

# Normed Vector Spaces Lecture Notes

*Hand written summary from lectures*

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`http://farhi.bakir.free.fr/home/index-fr.html`

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## Disclaimer

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if you spot an error feel free to open an issue or submit a pull request, or contact me via gmail :

`kara.abderahmane@nhsm.edu.dz`

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*Main Writer*

CONTRIBUTORS

KARA ABDERAHMANE



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# 1 THE CONCEPT OF A NORM ON A REAL OR COMPLEX VECTOR SPACE

For all what follows  $\mathbb{K}$  denotes one of the two fields  $\mathbb{R}$  or  $\mathbb{C}$  and  $|\cdot|$  denotes the absolute value if  $\mathbb{K} = \mathbb{R}$  and the modulus if  $\mathbb{K} = \mathbb{C}$ .

## 1.1 Norm on a $\mathbb{K}$ -vector space

### Definition 1.1.1: Norm

Let  $E$  be a  $\mathbb{K}$ -vector space, we call a norm on  $E$  every map  $\|\cdot\| : E \rightarrow [0, \infty)$  satisfying the following properties:

- (i)  $\forall x \in E : \|x\| = 0 \implies x = 0_E$
- (ii)  $\forall x \in E, \forall \lambda \in \mathbb{K} : \|\lambda x\| = |\lambda| \|x\|$
- (iii)  $\forall x, y \in E : \|x + y\| \leq \|x\| + \|y\|$

### Remark

- A  $\mathbb{K}$ -vector space  $E$  equipped with a norm  $\|\cdot\|$  is called a **normed vector space** (abbreviated to N.V.S), it is written  $(E, \|\cdot\|)$  or simply  $E$  if there is no ambiguity about the norm  $\|\cdot\|$
- The equivalence " $\iff$ " in (i) can be replaced by the implication " $\implies$ " because the implication  $(x = 0_E \implies \|x\| = 0)$  can be obtained from property (ii) by taking  $\lambda = 0$
- Inequality in (iii) is called "**The Triangle Inequality**" or "**The Convex Inequality**", it is equivalent to say that the norm  $\|\cdot\|$  is a convex function on  $E$ , that is:

$$\forall t \in (0, 1), \forall x, y \in E : \|tx + (1 - t)y\| \leq t\|x\| + (1 - t)\|y\|$$

Indeed, we have:

$$\begin{aligned}\|tx + (1-t)y\| &\leq \|tx\| + \|(1-t)y\| \\ &\leq |t| \|x\| + |1-t| \|y\| && \leq t\|x\| + (1-t)\|y\|\end{aligned}$$

$t = \frac{1}{2}$ : we get it

- if  $E$  is a  $\mathbb{K}$ -vector space and  $\|\cdot\| : E \rightarrow [0, \infty)$  satisfies just properties (i) and (ii) then  $\|\cdot\|$  is called a **seminorm** on  $E$  (so seminorm could assign 0 to non-zero vectors), the pair  $(E, \|\cdot\|)$  is then called a **Seminormed Vector Space**.

## 1.2 Metric Associated to a Norm

### Definition 1.2.1:

Let  $(E, \|\cdot\|)$  be a N.V.S, Define:

$$\begin{aligned}d : E^2 &\longrightarrow [0, \infty) \\ (x, y) &\longmapsto d(x, y) = \|x - y\|\end{aligned}$$

we can easily verify that  $d$  is a metric on  $E$ , and it is called **The Metric Associated To The Norm  $\|\cdot\|$  or The Generated Metric By The Norm**

### Remark

- Thanks to the concept of the metric generated by a norm, a N.V.S is seen as a particular metric space, which is a particular topological space.
- The definition of the open ball, a closed ball, a sphere, an open set, a closed set, a neighborhood, the interior of a set, limit, the closure of a set, etc... in a N.V.S are those related to the metric generated by the norm.
- Every metric  $d$  generated by a norm (in a given N.V.S  $E$ ) is invariant by translation, that is:

$$\forall x, y, z \in E : d(x + z, y + z) = d(x, y)$$

- There exist natural metrics that are not generated by any norm (like discrete distance).

