Contents

Chapter 1

Overview of Hilbert and Banach spaces

Lecture 1: Overview `08:22 AM Tue, Sep 23 2025`

Let E be a given set, and let $\mathbb{K} = \mathbb{C}, \mathbb{K} = \mathbb{R}$ be the know fields. let us define two operations on $E \times E$ and $\mathbb{K} \times E$ by the :

$$\begin{array}{rccc} + : & E \times E & \longrightarrow & E \\ & (x,y) & \longmapsto & x+y \end{array}$$

$$\begin{array}{rccc} \cdot : & \mathbb{K} \times E & \longrightarrow & E \\ & (\lambda,x) & \longmapsto & \lambda \cdot x \end{array}$$

compact $\Longleftrightarrow$ closed and bounded (finite dimensional)

compact $\Longrightarrow$ closed bounded (generally)

compactness is important because most important theorems use compact by assumption

more open sets = less compacts

Definition 1.0.1 (Vector space) : *A vector space E over $\mathbb{K}$ is a set equipped by two operations, called additions and scalar multiplication which verify the following axioms*

1. $\forall x, y \in E : \quad x + y = y + x$
2. $\forall x, y, z \in E : \quad (x+y)+z = x+(y+z)$
3. $\exists 0_E \in E : \quad x + 0_E = x$
4. $\forall x \in E, \exists y \in E : \quad x + y = 0_E$ *(denote y = -x)*
5. $\forall \lambda, \beta \in \mathbb{K}, \forall x \in E : \quad (\alpha\beta)x = \alpha(\beta x)$

6. $\forall x \in E: \quad 1_{\mathbb{K}} \cdot x = x$

7. $\forall x, y \in E, \forall \lambda \in \mathbb{K}: \quad \lambda(x+y) = \lambda x + \lambda y$

8. $\forall \alpha, \beta \in \mathbb{K}, \forall x \in E: \quad (\alpha+\beta)x = \alpha x + \beta x$

Definition 1.0.2 (Vector subspace) : *A subset $F \subset E$ is called a subvector space if:*

1. $0_E \subset F$

2. $\forall \lambda \in \mathbb{K}, \forall x, y \in E \implies \lambda x + y \in E$

Definition 1.0.3 (Inner product) : *An inner product on a vector space E is a function:*

$$\begin{aligned} \langle \cdot \rangle : \quad E \times E &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto \langle x, y \rangle \end{aligned}$$

which verifies:

1. *Linearity in the first argument :*

$$\begin{aligned} \forall x, y, z \in E, \forall \alpha \in \mathbb{K}: \quad \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \\ \langle x+y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle \end{aligned}$$

2. *Conjugate symmetry:*

$$\forall x, y \in E: \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

3. *Positive difinitness:*

$$\begin{aligned} \langle x, x \rangle &\geq 0 \\ \langle x, x \rangle = 0 &\iff x = 0_E \end{aligned}$$

Lecture 2

`08:07 AM Tue, Sep 30 2025`

Definition 1.0.4 : *Let E be a set, and d be a metric on E, we said that (E, d) is complete if and only if every cauchy sequence is convergent*

Definition 1.0.5 : *A Banach space is a complete vectorial normed space*

Example:

- $\mathbb{K}(\mathbb{R}, \mathbb{C})$ *is a Banach space*
- $\mathbb{K}^n$ *is a Banach space*
- *if E is a normed space with finite dimension, then E is Banach*
- $\ell^p(\mathbb{N}, \mathbb{K})$ *with* $1 \leq p \leq +\infty$ *is Banach space (Exercise).*
- *Every finite set product of Banach spaces is a Banach space too.*

Proposition 1.0.1 : *The limit of a sequence in a Banach space (if it exists) is unique.*

Proposition 1.0.2 : *Let F be a closed subset of E. if $(x_n)_{n\in\mathbb{N}} \subset F$ is convergent to $x \in E$, then $x \in F$.*

REMARK:

The importance of the notion of equivallent norms on a normed vector space is:
Let $\|.\|_1$ and $\|.\|_2$ be two equivallent norms on $V = E$. one has:

$$(x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_1 \iff (x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_2$$

Proposition 1.0.3 (Definition) : *Let V and W be two norms on V (resp. W), and let $f : V \longrightarrow W$ be a function. the following statments are equivallent:*

1. *f is continuous from V into W*
2. *and:*
$$\forall x_0 \in V, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in V : \quad \|x - x_0\| \leq \delta \implies \|f(x) - f(x_0)\| \leq \varepsilon$$
3. *and:*
$$\forall x \in V, \forall (x_n)_{n\in\mathbb{N}} \in V : \quad x_n \to x \implies f(x_n) \to f(x)(\|f(x_n - f(x))\| \to 0 \quad n \to \infty)$$

REMARK:

On finite dimension vector space, all the norm are equivallents.

