$



Proof. $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges implies that $a_n(z-z_0)^n \to 0$ as $n \to +\infty$, which implies that the sequence $\{a_n(z_1-z_0)^n\}_{n\geq 0}$ is bounded. *Proposition 1.0.1* permits us to conclude the required result. □

Theorem 1.0.3 (Radius of Convergence) : *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Then there exists a unique $R \in [0,\infty]$, called the radius of convergence with the following properties:*

① *The power series converges absolutely for every $z \in \mathbb{C}$ satisfying $|z-z_0| < R$.*

② *The power series diverges for every $z \in \mathbb{C}$ satisfying $|z-z_0| > R$. The disk $D(z_0,R) = \{z \in \mathbb{C} : \quad |z-z_0| < R\}$ is called the disk of convergence.*

Proof. Define the set $A \subset \mathbb{R}_{\geq 0}$ of nonegative real numbers for which the sequence $\{|a_n|\, r^n\}_{n\in\mathbb{N}_0}$ is bounded.

$$A := \left\{ r \geq 0 : \quad \sup_{n\in\mathbb{N}_0} |a_n|\, r^n < \infty \right\}$$

we have $A \neq \emptyset$ because $0 \in A$. Define $R := \sup A \in [0,\infty]$, we now show that R has the stated properties.

➻① Let $z \in D(z_0,R)$. By definition of the supremum, there exists $r \in A$, (i.e., $|a_n|\, r^n$ is bounded) such that $|z-z_0| < r \leq R$. Since $|z-z_0| < r$ and $\{|a_n|\, r^n\}_{n\geq 0}$ is bounded, then by Abel's lemma, we deduce that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely.

➻② Let $z \in \mathbb{C}$ such that $|z-z_0| > R$, suppose for contradictions that the power series converges at z. Then by the *Corollary 1.0.2*, it would converge absolutely for any ω with $|\omega - z_0| < |z-z_0|$. In particular, for any r such that:

$$R < r < |z-z_0|$$

the series would converge at points on the circle $C(z_0,r)$, implying $r \in A$. This contradicts the fact that $R = \sup A$. Therefore, the power series diverges.

➼ The Uniqueness of R:

If another $R' \in [0, \infty]$ satisfies the same properties, a point z such that $|z - z_0|$ lies between R and R' would lead to a contradiction regarding the convergence or divergence of the power series. □

1.1 Formulas for Calculating the Radius of Convergence

Proposition 1.1.1 (Hadamard's Formula) : *Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series centered at $z_0 \in \mathbb{C}$. Denote by R its radius of convergence. Then:*

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$

Proof. Let $L := \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} \in [0, \infty]$. We must show that $R = \frac{1}{L}$. Let $z \in \mathbb{C} \setminus \{z_0\}$, we distinguish three cases:

➼① If $L = 0$. In this case, we have:

$$0 \le \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \le \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$$

Thus, $\lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$. This implies that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and equals to 0, so for all n sufficiently large, we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{2\,|z - z_0|};$$

That is,

$$|a_n(z - z_0)^n| < \frac{1}{2^n}.$$

Since the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges then the series $\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$ converges $\forall z \in \mathbb{C}$, thus $R = +\infty = \frac{1}{L}$

➼ ② If $L = +\infty$, we have $L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = +\infty$ is equivallent to the fact that the sequence $\left\{ |a_n|^{\frac{1}{n}} \right\}_{n\in\mathbb{N}}$ is bounded. Therefore, the sequence:

$$|a_n(z - z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}} \, |z - z_0|$$

is also unbounded. This implies that $|a_n(z - z_0)^n|$ is unbounded, thus $|a_n(z - z_0)^n|$ does not converge to 0 as $n \to \infty$. Hence $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges. Hence $R = 0$.