## 1.3 Examples of some concepts on a N.V.S derived from its metric structure

1. Let  $(E, \|\cdot\|)$  be a N.V.S,  $(x_n)_{n \in \mathbb{N}}$  be a sequence of  $E$ , and  $x$  be a vector of  $E$ .

- We say that  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$  if we have  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ , equivalently:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n \in \mathbb{N} \quad (n > N \implies \|x_n - x\| < \varepsilon)$$

in this case we write  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  on  $n \rightarrow \infty$

- We say that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence if we have  $\lim_{p, q \rightarrow \infty} \|x_p - x_q\| = 0$ , equivalently:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall p, q \in \mathbb{N} \quad (p > q > N \implies \|x_p - x_q\| < \varepsilon)$$

2. Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be two N.V.S over the same field  $\mathbb{K}$ ,  $f : E \rightarrow F$  be a map from  $E$  to  $F$ , Let  $x_0 \in E$  and  $y_0 \in F$ ,

- We say that  $f(x)$  tends to  $y_0$  when  $x$  tends to  $x_0$  (and we write  $\lim_{x \rightarrow x_0} f(x) = y_0$  or  $f(x) \rightarrow y_0$  as  $x \rightarrow x_0$ )

$$\begin{cases} \forall \varepsilon > 0, \exists \eta > 0, \quad \text{s.t.} \quad \forall x \in E : \\ \|x - x_0\|_E < \eta \implies \|f(x) - y_0\|_F < \varepsilon \end{cases}$$

- We say that  $f$  is continuous at  $x_0$  if we have:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

that is,

$$\begin{cases} \forall \varepsilon > 0, \exists \eta > 0, \quad \text{s.t.} \quad \forall x \in E : \\ \|x - x_0\|_E < \eta \implies \|f(x) - f(x_0)\|_F < \varepsilon \end{cases}$$

- We say that  $f$  is continuous on  $E$  if it is continuous at all vector  $x$  of  $E$ .
- We say that  $f$  is uniformly continuous on  $E$  if we have  $\forall \varepsilon > 0, \exists \eta > 0$  such that  $\forall x, y \in E$ :

$$\|x - y\|_E < \eta \implies \|f(x) - f(y)\|_F < \varepsilon$$

- Let  $M > 0$ , we say that  $f$  is  $M$ -Lipchitz if we have:

$$\forall x, y \in E : \quad \|f(x) - f(y)\|_F \leq M \|x - y\|_E$$

- We say that  $f$  is a contraction if it is  $M$ -Lipchitz for some constant  $M \in (0, 1)$ , Note that/

$$\text{Lipchitz Continuous} \implies \text{Uniformly Continuous} \implies \text{Continuous}$$

## 1.4 Equivalent and Topologically Equivalent Norms

### Definition 1.4.1:

Let  $E$  be a  $\mathbb{K}$ -vector space and  $N_1$  and  $N_2$  two norms on  $E$ :

- We say that  $N_1$  and  $N_2$  are topologically equivalent if their associated metrics are topologically equivalent, that is they induce the same topology on  $E$ .
- We say that  $N_1$  and  $N_2$  are equivalent if their associated metrics are equivalent, that is there exist  $\alpha, \beta > 0$  such that:

$$\alpha N_1 \leq N_2 \leq \beta N_1 \quad (\text{i.e. } \forall x \in E : \alpha N_1(x) \leq N_2(x) \leq \beta N_1(x))$$

### Remark

- It is known that two equivalent metrics (on a given non-empty set) are topologically equivalent but the inverse is generally false.
- Note that in a  $\mathbb{K}$ -vector space, the two concepts "equivalent norms" and "topologically equivalent norms" coincide
- We will show later that two norms on a  $\mathbb{K}$ -vector space are topologically equivalent if and only if they are equivalent.
- We will show also that: Any two norms on a finite-dimensional vector space (over  $\mathbb{K}$ ) are equivalent

