Contents

Chapter 1

Overview of Hilbert and Banach spaces

Lecture 1: Overview 08:22 AM Tue, Sep 23 2025

Let E be a given set, and let $\mathbb{K} = \mathbb{C}, \mathbb{K} = \mathbb{R}$ be the know fields. let us define two operations on $E \times E$ and $\mathbb{K} \times E$ by the :

$$\begin{aligned} + : \quad E \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y \end{aligned}$$

$$\begin{aligned} \cdot : \quad \mathbb{K} \times E &\longrightarrow E \\ (\lambda, x) &\longmapsto \lambda \cdot x \end{aligned}$$

Definition 1.0.1 (Vector space) : *A vector space E over $\mathbb{K}$ is a set equipped by two operations, called additions and scalar multiplication which verify the following axioms*

1. $\forall x, y \in E : \quad x + y = y + x$
2. $\forall x, y, z \in E : \quad (x + y) + z = x + (y + z)$
3. $\exists 0_E \in E : \quad x + 0_E = x$
4. $\forall x \in E, \exists y \in E : \quad x + y = 0_E$ *(denote y = -x)*
5. $\forall \lambda, \beta \in \mathbb{K}, \forall x \in E : \quad (\alpha\beta)x = \alpha(\beta x)$
6. $\forall x \in E : \quad 1_{\mathbb{K}} \cdot x = x$

compact $\iff$ closed and bounded (finite dimensional)

compact $\implies$ closed bounded (generally)

compactness is important because most important theorems use compact by assumption

more open sets = less compacts

7. $\forall x, y \in E, \forall \lambda \in \mathbb{K}: \quad \lambda(x+y) = \lambda x + \lambda y$

8. $\forall \alpha, \beta \in \mathbb{K}, \forall x \in E: \quad (\alpha+\beta)x = \alpha x + \beta x$

Definition 1.0.2 (Vector subspace) : *A subset $F \subset E$ is called a subvector space if:*

1. $0_E \subset F$

2. $\forall \lambda \in \mathbb{K}, \forall x, y \in E \implies \lambda x + y \in E$

Definition 1.0.3 (Inner product) : *An inner product on a vector space E is a function:*

$$\begin{array}{rccc} \langle\cdot\rangle : & E \times E & \longrightarrow & \mathbb{K} \\ & (x, y) & \longmapsto & \langle x, y\rangle \end{array}$$

which verifies:

1. *Linearity in the first argument :*

$$\begin{aligned} \forall x, y, z \in E, \forall \alpha \in \mathbb{K}: \quad & \langle \alpha x, y\rangle = \alpha \langle x, y\rangle \\ & \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \end{aligned}$$

2. *Conjugate symmetry:*

$$\forall x, y \in E: \quad \langle x, y\rangle = \overline{\langle y, x\rangle}$$

3. *Positive difinitness:*

$$\langle x, x\rangle \geq 0$$
$$\langle x, x\rangle = 0 \iff x = 0_E$$

Lecture 2

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Definition 1.0.4 : *Let E be a set, and d be a metric on E, we said that (E, d) is complete if and only if every cauchy sequence is convergent*

Definition 1.0.5 : *A Banach space is a complete vectorial normed space*

Example:

- $\mathbb{K}(\mathbb{R},\mathbb{C})$ *is a Banach space*
- $\mathbb{K}^n$ *is a Banach space*
- *if E is a normed space with finite dimension, then E is Banach*
- $\ell^p(\mathbb{N},\mathbb{K})$ *with* $1 \leq p \leq +\infty$ *is Banach space (Exercise).*
- *Every finite set product of Banach spaces is a Banach space too.*

Proposition 1.0.1 : *The limit of a sequence in a Banach space (if it exists) is unique.*

Proposition 1.0.2 : *Let F be a closed subset of E. if $(x_n)_{n\in\mathbb{N}} \subset F$ is convergent to $x \in E$, then $x \in F$.*

REMARK:

The importance of the notion of equivalent norms on a normed vector space is:
Let $\|.\|_1$ and $\|.\|_2$ be two equivalent norms on $V = E$. one has:

$$(x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_1 \iff (x_n)_{n\in\mathbb{N}} \text{ CV for } \|.\|_2$$

