**Normed Vector Spaces Lecture**

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Comprehensive document for

The Subject Normed Vector Spaces

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Last Update : 2025-05-08

# CONTENTS

# 1 THE CONCEPT OF A NORM ON A REAL OR COMPLEX VECTOR SPACE

For all what follows $\mathbb{K}$ denotes one of the two feild $\mathbb{R}$ or $\mathbb{C}$ and $|.|$ denotes the absolute value if $\mathbb{K} = \mathbb{R}$ and the modulus if $\mathbb{K} = \mathbb{C}$.

## 1.1 Norm on a $\mathbb{K}$-vector space

**Definition 1.1.1: Norm**

Let $E$ be a $\mathbb{K}$-vector space, we call a norm on $E$ every map $\|.\| : E \longrightarrow [0, \infty)$ satisfiying the following properties:

(i) $\forall x \in E : \quad \|x\| = 0 \implies x = 0_E$

(ii) $\forall x \in E, \forall \lambda \in \mathbb{K} : \quad \|\lambda x\| = |\lambda| \, \|x\|$

(iii) $\forall x, y \in E : \quad \|x + y\| \leq \|x\| \|y\|$

**Remark**

- A $\mathbb{K}$-vector space E equipped with a norm $\|.\|$ is called **a normed vector spcae** (abbreviated to N.V.S), it is written $(E, \|.\|)$ or simply $E$ if there is no ambiguity about the norm $\|.\|$
- The equivallence " $\iff$ " in (i) can be replaced by the implication " $\implies$ " because the implication $(x = 0_E \implies \|x\| = 0)$ can be obtained from property (ii) by taking $\lambda = 0$
- Inequality in (iii) is called **"The Triangle Inequlity"** or **"The Convex Inequality"**, it is equivalent to say that the norm $\|.\|$ is a convex function on $E$, that is:

$$\forall t \in (0, 1), \forall x, y \in E : \quad \|tx + (1 - t)y\| \leq t\|x\| + (1 - t)\|y\|$$

Indeed, we have:

$$\|tx + (1-t)y\| \leq \|tx\| + \|(1-t)y\|$$
$$\leq |t|\,\|x\| + |1-t|\,\|y\| \qquad \leq t\|x\| + (1-t)\|y\|$$

$t = \frac{1}{2}$: we get it

- if $E$ is a $\mathbb{K}$-vector space and $\|.\| : E \longrightarrow [0, \infty)$ satisfies just properties (i) and (ii) then $\|.\|$ is called a **seminorm on** $E$ (so seminorm could assing 0 to non-zero vectors), the pair $(E, \|.\|)$ is then called a **Seminormed Vector Space**.

## 1.2 Metric Associated to a Norm

Definition 1.2.1:

Let $(E, \|.\|)$ be a N.V.S, Define:

$$\begin{array}{rccc} d: & E^2 & \longrightarrow & [0, \infty) \\ & (x, y) & \longmapsto & d(x, y) = \|x - y\| \end{array}$$

we can easily verify that $d$ is a metric on $E$, and it is called **The Metric Associated To The Norm** $\|.\|$ or **The Generated Metric By The Norm**

Remark

- Thanks to the concept of the metric generated by a norm, a N.V.S is seen as a particular metric space, which is a particular topological space.
- The definition of the open ball, a closed ball, a sphere, an open set, a closed set, a neighborhood,the interior of a set, limit, the closure of a set,etc... in a N.V.S are those related to the metric generated by the norm.
- Every metric $d$ generated by a norm (in a given N.V.S $E$) is invarient by translation, that is:

$$\forall x, y, z \in E: \quad d(x + z, y + z) = d(x, y)$$

- There exist natural metric that are not generated by any norm (like discrect distance).

## 1.3 Examples of some concepts on a N.V.S derived from its metric structure

1. Let $(E, \|.\|)$ be a N.V.S, $(x_n)_{n\in\mathbb{N}}$ be a sequence of $E$, and $x$ be a vector of $E$.

    - We say that $(x_n)_{n\in\mathbb{N}}$ converges to $x$ if we have $\lim_{n\to\infty}\|x_n - x\| = 0$, equivallently:

    $$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \quad \text{s.t} \quad \forall n \in \mathbb{N} \quad (n > N \implies \|x_n - x\|)$$

    in this case we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ on $n \to \infty$

    - We say that $(x_n)_{n\in\mathbb{N}}$ is a cuachy sequence if we have $\lim_{p,q\to\infty}\|x_p - x_q\| = 0$, equivalantly:

    $$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ \forall p, q \in \mathbb{N} \quad (p > q > N \implies \|x_p - x_q\| < \varepsilon)$$

2. Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two N.V.S over the same field $\mathbb{K}$, $f : E \longrightarrow F$ be a map from $E$ to $F$, Let $x_0 \in E$ and $y_0 \in F$,

    - We say that $f(x)$ tends to $y_0$ when x tends to $x_0$ (and we write $\lim_{x\to x_0} f(x) = y_0$ or $f(x) \to y_0$ as $x \to x_0$ )

    $$\begin{cases} \forall \varepsilon > 0,\ \exists \eta > 0, \quad \text{s.t} \quad \forall x \in E : \\ \|x - x_0\|_E < \eta \implies \|f(x) - y_0\|_F < \varepsilon \end{cases}$$

    - We say that $f$ is continious at $x_0$ if we have:

    $$\lim_{x\to x_0} f(x) = f(x_0)$$

    that is,

    $$\begin{cases} \forall \varepsilon > 0,\ \exists \eta > 0, \quad \text{s.t} \quad \forall x \in E : \\ \|x - x_0\|_E < \eta \implies \|f(x) - f(x_0)\|_F < \varepsilon \end{cases}$$

    - We say that $f$ is continious on E if it is continious at all vector $x$ of $E$.

    - We say that $f$ is uniformally continious on E if we have $\forall \varepsilon > 0, \exists \eta > 0$ such that $\forall x, y \in E$:

    $$\|x - y\|_E < \eta \implies \|f(x) - f(y)\|_F < \varepsilon$$

    - Let $M > 0$, wz say that $f$ is M-lipchitz if we have:

    $$\forall x, y \in E : \quad \|f(x) - f(y)\|_F \leq M\|x - y\|_E$$

    - We say that $f$ is a contraction if it is M-lipchitz for some constant $M \in (0, 1)$, Note that/

    $$\text{Lipchitz Continious} \implies \text{Uniformally Continious} \implies \text{Continious}$$

## 1.4 Equivalent and Topologically Equivalent Norms

**Definition 1.4.1:**

Let $E$ be a $\mathbb{K}$-vector space and $N_1$ and $N_2$ two norms on $E$:

- We say that $N_1$ and $N_2$ are topologically equivallent if their associated ,etrics are topologically equivalent, that is they induce the same topology on $E$.
- We say that $N1$ and $N2$ are equivalent if their associated metrics are equivalent, that is their exist $\alpha, \beta > 0$ such that:
$$\alpha N_1 \leq N_2 \leq \beta N_1 \qquad (\text{i.e} \quad \forall x \in E: \quad \alpha N_1(x) \leq N_2(x) \leq \beta N_1(x))$$

**Remark**

- It is known that two equivalent metrics (on a given non-empty set) are topologically equivalent but the inverse is generally false.
- Note that in a $\mathbb{K}$-vector space, the two concepts ”equivalent norms” and ”topologically equivalent norms” coincide
- We will show later that two norms on a $\mathbb{K}$-vector space are topologically equivalent if and only if they are equivalent.
- We will show also that: Any two norms on a finite-dimensional vetor space(over $\mathbb{K}$) are equivalent

## 1.5 Examples of norms on $\mathbb{R}^n$ and $\mathbb{C}^n$

**Example**

1. In $\mathbb{R}$ (Considered as $\mathbb{R}$ vector space), the usual norm is the absolute value, in $\mathbb{C}$ (Considered as $\mathbb{C}$ vector space), the usual norm is the modulus.
2. Let $n \geq 2$ be an integer, we may define on $\mathbb{K}^n$ (Considered as $\mathbb{K}$ vector space), several norms including. $\{\|\|_1, \|\|_2, \|\|_p\}$, with $(p \geq 1)$, and $\|\|_\infty$, the norms we just stated are the

most widely used, they are defined by :

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

$$\|x\|_\infty := \max_{1\leq i\leq n} |x_i|$$

Both in $\mathbb{R}^n$ and in $\mathbb{C}^n$ $(n \in \mathbb{N})$, the norm $\|.\|_2$ is called the euclidean norm, and the norm $\|.\|_p$ $(p \geq 1)$ is called the Holder norm of exponent $p$ (or simpley, the $p$-norm).
Remark that $\|.\|_1$ and $\|.\|_2$ are special cases of $\|.\|_p$. We can also show that :

$$\lim_{n\to\infty} \|.\|_p = \|.\|_\infty$$

Further, it's easy to show that the norms

$$\|.\|_p \quad \forall p \geq 1 \text{ are equivallent}$$

Prove that $\|.\|_1 \sim \|.\|_\infty$ and $\|.\|_2 \sim \|.\|_1$. (Hint : $n\left((\max|x_i|)^2\right)^{1/2}$)

Furthermore, its easy to show that the norms $\|.\|_p\,(p \geq 1)$, are all equivalent (they are even equal for $n = 1$), To show that $\|.\|_p\,(p \geq 1$ arbitrary ), is really a norm on $\mathbb{K}^n$, only the triangle inequality that poses a problem, (The special cases $p = 1$, and $p = \infty$ are easy), we fix this problem by solving the following exercise below !

**Consider the following exercise :**

Let $n$ be a positive integer and let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$.

(i) By using the connexity of the exponential function, show that for all positive real numbers $a$ and $b$, we have

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

(Known as The Young Inequality)

(ii) Deduce that for all positive real numbers $x_1, x_2, ..., x_n, y_1, y_2, \ldots, y_n$, we have :

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}$$

(Known as the Holder Inequality)

(iii) Deduce that for all positive real numbers $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$, we have :

$$\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n}x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n}y_i^p\right)^{1/p}$$

(Called the Minkowski Inequality)

(iv) Conclude that $\|.\|$ is really a norm on $\mathbb{K}^n$ where ($K = \mathbb{R}$ or $\mathbb{C}$)

**Solution :**

(i) Since the function $u \to e^u$ is convex on $\mathbb{R}$ because $\left((e^u)'' = e^u > 0\right)$, then we have for all $t \in [0,1]$ and for all $x, y \in \mathbb{R}$ :

$$e^{tx+(1-t)y} \leq te^x + (1-t)\,e^y$$

We apply the above for $t = \frac{1}{p}$ so $(1-t) = 1 - \frac{1}{p} = \frac{1}{q}$, and for $x, y$ such that $e^x = a^p$ (i.e. $x = p\ln(a)$ ), and $e^y = b^q$ (i.e. $y = q\ln(b)$ ) we obtain that :

$$\begin{aligned}(a^p)^{1/p}(b^q)^{1/q} &\leq \frac{a^p}{p} + \frac{b^q}{q}\\ a \cdot b &\leq \frac{a^p}{p} + \frac{b^q}{q}\end{aligned}$$

as required.

(ii) Let $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n > 0$, for $i \in \{1, 2, \ldots, n\}$, by applying the Young inequality proved above for $a = \frac{x_i}{(\sum_{i=1}^{n} x_j^p)^{1/p}}$ and $b = \frac{y_i}{\left(\sum_{j=1}^{n} y_j^q\right)^{1/q}}$ we get :

$$\frac{x_i y_i}{\left(\sum_{j=1}^{n} x_j^p\right)^{1/p}\left(\sum_{j=1}^{n} y_j^q\right)^{1/q}} \leq \frac{1}{p}\left[\frac{x_i^p}{\sum_{j=1}^{n} x_j^p}\right] + \frac{1}{q}\left[\frac{y_i^q}{\sum_{j=1}^{n} y_j^q}\right]$$

Next, by taking the summation from $i = 1$ to $n$, in the two sides of his last inequality , we get :

$$\frac{\sum_{i=1}^{n} x_i y_i}{\left(\sum_{j=1}^{n} x_j^p\right)^{1/p}\left(\sum_{j=1}^{n} y_j^q\right)^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\sum_{j=1}^{n} y_j^q\right)^{1/q}$$

As required

Remark that the Holder inequality generalizes, the Cauchy-Schawrtz Inequality for the usual inner product of $\mathbb{R}^n$ (take $p = q = 2$).

(iii) Let $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n > 0$, we have :

$$\begin{aligned}\sum_{i=1}^{n}(x_i+y_i)^p &= \sum_{i=1}^{n}(x_i+y_i)(x_i+y_i)^{p-1}\\ &= \sum_{i=1}^{n}x_i(x_i+y_i)^{p-1}+\sum_{i=1}^{n}y_i(x_i+y_i)^{p-1}\end{aligned}$$

Then by applying the Holder inequality, for each of the two sums $\sum_{i=1}^{n}x_i(x_i+y_i)^{p-1}$ and $\sum_{i=1}^{n}y_i(x_i+y_i)^{p-1}$ we derive that :

$$\sum_{i=1}^{n}(x_i+y_i)^p \le \left(\sum_{i=1}^{n}x_i^p\right)^{1/p}\left(\sum_{i=1}^{n}(x_i+y_i)^{(p-1)q}\right)^{1/q}+\left(\sum_{i=1}^{n}y_i^p\right)^{1/p}\left(\sum_{i=1}^{n}(x_i+y_i)^{(p-1)q}\right)^{1/q}$$

And since $(p-1)q=p$ (Because $\frac{1}{p}+\frac{1}{q}=1$), it follows that :

$$\sum_{i=1}^{n}(x_i+y_i)^p \le \left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1/q}\left(\left(\sum_{i=1}^{n}x_i^p\right)^{1/p}+\left(\sum_{i=1}^{n}y_i^p\right)^{1/p}\right)$$

$$\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1-\frac{1}{q}} \le \left(\sum_{i=1}^{n}x_i^p\right)^{1/p}+\left(\sum_{i=1}^{n}y_i^p\right)^{1/p}$$

$$\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1/p} \le \left(\sum_{i=1}^{n}x_i^p\right)^{1/p}+\left(\sum_{i=1}^{n}y_i^p\right)^{1/p}$$

(iv) The two first properties of a norm ( i.e. , $(i)$ and $(ii)$)), are clearly satisfied by $\|.\|_p$ , so it remains to shows the triangle inequality $\left(\|x+y\|_p \le \|x\|_p+\|y\|_p \quad \forall x,y \in \mathbb{K}^n\right)$. First, remark that the Minkowski Inequality (proved above), remains true for $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n > 0$ (That is if some if the $x_i$'s and $y_i$'s are zero), This can be justified by the continuity for example now, for

$$X := \begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} \quad Y := \begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix} \in \mathbb{K}^n$$

We have that :

$$\|x+y\|_p = \left(\sum_{i=1}^{n}\|x_i+y_i\|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}\underbrace{(|x_i|+|y_i|)^p}_{\in[0,\infty)}\right)^{1/p} \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$$

According to the Minkowsky Inequality we get it equal

$$\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p} = \|x\|_p+\|y\|_p$$

As required, Consequently, $\|.\|_p$ is a norm on $\mathbb{C}^n$

## 1.6 Finite product of normed vector spaces

Let $(E_1, N_1), (E_2, N_2), \ldots, (E_k, N_k) \quad (k \geq 1)$, be normed vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and set $E := E_1 \times E_2 \times \ldots \times E_k$.

We may define on $E$ several norms which are expressed in terms of $N_1, N_2, \ldots, N_k$. Among these norms we set :

- $\|.\|_1 : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_1 := \sum_{i=1}^k N_i(x_i)$
- $\|.\|_2 : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_2 := \left(\sum_{i=1}^k N_i(x_i)^2\right)^{1/2}$
- $\|.\|_p : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_p := \left(\sum_{i=1}^k N_i(x_i)^p\right)^{1/p}$
- $\|.\|_\infty : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_\infty := \max_{1 \leq i \leq k} N_i(x_i)$

We can show that all the norms $\|.\|_p \quad (1 \leq p \leq \infty)$ are equivalent, and that the common toplogy generated by them is the product topology of $E$. This allows us to affirm that, A toplogical product of a finite number of N.V.S is a N.V.S.

Note that this last result is in general false for a toplogical product of an infinite number of normed vector spaces.

## 1.7 Exampels of norms of an infinite-dimensional vector space

Let $a, b \in \mathbb{R}$ with $a < b$, The $\mathbb{R}-$vector space

$$E := \mathcal{C}^a\left([a,b], \mathbb{R}\right) \quad \text{Contituted of continiuous functions on } [a,b]$$

Can be equipped with several importants norms, including $\|.\|_1, \|.\|_2, \|.\|_p \quad (p \geq 1)$ and $\|.\|_\infty$

## 1.8 Examples of norms of an infinite dimensional vector spaces

let $a, b \in \mathbb{R}$ with $a < b$. The $\mathbb{R}$-vector space $E := \mathcal{C}^0\left([a,b], \mathbb{R}\right)$, (Constituted of continious real functions on $[a,b]$). can be equipped with several important norms, including $\|.\|_1, \|.\|_2, \|.\|_p \quad (p \geq 1)$, and $\|.\|_\infty$ defined by

$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

$$\|f\|_2 = \sqrt{\int_a^b |f(t)^2|\, dt}$$

$$\|f\|_p = \left(\int_a^b |f(t)|^p\, dt\right)^{1/p}$$

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)| = \max_{t\in[a,b]} |f(t)|$$

The norm $\|.\|_2$ is called the euclidean norm, the norm $\|.\|_p$ with $(p \geq 1)$ is called the Holder norm of exponent $p$ (or simply the $p$-norm), and the norm $\|.\|_\infty$ is called the uniform norm say that a sequence of functions $(f_n)_{n\in\mathbb{N}}$, belonging to $\mathcal{C}^0\left([a,b], \mathbb{R}\right)$, converges to $f \in \mathcal{C}^0\left([a,b], \mathbb{R}\right)$ in the sense of the norm $\|.\|_\infty$ is equivalent to say that $(f_n)_{n\in\mathbb{N}}$ converges uniformaly to $f$ on $[a,b]$, we can show that we have $\lim_{p\to\infty} \|.\|_p = \|.\|_\infty$ Further, it's important to note that these norms are not equivalent.

**Exercise :**

Show that $\|.\|_p \quad (p \geq 1)$, is really a norm on $E := \mathcal{C}^0([a,b], \mathbb{R})$.

*Hint : Take inspiration from the solution of the previous exercise.*

## 1.9 Banach Spaces :

**Definition 1.9.1:**

A banach space is a normed $\mathbb{K}$-vector space which is complete for the metric induced by it's norm.

**Example**

In finite dimensional, let $n \in \mathbb{N}$ :

$$\mathbb{R} - NVS \quad (\mathbb{R}, \|.\|) \quad (\mathbb{R}^n, \|.\|_1) \quad (\mathbb{R}^n, \|.\|_2) \quad (\mathbb{R}^n, \|.\|_\infty)$$

they are all banach spaces, the same is for the :

$$\mathbb{C} - NVS \quad (\mathbb{C}, \|.\|) \quad (\mathbb{C}^n, \|.\|_1) \quad (\mathbb{C}^n, \|.\|_2) \quad (\mathbb{C}^n, \|.\|_\infty)$$

Later, we will show a more general result stating that :

*Any finite-dimensional normal vector space is Banach*

**Theorem 1.9.1:**

The $\mathbb{R}$-vector space $E := \mathcal{C}^0([0,1]), \mathbb{R}$, equipped with it's uniform norm $\|.\|_\infty$, is Banach.

*Proof.* We have to show that $(E, \|.\|_\infty)$ is complete, that is every cauchy sequence of $(E, \|.\|_\infty)$ converges in $(E, \|.\|_\infty)$, so let $(f_n)_{n\in\mathbb{N}}$ be a cauchy sequence of $(E, \|.\|_\infty)$ and let us show that it converges in $(E, \|.\|_\infty)$, By hypothesis, we have :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q > N \implies \|f_p - f_q\|_\infty < \varepsilon$$

that is (according to the definition of $\|.\|_\infty$ ) :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q > N \implies \sup_{x\in[0,1]} \left|f_p(x) - f_q(x)\right| < \varepsilon$$

or equivalently

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q > N \implies \forall x \in [0,1] : \quad \left|f_p(x) - f_q(x)\right| < \varepsilon$$

□

Property $(1)$ shows that for all $x \in [0,1]$, the real sequence $(f_n)_{n\in\mathbb{N}}$, is Cauchy in $(\mathbb{R}, \|.\|)$. But since is banach (i.e, complete) we derive that, for all $x \in [0,1]$, the real sequence $(f_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, so we can define

$$\begin{array}{rccc} f: & [0,1] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) := \lim_{n\to\infty} f_n(x) \end{array} \quad (\forall x \in [0,1])$$

on the other words, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $f$. Now we are going to show that $f \in E$ and that $(f_n)_{n\in\mathbb{N}}$ converges in $(E, \|.\|_\infty)$ to $f$ (i.e., $(f_n)_{n\in\mathbb{N}}$ converges uniformally to $f$), by taking in $(1)$.

$$q = n > N \quad \text{and} \quad p \to \infty$$

we will obtain :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}: \quad n > N \implies \forall x \in [0,1]: \quad |f_n(x) - f(x)| < \varepsilon$$

which is equivalent to

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}: \quad n > N \implies \sup_{x\in[0,1]} |f_n(x) - f(x)| \leq \varepsilon$$

Showing that, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformally to $f$ on $[0,1]$.

*Recall a theorem in* ***Analysis 3****, Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continiuous functions on a closed interval $[a,b]$ where $(a, b \in \mathbb{R}, a < b)$, that converges uniformaly to a function $f$ on $[a,b]$. Then $f$ is also continous on $[a,b]$.*

By applying this result of analysis 3, we derive that $f$ is also continuous on $[0,1]$, that is $f \in E$, and $(f_n)_{n\in\mathbb{N}}$ is convergent in $(E, \|.\|_\infty)$ to $f$, we conclude that $\left(\mathcal{C}^0\left([0,1], \mathbb{R}\right), \|.\|_\infty\right)$ is Banach.

## 1.10 Bounded subset and bounded map on N.V.S :

The concepts of "bounded subsets" and "bounded maps" (or "bounded functions"), are in general defined in a metric space, however, the use of norms allows to simplify them as stated by the following propositions :

**Theorem 1.10.1:**

A non empty subset $A$ of a N.V.S $E$ is bounded if and only if there is a positive real number $M$ such that :

$$\forall x \in A : \quad \|x\| \leq M$$

*Proof.* Let $E$ be a N.V.S and $A$ be a non empty subset of $E$.

$$( \implies )$$

Suppose that $A$ is bounded, that is $\delta(A) < +\infty$, and let $x_0 \in A$ be fixed. For all $x \in A$, we have

$$\begin{aligned} \|x\| = \|x - x_0 + x_0\| &\leq \|x - x_0\| + \|x_0\| \\ &\leq \delta(A) + \|x_0\| \end{aligned}$$

So it sufficies to take $M = \delta(A) + \|x_0\|$, to obtain the required property.

$$( \impliedby )$$

Conversly, suppose that there exist $M > 0$ so that we have

$$\forall x \in A : \quad \|x\| \leq M$$

but this is equivalent to say that

$$A \subset \overline{B}(0_E, M)$$

implying that $A$ is bounded this completes the proof of the proposition □

**Theorem 1.10.2:**

Let $X$ be a non empty set, $E$ be a N.V.S and

$$f : X \longrightarrow E$$

be a map, then $f$ is bounded if and only if $\exists M > 0$ such that :

$$\forall x \in X : \quad \|f(x)\| \leq M$$

*Proof.* By definition, we say that $f$ is bounded, it's equivalent to say that $f(X)$ is bounded, which is equivalent to say (according to the previous propsition), that $\exists M > 0$ such that :

$$\forall y \in f(X) : \quad \|y\| \leq M$$

equivalent to

$$\forall x \in X : \quad \|f(x)\| \leq M$$

This complets the proof. □

# 2 CONTINIOUS LINEAR MAPPINGS BETWEEN TWO N.V.S

**Theorem 2.0.1: Fundamental**

Let $E$ and $F$ be two N.V.S on the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $f : E \longrightarrow F$, be a linear mapping then the following properties are equivalent

(i) $f$ is continiuous on $E$

(ii) $f$ is continious at the same $x_0 \in E$

(iii) $f$ is bounded on $\overline{B}\left(0_E, 1\right)$, i.e. :

$$\exists M > 0, \forall x \in \overline{B}\left(0_E, 1\right) : \quad \|f(x)\|_F \leq M$$

(iv) $f$ is bounded on $S\left(0_E, 1\right)$

(v) $\exists M > 0$ such that :

$$\forall x \in E : \quad \|f(x)\|_F \leq M\|x\|_E$$

(vi) $f$ is Lipchitz continious

*Proof.* We will show the following implications :

$$(i) \implies (ii) \implies (iii) \implies (iv) \implies (v) \implies (vi) \implies (i)$$

$$(i) \implies (ii)$$

This is obvious

$$(ii) \implies (iii)$$

Suppose that $f$ is continious at some $x_0 \in E$, so $\exists \mu > 0$ such that :

$$\forall x \in E : \quad \|x - x_0\| < \mu \implies \|f(x) - f(x_0)\|_F < 1 \tag{2.1}$$

now, giving $y \in \overline{B}\,(0_E, 1)$ arbitrary, putting $x = \frac{\mu}{2}y + x_0$, we have :

$$\|x - x_0\|_E = \|\frac{\mu}{2}y\|_E = \frac{\mu}{2}\|y\|_E \leq \frac{\mu}{2} < \mu$$

then $\|x - x_0\| < \mu$, thus according to $(1)$ $\|f(x) - f(x_0)\| < 1$ but $f$ is linear

$$\begin{aligned}\|f(x) - f(x_0)\|_F = \|f(x - x_0)\|_F = \|f\left(\frac{\mu}{2}y\right)\|_F &= \|\frac{\mu}{2}f\,(y)\,\|_F \\ &= \frac{\mu}{2}\|f\,(y)\,\|_F\end{aligned}$$

hence

$$\frac{\mu}{2}\|f(y)\|_F < 1$$

implying that

$$\|f(y)\|_F < \frac{2}{\mu} \quad (\forall y \in \overline{B}\,(0_E, 1))$$

this shows that $f$ is bounded on $\overline{B}(0_E, 1)$

$$(iii) \implies (iv)$$

This is obvious since $S_E\,(0_E, 1) \subset \overline{B}_E\,(0_E, 1)$, that is :

$$\exists M > 0, \forall u \in S_E\,(0_E, 1) : \quad \|f(x)\|_F \leq M$$

so, for any $x \in E \backslash \{0_E\}$, since $\frac{x}{\|x\|_E} \in S_E\,(0_E, 1)$, we have :

$$\|f\left(\frac{x}{\|x\|_E}\right)\| \leq M$$

which gives

$$\|f(x)\|_F \leq M\|x\|_E$$

as required, remark that this last inequality is also valid for $x = 0_E$

$$(iv) \implies (v)$$

Suppose that $\exists M > 0$, satisfying the property :

$$\forall x \in E : \quad \|f(x)\|_F \leq M\|x\|_E$$

then, for all $x, y \in E$, we have :

$$\|f(x) - f(y)\|_F = \|f(x - y)\| \leq M\| \|x - y\|_E\|$$

implying that $f$ is $M$-Lipschitz

$$(iv) \implies (v)$$

this is known to be true in metric spaces, (in general). This proof is complete $\square$

**Theorem 2.0.2:**

Let $E$ be a $\mathbb{K}$-Vector space and let $N_1$ and $N_2$ be two norms on $E$, then we have equivalence between :

(i) $N_1$ and $N_2$ are toplogically equivalent

(ii) $N_1$ and $N_2$ are equivalent

*Proof.* we have

$$\begin{aligned} id_E : \quad (E, N_1) &\longrightarrow (E, N_2) \\ x &\longmapsto x \end{aligned}$$

is bicontinious, and it's bi-Lipschitz continious. But since $id_E : (E, N_1) \longrightarrow (E, N_2)$ and it's inverse $id_E^{-1} : (E, N_2) \longrightarrow (E, N_1)$, are obviously linear, then (by the above theorem we have the equivallence), between " $id_E$ is bicontinious ", and " $id_E$ is bi-Lipschitz continious", hence they are equivallent, as required. $\square$

*Notation :* let $E$ and $F$ be two N.V.S over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we let $L(E, F)$ denote the $\mathbb{K}$-vector space of linear maps from $E$ to $F$, and $\mathcal{L}(E, F)$ denote the $\mathbb{K}$-vector space of continious linear maps, from $E$ to $F$, In general we have :

$$\mathcal{L}(E, F) \not\subset L(E, F)$$

**Example**

Let $E := \mathcal{C}^0([0, 1]), \mathbb{R}$, considered as an $\mathbb{R}$-vector space, we consider in $E$ the two norms $\|.\|_1$

and $\|.\|_\infty$ defined previously, let

$$\begin{array}{rccl} \delta: & E & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \delta(f) := f(0) \end{array} \quad (\mathbb{R}, \|.\|)$$

$\delta$ is called the Dirac operator, it's clear that $\delta$ is linear. We shall prove that $\delta$ is continious with respect to $\|.\|_\infty$ but it's not continious with respect to $\|.\|_1$. - **For** $\|.\|_\infty$ **:**

$\forall f \in E$, we have :

$$|\delta(f)| = |f(0)| \leq \sup_{t\in[0,1]} |f(t)| = \|f\|_\infty$$

This shows according to the above theorem, that $\delta$ is continious in $(E, \|.\|_\infty)$

- **For** $\|.\|_1$ **:**

Consider the sequence of functions $(f_n)_{n\geq 1}$ of $E$, defined by $\forall n \in \mathbb{N}$ :

$$f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0, \frac{1}{n}) \\ 0 & \text{if } x \in [\frac{1}{n}, 1] \end{cases}$$

we have for all $n \geq 1$:

$$\begin{aligned} |\delta(f_n)| &= |f_n(0)| = 1 \\ \|f_n\|_1 &= \int_0^1 |f_n(x)|\, dx = \int_0^{1/n} (1 - nx)\, dx + \int_{1/n}^1 0dx \\ &= \left(x - \frac{n}{2}x^2\right)^{1/2} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \end{aligned}$$

thus $\forall n \in \mathbb{N}$, we have :

$$\frac{|\delta(f_n)|}{\|f_n\|1} = U_n \to \infty \quad \text{as} \quad n \to \infty$$

implying that $\frac{|\delta(f)|}{\|f\|_1}$, where $(f \in E \setminus \{0_E\})$, is unbounded from above, thus the diract operator $\delta$ is not continious on $(E, \|.\|_1)$.

**Remark**

If $E$ is an infinite dimensional $N.V.S$, we can show that we have

$$\mathcal{L}(E, F) \quad \not\subset \quad L(E, F)$$

That is there exist a linear map from $E$ to $F$ which is not continious.

