

Complex Analysis Lecture Notes

Hand written summary from lectures

Acknowledgment

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<http://farhi.bakir.free.fr/home/index-fr.html>

Disclaimer

These notes were written in real-time during the lectures, this is not the final version, yet. so they may contain :

- Incomplete or incorrect information.
- Typos, transcription mistakes, or missing content.
- Interpretations or notations that reflect my own understanding. at the moment

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kara.abderahmane@nhsm.edu.dz

Notes on Contribution :

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CONTRIBUTORS

Main Writer

KARA ABDERAHMANE

Main Drawer

Haddar Noureddine

ARMWRESTLING4EVER

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Chapter 1

Power Series

Lecture 1

08:06 AM Mon, Sep 29 2025

Definition 1.0.1 (Power Series) : A power series is a formal series of the form $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$.

More generally, given $z_0 \in \mathbb{C}$, a power series centered at z_0 is a formal series of the form:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n \in \mathbb{C} \quad (\forall n \in \mathbb{N}_0)$

Remark

The set of all complex power series (centered at 0) is denoted by $\mathbb{C}[[z]]$. More generally, given $z_0 \in \mathbb{C}$, the set of all complex power series centered at z_0 is denoted by $\mathbb{C}[[z - z_0]]$.

Operations on Formal Power Series:

Given $z_0 \in \mathbb{C}$, we equip $\mathbb{C}[[z - z_0]]$ with the following operations:

① **Additions:** For all $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} (a_n + b_n) (z - z_0)^n.$$

② **Multiplication**

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \times \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

where, $c_n := \sum_{k=0}^n a_k b_{n-k}$ for all $n \in \mathbb{N}_0$. Also $(c_n)_{n \in \mathbb{N}}$ is called the convolution of the two sequences $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$.

③ **Scalar Multiplication:** For all $\lambda \in \mathbb{C}$, and all $(a_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$:

$$\lambda \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} (\lambda a_n) (z - z_0)^n.$$

It's straightforward to verify that $\mathbb{C}[[z - z_0]]$ equipped with these operations forms a commutative algebra over \mathbb{C} . The Multiplicative identity is the constant power series:

$$1 = 1 + 0 \cdot (z - z_0) + 0 \cdot (z - z_0)^2 + \dots$$

Definition 1.0.2 (Domain of Convergence): The domain of convergence of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is the set of all points $z \in \mathbb{C}$ for which the series converge. The structure of this domain is very specific. It's a disk (possibly with some points in its boundary) centered at z_0 .

Proposition 1.0.1 (Abel's Lemma): Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series and let $z_1 \in \mathbb{C} \setminus \{z_0\}$. Suppose that the sequence $\{a_n (z_1 - z_0)^n\}_{n \in \mathbb{N}_0}$ is bounded. Then, the power series in question converges absolutely (so converges) for every $z \in \mathbb{C}$, such that:

$$|z - z_0| < |z_1 - z_0|$$

Proof. By hypothesis, $\exists M > 0$ such that $\forall n \in \mathbb{N}_0$:

$$|a_n (z_1 - z_0)^n| \leq M$$

Then, for all $z \in \mathbb{C}$ such that $|z - z_0| < |z_1 - z_0|$ we have:

$$\begin{aligned} |a_n (z - z_0)^n| &= \underbrace{|a_n (z_1 - z_0)^n|}_{\leq M} \cdot \underbrace{\left| \frac{z - z_0}{z_1 - z_0} \right|^n}_{< 1} \\ &\leq M \underbrace{\left| \frac{z - z_0}{z_1 - z_0} \right|^n}_{< 1}. \end{aligned}$$

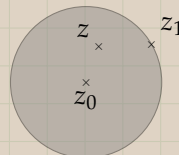
Since $\left| \frac{z - z_0}{z_1 - z_0} \right| < 1$ then the geometric series

$$\sum_{n=0}^{\infty} M \left| \frac{z - z_0}{z_1 - z_0} \right|^n \text{ Converges.}$$