## 1.5 Examples of norms on $\mathbb{R}^n$ and $\mathbb{C}^n$

### Example

1. In  $\mathbb{R}$  (Considered as  $\mathbb{R}$  vector space), the usual norm is the absolute value, in  $\mathbb{C}$  (Considered as  $\mathbb{C}$  vector space), the usual norm is the modulus.
2. Let  $n \geq 2$  be an integer, we may define on  $\mathbb{K}^n$  (Considered as  $\mathbb{K}$  vector space), several norms including  $\{\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p\}$ , with  $(p \geq 1)$ , and  $\|\cdot\|_\infty$ , the norms we just stated are the



most widely used, they are defined by :

$$\|x\|_1 := \sum_{i=1}^n |x_i|$$

$$\|x\|_2 := \sqrt{\sum_{i=1}^n |x_i|^2} = \left( \sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}$$

$$\|x\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$$

Both in  $\mathbb{R}^n$  and in  $\mathbb{C}^n$  ( $n \in \mathbb{N}$ ), the norm  $\|\cdot\|_2$  is called the euclidean norm, and the norm  $\|\cdot\|_p$  ( $p \geq 1$ ) is called the Holder norm of exponent  $p$  (or simply, the  $p$ -norm).

Remark that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are special cases of  $\|\cdot\|_p$ . We can also show that :

$$\lim_{n \rightarrow \infty} \|\cdot\|_p = \|\cdot\|_\infty$$

Further, it's easy to show that the norms

$$\|\cdot\|_p \quad \forall p \geq 1 \text{ are equivalent}$$

Prove that  $\|\cdot\|_1 \sim \|\cdot\|_\infty$  and  $\|\cdot\|_2 \sim \|\cdot\|_1$ . (Hint :  $n ((\max |x_i|)^2)^{1/2}$ )

Furthermore, it's easy to show that the norms  $\|\cdot\|_p$  ( $p \geq 1$ ), are all equivalent (they are even equal for  $n = 1$ ). To show that  $\|\cdot\|_p$  ( $p \geq 1$  arbitrary), is really a norm on  $\mathbb{K}^n$ , only the triangle inequality that poses a problem, (The special cases  $p = 1$ , and  $p = \infty$  are easy), we fix this problem by solving the following exercise !

**Consider the following exercise :**

Let  $n$  be a positive integer and let  $p, q > 1$ , such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

- (i) By using the connexity of the exponential function, show that for all positive real numbers  $a$  and  $b$ , we have

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

(Known as The Young Inequality)

- (ii) Deduce that for all positive real numbers  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ , we have :

$$\sum_{i=1}^n x_i y_i \leq \left( \sum_{i=1}^n x_i^p \right)^{1/p} + \left( \sum_{i=1}^n y_i^q \right)^{1/p}$$

(Known as the Holder Inequality)

(iii) Deduce that for all positive real numbers  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ , we have :

$$\left( \sum_{i=1}^n (x_i + y_i)^p \right)^{1/p} \leq \left( \sum_{i=1}^n x_i^p \right)^{1/p} + \left( \sum_{i=1}^n y_i^p \right)^{1/p}$$

(Called the Minkowski Inequality)

(iv) Conclude that  $\|\cdot\|$  is really a norm on  $\mathbb{K}^n$  where ( $K = \mathbb{R}$  or  $\mathbb{C}$ )

### Solution :

(i) Since the function  $u \rightarrow e^u$  is convex on  $\mathbb{R}$  because  $((e^u)'' = e^u > 0)$ , then we have for all  $t \in [0, 1]$  and for all  $x, y \in \mathbb{R}$  :

$$e^{tx+(1-t)y} \leq te^x + (1-t)e^y$$

We apply the above for  $t = \frac{1}{p}$  so  $(1-t) = 1 - \frac{1}{p} = \frac{1}{q}$ , and for  $x, y$  such that  $e^x = a^p$  (i.e.  $x = p \ln(a)$ ), and  $e^y = b^q$  (i.e.  $y = q \ln(b)$ ) we obtain that :

$$\begin{aligned} (a^p)^{1/p} (b^q)^{1/q} &\leq \frac{a^p}{p} + \frac{b^q}{q} \\ a \cdot b &\leq \frac{a^p}{p} + \frac{b^q}{q} \end{aligned}$$

as required.

