

Chapter 1

Overview of Hilbert and Banach spaces

Lecture 1: Overview

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Let E be a given set, and let $\mathbb{K} = \mathbb{C}, \mathbb{K} = \mathbb{R}$ be the known fields. Let us define two operations on $E \times E$ and $\mathbb{K} \times E$ by the :

$$\begin{aligned} + : E \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y \end{aligned}$$

$$\begin{aligned} \cdot : \mathbb{K} \times E &\longrightarrow E \\ (\lambda, x) &\longmapsto \lambda \cdot x \end{aligned}$$

Definition 1.0.1 (Vector space) : A vector space E over \mathbb{K} is a set equipped by two operations, called additions and scalar multiplication which verify the following axioms

1. $\forall x, y \in E : x + y = y + x$
2. $\forall x, y, z \in E : (x + y) + z = x + (y + z)$
3. $\exists 0_E \in E : x + 0_E = x$
4. $\forall x \in E, \exists y \in E : x + y = 0_E$ (denote $y = -x$)
5. $\forall \lambda, \beta \in \mathbb{K}, \forall x \in E : (\alpha\beta)x = \alpha(\beta x)$

compact \iff
closed and
bounded
(finite dimensional)

compact
 \implies closed
bounded (generally)

compactness
is important
because most
important
theorems use
compact by
assumption

more open
sets = less
compacts

$$6. \forall x \in E : 1_{\mathbb{K}} \cdot x = x$$

$$7. \forall x, y \in E, \forall \lambda \in \mathbb{K} : \lambda(x + y) = \lambda x + \lambda y$$

$$8. \forall \alpha, \beta \in \mathbb{K}, \forall x \in E : (\alpha + \beta)x = \alpha x + \beta x$$

Definition 1.0.2 (Vector subspace) : A subset $F \subset E$ is called a subvector space if:

$$1. 0_E \in F$$

$$2. \forall \lambda \in \mathbb{K}, \forall x, y \in E \implies \lambda x + y \in E$$

Definition 1.0.3 (Inner product) : An inner product on a vector space E is a function:

$$\begin{aligned} \langle \cdot \rangle : E \times E &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto \langle x, y \rangle \end{aligned}$$

which verifies:

1. Linearity in the first argument :

$$\begin{aligned} \forall x, y, z \in E, \forall \alpha \in \mathbb{K} : \quad \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \\ \langle x + y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle \end{aligned}$$

2. Conjugate symmetry:

$$\forall x, y \in E : \langle x, y \rangle = \overline{\langle y, x \rangle}$$

3. Positive definiteness:

$$\begin{aligned} \langle x, x \rangle &\geq 0 \\ \langle x, x \rangle = 0 &\iff x = 0_E \end{aligned}$$

Lecture 2

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Definition 1.0.4 : Let E be a set, and d be a metric on E , we said that (E, d) is complete if and only if every cauchy sequence is convergent

Definition 1.0.5 : A Banach space is a complete vectorial normed space

Example:

- $\mathbb{K}(\mathbb{R}, \mathbb{C})$ is a Banach space
- \mathbb{K}^n is a Banach space
- if E is a normed space with finite dimension, then E is Banach
- $\ell^p(\mathbb{N}, \mathbb{K})$ with $1 \leq p \leq +\infty$ is Banach space (Exercise).
- Every finite set product of Banach spaces is a Banach space too.

Proposition 1.0.1 : The limit of a sequence in a Banach space (if it exists) is unique.

Proposition 1.0.2 : Let F be a closed subset of E . if $(x_n)_{n \in \mathbb{N}} \subset F$ is convergent to $x \in E$, then $x \in F$.