Thus, the series $\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$ also converges, that is $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is absolutely convergent. \square

Corollary 1.0.2 : Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series which converges at some $z = z_1 \in \mathbb{C} \setminus \{z_0\}$. Then the power series in question converges absolutely (so converges), for every $z \in \mathbb{C}$ such that:

$$|z - z_0| < |z - z_1|$$



Proof. $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges implies that $a_n(z - z_0)^n \rightarrow 0$ as $n \rightarrow +\infty$, which implies that the sequence $\{a_n(z_1 - z_0)^n\}_{n \geq 0}$ is bounded. *Proposition 1.0.1* permits us to conclude the required result. \square

Theorem 1.0.3 (Radius of Convergence) : Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Then there exists a unique $R \in [0, \infty]$, called the radius of convergence with the following properties:

- ① The power series converges absolutely for every $z \in \mathbb{C}$ satisfying $|z - z_0| < R$.
- ② The power series diverges for every $z \in \mathbb{C}$ satisfying $|z - z_0| > R$. The disk $D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$ is called the disk of convergence.

Proof. Define the set $A \subset \mathbb{R}_{\geq 0}$ of nonnegative real numbers for which the sequence $\{|a_n| r^n\}_{n \in \mathbb{N}_0}$ is bounded.

$$A := \left\{ r \geq 0 : \sup_{n \in \mathbb{N}_0} |a_n| r^n < \infty \right\}$$

we have $A \neq \emptyset$ because $0 \in A$. Define $R := \sup A \in [0, \infty]$, we now show that R has the stated properties.

•① Let $z \in D(z_0, R)$. By definition of the supremum, there exists $r \in A$, (i.e., $|a_n| r^n$ is bounded) such that $|z - z_0| < r \leq R$. Since $|z - z_0| < r$ and $\{|a_n| r^n\}_{n \geq 0}$ is bounded, then by Abel's lemma, we deduce that the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely.

•② Let $z \in \mathbb{C}$ such that $|z - z_0| > R$, suppose for contradictions that the power series converges at z . Then by the *Corollary 1.0.2*, it would converge absolutely for any ω with $|\omega - z_0| < |z - z_0|$. In particular, for any r such that:

$$R < r < |z - z_0|$$

the series would converge at points on the circle $C(z_0, r)$, implying $r \in A$. This contradicts the fact that $R = \sup A$. Therefore, the power series diverges.

❖ The Uniqueness of R :

If another $R' \in [0, \infty]$ satisfies the same properties, a point z such that $|z - z_0|$ lies between R and R' would lead to a contradiction regarding the convergence or divergence of the power series. \square

1.1 Formulas for Calculating the Radius of Convergence

Proposition 1.1.1 (Hadamard's Formula) : Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series centered at $z_0 \in \mathbb{C}$. Denote by R its radius of convergence. Then:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$

Proof. Let $L := \lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}} \in [0, \infty]$. We must show that $R = \frac{1}{L}$. Let $z \in \mathbb{C} \setminus \{z_0\}$, we distinguish three cases:

❖① If $L = 0$. In this case, we have:

$$0 \leq \liminf_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 0$$

Thus, $\lim_{n \rightarrow \infty} \inf |a_n|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}} = 0$. This implies that $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$ exists and equals to 0, so for all n sufficiently large, we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{2|z - z_0|};$$

That is,

$$|a_n(z - z_0)^n|^{\frac{1}{n}} < \frac{1}{2^n}.$$

Since the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges then the series $\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$ converges $\forall z \in \mathbb{C}$, thus $R = +\infty = \frac{1}{L}$

❖② If $L = +\infty$, we have $L = \lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}} = +\infty$ is equivalent to the fact that the sequence $\left\{ |a_n|^{\frac{1}{n}} \right\}_{n \in \mathbb{N}}$ is bounded. Therefore, the sequence:

$$|a_n(z - z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}} |z - z_0|$$

is also unbounded. This implies that $|a_n(z - z_0)^n|$ is unbounded, thus $|a_n(z - z_0)^n|$ does not converge to 0 as $n \rightarrow \infty$. Hence $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges. Hence $R = 0$.