Let $E$ and $F$ be two N.V.S over $\mathbb{K}$, for $f \in \mathcal{L}(E,F)$, we define $|||\ f\ |||$ by :

$$|||\ f\ |||:= \sup_{x\in E\setminus\{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E}$$

According to item $(v)$ of the above theorem, we have that $|||\ f\ ||| \in [0,\infty)$ i.e., $|||\ f\ |||$ is a non negative real number, so $|||\ .\ |||$ constitues a map from $\mathcal{L}(E,F)$ to $[0,\infty)$

**Theorem 2.0.3:**

The map $|||\ .\ |||$ defined above is a norm $\mathcal{L}(E,F)$ (seen as a $\mathbb{K}$ vector space)

*Proof.* Let us show that $|||\ .\ |||$ satisfies the three axioms of a norm on $\mathcal{L}(E,F)$

(i) $1^{st}$ axiom :

For all $f \in \mathcal{L}(E,F)$ we have

$$\begin{aligned}
|||\ f\ |||=0 &\iff \sup_{x\in E\setminus\{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} = 0 \\
&\iff \forall x \in E\setminus\{0_E\} : \quad \|f(x)\|_F = 0 \\
&\iff \forall x \in E\setminus\{0_E\} : \quad f(x) = 0_F \\
&\iff \forall x \in E : \quad f(x) = 0_F \\
&\iff f = 0_{\mathcal{L}(E,F)}
\end{aligned}$$

(ii) $2^{nd}$ axiom : $\forall f \in \mathcal{L}(E,F)$, we have

$$\begin{aligned}
|||\ \lambda f\ ||| &= \sup_{x\in E\setminus\{0_E\}} \frac{\|(\lambda f)(x)\|_F}{\|x\|_E} \\
&= \sup_{x\in E\setminus\{0_E\}} \frac{\|\lambda f(x)\|_F}{\|x\|_E} \\
&= \sup_{x\in E\setminus\{0_E\}} \frac{|\lambda|\ \|f(x)\|_F}{\|x\|_E} \\
&= |\lambda| \sup_{x\in E\setminus\{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} = |\lambda|\ |||\ f\ |||
\end{aligned}$$

As required

(iii) $3^{rd}$ axiom :

let $f,g \in \mathcal{L}(E,F)$, we have for all $x \in E\setminus\{0_E\}$ :

$$\begin{aligned}
\|(f+g)(x)\|_F &= \|f(x)+g(x)\|_F \\
&\leq \|f(x)\|_F + \|g(x)\|_F
\end{aligned}$$

Thus (by dividing by $\|x\|_E$ ) :

$$\begin{aligned}\frac{\|(f+g)(x)\|_F}{\|x\|_E} &\leq \frac{\|f(x)\|_F}{\|x\|_E} + \frac{\|g(x)\|_F}{\|x\|_E} \\ &\leq ||| f ||| + ||| g |||\end{aligned}$$

So all $x \in E \setminus \{0_E\}$

$$\frac{\|(f+g)(x)\|_F}{\|x\|_E} \leq ||| f ||| + ||| g |||$$

Hence, by taking the supremum over $x \in E \setminus \{0_E\}$ :

$$||| f+g ||| \leq ||| f ||| + ||| g |||$$

as required, consequently, $||| \,.\, |||$ is a norm on $\mathcal{L}(E,F)$

□

*Terminology :*

Let $E$ and $F$ be two N.V.S over $\mathbb{K}$, then the norm $||| \,.\, |||$ of $\mathcal{L}(E,F)$ (constituted from the two norms $\|.\|_E$ of $E$ and $\|.\|_F$ of $F$), is called the subordinate norm induced by the norms $\|.\|_E$ of $E$ and $\|.\|_F$ of $F$.

**Theorem 2.0.4:**

Let $E$ and $F$ be two N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then for all $f \in \mathcal{L}(E,F)$, we have :

$$\begin{aligned}||| f ||| &:= \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} \\ &= \sup_{x \in S_E(0_E,1)} \|f(x)\|_F \\ &= \sup_{x \in B_E(0_E,1)} \|f(x)\|_F \\ &= \sup_{x \in \overline{B_E}(0_E,1)} \|f(x)\|_F\end{aligned}$$

*Proof.* We have to show the following multiple inequality :

$$\begin{aligned}\sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} \leq_1 \sup_{x \in S_E(0_E,1)} \|f(x)\|_F \leq_2 & \sup_{x \in B_E(0_E,1)} \|f(x)\|_F \\ \leq_3 & \sup_{x \in \overline{B_E}(0_E,1)} \|f(x)\|_F \\ \leq_4 & \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E}\end{aligned}$$

Since this inequality $\leq_3$ is obvious, because $B_E(0_E, 1) \subset \overline{B_E}(0_E, 1)$ , we have to show the three inequalitys

$$\leq_1 \quad \leq_2 \quad \leq_4$$

Let us show $\leq_1$ for all $x \in E\setminus\{0_E\}$, we have :

$$\frac{\|f(x)\|_F}{\|x\|_E} = \|f\left(\frac{x}{\|x\|_E}\right)\|_F \leq \sup_{y \in S_E(0_E,1)} \|f(y)\|_F$$

so for all $x \in E\setminus\{0_E\}$ :

$$\frac{\|f(x)\|_F}{\|x\|_E} \leq \sup_{y \in S_E(0_E,1)} \|f(y)\|_F$$

Thus by taking the supremum over $x$, we get the required result, Now let us agains show the second inequality $\leq_2$, for all $x \in S_E(0_E, 1, 1)$, we have

$$\|f(x)\|_F = \frac{1}{r}\|f(\underbrace{rx}_{\in B_E(0_E,1)})\|_F \leq \frac{1}{r} \sup_{y \in B_E(0_E,1)} \|f(y)\|_F$$

so

$$\forall x \in S_E(0_E, 1), \forall r \in (0, 1): \quad \|f(x)\|_F \leq \frac{1}{r} \sup_{y \in B_E(0_E,1)} \|f(y)\|_F$$

So, by taking $r \to^{<} 1$, we get

$$\forall x \in S_E(0_E, 1): \quad \|f(x)\|_F \leq \sup_{y \in B_E(0_E,1)} \|f(y)\|_F$$

then by taking the supremum over $x$ :

$$\sup_{x \in S_E(0_{E,1})} \|f(x)\|_F \leq \sup_{y \in B_E(0_E,1)} \|f(y)\|_F$$

as required, now let us show the $\leq_4$, we have for all $x \in \overline{B_E}(0_E, 1) \setminus \{0_E\}$, we have :

$$0 < \|x\|_E \leq 1 \implies \frac{1}{\|x\|} \geq 1$$

so we get :

$$\begin{aligned}\|f(x)\|_F &\leq \frac{\|f(x)\|_F}{\|x\|_E} \\ &\leq \sup_{y \in E\setminus\{0_E\}} \frac{\|f(y)\|_F}{\|y\|_E} = \|f\|\end{aligned}$$

So $\forall x \in \overline{B_E}(0_E, 1) \setminus \{0_E\}$ :

$$\|f(x)\|_F \leq ||| f |||$$

which is also true for $x = 0_E$ since $f$ is linear, so

$$\forall x \in \overline{B_E}(0_E, 1) : \|f(x)\|_F \leq ||| f |||$$

then by taking the supremum over $x$ :

$$\sup_{x\in\overline{B_E}(0_E,1)} \|f(x)\|_F \leq ||| f |||$$

as required, this completes the proof. □

This following proposition is an immediate consequence of the definition of a subordinate norm

**Theorem 2.0.5:**

Let $E$ and $F$ be two N.V.S over $\mathbb{K}=\mathbb{R}$, or $\mathbb{C}$ and $f\in\mathcal{L}(E,F)$, we have :

1. $$\forall x\in E: \quad \|f(x)\|_F \leq ||| f ||| \cdot \|x\|_E$$

2. if $M\in[0,\infty)$ satisfies :

$$\|f(x)\|_F \leq M\|x\|_E \quad (\forall x\in E)$$

then

$$||| f ||| \leq M$$

By applying theorem 5, we obtain a remarkable inequality concerning the subordinate norm of a composition of two continious linear mappings betewen $N.V.S$

**Theorem 2.0.6:**

Let $E, F$ and $G$ be three N.V.S over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, and let $f: E\longrightarrow F$ and $g: F\longrightarrow G$ be two continious linear mappings then we have :

$$||| g\circ f ||| \leq ||| g ||| \cdot ||| f |||$$

*Proof.* Since $f: E\longrightarrow F$ and $g: F\longrightarrow G$ and both linear then $g\circ f: E\longrightarrow G$ is also linear, similarly, since $f$ and $g$ are both continious then $g\circ f$ is continious therefore $g\circ f\in\mathcal{L}(E,G)$. Next, using twice successively the inequality of item $(1)$, of proposition $(5)$, we have for all $x\in E$ :

$$\begin{aligned}\|(g\circ f)(x)\|_G = \|g(f(x))\|_G &\leq ||| g ||| \cdot \|f(x)\|_F \\ &\leq ||| g ||| \cdot ||| f ||| \cdot \|x\|_E\end{aligned}$$

This implies according to item $(2)$ of proposition $(5)$, that :

$$||| g\circ f ||| \leq ||| g ||| \cdot ||| f |||$$

as required, this completes the proof. □

## 2.1 Normed Algebra

**Definition 2.1.1:**

Let $\mathbb{K}$ be a filed, an algebra over $\mathbb{K}$ or simply a $\mathbb{K}$-algebra is a $\mathbb{K}$-vector space a $\mathcal{A}$ or $(\mathcal{A}, +, .)$ equipped with a bilinear multiplication operation, $\times : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ such that $(A, +, \times)$ is a ring and $'' \times ''$ is a compatible with scalar multiplication, that is

$$\forall \lambda \in \mathbb{K}, \forall x, y \in \mathcal{A} : \quad (\lambda \cdot x) \times y = x \times (\lambda \cdot y) = \lambda \cdot (x \times y)$$

**Example**

For any field $\mathbb{K}$ and a ny $n \in \mathbb{N}$, $\mathcal{M}_n(\mathbb{K})$ is $\mathbb{K}$-algebra

**Definition 2.1.2:**

let $(\mathcal{A}, +, \times, \cdot)$ be a $\mathbb{K}$-algebra, an *algebra-norm* on $\mathcal{A}$ is a norm $|||\ .\ |||$ on the $\mathbb{K}$-vector space $(\mathcal{A}, +, \cdot)$ which satisfies in addition the property :

$$|||\ y \times x\ ||| \leq |||\ x\ ||| \cdot |||\ y\ |||$$

we say that $|||\ .\ |||$ is submultiplicative.

here are the following axioms of the algebra-norm

1. $|||\ x\ ||| = 0 \implies x = 0_{\mathcal{A}}$
2. $|||\ \lambda x\ ||| = |\lambda| \cdot |||\ x\ ||| \quad \forall \lambda \in \mathbb{K}, \forall x \in \mathcal{A}$
3. $|||\ x + y\ ||| \leq |||\ x\ ||| + |||\ y\ ||| \quad \forall x, y \in \mathcal{A}$
4. $|||\ x \times y\ ||| \leq |||\ x\ ||| \cdot |||\ y\ ||| \quad \forall x, y \in \mathcal{A}$

**Example**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then $\mathcal{L}(E, E)$ with the laws $+, \cdot, \circ$ equipped with the subordinate norm $|||\ .\ |||$ induced by $\|.\|_E$ is a normed algebra according to the above proposition

## 2.2 An important particular case (matrix norm)

**Definition 2.2.1:**

Let $n \in \mathbb{N}$, a matrix norm on $\mathcal{M}_n(\mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a map $||| \, . \, |||: \mathcal{M}_n(\mathbb{K}) \longrightarrow [0, \infty)$ which satisfies :

(i) $\forall A \in \mathcal{M}_n(\mathbb{K}): \quad ||| \, A \, ||| = 0 \implies A = 0_{\mathcal{M}_n(\mathbb{K})}$

(ii) $\forall A \in \mathcal{M}_n(\mathbb{K}), \forall \alpha \in \mathbb{K}: \quad ||| \, \alpha A \, ||| = |\alpha| \cdot ||| \, A \, |||$

(iii) $\forall A, B \in \mathcal{M}_n(\mathbb{K}): \quad ||| \, A + B \, ||| \leq ||| \, A \, ||| + ||| \, B \, |||$

(iv) $\forall A, B \in \mathcal{M}_n(\mathbb{K}): \quad ||| \, AB \, ||| \leq ||| \, A \, ||| \cdot ||| \, B \, |||$

in other words, a matrix norm is an algebra norm on $(\mathcal{M}_n(\mathbb{K}), +, \times, \cdot)$ where $\times$ is matrix multiplication and $\cdot$ is scalar multiplication.

**Remark**

Let $n \in \mathbb{N}$, any norm $\|.\|$ on the $\mathbb{K}$-vector space $\mathbb{K}^n$ iduces a matrix norm $||| \, . \, |||$ on $\mathcal{M}_n(\mathbb{K})$, whixh is defined by :

$$||| \, A \, ||| := \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|Ax\|}{\|x\|} = \sup_{x \in \mathbb{K}, \|x\|=1} \|Ax\|$$

This particular matrix norm is called

*"The subordinate norm induced by $\|.\|$"*

**Example**

let $n \in \mathbb{N}$.

- the subordinate norm on $\mathcal{M}_n(\mathbb{K})$ induced by the norm of $\|.\|_1$ on $\mathbb{K}^n$ is given by

$$\|A\|_1 := \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|Ax\|}{\|x\|}$$

- the subordinate matrix norm on $\mathcal{M}_n(\mathbb{K})$ induced by the norm $\|.\|_\infty$ on $\mathbb{K}^n$ is given by :

$$||| \, A \, |||_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| = ||| \, A^T \, |||_1$$

- the subordinate norm on $\mathcal{M}_n(\mathbb{K})$ induced by the norm $\|.\|_2$ of $\mathbb{K}^n$ is given by :

$$||| A |||_2 = \sqrt{\rho\left(A^T A\right)} \quad (\forall A \in \mathcal{M}_n(\mathbb{K}))$$

where $\rho$ denotes the spectral radius of a square matrix $M$ of $\mathcal{M}_n(\mathbb{K})$

$$(\rho(M) := \max\{|\lambda|, \lambda \in \sigma_{\mathbb{C}}(M)\})$$

the square root of the eigen values of the positive semi definite matrix $A^T A$ are called singular values of $A$

$$||| A |||_2 = \max S.V(A) \qquad \text{(the largest singular value of } A\text{)}$$

- suppose that $n \geq 2$, we define

$$\begin{array}{rccc} N: & \mathcal{M}_n(\mathbb{K}) & \longrightarrow & [0, \infty) \\ & A & \longmapsto & N(A) := \max_{1\leq i,j\leq n} \left|a_{ij}\right| \end{array}$$

it's clear that $N$ is a clear norm on $\mathcal{M}_n(\mathbb{K})$ but it's not a matrix norm on it because we have for example

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

we have

$$A^2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = n \times A$$

so $N\left(A^2\right) = n$ and $N(A)^2 = 1^2 = 1$ then

$$N(A^2) \quad \not\leq \quad N(A)^2$$

thus $N$ is not a matrix norm.

**Remark**

let $n \in \mathbb{N}$, for any matrix norm $||| \,.\, |||$ on $\mathcal{M}_n(\mathbb{K})$, we have $||| I_n ||| \geq 1$. Indeed,

$$||| I_n^2 ||| \leq ||| I_n |||^2$$

that is

$$||| I_n ||| \leq ||| I_n |||^2$$

hence $||| I_n ||| \geq 1$

**Definition 2.2.2:**

let $n \in \mathbb{N}$, if a matrix norm $||| \, . \, |||$ on $\mathcal{M}_n(\mathbb{K})$ satisfies $||| I_n ||| = 1$ then it's said to be unital

**Example**

Any suboridnate matrix norm $||| \, . \, |||$ on $\mathcal{M}_n(\mathbb{K})$ where $(n \in \mathbb{N})$ induced by a norm $\|.\|$ on $\mathbb{K}^n$ is unital, indeed, in such a case, we have :

$$||| I_n ||| = \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|I_n x\|}{\|x\|} = \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|x\|}{\|x\|} = 1$$

note that there exist *unital matrix norms* on $\mathcal{M}_n(\mathbb{K})$ which are not subordinate, (i.e., not induced by any vector space norm $\mathbb{K}^n$)

## 2.3 The spectral radius of a complex square matrix

**Definition 2.3.1:**

Let $n \in \mathbb{N}$ and $A \in \mathcal{M}_n(\mathbb{C})$ the spectral radius of $A$, denoted $\rho(A)$, is the maximum of the modulus of the eigen values of $A$, that is

$$\rho(A) := \max \{|\lambda| : \quad \lambda \in \sigma_{\mathbb{C}}(A)\}$$

we have the following theorem

**Theorem 2.3.1:**

let $n \in \mathbb{N}$ and let $||| \, . \, |||$ be a matrix norm on $\mathcal{M}_n(\mathbb{C})$, then for any $A \in \mathcal{M}_n(\mathbb{C})$, we have :

$$\rho(A) \quad \leq \quad ||| A |||$$

*Proof.* let $A \in \mathcal{M}_n(\mathbb{C})$ and let $\lambda \in \mathbb{C}$ be an arbitrary eigen value of $A$, so $\exists x \in \mathbb{C}^n \setminus \{0_{\mathbb{C}^n}\}$ such that $Ax = \lambda x$ consider :

$$B := (X \backslash 0_{\mathbb{C}^n} \backslash \ldots \backslash 0_{\mathbb{C}^n}) \quad M_n(\mathbb{C}) \setminus \left\{0_{\mathcal{M}_n(\mathbb{C})}\right\}$$

then we have :

$$\begin{aligned} AB &= (Ax \mid A0_{\mathbb{C}^n} \mid \dots \mid A0_{\mathbb{C}^n}) \\ &= (\lambda x \mid 0_{\mathbb{C}^n} \mid \dots \mid 0_{\mathbb{C}^n}) \\ &= \lambda\,(x \mid 0_{\mathbb{C}n} \mid \dots \mid 0_{\mathbb{C}^n}) \\ &= \lambda B \end{aligned}$$

thus

$$||| \; AB \; ||| = ||| \; \lambda B \; ||| = |\lambda| \, |B|$$

so

$$\lambda \; ||| \; B \; ||| = ||| \; AB \; ||| \leq ||| \; A \; ||| \cdot ||| \; B \; |||$$

thus

$$|\lambda| \leq ||| \; A \; ||| \qquad (\forall \lambda \in \sigma_{\mathbb{C}}(A))$$

hence

$$\max_{\lambda \in \sigma_{\mathbb{C}}(A)} |\lambda| \leq ||| \; A \; ||| \implies (\rho(A)) \leq ||| \; A \; |||$$

as required ☐

**Theorem 2.3.2: Gelfond's formula**

Let $n \in \mathbb{N}$ and $||| \; . \; |||$ be a matrix norm on $\mathcal{M}_n(\mathbb{C})$ then for every $A \in \mathcal{M}_n(\mathbb{C})$, we have

$$\rho(A) = \lim_{k \to \infty} ||| \; A^k \; |||^{1/k}$$

# 3 PROPERTIES OF FINITE-DIMENSIONAL N.V.S

## 3.1 Norms on a finite-dimensional $\mathbb{K}$-vector space

Let $n \in \mathbb{N}$ and $E$ be an $n$-dimensional vector space over $\mathbb{K}$, let also $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be a basis of $E$, using $\mathcal{B}$ we can construct on $E$ several norms including :

$$\|.\|_{1,\mathcal{B}} \quad \|.\|_{2,\mathcal{B}} \quad \|.\|_{p,\mathcal{B}} \quad (p \geq 1) \quad \text{and} \quad \|.\|_{\infty,\mathcal{B}}$$

defined by

$$\begin{aligned} \|x\|_{1,\mathcal{B}} &:= \sum_{i=1}^{n} |x_i| \\ \|x\|_{2,\mathcal{B}} &:= \sqrt{\sum_{i=1}^{n} |x_i|^2} \\ \|x\|_{3,\mathcal{B}} &:= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \\ \|x\|_{\infty,\mathcal{B}} &:= \max_{1 \leq i \leq n} \|x_i\| \end{aligned}$$

we easily show that these norms on $E$ are all equivalent, lets consider in particular the norm $\|.\|_{\infty,\mathcal{B}}$, it's immediate that the map

$$\begin{array}{ccc} (\mathbb{K}^n, \|.\|_\infty) & \longrightarrow & (E, \|.\|_{\infty,\mathcal{B}}) \\ (x_1, x_2, \ldots, x_n) & \longmapsto & x_1 e_1 + \ldots + x_n e_n \end{array}$$

this map is an isometry (bijective), since the distances are conserved we call it *isomorphism isometric* , it's an homeomorphism because it's lipschitz, consequently, the $\mathbb{K}$-N.V.S, $(E, \|.\|_{\infty,\mathcal{B}})$ and $(\mathbb{K}^n, \|.\|_\infty)$ have the same toplogical and metric properties, in particular, we derive that :

(1) The N.V.S $(E, \|.\|_{\infty,\mathcal{B}})$ is complete (i.e., a Banach space)

(2) The compact parts of $(E, \|.\|_{\infty,\mathcal{B}})$ are exactly bounded parts in particular

$$S_E\left(0_E, 1\right)|_{\|.\|_{\infty,\mathcal{B}}} \quad \text{is compact in} \quad (E, \|.\|_{\infty,\mathcal{B}})$$

these two properties are used to prove the following fundamental theorem

**Theorem 3.1.1:**

On a finite-dimensional vector space $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, all norms are equivalent

*Proof.* let $n \in \mathbb{N}$ and $\mathbb{E}$ an $n$-dimensional vector space over $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$, let also $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be a fixed basis of $E$, we are going to show that every norm on $E$ is equivalent to the norm $\|.\|_{\infty,\mathcal{B}}$, let $N$ be an arbitrary norm on $E$ and let us show that $N \sim \|.\|_{\infty,\mathcal{B}}$ on the one hand, by using the properties of $N$ as a norm on $E$, we have for all $x = x_1e_1 + \ldots + x_ne_n$ with $(x_1, \ldots, x_n \in \mathbb{K})$, we have :

$$\begin{aligned}
N\left(x\right) &= N\left(x_1e_1 + \ldots + x_ne_n\right) \\
&\leq N\left(x_1e_1\right) + \ldots + N\left(x_ne_n\right) \\
&= |x_1|\, N\left(e_1\right) + |x_2|\, N\left(e_2\right) + \ldots + |x_n|\, N\left(e_n\right) \\
&\leq \left(\max_{1\leq i\leq n} |x_i|\right) \sum_{i=1}^{n} N\left(e_i\right) = \left(\sum_{i=1}^{n} N\left(e_i\right)\right) \|x\|_{\infty,\mathcal{B}}
\end{aligned}$$

so by setting $\beta = \sum_{i=1}^{n} N\left(e_i\right) > 0$, we have

$$N\left(x\right) \leq \beta \|x\|_{\infty,\beta} \quad (\forall x \in E)$$

$\square$

some recap, we have

$$\begin{aligned}
(\mathbb{K}^n, \|.\|_{\infty}) &\longrightarrow \quad (E, \|.\|_{\infty,\mathcal{B}}) \\
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &\longmapsto \quad x_1e_1 + \ldots + e_nx_n
\end{aligned}$$

1. we deduce that $(\mathbb{E}, \|.\|_{\infty,\mathcal{B}})$ is banach

2. the compact parts of $(E, \|.\|_{\infty})$ are exactly closed and bounded parts in particular :

$$S_E\left(0_E, 1\right) \quad \text{is compact}$$

**Theorem 3.1.2:**

On a finite dimensional vector space on $\mathbb{R}$ or $\mathbb{C}$, all norms are equivalent.

*Proof.* Let $N$ be an arbitrary norm on $E$, we want to show that

$$N \sim \|.\|_{\infty,\mathcal{B}}$$

we have

$$\begin{aligned} N(x) = N(x_1e_1 + \ldots + x_ne_n) &\leq \sum_{i=1}^{n} N(x_ie_i) \\ &= \sum_{i=1}^{n} |x_i| N(e_i) \\ &\leq \left( \sum_{i=1}^{n} N(e_i) \right) \|x\|_{\infty,\mathcal{B}} \end{aligned}$$

On the other hand, according to a well known property of the norms pon a $\mathbb{K}$-vector space, (See Ex 1.1), we have for all $x, y \in E$ :

$$|N(x) - N(y)| \leq N(x-y)$$

but since $N \leq \beta\|.\|_{\infty,\mathcal{B}}$, we derive that for all $x, y \in E$ :

$$|N(x) - N(y)| \leq \beta\|x-y\|_{\infty,\mathcal{B}}$$

implying that the map :

$$\begin{array}{rccc} N: & (E, \|.\|_{\infty,\mathcal{B}}) & \longrightarrow & (\mathbb{R}, \|.\|) \\ & x & \longmapsto & N(x) \end{array}$$

is $\beta$-Lipschitz, so continuous on $(E, \|.\|_{\infty,\mathcal{B}})$ , next, giving that the unit sphere $S_E(0_E, 1)$, of $(E, \|.\|_{\infty,\beta})$, is compact in $(E, \|.\|_{\infty,\mathcal{B}})$, see properties of the N.V.S $(E, \|.\|_{\infty,\mathcal{B}})$ cited above, it follows according to the extreme value theorem, recall

*Let $X$ be a compact toplogical space and, $f : X \longrightarrow \mathbb{R}$ be a continuous map, then $f$ is bounded on $X$ and attains its bounds, meaning there exist points $x_{min}, x_{\max} \in X$ such that :*

$$f(x_{min}) = \inf_{x\in X} f(x) \quad \text{and} \quad f(x_{\max}) = \sup_{x\in X} f(x)$$

that the map $N$ above is bounded on the sphere $S_E(0_E, 1)\mid_{\|.\|_{\infty,\mathcal{B}}}$, and attains it's supremum and infinimum in that sphere, so there exist $x_0 \in S_E(0_E, 1)\mid_{\|.\|_{\infty,\mathcal{B}}}$ such that

$$N(x) \geq N(x_0) \quad \left(\forall x \in S_E(0_E, 1)\mid_{\|.\|_{\infty,\mathcal{B}}}\right)$$

put $\alpha := N(x_0) \geq 0$ , if we suppose that $\alpha = 0$, we obtain (since $N$ is a norm on $E$) that, $x_0 = 0_E \notin S_E(0_E, 1)\mid_{\|.\|_{\infty,\mathcal{B}}}$, which is a contradiction, thus $\alpha > 0$, and we have :

$$\forall x \in S_E(0_E, 1)\mid_{\|.\|_{\infty,\mathcal{B}}} : \quad N(x) \geq \alpha$$

finally, giving $x \in E\backslash \{0_E\}$, by applying the last inequality for

$$\frac{x}{\|x\|_{\infty,\mathcal{B}}} \in S_E(0_E, 1)$$

we obtain

$$N\left(\frac{x}{\|x\|_{\infty,\mathcal{B}}}\right) \geq \alpha$$

that is

$$N(x) \geq \alpha\|x\|_{\infty,\mathcal{B}} \quad (\forall x \in E\backslash \{0_E\})$$

this inequality, is also true for $x = 0_E$ , hence we get

$$N(x) \geq \alpha\|x\|_{\infty,\mathcal{B}} \quad (\forall x \in E)$$

hence we have show that $N$ is equivalent to $\|.\|_{\infty,\mathcal{B}}$, as required, this completes the proof □

## 3.2 Toplogical and metric properties of a finite-dimensional N.V.S

From Theorem 1, we derive several important corollaries.

**Theorem 3.2.1:**

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ , we have :

(1) Every finite-dimensional N.V.S over $\mathbb{K}$ is banach

(2) The compact parts of a finite-dimensional N.V.S over $\mathbb{K}$ are exactly those which are both closed and bounded.

*Proof.* Let $(E, \|.\|)$ be a finite dimensional $N.V.S$, over $\mathbb{K}$, and $n := dim\,(E)$, since the case for $n = 0$ is trivial, we may suppose that $n \geq 1$, next let $\mathcal{B} = (e_1, e_2, \ldots)$ be a basis of $E$, since

$$\|.\| \sim \|.\|_{\infty,\mathcal{B}} \quad \text{by above Theorem}$$

then $(E, \|.\|)$ has the same toplogical and metric properties as $(E, \|.\|_{\infty,\mathcal{B}})$ so since properties $(1)$ and $(2)$ of the corollary hold for $(E, \|.\|_{\infty,\mathcal{B}})$ then they also hold for $(E, \|.\|)$, as required this achieves the proof. □

**Theorem 3.2.2:**

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $E$ and $F$ be two $\mathbb{K}$-N.V.S with $E$ is finite-dimensional, then every linear mapping from $E$ to $F$ is continuous

$$\mathcal{L}\,(E, F) = L\,(E, F)$$

*Proof.* Put $n = dim\,(E)$ since the case $n = 0$ is trivial, suppose that $n \geq 1$, fix a basis

$$\mathcal{B} = (e_1, \ldots, e_n)$$

of $E$, let $f : (E, \|.\|_E) \longrightarrow (F, \|.\|_F)$ be a linear mapping and we will show that it's continuous, according to Theorem 1, all norms on $E$ are equivalent then in particular

$$\|\cdot\|_E \sim \|\cdot\|_E$$

so there exist a positive constant $c$ such that

$$\|\cdot\|_{E,\mathcal{B},\infty} \leq c\|\cdot\|_E$$

using this last inequality together with the linearity of $f$ and the properties of a norm on a vector space, we have for every

$$x = x_1e_1 + \ldots + x_ne_n \in E \quad (x_1, x_2, \ldots, x_n) \in \mathbb{K}$$

we have

$$\begin{aligned}
\|f(x)\|_F = \|f\,(x_1e_1 + \ldots + x_ne_n)\,\|_F &= \|x_1f\,(e_1) + \ldots + x_nf\,(e_n)\,\|_F \\
&\leq \sum_{i=1}^{n} \|x_if(e_i)\|_F \\
&= \sum_{i=1}^{n} |x_i|\,\|f(e_i)\|_F \\
&\leq \left(\sum_{i=1}^{n} \|f(e_i)\|_F\right) \|x\|_{E,\infty,\mathcal{B}} \\
&\leq \left(c\sum_{i=1}^{n} \|\,(f(e_i))\,\|_F\right) \|x\|_E
\end{aligned}$$

that is

$$\|f(x)\|_F \le \left( c \sum_{i=1}^{n} (f(e_i))_F \right) \|x\|_E \quad (\forall x \in E)$$

showing that $f$ is continuous, as required □

we have also the following important theorem

**Theorem 3.2.3:**

Let $E$ and $F$ be two N.V.S over $\mathbb{K}\left(\{\mathbb{R}, \mathbb{C}\}\right)$, with $F$ is Banach, then the $\mathbb{K}$-N.V.S $\mathcal{L}(E,F)$ is Banach.

