Chapter 1

Overview of Hilbert and Banach spaces

Lecture 1: Overview 08:22 AM Tue, Sep 23 2025

Let E be a given set, and let $\mathbb{K} = \mathbb{C}, \mathbb{K} = \mathbb{R}$ be the know fields. let us define two operations on $E \times E$ and $\mathbb{K} \times E$ by the :

$$\begin{array}{rccc} + : & E \times E & \longrightarrow & E \\ & (x, y) & \longmapsto & x + y \end{array}$$

$$\begin{array}{rccc} \cdot : & \mathbb{K} \times E & \longrightarrow & E \\ & (\lambda, x) & \longmapsto & \lambda \cdot x \end{array}$$

compact $\iff$ closed and bounded (finite dimensional)

compact $\implies$ closed bounded (generally)

compactness is important because most important theorems use compact by assumption

more open sets = less compacts

Definition 1.0.1 (Vector space) : *A vector space E over $\mathbb{K}$ is a set equipped by two operations, called additions and scalar multiplication which verify the following axioms*

1. $\forall x, y \in E : \quad x + y = y + x$
2. $\forall x, y, z \in E : \quad (x + y) + z = x + (y + z)$
3. $\exists 0_E \in E : \quad x + 0_E = x$
4. $\forall x \in E, \exists y \in E : \quad x + y = 0_E$ *(denote y = -x)*
5. $\forall \lambda, \beta \in \mathbb{K}, \forall x \in E : \quad (\alpha\beta)x = \alpha(\beta x)$

6. $\forall x \in E: \quad 1_{\mathbb{K}} \cdot x = x$

7. $\forall x, y \in E, \forall \lambda \in \mathbb{K}: \quad \lambda(x+y) = \lambda x + \lambda y$

8. $\forall \alpha, \beta \in \mathbb{K}, \forall x \in E: \quad (\alpha+\beta)x = \alpha x + \beta x$

Definition 1.0.2 (Vector subspace) : *A subset $F \subset E$ is called a subvector space if:*

1. $0_E \subset F$

2. $\forall \lambda \in \mathbb{K}, \forall x, y \in E \implies \lambda x + y \in E$

Definition 1.0.3 (Inner product) : *An inner product on a vector space E is a function:*

$$\begin{aligned} \langle \cdot \rangle : \quad E \times E &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto \langle x, y \rangle \end{aligned}$$

which verifies:

1. *Linearity in the first argument :*

$$\begin{aligned} \forall x, y, z \in E, \forall \alpha \in \mathbb{K}: \quad \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \\ \langle x+y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle \end{aligned}$$

2. *Conjugate symmetry:*

$$\forall x, y \in E: \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

3. *Positive difinitness:*

$$\langle x, x \rangle \geq 0$$
$$\langle x, x \rangle = 0 \iff x = 0_E$$

Lecture 2

`08:07 AM Tue, Sep 30 2025`

Definition 1.0.4 : *Let E be a set, and d be a metric on E, we said that (E, d) is complete if and only if every cauchy sequence is convergent*

Definition 1.0.5 : *A Banach space is a complete vectorial normed space*

Example:

- $\mathbb{K}(\mathbb{R}, \mathbb{C})$ *is a Banach space*
- $\mathbb{K}^n$ *is a Banach space*
- *if E is a normed space with finite dimension, then E is Banach*
- $\ell^p(\mathbb{N}, \mathbb{K})$ *with* $1 \leq p \leq +\infty$ *is Banach space (Exercise).*
- *Every finite set product of Banach spaces is a Banach space too.*

Proposition 1.0.1 : *The limit of a sequence in a Banach space (if it exists) is unique.*

Proposition 1.0.2 : *Let F be a closed subset of E. if $(x_n)_{n\in\mathbb{N}} \subset F$ is convergent to $x \in E$, then $x \in F$.*

REMARK:

The importance of the notion of equivallent norms on a normed vector space is:
Let $\|.\|_1$ and $\|.\|_2$ be two equivallent norms on $V = E$. one has:

$$(x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_1 \iff (x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_2$$

Proposition 1.0.3 (Definition) : *Let V and W be two norms on V (resp. W), and let $f : V \longrightarrow W$ be a function. the following statments are equivallent:*

1. *f is continuous from V into W*
2. *and:*
$$\forall x_0 \in V, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in V : \quad \|x - x_0\| \leq \delta \implies \|f(x) - f(x_0)\| \leq \varepsilon$$
3. *and:*
$$\forall x \in V, \forall (x_n)_{n\in\mathbb{N}} \in V : \quad x_n \to x \implies f(x_n) \to f(x)(\|f(x_n - f(x))\| \to 0 \quad n \to \infty)$$

REMARK:

On finite dimension vector space, all the norm are equivallents.