❖ ③ If $L \in (0, \infty)$. Let $z \in \mathbb{C}$. We consider two subcases:

❶ If $|z - z_0| < \frac{1}{L}$. Choose r such that $|z - z_0| < r < \frac{1}{L}$, thus $L < \frac{1}{r}$. By definition of a $\lim_{n \rightarrow \infty} \sup$, for all n sufficiently large we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{r},$$

which implies that:

$$|a_n(z - z_0)^n| < \underbrace{\left(\frac{|z - z_0|}{r}\right)^n}_{<1}.$$

Since $\left|\frac{z - z_0}{r}\right| < 1$, the geometric series $\sum_{n=0}^{\infty} \left|\frac{z - z_0}{r}\right|^n$ converges. By comparison, the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely.

❷ If $(|z - z_0| > \frac{1}{L})$. In this case, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup |a_n(z - z_0)^n|^{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \sup \left(|a_n|^{\frac{1}{n}} |z - z_0| \right) \\ &= L |z - z_0| > 1 \end{aligned}$$

Thus, $\{a_n(z - z_0)^n\}_{n \in \mathbb{N}}$ is unbounded, hence $|a_n(z - z_0)^n|$ does not converge to zero as $n \rightarrow \infty$, implying that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges. Therefore:

$$R = \frac{1}{L}.$$

□

Lecture 2

08:00 AM Mon, Oct 06 2025

Proposition 1.1.2 (Ratio Test Formula) : Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Suppose that the limit

$$\alpha = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists (i.e., $\in [0, \infty]$). Then the radius of convergence R of the power series in question is $R = \alpha$.

Proof. We use the d'Allembert rule for the series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (z \in \mathbb{C} \setminus \{z_0\}).$$

Let $z \in \mathbb{C} \setminus \{z_0\}$. we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(z - z_0)^{n+1}}{a_n(z - z_0)^n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |z - z_0| \\ &= |z - z_0| \cdot \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \frac{|z - z_0|}{\alpha} \end{aligned}$$

By the d'Allembert rule, we have:

∞ The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges if

$$\frac{|z - z_0|}{\alpha} < 1 \quad \text{i.e.} \quad |z - z_0| < \alpha.$$

∞ The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ diverges if

$$\frac{|z - z_0|}{\alpha} > 1 \quad \text{i.e.} \quad |z - z_0| > \alpha.$$

Hence $R = \alpha$.

□

Example: Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ where $z_0 = 0$.

1st METHOD: (BY HADAMARD FORMULA)

We must compute $\lim_{n \rightarrow \infty} \sup \left(\frac{1}{n!} \right)^{\frac{1}{n}}$. By the stirling formula, we have that:

$$n! \sim_{+\infty} n^n e^{-n} \sqrt{2\pi n}.$$

Thus we get:

$$(n!)^{\frac{1}{n}} \sim_{+\infty} n e^{-1} (2\pi n)^{\frac{1}{2n}}.$$

Thus

$$\left(\frac{1}{n!} \right)^{\frac{1}{n}} \sim_{+\infty} \frac{e}{n} (2\pi n)^{-\frac{1}{2n}} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Thus $R = \frac{1}{0} = +\infty$.

This means that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$.

2nd METHOD:

We use Proposition 2 . we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \\ &= \lim_{n \rightarrow \infty} (n+1) = +\infty. \end{aligned}$$

Thus $R = +\infty$

1.2 Analytic Functions

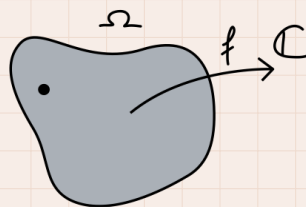
Definition 1.2.1 : Let Ω be a non empty open subset of \mathbb{C} and let $z_0 \in \Omega$.