(ii) Let  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0$ , for  $i \in \{1, 2, \dots, n\}$ , by applying the Young inequality proved above for  $a = \frac{x_i}{(\sum_{j=1}^n x_j^p)^{1/p}}$  and  $b = \frac{y_i}{(\sum_{j=1}^n y_j^q)^{1/q}}$  we get :

$$\frac{x_i y_i}{\left( \sum_{j=1}^n x_j^p \right)^{1/p} \left( \sum_{j=1}^n y_j^q \right)^{1/q}} \leq \frac{1}{p} \left[ \frac{x_i^p}{\sum_{j=1}^n x_j^p} \right] + \frac{1}{q} \left[ \frac{y_i^q}{\sum_{j=1}^n y_j^q} \right]$$

Next, by taking the summation from  $i = 1$  to  $n$ , in the two sides of his last inequality , we get :

$$\begin{aligned} \frac{\sum_{i=1}^n x_i y_i}{\left( \sum_{j=1}^n x_j^p \right)^{1/p} \left( \sum_{j=1}^n y_j^q \right)^{1/q}} &\leq \frac{1}{p} + \frac{1}{q} = 1 \\ \sum_{i=1}^n x_i y_i &\leq \left( \sum_{i=1}^n x_i^p \right)^{1/p} \left( \sum_{j=1}^n y_j^q \right)^{1/q} \end{aligned}$$

As required

Remark that the Holder inequality generalizes, the Cauchy-Schawrtz Inequality for the usual inner product of  $\mathbb{R}^n$  (take  $p = q = 2$ ).

(iii) Let  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0$ , we have :

$$\begin{aligned} \sum_{i=1}^n (x_i + y_i)^p &= \sum_{i=1}^n (x_i + y_i) (x_i + y_i)^{p-1} \\ &= \sum_{i=1}^n x_i (x_i + y_i)^{p-1} + \sum_{i=1}^n y_i (x_i + y_i)^{p-1} \end{aligned}$$

Then by applying the Holder inequality, for each of the two sums  $\sum_{i=1}^n x_i (x_i + y_i)^{p-1}$  and  $\sum_{i=1}^n y_i (x_i + y_i)^{p-1}$  we derive that :

$$\sum_{i=1}^n (x_i + y_i)^p \leq \left( \sum_{i=1}^n x_i^p \right)^{1/p} \left( \sum_{i=1}^n (x_i + y_i)^{(p-1)q} \right)^{1/q} + \left( \sum_{i=1}^n y_i^p \right)^{1/p} \left( \sum_{i=1}^n (x_i + y_i)^{(p-1)q} \right)^{1/q}$$

And since  $(p-1)q = p$  (Because  $\frac{1}{p} + \frac{1}{q} = 1$ ), it follows that :

$$\begin{aligned} \sum_{i=1}^n (x_i + y_i)^p &\leq \left( \sum_{i=1}^n (x_i + y_i)^p \right)^{1/q} \left( \left( \sum_{i=1}^n x_i^p \right)^{1/p} + \left( \sum_{i=1}^n y_i^p \right)^{1/p} \right) \\ \left( \sum_{i=1}^n (x_i + y_i)^p \right)^{1-\frac{1}{q}} &\leq \left( \sum_{i=1}^n x_i^p \right)^{1/p} + \left( \sum_{i=1}^n y_i^p \right)^{1/p} \\ \left( \sum_{i=1}^n (x_i + y_i)^p \right)^{1/p} &\leq \left( \sum_{i=1}^n x_i^p \right)^{1/p} + \left( \sum_{i=1}^n y_i^p \right)^{1/p} \end{aligned}$$

(iv) The two first properties of a norm ( i.e. , (i) and (ii) ), are clearly satisfied by  $\|\cdot\|_p$  , so it remains to shows the triangle inequality ( $\|x + y\|_p \leq \|x\|_p + \|y\|_p \quad \forall x, y \in \mathbb{K}^n$ ). First, remark that the Minkowski Inequality (proved above), remains true for  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0$  (That is if some if the  $x_i$ 's and  $y_i$ 's are zero), This can be justified by the continuity for example now, for

$$X := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad Y := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{K}^n$$

We have that :

$$\|x + y\|_p = \left( \sum_{i=1}^n \|x_i + y_i\|^p \right)^{1/p} \leq \left( \sum_{i=1}^n \underbrace{(|x_i| + |y_i|)^p}_{\in [0, \infty)} \right)^{1/p} \leq \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} + \left( \sum_{i=1}^n |y_i|^p \right)^{1/p}$$

