Contents

Chapter 1

Overview of Hilbert and Banach spaces

Lecture 1: Overview `08:22 AM Tue, Sep 23 2025`

Let E be a given set, and let $\mathbb{K} = \mathbb{C}, \mathbb{K} = \mathbb{R}$ be the know fields. let us define two operations on $E \times E$ and $\mathbb{K} \times E$ by the :

$$
\begin{aligned}
+ : \quad E \times E &\longrightarrow E \\
(x, y) &\longmapsto x + y
\end{aligned}
$$

$$
\begin{aligned}
\cdot : \quad \mathbb{K} \times E &\longrightarrow E \\
(\lambda, x) &\longmapsto \lambda \cdot x
\end{aligned}
$$

> **Definition 1.0.1 (Vector space) :** *A vector space E over $\mathbb{K}$ is a set equipped by two operations, called additions and scalar multiplication which verify the following axioms*
>
> 1. $\forall x, y \in E : \quad x + y = y + x$
>
> 2. $\forall x, y, z \in E : \quad (x + y) + z = x + (y + z)$
>
> 3. $\exists 0_E \in E : \quad x + 0_E = x$
>
> 4. $\forall x \in E, \exists y \in E : \quad x + y = 0_E$ *(denote y = -x)*
>
> 5. $\forall \lambda, \beta \in \mathbb{K}, \forall x \in E : \quad (\alpha\beta)x = \alpha(\beta x)$

compact $\Longleftrightarrow$ closed and bounded (finite dimensional)

compact $\Longrightarrow$ closed bounded (generally)

compactness is important because most important theorems use compact by assumption

more open sets = less compacts

6. $\forall x \in E: \quad 1_{\mathbb{K}} \cdot x = x$

7. $\forall x, y \in E, \forall \lambda \in \mathbb{K}: \quad \lambda(x + y) = \lambda x + \lambda y$

8. $\forall \alpha, \beta \in \mathbb{K}, \forall x \in E: \quad (\alpha + \beta)x = \alpha x + \beta x$

Definition 1.0.2 (Vector subspace) : *A subset $F \subset E$ is called a subvector space if:*

1. $0_E \subset F$

2. $\forall \lambda \in \mathbb{K}, \forall x, y \in E \implies \lambda x + y \in E$

Definition 1.0.3 (Inner product) : *An inner product on a vector space E is a function:*

$$\langle \cdot \rangle : \quad E \times E \quad \longrightarrow \quad \mathbb{K}$$
$$(x, y) \quad \longmapsto \quad \langle x, y \rangle$$

which verifies:

1. *Linearity in the first argument :*

$$\forall x, y, z \in E, \forall \alpha \in \mathbb{K}: \quad \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$
$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

2. *Conjugate symmetry:*

$$\forall x, y \in E: \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

3. *Positive difinitness:*

$$\langle x, x \rangle \geq 0$$
$$\langle x, x \rangle = 0 \iff x = 0_E$$

Lecture 2 $\hspace{4cm}$ `08:07 AM Tue, Sep 30 2025`

Definition 1.0.4 : *Let E be a set, and d be a metric on E, we said that (E, d) is complete if and only if every cauchy sequence is convergent*

Definition 1.0.5 : *A Banach space is a complete vectorial normed space*

Example:

- $\mathbb{K}(\mathbb{R}, \mathbb{C})$ *is a Banach space*

- $\mathbb{K}^n$ *is a Banach space*

- *if E is a normed space with finite dimension, then E is Banach*

- $\ell^p(\mathbb{N}, \mathbb{K})$ *with* $1 \leq p \leq +\infty$ *is Banach space (Exercise).*

- *Every finite set product of Banach spaces is a Banach space too.*

Proposition 1.0.1 : *The limit of a sequence in a Banach space (if it exists) is unique.*

Proposition 1.0.2 : *Let F be a closed subset of E. if $(x_n)_{n\in\mathbb{N}} \subset F$ is convergent to $x \in E$, then $x \in F$.*

REMARK:

The importance of the notion of equivallent norms on a normed vector space is:

Let $\|.\|_1$ and $\|.\|_2$ be two equivalent norms on $V = E$. one has:

$$(x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_1 \iff (x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_2$$