*Proof.* We have to show that any Cauchy sequence of $\mathcal{L}\,(E,F)$ is convergent in $(\mathcal{L}(E,F))$ so let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of $\mathcal{L}(E,F)$ and let us show that it converges for some $f \in \mathcal{L}(E,F)$, by hypothesis, we have :

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall p,q \in \mathbb{N} : \quad p > q \ge N \implies ||| f_p - f_q ||| \le \varepsilon$$

it follows from the definition of the norm $|||\;.\;|||$ of $\mathcal{L}(E,F)$ that :

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall p,q \in \mathbb{N} : \quad p > q \ge N \implies \forall x \in E : \quad \|f_p(x) - f_q(x)\| \le \varepsilon \|x\|_E$$

for $x \in E\backslash\,\{0_E\}$ fixed, by taking instead of $\varepsilon$ the positive real number $\frac{\varepsilon}{\|x\|_E}$, we desire the following

$$\forall \varepsilon > 0, \quad N\,(\varepsilon, x) \in \mathbb{N}, \forall p,q \in \mathbb{N} : \quad p > q \ge N\,(\varepsilon, x) \implies \|f_p(x) - f_q(x)\|_F \le \varepsilon$$

show that, for all $x \in E\backslash\,\{0_E\}$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ of $F$ is Cauchy, since $F$ is Banach then for all $x \in E\backslash\,\{0_E\}$, the sequence $(f_n)_{n\in\mathbb{N}}$ of $F$ is convergent, remark that the same sequence $(f_n(x))_{n\in\mathbb{N}}$ of $F$ also converge for $x = 0_E$ to $0_F$, since $f_n(0_E) = 0_F$ , then for all $n \in \mathbb{N}$ , because the maps $f_n$ are all linear so let us define

$$\begin{array}{rccl} f : & E & \longrightarrow & F \\ & x & \longmapsto & f(x) := \lim_{n\to\infty} f_n(x) \end{array}$$

Now, we are going to show that $f \in \mathcal{L}(E,F)$, that is $f$ is linear and continuous, and that $f$ is the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(E,F)$

*is $f$ linear?*

for all $x, y \in E$, for all $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} f(\lambda x + y) &:= \lim_{n\to\infty} f_n(\lambda x + y) \\ &= \lim_{n\to\infty} (\lambda f_n(x) + f_n(y)) \text{ since } f_n \text{ is linear for all } n\mathbb{N} \\ &= \lambda \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} f_n(y) \text{ ( by the continiouty of law } + \text{ and . of } F \text{ )} \\ &= \lambda f(x) + f(y) \end{aligned}$$

implying that $f$ is linear

*is $f$ continuous?*

By taking in $\varepsilon = 1$ , $q = N = N(1) \in \mathbb{N}$, and by letting $p \to \infty$, we obtain according to the continiouty of the norm $\|.\|_F$, that

$$\|f(x) - f_N(x)\| \leq \varepsilon\|x\|_E \quad (\forall x \in E)$$
$$\|(f - f_N)(x)\| \leq \|x\|_E \quad (\forall x \in E)$$

which implies that the linear map $(f - f_N)$, from $E$ to $F$ is continuous, thus $f := f_N + (f - f_N)$ is also continuous as the sum of two continuous mappings, consequently :

$$f \in \mathcal{L}(E, F)$$

*is $f$ the limit of $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(E, F)$*

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq N \implies \forall x \in E : \|f_p(x) - f_q(x)\|_F \leq \varepsilon\varepsilon\|x\|_E$$

by letting $p \to \infty$, and taking into account the continiouty of the norm $\|.\|_F$ of $E$, we obtain that

$$\begin{aligned} \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall q \in \mathbb{N} : \quad q \geq N &\implies \forall x \in E : \|f_p(x) - f(x)\| \leq \varepsilon\|x\|_E \\ &\iff \forall x \in E : \quad \frac{\|(f_q - f)(x)\|_F}{\|x\|_E} \leq \varepsilon \end{aligned}$$

that is

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall q \in \mathbb{N} : \quad q \geq N \implies \sup_{x \in E\setminus\{0_E\}} \frac{\|(f_q - f)(x)\|_F}{\|x\|_E} \leq \varepsilon$$

that is

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall q \in \mathbb{N} : \quad q \geq N \implies ||| f_q - f ||| \leq \varepsilon$$

showing that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f$ in $\mathcal{L}(E, F)$, this completes the proof $\square$

**Definition 3.2.1:**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we call the algebraic dual space of $E$, denoted $E^*$, the $\mathbb{K}$-vector space of $E$ constituing of linear forms on $E$, that is

$$E^* := L(E, \mathbb{K})$$

We call the continious dual space of $E$, denoted $E^{'}$, the $\mathbb{K}$-normed vector subspace of $E$ contituted of continious linear forms on $E$, that is

$$E^{'} := \mathcal{L}(E, \mathbb{K}) \quad (|||\ .\ |||)$$

note that the contrary here is relative to the suboridnate norm of $\mathcal{L}(E, \mathbb{K})$ induced by the $\|.\|_E$ of $E$ and $|.|$ of $\mathbb{K}$

**Example**

Let $a, b \in \mathbb{R}$ with $(a, b) \neq (0, 0)$, and let $f$ be the linear form on $\mathbb{R}^2$ defined by :

$$f(x, y) := ax + by \quad \left(\forall (x, y) \in \mathbb{R}^2\right)$$

(1) Explain why $f$ is continuous.

(2) (a) Determine $|||\ f\ |||$ with respect to the norm $\|.\|_1$ of $\mathbb{R}^2$ and $|.|$ of $\mathbb{R}$

(b) Determine $|||\ f\ |||$ with respect to the norm $\|.\|_2$ of $\mathbb{R}^2$ and $|.|$ of $\mathbb{R}$

( **Solution** )

(1) Since $dim\mathbb{R}^2 = 2 < \infty$ then $\mathcal{L}\left(\mathbb{R}^2, \mathbb{R}\right) = L\left(\mathbb{R}^2, \mathbb{R}\right)$ i.e. we have :

$$\left(\mathbb{R}^2\right)^{'} = \left(\mathbb{R}^2\right)^*$$

every linear form on $\mathbb{R}^2$ is continious, in particular $f$ is continuous

(2) (a) By definition :

$$|||\ f\ ||| := \sup_{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|f(x, y)|}{\|(x, y)\|_1} = \sup_{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|ax + by|}{|x| + |y|}$$

we have for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$

$$|ax + by| \leq |ax| + |by| = \underbrace{|a|}_{\max(|a|,|b|)} |x| + \underbrace{|b|}_{\max(|a|,|b|)} |y|$$

$$\leq \max(|a|, |b|)(|x| + |y|)$$

$$\frac{|ax + by|}{|x| + |y|} \leq \max(|a|, |b|)$$

hence

$$||| f ||| \leq \max(|a|, |b|)$$

by definition, we have :

$$||| f ||| \geq \frac{|f(1,0)|}{\|(1,0)\|_1} = \frac{|a|}{1} = |a|$$

and

$$||| f ||| \geq \frac{|f(0,1)|}{\|(0,1)\|_1} = \frac{|b|}{1} = |b|$$

thus we have :

$$||| f ||| \geq \max(|a|, |b|)$$

from the above we have shown that :

$$||| f ||| = \max(|a|, |b|)$$

(b) we have

$$||| f ||| = \sup_{(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}} \frac{|f(x,y)|}{\|(x,y)\|_2} = \sup_{(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}} \frac{|ax+by|}{\sqrt{x^2+y^2}}$$

According to the cauchy-schawrz in the Pre-Hilbert space $\left(\mathbb{R}^2, \langle . \rangle_u\right)$, we have :

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$|ax+by| = \left| \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_u \right| \leq \| \begin{pmatrix} a \\ b \end{pmatrix} \|_2 \cdot \| \begin{pmatrix} x \\ y \end{pmatrix} \|_2 = \sqrt{a^2+b^2} \cdot \sqrt{x^2+y^2}$$

therefore we get

$$||| f ||| \leq \sqrt{a^2+b^2}$$

on the other hand, we have

$$||| f ||| \geq \frac{|f(a,b)|}{\|(a,b)\|_2} = \frac{\arcsin a^2+b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$$

hence

$$||| f ||| = \sqrt{a^2+b^2}$$

Let us consider another example, let $E$ be a real pre-Hilbert space and $a$ be a fixed non zero vector of $E$, let also $f$ be the linear form of $E$ defined by

$$f(x) = \langle a, x \rangle \quad (\forall x \in E)$$

(1) Show that $f$ is continuous and determine

( **Solution** )

According to the Cauchy-Schawrz inequality, we have for all $x \in E$

$$|f(x)| = |\langle a, x\rangle| \leq \|a\|\|x\|$$

implying that $f$ is continious and

$$||| f ||| \leq \|a\|$$

On the other hand, we have

$$||| f ||| \geq \frac{|f(a)|}{\|a\|} = \frac{|\langle a, a\rangle|}{\|a\|} = \|a\|$$

hence

$$||| f ||| = \|a\|$$

**Theorem 3.2.4:**

Let $E$ be a N.V.S over $\mathbb{K}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and let $f$ be a linear form on $E$ , that is $f \in E^* = L(E, \mathbb{K})$. Then $f$ is continious if and only if it's kernel $Ker(f)$ is a closed part of $E$

*Proof.* ( $\Longrightarrow$ ) Suppose that $f : (E, \|.\|) \longrightarrow (\mathbb{K}, |.|)$ is continuous, then the inverse image of any closed subset of $\mathbb{K}$ is closed in $E$. Next, $\{0\}$ is a finite subset of $(\mathbb{K}, |.|)$, which is a Hausdorff space, so $\{0\}$ is closed in $(\mathbb{K}, |.|)$, thus

$$f^{-1}(\{0\}) = Ker(f) \quad \text{is closed.}$$

( $\Longleftarrow$ ), we shall prove the contrapositive, that is

$$f \textit{ is not continious } \Longrightarrow Ker(f) \textit{ is not closed}$$

Suppose that $f$ is not continuous, so $f \neq 0_{\mathcal{L}(E,K)}$, that is there exist $u \in E$ such that $f(u) \neq 0$, so by setting $v = \frac{1}{f(u)} \cdot u$, we have $f(v) = 1$, Next $f$ is continuous which means that the quantity

$$\frac{|f|}{\|x\|_E} \quad (x \in E \setminus \{0_E\})$$

is not bounded, from above for every $n \in \mathbb{N}$, we can find $x_n \in E \setminus \{0_E\}$ such that

$$\frac{|f(x_n)|}{\|x_n\|} \geq n$$

that is

$$|f(x_n)| \geq n\|x_n\| > 0$$

next, let us consider the sequence $(y_n)_{n\in\mathbb{N}}$ of $E$, defined by :

$$y_n := v - \frac{1}{f(x_n)} \cdot x_n \quad \forall n \in \mathbb{N}$$

On the other hand, we have for all $n \in \mathbb{N}$

$$f(y_n) = f(v) - \frac{1}{f(x_n)} \cdot f(x_n) = 1 - 1 = 0$$

implying that $(y_n)_{n\in\mathbb{N}}$ is a sequence of $Ker(f)$, and we have for all $n \in \mathbb{N}$ :

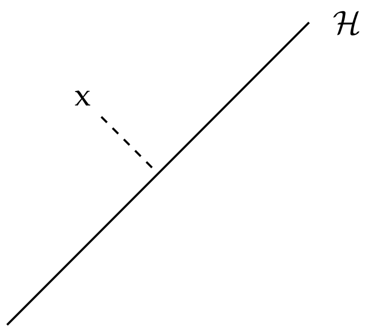
$$\|y_n - v\| = \| - \frac{1}{f(x_n)} x_n\| = \frac{\|x_n\|}{|f(x_n)|} \le \frac{1}{n}$$

so

$$\lim_{n\to\infty} \|y_n - v\| = 0$$

implying that $(y_n)_{n\in\mathbb{N}}$ converge to $v$, but we have $f(v) = 1 \neq 0$, so $v \notin ker(f)$, we can see that $(y_n)_{n\in\mathbb{N}}$ is a sequence of $Ker\,(f)$ which converges to $v \notin Ker(f)$ , this implies that $Ker(f)$ is not a closed set in $E$, as required, this completes the proof. □

## 3.3 The distance between a vector to a closed hyper plane of a N.V.S




**Theorem 3.3.1: (Ascoli)**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $f$ be a contunuous linear form on $E$, next let $a \in \mathbb{K}$ and

$$\mathcal{H} := \{x \in E : f(x) = a\}$$

then for all $u \in E$ , we have

$$d\,(u, H) = \frac{|f(u) - a|}{|||\, f\, |||}$$

To prove the above theorem, we use the following lemma, let $u \in E \backslash H$ be fixed, then for any $x \in E \backslash Ker(f)$ can be written as :

$$x = \lambda\,(u - h)$$

for some $\lambda \in \mathbb{K}^*$ and some $h \in H$

*Proof.* we will prove the lemma first, let $x \in E \backslash Ker(f)$, and put $h := u - \frac{f(u)-a}{f(x)} \cdot x$. then, we have

$$f(h) = f(u) - \frac{f(u)-a}{f(x)} \cdot f(x) = a$$

implying that $h \in H$, finally $h = u - \frac{f(u)-a}{f(x)} \cdot x$ gives

$$x = \frac{f(x)}{f(u)-a}(u-h)$$

putting

$$\lambda := \frac{f(x)}{f(u)-a} \in \mathbb{K}^*$$

we get $x = \lambda(u-h)$ , as required □

now after we warmed up, lets prove the theorem

*Proof.* The Ascoli formula is trivial when $u \in \mathcal{H}$, so let us prove the Ascoli formula for a fixed $u \in E \backslash H$, we have :

$$\begin{aligned} |||\, f \,||| := \sup_{x \in E \backslash \{0_E\}} \frac{|f(x)|}{\|x\|_E} &= \sup_{x \in E \backslash Ker(f)} \frac{|f(x)|}{\|x\|} \\ &= \sup_{\lambda \in \mathbb{K}^*, h \in H} \frac{|f(\lambda(u-h))|}{\|\lambda(u-h)\|} \\ &= \sup_{\lambda \in \mathbb{K}^*, h \in H} \frac{|\lambda|\,|f(u-h)|}{|\lambda|\,\|u-h\|} \\ &= \sup_{h \in H} \frac{|f(u)-f(h)|}{\|u-h\|} \\ &= \sup_{h \in H} \frac{|f(u)-a|}{\|u-h\|} \end{aligned}$$

after factoring out the $|f(u)-a|$ we get

$$\begin{aligned} |f(u)-a| \sup_{h \in H} \frac{1}{\|u-h\|} &= \frac{|f(u)-a|}{inf_{h \in H}\|u-h\|} \\ &= \frac{|f(u)-a|}{inf_{h \in H} d(u,h)} \\ &= \frac{|f(u)-a|}{d(u,h)} \end{aligned}$$

hence we get

$$|||\, f \,||| = \frac{|f(u)-a|}{d(u,H)}$$

which gives us the result

$$d(u,H) = \frac{|f(u)-a|}{|||\, f \,|||}$$

as required. □

In the euclidean place equipped with orthonormal basis, determine a closed formula for the distance between a point $(x_0, y_0)$ and a straight line of equation $ax + by + c = 0$, where $a, b, c \in \mathbb{R}$, where $(a, b) \neq (0, 0)$

**Solution**

we apply the Ascoli formula for $u = (x_0, y_0) \in \mathbb{R}^2$ and $H$ the straight line in the questio, so for the linear form $f$ defined by

$$f(x,y) = ax + by \quad \forall (x,y) \in \mathbb{R}^2$$

doing so we get :

$$\begin{aligned} d\left((x_0, y_0), H\right) &= \frac{|f(x_0, y_0) - (-c)|}{|||\, f\, |||} \\ &= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \end{aligned}$$

**Theorem 3.3.2: F.Riesz Theorem**

A N.V.S (over $\mathbb{R}$ or $\mathbb{C}$ ) is finite-dimensional if and only if $\overline{B}(0_E, 1)$ is compact.

*Proof.* First

$$(\implies)$$

Suppose that $E$ is finite-dimensional since $\overline{B}(0_E, 1)$ is both closed and bounded then by some theorem we wrote above, then it's compact as required

$$(\impliedby)$$

Suppose that $\overline{B}(0_E, 1)$ is a compact part of $E$ and let us show that $dimE < \infty$, obviously we have

$$\overline{B}(0_E, 1) \subset \bigcup_{x \in \overline{B}(0_E, 1)} B\left(x, \frac{1}{2}\right)$$

Since $\overline{B}(0_E, 1)$ is compact then

$$\exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in \overline{B}(0_E, 1) : \quad \overline{B}(0_E, 1) \subset \bigcup_{i=1}^{n} \overline{B}(x_i, 1/2)$$

we are going to show that

$$E = \langle x_1, \ldots, x_n \rangle$$

implying that

$$dimE \leq n < \infty$$

let us set

$$F := \langle x_1, \dots, x_n \rangle$$

and let us show that $E = F$, i.e $E \subset F$, let $x \in E$ be arbitrary and let us show that $x \in F$, to do so we will first show that for any vector $y \in F$, we choose close to $x$, that is another $y' \in F$ which is half closer, in other words $x$ satisfies the property

$$\forall y \in F, \exists y' \in F : \quad \|x - y'\| \leq \frac{1}{2}\|x - y\|$$

so let $y \in F$ be arbitrary and let us show the existence of $y' \in F$ which satisfies the above inequality, if $y = x$, it suffices to take $y' = y = x$ to have

$$\|x - y'\| \leq \frac{1}{2}\|x - y\|$$

Else if $y \neq x$, then we have $\|x - y\| \neq 0$, now we can define

$$z := \frac{x - y}{\|x - y\|}$$

since we have obviously that $z \in \overline{B}(0_E, 1)$, then according to the above there exist $i \in \{1, \dots, n\}$ such that $z \in B(x_i, \frac{1}{2})$, next set

$$y' := \underbrace{y}_{\in F} + \|x - y\| x_i$$

since $x_i, y \in F$ and $F$ is a vector subspace of $E$ then $y' \in F$. In addition we have

$$\begin{aligned} x - y' &= \underbrace{x - y}_{\|x-y\|z} - \|x - y\| x_i \\ &= \|x - y\|(z - x_i) \end{aligned}$$

Thus

$$\begin{aligned} \|x - y'\| &= \|x - y\| \underbrace{\|z - x_i\|}_{<1/2} \quad (z \in B(x_i, 1/2)) \\ &\leq \frac{1}{2}\|x - y\| \end{aligned}$$

so the property is confirmed. Now by re iterating (2) several times starting from $y = y_0 = 0_E$, we get

$$\begin{aligned} &\forall k \in \mathbb{N}, \exists y_k \in F : \quad \|x - y_k\| \leq \frac{1}{2^k}\|x - \underbrace{y_0}_{=0_E}\| \\ &\forall k \in \mathbb{N}, \exists y_k \in F : \quad \|x - y_k\| \leq \frac{1}{2^k}\|x\| \to 0 \text{ as } k \to \infty \end{aligned}$$

showing that the sequence $(y_k)_{k\in\mathbb{N}}$ of $F$ that converges to $x$, but since $F$ is closed because it's finite dimensional then $\lim_{k\to\infty} y_k = x \in F$, consequently we have $E = F$, thus $dimE = dimF < \infty$, this completes the proof □

**corollary 3.3.1: F.Riesz**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then the following properties are equivalent :

(i) $E$ is finite-dimensional

(ii) $\overline{B}(0_E, 1)$ is compact

(iii) The compact parts of $E$ we exactly its parts which are both closed and bounded

(iv) $E$ is locally compact

*Proof.* This equivalence $(i) \iff (iii)$ is provided theorem $0,8$. The implication $(i) \implies (iii)$ is provided by corollary $(2)$, The two implications $(iii) \implies (ii)$ and $(iii) \implies (iv)$ are trivial, To complete the proof it suffices to show that for example the implication

$$(iv) \implies (ii)$$

Suppose that $E$ is locally compact and show that $\overline{B}(0_E, 1)$ is locally compact and show that $\overline{B}(0_E, 1)$ is compact, by hypothesis, the zero vector $0_E$ of $E$ has atleast a compact neighborhood $V$, so $\exists r > 0$ such that $B(0_E, r) \subset V$, so :

$$\overline{B}(0_E, \frac{r}{2}) \subset B\,(0_E, r) \subset V$$

The inclusion $\overline{B}(0_{E,\frac{r}{2}}) \subset V$, implies that $\overline{B}(0_E, \frac{r}{2})$ is compact in $E$, since $\overline{B}(0_E, \frac{r}{2})$ is a closed part of $E$, included in the compact part $V$, Finally since $\overline{B}(0_E, 1)$ is the image of closed ball $\overline{B}(0_E, \frac{r}{2})$ by the continious map

$$\begin{array}{rccc} f: & E & \longrightarrow & E \\ & x & \longmapsto & \frac{2}{r}x \end{array}$$

we deduce that $\overline{B}(0_E, 1)$ is compact, as required this completes the proof □

# 4 CONTINUOUS MULTILINEAR MAPPING ON N.V.S

For simplicity we only study the continuous bilinear mappign N.V.S and we give with proofs the generalization of the obtained results to the continuous multilinear mapping on N.V.S let $\mathbb{K} = \mathbb{R}$ or $(\mathbb{C})$ and let $E, F$ and $G$ be three N.V.S on $\mathbb{K}$. The product toplogy of $E \times F$ can be induced by several norms on $E \times F$ one of these norms is defined by

$$\begin{array}{rccc} f: & E \times F & \longrightarrow & [0, \infty] \\ & (x, y) & \longmapsto & \max\left(\|x\|_E, \|y\|_E\right) \end{array}$$

For what all follows, we work with this norm which we denote $\|.\|_{E\times F}$

*The $\mathbb{K}$-vector space of the bilinear mappings from $E \times F$ to $G$ is denoted by*

$$L(E, F; G) \neq \mathcal{L}\left(E \times F; G\right)$$

*and the $\mathbb{K}$-vector space of the continuous bilinear mappings from $\mathbb{E} \times F$ to $G$ is denoted :*

$$\mathcal{L}(E, F; G)$$

**Theorem 4.0.1: Fundamental**

Let $f \in L(E, F; G)$, then the following properties are equivalent

(i) $f$ is continuous on $E \times F$

(ii) $f$ is continuous at $(0_E, 0_F)$

(iii) $f$ is bounded on $\overline{B}_E(0_E, 1) \times \overline{B}_F(0_F, 1)$

(iv) $f$ is bounded on $S_E(0_E, 1) \times S_F(0_F, 1)$

(v) $\exists M > 0$ such that

$$\forall (x,y) \in E \times F : \quad \|f(x,y)\|_G \leq M\|x\|_E\|y\|_F$$

*Proof.* we have to show the following implications :

$$(i) \implies (ii) \implies (iii) \implies (iv) \implies (v) \implies (i)$$

since the implication $(i) \implies (ii)$ and $(iii) \implies (iv)$ are obvious, we have just to show the three implications,

$$(ii) \implies (iii) \quad \text{and} \quad (iv) \implies (v) \quad \text{and} \quad (v) \implies (i)$$

$$((ii) \implies (iii))$$

Suppose that $f$ is continuous at $(0_E, 0_F)$ , so take $(\varepsilon = 1)$ there exist $\mu > 0$ such that

$$\forall (x,y) \in E \times F : \quad \|(x,y) - (0_E, 0_F)\| \leq \mu \implies \|f(x,y) - f(0_E, 0_F)\| \leq 1$$

That is,

$$\forall (x,y) \in E \times F : \quad (\|x\|_E \leq \mu \text{ and } \|y\|_F \leq \mu) \implies \|f(x,y)\|_G \leq 1 \qquad (1)$$

Now, let $(x,y) \in \overline{B_E}(0_E, 1) \times \overline{B_F}(0_F, 1)$ be arbitrary, then we have $\|\mu x\|_E \leq \mu$ and $\|\mu y\|_F \leq \mu$, implying according to (1) that

$$\|f(\mu x, \mu y)\|_G \leq 1 \iff \|f(x,y)\|_G \leq \frac{1}{\mu^2}$$

so, we have

$$\forall (x,y) \in \overline{B_E}(0_E, 1) \times \overline{B_F}(0_F, 1) : \|f(x,y)\|_G \leq \frac{1}{\mu^2}$$

This shows that $f$ is bounded on

$$\overline{B_E}(0_E, 1) \times \overline{B_F}(0_F, 1)$$

as required.

$$(\, (iv) \implies (v)\, )$$

Suppose that $f$ is bounded on $S_E(0_E, 1) \times S_F(0_F, 1)$ this means that there exist $M > 0$, such that,

$$\forall (x,y) \in S_E(0_E, 1) \times S_F(0_F, 1) : \|f(x,y)\|_G \leq M \qquad (2)$$

Now, let $(x,y) \in (E \setminus \{0_E\}) \times (F \setminus \{0_F\})$, then we have

$$\left( \frac{x}{\|x\|_E}, \frac{y}{\|y\|_F} \right) \in S_E(0_E, 1) \times S_F(0_F, 1)$$

implying according to (2) that,

$$\|f\left(\frac{x}{\|x\|_E}, \frac{y}{\|y\|_F}\right)\|_G \leq M$$

since we have that $f$ is bilinear we get

$$\|f(x,y)\|_G \leq M\|x\|_E\|y\|_F$$

as required.

*( This ineqaulity also holds for $x = 0_E$ and $y = 0_F$ )*

$$(v) \implies (i)$$

Suppose that there exist $M > 0$ such that

$$\forall (x,y) \in E \times F \quad \|f(x,y)\|_G \leq M\|x\|_E\|y\|_F$$

and let us show that $f$ is continuous on $E \times F$, that is $f$ is continuous at every $(x_0, y_0) \in E \times F$, so let $(x_0, y_0) \in E \times F$ be arbtirary and let us show that $f$ is continuous at $(x_0, y_0)$.

we have to show that,

$$\forall \varepsilon > 0, \exists \mu > 0 \text{ s.t. } \forall (x,y) \in E \times F : \|(x,y) - (x_0,y_0)\|_{E\times F} \leq \mu \implies \|f(x,y) - f(x_0,y_0)\|_G \leq \varepsilon \qquad (2)$$

let $\varepsilon > 0$ and take $\mu = \min\left\{1, \frac{\varepsilon}{M(1+\|x_0\|_E+\|y_0\|_F)}\right\}$, and let $(x,y) \in E \times F$ satisfying that,

$$\|(x,y) - (x_0,y_0)\|_{E\times F} \leq \mu$$

that is,

$$\|x - x_0\|_E \leq \mu \quad \text{and} \quad \|y - y_0\|_F \leq \mu$$

then we have,

$$\begin{aligned}
\|f(x,y) - f(x_0,y_0)\|_G &= \|f(x,y) - f(x_0,y) + f(x_0,y) - f(x_0,y_0)\|_G \\
&= \text{ bilinear } \|f(x-x_0,y) + f(x_0,y-y_0)\|_G \\
&\leq \underbrace{\|f(x-x_0,y)\|_G}_{\leq M\|x-x_0\|_F\|y\|_F} + \underbrace{\|f(x_0,y-y_0)\|_G}_{\leq M\|x_0\|_E\|y-y_0\|_F} \\
&\leq M\underbrace{\|x-x_0\|_E}_{\leq\mu}\|y\|_F + M\|x_0\|_E\underbrace{\|y-y_0\|_F}_{\leq\mu} \\
&\leq \mu M(\underbrace{\|y\|_F}_{\leq\|y-y_0\|_F+\|y_0\|_F\leq\mu+\|y_0\|_F} + \|x_0\|_E) \\
&\leq \mu M(\underbrace{\mu}_{\leq 1} + \|x_0\|_E + \|y_0\|_F) \\
&\leq \mu M(1 + \|x_0\|_E + \|y_0\|_F) \\
&\leq \varepsilon
\end{aligned}$$

Property (3) is then confirmed. Thus $f$ is continuous on $E \times F$, as required.
This completes the proof. $\square$

**Example 01**
Let $(E, \langle . \rangle)$ be a real pre-Hilbert space, prove that the inner product $\langle . \rangle : E^2 \longrightarrow \mathbb{R}$ is continuous on $E^2$.
**Solution 01**
$\langle . \rangle$ is bilinear form on $E^2$, we have according to the Cauchy schwarz inequality that for all $x, y \in E$,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

showing that according to item $(v)$ to the theorem, that $\langle . \rangle$ is continuous on $E^2$.
**Example 02**
Let $E, F$ and $G$ be there N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $f : E \times F \longrightarrow G$ be a continuous bilinear mapping, show that the mappings $f(x, .)(x \in E)$ and $f(., y)(y \in E)$ defined by,

$$\begin{array}{rccc} f(x,.): & F & \longrightarrow & G \\ & y & \longmapsto & f(x,y) \end{array}$$

and

$$\begin{array}{rccc} f(.,y): & E & \longrightarrow & G \\ & x & \longmapsto & f(x,y) \end{array}$$

are continuous.
**Solutions 02**
Since $f$ is bilinear then $f(x, .)(x \in E)$ and $f(., y)(y \in F)$ are all linear, next since $f : E \times F \longrightarrow G$ is bilinear and continuous, then there exist $M > 0$, such that for all $(x, y) \in E \times F$,

$$\|f(x,y)\|_G \leq M \|x\|_E \|y\|_F$$

now for $x \in E$ fixed, we have,

$$\forall y \in F, \|f(x,.)(y)\|_G \|f(x,y)\|_G \leq \underbrace{(M \cdot \|x\|_E)}_{\text{independent of } y} \|y\|_F$$

implying that $f(x, .)$ is continuous, we have,

$$\forall x \in E, \|f(.,y)(x)\|_G = \|f(x,y)\|_G \leq \underbrace{(M \cdot \|y\|_F)}_{\text{independent of } x} \cdot \|x\|_E$$

implying that $f(., y)$ is continuous on $E$.
**Question**
Is the converse of the result of **Example 02** true?? i.e.,

The partial continuity of a bilinear map with respect to each argument. $\stackrel{?}{\Longrightarrow}$ The continuity.

**Example 03**

let,

$$\ell^1 := \left\{ (x_n)_{n\in\mathbb{N}} \text{ real sequence such that } \sum_{n=1}^{\infty} |x_n| \text{ converges} \right\}$$

for $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, we define

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \qquad (\text{is a norm on } \ell^1\ )$$

consider,

$$\begin{array}{cccc} f: & \ell_1^2 & \longrightarrow & \mathbb{R} \\ & (x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}}) & \longmapsto & \sum_{n=1}^{\infty} x_n y_n \end{array}$$

(1) Show that $f$ is well-defined and that is symmetric and bilinear.

(2) Show that $f(x,.)\quad (x \in \ell^1)$ and $f(.,y)\quad (y \in \ell^1)$ are both continuous on $\ell^1$, but $f$ is not continuous.

**Solution 03**

(1) For all $x, y \in \ell^1$, we have,

$$\sum_{n=1}^{\infty} |x_n y_n| \le \sum_{n=1}^{\infty} |x_n|\,|y_n| \le \underbrace{\left(\sum_{n=1}^{\infty} |x_n|\right)}_{<\infty} \underbrace{\left(\sum_{n=1}^{\infty} |y_n|\right)}_{<\infty} < \infty$$

thus $\sum_{n=1}^{\infty} |x_n y_n|$ is convergent, that $\sum_{n=1}^{\infty} x_n y_n$ is absolutely convergent, so convergent. Hence $f$ is well-defined.