Let $f : \Omega \rightarrow \mathbb{C}$ be a map. then:

1. f is said to be analytic at z_0 if there exists $r > 0$ and a complex sequence $(a_n)_{n \in \mathbb{N}_0}$ such that $D(z_0, r) \subset \Omega$ and:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\forall z \in D(z_0, r)).$$

2. f is said to be analytic on Ω if its analytic at every point of Ω .



Example:

1. Every complex polynomial is analytic on \mathbb{C} . Indeed, let $P \in \mathbb{C}[Z]$, and $z_0 \in \mathbb{C}$. since

$P(z + z_0) \in \mathbb{C}[\mathbb{Z}]$, we can write:

$$P(z + z_0) = \sum_{n=0}^d a_n z^n \quad (d \in \mathbb{N}_0).$$

Substituting z by $(z - z_0)$, we get:

$$P(z) = \sum_{n=0}^d a_n (z - z_0)^n,$$

which is a power series centered at z_0 with infinite radius of convergence. Thus, P is analytic at z_0 . Since z_0 was arbitrary, P is analytic on \mathbb{C} .

2. The function $z \mapsto \frac{1}{z}$ is analytic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Indeed, let $z_0 \in \mathbb{C}^*$ arbitrary.

For $z \in D(z_0, |z_0|)$, we have:

$$\left| \frac{z - z_0}{z_0} \right| < 1.$$

We can write

$$\begin{aligned} \frac{1}{z} &= \frac{1}{z_0 + (z - z_0)} \\ &= \frac{1}{z_0} \cdot \frac{1}{1 + \frac{z - z_0}{z_0}} \\ &= \frac{1}{z_0} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - z_0}{z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z - z_0)^n, \end{aligned}$$

which is a power series centered at z_0 , valid on $D(z_0, |z_0|)$. Hence $z \mapsto \frac{1}{z}$ is analytic at z_0 .

Since $z_0 \in \mathbb{C}^*$ was arbitrary, then $z \mapsto \frac{1}{z}$ is analytic on \mathbb{C}^* .

1.2.1 Properties of Analytic Functions

Proposition 1.2.1 : Let Ω be a non empty open subset of \mathbb{C} and let $z_0 \in \Omega$. If $f, g : \Omega \rightarrow \mathbb{C}$ are analytic at z_0 , then the same is for $(f + g)$ and $(f \cdot g)$. Moreover, if f and g are represented by power series with radii of convergence R_f and R_g respectively then $(f + g)$ and $(f \cdot g)$ are represented by power series with radii of convergence $\geq \min(R_f, R_g)$.

Proof. Exercise. □

Corollary 1.2.2 : Let Ω be a non empty open subset of \mathbb{C} and let $f, g : \Omega \rightarrow \mathbb{C}$. If f and g are both analytic on Ω , then the same is for $(f + g)$ and $(f \cdot g)$.

Proposition 1.2.3 (Analyticity \implies Continuity) : Let Ω be a non empty open subset of \mathbb{C} and let $z_0 \in \Omega$. Let also $f : \Omega \rightarrow \mathbb{C}$ be a map. If f is analytic at z_0 then f is continuous at z_0

Proof. Suppose that f is analytic at z_0 then there exists $R > 0$ and a complex sequence $(a_n)_{n \in \mathbb{N}_0}$ such that $D(z_0, R) \subset \Omega$ and:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\forall z \in D(z_0, R))$$

In particular, $f(z_0) = a_0$. Thus for all $z \in D(z_0, R)$ we have:

$$\begin{aligned} f(z) - f(z_0) &= \sum_{n=1}^{\infty} a_n (z - z_0)^n \\ &= (z - z_0) \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \\ &= (z - z_0) \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \quad (1) \end{aligned}$$