Proposition 1.0.3 (Definition) : *Let V and W be two norms on V (resp. W), and let $f : V \longrightarrow W$ be a function. the following statments are equivalent:*

1. *f is continuous from V into W*
2. *and:*
$$\forall x_0 \in V, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in V : \quad \|x - x_0\| \leq \delta \implies \|f(x) - f(x_0)\| \leq \varepsilon$$
3. *and:*
$$\forall x \in V, \forall (x_n)_{n\in\mathbb{N}} \in V : \quad x_n \to x \implies f(x_n) \to f(x)(\|f(x_n - f(x))\| \to 0 \quad n \to \infty)$$

REMARK:

On finite dimension vector space, all the norm are equivalents.

1.1 Duality

Let V and W be two normed vector be two normed vector spaces

Proposition 1.1.1 : *Let $A : V \longrightarrow W$ be a linear function (Map). The following are equivallent:*

1. *A is continuous on V*
2. *A is continuous at 0*
3. *$\exists c > 0$: $\|A(v)\| \leq c\|v\| \quad \forall v \in V$*

Definition 1.1.1 : *We denote by $L(V, W)$ be the set of all linear functions, we denote $\mathcal{L}(V, W)$ the set of all continuous linear functions*

Proposition 1.1.2 : *$L(V, W)$ and $\mathcal{L}(V, W)$ are vector spaces*

Proposition 1.1.3 : *Let V be a normed space, and W be a Banach space then $\mathcal{L}(V, W)$ is Banach space.*

Definition 1.1.2 : *Let $(x_n)_{n\in\mathbb{N}}$ be sequence in a normed vector space E.*

- *we say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums $s_n = \sum_{k=1}^{n} x_k$ converges.*
- *we say that the series is normally (absolutely) convergent if the series of real positive numbers $\sum_{n=1}^{\infty} \|x_n\|$ converges*

Proposition 1.1.4 : *Let E be a normed vector space. The following are equivallents:*

1. *E is Banach.*
2. *Every normally convergent series is convergent.*

Definition 1.1.3 : *Let E and F be two normed vector spaces, A function (mapping) from E into F is called linear isomorphism if its linear, continuous bijective and its inverse is continuous (naturally linear)*

We denote by $\mathrm{ISO}(E,F)$ the set of all linear isomporphisms mapping from E into F.

Theorem 1.1.5 : *Let E and F be Banach spaces, then $ISO(E,F)$ is an open subset of $\mathcal{L}(E,F)$ and the map $u \longrightarrow u^{-1}$ from $ISO(E,F)$ into $ISO(F,E)$ is continuous. More, one has if $u_0 \in ISO(E,F)$ and $u \in B(u_0, \frac{1}{\|u_0^{-1}\|})$, then $u \in ISO(E,F)$, and:*

$$u^{-1} = \sum_{n=1}^{\infty} (id_E - u_0^{-1}u)^n \cdot u_0^{-1}$$

Corollary 1.1.6 (Von Neumann) : *Let E be a Banach space, and $u \in \mathcal{L}(E) = \mathcal{L}(E,E)$. if $\|u\| < 1$, then $id_E - u$ belongs to $ISO(E,E) = ISO(E)$ and it's inverse is given by:*

$$(Id_E - u)^{-1} = \sum_{n=1}^{\infty} u^n$$

NOTATIONS: $E' = \mathcal{L}(E)$, and $E^* = L(E)$.

1. E' toplogical dual space of E.
2. E^* algebraic dual space of E.

Proposition 1.1.7 : *Let E be a Banach space E_0 be normed vector space. let $p : E \longrightarrow E_0$ a continuous linear surjection mapping. If there exists $c > 0$ such that $\forall y_0 \in E_0, \exists x \in E$ such that $f(x) = y_0$ and $\|x\| \leq c\|y_0\|$ then E_0 is Banach.*

Theorem 1.1.8 : *Let E be a Banach space and let F be a closed subspace of E. Then $E_{|F}$ is Banach too.*

Proposition 1.1.9 : *Let $A \subset E$. A is compact if from every sequence of elements of A, we can extract a subsequence converging to an element in A.*

Theorem 1.1.10 (Riesz) : *The closed unit ball is compact if and only if $\dim E < +\infty$,*