*The symmetry and the bilinearity of $f$ are obvious.*

(2) Let $x \in \ell^1$ be fixed and let us show that the linear map $f(x,.)$ is continuous on $\ell^1$, for all $y \in \ell^1$, we have,

$$\begin{aligned} |f(x_i)(y)| = |f(x,y)| = \left| \sum_{n=1}^{\infty} x_n y_n \right| &\le \sum_{n=1}^{\infty} |x_n y_n| \\ &\le \sum_{n=1}^{\infty} |x_n|\,|y_n| \\ &\le \left(\sum_{n=1}^{\infty} |x_n|\right) \|y\|_\infty \end{aligned}$$

i.e.

$$|f(x_i)(y)| \leq \overbrace{\sum_{n=1}^{\infty} |x_n|}^{M} \|y\|_\infty$$

Since the series $\sum_{n=1}^{\infty} |x_n|$ converges, since $x \in \ell^1$, then the last inequality show that $f(x_i)$ is continious on $\ell^1 (\forall x \in \ell^1)$, By the same way or by symmetry, we show that $f(.,y)$ where $y$ is fixed in $\ell^1$, is continuous on $\ell^1$.

(3) Now Let us show that $f$ is not continuous for $n \in \mathbb{N}$ arbitrary, let,

$$u_n = \begin{cases} 1 \text{ if } 1 \leq n \leq N \\ 0 \text{ if } n > N \end{cases} \qquad (\forall n \in \mathbb{N})$$

where

$$v_n = u_n \quad (\forall n \in \mathbb{N})$$

put $u = (u_n)_{n\in\mathbb{N}}$ , $v = (v_n)_{n\in\mathbb{N}}$.

$$u = (1, 1, \ldots, 1, 0, 0, \ldots)$$
$$v = (1, 1, \ldots, 1, 0, 0, \ldots)$$

It's clear that $u, v \in \ell^1$, since

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} |v_n| = N < \infty$$

On the other hand, we have,

$$\frac{|f(u,v)|}{\|u\|_\infty \cdot \|v\|_\infty} \leq \frac{N}{1 \times 1} = N$$

hence,

$$\sup_{x,y\in\ell^1\setminus\{0_{\ell^1}\}} \frac{|f(x,y)|}{\|x\|_\infty \|y\|_\infty} = \infty$$

implying that $f$ is not continuous.

## 4.1 A norm on $\mathcal{L}(E, F; G)$

Let $E, F$ and $G$ be three N.V.S over a same field, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ for $f \in \mathcal{L}(E, F; G)$, we define $||| f |||$ by,

$$||| f ||| := \sup_{\substack{x \in E\setminus\{0_E\} \\ y \in F\setminus\{0_F\}}} \frac{\|f(x,y)\|_G}{\|x\|_E \|y\|_F}$$

According to item $(v)$ of theorem 1, we have that

$$||| f ||| \in [0, \infty) \quad \text{i.e. } (||| f ||| < \infty)$$

so $|||\ .\ |||$ constitues a map from $\mathcal{L}(E, F; G)$ to $[0, \infty)$

**Theorem 4.1.1:**

The map $|||\ .\ |||$ defined above is a norm on $\mathcal{L}(E, F; G)$

*Proof.* Exercise. □

**Terminology**

The norm $|||\ .\ |||$ defined above on $\mathcal{L}(E, F; G)$ is called the subordinate norm induced by the norm $\|.\|_E$ of $E$ and $\|.\|_F$ of $F$, and $\|.\|_G$ of $G$.

we have several variants of the definition of a subordinate norm, including the following, $\forall f \in \mathcal{L}(E, F; G)$,

$$\begin{aligned} ||| f ||| = \sup_{\substack{x \in \overline{B_E}(0_E, 1) \\ y \in \overline{B_F}(0_F, 1)}} \|f(x, y)\|_G &= \sup_{\substack{x \in \overline{B_E}(0_E, 1) \\ y \in \overline{B_F}(0_F, 1)}} \|f(x, y)\|_G \\ &= \sup_{\substack{x \in \overline{S_E}(0_E, 1) \\ y \in \overline{S_F}(0_F, 1)}} \|f(x, y)\|_G \\ &= \inf \{M > 0 \text{ such that } \|f(x, y)\|_G \leq M\|x\|_E\|y\|_F, \forall x, y \in E, F\} \end{aligned}$$

*Proof.* Exercise! □

we have the following proposition.

**Theorem 4.1.2:**

Let $E, F$ and $G$ be three N.V.S over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $f \in L(E, F; G)$ then we have,

(1) If $f$ is continuous then

$$\forall (x, y) \in E \times F, \|f(x, y)\|_G \leq ||| f ||| \cdot \|x\|_E \cdot \|y\|_F$$

(2) if $M > 0$ satisfies

$$\|f(x, y)\|_G \leq M\|x\|_E\|y\|_F \quad (\forall (x, y) \in E \times F)$$

then $f$ is continuous and $||| f ||| \leq M$

we also have the following propositions,

**Theorem 4.1.3:**

Let $E, F$ and $G$ be three N.V.S, over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ suppose that $E$ and $F$ are both dimensional, then every bilinear mapping from $E \times F$ to $G$ is continuous,

$$(\text{ i.e. } \mathcal{L}(E, F; G) = L(E, F; G)$$

*Proof.* (Exercise) □

**Theorem 4.1.4:**

Let $E, F$ and $G$ be three N.V.S, over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, suppose that $G$ is Banach, then the $\mathbb{K}$-N.V.S $\mathcal{L}(E, F; G)$ is Banach.

*Proof.* Exercise □

**Corollary**

Let $E, F$ be two N.V.S over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then $\mathcal{L}(E, F; \mathbb{K})$ is Banach, that space is called the space of continuous bilinear forms on $E \times F$

## 4.2 An important isomorphism isometric

Let $E, F$ and $G$ be three N.V.S over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then there exist a natural transformation from $\mathcal{L}(E, \mathcal{L}(F, G))$ to $\mathcal{L}\,(E, F; G)$, which is defined by

$$\begin{array}{rccc} i: & \mathcal{L}\,(E, \mathcal{L}\,(F, G)) & \longrightarrow & \mathcal{L}\,(F, F; G) \\ & f & \longmapsto & i\,(f): \begin{array}{ccc} E \times F & \longrightarrow & G \\ (x, y) & \longmapsto & i(f)(x, y) = f(x) f(y) \end{array} \end{array}$$

Its easy to show that its well defined, linear and bijective with $i^{-1}$ give :

$$\begin{array}{rccc} i^{-1}: & \mathcal{L}(E, F; G) & \longrightarrow & \mathcal{L}\,(E, \mathcal{L}(F, G)) \\ & g & \longmapsto & i^{-1}(g): \begin{array}{ccc} E & \longrightarrow & \mathcal{L}(F, G) \\ x & \longmapsto & i^{-1}(g)(x) \end{array} : \begin{array}{ccc} F & \longrightarrow & G \\ y & \longmapsto & i^{-1}(g)(x)(y) = g(x, y) \end{array} \end{array}$$

now let us show that $i$ is an isometry, with respect to the natural norms defined on $\mathcal{L}(E,F;G)$ and $\mathcal{L}(E,\mathcal{L}(F,G))$, for all $f \in \mathcal{L}(E,\mathcal{L}(F,G))$, we have

$$\begin{aligned}
\|i(f)\|_{\mathcal{L}(E,F;G)} = \sup_{\substack{x \in E\backslash 0_E \\ y \in F\backslash 0_F}} \frac{\|i(f)(x,y)\|_G}{\|x\|_E\|y\|_F} &= \sup_{\substack{x \in E\backslash 0_E \\ x \in F\backslash 0_F}} \frac{\|f(x)(y)\|_G}{\|x\|_E\|y\|_F} \\
&= \sup_{x\in E\backslash\{0_E\}} \frac{1}{\|x\|_E} \sup_{y\in F\backslash\{0_F\}} \frac{\|f(x,y)\|_G}{\|y\|_F} \\
&= \sup_{x\in E\backslash\{0_E\}} \frac{1}{\|x\|_E} \|f(x)_{\mathcal{L}(F,G)}\| \\
&= \|f\|_{\mathcal{L}(E,\mathcal{L}(F,G))}
\end{aligned}$$

that is $i$ is an isometry, because of the isomorphism isometric $i$ between $\mathcal{L}(E,\mathcal{L}(F,G))$ and $\mathcal{L}(E,F;G)$, we often identify $\mathcal{L}(E,\mathcal{L}(F,G))$ to $\mathcal{L}(E,F;G)$, This is used in particular in differential calculus on N.V.S (for defining second derivative)

## 4.3 An introduction to differential calculus in N.V.S

Let $E$ and $F$ be two N.V.S over the a same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $U$ be an open subset of $E$ and $a \in U$. Finally, let $f : U \longrightarrow F$ be a map

**Definition 4.3.1:**

We say that $f$ is differentiable at $a$ if there exist $g \in \mathcal{L}(E,F)$ so that we have in the neighborhood of $a$

$$\|f(x) - f(a) - g(x-a)\|_F = o\left(\|x-a\|_E\right)$$

**Remark**

(1) If $f$ is differentiable at $a$ then $f$ is continuous at $a$. Indeed, by letting $x \to a$, we obtain since ( $g$ is continuous at $0_E$ ), that $\lim_{x\to a} f(x) = f(a)$, showing that $f$ is continuous at $a$.

(2) If $f$ is idfferentiable at $a$ then the continuous linear mapping $g$ is unique.

*Proof.* Let $g_1, g_2 \in \mathcal{L}(E,F)$, each of them satisfies

$$\|f(x) - f(a) - g_1(x-a)\|_F = o\left(\|x-a\|_E\right)$$
$$\|f(x) - f(a) - g_2(x-a)\|_F = o\left(\|x-a\|_E\right)$$

when $x$ is in the neighborhood of $a$, so for all $h \in E$ ( in the neighborhood of $0_E$ , we have

$$\begin{aligned}\|(g_1 - g_2)(h)\|_F &= \|g_1(h) - g_2(h)\|_F \\ &= \| \,(f(a+h) - f(a) - g_2(h)) - (f(a+h) - f(a) - g_1(h))\, \| \\ &\leq \underbrace{\|f(a+h) - f(a) - g_2(h)\|_F}_{o(\|h\|_E)} + \underbrace{\|f(a+h) - f(a) - g_1(h)\|_F}_{o(\|h\|_E)} = o(\|h\|_E)\end{aligned}$$

Thus $\|(g_1 - g_2)(h)\|_F = o(\|h\|_E)$, in other words

$$\lim_{\|h\|_E \to 0} \frac{\|(g_1 - g_2)(h)\|_F}{\|h\|_E} = 0$$

now let $x \in E \setminus \{0_E\}$ be arbitrary, by taking $h = \varepsilon x$ and $(\varepsilon \to^{>} 0)$, we get

$$\lim_{\varepsilon \to^{>} 0} \frac{\|(g_1 - g_2)(\varepsilon x)\|_F}{\|\varepsilon x\|_E} = 0$$

thus we see

$$\frac{\|(g_1 - g_2)(x)\|_F}{\|x\|_E} = 0$$

thus we see that

$$g_1(x) = g_2(x) \quad (\forall x \in E \setminus \{0_E\})$$

which remains true for $x = 0_E$, hence $g_1(x) = g_2(x)$ for all $x \in E$, therefore $g_1 = g_2$, by the uniqueness of $g$ is then proved. □

**Definition 4.3.2:**

If $f$ is differentiable at $a$ then the continuous linear mapping $g$ satisfying

$$\|f(x) - f(a) - g(x-a)\|_F = o(\|x - a\|_E)$$

is called

*The derivative of $f$ at $a$, and it's denoted $f'(a)$*

## 4.4 Relationship with the classical case $E = F = \mathbb{R}$

If $E = F = \mathbb{R}$, and $U$ is an open subset of $\mathbb{R}$, $f : U \longrightarrow \mathbb{R}$, and $a \in U$ then the classical definition of the differentiability states that

*$f$ is differentiable at $a$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exsits (i.e. $\in \mathbb{R}$)*

So if its the case and we let

$$l := \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

we desire that

$$\lim_{x\to a} \left( \frac{f(x)-f(a)}{x-a} - l \right) = 0$$

that is

$$\lim_{x\to a} \frac{f(x)-f(a)-l(x-a)}{x-a} = 0$$

therefore we see

$$|f(x)-f(a)-l(x-a)| = o(|x-a|) \quad \text{when } x \to a$$

so hence

$$\begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & lx \end{array}$$

$$\in \mathcal{L}(\mathbb{R},\mathbb{R})$$

satisfies, so in the sense of Definition 2, $f$ is differnetiable at $a$ and

$$f'(a) = \left[ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & lx \end{array} \right]$$

By identifying the homothety of center $0$ and ratio $l$ to $l$, we obtain the equivalence between the classical case $(E = F = \mathbb{R})$, and the general case on N.V.S

$$\begin{array}{rccc} : & \mathbb{R} & \longrightarrow & \mathcal{L}(\mathbb{R},\mathbb{R}) \\ & l & \longmapsto & \mathcal{H}(0,l) : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & lx \end{array} \end{array}$$

is an isomorphism isometric.

In fact, we identify $\mathcal{L}(\mathbb{R},\mathbb{R})$ with $\mathbb{R}$.

**Definition 4.4.1:**

We say that $f$ is differentiable in $U$, if its differnetiable at every point of $U$.

- If $f$ is differentiable in $U$ then it's derivative is the map $f'$ defined by :

$$\begin{array}{rccc} f': & U & \longrightarrow & \mathcal{L}(E,F) \\ & a & \longmapsto & f'(a) \end{array}$$

In the particular case $E = \mathbb{R}$, we can identify $\mathcal{L}(E,F) = \mathcal{L}(\mathbb{R},F)$ to $F$, so we obtain $f' : U \longrightarrow F$ as in the classical case $E = F = \mathbb{R}$.

## 4.5 The Second Derivative

Let $E$ and $F$ be two N.V.S, and $U$ be an open subset of $E$, and $f : U \longrightarrow F$ suppose that $f$ is differentiable in $U$ and let $f' : U \longrightarrow \mathcal{L}(E, F)$ be it's derivative so we can ask if $f'$ is differentiable in $U$

**Definition 4.5.1:**

We say that $f$ is twice differentiable at $a \in U$ if $f'$ is differentiable at $a$. In this case we denote $f''(a)$ the derivative of $f'$ at $a$, so

$$f''(a) \in \mathcal{L}(E, \mathcal{L}(E, F))$$

called the second derivative of $f$ at $a$.

**Definition 4.5.2:**

We say that $f$ is twice differentiable in $U$ if its twice differentiable at every point of $U$. In such a case, the second derivative of $f$ is the map.

$$\begin{array}{rccc} f'': & U & \longrightarrow & \mathcal{L}(E, \mathcal{L}(E, F)) \\ & a & \longmapsto & f''(a) \end{array}$$

Then we often consider $f''(a)(a \in U)$, as an element of $\mathcal{L}(E, E; F)$ that is $f''(a)$ is a continuous bilinear map from $E \times E$ to $F$.

## 4.6 Generalization of the multilinear mappings

Let $n \in \mathbb{N}$, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ , and let $E_1, \ldots, E_n$ and $G$ be N.V.S over $\mathbb{K}$, the topological product space $E_1 \times E_2 \times \ldots \times E_n$, can be represented by several norms, the more simple is perhaps $\|.\|_\infty$ defined by :

$$\begin{array}{rccc} \|.\|_\infty : & E_1 \times E_2 \times \ldots \times E_n & \longrightarrow & [0, \infty) \\ (x_1, \ldots, x_n) & & \longmapsto & \max\left(\|x_1\|_{E_1}, \ldots, \|x_n\|_{E_n}\right) \end{array}$$

☞ *Let* $\mathbb{K}$*-Vector space of the multilinear mappings from* $E_1 \times E_2 \ldots \times E_n$ *to* $G$ *is denoted by* $L(E_1, \ldots, E_n; G)$ *and the* $\mathbb{K}$*-Vector space of the continuous multilinear mappings from* $E_1, \ldots, E_n$ *to* $G$ *is denoted by* $\mathcal{L}(E_1, \ldots, E_N; G)$.

**Theorem 4.6.1: Fundamental**

Let $f \in \mathcal{L}(E_1, \ldots, E_n)$, Then the following properties are equivalent :

(i) $f$ is continuous on $E_1 \times \ldots \times E_n$

(ii) $f$ is continuous on $(0_{E_1}, \ldots, 0_{E_n})$

(iii) $f$ is bounded on

$$\overline{B_{E_1}(0_{E_1}, 1)} \times \overline{B_{E_2}(0_{E_2}, 1)} \times \ldots \times \overline{B_{E_n}(0_{E_n}, 1)}$$

(iv) $f$ is bounded on

$$S_{E_1}(0_{E_1}, 1) \times \ldots \times S_{E_n}(0_{E_n}, 1)$$

(v) $\exists M > 0$ such that

$$\forall (x_1, \ldots, x_n) \in E_1 \times \ldots \times E_n \quad \|f(x_1, \ldots, x_n)\|_G \leq M\|x_1\|_{E_1} \times \ldots \|x_n\|_{E_n}$$

*Proof.* The same as that corresponding to the case where $n = 2$ □

A norm on $\overline{\mathcal{L}(E_1, \ldots, E_n; G)}$ :, for $f \in \mathcal{L}(E_1, \ldots, E_n : G)$, we define $|||\ f\ |||$ by :

$$|||\ f\ |||:= \sup_{x_1, \ldots, x_n \in E_1 \setminus \{0_E\}, \ldots E_n \setminus \{0_{E_n}\}} \frac{\|f(x_1, \ldots, x_n)\|}{\|x_1\|_{E_1} \ldots \|x_n\|_{E_n}}$$

according to item $(v)$ for the previous theorem, we have that $|||\ f\ ||| \in [0, \infty)$, i.e $|||\ f\ |||$ is a non negative real number, so $|||\ .\ |||$ constitutes a map from $\mathcal{L}(E_1, \ldots, E_n; G)$ to $[0, \infty)$ :

The map $|||\ .\ |||$ defined above is a norm on $\mathcal{L}(E_1, \ldots, E_n; G)$, it's called the subordinate norm induced by the norms $\|.\|_{E_1}$ of $E_1$, $\|.\|_{E_2}$ of $E_2, \ldots,$ $\|.\|_n$ of $E_n$, and $\|.\|_G$ of $G$

*Proof.* Exercise! □

**Remark**

All the proposition of $\mathcal{L}(E_1,\dots,E_n;G)$ seen previously for the case $n=2$ are easily and naturally generalizable for every $n$

An important example, let $n \in \mathbb{N}$ and take $E_1 = E_2 = \dots = E_n = \mathbb{R}^n$ and $G = \mathbb{R}$, and we get

$$\begin{array}{rccc} det: & \mathbb{R}^n \times \dots \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x_1,\dots,x_n) & \longmapsto & det(x_1,\dots,x_n) \end{array}$$

It's know that for determinant is multilinear.

Next, since $\mathbb{R}^n$ is finite-dimensional then $det$ is continuous let us equip $\mathbb{R}^n$ with it's eucledean norm $\|.\|_2$ and $\mathbb{R}$ with the absolute value $|.|$.

Then we propose to determine $|||\ det\ |||$, by definition we have

$$|||\ det\ ||| := \sup_{x_1,\dots,x_n \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}} \frac{|det(x_1,\dots,x_n)|}{\|x_1\|_2 \dots \|x_n\|_2}$$

so by taking in particular $(x_1,\dots,x_n) = (e_1,\dots,e_n)$, the canonical basis of $\mathbb{R}^n$, we have that,

$$|||\ det\ ||| \geq \frac{|det(e_1,\dots,e_n)|}{\|e_1\|_2 \dots \|e_n\|_2} = \frac{1}{1 \times 1 \dots \times 1} = 1$$

so

$$|||\ det\ ||| \geq 1$$

To conclude to the exact value of $|||\ det\ |||$, we use the following theorem

**Theorem 4.6.2: Hadamard's inequality**

For every $x_1,\dots,x_n \in \mathbb{R}^n$, we have

$$|det(x_1,\dots,x_n)| \leq \|x_1\|_2 \cdot \dots \|x_n\|_2$$

Besides, the inequality is attained if and only if $x_1,\dots,x_n$, and pairwise orthogonal with respect to the usual inner product of $\mathbb{R}^n$

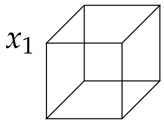
Hadamar's inequality implies immediately that $|||\ det\ ||| = 1$

## 4.7 The geometric sense of Hadamard's inequality

The geometric sense of Hadamard's inequality is the following

*In the Euclidean space of n dimension, the volume of the parallelopiped spanned by the n linearly independent vectors $x_1, \ldots, x_n$ of lenghts $l_1, \ldots, l_n$, is at most equal to $l_1 \cdot l_2 \cdot \ldots \cdot l_n$.*

In addition, this volume is optimal (i.e. Equal to $l_1 \cdot l_2 \ldots l_n$), if and only if the vectors $x_1, \ldots, x_n$ are linearly independent




*Proof.* If $x_1, \ldots, x_n$ are linearly dependent, the Hadamard inequality is trivial, suppose for the sequel that $x_1, \ldots, x_n$ are linearly independent, in other words $(x_1, \ldots, x_n)$ constitutes a basis of $\mathbb{R}^n$, We use the Gram-Schmidtz process to transform $(x_1, \ldots, x_n)$ to an orthogonal basis $(y_1, \ldots, y_n)$ of $\mathbb{R}^n$.

By The Gram-Schmidtz, there exist $\alpha_{ij} \in \mathbb{R}\,(1 \leq j < i \leq n)$ such that the vectors $y_1, \ldots, y_n$ of $\mathbb{R}^n$ defined by

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 + \alpha_{21}x_1 \\ y_3 = x_3 + \alpha_{31}x_1 + \alpha_{32}x_2 \\ \vdots \\ y_n = x_n + \alpha_{n1}x_1 + \alpha_{n2}x_2 + \ldots + \alpha_{n,n-1}x_{n-1} \end{cases}$$

are pairwise orthogonal, by putting the condition in addition for $i, j \in \{1, \ldots, n\}$

$$\alpha_{i,j} = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \quad \text{and} \quad T = \left(\alpha_{i,j}\right)_{1 \leq i,j \leq n} \in \mathcal{M}(\mathbb{R})$$

Which is a linear transformation with diagonal entries all equal to 1, as its non singular, specifically the system can be rewritten as

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

which gives

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = T^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$T^{-1}$ as $(T)$ is lower triangular with diagonal entries all equal to 1, now let

$$(\beta_{i,j})_{1\leq i,j\leq n} = T^{-1} \quad \beta_{i,j} = \begin{cases} 1 & i = j \\ 0 & j < j \end{cases}$$

and we have

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 + \beta_{21}y_1 \\ x_3 = y_3 + \beta_{31}y_1 + \beta_{32}y_2 \\ \vdots \\ x_n = y_1 + \beta_{n1}y_1 + \ldots + \beta_{n,n-1}y_{n-1} \end{cases}$$

Now, since the determinant is an alternating multi linear form then we desire from the above system, that

$$det(x_1, \ldots, x_n) = det(y_1, \ldots, y_n)$$

Next, by the pythagorean theorem, we have according to the system, the fact that $y_i's$ are all pairwise orthogonal, we get that :

$$\begin{cases} \|x_1\|^2 = \|y_1\|^2 \\ \|x_2\|^2 = \|y_2\|^2 + \beta_{21}^2\|y_1\|^2 \geq \|y_2\|^2 \\ \|x_3\|^2 = \|y_3\|^2 + \beta_{31}^2\|y_1\|^2 + \beta_{32}^2\|y_2\|^2 \geq \|y_3\|^2 \\ \vdots \\ \|x_1\|^2 = \|y_1\|^2 + \beta_{n1}^2\|y_1\|^2 + \ldots + \beta_{n,n-1}^2\|y_{n-1}\|^2 \geq \|y_n\|^2 \end{cases}$$

hence we get

$$\|x_1\|^2 \cdot \|x_2\|^2 \cdot \ldots \|x_n\|^2 \geq \|y_1\|^2 \cdot \|y_2\|^2 \ldots \|y_n\|^2$$

that is

$$\|x_1\| \cdot \|x_2\| \ldots \|x_n\| \geq \|y_1\| \cdot \|y_1\| \ldots \|y_n\|$$

now, we are goin to show that

$$|det(y_1, \ldots, y_n)| = \|y_1\| \cdot \|y_2\| \ldots \|y_n\|$$

Let $A = (y_1|y_2|\dots|y_n)$ $(\in \mathcal{M}_n(\mathbb{R}))$, so

$$A^T = \begin{pmatrix} y_1^T \\ \hline y_2^T \\ \hline \vdots \\ \hline y_n^T \end{pmatrix}$$

hence

$$A^T A = \begin{pmatrix} y_1^T \\ \hline y_2^T \\ \vdots \\ \hline y_n^T \end{pmatrix} \begin{pmatrix} y_1| & y_2| & \dots| & y_n \end{pmatrix}$$

which equals

$$\begin{pmatrix} \langle y_1, y_1\rangle & \dots & \langle y_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle y_n, y_1\rangle & \dots & \langle y_n, y_n\rangle \end{pmatrix} = \begin{pmatrix} \|y_1\|^2 & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & \|y_n\|^2 \end{pmatrix}$$

so

$$A^T A = diag(\|y_1\|^2, \dots, \|y_n\|^2)$$

then by taking the determinants

$$(detA)^2 = \|y_1\|^2 \dots \|y_n\|^2$$

then

$$|det(A)| = \|y_1\| \dots \|y_n\|$$

i.e

$$det(y_1, \dots, y_n) = \|y_1\| \dots \|y_n\|$$

confirming the formula, now we have according to 1 ,2 and 3

$$\begin{aligned} |det(x_1, \dots, x_n)| &= |det(y_1, \dots, y_n)| \\ &= \|y_1\|\|y_2\| \dots \|y_n\| \\ &= \|x_1\| \cdot \|x_2\| \dots \|x_n\| \end{aligned}$$

as required, in addition the equlaity

$$|det(x_1, \dots x_n)| = \|x_1\|\|x_2\| \dots \|x_n\|$$

hold if and only if

$$\|y_1\| \dots \|y_n\| = \|x_1\| \ \dots \|x_n\|$$

but this equivalent according to 3 to $\|x_i\| = \|y_i\|$ for all $i$, which is equivalent to $\beta_{i,j} = 0$ for all $i > j$ , that is $T = I_n$ which is equivalent to

$$(y_1, \dots, y_n) = (x_1, \dots, x_n)$$

which holds if and only if $x_1, \dots, x_n$ are pairwsie orthogonal, the proof is complete □

## 4.8 Series in N.V.S

**Definition 4.8.1:**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $(u_n)_{n\in\mathbb{N}}$.
The infinite sum $\sum_{k=0}^{\infty} u_k$, is called the series of $E$ with general term $u_k$, For $n \in \mathbb{N}$ fixed, the finite sum $S_n = \sum_{k=1}^{n} u_k$ is called the $n^{th}$ partial sum (or the partial sum of rank $n$) of the series $\sum_{k=1}^{n} u_k$, we say that the series $\sum_{k=1}^{\infty} u_k$ converges in $E$ if the sequence $(S_n)_{n\in\mathbb{N}}$ converges in $E$, In such a case, we call the limit $S$ of $(S_n)_{n\in\mathbb{N}}$, the sum of the series $\sum_{k=1}^{\infty} u_k$, and we write,

$$\sum_{k=1}^{\infty} u_k = S$$

- Besides for $n \in \mathbb{N}$, $R_n := S - S_n$ is called the $n^{th}$ remainder or the remainder of rank $n$ of the series $\sum_{k=1}^{\infty} u_k$, and we often write,

$$R_n = \sum_{k=n+1}^{\infty} u_k$$

- If a series of $E$is not convergent, we say taht its divergent

*The concept of series is rather important in a banach space, then in an arbitrary N.V.S*

**Definition 4.8.2: Cauchy Criterion**

Let $E$ be a Banach N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\sum_{k=1}^{\infty} u_k$ be a series of $E$. Then $\sum_{k=1}^{\infty} u_k$ is convergent if and only if it satisfies

$$\forall p s > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq \mathbb{N} \implies \| \sum_{k=q+1}^{p} u_k \| \leq \varepsilon$$

*Proof.* Let $(S_n)_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} u_k$, (i.e. $S_n = \sum_{k=1}^{n} u_k$, $\forall n \in \mathbb{N}$), so we have,

$$\begin{aligned}
\sum_{k=1}^{\infty} u_k \text{ is convergent } &\iff (S_n)_{n\in\mathbb{N}} \text{ is convergent} \\
&\iff (S_n)_{n\in\mathbb{N}} \text{ is Cauchy ( Since } E \text{ is Banach )} \\
&\iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq \mathbb{N} \implies \|S_p - S_q\| < ps \\
&\iff p > q \geq \mathbb{N} \implies \|\sum_{k=q+1}^{p} u_k\| < \varepsilon
\end{aligned}$$

as required. □

**Definition 4.8.3:**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, a series $\sum_{k=1}^{\infty} u_k$ of $E$ is said to be *normally convergent* if the real series (with nonegative terms) $\sum_{k=1}^{\infty} \|u_k\|$ converges. (in $\mathbb{R}$)

**Theorem 4.8.1:**

Let $E$ be a Banach N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, if a series $\sum_{k=1}^{\infty} u_k$ of $E$ is *normally convergent* then its convergent and we have in this case :

$$\|\sum_{k=1}^{\infty} u_k\| \leq \sum_{k=1}^{\infty} \|u_k\|$$

*Proof.* Let $\sum_{k=1}^{\infty} u_k$ be a series of $E$, suppose that $\sum_{k=1}^{\infty} u_k$ is normally convergent (i.e. the real series $\sum_{k=1}^{\infty} \|u_k\|$ converges), and let us prove that $\sum_{k=1}^{\infty} u_k$ is convergent for all $p, q \in \mathbb{N}$, with $p > q$ we have,

$$0 \leq \|\sum_{k=q+1}^{p} u_k\| \overset{I.I}{\leq} \sum_{k=q+1}^{q} \|u_k\| \tag{4.1}$$

but since $\sum_{k=1}^{\infty} \|u_k\|$ is assumed convergent in $\mathbb{R}$ then it satisfies the cauchy criterion i.e.,

$$\lim_{p,q\to\infty} \sum_{k=q+1}^{p} \|u_k\| = 0$$

Consequently by applying the squeeze theorem in (1), we get,

$$\lim_{p,q\to\infty} \|\sum_{k=q+1}^{p} u_k\| = 0$$

implying since $E$ is banach, that the series $\sum_{k=1}^{\infty} u_k$ is convergent, as required.