According to the Minkowsky Inequality we get it equal

$$\left( \sum_{i=1}^n |x_i|^p \right)^{1/p} + \left( \sum_{i=1}^n |y_i|^p \right)^{1/p} = \|x\|_p + \|y\|_p$$

As required, Consequently,  $\|\cdot\|_p$  is a norm on  $\mathbb{C}^n$

## 1.6 Finite product of normed vector spaces

Let  $(E_1, N_1), (E_2, N_2), \dots, (E_k, N_k)$  ( $k \geq 1$ ), be normed vector spaces over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and set  $E := E_1 \times E_2 \times \dots \times E_k$ .

We may define on  $E$  several norms which are expressed in terms of  $N_1, N_2, \dots, N_k$ . Among these norms we set :

$$\bullet \|\cdot\|_1 : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_1 := \sum_{i=1}^k N_i(x_i)$$

$$\bullet \|\cdot\|_2 : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_2 := \left( \sum_{i=1}^k N_i(x_i)^2 \right)^{1/2}$$

$$\bullet \|\cdot\|_p : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_p := \left( \sum_{i=1}^k N_i(x_i)^p \right)^{1/p}$$

$$\bullet \|\cdot\|_\infty : \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in E : \quad \|x\|_\infty := \max_{1 \leq i \leq k} N_i(x_i)$$

We can show that all the norms  $\|\cdot\|_p$  ( $1 \leq p \leq \infty$ ) are equivalent, and that the common topology generated by them is the product topology of  $E$ . This allows us to affirm that, A topological product of a finite number of N.V.S is a N.V.S.

Note that this last result is in general false for a topological product of an infinite number of normed vector spaces.

## 1.7 Exampels of norms of an infinite-dimensional vector space

Let  $a, b \in \mathbb{R}$  with  $a < b$ , The  $\mathbb{R}$ -vector space

$$E := \mathcal{C}^0([a, b], \mathbb{R}) \quad \text{Constituted of continiuous functions on } [a, b]$$

Can be equipped with several importants norms, including  $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p$  ( $p \geq 1$ ) and  $\|\cdot\|_\infty$

## 1.8 Examples of norms of an infinite dimensional vector spaces

let  $a, b \in \mathbb{R}$  with  $a < b$ . The  $\mathbb{R}$ -vector space  $E := \mathcal{C}^0([a, b], \mathbb{R})$ , (Constituted of continious real functions on  $[a, b]$ ). can be equipped with several important norms, including  $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p$  ( $p \geq 1$ ), and  $\|\cdot\|_\infty$  defined by

$$\begin{aligned} \|f\|_1 &= \int_a^b |f(t)| dt \\ \|f\|_2 &= \sqrt{\int_a^b |f(t)|^2 dt} \\ \|f\|_p &= \left( \int_a^b |f(t)|^p dt \right)^{1/p} \\ \|f\|_\infty &= \sup_{t \in [a, b]} |f(t)| = \max_{t \in [a, b]} |f(t)| \end{aligned}$$

The norm  $\|\cdot\|_2$  is called the euclidean norm, the norm  $\|\cdot\|_p$  with ( $p \geq 1$ ) is called the Holder norm of exponent  $p$  (or simply the  $p$ -norm), and the norm  $\|\cdot\|_\infty$  is called the uniform norm say that a sequence of functions  $(f_n)_{n \in \mathbb{N}}$ , belonging to  $\mathcal{C}^0([a, b], \mathbb{R})$ , converges to  $f \in \mathcal{C}^0([a, b], \mathbb{R})$  in the sense of the norm  $\|\cdot\|_\infty$  is equivalent to say that  $(f_n)_{n \in \mathbb{N}}$  converges uniformaly to  $f$  on  $[a, b]$ , we can show that we have  $\lim_{p \rightarrow \infty} \|\cdot\|_p = \|\cdot\|_\infty$  Further, it's important to note that these norms are not equivalent.

**Exercise :**

Show that  $\|\cdot\|_p$  ( $p \geq 1$ ), is really a norm on  $E := C^0([a, b], \mathbb{R})$ .