1.1 Duality

Let V and W be two normed vector be two normed vector spaces

Proposition 1.1.1 : *Let $A : V \longrightarrow W$ be a linear function (Map). The following are equivallent:*

1. *A is continuous on V*
2. *A is continuous at 0*
3. $\exists c > 0$: $\|A(v)\| \leq c\|v\| \quad \forall v \in V$

Definition 1.1.1 : *We denote by $L(V, W)$ be the set of all linear functions, we denote $\mathcal{L}(V, W)$ the set of all continuous linear functions*

Proposition 1.1.2 : *$L(V, W)$ and $\mathcal{L}(V, W)$ are vector spaces*

Proposition 1.1.3 : *Let V be a normed space, and W be a Banach space then $\mathcal{L}(V, W)$ is Banach space.*

Definition 1.1.2 : *Let $(x_n)_{n\in\mathbb{N}}$ be sequence in a normed vector space E.*

- *we say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums $s_n = \sum_{k=1}^{n} x_k$ converges.*
- *we say that the series is normally (absolutely) convergent if the series of real positive numbers $\sum_{n=1}^{\infty} \|x_n\|$ converges*

Proposition 1.1.4 : *Let E be a normed vector space. The following are equivallents:*

1. *E is Banach.*
2. *Every normally convergent series is convergent.*

Definition 1.1.3 : *Let E and F be two normed vector spaces, A function (mapping) from E into F is called linear isomorphism if its linear, continuous bijective and its inverse is continuous (naturally linear)*

We denote by $\mathrm{ISO}(E,F)$ the set of all linear isomporphisms mapping from E into F.

Theorem 1.1.5 : *Let E and F be Banach spaces, then $ISO(E,F)$ is an open subset of $\mathcal{L}(E,F)$ and the map $u \longrightarrow u^{-1}$ from $ISO(E,F)$ into $ISO(F,E)$ is continuous. More, one has if $u_0 \in ISO(E,F)$ and $u \in B(u_0, \frac{1}{\|u_0^{-1}\|})$, then $u \in ISO(E,F)$, and:*

$$u^{-1} = \sum_{n=1}^{\infty} (id_E - u_0^{-1}u)^n \cdot u_0^{-1}$$

Corollary 1.1.6 (Von Neumann) : *Let E be a Banach space, and $u \in \mathcal{L}(E) = \mathcal{L}(E,E)$. if $\|u\| < 1$, then $id_E - u$ belongs to $ISO(E,E) = ISO(E)$ and it's inverse is given by:*

$$(Id_E - u)^{-1} = \sum_{n=1}^{\infty} u^n$$

NOTATIONS: $E' = \mathcal{L}(E)$, and $E^* = L(E)$.

1. E' toplogical dual space of E.
2. E^* algebraic dual space of E.

Proposition 1.1.7 : *Let E be a Banach space E_0 be normed vector space. let $p : E \longrightarrow E_0$ a continuous linear surjection mapping. If there exists $c > 0$ such that $\forall y_0 \in E_0, \exists x \in E$ such that $f(x) = y_0$ and $\|x\| \leq c\|y_0\|$ then E_0 is Banach.*

Theorem 1.1.8 : *Let E be a Banach space and let F be a closed subspace of E. Then $E_{|F}$ is Banach too.*

Proposition 1.1.9 : *Let $A \subset E$. A is compact if from every sequence of elements of A, we can extract a subsequence converging to an element in A.*

Theorem 1.1.10 (Riesz) : *The closed unit ball is compact if and only if $\dim E < +\infty$,*

Chapter 2

Compactness

Lecture 3 08:12 AM Tue, Oct 07 2025

Definition 2.0.1 (Bolzano-Weirstrass Property) : *A set $A \subset V$, is compact if from every sequence of A we can extract a subsequence which is convergent in A.*

Theorem 2.0.1 (Riesz) : *The closed unit ball of V is compact if and only if dimension of V is finite.*

2.1 Equicontinuity

Definition 2.1.1 : *Let (X, d_1) and (Y, d_2) be two metric spaces.*
Consider $F \subset \{f : X \longrightarrow Y\}$, we say that F is equicontinuous at $x_0 \in X$. if:

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad d_1(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon, \quad \forall f \in F$$

Definition 2.1.2 : *If F is equicontinuous, at each point of X, then we say that F is equicontinuous on X.*

Theorem 2.1.1 (Ascoli-Arzela Theorem) : *Let (X, d) be a compact metric space. A family F of real valued continuous functions on X is relatively compact in $\mathcal{C}(X, \mathbb{R})$ with the sup norm if and only if:*