Now let us prove the inequality of the theorem in the case when the series $\sum_{k=1}^{\infty} u_k$ is normally convergent then for all $n \in \mathbb{N}$, we have,

$$\|\sum_{k=1}^{n} u_k\| \leq \sum_{k=1}^{n} \|u_k\|$$

by letting $n \to \infty$, and using the continuity of $\|.\|$, we get,

$$\|\sum_{k=1}^{\infty} u_k\| \leq \sum_{k=1}^{\infty} \|u_k\|$$

as required, This completes the proof □

- **An Important Example** *(Exponential of an operator of a Banach Space)*

Let $E$ be a Banach N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $f \in \mathcal{L}(E) := \mathcal{L}(E, E)$ consider the series $\sum_{n=0}^{\infty} \frac{f^n}{n!}$

in $(\mathcal{L}(E))$, then we have for all $n \in \mathbb{N}_0$, ☞ Note that $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{\text{n times}}$

$$||| \frac{f^n}{n!} |||= \frac{1}{n!} ||| f^n |||\leq \frac{1}{n!} ||| f |||^n$$

Since the real series $\sum_{k=1}^{\infty} \frac{1}{k!} ||| f |||^k$ converges to $\exp(||| f |||)$ then the real series $\sum_{k=1}^{\infty} ||| \frac{f^k}{k!} |||$ is also convergent, that is the series $\sum_{k=1}^{\infty} \frac{f^k}{k!}$ (of $\mathcal{L}(E)$ ) is normally convergent but since $\mathcal{L}(E)$ is Banach,(because $E$ is Banach) then according to the theorem, The series $\sum_{k=1}^{\infty} \frac{f^k}{k!}$ is convergent in $\mathcal{L}(E)$, and we have

$$||| \sum_{k=1}^{\infty} \frac{f^k}{k!} |||\leq e^{|||f|||} \tag{4.2}$$

**Definition 4.8.4:**

In the above situation (i.e. if $E$is a Banach space and $f \in \mathcal{L}(E)$) the sum of the convergent series $\sum_{k=1}^{\infty} \frac{f^k}{k!}$ is called the exponential of the operator $f$ and denoted by $e^f$ or $\exp(f)$, so we have according to (2),

$$||| e^f |||\leq e^{|||f|||} \quad (\forall f \in \mathcal{L}(E)) \tag{4.3}$$

**Remark**

If $E$ is a Banach space, and $f, g \in \mathcal{L}(E)$, the equality of eperators,

$$e^{f+g} = e^f \circ e^g$$

is in general false, but it becomes true when $f$ and $g$ *commute.*

$$\begin{aligned} e^x \cdot e^y &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} \frac{y^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{x^n}{k!} \frac{y^{n-k}}{(n-k)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n = e^{x+y} \end{aligned}$$

In particular, we have for all $f \in \mathcal{L}(E)$,

$$e^f \circ e^{-f} = e^{0_{\mathcal{L}(E)}} = id_E$$
$$e^{-f} \circ e^f = e^{0_{\mathcal{L}(E)}} = id_E$$

Consequently, for every $f \in \mathcal{L}(E)$, the operator $e^f (\in \mathcal{L}(E))$ is invertible (i.e., $e^f \in GL(E)$), and $\left(e^f\right)^{-1} = e^{-f}$.

- **A particular case :** let $n \in \mathbb{N}$, we take $E = \mathbb{K}^n$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and we verify identity $\mathcal{L}(E) = L(E)$ to $\mathcal{M}_n(\mathbb{K})$.

Since $E$ is finite dimensional then its Banach so, we can define the exponential of a matrix $A$ of $\mathcal{M}_n(\mathbb{K})$ by,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \in \mathcal{M}_n(\mathbb{K})$$

in general $e^{A+B} \neq e^A \cdot e^B$, for $A, B \in \mathcal{M}_n(\mathbb{K})$, but if $AB = BA$, then we have $e^{A+B} = e^A \cdot e^B = e^B \cdot e^A$.

**Exercise 01 :**

Let $n \in \mathbb{N}$, and let $\lambda_1, \ddots, \lambda_n \in \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, set $D = diag(\lambda_1, \dots, \lambda_n)$.

(1) Show that

$$e^D = diag(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}) = \begin{pmatrix} e^{\lambda_1} & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & e^{\lambda_n} \end{pmatrix}$$

*Proof.*

$$e^D = \sum_{k=0}^{\infty} \frac{D^k}{k!} = \sum_{k=1}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & \lambda_n^k \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \end{pmatrix}$$

□

**Exercise 02 :**

Llet $n \in \mathbb{N}$, and $P \in GL_n(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $A \in \mathcal{M}_n(\mathbb{K})$.

(1) Show that :

$$\exp\left(P^{-1}AP\right) = P^{-1}\exp(A)P$$

*Proof.*

$$\begin{aligned}\exp(P^{-1}AP)) &= \sum_{k=0}^{\infty}\frac{(P^{-1}AP)^k}{k!} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(P^{-1}A^kP\right)\\ &= P^{-1}\left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)P = P^{-1}e^AP\end{aligned}$$

□

**Theorem 4.8.2:**

Let $n \in \mathbb{N}$, and $x_0 \in \mathbb{R}^n$, and $A \in \mathcal{M}_n(\mathbb{R})$ and denote by $X$ a function of $t$ from $\mathbb{R}$ to $\mathbb{R}^n$, by

$$\begin{aligned} X: \ \mathbb{R} &\longrightarrow \mathbb{R}^n\\ t &\longmapsto X(t)\end{aligned}$$

then the solution of the linear differential system with initial condition

$$\begin{cases} X(0) = x_0\\ X'(t) = A\cdot X(t)\end{cases} \tag{4.4}$$

is the following :

$$X(t) = e^{tA}x_0$$

*Proof.* Put $Y(t) = e^{-tA}X(t)$, then

$$Y'(t) = -Ae^{-tA}X(t) + e^{-tA}X'(t)$$

so $X$ is a solution of (5), we have

$$\begin{cases} X(0) = x_0\\ X'(t) = AX(t)\end{cases} \iff \begin{cases} Y(0) = x_0\\ Y'(t) = 0_{\mathbb{R}^n}\end{cases} \iff Y(t) = x_0 \quad (\forall t \in \mathbb{R})$$

we deduce $X(t) = e^{tA}x_0$ □

- **Problem :** *(How to compute $e^A$ in general ?)*

- ***The Solution :***

*For $n \in \mathbb{N}$ , and $A \in \mathcal{M}_n(\mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, to compute $e^A$, we use the Dunford decomposition of $A$, we write $A$ as,*

$$A = U + N \quad (U, N \in \mathcal{M}_n(\mathbb{K}))$$

*with,*

- *U is diagonalizable in other words there exist* $P \in GL(\mathbb{K})$ *and* $D \in \mathcal{M}_n(\mathbb{K})$ *diagonal such that* $U = PDP^{-1}$.
- *N is nilpotent i.e. there exist* $k \in \mathbb{N}$ *such that.* $N^k = 0$
- *U commutes with N i.e.* $UN = NU$.

*So, since U and N commute with N, we have*

$$e^A = e^{U+N} = e^U \cdot e^N$$

*but we have*

$$e^U = e^{PDP^{-1}} = Pe^D P^{-1}$$

*and*

$$e^N = \sum_{l=0}^{\infty} = \frac{N^l}{l!} = \sum_{l=0}^{k-1} \frac{N^l}{l!}$$

*(since* $N^l = 0$ *for* $l \geq k$*), hence we obtain the closed form of* $e^A$.

*Note that the Dunford decomposition of A can be obtained by using the jordan form A.*

*By the same way, we can define* $\sin(f)$, $\cos(f)$, $\sinh(f)$, *etcetera, when f is continuous, linear operator of a Banach space*

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

- ***Exercise :*** *(Important) Let E be a Banach space, we denote by* $\mathcal{GL}(E)$, *the set of endomorphisms of g of E which are continuous, invertible, and for which* $g^{-1}$ *is continuous, we have*

$$\mathcal{GL}(E) \subset \mathcal{L}(E)$$

*(1) Let* $f \in \mathcal{L}(E)$ *satisfying* $||| f ||| < 1$

  *(a) Show that* $(id_E + f)$ *and* $(id_E - f)$ *are in in* $\mathcal{GL}(E)$

*(2) Deduce that* $\mathcal{GL}(E)$ *is an open subset of* $\mathcal{L}(E)$

*(3) Show that the map*

$$\begin{array}{ccc} \mathcal{GL}(E) & \longrightarrow^{\phi} & \mathcal{GL}(E) \\ f \to & & f^{-1} \end{array}$$

*is continuous*

- ***Solution :***

(1) *First, the continuity and the linearity of* $(id_E + f)$ *and* $(id_E - f)$ *are obvious, are obvious next consider the series*

$$\sum_{n=0}^{\infty} f^n \quad of \quad \mathcal{L}(E) \text{ We have}$$

*for all* $n \in \mathbb{N}_0$,

$$||| f^n ||| \quad \leq \quad ||| f |||^n$$

*Since* $||| f |||< 1$ *then the real geometric series* $\sum_{n=0}^{\infty} ||| f |||^n$ *is convergent, thus the real series* $\sum_{n=0}^{\infty} ||| f^n |||$ *is also convergence, in other words the series* $\sum_{n=0}^{\infty} f^n$ *of* $\mathcal{L}(E)$ *is normally convergent, since* $\mathcal{L}(E)$ *is Banach because E is banach, then* $\sum_{n=0}^{\infty} f^n$ *is convergent in* $\mathcal{L}(E)$*, set*

$$g = \sum_{n=0}^{\infty} f^n \in \mathcal{L}(E)$$

*we have for all* $n \in \mathbb{N}_0$,

$$(id_E - f) \circ \sum_{n=0}^{N} f^n = \sum_{n=0}^{N} \left(f^n - f^{n+1}\right) = id_E - f^{N+1}$$

*By letting* $N \to \infty$*, we get,*

$$(id_E - f) \circ g = id_E$$

*we prove by the same way that* $g \circ (id_E - f) = id_E$*, thus* $(id_E - f)$ *is invertible and* $(id_E - f)^{-1} = g = \sum_{n=0}^{\infty} f^n \in \mathcal{L}(E)$*, thus,*

$$(id_E - f) \in \mathcal{GL}(E)$$

*(motivation* $(1-x) \times \frac{1}{1-x} = 1$ *)*

*by replacing* $f$ *by* $-f$*, we find that* $(id_E + f)$ *is also invertible is also invertible and*

$$(id_E + f)^{-1} = \sum_{n=0}^{\infty} (-f)^n = \sum_{n=0}^{\infty} (-1)^n f^n \in \mathcal{L}(E)$$

*Consequently* $(id_E + f) \in \mathcal{GL}(E)$

*(2)* $\mathcal{GL}(E)$ *is an open subset of* $\mathcal{L}(E)$ *??*

we have to show that $\mathcal{GL}(E)$ is a neighborhood of all if elements so, let $f_0 \in \mathcal{GL}(E)$ arbitrary and let us show that $\exists r > 0$ such that $\mathcal{B}_{\mathcal{L}(E)}(f_0, \frac{1}{|||f_0^{-1}|||})$.

$$\text{That is } f \in \mathcal{L}(E) \text{ and } ||| f - f_0 |||< \frac{1}{|||f_0^{-1}|||}$$

let us show that $f \in \mathcal{GL}(E)$, we have

$$||| f_0^{-1} \circ f - id_E |||=||| f_0^{-1} \circ (f - f_0) ||| \leq ||| f_0^{-1} ||| \cdot \underbrace{||| f - f_0 |||}_{< \frac{1}{|||f_0^{-1}|||}}$$

$$< 1$$

thus according to the result of Question (1), we have

$$\left(f_0^{-1} \circ f - id_E\right) + id_E = f_0^{-1} \circ f \in \mathcal{GL}(E)$$

Thus,

$$f = f_0 \circ \left(f_0^{-1} \circ f\right) \in \mathcal{GL}(E)$$

as required, this confirms the inclusion, so $\mathcal{GL}(E)$ is a neighborhood of any $f_0 \in \mathcal{GL}(E)$, so $\mathcal{GL}(E)$ is an open subset of $\mathcal{L}(E)$.

$$\begin{aligned}
\mathcal{GL}(\mathbb{R}^n) &= GL(\mathbb{R}^n) \simeq GL_n(\mathbb{R}) \\
&= \{A \in \mathcal{M}_n(\mathbb{R}) : det(A) \neq 0\} \\
&= \{A \in \mathcal{M}_n(\mathbb{R}) : det(A) \in (-\infty, 0) \cup (0, \infty)\} \\
&= det^{-1}((-\infty, 0) \cup (0, \infty))
\end{aligned}$$

(3)

$$\begin{array}{ccc}
\mathcal{GL}(E) & \longrightarrow^{\phi} & \mathcal{GL}(E) \\
f & \longmapsto & f^{-1}
\end{array}$$

is continiuous ??, let us show the continuity of $\phi$ at some $f_0 \in \mathcal{GL}(E)$ arbitrary, for all $f \in \mathcal{GL}(E)$, such that

$$||| f - f_0 |||< \frac{1}{||| f_0^{-1} |||}$$

we have,

$$\begin{aligned}
f^{-1} - f_0^{-1} &= f_0^{-1} \circ \left(f_0 \circ f^{-1} - id_E\right) \\
&= f_0^{-1} \circ \left(f_0 \circ f^{-1} - id_E\right) \\
&= f_0^{-1} \circ \left(\left(f \circ f_0^{-1}\right)^{-1} - id_E\right) \\
&= f_0^{-1} \circ \left((f - f_0 + f_0) \circ f_0^{-1}\right)^{-1} - id_E \\
&= f_0^{-1} \circ \left[\left((f - f_0) \circ f_0^{-1} + id_E\right)^{-1} - id_E\right]
\end{aligned}$$

From Question (1),

$$f_0^{-1} \circ \left[ \sum_{n=0}^{\infty} (-1)^n \left( (f - f_0) \circ f_0^{-1} \right)^n - id_E \right]$$

Hence

$$||| f^{-1} - f_0^{-1} ||| \leq ||| f_0^{-1} ||| \sum_{n=0}^{\infty} ||| (-1)^n \left( (f - f_0) \circ f_0^{-1} \right)^n |||$$

Hence

$$\begin{aligned} ||| f^{-1} - f_0^{-1} ||| &\leq ||| f_0^{-1} ||| \sum_{n=1}^{\infty} ||| (-1)^n ((f - f_0) \circ f_0^{-1})^n ||| \\ &\leq ||| f_0^{-1} ||| \cdot \sum_{n=0}^{\infty} ||| f - f_0 |||^n ||| f_0^{-1} |||^n \end{aligned}$$

Thus,

$$||| f^{-1} - f_0^{-1} ||| \leq ||| f_0^{-1} ||| \cdot \left[ \frac{||| f - f_0 ||| \cdot ||| f_0^{-1} |||}{1 - ||| f - f_0 ||| \cdot ||| f_0^{-1} |||} \right]$$

This shows that,

$$\lim_{f \to f_0} ||| f^{-1} - f_0^{-1} ||| = 0$$

That is $f^{-1} \to f_0^{-1}$, $f \to f_0$, hence consequently $\phi$ is continiuous

*Definition 4.8.5:*

*Let $E$ be a N.V.S, A series $\sum_{n=1}^{\infty} x_n$ of $E$ is said to be unconditionally convergent if for every permutation $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$, the series $\sum_{n=0}^{\infty} x_{\sigma(n)}$ converges to the same sum (in particular, the series $\sum_{n=0}^{\infty} x_n$ converges).*

☞ *Recall* Let $E$ be a N.V.S $\sum_{n=0}^{\infty} x_n$ is unconditionally convergent if and only if $\forall\ \sigma : \mathbb{N} \longrightarrow \mathbb{N}$ a bijective, the series $\sum_{n=0}^{\infty} x_{\sigma(n)}$ is convergent to the same sum.

Example

In $\mathbb{R}$, the series,

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

which is convergent to $\ln(2)$, is conditionally convergent, consider the permutation of $\mathbb{N}$, that is given by,

$$(1, 2, 3, 5, 4, 7, 9, 11, 6, \dots)$$

therefore it transforms to

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$$

transform it to a divergent series also the permutation,

$$(1, 2, 4, 3, 6, 8, \dots) = (n, 2n, 2n+2)$$

transforms the series to,

$$\begin{aligned}
\frac{1}{1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots &= \sum_{n=0}^{\infty} \left( \frac{1}{2n+1} - \frac{1}{2(2n+1)} - \frac{1}{2(2n+2)} \right) \\
&= \sum_{n=0}^{\infty} \left( \frac{1}{2(2n+1)} - \frac{1}{2(2n+2)} \right) \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2n+1} - \frac{1}{2n+2} \\
&= \frac{1}{2} \left[ \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots \right] \\
&= \frac{1}{2} \ln(2) \neq \ln(2)
\end{aligned}$$

**Theorem 4.8.3: The Riemann rearrangement**

If a real series is conditionally convergent then its terms can be rearranged so that the new series converges to an arbitrary real number, or diverges

**Theorem 4.8.4:**

Let $E$ be a Banach space, then any normally convergent series of $E$ is unconditionally convergent

*Proof.* Let $\sum_{n=0}^{\infty} x_n$ be a normally convergent series of $E$ (i.e. the real series $\sum_{n=0}^{\infty} \|x_n\|$ is convergent), then for the permutation $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ we have for all $n \in \mathbb{N}$, we will consider the series,

$$\begin{aligned}
\sum_{n=0}^{N} \|x_{\sigma(n)}\| = \sum_{k \in \{\sigma(0), \dots, \sigma(N)\}} \|x_k\| &\leq \sum_{k=1}^{\max(\sigma(i)), 1 \leq i \leq N} \|x_k\| \\
&\leq \sum_{k=1}^{\infty} \|x_k\|
\end{aligned}$$

This implies that the nonegative real series $\sum_{n=0}^{\infty} \|x_{\sigma(n)}\|$ is convergent, that is the series $\sum_{n=0}^{\infty} x_{\sigma(n)}$ of $E$ is normally convergent, since $E$ is Banach so we conclude that the series $\sum_{n=0}^{\infty} x_{\sigma(n)}$ is convergent, as required.

Now let us show that $\sum_{n=0}^{\infty} x_{\sigma(n)}$ has the same sum as $\sum_{n=0}^{\infty} x_n$ let us define for all $n \in \mathbb{N}$.

$$a_n = \begin{cases} \min\left(A = \{1,2,\dots,n\} \Delta \{\sigma(1),\dots,\sigma(n)\}\right) & \text{if } A \neq \emptyset \\ n & \text{if } A = \emptyset \end{cases}$$

and let us admit for the moment that

$$\lim_{n\to\infty} a_n = \infty$$

then we have for all $n \in \mathbb{N}$,

$$\begin{aligned} \|\sum_{n=1}^{N} x_{\sigma(n)} - \sum_{n=1}^{N} x_n\| &= \|\sum_{i\in\{\sigma(1),\dots,\sigma(n)\}} x_i - \sum_{i\in\{1,\dots,N\}} x_i\| \\ &= \|\sum_{i\in\{\sigma(1),\dots,\sigma(N)\}\setminus\{1,2,\dots,N\}} x_i - \sum_{i\in\{1,\dots,N\}\setminus\{\sigma(1),\dots,\sigma(N)\}} \| \\ &\leq \sum_{i\in\{\sigma(1),\dots,\sigma(N)\}\setminus\{1,\dots,N\}} \|x_i\| + \sum_{i\in\{1,\dots,N\}\setminus\{\sigma(1),\dots,\sigma(N)\}} \|x_i\| \\ &= \sum_{i\in\{1,\dots,N\}\Delta\{\sigma(1),\dots,\sigma(N)\}} \|x_i\| \\ &\leq \sum_{i\geq a_N} \|x_i\| \end{aligned}$$

Then by letting $N \to \infty$, we get since $\sum_{i=1}^{\infty} \|x_i\|$ converge and $a_N \to \infty$ as $N \to \infty$, we get,

$$\sum_{n=0}^{\infty} x_{\sigma(n)} = \sum_{n=0}^{\infty} x_n$$

as required.

Now, it remains to prove that $\lim_{n\to\infty} a_n = \infty$, this is equivalent to show that for all $k \in \mathbb{N}$, there exist $N_k$ such that $\forall n \in \mathbb{N} : n \geq N_k \implies a_n \geq k$, now let $k \in \mathbb{N}$, and take $N_k := \max\left\{1,\dots,k,\sigma^{-1}(1),\dots,\sigma^{-1}(k)\right\}$, then for any $n \in \mathbb{N}$, we have in one hand:

$$N \geq N_k \implies N \geq k \quad (\text{ since } N_k \geq k\ ) \implies \{\sigma(1),\dots,\sigma(N)\} \setminus \{1,\dots,N\} \subset \{k+1,k+2,\dots\}$$

On the other hand,

$$N \geq N_k \implies \sigma\sigma^{-1}(1),\sigma^{-1}(2),\dots,\sigma^{-1}(k) \leq N_k \leq N$$

which implies,

$$\begin{aligned}
&\Longrightarrow \sigma^{-1}(1), \sigma^{-1}(2), \dots, \sigma^{-1}(k) \in \{1, \dots, N\} \\
&\Longrightarrow 1, \dots, k \in \{\sigma(1), \dots, \sigma(N)\} \\
&\Longrightarrow \{1, \dots, N\} \setminus \{\sigma(1), \dots, \sigma(N)\} \subset \{k+1, k+2, \dots\}
\end{aligned}$$

so from the two hands, we get $\forall n \in \mathbb{N}$,

$$\begin{aligned}
N \geq N_k &\Longrightarrow \{1, \dots, N\} \,\Delta\, \{\sigma(1), \dots, \sigma(N)\} \subset \{k+1, k+2, \dots\} \\
&\Longrightarrow a_N \geq k \quad \text{(also true for } a_N = N, \text{ since } N \geq N_k \geq k)
\end{aligned}$$

as required. Thus $a_n \to \infty$ as $n \to \infty$. which completes the proof. □

## 4.9 The summability of general series

We call a general series any infinite sum of element of a N.V.S, that is a $\sum_{i \in I} x_i$, where $I$ is infinite.

**Definition 4.9.1: Generalize the unconditional convergence**

Let $E$ be a N.V.S. A general series $\sum_{i \in I} x_i$ of $E$ is said to be summable with sum $S \in E$, if it satisfies the following property,
$\forall \varepsilon > 0, \exists I_\varepsilon \subset I$ finite, s.t. $\forall J$ a finite subset of $I$, we have

$$I_\varepsilon \subset J \Longrightarrow \|\sum_{i \in J} x_i - S\| < \varepsilon$$

Let $E$ be a N.V.S. If a general series $\sum_{i \in I} x_i$ is summable then it has a unique sum,

*Proof.* Let $\sum_{i \in I} x_i$ be a general summable series with sums $S$ and $S'$ $(S, S' \in E)$, and let us prove that $S = S'$. Let $\varepsilon > 0$ arbitrary, By definition $\exists I_\varepsilon \subset I$, with $I_\varepsilon$ finite, such that,
$\forall J$ a finite subset of $I$, we have

$$I_\varepsilon \subset J \Longrightarrow \|\sum_{i \in J} x_i - S\| < \frac{\varepsilon}{2}$$

Similarly, $\exists I_\varepsilon \subset I$, with $I_\varepsilon$ finite, such that

$\forall J$ a finite subset of $I$, we have,

$$I_\varepsilon \subset J \implies \|\sum_{i\in J} x_i - S'\| < \frac{\varepsilon}{2}$$

So, by taking $J = I_\varepsilon \cup I'_\varepsilon$ which if a finite subset of $I$ and contains both $I_\varepsilon$ and $I'_\varepsilon$, we have, $\|\sum_{i\in J} x_i - S\| < \frac{\varepsilon}{2}$ and $\|\sum_{i\in J} x_i - S'\| < \frac{\varepsilon}{2}$. Hence,

$$\begin{aligned}\|S - S'\| &= \|S - \sum_{i\in J} x_i + \sum_{i\in J} x_i - S'\| \\ &\leq \|S - \sum_{i\in J} x_i\| + \|\sum_{i\in J} x_i - S'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad (= \varepsilon)\end{aligned}$$

Thus $\|S - S'\| < \varepsilon$ for all $\varepsilon > 0$, implying that $S = S'$, as required. □

*The Cauchy Criterion*

Let $E$ be a N.V.S. We say that a general series $\sum_{i\in I} x_i$. Satisfies the Cauchy Criterion if,

$\forall \varepsilon > 0, \exists I_\varepsilon \subset I$, with $I_\varepsilon$ finite, s.t. $\forall J$ a finite subset of $I$, disjoint with $I_\varepsilon$, we have

$$\|\sum_{i\in J} x_i\| < \varepsilon$$

$\sum_{i\in\mathbb{N}} x_i$ is Cauchy if and only if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ s.t. } \forall p, q \in \mathbb{N} : p > q > N_\varepsilon \implies \|\sum_{i=q+1}^{p} x_i\| < \varepsilon$$

which implies that

$$\forall \varepsilon > 0, \exists I_\varepsilon = \{1, \ldots, N_\varepsilon\} \subset \mathbb{N} \text{ finite s.t. } \forall J = \{q+1, \ldots, p\} \subset \mathbb{N} \text{ finite}$$

and

$$J \cap I_\varepsilon = \emptyset \implies \|\sum_{i\in J} x_i\| < \varepsilon$$

**Theorem 4.9.1:**

Let $E$ be a Banach Space. Then every general series $\sum_{i\in I} x_i$ of $E$ which satisfies the cauchy criterion is summable.

*Proof.* Let $\sum_{i\in I} x_i$ be a general series of $E$. Which satisfies the Cauchy criterion then for all $n \in \mathbb{N}$, there exist $I_n \subset I$ with $I_n$ finite, such that $\forall J$ a finite subset of $I$, with $J \cap I_n = \emptyset$, we have $\|\sum_{i\in J} x_i\| <$

$\frac{1}{n}$ , let us define for all $n \in \mathbb{N}$,

$$S_n := \sum_{i \in I_1 \cup I_2 \cup \ldots \cup I_n} x_i \qquad \text{( a finite sum )}$$

$(S_n)_{n \in \mathbb{N}}$ is a sequence of $E$

we have for any $p, q \in \mathbb{N}$, with $p > q$,

$$\|S_p - S_q\| = \| \sum_{i \in \underbrace{I_1 \cup \ldots \cup I_p \backslash I_1 \cup \ldots \cup I_q}_{\text{disjoint } (I_p, I_q)}} x_i \| < \frac{1}{q} \to 0 \text{ as } q \to \infty$$

Thus $(S_n)_{n \in \mathbb{N}}$ is Cauchy. Since $E$ is Banach then $(S_n)_{n \in \mathbb{N}}$ is convergent. Let $S = \lim_{n \to \infty} S_n \in E$ , and let us show that the general series $\sum_{i \in I} x_i$ is sommable with sum $S$ □

**Theorem 4.9.2:**

Let $E$ be a Banach space. Then every general series $\sum_{i \in I} x_i$ of $E$ which satisfies Cauchy criterion is summable.

*Proof.* Let $\sum_{i \in I} x_i$ be a general series $E$ which satisfies the Cauchy criterion, Then for all $n \in \mathbb{N}$, $\exists I_n \subset I$, with $I_n$ finite, such that $\forall J$ a finite subset of $I$, with $J \cap I_n = \emptyset$, we have,

$$\| \sum_{i \in J} x_i \| < \frac{1}{n}$$

Let us define for all $n \in \mathbb{N}$,

$$S_n := \sum_{i \in I_1 \cup I_2 \cup \ldots \cup I_n} x_i \, (\in E)$$

Clearly, $(S_n)_{n \in \mathbb{N}}$ is a sequence of $E$.

we have for any $p, q \in \mathbb{N}$, with $p > q$,

$$\|S_p - S_q\| = \| \sum_{i \in I_1 \cup \ldots \cup I_p} x_i - \sum_{i \in I_1 \cup \ldots \cup I_q} x_i \| = \| \sum_{i \in \underbrace{(I_1 \cup \ldots I_p) \backslash (I_1 \cup \ldots \cup I_q)}_{\text{finite, disjoint with } I_q}} x_i \| < \frac{1}{q}$$

Hence $\lim_{p,q \to \infty} \|S_p - S_q\| = 0$, implying that $(S_n)_{n \in \mathbb{N}}$ is Cauchy since $E$ is Banach then $(S_n)_{n \in \mathbb{N}}$ is convergent. Let $S := \lim_{n \to \infty} S_n \in E$, and let us show that the general series $\sum_{i \in I} x_i$ is summable with sum $S$ $\forall \varepsilon > 0, \exists I_\varepsilon \subset I$, with $I_\varepsilon$ finite , $\forall J \subset I$, J finite

$$I_\varepsilon \subset J \implies \| \sum_{i \in J} x_i - S \| < \varepsilon$$

Let $\varepsilon > 0$ arbitrary then since $S_n \to S$ in $E$ and $\frac{1}{n} \to 0$ as $n \to \infty$ in $\mathbb{R}$, then $\exists n_0 \in \mathbb{N}$ , such that,

$$\|S_{n_0} - S\| < \frac{\varepsilon}{2} \text{ and } \frac{1}{n_0} < \frac{\varepsilon}{2}$$

take $I_\varepsilon = I_1 \cup \ldots \cup I_{n_0}$, For any subset $J$ of $I$ whixh is finite and contains $I_\varepsilon$, we have,

$$\begin{aligned}\|\sum_{i\in J} x_i - S\| = \|\sum_{i\in I_1\cup\ldots\cup I_{n_0}} x_i + \sum_{i\in J\setminus(I_1\cup\ldots\cup I_{n_0})} x_i - S\| &= \|S_{n_0} - S + \sum_{i\in J\setminus(I_1\cup\ldots\cup I_{n_0})} x_i\| \\ &\leq \underbrace{\|S_{n_0} - S\|}_{<\varepsilon/2} + \|\sum_{i\in J\setminus(I_1\cup\ldots\cup I_{n_0})} x_i\| \\ &< \varepsilon\end{aligned}$$

Thus $\sum_{i\in I} x_i$ is summable with sum $S$, hence the proof is complete. □

Let $E$ be N.V.S prove that if a general series of $E$ is summable then it satisfies the Cauchy criterion

**Definition 4.9.2:**

Let $E$ be a N.V.S and $\sum_{i\in I} x_i$ be a general series of $E$, We say that $\sum_{i\in I} x_i$ is normally summable if the real general series $\sum_{i\in I} \|x_i\|$ is summable.

**Theorem 4.9.3:**

Let $E$ be a Banach Space and $\sum_{i\in I} x_i$ be a general series, if $\sum_{i\in I} x_i$ is normally summable then its summable and we have

$$\|\sum_{i\in I} x_i\| \leq \sum_{i\in I} \|x_i\|$$

*Proof.* Suppose that $\sum_{i\in I} x_i$ is normally summable, that is, the real general series $\sum_{i\in I} \|x_i\|$ is summable, Thus $\sum_{i\in I} \|x_i\|$ satisfies the Cauchy criterion (see Previous exercise).

It follows that $\sum_{i\in I} x_i$ also satisfies the Cauchy criterion $\forall\varepsilon > 0, \exists I_\varepsilon \subset I$ finite , $\forall J \subset$, J finite , $J \cap I_\varepsilon = \emptyset$

$$\implies \sum_{i\in J} \|x_i\| < \varepsilon \implies \|\sum_{i\in J} x_i\| < \varepsilon$$

Thus according to the previous theorem, The general series $\sum_{i\in I} x_i$ is summable as required.