Proposition 1.0.3 (Definition) : *Let V and W be two norms on V (resp. W), and let $f : V \longrightarrow W$ be a function. the following statments are equivallent:*

1. *f is continuous from V into W*

2. *and:*

$$\forall x_0 \in V, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in V : \quad \|x - x_0\| \leq \delta \implies \|f(x) - f(x_0)\| \leq \varepsilon$$

3. *and:*

$$\forall x \in V, \forall (x_n)_{n\in\mathbb{N}} \in V : \quad x_n \to x \implies f(x_n) \to f(x)(\|f(x_n - f(x))\| \to 0 \quad n \to \infty)$$

REMARK:

On finite dimension vector space, all the norm are equivalents.

1.1 Duality

Let V and W be two normed vector be two normed vector spaces

Proposition 1.1.1 : *Let $A : V \longrightarrow W$ be a linear function (Map). The following are equivalent:*

1. *A is continuous on V*

2. *A is continuous at 0*

3. *$\exists c > 0$: $\|A(v)\| \leq c\|v\|$ $\forall v \in V$*

Definition 1.1.1 : *We denote by $L(V, W)$ be the set of all linear functions, we denote $\mathcal{L}(V, W)$ the set of all continuous linear functions*

Proposition 1.1.2 : *$L(V, W)$ and $\mathcal{L}(V, W)$ are vector spaces*

Proposition 1.1.3 : *Let V be a normed space, and W be a Banach space then $\mathcal{L}(V, W)$ is Banach space.*

Definition 1.1.2 : *Let $(x_n)_{n \in \mathbb{N}}$ be sequence in a normed vector space E.*

- *we say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums $s_n = \sum_{k=1}^{n} x_k$ converges.*

- *we say that the series is normally (absolutely) convergent if the series of real positive numbers $\sum_{n=1}^{\infty} \|x_n\|$ converges*

Proposition 1.1.4 : *Let E be a normed vector space. The following are equivalents:*

1. *E is Banach.*

2. *Every normally convergent series is convergent.*

Definition 1.1.3 : *Let E and F be two normed vector spaces, A function (mapping) from E into F is called linear isomorphism if its linear, continuous bijective and its inverse is continuous (naturally linear)*

We denote by $\text{ISO}(E, F)$ the set of all linear isomporphisms mapping from E into F.

Theorem 1.1.5 : *Let E and F be Banach spaces, then $\text{ISO}(E, F)$ is an open subset of $\mathcal{L}(E, F)$ and the map $u \longrightarrow u^{-1}$ from $\text{ISO}(E, F)$ into $\text{ISO}(F, E)$ is continuous. More, one has if $u_0 \in \text{ISO}(E, F)$ and $u \in B(u_0, \frac{1}{\|u_0^{-1}\|})$, then $u \in \text{ISO}(E, F)$, and:*

$$u^{-1} = \sum_{n=1}^{\infty} (id_E - u_0^{-1}u)^n \cdot u_0^{-1}$$

Corollary 1.1.6 (Von Neumann) : *Let E be a Banach space, and $u \in \mathcal{L}(E) = \mathcal{L}(E, E)$. if $\|u\| < 1$, then $id_E - u$ belongs to $\text{ISO}(E, E) = \text{ISO}(E)$ and it's inverse is given by:*

$$(Id_E - u)^{-1} = \sum_{n=1}^{\infty} u^n$$

<u>NOTATIONS:</u> $E' = \mathcal{L}(E)$, and $E^* = L(E)$.

1. E' toplogical dual space of E.

2. E^* algebraic dual space of E.

Proposition 1.1.7 : *Let E be a Banach space E_0 be normed vector space. let $p : E \longrightarrow E_0$ a continuous linear surjection mapping. If there exists $c > 0$ such that $\forall y_0 \in E_0, \exists x \in E$ such that $f(x) = y_0$ and $\|x\| \leq c\|y_0\|$ then E_0 is Banach.*

Theorem 1.1.8 : *Let E be a Banach space and let F be a closed subspace of E. Then $E_{|F}$ is Banach too.*

Proposition 1.1.9 : *Let $A \subset E$. A is compact if from every sequence of elements of A, we can extract a subsequence converging to an element in A.*

Theorem 1.1.10 (Riesz) : *The closed unit ball is compact if and only if $\dim E < +\infty$,*