REMARK:

The importance of the notion of equivalent norms on a normed vector space is:

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on $V = E$. one has:

$$(x_n)_{n \in \mathbb{N}} \text{ CV for } \|\cdot\|_1 \iff (x_n)_{n \in \mathbb{N}} \text{ CV for } \|\cdot\|_2$$

Proposition 1.0.3 (Definition) : Let V and W be two norms on V (resp. W), and let $f : V \longrightarrow W$ be a function. the following statements are equivalent:

1. f is continuous from V into W
2. and:

$$\forall x_0 \in V, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in V : \|x - x_0\| \leq \delta \implies \|f(x) - f(x_0)\| \leq \varepsilon$$

3. and:

$$\forall x \in V, \forall (x_n)_{n \in \mathbb{N}} \in V : x_n \rightarrow x \implies f(x_n) \rightarrow f(x) (\|f(x_n) - f(x)\| \rightarrow 0 \text{ as } n \rightarrow \infty)$$

REMARK:

On finite dimension vector space, all the norm are equivalent.

1.1 Duality

Let V and W be two normed vector be two normed vector spaces

Proposition 1.1.1 : Let $A : V \longrightarrow W$ be a linear function (Map). The following are equivalent:

1. A is continuous on V
2. A is continuous at 0
3. $\exists c > 0: \|A(v)\| \leq c\|v\| \quad \forall v \in V$

Definition 1.1.1 : We denote by $L(V, W)$ be the set of all linear functions, we denote $\mathcal{L}(V, W)$ the set of all continuous linear functions

Proposition 1.1.2 : $L(V, W)$ and $\mathcal{L}(V, W)$ are vector spaces

Proposition 1.1.3 : Let V be a normed space, and W be a Banach space then $\mathcal{L}(V, W)$ is Banach space.

Definition 1.1.2 : Let $(x_n)_{n \in \mathbb{N}}$ be sequence in a normed vector space E .

- we say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums $s_n = \sum_{k=1}^n x_k$ converges.
- we say that the series is normally (absolutely) convergent if the series of real positive numbers $\sum_{n=1}^{\infty} \|x_n\|$ converges

Proposition 1.1.4 : Let E be a normed vector space. The following are equivalent:

1. E is Banach.
2. Every normally convergent series is convergent.

Definition 1.1.3 : Let E and F be two normed vector spaces, A function (mapping) from E into F is called linear isomorphism if its linear, continuous bijective and its inverse is continuous (naturally linear)

We denote by $\text{ISO}(E, F)$ the set of all linear isomorphisms mapping from E into F .

Theorem 1.1.5 : Let E and F be Banach spaces, then $\text{ISO}(E, F)$ is an open subset of $\mathcal{L}(E, F)$ and the map $u \rightarrow u^{-1}$ from $\text{ISO}(E, F)$ into $\text{ISO}(F, E)$ is continuous. More, one has if $u_0 \in \text{ISO}(E, F)$ and $u \in B(u_0, \frac{1}{\|u_0^{-1}\|})$, then $u \in \text{ISO}(E, F)$, and:

$$u^{-1} = \sum_{n=1}^{\infty} (id_E - u_0^{-1}u)^n \cdot u_0^{-1}$$

Corollary 1.1.6 (Von Neumann) : Let E be a Banach space, and $u \in \mathcal{L}(E) = \mathcal{L}(E, E)$. if $\|u\| < 1$, then $id_E - u$ belongs to $\text{ISO}(E, E) = \text{ISO}(E)$ and it's inverse is given by:

$$(Id_E - u)^{-1} = \sum_{n=1}^{\infty} u^n$$

NOTATIONS: $E' = \mathcal{L}(E)$, and $E^* = L(E)$.

1. E' topological dual space of E .
2. E^* algebraic dual space of E .

Proposition 1.1.7 : Let E be a Banach space E_0 be normed vector space. let $p : E \rightarrow E_0$ a continuous linear surjection mapping. If there exists $c > 0$ such that $\forall y_0 \in E_0, \exists x \in E$ such that $f(x) = y_0$ and $\|x\| \leq c\|y_0\|$ then E_0 is Banach.

Theorem 1.1.8 : Let E be a Banach space and let F be a closed subspace of E . Then $E|_F$ is Banach too.

Proposition 1.1.9 : Let $A \subset E$. A is compact if from every sequence of elements of A , we can extract a subsequence converging to an element in A .