Now, let us prove the inequality of the theorem, Let $S := \sum_{i\in I} x_i$ and $S' := \sum_{i\in I} \|x_i\| \in \mathbb{R}$, we have to show that $\|S\| \leq S'$, For all $\varepsilon > 0$, there exist $I_\varepsilon \subset I$, with $I_\varepsilon$ finite such that $\forall J \subset I$, such that $J$ finite,

$$I_\varepsilon \subset J \implies \|\sum_{i\in J} x_i - S\| < \varepsilon$$

Similarly, for all $\varepsilon > 0$, there exist $I'_\varepsilon \subset I$, with $I'_\varepsilon$ finite, such that $\forall J \subset I$, with $J$ finite, with $J$ finite,

$$I'_\varepsilon \subset J \implies \|\sum_{i\in J} \|x_i\| - S'\| < \varepsilon$$

For $\varepsilon > 0$, by taking $J = I_\varepsilon \cup I'_\varepsilon$, we have

$$\|\sum_{i\in J} x_i - S\| < \varepsilon$$

$$\|\sum_{i\in J} \|x_i\| - S'\| < \varepsilon$$

Hence, using the above inequalitys, we have,

$$\begin{aligned}\|S\| &= \|S - \sum_{i\in J} x_i + \sum_{i\in J} x_i\| \\ &\leq \underbrace{\|S - \sum_{i\in J} x_i\|}_{<\varepsilon} + \underbrace{\sum_{i\in J} \|x_i\|}_{<S'+\varepsilon} \\ &< S' + 2\varepsilon\end{aligned}$$

Thus $\|S\| < S' + 2\varepsilon$ for all $\varepsilon > 0$, by taking $\varepsilon \to 0^+$ gives $\|S\| \leq S'$, as required. this completes the proof. $\square$

The following theorem shows that every generla series of a N.V.S, can always be reduced to an ordinary series i.e $I = \mathbb{N}$.

**Theorem 4.9.4:**

Let $E$ be a N.V.S and $\sum_{i\in I} x_i$, be a general series of $E$, Suppose that $\sum_{i\in I} x_i$ is summable. then the set

$$I' := \{i \in I : x_i \neq 0_E\}$$

is at most countable. In addition, the general series $\sum_{i\in I'} x_i$ is summable and we have

$$\sum_{i\in I'} x_i = \sum_{i\in I} x_i$$

*Proof.* for all $n \in \mathbb{N}$, put

$$I'_n := \left\{i \in I : \|x_i\| > \frac{1}{n}\right\}$$

So, we have that

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} I'_n &= \left\{i \in I : \exists n \in \mathbb{N} \text{ such that } \|x_i\| > \frac{1}{n}\right\} \\ &= \{i \in I : x_i \neq 0_E\} = I'\end{aligned}$$

$$I = \bigcup_{n\in\mathbb{N}} I'_n$$

Next, let us prove that $I'_n$ is finite for every $n \in \mathbb{N}$. So let $n \in \mathbb{N}$ , since $\sum_{i\in I} x_i$ is assumed to be summable then it satisfies the Cauchy criterion, So $\exists I_n \subset I$, with $I_n$ finite, such that $\forall J \subset I$, with $J$ finite,

$$J \cap I_n = \emptyset \implies \|\sum_{i\in J} x_i\| < \frac{1}{n}$$

(Cauchy criterion for $\varepsilon = \frac{1}{n}$)

In Particular, for every $j \in I$, we have for $J = \{j\}$,

$$\forall j \in I, \{j\} \cap I_n = \emptyset \implies \|x_j\| < \frac{1}{n}$$

Equivalently,

$$\begin{aligned}\forall j \in I, j \notin I_n &\implies \|x_j\| < \frac{1}{n}\\ &\implies j \notin I'_n\end{aligned}$$

$$\forall j \in I, j \notin I_n \implies j \notin I'_n$$

By the contrapositive we have,

$$\forall j \in I, j \in I'_n \implies j \in I_n$$

Thus,

$$I'_n \subset I_n$$

Since $I_n$ is finite, we derive that $I'_n$ is finite.

Consequently according to the above, $I'$ is a countable union of finite sets, implying that $I'$ is at most countable, as required.

Now, let us prove the second part of the theorem, set $S := \sum_{i\in I} x_i$ then $\forall \varepsilon > 0, \exists I_\varepsilon \subset I,$ with $I_\varepsilon$ finite , $\forall J \subset I,$ with J finite, we have,

$$I_\varepsilon \subset J \implies \|\sum_{i\in J} x_i - S\| < \varepsilon$$

Let $\varepsilon > 0$ be arbitrary, by putting $I'_\varepsilon = I_\varepsilon \cap I'$, which is finite since $I_\varepsilon$ is finite and $\subset I'$, we have for any finite subset $J'$ of $I'$, containing $I'_\varepsilon$,

$$\begin{aligned}\sum_{i\in J'} x_i &= \sum_{i\in J'\cup I'_\varepsilon} x_i \qquad \text{since } I'_\varepsilon \subset J'\\ &= \sum_{i\in (J'\cup I_\varepsilon)\cap I'} x_i\\ &= \sum_{i\in J'\cup I'_\varepsilon} x_i\end{aligned}$$

But since $J' \cup I_\varepsilon$ is finite and contains $I_\varepsilon$ it fololws that

$$\|\sum_{i\in J'} x_i - S\| = \|\sum_{i\in J'\cup I_\varepsilon} x_i - S\| < \varepsilon$$

This concludes that the general series $\sum_{i\in I'} x_i$ is summable and we have

$$\sum_{i\in I'} x_i = \sum_{i\in I} x_i$$

This completes the proof. □

**Theorem 4.9.5:**

Let $E$ be a N.V.S and $\sum_{i\in I} x_i$ be a general series of $E$. Suppose that $\sum_{i\in I} x_i$ is summable. then for all other set $L$ equinumerous, with $I$ (I forgot about 2 words here) all bijection $\sigma : L \longrightarrow I$ the general series $\sum_{l\in L} x_{\sigma(l)}$, is summable and we have

$$\sum_{l\in L} x_{\sigma(l)} = \sum_{i\in I} x_i$$

*Proof.* Set $S := \sum_{i\in I} x_I$ and let $\varepsilon > 0$, be arbitrary, then $\exists I_\varepsilon \subset I$ , with $I_\varepsilon$ finite, such that for all $J \subset I$, with $J$ finite, and

$$I_\varepsilon \subset J \implies \|\sum_{i\in J} x_i - S\| < \varepsilon$$

Does? $\exists L_\varepsilon \subset L$, with $L_\varepsilon$ finite such that $\forall K \subset L$, with $K$ finite, and,

$$\underbrace{L_\varepsilon \subset K}_{\text{I didint see this clearly from the table, could be wrong}} \implies \|\sum_{l\in K} x_{\sigma(l)} - S\| < \varepsilon$$

Define $L_\varepsilon = \sigma^{-1}(I_\varepsilon)$ since $I_\varepsilon \subset I$ then, $L_\varepsilon \subset L$ , $L_\varepsilon$ is finite ( Since $I_\varepsilon$ is finite and $\sigma$ is bijective), Next for all $K \subset L$, with $K$ is finite, and $L_\varepsilon \subset K$, and we have

$$\sum_{l\in K} x_{\sigma(l)} = \sum_{i\in\sigma(K)} x_i \qquad (i = \sigma(l))$$

Since $L_\varepsilon \subset K$, then $I_\varepsilon = \sigma(L_\varepsilon) \subset \sigma(K)$, implying that

$$\|\sum_{i\in\sigma(K)} x_i - S\| < \varepsilon \text{ i.e. } \|\sum_{l\in K} x_{\sigma(l)} - S\| < \varepsilon$$

this shows that the general series $\sum_{l\in L} x_{\sigma(l)}$ is summable and we have

$$\sum_{l\in L} x_{\sigma(l)} = \sum_{i\in I} x_i$$

the proposition is proved. □

*Corollary* , Let $E$ be a N.V.S. Then every summable general series can be transformed either into a finite sum or into an arbitrary series

*Proof.* Let $\sum_{i\in I} x_i$ be a summable general series of $E$. Let

$$I' := \{i \in I : x_i \neq 0_E\}$$

Its proved previously that $I'$ is at most countable and that

$$\sum_{i\in I} x_i = \sum_{i\in I'} x_i$$

We distinguish two cases.

1. If $I'$ is finite, in this case $\sum_{i\in I} x_i$ is transformed to the finite sum $\sum_{i\in I'} x_i$
2. If $I'$ is countably infinite. In this case $\exists\, \sigma : \mathbb{N} \longrightarrow I'$ a bijection. So, by the previous proposition, we have
$$\sum_{i\in I'} x_i = \sum_{l\in\mathbb{N}} x_{\sigma(l)} = \sum_{l=1}^{\infty} x_{\sigma(l)}$$
which is an ordinary series of $E$.

The corollary is proved. □

*Exercise : (Summation by Packet)* Let $E$ be a Banach Space. then $\sum_{i\in I} x_i$ be asummable general series of $E$, and $(I_\alpha)_{\alpha\in A}$ be a partition of $I$,

1. Show that for every $\alpha \in A$, the general $\sum_{i\in I_\alpha} x_i$ is summable
2. Show that the general series
$$\sum_{\alpha\in A}\left(\sum_{i\in I_\alpha} x_i\right)$$
is summable with sum equal to $\sum_{i\in I} x_i$.

**Remainder** : *(Separable spaces)*A toplogical space is said to be separable if it contains a countable dense subset.

**Example**

$\mathbb{R}$ equipped with its usual toplogy is separable since $Q \subset \mathbb{R}$ is countable dense subset of $\mathbb{R}$, is a countable dense subset of $\mathbb{R}$. More generally, $\mathbb{R}^n$ is seperable for all $n \in \mathbb{N}$ (consider the subset $Q^n$ of $\mathbb{R}^n$)

**Generalization** Every finite dimensional N.V.S (Over $\mathbb{R}$ or $\mathbb{C}$ ) is separable, since,

$$E \simeq \mathbb{K}^n \simeq \mathbb{R}^n \simeq \mathbb{C}^n$$

**An important exmaple**

**Theorem 4.9.6: The weirstrass approximation theorem**

Let $a, b \in \mathbb{R}$ with $a < b$, then for every real valued continuous function on $[a, b]$, there exist a real polynomial sequence $(P_n)_{n\in\mathbb{N}}$ which uniformally converges to $f$ on $[a, b]$, in other words, for every $\varepsilon > 0$, there exist a real polynomial $P$ such that

$$|f(x) - p(x)| < \varepsilon \quad (\forall x \in [a, b])$$

If $[a, b] = [0, 1]$, we can take

$$P_n(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}$$

The Bernestein polynomials associated to $f$

*Consequence :* let $a, b \in \mathbb{R}$, with $a < b$, then N.V.S $(\mathcal{C}^0([a, b], \mathbb{R}, \|.\|_\infty))$, is separable. Indeed, the subset of polynomail functions with rational coefficients on $[a, b]$ is countable and desne in $\left(\mathcal{C}^0([a, b], \mathbb{R}), \|.\|_\infty\right)$

**Definition 4.9.3:**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

1. A subset $S$ of $E$ is said to be total if its span (i.e., the set of finite linear combinations of elements of $E$ ) is dense.

2. A Hamel basis of $E$ is linearly independent subset of $E$ which spans $E$ (The concept already known in Linear Algebra-Algebra2) It follows from Zorn's lemma that every vector space has a Hamel basis and that two Hamel bases of a same vector space are necessarily

equinumerous.

3. A schauder basis of $E$ is a sequence $(l_n)_{n\in\mathbb{N}}$ of $E$ such that for each vector $x \in E$, there exists a unique sequence $(\lambda_n)_{n\in\mathbb{N}}$ of scalars such that

$$x = \sum_{n=0}^{\infty} \lambda_n l_n$$

that is,

$$\|x - \sum_{n=1}^{N} l_n \lambda_n\| \to 0 \quad 0 \text{ as } N \to \infty$$

**Remark :**

1. Its easy to show that if a N.V.S $E$ has a Schauder basis then its separable (Exercise)
2. A Hamel basis (if its finite or countable) of a N.V.S is always Schauder basis (obvious) but the converse is false (see below!)
3. In a finite dimensional N.V.S the concept of Hamel basis and Schauder basis coincides

Example

1. (In relation with Fourier series let $p > 1$, It's show showed that the trigonometric,

$$1, \cos(x), \sin(x), \ldots$$

is a Schauder basis of the $\mathbb{R}$-N.V.S $L^p([0, 2\pi])$,

$$L^p([0,2\pi]) = \left\{ f : [0, 2\pi] \to \mathbb{R} \text{ s.t. } \int_0^{2\pi} |f(x)|^p \, d(\mu(x)) < \infty \right\}$$

with the norm $\|.\|_p$)

2. Let $C_0$ denote the $\mathbb{R}$-vector space of real sequences which converge to 0 and let

$$\begin{array}{rccc} \|.\|_\infty : & C_0 & \longrightarrow & [0, \infty] \\ & x = (x_n)_{n\in\mathbb{N}} & \longmapsto & \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| \end{array}$$

It's obvious that $\|.\|_\infty$ is a norm on $C_0$ (In fact $C_0$ is a normed subspace of $(l^\infty, \|.\|_\infty)$), where,

$$l^\infty = \{ \text{ the real bounded sequences } \}$$

for all $n \in \mathbb{N}$, let,

$$l^{(n)} = (l_i^{(n)})_{i\in\mathbb{N}}$$

be the real sequence of $C_0$ defined by,

$$l_i^{(n)} := \{10\} \begin{matrix} i = n \\ \\ 0 \text{ else} \end{matrix} = (0, 0, \ldots, 0, 0, \ddots) \in C_0$$

Its clear that $(e^{(n)})_{n\in\mathbb{N}}$ is linearly independent and is not a Hamel basis of $C_0$. Because

$$\left\langle e^{(n)}, n \in \mathbb{N} \right\rangle = C_{00} \neq C_0$$

where

$$C_{00} = \{ \text{ real sequences } (u_N)_{n\in\mathbb{N}}, \text{ for } u_n = 0 \text{ for } n \text{ sufficiently large } \}$$

$C_{00} \neq C_0$ since we have for example $(\frac{1}{n})_{n\in\mathbb{N}} \in C_0$, but $(\frac{1}{n})_{n\in\mathbb{N}} \notin C_{00}$.
Next, for any $x = (x_n)_{n\in\mathbb{N}} \in C_0$, we have for $n \in \mathbb{N}$,

$$\begin{aligned}\|x - \sum_{n=1}^{N} x_n e^{(n)}\|_\infty &= \|(x_1, x_2, \ldots) - (x_1, \ldots, x_N, 0, \ldots)\|_\infty \\ &= \|(0, \ldots, 0, x_{N+1}, \ldots)\|_\infty \\ &= \sup_{n \geq N+1} |x_n|\end{aligned}$$

hence,

$$\begin{aligned}\lim_{n\to\infty} \|x - \sum_{n=1}^{N} x_n e^{(n)}\| &= \lim_{n\geq N+1} \sup |x_n| \\ &= \overline{\lim_{n\to\infty}} |x_n| \\ &= \lim_{n\to\infty} |x_n| = 0\end{aligned}$$

This implies that the sequence $\left(\sum_{n=1}^{N} x_n e^{(n)}\right)_{n\in\mathbb{N}}$ of $C_0$ is convergent to $x$.
Equivalently, the series $\sum_{n=0}^{\infty} x_n e^{(n)}$ of $E$ is convergent to $x$, i.e.

$$x = \sum_{n=0}^{\infty} x_n e^{(n)} \quad (\text{in } C_0 \text{ })$$

Let us show the uniqueness of a such representation of $x \in C_0$. Suppose that $x \in C_0$ is representable as

$$x = \sum_{n=0}^{\infty} \alpha_n e^{(n)} = \sum_{n=0}^{\infty} \beta_n e^{(n)} \quad (\alpha_n, \beta_n \in \mathbb{R}, \forall n \in \mathbb{N})$$

we have for $n \in \mathbb{N}$,

$$\begin{aligned}&\|\sum_{i=1}^{N} \alpha_i e^{(i)} - \sum_{i=1}^{N} \beta_i e^{(i)}\| \\ &\qquad = \|\sum_{i=1}^{N} (\alpha_i - \beta_i)\, e^i\| = \max_{1\leq i\leq N} |\alpha_i - \beta_i|\end{aligned}$$

So for all $n, N \in \mathbb{N}$, with $n \leq N$, we have,

$$|\alpha_n - \beta_n| \leq \max_{1 \leq i \leq N} |\alpha_i - b_i| = \| \sum_{i=1}^{N} \alpha_i e^{(i)} - \sum_{i=1}^{N} \beta_i e^{(i)} \|_\infty \text{ By taking } N \to \infty$$

we get that, $|\alpha_n - \beta_n| \leq 0$, thus we have that,

$$\alpha_n = \beta_n \qquad (\forall n \in \mathbb{N})$$

Thus, the representation of $x$, $\sum_{n=1}^{\infty} x_n e^{(n)}$ is unique.

Consequently, $(e^{(n)})_{n \in \mathbb{N}}$ is a Schauder basis of $C_0$

# 5 FUNDAMENTAL THEOREMS ON BANACH SPACES :

- The open mapping theorem.
- The closed graph theorem.
- The Banach-SteinHauns Theorem
- The Hahn-Banach

## 5.1 The open mapping theorem

**Reminders :** A mapping $f$ from a toplogical space $X$ into a toplogical space $Y$ is said to be an open mapping. if the image by $f$ of every open subset of $X$ is an open subset of $Y$

**Theorem 5.1.1: (The open mapping theorem-Schaunder**

Let $f$ be a continious linear mapping from a Banach space $E$ to a Banach space $F$. Then the two following properties are equivalent,

i $f$ is surjective

ii $f$ is an open mapping

*Proof.* $(ii) \implies (i)$

We argue by contradiction. Suppose that $f$ is an open mapping that $f$ is not surjective ( i.e. .$f(E) \neq F$), so $f(E)$, is a proper subspace of $F$, implying (see the tutorial worksheet number 1 ), that

$$int(f(E)) = \emptyset$$

On the other hand, since $f$ is an open mapping and $E$ is open in $E$ then $f(E)$ is open in $F$, thus $int(f(E)) = f(E)$, Hence $f(E) = \emptyset$, which is a contradiction.

$$(i) \implies (ii)$$

we need preliminarry results.

**Theorem 5.1.2:**

Let $E$ and $F$ be two N.V.S and $f : E \longrightarrow F$ be a linear mapping then the two following properties are equivalent,

i $f$ is an open mapping

ii $\exists r > 0$ such that
$$B_F(0_F, r) \subset f\left(B_E(0_E, 1)\right)$$

*Proof.* $(i) \implies (ii)$

Suppose that $f$ is an open mapping. Since $B_E(0_E, 1)$ is an open subset of $E$, then $f(B_E(0_E, 1))$ is an open subset of $F$. So since,
$$0_F = f(0_E) \in f(B_E(0_E, 1))$$
then $f(B_E(0_E, 1))$ is a neighborhood of $0_F$, that is $\exists r > 0$ such that
$$B_F(0_F, r) \subset f(B_E(0_E, 1))$$
as required. □

□

**Theorem 5.1.3: (The open mapping theorem)**

Let $E, F$ be two Banach spaces. and let $f \in \mathcal{L}(E, F)$, then the following assertions are equivalent,

(i) $f$ is surjective

(ii) $f$ is an open mapping

*Proof.* Last time we have proved that $(ii) \implies (i)$, now

$$(i) \implies (ii)$$

*Proposition 01:* let $E, F$ be two N.V.S. and $f : E \longrightarrow F$ be a linear map, then,

(a) $f$ is an open mapping

(b) $\exists r > 0$ such that $f\left(B_E(0_E, 1)\right) \supset B_F(0_F, r)$

*Proof.* $(\alpha) \implies (\beta)$

Suppose that $f$ is an open mapping $B_E(0_E, 1)$ is open in $E$, then $f(B_E(0_E, 1))$ is open in $F$.

$$0_F = f(0_E) \in f(B_E(0_E, 1))$$

Thus there exist $r > 0$ such that

$$f(B_E(0_E, 1)) \in \mathcal{V}(0_F)$$

Therefore

$$B_F(0_F, r) \subset f(B_E(0_E, 1))$$

$$(\beta) \implies (\alpha)$$

**Notation :** For a given non empty subsets $A$ and $B$ of a N.V.S $V$ , then $x_0 \in V$, and a given scalar $\lambda$, we let $(A + B)$, $A + x_0$, and $\lambda A$, respectively, denote the following subsets of $V$ :

$$A + B := \{a + b : a \in A, b \in B\}$$
$$A + x_0 := A + \{x_0\} = \{a + x_0 : a \in A\}$$
$$\lambda A := \{\lambda a, a \in A\}$$

Note that $2A \neq A + A$ because,

$$\{2a : a \in A\} \subset \{a + b : a, b \in A\}$$

Suppose that $\exists r > 0$ such that

$$B_F(0_F, r) \subset f(B_E(0_E, 1))$$

Let $\mathcal{O}$ be an open subset of $E$, and let us show that $f(\mathcal{O})$ is an open subset of $F$, we have to show that $f(\mathcal{O})$ is a neighborhood of every element of $f(\mathcal{O})$.

Let $y \in f(\mathcal{O})$ arbitrary and show that $f(\mathcal{O})$ is a neighborhood of $y$.

$y \in f(\mathcal{O})$, which means that $\exists x \in \mathcal{O}$ such that $y = f(x)$. But since $\mathcal{O}$ is an open set in $E$, and $x \in \mathcal{O}$, then $\exists \varepsilon > 0$ such that

$$B_E(x, \varepsilon) \subset \mathcal{O}$$

Hence

$$f\left(B_E(x, \varepsilon)\right) \subset f(\mathcal{O})$$

Since $f$ is linear, then we have

$$\begin{aligned} f\left(B_E(x,\varepsilon)\right) &= f\left(\varepsilon B_E(0_E,1)+x\right) \\ &= \varepsilon \underbrace{f\left(B_E(0_E,1)\right)}_{\supset B_F(0_F,r)} + f(x) \supset \varepsilon B_F(0_F,r)+f(x) \\ &= B_F(f(x),\varepsilon r) \\ &= B_F(y,\varepsilon r) \end{aligned}$$

Hence $f(\mathcal{O}) \supset B_F(y,\varepsilon r)$ implying that $f(\mathcal{O})$ is a neighborhood of $y$. Thus since $y$ is arbitrary in $f(\mathcal{O})$, then $f(\mathcal{O})$ is open in $F$. Consequently, $f$ is an open mapping, as required, this completes the proof. $\square$

**Theorem 5.1.4:**

Let $E$ be a Banach space, and $F$ be an arbitrary N.V.S. And $f \in \mathcal{L}(E,F)$ let $\varepsilon \in (0,1)$ and $A$ be a *bounded* subset of $F$, satisfying

$$A \subset f(B_E(0_{E,1})) + \varepsilon A$$

Then we have

$$A \subset \frac{1}{1-\varepsilon} f(B_E(0_E,1))$$

*Proof.* Let $a_0 \in A$ and let us show that $a_0 \in \frac{1}{A-\varepsilon} f(B_E(0_E,1))$ and let us show that $a_0 \in \frac{1}{1-\varepsilon} f(B_E(0_E,1))$. Since $a_0 \in A$ and $A \subset f(B_E(0_E,1)) + \varepsilon A$, then $a_0 \in f(B_E(0_E,1)) + \varepsilon A$, this $\exists x_0 \in B_E(0_E,1)$ and $\exists a_1 \in A$ such that,

$$a_0 = f(x_0) + \varepsilon a_1$$

Similarly, since $a_1 \in A$ and

$$A \subset f(B_E(0_E,1)) + \varepsilon A$$

then $a_1 \in f(B_E(0_E,1)) + \varepsilon A$. Thus there exist $x_1 \in B_E(0_E,1)$ and there exist $a_2 \in A$, such that

$$a_1 = f(x_1) + \varepsilon a_2$$

By iterating the process, we get a sequence $(x_n)_{n\in\mathbb{N}_0}$ of $B_E(0_E,1)$ and a sequence $(a_n)_{n\in\mathbb{N}_0}$ of $A$ such that

$$a_n = f(x_n) + \varepsilon a_{n+1} \qquad (\forall n \in \mathbb{N}_0)$$

*Thus,*

$$\begin{aligned}
a_0 &= f(x_0) + \varepsilon a_1 \\
&= f(x_0) + \varepsilon(f(x_1) + \varepsilon a_2) \\
&= f(x_0 + \varepsilon x_1) + \varepsilon^2 a_2 \\
&= f(x_0 + \varepsilon x_1) + \varepsilon^2(f(x_2) + a_3) \\
&= f(x_0 + \varepsilon x_1 + \varepsilon^2 x_2) + \varepsilon^3 a_3 \\
&= f(x_0 + \varepsilon x_1 + \ldots + \varepsilon^n x_n) + \varepsilon^{n+1} a_{n+1}
\end{aligned}$$

Since the series $\sum_{n=0}^{\infty} \varepsilon^n x_n$ of $E$ is normally convergent (because for every $n \in \mathbb{N}_0$), we have

$$\|\varepsilon^n x_n\|_E = \varepsilon^n \|x_n\|_E < \varepsilon^n$$

and the real geometric series $\sum_{n=0}^{\infty} \varepsilon^n$ converges since its ratio $\varepsilon \in (0,1)$, then we derive that $\sum_{n=0}^{\infty} \varepsilon^n x_n$ is convergent in $E$, and since $E$ is Banach. So setting

$$x := \sum_{n=0}^{\infty} \varepsilon^n x_n \in E$$

and letting $n \to \infty$, we get,

$a_0 = f(x)$ (since $f$ is continuous and $\varepsilon^{n+1} a_{n+1} \to 0$ as $n \to \infty$ , because $A$ is bounded and $0 < \varepsilon < 1$)

finally, we observe that,

$$\begin{aligned}
\|x\|_E = \| \sum_{n=0}^{\infty} \varepsilon^n x_n \|_E \leq \leq \sum_{n=0}^{\infty} \|\varepsilon^n x_n\|_E \\
= \sum_{n=0}^{\infty} \varepsilon^n \|x_n\|_E < 1
\end{aligned}$$

*Thus,*

$$\|x\|_E < \sum_{n=0}^{\infty} \varepsilon^n = \frac{1}{1-\varepsilon}$$

by setting $u = (1-\varepsilon)x$, we get,

$$\|u\|_E < 1 \quad \text{i.e.} \quad u \in B_E(0_E, 1)$$

Hence,

$$\begin{aligned}
a_0 = f(x) &= f\left(\frac{1}{1-\varepsilon} u\right) \\
&= \frac{1}{1-\varepsilon} f(u) \\
&\in \frac{1}{1-\varepsilon} f(B_E(0_E, 1))
\end{aligned}$$

consequently $A \subset \frac{1}{1-\varepsilon} f(B_E(0_E, 1))$, as required. □

**Theorem 5.1.5:**

Let $E$ be a Banach space, and $F$ be an arbitrary N.V.S. Next, let $f \in \mathcal{L}(E,F)$ and $r,s > 0$, suppose that,

$$\overline{f\left(B_E(0_E,r)\right)} \supset B_F(0_F,s)$$

then,

$$f(B_E(0_E,r)) \supset B_F(0_F,s)$$

**Remark :** In the context of Proposition 3 (i.e. above theorem), we have,

$$\begin{aligned} f(B_E(0_E,r)) \supset B_F(0_F,s) &\iff rf(B_E(0_E,1)) \supset sB_F(0_F,1) \\ &\iff \frac{r}{s}f(B_E(0_E,1)) \supset B_F(0_F,1) \end{aligned}$$

*similarly,*

$$\begin{aligned} f(B_E(0_E,r)) \supset B_F(0_F,s) &\iff r\overline{f(B_E(0_E,1))} \supset sB_F(0_F,1) \\ &\iff \frac{r}{s}\overline{f(B_E(0_E,1))} \supset B_F(0_F,1) \end{aligned}$$

if we put $g = \frac{r}{s}f \in \mathcal{L}(E,F)$, the proposition becomes,

$$''\overline{g(B_E(0_E,1))} \supset B_F(0_F,1) \implies g(B_E(0_E,1)) \supset B_F(0_F,1)''$$

*Proof.* By replacing if necessary $f$ by $\frac{r}{s}f$, we may suppose that $r = s = 1$. So, we haev to show the implication,

$$B_F(0_F,1) \subset \overline{f(B_E(0_E,1))} \implies B_F(0_F,1) \subset f(B_E(0_E,1))$$

suppose that

$$B_F(0_F,1) \subset \overline{f(B_E(0_E,1))}$$

and let us shwo that $B_F(0_F,1) \subset f(B_E(0_E,1))$ for all $\varepsilon \in (0,1)$, we have,

$$\overline{f(B_E(0_E,1))} \subset f(B_E(0_E,1)) + \varepsilon B_F 0_F, 1$$

Indeed, for any $y \in \overline{f(B_E(0_E,1))}$, we have $B_F(y,\varepsilon) \cap f(B_E(0_E,1)) \neq \emptyset$, so, by considering $u \in B_F(y,\varepsilon) \cap f(B_E(0_E,1))$, we have

$$y = u + \underbrace{(y-u)}_{\in B_F(0_F,\varepsilon) = \varepsilon B_F(0_F,1)} \in f(B_E(0_E,1)) + \varepsilon B_F(0_F,1)$$

Thus the claimed inclusion is proved.