Chapter 2

Compactness

Lecture 3 08:12 AM Tue, Oct 07 2025

Definition 2.0.1 (Bolzano-Weirstrass Property) : *A set $A \subset V$, is compact if from every sequence of A we can extract a subsequence which is convergent in A.*

Theorem 2.0.1 (Riesz) : *The closed unit ball of V is compact if and only if dimension of V is finite.*

2.1 Equicontinuity

Definition 2.1.1 : *Let (X, d_1) and (Y, d_2) be two metric spaces.*
Consider $F \subset \{f : X \longrightarrow Y\}$, we say that F is equicontinuous at $x_0 \in X$. if:

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad d_1(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon, \quad \forall f \in F$$

Definition 2.1.2 : *If F is equicontinuous, at each point of X, then we say that F is equicontinuous on X.*

Theorem 2.1.1 (Ascoli-Arzela Theorem) : *Let (X, d) be a compact metric space. A family F of real valued continuous functions on X is relatively compact in $\mathcal{C}(X, \mathbb{R})$ with the sup norm if and only if:*

1. *F is uniformally bounded*
2. *F is equicontinuous on X*

Theorem 2.1.2 (Bernestein-Weirstrass) : *Let $[a, b]$ and consider $f : [a, b] \longrightarrow \mathbb{R}$ continuous function, then there exists a sequence of polynomial functions $(f_n)_{n\in\mathbb{N}}$ such that f_n converges uniformally to f on $[a, b]$.*

Theorem 2.1.3 (Riesz representation) : *Let H be Hilbert space, and consider:*

$$H' = \{f : H \longrightarrow \mathbb{C} : \quad f \text{ continuous } \}$$

then:

$$\forall f \in H', \quad \exists ! y \in H : \quad f(x) = \langle x, y\rangle$$

Theorem 2.1.4 : *Let $(E, \|.\|)$ be a normed vector space, then there exists an inner product on E, if and only if $\|.\|$ obeys the parallelogram law.*

$$✎\ \|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

2.2 Hahn-Banach Theorems

Theorem 2.2.1 : *Let E be an $\mathbb{R}$-vector space and let $p : E \longrightarrow \mathbb{R}$ be a function such that:*

1. $p(\lambda x) = \lambda p(x), \quad \forall \lambda > 0, \quad \forall x \in E$
2. $p(x + y) \leq p(x) + p(y), \quad \forall x, y \in E$

Let G be a vector subspace of E, and let $g : G \longrightarrow \mathbb{R}$be a linear function such that:

$$g(x) \leq p(x), \quad \forall x \in G$$

☞ *then there exists $f : E \longrightarrow \mathbb{R}$ linear such that:*

$$\begin{cases} f(x) = g(x), & \forall x \in G \\ f(x) \leq p(x), & \forall x \in \end{cases}$$

Theorem 2.2.2 : *Let $G \subset E$ be a vector subspace of E. if $g : G \longrightarrow \mathbb{R}$ is a linear continuous function then there exists $f \in E'$. such that:*

① $f(x) = g(x), \quad \forall x \in G$

② $\|f\|_{E'} = \|g\|_{G'}$

$$\boxed{☞ \ \|f\|_{E'} = \|f\|_{\mathcal{L}(E,\mathbb{R})} = \sup_{\|x\| \leq 1, x \in E} |f(x)|}$$

Definition 2.2.1 : *Let E be a normed space, an affine hyperplane is a subset of E. denoted by H, and defined by:*

$$H = \{x \in E : \quad f(x) = \alpha\}$$

where α is a real number and f is a linear form in E. sometimes we write:

$$[H = \alpha]$$

Definition 2.2.2 : *Let A be a set, we say that A is convex, if for all $x, y \in A$ and for all $t \in [0, 1]$.*

$$\Longrightarrow \quad tx + (1 - t)y \in A$$

Theorem 2.2.3 : *An affine hyperplane.*

$$H = \{x \in E : \quad f(x) = \alpha\}$$

is closed if and only if f is continuous.