*Hint : Take inspiration from the solution of the previous exercise.*

## 1.9 Banach Spaces :

### Definition 1.9.1:

A banach space is a normed  $\mathbb{K}$ -vector space which is complete for the metric induced by it's norm.

### Example

In finite dimensional, let  $n \in \mathbb{N}$  :

$$\mathbb{R} - \text{NVS} \quad (\mathbb{R}, \|\cdot\|) \quad (\mathbb{R}^n, \|\cdot\|_1) \quad (\mathbb{R}^n, \|\cdot\|_2) \quad (\mathbb{R}^n, \|\cdot\|_\infty)$$

they are all banach spaces, the same is for the :

$$\mathbb{C} - \text{NVS} \quad (\mathbb{C}, \|\cdot\|) \quad (\mathbb{C}^n, \|\cdot\|_1) \quad (\mathbb{C}^n, \|\cdot\|_2) \quad (\mathbb{C}^n, \|\cdot\|_\infty)$$

Later, we will show a more general result stating that :

*Any finite-dimensional normed vector space is Banach*

### Theorem 1.9.1:

The  $\mathbb{R}$ -vector space  $E := C^0([0, 1], \mathbb{R})$ , equipped with it's uniform norm  $\|\cdot\|_\infty$ , is Banach.

*Proof.* We have to show that  $(E, \|\cdot\|_\infty)$  is complete, that is every cauchy sequence of  $(E, \|\cdot\|_\infty)$  converges in  $(E, \|\cdot\|_\infty)$ , so let  $(f_n)_{n \in \mathbb{N}}$  be a cauchy sequence of  $(E, \|\cdot\|_\infty)$  and let us show that it converges in  $(E, \|\cdot\|_\infty)$ , By hypothesis, we have :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q > N \implies \|f_p - f_q\|_\infty < \varepsilon$$

that is (according to the definition of  $\|\cdot\|_\infty$ ) :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q > N \implies \sup_{x \in [0, 1]} |f_p(x) - f_q(x)| < \varepsilon$$

or equivalently

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q > N \implies \forall x \in [0, 1] : \quad |f_p(x) - f_q(x)| < \varepsilon$$

□

Property (1) shows that for all  $x \in [0, 1]$ , the real sequence  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in  $(\mathbb{R}, \|\cdot\|)$ . But since  $\mathbb{R}$  is Banach (i.e., complete) we derive that, for all  $x \in [0, 1]$ , the real sequence  $(f_n)_{n \in \mathbb{N}}$  converges in  $\mathbb{R}$ , so we can define

$$\begin{aligned} f : [0, 1] &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) := \lim_{n \rightarrow \infty} f_n(x) \end{aligned} \quad (\forall x \in [0, 1])$$

on the other words, the sequence of functions  $(f_n)_{n \in \mathbb{N}}$  converges pointwise to  $f$ . Now we are going to show that  $f \in E$  and that  $(f_n)_{n \in \mathbb{N}}$  converges in  $(E, \|\cdot\|_\infty)$  to  $f$  (i.e.,  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$ ), by taking in (1).

$$q = n > N \quad \text{and} \quad p \rightarrow \infty$$

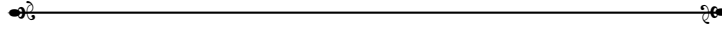
we will obtain :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : \quad n > N \implies \forall x \in [0, 1] : \quad |f_n(x) - f(x)| < \varepsilon$$

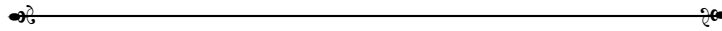
which is equivalent to

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : \quad n > N \implies \sup_{x \in [0, 1]} |f_n(x) - f(x)| \leq \varepsilon$$

Showing that, the sequence of functions  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$  on  $[0, 1]$ .



Recall a theorem in **Analysis 3**, Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of continuous functions on a closed interval  $[a, b]$  where  $(a, b \in \mathbb{R}, a < b)$ , that converges uniformly to a function  $f$  on  $[a, b]$ . Then  $f$  is also continuous on  $[a, b]$ .



By applying this result of analysis 3, we derive that  $f$  is also continuous on  $[0, 1]$ , that is  $f \in E$ , and  $(f_n)_{n \in \mathbb{N}}$  is convergent in  $(E, \|\cdot\|_\infty)$  to  $f$ , we conclude that  $(\mathcal{C}^0([0, 1], \mathbb{R}), \|\cdot\|_\infty)$  is Banach.