By the Hadamard formula, we see that the power series $\sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n$ has the same radius of convergence as the original power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. Consequently, the power series $\sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n$ converges absolutely for $|z - z_0| < R$. Let $r \in \mathbb{R}$ such that $0 < r < R$. Then for all $z \in D(z_0, r)$, we have from (1) the estimate:

$$\begin{aligned} |f(z) - f(z_0)| &= |z - z_0| \cdot \left| \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \right| \\ &\leq |z - z_0| \sum_{n=0}^{\infty} |a_{n+1}| |z - z_0|^n \\ &\leq |z - z_0| \underbrace{\sum_{n=0}^{\infty} |a_{n+1}| \cdot r^n}_{< +\infty \text{ since } r < R} \end{aligned}$$

Taking the limit as $z \rightarrow z_0$, we conclude that $\lim_{z \rightarrow z_0} f(z) = f(z_0)$, so f is continuous at z_0 . □

Corollary 1.2.4 (Immediate) : Let Ω be a non empty open subset of \mathbb{C} and $f : \Omega \rightarrow \mathbb{C}$. If f is analytic on Ω , then f is continuous on Ω .

Proposition 1.2.5 (Composition of Analytic functions) : Let Ω_1 and Ω_2 be two nonempty open subsets of \mathbb{C} and let $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_2 \rightarrow \mathbb{C}$ be two maps. Let also $z_0 \in \Omega_1$. If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)$ is analytic at z_0 .

Proof. Exercise □

Corollary 1.2.6 (Immediate) : Let Ω_1 and Ω_2 be two nonempty open subsets of \mathbb{C} and let $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_2 \rightarrow \mathbb{C}$ be two maps. If f is analytic on Ω_1 and g is analytic on Ω_2 then $(g \circ f)$ is analytic on Ω_1 .

Proposition 1.2.7 (Quotient of Analytic Functions) : Let Ω be a nonempty open subsets of \mathbb{C} and let $z_0 \in \Omega$. Let also $f, g : \Omega \rightarrow \mathbb{C}$ be two functions which are both analytic at z_0 and such that $g(z_0) \neq 0$. Then the function $\frac{f}{g}$ is analytic at z_0 .

Proof. Since $g(z_0) \neq 0$ then the function $h : w \rightarrow \frac{1}{w}$ is analytic at $g(z_0)$ (as seen in previous examples). Therefore, by Proposition 1.2.5, the function $\frac{1}{g} = h \circ g$ is analytic at z_0 .

It then follows from Proposition 1.2.1 that the product $f \cdot \left(\frac{1}{g}\right)$ is analytic at z_0 . □

Corollary 1.2.8 (Immediate) : Let Ω be a non empty open subset of \mathbb{C} and let $f, g : \Omega \rightarrow \mathbb{C}$ be two analytic functions on Ω such that $g(z) \neq 0$ for every $z \in \Omega$. Then the function $\frac{f}{g}$ is analytic on Ω .

Example: Every rational function is analytic on its domain of definition. This is because a rational function is a quotient of two polynomials, and polynomials are analytic on \mathbb{C} .

1.3 Power series define Analytic functions

Theorem 1.3.1 : A power series with a positive radius of converges defines an analytic function on its disk of convergence.

Proof. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series ($z_0 \in \mathbb{C}, (a_n)_{n \in \mathbb{N}} \subset \mathbb{C}$) with radius of convergence $R > 0$. Define the function f on the disk $D(z_0, R)$ by:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

We must show that f is analytic on $D(z_0, R)$. Let $z_1 \in D(z_0, R)$ arbitrary. We will show that f is analytic at z_1 . For $z \in D(z_1, R - |z_1 - z_0|)$, we have

$$|z - z_0| \stackrel{T.I.}{\leq} \underbrace{|z - z_1|}_{< R - |z_1 - z_0|} + |z_1 - z_0| < R$$

Thus $D(z_1, R - |z_1 - z_0|) \subset D(z_0, R)$, so the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely. so:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n(z - z_0)^n \\ &= \sum_{n=0}^{\infty} a_n((z - z_1) + (z_1 - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^n \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0)^{n-k} \right) (z - z_1)^k \end{aligned}$$

The interchange of summation is justified by the absolute convergence of the double series for $z \in D(z_1, R - |z_1 - z_0|)$. This express $f(z)$ as a power series in $(z - z_1)$ in the disk $D(z_1, R - |z_1 - z_0|)$, proving that f is analytic at z_1 . Since z_1 was arbitrary in $D(z_0, R)$, then f is analytic on $D(z_0, R)$. \square

Lecture 3

08:14 AM Mon, Oct 13 2025

Example: The power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $R = +\infty$. Therefore (by the previous Theorem), it defines an analytic function on the whole complex plane \mathbb{C} .