1.1 Duality

Let V and W be two normed vector be two normed vector spaces

Proposition 1.1.1 : *Let $A : V \longrightarrow W$ be a linear function (Map). The following are equivallent:*

1. *A is continuous on V*
2. *A is continuous at 0*
3. *$\exists c > 0$: $\|A(v)\| \leq c\|v\| \quad \forall v \in V$*

Definition 1.1.1 : *We denote by $L(V, W)$ be the set of all linear functions, we denote $\mathcal{L}(V, W)$ the set of all continuous linear functions*

Proposition 1.1.2 : *$L(V, W)$ and $\mathcal{L}(V, W)$ are vector spaces*

Proposition 1.1.3 : *Let V be a normed space, and W be a Banach space then $\mathcal{L}(V, W)$ is Banach space.*

Definition 1.1.2 : *Let $(x_n)_{n\in\mathbb{N}}$ be sequence in a normed vector space E.*

- *we say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums $s_n = \sum_{k=1}^{n} x_k$ converges.*
- *we say that the series is normally (absolutely) convergent if the series of real positive numbers $\sum_{n=1}^{\infty} \|x_n\|$ converges*

Proposition 1.1.4 : *Let E be a normed vector space. The following are equivallents:*

1. *E is Banach.*
2. *Every normally convergent series is convergent.*

Definition 1.1.3 : *Let E and F be two normed vector spaces, A function (mapping) from E into F is called linear isomorphism if its linear, continuous bijective and its inverse is continuous (naturally linear)*

We denote by $\mathrm{ISO}(E, F)$ the set of all linear isomporphisms mapping from E into F.

Theorem 1.1.5 : *Let E and F be Banach spaces, then $ISO(E, F)$ is an open subset of $\mathcal{L}(E, F)$ and the map $u \longrightarrow u^{-1}$ from $ISO(E, F)$ into $ISO(F, E)$ is continuous. More, one has if $u_0 \in ISO(E, F)$ and $u \in B(u_0, \frac{1}{\|u_0^{-1}\|})$, then $u \in ISO(E, F)$, and:*

$$u^{-1} = \sum_{n=1}^{\infty} (id_E - u_0^{-1} u)^n \cdot u_0^{-1}$$

Corollary 1.1.6 (Von Neumann) : *Let E be a Banach space, and $u \in \mathcal{L}(E) = \mathcal{L}(E, E)$. if $\|u\| < 1$, then $id_E - u$ belongs to $ISO(E, E) = ISO(E)$ and it's inverse is given by:*

$$(Id_E - u)^{-1} = \sum_{n=1}^{\infty} u^n$$

NOTATIONS: $E' = \mathcal{L}(E)$, and $E^* = L(E)$.

1. E' toplogical dual space of E.
2. E^* algebraic dual space of E.

Proposition 1.1.7 : *Let E be a Banach space E_0 be normed vector space. let $p : E \longrightarrow E_0$ a continuous linear surjection mapping. If there exists $c > 0$ such that $\forall y_0 \in E_0, \exists x \in E$ such that $f(x) = y_0$ and $\|x\| \leq c\|y_0\|$ then E_0 is Banach.*

Theorem 1.1.8 : *Let E be a Banach space and let F be a closed subspace of E. Then $E_{|F}$ is Banach too.*

Proposition 1.1.9 : *Let $A \subset E$. A is compact if from every sequence of elements of A, we can extract a subsequence converging to an element in A.*

Theorem 1.1.10 (Riesz) : *The closed unit ball is compact if and only if dim $E < +\infty$,*