

Chapter 1

Overview of Hilbert and Banach spaces

Lecture 1: Overview

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Let E be a given set, and let $\mathbb{K} = \mathbb{C}, \mathbb{K} = \mathbb{R}$ be the known fields. Let us define two operations on $E \times E$ and $\mathbb{K} \times E$ by the :

$$\begin{aligned} + : E \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y \\ \cdot : \mathbb{K} \times E &\longrightarrow E \\ (\lambda, x) &\longmapsto \lambda \cdot x \end{aligned}$$

Definition 1.0.1 (Vector space) : A vector space E over \mathbb{K} is a set equipped by two operations, called additions and scalar multiplication which verify the following axioms

1. $\forall x, y \in E : x + y = y + x$
2. $\forall x, y, z \in E : (x + y) + z = x + (y + z)$
3. $\exists 0_E \in E : x + 0_E = x$
4. $\forall x \in E, \exists y \in E : x + y = 0_E$ (denote $y = -x$)
5. $\forall \lambda, \beta \in \mathbb{K}, \forall x \in E : (\alpha\beta)x = \alpha(\beta x)$

compact \iff
closed and
bounded
(finite dimensional)

compact
 \implies closed
bounded (generally)

compactness
is important
because most
important
theorems use
compact by
assumption

more open
sets = less
compacts

6. $\forall x \in E : 1_{\mathbb{K}} \cdot x = x$
7. $\forall x, y \in E, \forall \lambda \in \mathbb{K} : \lambda(x + y) = \lambda x + \lambda y$
8. $\forall \alpha, \beta \in \mathbb{K}, \forall x \in E : (\alpha + \beta)x = \alpha x + \beta x$

Definition 1.0.2 (Vector subspace) : A subset $F \subset E$ is called a subvector space if:

1. $0_E \subset F$
2. $\forall \lambda \in \mathbb{K}, \forall x, y \in E \implies \lambda x + y \in F$

Definition 1.0.3 (Inner product) : An inner product on a vector space E is a function:

$$\begin{aligned}\langle \cdot \rangle : E \times E &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto \langle x, y \rangle\end{aligned}$$

which verifies:

1. Linearity in the first argument :

$$\begin{aligned}\forall x, y, z \in E, \forall \alpha \in \mathbb{K} : \quad \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \\ \langle x + y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle\end{aligned}$$

2. Conjugate symmetry:

$$\forall x, y \in E : \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

3. Positive definiteness:

$$\begin{aligned}\langle x, x \rangle &\geq 0 \\ \langle x, x \rangle = 0 &\iff x = 0_E\end{aligned}$$

Lecture 2

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Definition 1.0.4 : Let E be a set, and d be a metric on E , we said that (E, d) is complete if and only if every cauchy sequence is convergent

Definition 1.0.5 : A Banach space is a complete vectorial normed space

Example:

- $\mathbb{K}(\mathbb{R}, \mathbb{C})$ is a Banach space
- \mathbb{K}^n is a Banach space
- if E is a normed space with finite dimension, then E is Banach
- $\ell^p(\mathbb{N}, \mathbb{K})$ with $1 \leq p \leq +\infty$ is Banach space (Exercise).
- Every finite set product of Banach spaces is a Banach space too.

Proposition 1.0.1 : The limit of a sequence in a Banach space (if it exists) is unique.

Proposition 1.0.2 : Let F be a closed subset of E . if $(x_n)_{n \in \mathbb{N}} \subset F$ is convergent to $x \in E$, then $x \in F$.