Theorem 1.1.10 (Riesz) : The closed unit ball is compact if and only if $\dim E < +\infty$,

Chapter 2

Compactness

Lecture 3

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Definition 2.0.1 (Bolzano-Weirstrass Property) : A set $A \subset V$, is compact if from every sequence of A we can extract a subsequence which is convergent in A .

Theorem 2.0.1 (Riesz) : The closed unit ball of V is compact if and only if dimension of V is finite.

2.0.1 Equicontinuity

Definition 2.0.2 : Let (X, d_1) and (Y, d_2) be two metric spaces.

Consider $F \subset \{f : X \rightarrow Y\}$, we say that F is equicontinuous at $x_0 \in X$. if:

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad d_1(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon, \quad \forall f \in F$$

Definition 2.0.3 : If F is equicontinuous, at each point of X , then we say that F is equicontinuous on X .

Theorem 2.0.2 (Ascoli-Arzelà Theorem) : Let (X, d) be a compact metric space. A family F of real valued continuous functions on X is relatively compact in $\mathcal{C}(X, \mathbb{R})$ with the sup norm if and only if:

1. F is uniformly bounded
2. F is equicontinuous on X

Theorem 2.0.3 (Bernstein-Weierstrass) : Let $[a, b]$ and consider $f : [a, b] \rightarrow \mathbb{R}$ continuous function, then there exists a sequence of polynomial functions $(f_n)_{n \in \mathbb{N}}$ such that f_n converges uniformly to f on $[a, b]$.

Theorem 2.0.4 (Riesz representation) : Let H be Hilbert space, and consider:

$$H' = \{f : H \rightarrow \mathbb{C} : f \text{ continuous} \}$$

then:

$$\forall f \in H', \quad \exists ! y \in H : f(x) = \langle x, y \rangle$$

Theorem 2.0.5 : Let $(E, \|\cdot\|)$ be a normed vector space, then there exists an inner product on E , if and only if $\|\cdot\|$ obeys the parallelogram law.

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

2.1 Hahn-Banach Theorems

Theorem 2.1.1 : Let E be an \mathbb{R} -vector space and let $p : E \rightarrow \mathbb{R}$ be a function such that:

1. $p(\lambda x) = \lambda p(x), \quad \forall \lambda > 0, \quad \forall x \in E$
2. $p(x + y) \leq p(x) + p(y), \quad \forall x, y \in E$

Let G be a vector subspace of E , and let $g : G \rightarrow \mathbb{R}$ be a linear function such that:

$$g(x) \leq p(x), \quad \forall x \in G$$

then there exists $f : E \rightarrow \mathbb{R}$ linear such that:

$$\begin{cases} f(x) = g(x), & \forall x \in G \\ f(x) \leq p(x), & \forall x \in E \end{cases}$$

Theorem 2.1.2 : Let $G \subset E$ be a vector subspace of E . if $g : G \rightarrow \mathbb{R}$ is a linear continuous function then there exists $f \in E'$ such that:

- ① $f(x) = g(x), \quad \forall x \in G$
- ② $\|f\|_{E'} = \|g\|_{G'}$

$$\|f\|_{E'} = \|f\|_{\mathcal{L}(E, \mathbb{R})} = \sup_{\|x\| \leq 1, x \in E} |f(x)|$$

Definition 2.1.1 : Let E be a normed space, an affine hyperplane is a subset of E . denoted by H , and defined by:

$$H = \{x \in E : f(x) = \alpha\}$$

where α is a real number and f is a linear form in E . sometimes we write:

$$[H = \alpha]$$

Definition 2.1.2 : Let A be a set, we say that A is convex, if for all $x, y \in A$ and for all $t \in [0, 1]$.

$$\implies tx + (1 - t)y \in A$$

Theorem 2.1.3 : *An affine hyperplane.*

$$H = \{x \in E : f(x) = \alpha\}$$

is closed if and only if f is continuous.

Definition 2.1.3 : *Let A, B be two subsets of a vector space E , we say that $H = \{x \in E : f(x) = \alpha\}$ separates A and B , if:*

$$f(x) \geq \alpha, \quad \forall x \in A$$

and

$$f(x) \leq \alpha, \quad \forall x \in B$$