Definition 2.2.3 : *Let A, B be two subsets of a vector space E, we say that $H = \{x \in E : \quad f(x) = \alpha\}$ seperates A and B, if:*

$$f(x) \geq \alpha, \quad \forall x \in A$$

and

$$f(x) \leq \alpha, \quad \forall x \in B$$

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Theorem 2.2.4 (Hahn-Banach) : *Let $A \subset E$, $B \subset E$ be two convex subsets such that $A \cap B = \emptyset$. Assume that one of them is open. Then there exists an affine hyperplane (closed) which seperates A and B. i.e*

$$\exists f \in E', \quad \alpha \in \mathbb{K}$$

such that

$$f(x) \geq \alpha, \forall x \in A, \text{ and } f(x) \leq \alpha, \forall x \in B.$$

Proposition 2.2.5 : *Let $C \subset E$ be a nonempty convex open subset and let $x_0 \in E \backslash C$. then there exists $f \in E'$, such that*

$$f(x) < f(x_0) \qquad \forall (x \in C).$$

In particular $[f(x_0) = \alpha]$ seperates $\{x_0\}$ and C.

Theorem 2.2.6 (Hahn-Banach) : *Let $A \subset E$, $B \subset E$ be tw oconvex nonempty subsets such that $A \cap B = \emptyset$ we suppose that one is compact and the second is closed. Then there exists a closed affine hyperplane which seperates strictly A and B. i.e. there exists $f \in E'$, and $\alpha \in \mathbb{K}(\mathbb{R})$. such that*

$$f(x) < \alpha \qquad (\forall x \in A),$$

$$f(x) < \alpha, \qquad (\forall x \in B).$$

Corollary 2.2.7 : *Let F be a subspace of E, such that $\overline{F} \neq E$. Then there exists $f \in E', f \neq 0_{E'}$ where $f(x) = 0, \quad \forall x \in F$.*

We apply the second version of Geometric Hahn-Banach (second version) to $A = \overline{F}, B = \{x_0\}$, $x_0 \notin \overline{F}$. we have A is a subspace so it's convex and closed, and B is convex and compact $A \cap B = \emptyset$. `Well the idea is pretty simple; Do it as an exercise.`

Theorem 2.2.8 (Banach-Steinhauss) : *Let E and F be two Banach spaces, and let $(T_i)_{i\in I}$ be a family not necessarly countable of continuous linear functions from E into F,*

$$T_i \in \mathcal{L}(E, F).$$

Assume that

$$\sup_{i\in I} \|T_i x\| < +\infty, \qquad (\forall x \in E).$$

Then

$$\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)} < +\infty,$$

which means there exists $\exists c > 0$, such that

$$\|T_i x\| \leq c\|x\| \qquad (\forall x \in E), \quad \forall i \in I$$

Theorem 2.2.9 : *Let E, F be two Banach spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}(E,F)$ such that $\forall x \in E$, $T_n x$ converges to Tx, when n converge to $+\infty$, then we have:*

- $$\sup_n \|T_n\|_{\mathcal{L}(E,F)} < +\infty$$
- $$T \in \mathcal{L}(E,F)$$
- $$\|T\|_{\mathcal{L}(E,F)} \leq \liminf_{n\to\infty} \|T_n\|_{\mathcal{L}(E,F)}$$

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Corollary 2.2.10 : *Let G be a vector space, and let B be a subset of G. Assume that:*

$$\forall f \in G' : \quad f(B) = \{f(b) : b \in B\} \text{ is bounded} \implies B \text{ is bounded.}$$

Proof. Consider:

$$\begin{array}{rcccc} T_b : & G' & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & T_b(f) = \langle f, b\rangle = f(b) \end{array}$$

Then using (B.S) principle, we get

$$|\langle f, b\rangle| \leq c\|f\|.$$

Thus we get

$$\|b\| \leq c, \quad \forall b \in B,$$

which means that B is bounded. □

Corollary 2.2.11 : *Let G be a banach space and let B' be a subset of G' assume that:*

$$\forall x \in G : \quad \langle B', x\rangle = \{\langle f, x\rangle : f \in G'\} \text{ is bounded} \implies B' \text{ is bounded.}$$

Theorem 2.2.12 (Open mapping theorem) : *Let E, and F be two Banach spaces, and let $T \in \mathcal{L}(E,F)$ assume that T is onto, then there exists $c > 0$, such that:*

$$B_F(0,c) \subset T(B_E(0,1)) \qquad (*)$$

$(*)$ can be translated into image of open set is open.