REMARK:

The importance of the notion of equivalent norms on a normed vector space is:

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on $V = E$. one has:

$$(x_n)_{n \in \mathbb{N}} \text{ CV for } \|\cdot\|_1 \iff (x_n)_{n \in \mathbb{N}} \text{ CV for } \|\cdot\|_2$$

Proposition 1.0.3 (Definition) : Let V and W be two norms on V (resp. W), and let $f : V \rightarrow W$ be a function. the following statements are equivalent:

1. f is continuous from V into W

2. and:

$$\forall x_0 \in V, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in V : \|x - x_0\| \leq \delta \implies \|f(x) - f(x_0)\| \leq \varepsilon$$

3. and:

$$\forall x \in V, \forall (x_n)_{n \in \mathbb{N}} \in V : x_n \rightarrow x \implies f(x_n) \rightarrow f(x) (\|f(x_n) - f(x)\| \rightarrow 0 \quad n \rightarrow \infty)$$

REMARK:

On finite dimension vector space, all the norm are equivalents.

1.1 Duality

Let V and W be two normed vector be two normed vector spaces

Proposition 1.1.1 : Let $A : V \rightarrow W$ be a linear function (Map). The following are equivalent:

1. A is continuous on V
2. A is continuous at 0
3. $\exists c > 0 : \|A(v)\| \leq c\|v\| \quad \forall v \in V$

Definition 1.1.1 : We denote by $L(V, W)$ be the set of all linear functions, we denote $\mathcal{L}(V, W)$ the set of all continuous linear functions

Proposition 1.1.2 : $L(V, W)$ and $\mathcal{L}(V, W)$ are vector spaces

Proposition 1.1.3 : Let V be a normed space, and W be a Banach space then $\mathcal{L}(V, W)$ is Banach space.

Definition 1.1.2 : Let $(x_n)_{n \in \mathbb{N}}$ be sequence in a normed vector space E .

- we say that the series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums $s_n = \sum_{k=1}^n x_k$ converges.
- we say that the series is normally (absolutely) convergent if the series of real positive numbers $\sum_{n=1}^{\infty} \|x_n\|$ converges

Proposition 1.1.4 : Let E be a normed vector space. The following are equivalents:

1. E is Banach.
2. Every normally convergent series is convergent.

Definition 1.1.3 : Let E and F be two normed vector spaces, A function (mapping) from E into F is called linear isomorphism if its linear, continuous bijective and its inverse is continuous (naturally linear)

We denote by $\text{ISO}(E, F)$ the set of all linear isomorphisms mapping from E into F .

Theorem 1.1.5 : Let E and F be Banach spaces, then $\text{ISO}(E, F)$ is an open subset of $\mathcal{L}(E, F)$ and the map $u \rightarrow u^{-1}$ from $\text{ISO}(E, F)$ into $\text{ISO}(F, E)$ is continuous. More, one has if $u_0 \in \text{ISO}(E, F)$ and $u \in B(u_0, \frac{1}{\|u_0^{-1}\|})$, then $u \in \text{ISO}(E, F)$, and:

$$u^{-1} = \sum_{n=1}^{\infty} (\text{id}_E - u_0^{-1}u)^n \cdot u_0^{-1}$$

Corollary 1.1.6 (Von Neumann) : Let E be a Banach space, and $u \in \mathcal{L}(E) = \mathcal{L}(E, E)$. if $\|u\| < 1$, then $\text{id}_E - u$ belongs to $\text{ISO}(E, E) = \text{ISO}(E)$ and it's inverse is given by:

$$(\text{id}_E - u)^{-1} = \sum_{n=1}^{\infty} u^n$$

NOTATIONS: $E' = \mathcal{L}(E)$, and $E^* = L(E)$.

1. E' toplogical dual space of E .
2. E^* algebraic dual space of E .

Proposition 1.1.7 : Let E be a Banach space E_0 be normed vector space. let $p : E \rightarrow E_0$ a continuous linear surjection mapping. If there exists $c > 0$ such that $\forall y_0 \in E_0, \exists x \in E$ such that $f(x) = y_0$ and $\|x\| \leq c\|y_0\|$ then E_0 is Banach.

Theorem 1.1.8 : Let E be a Banach space and let F be a closed subspace of E . Then $E|_F$ is Banach too.