➽ ③ If$L \in (0, \infty)$. Let $z \in \mathbb{C}$. We consider two subcases:

❶ If $|z - z_0| < \frac{1}{L}$. Choose r such that $|z - z_0| < r < \frac{1}{L}$, thus $L < \frac{1}{r}$. By defintion of a $\lim_{n\to\infty} \sup$, for all n sufficiently large we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{r},$$

which implies that:

$$|a_n(z - z_0)^n| < \underbrace{\left(\frac{|z - z_0|}{r}\right)^n}_{<1}.$$

Since $\left|\frac{z-z_0}{r}\right| < 1$, the geometric series $\sum_{n=0}^{\infty} \left|\frac{z-z_0}{r}\right|^n$ converges. By comparison, the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely.

❷ If ($|z - z_0| > \frac{1}{L}$). In this case, we have:

$$\begin{aligned} \lim_{n\to\infty} \sup |a_n(z - z_0)^n|^{\frac{1}{n}} &= \lim_{n\to\infty} \sup \left(|a_n|^{\frac{1}{n}} |z - z_0|\right) \\ &= L\,|z - z_0| > 1 \end{aligned}$$

Thus, $\{a_n(z - z_0)^n\}_{n\in\mathbb{N}}$ is unbounded, hence $|a_n(z - z_0)^n|$ does not converge to zero as $n \to \infty$, implying that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges. Therefore:

$$R = \frac{1}{L}.$$

□

Lecture 2 08:00 AM Mon, Oct 06 2025

Proposition 1.1.2 (Ratio Test Formula) : *Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Suppose that the limit*

$$\alpha = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists (i.e., $\in [0, \infty]$). Then the radius of convergence R of the power series in question is $R = \alpha$.

Proof. We use the d'Allembert rule for the series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad (z \in \mathbb{C} \setminus \{z_0\}).$$

Let $z \in \mathbb{C}\setminus\{z_0\}$. we have:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot|z-z_0| \\ &= |z-z_0|\cdot\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \frac{|z-z_0|}{\alpha}\end{aligned}$$

By the d'Allembert rule, we have:

- The series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges if

$$\frac{|z-z_0|}{\alpha} < 1 \quad \text{i.e.} \quad |z-z_0| < \alpha.$$

- The series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges if

$$\frac{|z-z_0|}{\alpha} > 1 \quad \text{i.e.} \quad |z-z_0| > \alpha.$$

Hence $R = \alpha$.

$\square$

Example: *Determine the radius of convergence of the power series* $\sum_{n=0}^{\infty} \frac{z_n}{n!}$ *where* $z_0 = 0$.

1^{st} METHOD: (BY HADAMARD FORMULA)

We must compute $\lim_{n\to\infty}\sup\left(\frac{1}{n!}\right)^{\frac{1}{n}}$. *By the stirling formula, we have that:*

$$n! \sim_{+\infty} n^n e^{-n}\sqrt{2\pi n}.$$

Thus we get:

$$(n!)^{\frac{1}{n}} \sim_{+\infty} ne^{-1}(2\pi n)^{\frac{1}{2n}}.$$

Thus

$$\left(\frac{1}{n!}\right)^{\frac{1}{n}} \sim_{+\infty} \frac{e}{n}(2\pi n)^{-\frac{1}{2n}} \to 0 \text{ as } n \to +\infty .$$

Thus $R = \frac{1}{0} = +\infty$.

This means that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ *converges for all* $z \in \mathbb{C}$.

2 nd METHOD:

We use Proposition 2 . we have:

$$\lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!}$$
$$= \lim_{n\to\infty}(n+1) = +\infty.$$

Thus $R = +\infty$

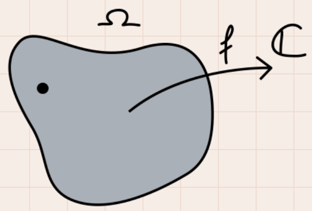
1.2 Analytic Functions

Definition 1.2.1 : *Let Ω be a non empty open subset of $\mathbb{C}$ and let $z_0 \in \Omega$.*
Let $f : \Omega \longrightarrow \mathbb{C}$ be a map. then:

1. *f is said to be analytic at z_0 if there exists $r > 0$ and a complex sequence $(a_n)_{n\in\mathbb{N}_0}$ such that $D(z_0, r) \subset \Omega$ and:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad (\forall z \in D(z_0, r)).$$