Corollary 2.2.13 : *Let $(E, \|.\|_1)$, and $(E, \|.\|_2)$ be Banach spaces. If there exists a constant $c > 0$, such that:*

$$\|x\|_1 \leq c\|x\|_2 \quad (\forall x \in E)$$

then $\|.\|_1$, and $\|.\|_2$ are equivalents.

Corollary 2.2.14 : *Let E, and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$ assume that T is bijective, then*

$$T^{-1} \in \mathcal{L}(F, E)$$

Chapter 3

Weak topologies

3.1 Introduction

Let X be a given set, and let (Y_i, σ_i) be a familly of toplogical spaces, I is an arbitrary set of index. Let $\varphi_i : X \longrightarrow Y_i$ be a familly of maps. we search a toplogy on X, which is the smallest one, and makes $\varphi_i, \forall i \in I$ continuous. Let be τ this toplogy. we can prove that:

$$\tau = \left\{ \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \varphi_i^{-1}(w_i), i \right\}$$

defines the coarest toplogy, such that that $\varphi_i^{-1}(w_i)$ is an elemetary open of τ

✎Remark:

A neighborhood of $x \in X$ is defined by:

$$U = \bigcap_{\text{finite}} \varphi_i^{-1}(V_i)$$

where V_i is a neighborhood of $\varphi_i(x)$.

Proposition 3.1.1 : *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*

$$x_n \overset{n\to\infty,\tau}{\longrightarrow} x \iff \varphi_i\,(x_n) \overset{\forall i\in I, n+\infty}{\longrightarrow} \varphi_i(x_n)$$

Proof.

$$(\implies)\text{ Easy and known ❄}$$

$$(\impliedby)$$

We suppose that $\varphi_i(x_n) \overset{\forall i\in I, n+\infty}{\longrightarrow} \varphi_i(x)$ and we need to prove that

$$x_n \overset{n+\infty,\tau}{\longrightarrow} x$$

Let U be a neighborhood of x. then:

$$U = \bigcap_{i\in J, J\subset I, J \text{ finite}} \varphi_i(V_i)$$

Where V_i is a neighborhood of $\varphi_i(x)$. So for each $i \in J$, V_i is a neighborooh of $\varphi_i(x)$. $\square$

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Proposition 3.1.2 : *Let $(x_n)_{n\in\mathbb{N}} \subset X$, then we have:*

$$x_n \overset{\tau}{\rightarrow}_{n\to+\infty} x \iff \varphi_i(x_n) \rightarrow_{n\to+\infty} \varphi_i(x)$$

Proof. ➽ If $\varphi_i, i \in I$ is continuous, and $x_n \to x$ then $\varphi_i(x_n) \to \varphi_i(x)$ for all $i \in I$.

➽ Conversely, suppose $\varphi_i(x_n) \to \varphi_i(x)$ and do we have $x_n \overset{\tau}{\to} x$?. we have,

$$\varphi_i(x_n) \to \varphi_i(x) \iff \forall V_i \in \mathcal{V}(\varphi_i(x)), \exists N_i \in \mathbb{N} : \forall n \geq N_i \to \varphi_i(x_n) \in V_i,$$

Thus we get $x_n \in \varphi_i^{-1}(V_i)$, so $\forall n = \sup_i N_i \implies x_n \in \bigcap_i \varphi_i^{-1}(V_i)$ which is a neighborhood of x, thus $x_n \overset{\tau}{\to} x$. $\square$

Proposition 3.1.3 : *Let Z be a toplogical space, and let $\Psi : Z \longrightarrow X$ be a mapping. Then Ψ is continuous if and only if $\varphi \circ \Psi$ is continuous.*

Proof. ➽ ($\implies$) (evident). If φ_i and Ψ are continuous then $\varphi_i \circ \Psi$ is continuous. ➽ ($\impliedby$) Suppose that $\varphi_i \circ \Psi$ is continuous and φ_i is continuous, how about Ψ?. Ψ is continious $\iff$ all open of X $\implies$ inverse image by Ψ is open in Z. Let U be an open set in τ, so $U = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \varphi_i^{-1}(W_i)$, such that W_i is open in $\mathcal{O}_i$. Then

$$\begin{aligned}\Psi^{-1}(U) &= \Psi^{-1}\left(\bigcup_{\text{arb}} \bigcap_{\text{finite}} \varphi_i^{-1}(W_i)\right) \\ &= \bigcup_{\text{arb}} \bigcap_{\text{finite}} (\varphi \circ \Psi)^{-1}(W_i), \quad \text{open in } Z.\end{aligned}$$

$\square$

3.2 Weak topology

Let E be a Banach space, and let $f \in E'$. We denote by $\varphi_f : E \longrightarrow \mathbb{R}$ the linear form defined by,

$$\varphi_f(x) = f(x), \forall x \in E, f \in E'.$$

We have a familly $(\varphi_f)_{f \in E'}$ of maps from E into $\mathbb{R}$.