Definition 1.3.1 : The analytic function on \mathbb{C} defined by:

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is called the exponential function.

Definition 1.3.2 (Entire function) : A complex function $f : \mathbb{C} \rightarrow \mathbb{C}$ which is analytic on the whole complex plane \mathbb{C} is called an entire function.

Example:

- ① Every complex polynomial is an entire function.
- ② The exponential function $\exp(z)$ is an entire function.

1.3.1 Properties of the exponential function

Proposition 1.3.2 : The exponential function defines the following properties:

- ① $\forall z_1, z_2 \in \mathbb{C}$, we have:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} \text{ and } e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}.$$

- ② for all $z \in \mathbb{C}$, we have $e^z \neq 0$.

- ③ (EULER'S FORMULA): $\forall \theta \in \mathbb{R}$, we have:

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

- ④ $\forall z \in \mathbb{C}$, we have:

$$e^z = 1 \iff z \in 2\pi i\mathbb{Z}.$$

More generally, for all $z, z' \in \mathbb{C}$, we have:

$$e^z = e^{z'} \iff z - z' \in 2\pi i\mathbb{Z}.$$

So, the exponential function is periodic with period $2\pi i$.

Proof.

❖ ① $\forall z_1, z_2 \in \mathbb{C}$, we have

$$\begin{aligned}
 e^{z_1} \cdot e^{z_2} &= \sum_{k=0}^{+\infty} \frac{z_1^k}{k!} \cdot \sum_{\ell=0}^{+\infty} \frac{z_2^\ell}{\ell!} \\
 &= \sum_{k, \ell \in \mathbb{N}_0} \frac{z_1^k z_2^\ell}{k! \ell!} \\
 &= \sum_{n=0}^{+\infty} \left(\sum_{k, \ell \in \mathbb{N}_0, k+\ell=n} \frac{z_1^k z_2^\ell}{k! \ell!} \right) \\
 &= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n \frac{z_1^k z_2^{n-k}}{k! (n-k)!} \right) \\
 &= \sum_{n=0}^{+\infty} \frac{1}{n!} \underbrace{\left(\sum_{k=0}^n \frac{n!}{k! (n-k)!} z_1^k z_2^{n-k} \right)}_{= \sum_{k=0}^n \binom{n}{k} z_1^k z_2^{n-k} = (z_1 + z_2)^n} \\
 &= \sum_{n=0}^{+\infty} \frac{1}{n!} (z_1 + z_2)^n = e^{z_1 + z_2},
 \end{aligned}$$

next, we have:

$$e^{z_1 - z_2} \cdot e^{z_2} \stackrel{\text{by the first formula}}{=} e^{z_1 - z_2 + z_2} = e^{z_1}.$$

Hence $e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}$, as required.

❖ ② For all $z \in \mathbb{C}$, we have:

$$e^z \cdot e^{-z} \stackrel{(1)}{=} e^{z-z} = e^0 = 1.$$

Thus $e^z \neq 0$.

❖ ③ (EULER'S FORMULA).

For all $\theta \in \mathbb{R}$, we have:

$$\begin{aligned}
 e^{i\theta} &= \sum_{n=0}^{+\infty} \frac{(i\theta)^n}{n!} \\
 &= \sum_{n=0}^{+\infty} i^n \frac{\theta^n}{n!} \\
 &= \sum_{n \in \mathbb{N}_0, n \text{ is even}} i^n \frac{\theta^n}{n!} + \sum_{n \in \mathbb{N}_0, n \text{ is odd}} i^n \frac{\theta^n}{n!} \\
 &= \sum_{k=0}^{+\infty} i^{2k} \frac{\theta^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} i^{2k+1} \frac{\theta^{2k+1}}{(2k+1)!} \\
 &= \underbrace{\sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}}_{\cos \theta} + i \underbrace{\sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}}_{\sin \theta} \\
 &= \cos \theta + i \sin \theta,
 \end{aligned}$$

as required.