Chapter 2

Compactness

Definition 2.0.1 (Bolzano-Weirstrass Property) : *A set $A \subset V$, is compact if from every sequence of A we can extract a subsequence which is convergent in A.*

Theorem 2.0.1 (Riesz) : *The closed unit ball of V is compact if and only if dimension of V is finite.*

2.1 Equicontinuity

Definition 2.1.1 : *Let (X, d_1) and (Y, d_2) be two metric spaces.*

Consider $F \subset \{f : X \longrightarrow Y\}$, we say that F is equicontinuous at $x_0 \in X$. if:

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad d_1(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon, \quad \forall f \in F$$

Definition 2.1.2 : *If F is equicontinuous, at each point of X, then we say that F is equicontinuous on X.*

7

Theorem 2.1.1 (Ascoli-Arzela Theorem) : *Let (X, d) be a compact metric space. A family F of real valued continuous functions on X is relatively compact in $\mathcal{C}(X, \mathbb{R})$ with the sup norm if and only if:*

1. *F is uniformally bounded*

2. *F is equicontinuous on X*

Theorem 2.1.2 (Bernestein-Weirstrass) : *Let $[a, b]$ and consider $f : [a, b] \longrightarrow \mathbb{R}$ continuous function, then there exists a sequence of polynomial functions $(f_n)_{n \in \mathbb{N}}$ such that f_n converges uniformly to f on $[a, b]$.*

Theorem 2.1.3 (Riesz representation) : *Let H be Hilbert space, and consider:*

$$H' = \{f : H \longrightarrow \mathbb{C} : \quad f \text{ continuous }\}$$

then:

$$\forall f \in H', \quad \exists! y \in H : \quad f(x) = \langle x, y \rangle$$

Theorem 2.1.4 : *Let $(E, \|.\|)$ be a normed vector space, then there exists an inner product on E, if and only if $\|.\|$ obeys the parallelogram law.*

$$\text{✎ } \|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

2.2 Hahn-Banach Theorems

Theorem 2.2.1 : *Let E be an $\mathbb{R}$-vector space and let $p : E \longrightarrow \mathbb{R}$ be a function such that:*

   1. $p(\lambda x) = \lambda p(x), \quad \forall \lambda > 0, \quad \forall x \in E$

   2. $p(x + y) \leq p(x) + p(y), \quad \forall x, y \in E$

Let G be a vector subspace of E, and let $g : G \longrightarrow \mathbb{R}$ be a linear function such that:

$$g(x) \leq p(x), \quad \forall x \in G$$

☞ *then there exists $f : E \longrightarrow \mathbb{R}$ linear such that:*

$$\begin{cases} f(x) = g(x), & \forall x \in G \\ f(x) \leq p(x), & \forall x \in \end{cases}$$

Theorem 2.2.2 : *Let $G \subset E$ be a vector subspace of E. if $g : G \longrightarrow \mathbb{R}$ is a linear continuous function then there exists $f \in E'$. such that:*

   ① *$f(x) = g(x), \quad \forall x \in G$*

   ② *$\|f\|_{E'} = \|g\|_{G'}$*

$$\boxed{☞\ \|f\|_{E'} = \|f\|_{\mathcal{L}(E,\mathbb{R})} = \sup_{\|x\| \leq 1, x \in E} |f(x)|}$$

Definition 2.2.1 : *Let E be a normed space, an affine hyperplane is a subset of E. denoted by H, and defined by:*

$$H = \{x \in E : \quad f(x) = \alpha\}$$

where α is a real number and f is a linear form in E. sometimes we write:

$$[H = \alpha]$$

Definition 2.2.2 : *Let A be a set, we say that A is convex, if for all $x, y \in A$ and for all $t \in [0, 1]$.*

$$\implies tx + (1-t)y \in A$$

Theorem 2.2.3 : *An affine hyperplane.*

$$H = \{x \in E : \quad f(x) = \alpha\}$$

is closed if and only if f is continuous.

Definition 2.2.3 : *Let A, B be two subsets of a vector space E, we say that $H = \{x \in E : \quad f(x) = \alpha\}$ seperates A and B, if:*

$$f(x) \geq \alpha, \quad \forall x \in A$$

and

$$f(x) \leq \alpha, \quad \forall x \in B$$