Remark: On E we have already a toplogy which is defined by the norm (and it's called strong toplogy). On E, we will define a new toplogy called weak, denoted by $\sigma(E, E')$ and defined by:

Definition 3.2.1 : *The weak toplogy $\sigma(E, E')$ on E is the "Cheapest", the* coarset toplogy associated to $(\varphi_f), f \in E'$, in the sense of the previous section where;

$$X = E, Y_i = \mathbb{R} \quad \forall i, I = E'.$$

Remark: We denote by

$$x_n \rightharpoonup x$$

to mean weak convergence, or

$$x_n \overset{\sigma(E,E')}{\longrightarrow} x$$

x_n converges to x weakly.

Proposition 3.2.1 : *The weak toplogy $\sigma(E, E')$ is Hausdorff.*

Proof. Let $x_1, x_2 \in E$ such that $x_1 \neq x_2$, is there $U_1 \in \mathcal{V}(x_1), U_2 \in \mathcal{V}(x_2)$ of $\sigma(E, E')$, such that $U_1 \cap U_2 = \emptyset$. Let consider $\{x_1\} = A$ and $\{x_2\} = B$. Using second theorem of H.B, there exists $\alpha \in \mathbb{R}, f \in E'$ such that

$$f(x_1) < \alpha < f(x_2)$$

Let

$$U_1 = \{x \in E : f(x) < \alpha\}$$

$$U_2 = \{x \in E : f(x) > \alpha\}$$

so

$$U_1 = f^{-1}((-\infty, \alpha))$$

$$U_2 = f^{-1}((\alpha, +\infty))$$

and $U_1 \cap U_2 = \emptyset$, also $U_1, U_2 \in \sigma(E, E')$. □

Theorem 3.2.2 : *Let $x_0 \in E$, and let $\varepsilon > 0$. Let $\{f_1, f_2, \dots, f_k\} \subset E'$. Consider,*

$$V = \left\{x \in E : |\langle f_i, x - x_0\rangle| < \varepsilon, i = \overline{1, k}\right\}$$

then V is a neighborhood of x_0 in the toplogy $\sigma(E, E')$.

Proof. One has $V = \bigcap_{i=1}^{k} \varphi^{-1}(f_i)\left((a_i - \varepsilon, a_i + \varepsilon)\right)$, where $a_i = \langle f_i, x_0\rangle$ which is open for $\sigma(E, E')$, and $x_0 \in V$. Conversely, Let U be a neighborhood of x_0 for the topology $\sigma(E, E')$. So it takes the form:

$$\bigcap_{\text{finite}} \varphi_f^{-1}(W_i) = W.$$

where $x_0 \in W$, and $W \subset U$ and W_i is open in $\mathbb{R}$. W_i is open in $\mathbb{R}$, so $\exists \varepsilon > 0$ such that $(a_i - \varepsilon, a_i + \varepsilon) \subset W_i$. as it follows that $x_0 \in W \subset U$. □

Theorem 3.2.3 : *Let E be a Banach space, and let $(x_n)_{n\in\mathbb{N}} \subset E$ be a sequence. Then:*

- $x_n \overset{\sigma(E,E')}{\longrightarrow} x \iff f(x_n) \overset{|\cdot|}{\longrightarrow}_{n+\infty} f(x), \forall f \in E.$
- $x_n \overset{\|\cdot\|}{\longrightarrow} x \implies x_n \rightharpoonup x$
- $x_n \rightharpoonup x \implies (x_n)_{n\in\mathbb{N}}$ *is bounded and* $\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|$
- $x_n \rightharpoonup x$, *and* $f_n \to f \implies \langle f_n, x_n\rangle \rightharpoonup \langle f, x\rangle$

Proof. ➻ ① Definition of coarset toplogy + Proposition 1.