## 1.10 Bounded subset and bounded map on N.V.S :

The concepts of "bounded subsets" and "bounded maps" (or "bounded functions"), are in general defined in a metric space, however, the use of norms allows to simplify them as stated by the following propositions :

**Theorem 1.10.1:**

A non empty subset  $A$  of a N.V.S  $E$  is bounded if and only if there is a positive real number  $M$  such that :

$$\forall x \in A : \quad \|x\| \leq M$$

*Proof.* Let  $E$  be a N.V.S and  $A$  be a non empty subset of  $E$ .

$$( \implies )$$

Suppose that  $A$  is bounded, that is  $\delta(A) < +\infty$ , and let  $x_0 \in A$  be fixed. For all  $x \in A$ , we have

$$\begin{aligned} \|x\| &= \|x - x_0 + x_0\| \leq \|x - x_0\| + \|x_0\| \\ &\leq \delta(A) + \|x_0\| \end{aligned}$$

So it suffices to take  $M = \delta(A) + \|x_0\|$ , to obtain the required property.

$$( \Longleftarrow )$$

Conversly, suppose that there exist  $M > 0$  so that we have

$$\forall x \in A : \quad \|x\| \leq M$$

but this is equivalent to say that

$$A \subset \overline{B}(0_E, M)$$

implying that  $A$  is bounded this completes the proof of the proposition □

**Theorem 1.10.2:**

Let  $X$  be a non empty set,  $E$  be a N.V.S and

$$f : X \longrightarrow E$$

be a map, then  $f$  is bounded if and only if  $\exists M > 0$  such that :

$$\forall x \in X : \quad \|f(x)\| \leq M$$

*Proof.* By definition, we say that  $f$  is bounded, it's equivalent to say that  $f(X)$  is bounded, which is equivalent to say (according to the previous proposition), that  $\exists M > 0$  such that :

$$\forall y \in f(X) : \quad \|y\| \leq M$$

equivalent to

$$\forall x \in X : \quad \|f(x)\| \leq M$$

This completes the proof. □





# 2

## CONTINUOUS LINEAR MAPPINGS BETWEEN TWO N.V.S

### Theorem 2.0.1: Fundamental

Let  $E$  and  $F$  be two N.V.S on the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $f : E \rightarrow F$ , be a linear mapping then the following properties are equivalent

- (i)  $f$  is continuous on  $E$
- (ii)  $f$  is continuous at the same  $x_0 \in E$
- (iii)  $f$  is bounded on  $\overline{B}(0_E, 1)$ , i.e. :

$$\exists M > 0, \forall x \in \overline{B}(0_E, 1) : \|f(x)\|_F \leq M$$

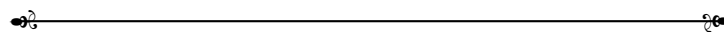
- (iv)  $f$  is bounded on  $S(0_E, 1)$
- (v)  $\exists M > 0$  such that :

$$\forall x \in E : \|f(x)\|_F \leq M\|x\|_E$$

- (vi)  $f$  is Lipchitz continuous

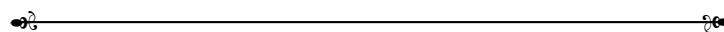
*Proof.* We will show the following implications :

$$(i) \implies (ii) \implies (iii) \implies (iv) \implies (v) \implies (vi) \implies (i)$$



$$(i) \implies (ii)$$

This is obvious



$$(ii) \implies (iii)$$

Suppose that  $f$  is continuous at some  $x_0 \in E$ , so  $\exists \mu > 0$  such that :

$$\forall x \in E : \|x - x_0\| < \mu \implies \|f(x) - f(x_0)\|_F < 1 \quad (2.1)$$

now, giving  $y \in \overline{B}(0_E, 1)$  arbitrary, putting  $x = \frac{\mu}{2}y + x_0$ , we have :

$$\|x - x_0\|_E = \left\| \frac{\mu}{2}y \right\|_E = \frac{\mu}{2}\|y\|_E \leq \frac{\mu}{2} < \mu$$

then  $\|x - x_0\| < \mu$ , thus according to (1)  $\|f(x) - f(x_0)\| < 1$  but  $f$  is linear

$$\begin{aligned} \|f(x) - f(x_0)\|_F &= \|f(x - x_0)\|_F = \left\| f\left(\frac{\mu}{2}y\right) \right\|_F = \left\| \frac{\mu}{2}f(y) \right\|_F \\ &= \frac{\mu}{2}\|f(y)\|_F \end{aligned}$$