◆ ④ Let $z \in \mathbb{C}$ and write

$$z = x + iy \quad (x, y \in \mathbb{R}).$$

we have

$$\begin{aligned} e^z &= e^{x+iy} \\ &\stackrel{(1)}{=} e^x \cdot e^{iy} \\ &\stackrel{(3)}{=} e^x (\cos y + i \sin y) \\ &= e^x \cos y + i e^x \sin y. \end{aligned}$$

Thus

$$\begin{aligned} e^z = 1 &\iff \begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0 \end{cases} \iff \begin{cases} \cos y = e^{-x} > 0 \\ \sin y = 0 \end{cases} \\ &\iff \begin{cases} \exists k \in \mathbb{Z} : y = 2\pi k \\ e^{-x} = \cos 2\pi k = 1 \end{cases} \iff \begin{cases} \exists k \in \mathbb{Z} : y = 2\pi k \\ x = 0 \end{cases} \\ &\iff z = 2\pi ki \quad (k \in \mathbb{Z}) \\ &\iff z \in 2\pi\mathbb{Z}i, \end{aligned}$$

as required. □

1.3.2 Trigonometric and hyperbolic functions

Definition 1.3.3 (Complex Trigonometric functions) : We define the trigonometric functions cosine and sine by:

$$\begin{aligned} \cos z &:= \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \\ \sin z &:= \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (\forall z \in \mathbb{C}). \end{aligned}$$

Clearly, these functions extend the real functions \cos and \sin . The power series defining \cos and \sin have infinite radius of convergence, thus (By a previous theorem) \cos and \sin are analytic on \mathbb{C} ; that is, \cos and \sin are entire functions.

Remark

We easily verify the extended Euler's formula:

$$e^{iz} = \cos z + i \sin z \quad (\forall z \in \mathbb{C}).$$

From this formula, we derive:

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2}, \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \quad (\forall z \in \mathbb{C}). \end{aligned}$$

Exercise

Using property ④ of Proposition 1.3.2 and Euler's formula, show the following properties:

- ① The functions \cos and \sin are both 2π -periodic.
- ② The set of zeros of $z \mapsto \cos z$ is $(\frac{\pi}{2} + \pi\mathbb{Z})$, while the set of zeros of $z \mapsto \sin z$ is $\pi\mathbb{Z}$.
- ③ For all $z \in \mathbb{C}$, we have

$$\cos^2 z + \sin^2 z = 1.$$

FOR EXAMPLE, FOR ③: By the Euler formula, we have for all $z \in \mathbb{C}$:

$$\begin{aligned} \cos^2 z + \sin^2 z &= \left(\frac{e^{iz} + e^{-iz}}{2} \right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i} \right)^2 \\ &= \frac{4}{4} = 1 \end{aligned}$$

Definition 1.3.4 (Complex hyperbolic functions): We define the hyperbolic functions \cosh and \sinh by:

$$\begin{aligned} \cosh z &:= \sum_{n=0}^{+\infty} \frac{z^{2n}}{(2n)!} = \frac{e^z + e^{-z}}{2} = \cos(iz), \\ \sinh z &:= \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{(2n+1)!} = \frac{e^z - e^{-z}}{2} = -i \sin(iz) \quad (\forall z \in \mathbb{C}). \end{aligned}$$

Clearly, these definitions extend the real functions \cosh and \sinh . Like the trigonometric functions \cos and \sin , the hyperbolic functions \cosh and \sinh are also entire functions.

These functions are not bounded in \mathbb{C} , when you replace $x \leftarrow ix$, you get $\cos ix = \cosh x$.