From $B_F(0_F,1) \subset \overline{f(B_E(0_E,1))}$ and

$$\overline{f(B_E(0_E,1))} \subset f(B_E(0_E,1)) \subset f(B_E(0_E,1)) + \varepsilon B_F(0_F,1)$$

we deduce the inclusion

$$B_F(0_F,1) \subset f(B_E(0_E,1)) + \varepsilon B_F(0_F,1)$$

so, by applying one of the above theorems (find it!) for $A = B_F(0_F,1)$, we desire,

$$B_F(0_F,1) \subset \frac{1}{1-\varepsilon} f(B_E(0_E,1))$$

Now let $y \in B_F(0_F,1)$ arbitrary, so $\|y\|_F < 1$ , thus

$$\exists \varepsilon \in (0,1) \text{ s.t. } \quad \|y\|_F < 1-\varepsilon < 1$$

implying that $\frac{1}{1-\varepsilon}y \in B_F(0_F,1)$, so by the above inclusion,

$$\frac{1}{1-\varepsilon}y \in \frac{1}{1-\varepsilon} f(B_E(0_E,1))$$

thus $y \in f(B_E(0_E,1))$. Hence the inclusion

$$B_F(0_F,1) \subset f(B_E(0_E,1))$$

as required. □

Lets finish the proof that we initially started, suppose that $f$ is surjective and let us show that $f$ is an open mapping. According to Theorem 1, it sufficies to show that $\exists r > 0$, such that

$$f(B_E(0_E,1)) \supset B_F(0_F,1)$$

Next, according to Proposition 03, it sufficies to show $\exists r > 0$, such that

$$\overline{f(B_E(0_E,1))} \supset B_F(0_F,r)$$

we have obviously

$$E = \bigcup_{n=1}^{\infty} B_E(0_E,n)$$

thus,

$$F = f(E) = \bigcup_{n=1}^{\infty} f(B_E(0_E,n)) \quad \text{(since } f \text{ is surjective)}$$

in other words,

$$F = \bigcup_{n=1}^{\infty} f(B_E(0_E,n))$$

by inserting the closure on both sides,

$$F = \bigcup_{n=1}^{\infty} \overline{f(B_E(0_E,n))}$$

we get

$$int(F) = F \neq \emptyset \text{ so } \bigcup_{n=1}^{\infty} \overline{f(B_E(0_E,1))} \neq \emptyset$$

It follows according to the Baire theorem, that there exist $n_0 \in \mathbb{N}$ such that

$$int(\overline{f(B_E(0_E,n_0))}) \neq \emptyset$$

But

$$\overline{f(B_E(\overset{\circ}{0}_E,n_0))} = n_0\overline{f(B_E(\overset{\circ}{0}_E,1))}$$

Hence

$$\overline{f(B_E(\overset{\circ}{0}_E,1))} \neq \emptyset$$

Consequently, there exist $y \in \overline{f(B_E(0_E,1))}$, and ther eexist $r > 0$ such that

$$B_F(y,r) \subset \overline{f(B_F(0_E,1))}$$

Now by using the above inclusion, and the immediate fact that the set $\overline{f(B_E(0_E,1))}$ is convex and symmetric, since

$$\begin{aligned} B_E(0_E,1) \text{ is convex } &\Longrightarrow f \text{ is linear therefore } f(B_E(0_E,1)) \text{ is convex} \\ &\Longrightarrow \overline{f(B_E(0_E,1))} \text{ is convex} \end{aligned}$$

$\overline{f(B_E(0_E,1))}$ is symmetric ( $\forall a \in A, -a \in A$ ), since $B_E(0_E,1)$ is symmetric.

$$\begin{aligned} &\Longrightarrow f \text{ is linear therefore } f(B_E(0_E,1)) \text{ is symmetric} \\ &\Longrightarrow \overline{f(B_E(0_E,1))} \end{aligned}$$

we have for all $z \in B_F(0_F,r)$,

$$z+y, -z+y \in B_F(y,r)$$

thus we get,

$$z+y, -z+y \in \overline{f}(B_E(0_E,1))$$

thus (since $\overline{f(B_E(0_E,1))}$ is symmetric),

$$z+y, z-y \in \overline{f(B_E(0_E,1))}$$

thus (since $\overline{f(B_E(0_E,1))}$ is convex),

$$\frac{1}{2}((z+y)+(z-y)) = z \in \overline{f(B_E(0_E,1))}$$

hence the required inclusion,

$$B_F(0_E,r) \subset \overline{f(B_E(0_E,1))}$$

This completes the proof. □

*We can derive a bunch of theorems from the latter.*

**Theorem 5.1.6: (The Banach Isomorphism Theorem)**

Let $E$ and $F$ be two Banach spaces, and let $f \in \mathcal{L}(E, F)$ bijective, then then $f$ is an isomorphism of N.V.S (i.e. $f^{-1}$ is continiuous)

*Proof.* Since $f$ is surjective, then (accroding to the open mapping theorem) $f$ is open; that is the image (by $f$) of an open subset of $E$ is an open subset of $F$. Equivalently, the preimage by $f^{-1}$ of any open subset of $E$ is open in $F$. this shows that $f^{-1}$ is continuous thus $f$ is an isomorphism of N.V.S. □

**Theorem 5.1.7:**

Let $N_1$ and $N_2$ be two norms on $\mathbb{K}$-vector space $E$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, such that the two N.V.S $(E, N_1)$ and $(E, N_2)$ are both Banach. Then for $N_1$ and $N_2$ to be equivalent, it sufficies to have $N_2 \leq \alpha N_1$ or the converse for some $\alpha > 0$

*Proof.* Suppose that $\exists \alpha > 0$, such that $N_2 \leq \alpha N_1$. So the identity map of $E$,

$$\begin{array}{rccc} Id_E : & (E, N_1) & \longrightarrow & (E, N_2) \\ & x & \longmapsto & x \end{array}$$

$$\begin{aligned} N_2 \leq \alpha N_1 &\implies id_E \quad \text{is } \alpha\text{-Lipschitz} \\ &\implies id_E \text{ is continuous} \end{aligned}$$

$id_E$ is linear, bijective, and continuous this implies (according to the above theorem), that $id_E$ is an isomorphism of N.V.S, i.e., so $id_E^{-1}$ is continuous, so Lipschitz continuous, so $\exists \beta > 0$ such that $N_1 \leq \beta N_2$, Hence $N_1$ and $N_2$ are equivalent. □

**Theorem 5.1.8: (The closed graph theorem)**

Let $E$ and $F$ be two Banach spaces over some field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $f : E \longrightarrow F$ be a linear mapping, then $f$ is continuous if and onyl if its graph $G(f)$ is closed in the Banach space $E \times F$, Recall that

$$G(f) := \{(x, f(x)) : x \in E\}$$

*Proof.*

$$(\implies)$$

Suppose that $f$ is continuous and show that $G(f)$ is closed in $E \times F$. So, let $\{(x_n, f(x_n))\}_{n \in \mathbb{N}}$, be an arbitrary sequence of $G(f)$, converging in $E \times F$ to some $(x, y) \in E \times F$ and let show that

$$(x, y) \in G(f) \quad y = f(x)$$

since the projections are continuous

$$\begin{array}{rccc} \pi_1 : & E \times F & \longrightarrow & E \\ & (u, v) & \longmapsto & u \end{array}$$

and

$$\begin{array}{rccc} \pi_2 : & E \times F & \longrightarrow & F \\ & (u, v) & \longmapsto & v \end{array}$$

are both continuous, then the fact

$$(x_n, f(x_n)) \to (x, y) \quad \text{as} \quad n \to \infty$$

implies

$$x_n \to x \quad f(x_n) \to y \quad \text{as} \quad n \to \infty$$

But on the other hand, we have since $f$ is continuous, we have

$$x_n \to x \text{ (in E)} \implies f(x_n) \to f(x) \text{ (in F)} \quad \text{as } n \to \infty$$

It follows according to the uniqueness of the limit that $y = f(x)$, as required.

$$(\Longleftarrow)$$

Conversely, suppose that $G(f)$ is closed in $E \times F$. This implies that the vector subspace $G(f)$ of $E \times F$ is Banach (a closed susbet of complete space is complete). Next, consider the two maps,

$$p_1 = \pi_1|_{G(f)} \quad p_2 = \pi_2|_{G(f)}$$

where

$$\begin{array}{rccc} p_1 : & G(f) & \longrightarrow & E \\ & (u, f(u)) & \longmapsto & u \end{array}$$

and

$$\begin{array}{rccc} p_2 : & G(f) & \longrightarrow & F \\ & (u, f(u)) & \longmapsto & f(u) \end{array}$$

Since $\pi_1$ and $\pi_2$ are linear and continiuous then $p_1$ and $p_2$ are also linear and continuous, Besides $p_1$ is clearly bejictive. So according to the Banach Isomorphism theorem we get that $p_1^{-1}$ is continuous, then,

$$\begin{array}{rccccc} f : & E & \longrightarrow & G(f) & \longrightarrow & F \\ & u & \longmapsto & (u, f(u)) & \longrightarrow & f(u) \end{array}$$

clearly

$$f = p_2 \circ p_1^{-1}$$

is continuous, since its a composition of two continuous maps, as required. this completes the proof of the theorem. □

*The Banach-Steinhans Theorem*

**Definition 5.1.1: Meager Sets**

Let $E$ be a toplogical space and $X$ be a subset of $E$. Then $X$ is said to be meager if it can be included in a countable union of closed subsets of $E$ of empty interior.
Equivalently, $X$ is meager if its a countable union of subsets whose closure has empty interior.

A set that is not meager is said to be nonmeager

**Example**

1. $Q$ is meager in $\mathbb{R}$ equipped with its usual toplogy. Indeed we can write,

$$Q = \bigcup_{n \in Q} \{n\}$$

$\{x\}$ is closed in $\mathbb{R}$, and $\overset{\circ}{\overline{\{x\}}} = \emptyset$, Other method is,

$$Q = \mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z} \cup \dots$$

for all $n \in \mathbb{N}$, we have

$$\overset{\circ}{\overline{\frac{1}{n}\mathbb{Z}}} = \frac{1}{n}\overset{\circ}{\overline{\mathbb{Z}}} = \emptyset$$

since $\overline{\mathbb{Z}} = \mathbb{Z}$ and $\mathring{\mathbb{Z}} = \emptyset$

2. Let $E$ be Baire space (i.e., a toplogical space that satisfies the Baire property).

   - $E$ is nonmeager in $E$.

     *Proof. Indeed if $E = \bigcup_{n=0}^{\infty} F_n$, where $F_n = \emptyset, \forall n \in \mathbb{N}$, then since $E$ is Baire we get $\mathring{E} = \emptyset$, which is a contradiction .* □

   - More generally, if $A$ is a meager subset of $E$, then $E \backslash A$ is dense in $E$

     *Proof.* Since $A$ is meager then we have

$$A \subset \bigcup_{n=1}^{\infty} F_n \quad \mathring{F}_n = \emptyset \quad \forall n \in \mathbb{N}$$

Since $E$ is Biare then $\bigcup_{n=1}^{\infty} \mathring{F_n} = \emptyset$. Thus $\mathring{A} \subset \bigcup_{n=1}^{\infty} = \emptyset$, thus $\mathring{A} = \emptyset$, hence

$$\overline{E \backslash A} = E \backslash \mathring{A} = E \backslash \emptyset = E$$

that is $X \backslash A$ is dense in $E$ □

**Theorem 5.1.9: Banach-Steinhaus 1927**

Let $E$ and $F$ be two N.V.S for a family of continuous mappings from $E$ to $F$ to be uniformally bounded on the unit ball of $E$, it sufficies that it be pointwise bounded on a noneager subset of $E$.

**Definition 5.1.2: (Uniformally bounded in Unit ball)**

$(f_i)_{i \in I}$ linear continuous.

$$\exists M > 0, \forall x \in B_E(0_E, 1) \|f_i(x)\| \leq M$$

**Definition 5.1.3: (Pointwise bounded on A)**

Pointwise bounded on $A$, for all $x \in A$, $\exists M_x$ such that,

$$\forall i \in I : \quad \|f_i(x)\| \leq M_x$$

More explicitly, let $A \subset \mathcal{L}(E, F)$, and for all $x \in E$, let

$$A_x := \{f(x), f \in A\}$$

Finally, let

$$B := \{x \in E, A_x \text{ is bounded in } F\}$$

Suppose that $B$ is nonmeager in $E$, then $A$ is bounded in $\mathcal{L}(E, F)$, In particular $B = E$

*Proof.* We can write $B$ as,

$$\begin{aligned} B &= \bigcup_{n=1}^{\infty} \{x \in E, A_x \text{ is bounded by } n \text{ in } F\} \\ &= \bigcup_{n=1}^{\infty} \{x \in E : \|f(x)\|_F \leq n, \forall f \in A\} \end{aligned}$$

next for all $n \in \mathbb{N}$, we have

$$B_n = \bigcap_{f \in A} \underbrace{\{x \in E : \|f(x)\|_F \leq n\}}_{B_{n,f}}$$

since for any $n \in \mathbb{N}$ and any $f \in A$, $B_{n,f}$ is the preimage of the closed subset $(-\infty, n]$ of $\mathbb{R}$ by the continuous map

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \|f(x)\| = \|\| \circ f \end{array}$$

then $B_{n,f}$ is closed in $E$ for all $n \in \mathbb{N}, \forall f \in A$, thus $B_n (n \in \mathbb{N})$ is closed in $E$ as its the intersction of closed subsets of $E$,but since $B$ is non meager and $B = \bigcup_{n=1}^{\infty} B_n$ , where $B_n$ is closed for all $n$, there exist $N \in \mathbb{N}$ such that

$$\mathring{B_N} \neq \emptyset$$

therefore $\exists x_0 \in E, \exists r > 0$ such that

$$B_E(x_0, r) \subset B_N$$

Now, for all $f \in A$ and for all $x \in B_E(0_E, 1)$, we have that

$$x_0(+/-)rx \in B_E(x_0, r) \subset B_N$$

implying that

$$\|f(x_0(+/-)rx)\|_F \leq N$$

consequently, we have

$$\forall f \in A, \forall x \in B_E(0_E, 1) \quad f(x) = f\left(\frac{1}{2r}[(x_0 + rx) - (x_0 - rx)]\right)$$

since $f$ is linear we get

$$f(x) = \frac{1}{2r}\left[f(x_0 + rx) - f(x_0 - rx)\right]$$

thus

$$\forall f \in A, \forall x \in B_E(0_E, 1)$$

we get

$$\begin{aligned} \|f(x)\|_F &\leq \frac{1}{2r}\left[\|f(x_0 + rx)\|_F + \|f(x_0 - rx)\|_F\right] \\ &\leq \frac{N}{r} \end{aligned}$$

implying that

$$||| \, f \, ||| \leq \frac{N}{r} \quad (f \in A)$$

showing that $A$ is bounded in $\mathcal{L}(E, F)$, as required.

$\square$

before we continue the main proof, we will add some small theorems

**Theorem 5.1.10: 1**

Let $E$ be a Banach space and $F$ be an arbitrary N.V.S. Let also $A$ be a subset of $\mathcal{L}(E,F)$. Then the two following properties are equivalent,

(i) $A$ is bounded in $\mathcal{L}(E,F)$

(ii) for all $x \in E$, the subset

$$\{f(x), f \in A\} \text{ of } F \text{ is bounded.}$$

*Proof.* Since $E$ is Banach then its Baire, hence $E$ is nonmeager in it self the result of the corollary then follows from the previous proof. □

**Theorem 5.1.11: 2**

Let $E$ be a Banach soace, and $F$ be an arbitrary N.V.S. Let also $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}(E,F)$, suppose that for all $x \in E$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in $F$ and denote by $f(x)$ its limit, then

- $(f_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}(E,F)$
- $f \in \mathcal{L}(E,F)$
- $||| f ||| \leq \lim_{n\to\infty} inf ||| f_n |||$

*Proof.* The Boundedness of $f$

for all $x \in E$, since the sequence $(f_n)_{n\in\mathbb{N}}$ of $F$ is assumed convergent, then its bounded. this implies implies according to the Theorem 1, that the sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{L}(E,F)$ is bounded.

The Linearity of $f$ (obvious)

for all $\lambda \in \mathbb{K}$, $\forall x, y \in E$, we have,

$$\begin{aligned} f(\lambda x + y) &= \lim_{n\to\infty} f_n(\lambda x + y) \\ &= \lim_{n\to\infty} (\lambda f_n(x) + f_n(y)) \\ &= \lambda \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} f_n(y) \\ &= \lambda f(x) + f(y) \end{aligned}$$

showing that $f$ is linear.

The continuity of $f$ and the estimate of $||| f |||$,

$\forall x \in E$, we have

$$\begin{aligned}
\|f(x)\|_F &= \| \lim_{n\to\infty} f_n(x)\|_F \\
&= \lim_{n\to\infty} \|f_n(x)\|_F \\
&= \lim_{n\to\infty} inf \|f_n(x)\|_F \\
&\leq \lim_{n\to\infty} inf \left(||| f_n ||| \, \|x\|_E\right) = \left(\lim_{n\to\infty} inf \; ||| f_n |||\right) \|x\|_E
\end{aligned}$$

implying that $f$ is continuous and that

$$||| f ||| \leq \lim_{n\to\infty} inf \; ||| f_n |||$$

This completes the proof. □

*Corollary :* Let $E$ be a Banach space and $F$ an $G$ be two arbitrary N.V.S. let also $h : E \times F \longrightarrow G$ be a bilinear mapping that is separately continuous, that is $h$ satisfies the following properties,

(1) for all $y \in F$, the linear mapping

$$\begin{array}{rccc} h(.,y): & E & \longrightarrow & G \\ & x & \longmapsto & h(x,y) \end{array}$$

is continuous

(2) for all $x \in E$, the linear mapping

$$\begin{array}{rccc} h(x,.): & F & \longrightarrow & G \\ & y & \longmapsto & h(x,y) \end{array}$$

is continuous

Then $h$ is *continuous*

*Proof.* Define

$$A = \left\{h(.,y) : y \in \overline{B_F}(0_F,1)\right\} \subset \mathcal{L}(E,G)$$

and for all $x \in E$,

$$\begin{aligned}
A_x &:= \{f(x), f \in A\} \\
&= \left\{h(x,y), y \in \overline{B_F}(0_F,1)\right\} \\
&= \left\{h(x,.)(y), y \in \overline{B_F}(0_F,1)\right\}
\end{aligned}$$

Giving $x \in E$, since the linear mapping $h(x,.)$ is continuous by hypothesis then the last inequality shows that the subset $A_x$ of $G$ is bounded. Thus (by Banach Steinhaus theorem), the subset $A$ of $\mathcal{L}(E,G)$ is bounded (say by a pointwise constant $M$). Hence, we have for all $x \in \overline{B_F}(0_E,1)$ and $y \in \overline{B_F}(0_F,1)$,

$$\begin{aligned} \|h(x,y)\|_G &= \|h(.,y)(x)\|_G \\ &\leq ||| \underbrace{h(.,y)}_{\in A} |||_{\mathcal{L}(E,G)} \cdot \|x\|_E \leq M \end{aligned}$$

implying that $h$ is continuous, hence the corollary is proved. □

# $\int$ QUOTIENT VECTOR NORMED SPACES 6

Let $E$ be a N.V.S. and $H$ be a vector subspace of $E$. Recall that the quotient vector space of $E$ on $H$ is given by,

$$E_{\backslash H} = \{x + H, x \in E\}$$

Consider the map

$$\begin{array}{rccc} \|.\|_{E_{\backslash H}}: & E_{\backslash H} & \longrightarrow & [0, \infty) \\ & C & \longmapsto & \inf_{x \in C} \|x\|_E \end{array}$$

the map $\|.\|_{E_{\backslash H}}$ defines a seminorm on $E_{\backslash H}$. In addition, $\|.\|_{E \backslash H}$ becomes a norm on $E_{\backslash H}$ if and only if $H$ is closed in $E$.

*Proof.* Let us show that the map $\|.\|_{E_{\backslash H}}$ satisfies the three properties of a seminorm on the quotient vector space $E_{\backslash H}$.

1. The zero vector of the quotient vector space $E_{\backslash H}$ is $C(0_E) = 0_E + H = H$, and we have,

   $$\|H\|_{E_{\backslash H}} = \inf_{x \in H} \|x\|_E \leq \|0_E\|_E$$

   Thus, $\|H\|_{E \backslash H} = 0$, as required.

2. Let $\lambda \in \mathbb{K}$ and $C \in E_{\backslash H}$, since $\lambda C = \{\lambda x, x \in C\}$ then we have,

   $$\begin{aligned} \|\lambda C_{E_{\backslash H}}\| &= \inf_{x \in C} \|\lambda x\|_E \\ &= \inf_{x \in C} (|\lambda| \, \|x\|_E) \\ &= |\lambda| \left( \inf_{x \in C} \|x\|_E \right) = |\lambda| \, \|C\|_{E_{\backslash H}} \end{aligned}$$

   as required.

3. Let $C_1, C_2 \in E_{\backslash H}$ which we can write as

   $$C_1 = x_1 + H \quad C_2 = x_2 + H$$

where $x_1, x_2 \in E$,

$$\|C_1 + C_2\|_{E_{\setminus H}} \overset{?}{\leq} \|C_1\|_{E_{\setminus H}} + \|C_2\|_{E_{\setminus H}}$$

then $C_1 + C_2 = x_1 + x_2 + H$, By the triangle inequality in $E$, we have for all $h_1, h_2 \in H$,

$$(x_1 + h_1) + (x_2 + h_2) \leq \|x_1 + h\|_E + \|x_2 + h_2\|_E$$

taking in the two sides of this inequality the infimum where $h_1, h_2 \in H$, we obtain since $(\{h_1 + h_2, h_1, h_2 \in H\} = H)$

$$\inf_{h \in H} \|x_1 + x_2 + h\|_E \leq \inf_{h_1 \in H} \|x + h_1\| + \inf_{h_2 \in E} \|x + h_2\|_E$$

That is

$$\|C_1 + C_2\|_{E_{\setminus H}} \leq \|C_1\|_{E_{\setminus H}} + \|C_2\|_{E_{\setminus H}}$$

as required. Consequently, $\|.\|_{E_{\setminus H}}$ defines a seminorm on $E\setminus H$.
Next, denoting by $d$ the metric associated to the norm of $E$, we have for all $x \in E$,

$$\begin{aligned}
\|x + H\|_{E\setminus H} &= \inf_{h \in H} \|x + h\|_E \\
&= \inf_{h \in H} \|x - h\|_E \\
&= \inf_{h \in H} d(x, H) \\
&= d(x; H)
\end{aligned}$$

It follows according to the well-known results on metric spaces, that for all $x \in E$,

$$\begin{aligned}
\|x + H\|_{E\setminus H} = 0 &\iff d(x, H) = 0 \\
&\iff x \in \overline{H}
\end{aligned}$$

Therefore, $\|.\|_{E_{\setminus H}}$ defines a norm on $E_{\setminus H}$ if and only if $\overline{H} = 0_{E\setminus H} = H$, that is if and only if $H$ is closed in $E$, the proof is complete.

□

**Terminology :**

The map $\|.\|_{E_{\setminus H}}$ defined above is called the quotient seminorm of $E_{\setminus H}$, if $H$ is closed in $E$, its called the quotient norm of $E_{\setminus H}$.

☞ *NB :* whenever the quotient space $E_{\backslash H}$ is mentioned (where $E$ is N.V.S. and $H$ is closed vector subspace of $E$ ) its completely assumed that $E\backslash H$ is equipped with the quotient norm $\|.\|_{E_{\backslash H}}$ defined previously.

**Theorem 6.0.1:**

Let $E$ be a N.V.S. and $H$ be a closed *proper* subspace of $E$. then the quotient map

$$\begin{array}{rcl} \Pi : \; E & \longrightarrow & E_{\backslash H} \\ x & \longmapsto & x + H \end{array}$$

is continuous, and satisfies $||| \; \pi \; ||| = 1$

*Proof.* Recall that $\pi$ is linear. Next, for all $x \in E$, we have,

$$\begin{aligned} \|\pi(x)\|_{E\backslash H} = \|x + H\|_{E_{\backslash H}} := \inf_{h \in H} \|x + h\|_E \\ \leq \|x + 0_E\|_E = \|x\|_E \end{aligned}$$

implying that $\pi$ is continuous and that

$$||| \; \pi \; ||| \leq 1$$

Now, let us show that

$$||| \; \pi \; ||| \geq 1$$

To do so, fix $a \in E\backslash H$, thus $\pi(a) \neq H = 0_{E_{\backslash H}}$, implying that $\|\pi(a)\|_{E_{\backslash H}} > 0$, by definition of $\|\pi(a)\|_{E_{\backslash H}}$ and the characterization of the infimum of a subset of $\mathbb{R}$,

$$\|\pi(a)\|_{E_{\backslash H}} = \inf_{x \in \pi(a)} \|x\|_E$$

for all $\varepsilon > 0$, there exist $x_E \in \pi(a)$ such that,

$$\begin{aligned} \|\pi(a)\|_{E_{\backslash H}} &\leq \|x_\varepsilon\|_E \\ &\leq \|\pi(a)\|_{E_{\backslash H}} + \varepsilon \end{aligned}$$

implying that,

$$\frac{\|\pi(x_a)\|_{E_{\backslash H}}}{\|x_\varepsilon\|_E} \geq 1 - \frac{me}{\|x_\varepsilon\|_E} \geq 1 - \frac{\varepsilon}{\|\pi(a)\|_{E_{\backslash H}}}$$

Thus,

$$\begin{aligned} ||| \; \pi \; ||| = \sup_{x \in E \backslash \{0_E\}} \frac{\|\pi(x)\|_{E_{\backslash H}}}{\|x\|_E} &\geq \frac{\|\pi(x_\varepsilon)\|_{E_{\backslash H}}}{\|x_\varepsilon\|_E} \\ &\geq 1 - \frac{\varepsilon}{\|\pi(a)\|_{E_{\backslash H}}} \end{aligned}$$

hence

$$||| \, \pi \, ||| \geq 1 - \frac{\varepsilon}{\|\pi(a)\|_{E_{\setminus H}}}$$

by taking $\varepsilon \to 0^+$ gives $||| \, \pi \, ||| \geq 1$, as required here $||| \, \pi \, ||| = 1$, completing this proof. □

**Theorem 6.0.2:**

Let $E$ be a Banach N.V.S. and $H$ be a closed vector subspace of $E$, then $E_{\setminus H}$ is Banach.

*Proof.* To show that $E \setminus H$ is Banach, we will prove that every normally convergent series in $E_{\setminus H}$ is convergent, Let $\sum_{n=1}^{\infty} C_n$ be a normally convergent series in $E_{\setminus H}$, This means that the real series $\sum_{n=1}^{\infty} \|C_n\|_{E_{\setminus H}}$ is convergent, by the definition of $\|C_n\|_{E_{\setminus H}} (= \inf_{x \in C_n} \|x\|_E)$, and the chracterzation of the infimum of a subset of $\mathbb{R}$, for all $n \in \mathbb{N}$, there exist $x_n \in C_n$ such that

$$\|x_n\|_E \leq \|C_n\|_{E_{\setminus H}} + \frac{1}{2^n}$$

This implies that the real series

$$\sum_{n=1}^{\infty} \|x_n\|_E$$

converges, namely the series $\sum_{n=1}^{\infty} x_n$ is normally convergent in $E$, but since $E$ is Banach, it follows that the series $\sum_{n=1}^{\infty} x_n$ is convergent in $E$. Finally, since $\pi$ is continuous (according to proposition 2), we conclude that the series $\sum_{n=1}^{\infty} \pi(x_n) = \sum_{n=1}^{\infty} C_n$ is convergent in $E \setminus H$, as required therefore $E_{\setminus H}$ is Banach, completing the proof. □

*The Hahn-Banach theorem*

**PreLiminaries :**

**Theorem 6.0.3: Zorn's Lemma**

Let $X$ be partially ordered suppose that every *chain* $\mathcal{C}$ in $X$, (That is, every totally ordered subset of $X$), has an upper bound in $X$. Then $X$ contains atleast one maximal element

*Note :* $m$ is upper-bound

$$\forall x \in A, x \leq m$$

Example

**Theorem 6.0.4:**

Every vector space has a basis. (Teacher provided a Skratch proof, we may prove it next time)

**Theorem 6.0.5: Zorn's Lemma**

Let $X$ be a partially ordered set, suppose that every chain in $\mathcal{C}$ in $X$, that is every totally ordered subset of $X$, has an upper bound in $X$. Then $X$ contains atleast one maximal element.

**Theorem 6.0.6:**

Every vector space has (atleast) a basis.

*Proof.* Let $E$ be a vector space over some field $\mathbb{K}$, (not necessarly $\mathbb{R}$ or $\mathbb{C}$), if $E = \{0_E\}$ then $\emptyset$ is the basis of $E$. Now suppose that $E \neq \{0_E\}$, Consider $X$ the set of all linearly independent subsets of $X$ of $E$, we have $X \neq \emptyset$ because every nonzero vector of $E$ is a linearly independent subset of $E$. we equip $X$ with the partial order of set inclusion

$$(X, \subset)$$

for every chain $\mathcal{C}$ of $X$ we claim that the set $\bigcup_{s\in\mathcal{C}} S$ is linearly independent. (i.e. $\in X$), so $\bigcup_{s\in\mathcal{S}} S$ constitutes an upper bound of $\mathcal{C}$ in $X$, let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and $x_1, \ldots, x_n \in \bigcup_{S\in\mathcal{C}} S$ such that,

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0_E$$

and show that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0_{\mathbb{K}}$$

by hypothesis, for all $i \in \{1, 2, \ldots, n\}$ there exists $S_i \in \mathcal{C}$ such that $x_i \in S_i$. Next, since $\mathcal{C}$ is totally ordered, there exists a bijection from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ such that

$$S_{\sigma(1)} \subset S_{\sigma(2)} \subset \ldots \subset S_{\sigma(n)}$$

consequently, we have

$$x_1, \ldots, x_n \in S_{\sigma(n)}$$

But since $S_{\sigma(n)}$ is linearly independent, then the equality

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0_E$$

implies that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0_{\mathbb{K}}$$

as required, our claim is confirmed.
So we can apply the zorn lemma which ensures that $X$ contains atleast one maximal element. Let $B$ be a maximal element of $X$ so $B$ is a linearly indepdent subset of $E$. Next, for every vector $x \in E$, we have either $x \in B$, thus ( $x \in \langle B \rangle$) or $x \notin B$, that is $B \subsetneq B \cup \{x\}$, (implying according to the maximality of $B$ in $X$) that

$$B \cup \{x\} \notin X$$

that is, $B \cup \{x\}$ is linearly dependent, hence $x \in \langle B \rangle$. So, we have for all $x \in E$, $x \in \langle B \rangle$. Thus $\langle B \rangle = E$, Consequently, $B$ is both linearly independent and spans $E$; that is, $B$ is a a basis of $E$.
Hence the proof is complete. $\square$

## 6.1 The problem of the extension of continuous linear forms on N.V.S

**Problem 01:** Let $E$ and $F$ be two vector spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $H$ be a proper subspace of $E$, If $f : H \longrightarrow F$ is a linear mapping from $H$ to $F$ can we extend it to a linear mapping $f^{\sim} : E \longrightarrow F$.

$$\begin{array}{cccc} f^{\sim} : & E & {}^{\pi}\longrightarrow^{H} & F \\ & x & \longmapsto & f^{\sim}(x) \end{array}$$

**Answer: Yes!**
It sufficies to consider a complementory subspace $G$ of $H$ in $E$, i.e.

$$G \oplus H = E$$

$$\begin{array}{cccc} f^{\sim} : & E & \longrightarrow & F \\ & x = h + g (h \in H, g \in G) & \longmapsto & f(h) \end{array}$$

In other words, we have $f^{\sim} = f \circ \pi$, where $\pi$ is the projection of $E$ into $H$ parallel to $G$

$$\begin{array}{cccccc} f : & E & \longrightarrow^{\pi} & H & \longrightarrow^{f} & F \\ & x = h + g & \longmapsto & h & \longmapsto & f(h) \end{array}$$

since $\pi$ is linear then $f^{\sim} = f \circ \pi$ is linear and since $\pi(h) = h (\forall h \in H)$, then

$$f^{\sim}_{|H} = f$$

that is $f^{\sim}$ extends $f$
**Problem 02:**

Now, suppose that $E$ and $F$ are two N.V.S over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $H$ be a proper normed vector subspace of $E$ and $f : H \longrightarrow F$ *linear* and *continuous* . Is't possible to extend $f$ to some *linear* and *continuous* mapping $f^{\sim} : E \longrightarrow F$

**Answer : No, in general !**

Note that the method used to solve **Problem 01** fails because the considered projection $\pi$ is in general not continuous.