➻ ② One has $x_n \rightharpoonup x \iff f(x_n) \to f(x), \forall f \in E'$.

$$|\langle f, x_n - x\rangle| \leq \|f\| \cdot \|x_n - x\|$$

when $x_n \overset{\|\cdot\|}{\to} x$, so $|\langle f, x_n - x\rangle| \to_{n+\infty} 0$, means $x_n \rightharpoonup x$. ➻ ③ $x_n \rightharpoonup x \iff f(x_n) \to f(x), \forall f \in E'$, we have $\langle f, x_n\rangle$ converges to $\langle f, x\rangle$ for all $f \in E'$. Consider $(\langle f, x_n\rangle)_{n\in\mathbb{N}}$ is convergent so bounded. Let E'' be the bidual of E, and consider $J : E \longrightarrow E''$ cannonical injection, such that $J(x)(f) = \langle f, x\rangle$. So

$$(J(x_n)(f))_{n\in\mathbb{N}} = (\langle f, x_n\rangle)_{n\in\mathbb{N}}$$

$J(x_n)$ is bounded, using uniform boundedness principle we get

$$\|J(x_n)\| < +\infty$$

but $\|J(x_n)\| = \|x_n\| < +\infty$, and

$$|\langle f, x_n\rangle| \leq \|f\|\|x_n\|$$

and thus

$$|\langle f, x\rangle| \leq \|f\| \liminf_{n\to\infty} \|x_n\|$$

which implies

$$\|x\| \leq \liminf_{n\to\infty} \|x\|.x \rightharpoonup x, f_n \xrightarrow{\|\cdot\|} f$$

$$|\langle f_n - f, x_n - x\rangle| = |\langle f_n, x_n - x\rangle - \langle f, x_n - x\rangle|$$

`Proof is not complete, next time ❄` □

Lecture 7

`08:01 AM Tue, Nov 04 2025`

Proof. `Continuation of the proof.`

$$(\text{if } x_n \rightharpoonup x, \text{ and } f_n \to f \text{ then } \langle f_n, x_n\rangle \to \langle f, x\rangle)$$

$$\begin{aligned} |\langle f_n, x_n\rangle - \langle f, x\rangle| &= |\langle f_n, x_n\rangle - \langle f, x_n\rangle + \langle f, x_n\rangle - \langle f, x\rangle| \\ &\leq |\langle f_n - f, x_n\rangle| + |\langle f, x_n - x\rangle| \\ &\leq \|x_n\|\|f_n - f\| + |\langle f, x_n - x\rangle| \to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

□

Proposition 3.2.4 : *Let C be a convex set of E. Then C is closed for the strong toplogy, if and only if C is closed for the weak toplogy.*

few people see how strong this proposition is.

Proof. Suppose that C is closed for the strong toplogy, and let us show that its closed for the weak toplogy. If C is closed for the weak toplogy, so C^c is open for the weak topology.
$(\implies)$: Let $x_0 \in \mathbb{C}^c$, C is closed, C is convex, applying Hahn-Banach theorem there exists $f \in E'$, and $\alpha \in \mathbb{R}$ such that:

$$f(x_0) < \alpha < f(X), \quad \forall x \in C.$$

from $f(x_0) < \alpha$, one can set:

$$V = \{x \in E : f(x) < \alpha\}$$

remark that $x_0 \in V$. So V is a neighborhood of x_0 in weak toplogy. we have $V \cap C = \emptyset$, which implies $V \subset C^c$. means that C^c is weak neighborhood of all it's element. this means once again that C^c is weakly open, so C is weakly closed. □

3.3 Weak star toplogy $\sigma(E', E)$

Let us consider $E' = \mathcal{L}(E, \mathbb{R})$ on E' we can easily define two toplogies.

What we did on E, we will do it on E'. The same but in reverse.

1. The strong toplogy defined by mean of norms.
2. The weak toplogy defined as in previous sections by $\sigma(E', E'')$. The definition of $\sigma(E', E'')$ can be done by:

$$\begin{array}{rccc} \varphi_f : & E' & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \varphi_x(f) = f(x) = \langle f, x \rangle \end{array}$$

when x varies in E, we get a familly of $(\varphi_x)_{x \in E}$, and the weak toplogy on $E'(\sigma(E', E''))$ is the easier toplogy defined on E' which makes all $(\varphi_x)_x$ continuous in the sense (definition).