1. *F is uniformally bounded*
2. *F is equicontinuous on X*

Theorem 2.1.2 (Bernestein-Weirstrass) : *Let $[a, b]$ and consider $f : [a, b] \longrightarrow \mathbb{R}$ continuous function, then there exists a sequence of polynomial functions $(f_n)_{n\in\mathbb{N}}$ such that f_n converges uniformally to f on $[a, b]$.*

Theorem 2.1.3 (Riesz representation) : *Let H be Hilbert space, and consider:*

$$H' = \{f : H \longrightarrow \mathbb{C} : \quad f \text{ continuous } \}$$

then:

$$\forall f \in H', \quad \exists! y \in H : \quad f(x) = \langle x, y \rangle$$

Theorem 2.1.4 : *Let $(E, \|.\|)$ be a normed vector space, then there exists an inner product on E, if and only if $\|.\|$ obeys the parallelogram law.*

$$✎ \ \|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

2.2 Hahn-Banach Theorems

Theorem 2.2.1 : *Let E be an $\mathbb{R}$-vector space and let $p : E \longrightarrow \mathbb{R}$ be a function such that:*

1. $p(\lambda x) = \lambda p(x), \quad \forall \lambda > 0, \quad \forall x \in E$
2. $p(x+y) \leq p(x) + p(y), \quad \forall x, y \in E$

Let G be a vector subspace of E, and let $g : G \longrightarrow \mathbb{R}$be a linear function such that:

$$g(x) \leq p(x), \quad \forall x \in G$$

☞ *then there exists $f : E \longrightarrow \mathbb{R}$ linear such that:*

$$\begin{cases} f(x) = g(x), & \forall x \in G \\ f(x) \leq p(x), & \forall x \in \end{cases}$$

Theorem 2.2.2 : *Let $G \subset E$ be a vector subspace of E. if $g : G \longrightarrow \mathbb{R}$ is a linear continuous function then there exists $f \in E'$. such that:*

① $f(x) = g(x), \quad \forall x \in G$

② $\|f\|_{E'} = \|g\|_{G'}$

$$\boxed{☞ \ \|f\|_{E'} = \|f\|_{\mathcal{L}(E,\mathbb{R})} = \sup_{\|x\| \leq 1, x \in E} |f(x)|}$$

Definition 2.2.1 : *Let E be a normed space, an affine hyperplane is a subset of E. denoted by H, and defined by:*

$$H = \{x \in E : \quad f(x) = \alpha\}$$

where α is a real number and f is a linear form in E. sometimes we write:

$$[H = \alpha]$$

Definition 2.2.2 : *Let A be a set, we say that A is convex, if for all $x, y \in A$ and for all $t \in [0, 1]$.*

$$\Longrightarrow \quad tx + (1-t)y \in A$$

Theorem 2.2.3 : *An affine hyperplane.*

$$H = \{x \in E : \quad f(x) = \alpha\}$$

is closed if and only if f is continuous.

Definition 2.2.3 : *Let A, B be two subsets of a vector space E, we say that $H = \{x \in E : \quad f(x) = \alpha\}$ seperates A and B, if:*

$$f(x) \geq \alpha, \quad \forall x \in A$$

and

$$f(x) \leq \alpha, \quad \forall x \in B$$

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08:16 AM Tue, Oct 14 2025

Theorem 2.2.4 (Hahn-Banach) : *Let $A \subset E$, $B \subset E$ be two convex subsets such that $A \cap B = \emptyset$. Assume that one of them is open. Then there exists an affine hyperplane (closed) which seperates A and B. i.e*

$$\exists f \in E', \quad \alpha \in \mathbb{K}$$

such that

$$f(x) \geq \alpha, \forall x \in A, \text{ and } f(x) \leq \alpha, \forall x \in B.$$

Proposition 2.2.5 : *Let $C \subset E$ be a nonempty convex open subset and let $x_0 \in E \backslash C$. then there exists $f \in E'$, such that*

$$f(x) < f(x_0) \qquad \forall (x \in C).$$

In particular $[f(x_0) = \alpha]$ seperates $\{x_0\}$ and C.

Theorem 2.2.6 (Hahn-Banach) : *Let $A \subset E$, $B \subset E$ be tw oconvex nonempty subsets such that $A \cap B = \emptyset$ we suppose that one is compact and the second is closed. Then there exists a closed affine hyperplane which seperates strictly A and B. i.e. there exists $f \in E'$, and $\alpha \in \mathbb{K}(\mathbb{R})$. such that*

$$f(x) < \alpha \qquad (\forall x \in A),$$

$$f(x) < \alpha, \qquad (\forall x \in B).$$

Corollary 2.2.7 : *Let F be a subspace of E, such that $\overline{F} \neq E$. Then there exists $f \in E', f \neq 0_{E'}$ where $f(x) = 0, \quad \forall x \in F$.*

We apply the second version of Geometric Hahn-Banach (second version) to $A = \overline{F}, B = \{x_0\}$, $x_0 \notin \overline{F}$. we have A is a subspace so it's convex and closed, and B is convex and compact $A \cap B = \emptyset$.