2. *f is said to be analytic on Ω if its analytic at every point of Ω.*



Example:

1. *Every complex polynomial is analytic on $\mathbb{C}$. Indeed, let $P \in \mathbb{C}[\mathbb{Z}]$, and $z_0 \in \mathbb{C}$. since*

$P(z+z_0) \in \mathbb{C}[\mathbb{Z}]$, we can write:

$$P(z+z_0) = \sum_{n=0}^{d} a_n z^n \quad (d \in \mathbb{N}_0).$$

Substituting z by $(z-z_0)$, we get:

$$P(z) = \sum_{n=0}^{d} a_n (z-z_0)^n,$$

which is a power series centered at z_0 with infinite randius of convergence. Thus, P is analytic at z_0. Since z_0 was arbitrary, P is analytic on $\mathbb{C}$.

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2. *The function $z \longrightarrow \frac{1}{z}$ is analytic on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Indeed, let $z_0 \in \mathbb{C}^*$ arbitrary.*

   For $z \in D(z_0, |z_0|)$, we have:

$$\left|\frac{z-z_0}{z_0}\right| < 1.$$

   We can write

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{z_0 + (z-z_0)} \\
&= \frac{1}{z_0} \cdot \frac{1}{1+\frac{z-z_0}{z_0}} \\
&= \frac{1}{z_0} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-z_0}{z_0}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z-z_0)^n,
\end{aligned}$$

   which is a power series centered at z_0, valid on $D(z_0, |z_0|)$. Hence $z \longrightarrow \frac{1}{z}$ is analytic at z_0. Since $z_0 \in \mathbb{C}^*$ was arbitrary, then $z \longrightarrow \frac{1}{z}$ is analytic on $\mathbb{C}^*$.

1.2.1 Properties of Analytic Functions

Proposition 1.2.1 : *Let Ω be a non empty open subset of $\mathbb{C}$ and let $z_0 \in \Omega$. If $f, g : \Omega \longrightarrow \mathbb{C}$ are analytic at z_0, then the same is for $(f+g)$ and $(f \cdot g)$. Moreover, if f and g are represented by power series with radii of convergence R_f and R_g respectively then $(f+g)$ and $(f \cdot g)$ are represented by power series with radii of convergence $\geq \min(R_f, R_g)$.*

Proof. Exercise. $\square$

Corollary 1.2.2 : *Let Ω be a non empty open subset of $\mathbb{C}$ and let $f, g : \Omega \longrightarrow \mathbb{C}$. If f and g are both analytic on Ω, then the same is for $(f+g)$ and $(f \cdot g)$.*

Proposition 1.2.3 (Analyticity $\Longrightarrow$ Continuity) : *Let Ω be a non empty open subset of $\mathbb{C}$ and let $z_0 \in \Omega$. Let also $f : \Omega \longrightarrow \mathbb{C}$ be a map. If f is analytic at z_0 then f is continuous at z_0*

Proof. Suppose that f is analytic at z_0 then there exists $R > 0$ and a complex sequence $(a_n)_{n\in\mathbb{N}_0}$ such that $D(z_0, R) \subset \Omega$ and:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\forall z \in D(z_0, R))$$

In particular, $f(z_0) = a_0$. Thus for all $z \in D(z_0, R)$ we have:

$$\begin{aligned} f(z) - f(z_0) &= \sum_{n=1}^{\infty} a_n (z - z_0)^n \\ &= (z - z_0) \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \\ &= (z - z_0) \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \qquad (1) \end{aligned}$$

By the Hadamard formula, we see that the power series $\sum_{n=0}^{\infty} a_{n+1}(z-z_0)^n$ has the same radius of convergence as the original power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. Consequently, the power series $\sum_{n=0}^{\infty} a_{n+1}(z - z_0)^n$ converges absolutely for $|z - z_0| < R$. Let $r \in \mathbb{R}$ such that $0 < r < R$. Then for all $z \in D(z_0, r)$, we have from (1) the estimate:

$$\begin{aligned} |f(z) - f(z_0)| &= |z - z_0| \cdot \left| \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \right| \\ &\le |z - z_0| \sum_{n=0}^{\infty} |a_{n+1}| \, |z - z_0|^n \\ &\le |z - z_0| \underbrace{\sum_{n=0}^{\infty} |a_{n+1}| \cdot r^n}_{<+\infty \text{ since } r<R}. \end{aligned}$$