✎Remark:

In general one can identify E to a proper part of E''. In some cases we have by identification $E = E''$ (which is called reflexive space). examples:

- $(L^p)'' = L^p \quad 1 < p < +\infty)$ reflexive .
- $(C_0)'' \equiv \ell^{+\infty}$
- Any Hilbert space is reflexive.

✎Remark:

Since $E \subset E''$, that the $\sigma(E', E)$ is weaker than $\sigma(E', E'')$.

Definition 3.3.1 : *We call weak star toplogy defined on E', the toplogy $\sigma(E', E)$ which is associated to the familly $(\varphi_x)_x$ in the sense of the previous coarser toplogy.*

Proposition 3.3.1 : *The toplogy $\sigma(E', E)$ is Hausdorff.*

Proof. Indeed, let $f_1, f_2 \in E'$: $f_1 \neq f_2 \implies \exists x_0 \in E : f_1(x_0) \neq f_2(x_0)$, there there exists $\alpha \in \mathbb{R}$:

$$f_1(x_0) < \alpha < f_2(x_0)$$

let

$$V_1 = \{f \in E' : f(x_0) < \alpha\} = \varphi_{x_0}^{-1}(] - \infty, \alpha[)$$

and

$$V_2 = \{f \in E' : f(x_0) > \alpha\} = \varphi_{x_0}^{-1}([\alpha, +\infty[)$$

we have $f_1 \in V_1, f_2 \in V_2$. □

Proposition 3.3.2 : *Let $f_0 \in E'$, if there exists $\{x_1, \ldots, x_k\}$ and $\varepsilon > 0$, the set :*

$$V = \left\{f \in E' : |\langle f - f_0, x_i\rangle| < \varepsilon, i = \overline{1,k}\right\}.$$

is a neighborhood of f_0 in $\sigma(E', E)$.

Notation: We denote by $f_n \overset{*}{\rightharpoonup} f$ for $\sigma(E', E)$.

Theorem 3.3.3 : *Let $(f_n)_{n\in\mathbb{N}_0} \subset E'$, and let $(x_n)_{n\in\mathbb{N}_0} \subset E$. we have:*

① $f_n \overset{*}{\rightharpoonup} f \iff \langle f_n, x\rangle \overset{n+\infty}{\longrightarrow} \langle f, x\rangle$.

② $f_n \to f \implies f_n \rightharpoonup f \implies f_n \overset{*}{\rightharpoonup} f$.

③ $f_n \overset{*}{\rightharpoonup} f \implies$ *that $(f_n)_{n\in\mathbb{N}}$ is bounded and:* $\|f\| \leq \lim_{n\to\infty} \inf \|f_n\|$.

④ $f_n \overset{*}{\rightharpoonup} f$, *and* $x_n \to x$.

Then:

$$\langle f_n, x_n\rangle \to \langle f, x\rangle$$

but if $f_n \overset{*}{\rightharpoonup} f$, $x_n \rightharpoonup x$ we can not conclude.

Corollary 3.3.4 (Algebraic) : *Let X be a vector space. and let $\Psi, \varphi_1, \ldots, \varphi_k$ be $k+1$ linear functionals defined on X, such that:*

$$\left(\varphi_i(v) = 0, i = \overline{1,k}\right) \implies \Psi(v) = 0, \quad \forall v \in X$$

then there exists $\lambda_1, \ldots, \lambda_k$, such that:

$$\Psi = \sum_{i=1}^{k} \lambda_i \varphi_i$$

Proof. Let us consider:

$$\begin{array}{rccc} F: & X & \longrightarrow & \mathbb{R}^{k+1} \\ & u & \longmapsto & F(u) = (\Psi(u), \varphi_1(u), \ldots, \varphi_k(u)) \end{array}$$

remark that $(1, 0, \ldots) \notin F(X)$. Let us denote by $x_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{k+1}$. We apply Hahn banach for $\{x_0\}$, and $F(X)$. so, there exists $\alpha \in \mathbb{R}$ and $f \in (\mathbb{R}^{k+1})'$. □