Definition 6.1.1:

Let $E$ be an $\mathbb{R}$-N.V.S, and $p : E \longrightarrow \mathbb{R}$ be a map, we say that $p$ is sublinear if it satisfies :

(i) $p(x+y) \leq p(x) + p(y) \qquad (\forall x, y \in \mathbb{R})$

(ii) $p(\lambda x) = \lambda p(x) \qquad (\forall \lambda \geq 0, \forall x \in E)$

Theorem 6.1.1: The Hahn-Banach Theorem

Let $E$ be an $\mathbb{R}$-vector space and $p : E \longrightarrow \mathbb{R}$ be a *sublinear* function. Then any lineart form $f$ on a vector subspace of $H$ of $E$ that is dominated above by $p$ has at least one linear extension to all $E$ that is also dominated above by $p$. More explicitly, for every linear form $f : H \longrightarrow \mathbb{R}$ satisfying

$$f(x) \leq p(x) \qquad (\forall x \in H)$$

there exists a linear form $f^{\sim} : E \longrightarrow \mathbb{R}$ such that

$$f^{\sim}_{|H} = f \text{ and } f^{\sim}(x) \leq f(x) \qquad (\forall x \in E)$$

*Proof.* Let $H$ be a vector subspace of $E$ and $f : H \longrightarrow \mathbb{R}$ be a linear form on $H$ that is dominated above by $p$ since the result of the theorem is trivial for $H = E$ suppose for the sequel that $H \neq E$.

$1^{st}$ **Step**

let $u \in E \backslash H$ be fixed we are going to show that there exist a linear form $g : H \oplus \mathbb{R}_u \longrightarrow \mathbb{R}$, extending $f$ and satisfying $g(x) \leq p(x)$ for all $x \in H + \mathbb{R}_u$, the determination of such a $g$ is clearly equivalent to the determination of its value at $u$, that is the determination of $\lambda := g(u) \in \mathbb{R}$ so that we have for all $h \in H$ and all $t \in \mathbb{R}$,

$$g(h+tu) \leq p(h+tu)$$

that is, since $g$ should be linear and extend $f$,

$$g(h) + tg(u) \leq p(h+tu)$$

i.e.,

$$f(h) + t\lambda \leq p(h + tu) \qquad (\forall h \in H, \forall t \in \mathbb{R}) \qquad (1)$$

since $(1)$ is obviously satisfied for $t = 0$, then we have

$$(1) \iff \begin{cases} f(\frac{1}{t}h) + \lambda \leq p(\frac{1}{h}h + u) & \text{if } t > 0 \qquad (2) \\ f(\frac{1}{t}h)L + \lambda \leq -p(-\frac{1}{t}h - u) & \text{if } t < 0 \qquad (3) \end{cases}$$

and we have

$$(2) \iff \lambda \leq p(x + u) - f(x) \qquad (\forall x \in H)$$
$$(3) \iff \lambda \geq f(y) - p(y - u)(\forall y \in H)$$

thus

$$\begin{aligned} (1) &\iff f(y) - p(y - u) \leq \lambda \leq p(x + u) - f(x) \qquad (\forall x, y \in H) \\ &\iff \sup_{y \in H} \{f(y) - p(y - u)\} \leq \lambda \leq \inf_{x \in H} \{p(x + u) - f(x)\} \qquad (4) \end{aligned}$$

the existence of $\lambda$ is then equivalent to

$$\sup_{y \in H} \{f(y) - p(y - u)\} \leq \inf_{x \in H} \{p(x + u) - f(x)\} \qquad (*)$$

Let us show (*), for all $x, y \in H$, we have according to the assumption made on $f$ and $p$,

$$\begin{aligned} f(x) + f(y) = f(x + y) \leq p(x + y) &= p((y - u) + (x + u)) \\ &\leq p(y - u) + p(x + u) \end{aligned}$$

hence

$$f(y) - p(y - u) \leq p(x + u) - f(x) \qquad (\forall x, y \in H)$$

thus,

$$\sup_{y \in H} \{f(y) - p(y - u)\} \leq \inf_{x \in H} \{p(x + u) - f(x)\}$$

confirming $(*)$, Hence the existence of $\lambda$ as required and then the existence of $g$ as required.

$2^{nd}$ **Step**

Consider the set $X$ of the pairs $(F, \varphi)$, where $F$ is a subspace of $E$ containing $H$ and $FF$ is a linear form on $F$ extending $f$ and satisfying

$$\varphi(x) \leq p(x) \qquad (\forall x \in F)$$

Since $(H, f) \in X$ then $X \neq \emptyset$, we equip $X$ with the binary relation $\mathcal{R}$ defined by

$$(F_1, \varphi_1)\mathcal{R}(F_2, \varphi_2) \iff F_1 \subset F_2 \text{ and } \varphi_{2|F_1} = \varphi_1$$

we easily check that $\mathcal{R}$ is a partial order on $X$.

Next for every chain $((F_i, \varphi_i))_{i\in I}$ of $X$, the pair $(F, \varphi)$ given by

$$F = \bigcup_{i\in I} F_i \quad \varphi(x) = \varphi_i(x) \qquad (\forall i \in I, \forall x \in F_i)$$

Clearly

The zorn lemma to desire that $(X, \mathcal{R})$ has at least 1 maximal element $(F^\sim, \varphi^\sim)$ but if $F^\sim \neq E$ and $u \in E\backslash F^\sim$, by the $1^{st}$ step, we can construct a pair

$$(F^\sim \oplus \mathbb{R}_u, \quad \Psi) \in X$$

which we strictly greater

Thus $F^\sim = E$. So it sufficies to take $f^\sim = \varphi^\sim$ to conclude to the resulmt of the theorem:w

□

**Theorem 6.1.2: (Hahn-Banach)**

Let $E$ be a $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $H$ be a vector subspace of $E$ let also $N : E \longrightarrow [0, \infty)$ be a seminorm on $E$ and $f : H \longrightarrow \mathbb{K}$ be a linear form on $H$, satisfying

$$|f(x)| \leq N(x) \qquad (\forall x \in E)$$

then there exist a $\mathbb{K}$-linear form $\overset{\sim}{f} : E \longrightarrow \mathbb{K}$, extending $f$ and satisfying

$$\overset{\sim}{f} \leq N(x) \qquad (\forall x \in E)$$

*Proof.* **Case 01 :**

If $\mathbb{K} = \mathbb{R}$ since we have for all $x \in H$,

$$f(x) \leq |f(x)| \leq N(x)$$

then by applying Theorem 1 for $p = N$, we find that there exist a linear form $\overset{\sim}{f} : E \longrightarrow \mathbb{R}$ extending $f$ and satisfying

$$\forall x \in E : \overset{\sim}{f}(x) \leq N(x) \qquad (1)$$

By applying $(1)$ for $(-x)$ instead of $x$, we get,

$$\begin{aligned}\widetilde{f}(-x) &\leq N(-x) = N(x)\\ -\widetilde{f}(x) &\leq N(x)\\ \widetilde{f}(x) &\geq -N(x) \qquad (2)\end{aligned}$$

from $(1)$ and $(2)$, we have

$$\begin{aligned}&\iff -N(x) \leq \widetilde{f}(x)\\ &\iff \left|\widetilde{f}(x)\right| \leq N(x)\end{aligned}$$

**Case 02:**

Define

$$\begin{array}{rccc} g: & H & \longrightarrow & \mathbb{R}\\ & x & \longmapsto & g(x) := Ref(x) = \frac{f(x)+\overline{f(x)}}{2}\end{array}$$

Its clear that $g$ is an $\mathbb{R}$-linear form on $H$, next we have for all $x \in H$,

$$\begin{aligned}|g(x)| = |Re(f(x))| &\leq |f(x)|\\ &\leq N(x)\end{aligned}$$

for all $x \in H$,

$$|g(x)| \leq N(x)$$

so we can apply the result of the first case, for the linear form $g$ on $H$, we find that $\exists\widetilde{g} : E \longrightarrow \mathbb{R}$ an $\mathbb{R}$-linear extending $g$, and satisfying,

$$\forall x \in E : \quad \left|\widetilde{g}(x)\right| \leq N(x)$$

Furthermore, we have, for all $x \in H$,

$$\begin{aligned}g(ix) &= Re(\overline{f}(ix))\\ &= Re(if(x))\\ &= -Imf(x)\\ &\implies Imf(x) = -g(ix)\end{aligned}$$

Then for all $x \in H$,

$$\begin{aligned}f(x) &= Ref(x) + iImf(x)\\ &= g(x) - ig(ix)\end{aligned}$$

Thus, we have for all $x \in H$,

$$f(x) = g(x) - ig(ix) \qquad (1)$$

therefore define, $\widetilde{f} : E \longrightarrow \mathbb{C}$, by,

$$\widetilde{f}(x) = \widetilde{g}(x) - i\widetilde{g}(ix)$$

**We will prove that it's an extension**

(1) Show that $\widetilde{f}$ extends $f$.

(2) Show that $\widetilde{f}$ is $\mathbb{C}$-linear.

*Proof.* Since $\widetilde{g}$ is $\mathbb{R}$-linear then $\widetilde{f}$ is obviously $\mathbb{R}$-linear. So, to show that $\widetilde{f}$ is $\mathbb{C}$-linear it sufficies to show that

$$\widetilde{f}(ix) = i\widetilde{f}(x) \qquad (\forall x \in E)$$

for all $x \in E$, we have,

$$\begin{aligned}
\widetilde{f}(ix) &= \widetilde{g}(ix) - i\widetilde{g}(-x) \\
&= \widetilde{g}(ix) + i\widetilde{g}(x) \\
&= i\left(\widetilde{g}(x) - i\widetilde{g}(ix)\right) = i\widetilde{f}(x)
\end{aligned}$$

as required, then $\overset{f}{\sim}$ is $\mathbb{C}$-linear. $\square$

Now we have to show that

$$\left|\widetilde{f}(x) \leq N(x) \qquad (\forall x \in E)\right|$$

Finally, for all $x \in E$, by writting the complex number $\widetilde{f}(x)$ in it exponential form, say,

$$\widetilde{f}(x) = \left|\widetilde{f}(x)\right| e^{i\theta} \qquad (\theta \in \mathbb{R})$$

we have,

$$\begin{aligned}
\left|\widetilde{f}(x)\right| &= \widetilde{f}(x)e^{-i\theta} \\
&= \widetilde{f}(xe^{-i\theta}) \\
&= Re\widetilde{f}(xe^{-i\theta}) \\
&= \widetilde{g}(xe^{-i\theta}) \\
&\leq N(xe^{-i\theta}) = \left|e^{-i\theta}\right| N(x) = N(x)
\end{aligned}$$

Thus

$$\left|\widetilde{f}(x)\right| \leq N(x) \qquad (\forall x \in E)$$

as required, thus this completes the proof. $\square$

**Theorem 6.1.3: Hahn-Banach**

Let $E$ be a N.V.S over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $H$ be a non zero subspace of $E$, then for all $f \in H' = \mathcal{L}(H, K)$ there exists

$$\widetilde{f} \in E' = \mathcal{L}(E, \mathbb{K})$$

extending $f$ and satisfying,

$$|||\, \widetilde{f}\, |||_{E'} = |||\, f\, |||_{H'}$$

*Proof.* let $f \in H'$. By applying Theorem 2 for $N(x) = |||\, f\, ||| \cdot \|x\|$, let us verify that $f$ is dominated by $N$ $N$ on $H$ we have for all $x \in H$,

$$|f(x)| \leq |||\, f\, |||\, \|x\| = N(x)$$

we find that there exist $\widetilde{f} : E \longrightarrow \mathbb{K}$ linear and extending $f$ and satisfying for all $x \in E$,

$$\left|\widetilde{f}(x)\right| \leq N(x) = |||\, f\, |||_{H'} \cdot \|x\|$$

implying that $\widetilde{f}$ is continuous, thus $\widetilde{f} \in E'$ and that

$$|||\, \widetilde{f}\, ||| \leq |||\, f\, |||$$

On the other hand, we have,

$$\begin{aligned} |||\, \widetilde{f}\, |||_E = \sup_{x \in E \setminus \{0_E\}} \frac{\left|\widetilde{f}(x)\right|}{\|x\|} &\geq \sup_{x \in H \setminus \{0_E\}} \frac{\left|\widetilde{f}(x)\right|}{\|x\|} \\ &= \sup_{x \in H \setminus \{0_E\}} \frac{|f(x)|}{\|x\|} \\ &= |||\, f\, |||_{H'} \end{aligned}$$

Hence

$$|||\, \widetilde{f}\, |||_{E'} = |||\, f\, |||_{H'}$$

completing this proof. □

**Some Theorems**

**Theorem 6.1.4:**

Let $E$ be a nonzero N.V.S over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, Then

(1) for all $x \in E \setminus \{0_E\}$, there exist a continuous linear form $f$ on $E$ such that

$$f(x) = \|x\|_E \quad \text{and} \quad |||\, f\, ||| = 1$$

In particular

$$E' \neq \{0_{E'}\}$$

(2) Let $x, y \in E$ such that,

$$f(x) = f(y) \qquad (\forall f \in E') \implies x = y$$

*Proof.* (1) Consider $H := \langle x \rangle$, $H$ is a subspace of $E$, and

$$\begin{array}{rccc} h: & H & \longrightarrow & \mathbb{K} \\ & \lambda x & \longmapsto & \lambda\|x\| \end{array} \qquad (\forall \lambda \in \mathbb{K})$$

It's clear that $h$ is linear, $h(x) = \|x\|$ by taking $\lambda = 1$, $h$ is continuous because $(dim(H) = 1 < \infty)$, by Theorem 3, there exists $f : E \longrightarrow \mathbb{K}$, linear continuous and satisfies

$$\begin{aligned} |||\, f \,|||_{E'} = |||\, h \,|||_{H'} &:= \sup_{\lambda \in \mathbb{K}^*} \frac{|h(\lambda x)|}{|||\, \lambda x \,|||} \\ &= \frac{|h(x)|}{\|x\|} = 1 \end{aligned}$$

so $f$ extends $h$ and $x \in H$, we have

$$f(x) = h(x) = \|x\|$$

this completes the proof of $(1)$.

(2) Let us show the contrapositive, i.e.

$$\forall x, y \in E : \big(x \neq y \implies \exists f \in E' : f(x) \neq f(y)\big)$$

let $x, y \in E$ such that $x \neq y$ , and set $z := x - y \in E \setminus \{0_E\}$, by applying the result of $(1)$ for $z$, we find that there exist $f \in E'$ such that,

$$f(z) = \|z\| \neq 0$$

hence we have,

$$f(x - y) = f(x) - f(y)$$

thus there exist $f \in E'$ such that

$$f(x) \neq f(y)$$

as required. Hence this completes the proof.

□

**Remark :**

The property of item 2 of Theorem 1 is expressed literally by saying that,

*"The continuous linear forms on E separate the vectors of E "*

*Remark by the Writter : Sometimes when i write $E\backslash H$ i mean quotient space not minus, understand from context.*

**Theorem 6.1.5: Theorem 2**

Let $E$ be a N.V.S over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $H$ be a subspace of $E$, and $x \in E\backslash\overline{H}$ then there exists a continuous, linear form $f$ on $E$ such that

$$||| f ||| \leq 1$$

and

$$f(x) = d(x, H) \neq 0 \quad f(H) = \{0\}$$

*Proof.* We apply Item 1 of Theorem 1 for the N.V.S Quotient space $E_{\backslash\overline{H}}$ and the non zero vector $cl(x) = x + \overline{H}$ , where

$$\left(cl(x) \neq 0_{E_{\backslash\overline{H}}} \neq 0_{E_{\backslash\overline{H}}} \text{ since } x \notin \overline{H}\right)$$

let, $\pi : E \longrightarrow E_{\backslash\overline{H}}$ be the quotient map. It's known that $\pi$ is continuous and that $||| \pi |||= 1$ , By Item 1 from Theorem 1, there exists a continuous linear form $\overline{f}$ on $E_{\backslash\overline{H}}$ if

$$\overline{f}(\pi(x)) = \|\pi(x)\|_{E_{\backslash\overline{H}}}$$

and

$$||| \overline{f} |||= 1$$

consider, $f : E \xrightarrow{\pi} E_{\backslash H} \xrightarrow{\overline{f}} \mathbb{K}$, i.e.

$$f = \overline{f} \circ \pi$$

$f$ is linear and continuous because its a composition of two linear and continuous maps. Then $f \in E'$, Next, we have

$$||| f |||=||| \overline{f} \circ \pi |||\leq \underbrace{||| \overline{f} |||}_{=1} \cdot \underbrace{||| \pi |||}_{=1}$$

thus,

$$||| f |||\leq 1$$

Next, we have,

$$\begin{aligned}
f(x) = \left(\overline{f} \circ \pi\right)(x) = \overline{f}(\pi(x)) &= \|\pi(x)\|_{E_{\setminus \overline{H}}} \\
&= \inf_{y \in \pi(x)} \|y\|_E \\
&= \inf_{h \in \overline{H}} \|x + h\|_E \\
&= \inf_{h \in \overline{H}} \|x - h\|_E \\
&= \inf_{h \in \overline{H}} d(x, h) \\
&= d(x, \overline{H}) = d(x, H)
\end{aligned}$$

Finally, we have,

$$\begin{aligned}
f(H) = \left(\overline{f} \circ \pi\right)(H) = \overline{f}&\left(\underbrace{\pi(H)}_{=\left\{0_{E_{\setminus \overline{H}}} \text{ since } H \subset \overline{H}\right\}}\right) \\
= \overline{f}(\left\{0_{E_{\setminus \overline{H}}}\right\}) = \{0\}
\end{aligned}$$

This completes the proof. $\square$

**Theorem 6.1.6: Theorem 3**

Let $E$ be a N.V.S over a filed $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $H$ be a subspace of $E$, Then the two following properties are equivallent,

(i) $H$ is dense in $E$

(ii) for all $f \in E'$, we have,

$$f_{|H} \text{ is zero} \implies f \text{ is zero}$$

*Proof.* Let's start proving!

$$(i) \implies (ii)$$

Already known !.

Suppose that $H$ is dense in $E$ (i.e. $\overline{H} = E$ ) and let $f \in E'$ such that $f_{|H} = 0$, that is $f(h) = 0$ for all $h \in H$.

Then, giving $x \in E$ since $H$ is dense in $E$, then there exist a sequence $(h_n)_{n \in \mathbb{N}}$ in $H$ converging to $x$, thus we have,

$$\begin{aligned} f(x) &= f(\lim_{n\to\infty} h_n) \\ &= \lim_{n\to\infty} f(h_n) \\ &= \lim_{n\to\infty} 0 = 0 \quad (\text{ since } h_n \in H \text{ and } f_{|H} \text{ is zero }) \end{aligned}$$

Thus $f = 0_E$, as required.

$$(ii) \implies (i)$$

let us show the contrapositive

$$\overline{(i)} \implies \overline{(ii)}$$

suppose that $\overline{(i)}$ i.e. $\overline{H} \neq E$ , thus there exists $x \in E\backslash\overline{H}$. By Theorem 2, there exists $f \in E'$ such that $f(H) = \{0\}$, and $f(x) = d(x, H) \neq 0$, in other words $d(x, H) \neq 0$ since $x \notin \overline{H}$ so $f \in E'$, and $f_{|H} = 0_{H'}$ and $f \neq 0_{E'}$ since $f(x) \neq 0$.

This completes the proof. □

**Theorem 6.1.7: Theorem 4**

Let $E$ be a N.V.S, $n$ be a positive integer, $x_1, \dots, x_n$ be $n$ vector linearly independent of $E$, and $c_1, \dots, c_n$ be $n$ scalars then there exists a continuous linear form on $f$ on $E$ such that

$$f(x_i) = c_i \qquad \forall i \in \{1, \dots, n\}$$

**Theorem 6.1.8:**

Let $E$ be a N.V.S, $n$ be a positive integer, $x_1, \dots, x_n$ be $n$ linearly independent vectors of $E$, and $c_1, \dots, c_n$ be $n$ scalars. Then there exist a continuous linear form $f$ on $E$ such that $f(x_i) = c_i$ for all $i \in \{1, \dots, n\}$.

*Proof.* Let

$$H := \langle x_1, \dots, x_n \rangle$$

and $h : H \longrightarrow \mathbb{K}$ be the linear form on $H$ defined by

$$h\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i c_i \qquad (\forall \lambda_i \in \mathbb{K} \forall i = 1, \dots, n)$$

so for all $i \in \{1, \dots, n\}$, we have $h(x_i) = c_i$, since $dim(H) = n < \infty$, then $h$ is continuous, so by the Hahn-Banach theorem, there exist $f \in E'$ extending $h$, so for all $i \in \{1, \dots, n\}$, we have that

$$f(x_i) = h(x_i) = c_i$$

hence the proof is complete. $\square$

## 6.2 The Geometric form of the Hahn-Banach Theorem

The geometric form of the Hahn-Banach Theorem deals with the separation of disjoint convex sets using affine hyperplanes.

**Reminders :**

Let $E$ be a N.V.S over $\mathbb{K}$ or $\mathbb{C}$. An affine hyperplane of $E$ is a subset $H$ of $E$, of the form,

$$H := \{x \in E : f(x) = \alpha\}$$

for some $f \in E^* \setminus \{0_{E^*}\}$ and $\alpha \in \mathbb{K}$, Its known that $H$ is closed if and only if $f$ is continuous.

**Theorem 6.2.1:**

Let $E$ be a N.V.S over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $C$ be an open and convex subset of $E$, containing $0_E$, for all $x \in E$, define,

$$p(x) := \inf \left\{\alpha > 0, \alpha^{-1}x \in C\right\}$$

then,

(i) $p$ is sublinear i.e.

$$\begin{cases} \text{Sub additive } \rightarrow p(x+y) \leq p(x) + p(y) \quad \forall x, y \in E \\ \text{Positively homogenous } \rightarrow p(\lambda x) = \lambda(x) \quad \forall \lambda \geq 0 \end{cases}$$

(ii) $\exists M > 0$ such that for all $x \in E$, we have,

$$p(x) \leq M\|x\|$$

(iii) and we have,

$$C = \{x \in E : p(x) < 1\}$$

we have that $p$ is called the Minkowski functional of $C$.

*Proof.* Let us first prove item $(ii)$, Since $C$ is open and contains $0_E$, then there exist $r > 0$, such that $B(0_E, r)$, Now for all $x \in E \setminus \{0_E\}$, we have

$$\frac{r}{2}\frac{x}{\|x\|} \in B(0_E, r) \subset C$$

implying that the positive real number, $\alpha = \frac{2}{r}\|x\|$ satisfies

$$\alpha^{-1}x \in C$$

thus, by definition of $p$,

$$p(x) \leq \frac{2}{r}\|x\|$$

This proves then the positive constant $M = \frac{2}{r}$.

Now let us prove then $(iii)$

$$C \subset \{x \in E : p(x) < 1\}$$

let $x \in C$, for $x = 0_E$, then we have clearly that

$$p(x) = p(0_E) = 0 < 1$$

suppose that $x \neq 0_E$ and let us show that $p(x) < 1$, since $C$ is open and $x \in C$, then $\exists \varepsilon > 0$ such that

$$B_E(x, \varepsilon) \subset C$$

so from,

$$(1 + \frac{\varepsilon}{2\|x\|})x \in B_E(x, \varepsilon) \subset C$$

we desire that $\alpha_0 = \left(1 + \frac{\varepsilon}{2\|x\|}\right)^{-1}$, satisfies that $\alpha_0^{-1}x \in C$, thus,

$$p(x) \leq \alpha_0 < 1$$

hence $p(x) < 1$ as required.

$$\{x \in E : p(x) < 1\} \subset C$$

let $x \in E$ such that $p(x) < 1$ and let us prove that $x \in C$. So by definition of $p(x)$ there exist $t \in (0, 1)$ such that $t^{-1}x \in C$ now since $C$ is convex and $0_E, t^{-1}x \in C$, then we have

$$t\left(t^{-1}x\right) + (1 - t)\, 0_E \in C$$

in other words,

$$x \in C$$

as required, Hence we have the equality,

$$C = \{x \in E : p(x) < 1\}$$

Finally let us prove $(i)$.

*Is p positively homogenous ?*

for all $\lambda > 0$, and $x \in E$, we have,

$$\begin{aligned} p(\lambda x) &:= \inf \left\{ \alpha > 0 : \alpha^{-1} \lambda x \in C \right\} \\ &= \lambda \left\{ \lambda^{-1} \alpha : \left( (\lambda^{-1} \alpha)^{-1} x \in C \right) \right. \\ &= \lambda \left\{ \beta > 0, \beta^{-1} x \in C \right\} \end{aligned}$$

*dotted*

thus,

$$\begin{aligned} p(\lambda x) &:= \inf \lambda \left\{ \beta > 0, \beta^{-1} x \in C \right\} \\ &= \lambda \underbrace{\inf \left\{ \beta > 0, \beta^{-1} \, x \in C \right\}}_{p(x)} \\ &= \lambda p(x) \end{aligned}$$

*Is p sub additive ?*

Let $x, y \in E$ be aribtrary, and show that,

$$p(x+y) \leq p(x) + p(y)$$

For $\varepsilon > 0$, we have from the posirive homogenity of $p$ that,

$$p\left( \frac{1}{p(x) + \varepsilon} x \right) = \frac{1}{p(x) + \varepsilon} p(x) < 1$$

implying then $(iii)$ already proved that,

$$\frac{1}{p(x) + \varepsilon} x \in C$$

similarly

$$\frac{1}{p(y) + \varepsilon} y \in C$$

so setting,

$$t = \frac{p(x) + \varepsilon}{p(x) + p(y) + 2\varepsilon} \in (0, 1)$$

we have from the convexity of $C$,

$$t\left( \frac{1}{p(x) + \varepsilon} \right) x + (1 - t) \left( \frac{1}{p(y) + \varepsilon} \right) y \in C$$

hence,

$$\frac{1}{p(x) + p(y) + 2\varepsilon} x + \frac{1}{p(x) + p(y) + 2\varepsilon} y \in C$$

twe get then,‘

$$\frac{1}{p(x)+p(y)+2\varepsilon}(x+y) \in C$$

hence

$$p\left(\frac{1}{p(x)+p(y)+2\varepsilon}(x+y)\right) < 1$$

by the positive homogenity of $p$, it follows that,

$$\frac{1}{p(x)+p(y)+2\varepsilon}p(x+y) < 1$$

i.e.

$$p(x+y) < p(x)+p(y)+2\varepsilon$$

by taking $\varepsilon \to 0^+$ it gives us, the inequality,

$$p(x+y) \leq p(x)+p(y)$$

as required. This completes the proof. □

**The geometric versions of the Hahn-Banach Theorem;**

**Theorem 6.2.2: The first geometric version of the Hahn-Banach Theorem**

Let $E$ be an $\mathbb{R}$ N.V.S, $A$ and $B$ be two *nonempty disjoint convex* subsets of $E$, Suppose that $A$ is open then. There exists affine hyperplane of $E$ which separates $A$ and $B$, that is there exists a non-zero continuous linear form $f$ on $E$ and a real number $\alpha$ such that,

$$f(x) \leq \alpha \leq f(y) \qquad (\forall x \in A, \forall y \in B)$$

**Theorem 6.2.3: The second geometric version of the Hahn-Banach Theorem**

let $E$ be on $\mathbb{R}$-N.V.S and $A$ and $B$ be two nonempty disjoint convex subsets of $E$, suppose that $A$ is closed and $B$ is compact, then there exists closed affine hyperplane of $E$ which separates strictly $A$ and $B$, that is, there exists a nonzero continuous linear form $f$ on $E$ and a real number $\alpha$ such that

$$f(x) < \alpha < f(y) \qquad (\forall x \in A, \forall y \in B)$$

To prove these theorems, we need the propositions

**corollary 6.2.1:**

Let $E$ be an $\mathbb{R}$-N.V.S, $C$ be a non empty open convex subset of $E$ and $x_0 \in E \backslash C$, then there exists a non zero continuous linear form $f$ on $E$ such that,

$$f(x) < f(x_0) \qquad (\forall x \in C)$$

In other words, the closed affine hyper plane of $E$ of equation

$$f(x) = f(x_0)$$

separates $\{x_0\}$ and $C$

*Proof.* By translating if necessary $C$ and $x$ by $a$ some vector of $(-C)$, suppose that $0_E \in C$ , and let $p$ denote the Minkowski functional of $C$, intrdouce

$$H := \langle x_0 \rangle$$

and $h : H \longrightarrow \mathbb{R}$ and $h(\lambda x_0) = \lambda$ for all $\lambda \in \mathbb{R}$, clearly $h$ is a linear form on $H$, Next since

$$C = \{x \in E, p(x) < 1\}$$

By item $(3)$ of the previous proposition, and $x_0 \notin C$ then $p(x_0) \geq 1$, then

$$h(x_0) = 1 \leq p(x_0)$$

it follows by distinguishing the cases $\lambda > 0$ and $\lambda \geq 0$ that,
if $\lambda > 0$, then we have,

$$h(\lambda x_0) = \lambda h(x_0) = \lambda$$
$$p(\lambda x_0) = \lambda p(x_0) \geq \lambda$$

so $h(\lambda x_0) \leq p(\lambda x_0)$.
if $\lambda \leq 0$, then we have

$$h(\lambda x_0) = \lambda h(x_0) = \lambda \leq 0$$

and

$$p(\lambda x_0) \geq 0$$

then

$$h(\lambda x_0) \leq p(\lambda x_0)$$

so for all $\lambda \in \mathbb{R}$, we have

$$h(\lambda x_0) \leq p(\lambda x_0)$$

i.e.,

$$\forall x \in H, h(x) \leq p(x)$$

(according to the Hahn Banach Theorem) there exists a lienar form $f$ on $E$, extending $h$ such that,

$$f(x) \leq p(x) \qquad (\forall x \in E)$$

Let us show that $f$ is continuous, by item $(ii)$ of the previous propostiions, there exists $M > 0$ constatnt such that

$$p(x) \leq M\|x\|$$

for all $x \in E$, thus

$$f(x) \leq p(x) \leq M\|x\| \qquad (\forall x \in E)$$

therefore

$$f(x) \leq M\|x\| \quad (\forall x \in E)$$

so by taking $(-x)$ instead of $x$, we get

$$f(-x) \leq M\| - x\| \, (\forall x \in E)$$

therefore

$$f(x) \geq -M\|x\|$$

thus,

$$-M\|x\| \leq f(x) \leq M\|x\| \qquad (\forall x \in E)$$

that is,

$$|f(x)| \leq M\|x\| \qquad (\forall x \in E)$$

Implying that $f$ is continuous.

1. Since $f$ extends $h$ and $x_0 \in H$, then,

$$f(x_0) = h(x_0) = 1 \neq 0$$

   thus $f$ is non-zero,

2. for all $x \in C$, we have $p(x) < 1$, thus

$$f(x) \leq p(x) < 1 = f(x_0)$$

   thus

$$\forall x \in C : f(x) < f(x_0)$$

   This completes the proof.

□