`Well the idea is pretty simple; Do it as an exercise.`

Theorem 2.2.8 (Banach-Steinhauss) : *Let E and F be two Banach spaces, and let $(T_i)_{i\in I}$ be a family not necessarly countable of continuous linear functions from E into F,*

$$T_i \in \mathcal{L}(E, F).$$

Assume that

$$\sup_{i\in I} \|T_i x\| < +\infty, \qquad (\forall x \in E).$$

Then

$$\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)} < +\infty,$$

which means there exists $\exists c > 0$, such that

$$\|T_i x\| \leq c\|x\| \qquad (\forall x \in E), \quad \forall i \in I$$

Theorem 2.2.9 : *Let E, F be two Banach spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}(E,F)$ such that $\forall x \in E$, $T_n x$ converges to Tx, when n converge to $+\infty$, then we have:*

- $$\sup_n \|T_n\|_{\mathcal{L}(E,F)} < +\infty$$
- $$T \in \mathcal{L}(E,F)$$
- $$\|T\|_{\mathcal{L}(E,F)} \leq \lim_{n\to\infty} \inf \|T_n\|_{\mathcal{L}(E,F)}$$

Lecture 5

08:05 AM Tue, Oct 21 2025

Corollary 2.2.10 : *Let G be a vector space, and let B be a subset of G. Assume that:*

$$\forall f \in G' : \quad f(B) = \{f(b) : b \in B\} \text{ is bounded} \implies B \text{ is bounded.}$$

Proof. Consider:

$$\begin{array}{rccc} T_b : & G' & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & T_b(f) = \langle f, b\rangle = f(b) \end{array}$$

Then using (B.S) principle, we get

$$|\langle f, b\rangle| \leq c\|f\|.$$

Thus we get

$$\|b\| \leq c, \quad \forall b \in B,$$

which means that B is bounded. □

Corollary 2.2.11 : *Let G be a banach space and let B' be a subset of G' assume that:*

$$\forall x \in G : \quad \langle B', x\rangle = \{\langle f, x\rangle : f \in G'\} \text{ is bounded} \implies B' \text{ is bounded.}$$

Theorem 2.2.12 (Open mapping theorem) : *Let E, and F be two Banach spaces, and let $T \in \mathcal{L}(E,F)$ assume that T is onto, then there exists $c > 0$, such that:*

$$B_F(0,c) \subset T(B_E(0,1)) \qquad (*)$$

$(*)$ can be translated into image of open set is open.

Corollary 2.2.13 : *Let $(E, \|.\|_1)$, and $(E, \|.\|_2)$ be Banach spaces. If there exists a constant $c > 0$, such that:*

$$\|x\|_1 \leq c\|x\|_2 \quad (\forall x \in E)$$

then $\|.\|_1$, and $\|.\|_2$ are equivalents.

Corollary 2.2.14 : *Let E, and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$ assume that T is bijective, then*

$$T^{-1} \in \mathcal{L}(F, E)$$

Chapter 3

Weak topologies

3.1 Introduction

Let X be a given set, and let (Y_i, σ_i) be a familly of toplogical spaces, I is an arbitrary set of index. Let $\varphi_i : X \longrightarrow Y_i$ be a familly of maps. we search a toplogy on X, which is the smallest one, and makes $\varphi_i, \forall i \in I$ continuous. Let be τ this toplogy. we can prove that:

$$\tau = \left\{ \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \varphi_i^{-1}(w_i), i \right\}$$

defines the coarest toplogy, such that that $\varphi_i^{-1}(w_i)$ is an elemetary open of τ

✎Remark:

A neighborhood of $x \in X$ is defined by:

$$U = \bigcap_{\text{finite}} \varphi_i^{-1}(V_i)$$

where V_i is a neighborhood of $\varphi_i(x)$.

Proposition 3.1.1 : *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*

$$x_n \overset{n\to\infty,\tau}{\to} x \iff \varphi_i(x_n) \overset{\forall i\in I, n+\infty}{\to} \varphi_i(x_n)$$

Proof.

$(\implies)$ Easy and known ❄

$(\impliedby)$

We suppose that $\varphi_i(x_n) \overset{\forall i \in I, n+\infty}{\to} \varphi_i(x)$ and we need to prove that

$$x_n \overset{n+\infty,\tau}{\to} x$$

Let U be a neighborhood of x. then:

$$U = \bigcap_{i \in J, J \subset I, J \text{ finite}} \varphi_i(V_i)$$

Where V_i is a neighborhood of $\varphi_i(x)$. So for each $i \in J$, V_i is a neighborooh of $\varphi_i(x)$. $\square$