Exercise

Using the expressions of \cosh and \sinh in terms of \cos and \sin , verify the following properties:

- ① The functions \cosh and \sinh are both 2π -periodic.
- ② The set of zeros of \cosh is $(\frac{\pi}{2}i + \pi i\mathbb{Z})$, while the set of zeros of \sinh is $\pi i\mathbb{Z}$.
- ③ For all $z \in \mathbb{C}$, we have

$$\cosh^2 z - \sinh^2 z = 1.$$

Definition 1.3.5 (Further trigonometric and hyperbolic functions) : We define the following functions:

$$\begin{aligned} \tan z &:= \frac{\sin z}{\cos z} & \left(\forall z \in \mathbb{C} \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z} \right) \right), \\ \cot z &:= \frac{\cos z}{\sin z} & (\forall z \in \mathbb{C} \setminus \pi\mathbb{Z}), \\ \tanh z &:= \frac{\sinh z}{\cosh z} & \left(\forall z \in \mathbb{C} \setminus \left(\frac{\pi}{2}i + \pi i\mathbb{Z} \right) \right), \\ \coth z &:= \frac{\cosh z}{\sinh z} & (\forall z \in \mathbb{C} \setminus \pi i\mathbb{Z}). \end{aligned}$$

This clearly extends the well-known real functions \tan , \cot , \tanh , and \coth . Note that each of these four functions is analytic in its domain of definition (according to the previous results on analytic functions).

1.4 Holomorph functions

Definition 1.4.1 : Let Ω be a nonempty open subset of \mathbb{C} and z_0 be a point in Ω . Let also $f : \Omega \rightarrow \mathbb{C}$ be a map.

- We say that f is holomorphic at z_0 if the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and belong to \mathbb{C} . In this case, the limit is called the derivative of f at the point z_0 and denoted by $f'(z_0)$.

- We say that f is holomorphic on Ω if it is holomorphic at every point in Ω .

In this case, the function

$$\begin{aligned} f' : \Omega &\longrightarrow \mathbb{C} \\ z &\longmapsto f'(z) \end{aligned}$$

is called the derivative of f .

Proposition 1.4.1 (Holomorphy of power series) : Let $z_0 \in \mathbb{C}$, $(a_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$, and S be the power series

$$S(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n.$$

Suppose that S has a positive radius of convergence R . Then S is holomorphic on $D(z_0, R)$ and we have for all $z \in D(z_0, R)$:

$$\begin{aligned} S'(z) &= \sum_{n=0}^{+\infty} n a_n (z - z_0)^{n-1} \\ &= \sum_{n=0}^{+\infty} (n+1) a_{n+1} (z - z_0)^n. \end{aligned}$$

Proof. For simplicity, suppose without loss of generality that $z_0 = 0$. First, remark that by using the Hadamard formula, the power series

$$\sum_{n=1}^{+\infty} n a_n z^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} z^n$$

has the same radius of convergence R as S . It follows that $\sum_{n=1}^{+\infty} n a_n z^{n-1}$ is absolutely convergent on $D(0, R)$; That is, for all $0 < r < R$, the series $\sum_{n=1}^{+\infty} n |a_n| z^{n-1}$ converges. Now, let $z_1 \in D(0, R)$ be arbitrary and show that S is holomorphic at z_1 . Choose $r \in \mathbb{R}$ such that $|z_1| < r < R$. For all $z \in D(0, r) \setminus \{z_1\}$, we have

$$\begin{aligned} \frac{S(z) - S(z_1)}{z - z_1} &= \frac{\sum_{n=0}^{+\infty} a_n z^n - \sum_{n=0}^{+\infty} a_n z_1^n}{z - z_1} \\ &= \sum_{n=0}^{+\infty} a_n \frac{z^n - z_1^n}{z - z_1} \\ &= \sum_{n=1}^{+\infty} a_n \sum_{k=0}^{n-1} z^k z_1^{n-1-k}. \end{aligned}$$

□