Taking the limit as $z \to z_0$, we conclude that $\lim_{z\to z_0} f(z) = f(z_0)$, so f is continuous at z_0. $\square$

Corollary 1.2.4 (Immediate) : *Let Ω be a non empty open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$. If f is analytic on Ω, then f is continuous on Ω.*

Proposition 1.2.5 (Composition of Analytic functions) : *Let Ω_1 and Ω_2 be two nonempty open subsets of $\mathbb{C}$ and let $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \mathbb{C}$ be two maps. Let also $z_0 \in \Omega_1$. If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)$ is analytic at z_0.*

Proof. Exercise □

Corollary 1.2.6 (Immediate) : *Let Ω_1 and Ω_2 be two nonempty open subsets of $\mathbb{C}$ and let $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \mathbb{C}$ be two maps. If f is analytic on Ω_1 and g is analytic on Ω_2 then $(g \circ f)$ is analytic on Ω_1.*

Proposition 1.2.7 (Quotient of Analytic Functions) : *Let Ω be a nonempty open subsets of $\mathbb{C}$ and let $z_0 \in \Omega$. Let also $f, g : \Omega \longrightarrow \mathbb{C}$ be two functions which are both analytic at z_0 and such that $g(z_0) \neq 0$. Then the function $\frac{f}{g}$ is analytic at z_0.*

Proof. Since $g(z_0) \neq 0$ then the function $h : w \longrightarrow \frac{1}{w}$ is analytic at $g(z_0)$ (as seen in previous examples). Therefore, by *Proposition* 1.2.5, the function $\frac{1}{g} = h \circ g$ is anayltic at z_0.
It then follows from *Proposition* 1.2.1 that the product $f \cdot \left(\frac{1}{g}\right)$ is analytic at z_0. □

Corollary 1.2.8 (Immediate) : *Let Ω be a non empty open subset of $\mathbb{C}$ and let $f, g : \Omega \longrightarrow \mathbb{C}$ be two analytic functions on Ω such that $g(z) \neq 0$ for every $z \in \Omega$. Then the function $\frac{f}{g}$ is analytic on Ω .*

Example: *Every rational function is analytic on its domain of definition. This is because a rational function is a quotient of two polynomials, and polynomials are analytic on $\mathbb{C}$.*

- -

1.3 Power series define Analytic functions

Theorem 1.3.1 : *A power series with a positive radius of converges defines an analytic function on its disk of convergence.*

Proof. Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series ($z_0 \in \mathbb{C}$, $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$) with radius of convergence

$R > 0$. Define the function f on the disk $D(z_0, R)$ by:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

We must show that f is analytic on $D(z_0, R)$. Let $z_1 \in D(z_0, R)$ arbitrary. We will show that f is analytic at z_1. For $z \in D(z_1, R - |z_1 - z_0|)$, we have

$$|z - z_0| \overset{T.I}{\leq} \underbrace{|z - z_1|}_{<R-|z_1-z_0|} + |z_1 - z_0| < R$$

Thus $D(z_1, R - |z_1 - z_0|) \subset D(z_0, R)$, so the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely. so:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ &= \sum_{n=0}^{\infty} a_n ((z - z_1) + (z_1 - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_k \binom{n}{k} (z_1 - z_0)^{n-k} \right) (z - z_1)^k \end{aligned}$$

The interchange of summation is justified by the absolute convergence of the double series for $z \in D(z_1, R - |z_1 - z_0|)$. This express $f(z)$ as a power series in $(z - z_1)$ in the disk $D(z_1, R - |z_1 - z_0|)$, proving that f is analytic at z_1. Since z_1 was arbitrary in $D(z_0, R)$, then f is analytic on $D(z_0, R)$. □