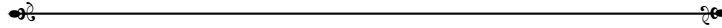
hence

$$\frac{\mu}{2}\|f(y)\|_F < 1$$

implying that

$$\|f(y)\|_F < \frac{2}{\mu} \quad (\forall y \in \overline{B}(0_E, 1))$$

this shows that  $f$  is bounded on  $\overline{B}(0_E, 1)$



$$(iii) \implies (iv)$$

This is obvious since  $S_E(0_E, 1) \subset \overline{B}_E(0_E, 1)$ , that is :

$$\exists M > 0, \forall u \in S_E(0_E, 1) : \|f(u)\|_F \leq M$$

so, for any  $x \in E \setminus \{0_E\}$ , since  $\frac{x}{\|x\|_E} \in S_E(0_E, 1)$ , we have :

$$\left\| f\left(\frac{x}{\|x\|_E}\right) \right\| \leq M$$

which gives

$$\|f(x)\|_F \leq M\|x\|_E$$

as required, remark that this last inequality is also valid for  $x = 0_E$



$$(iv) \implies (v)$$

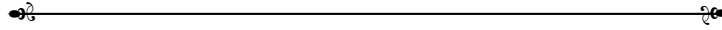
Suppose that  $\exists M > 0$ , satisfying the property :

$$\forall x \in E : \|f(x)\|_F \leq M\|x\|_E$$

then, for all  $x, y \in E$ , we have :

$$\|f(x) - f(y)\|_F = \|f(x - y)\|_F \leq M\|x - y\|_E$$

implying that  $f$  is  $M$ -Lipschitz



$$(iv) \implies (v)$$

this is known to be true in metric spaces, (in general). This proof is complete  $\square$

### Theorem 2.0.2:

Let  $E$  be a  $\mathbb{K}$ -Vector space and let  $N_1$  and  $N_2$  be two norms on  $E$ , then we have equivalence between :

- (i)  $N_1$  and  $N_2$  are topologically equivalent
- (ii)  $N_1$  and  $N_2$  are equivalent

*Proof.* we have

$$\begin{aligned} id_E : (E, N_1) &\longrightarrow (E, N_2) \\ x &\longmapsto x \end{aligned}$$

is bicontinuous, and it's bi-Lipschitz continuous. But since  $id_E : (E, N_1) \longrightarrow (E, N_2)$  and it's inverse  $id_E^{-1} : (E, N_2) \longrightarrow (E, N_1)$ , are obviously linear, then (by the above theorem we have the equivalence), between "  $id_E$  is bicontinuous ", and "  $id_E$  is bi-Lipschitz continuous ", hence they are equivalent, as required.  $\square$

*Notation :* let  $E$  and  $F$  be two N.V.S over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , we let  $L(E, F)$  denote the  $\mathbb{K}$ -vector space of linear maps from  $E$  to  $F$ , and  $\mathcal{L}(E, F)$  denote the  $\mathbb{K}$ -vector space of continuous linear maps, from  $E$  to  $F$ , In general we have :

$$\mathcal{L}(E, F) \subsetneq L(E, F)$$

### Example

Let  $E := \mathcal{C}^0([0, 1])$ ,  $\mathbb{R}$ , considered as an  $\mathbb{R}$ -vector space, we consider in  $E$  the two norms  $\|\cdot\|_1$

and  $\|\cdot\|_\infty$  defined previously, let

$$\begin{aligned}\delta : E &\longrightarrow \mathbb{R} \\ f &\longmapsto \delta(f) := f(0)\end{aligned} \quad (\mathbb{R}, \|\cdot\|)$$

$\delta$  is called the Dirac operator, it's clear that  $\delta$  is linear. We shall prove that  $\delta$  is continuous with respect to  $\|\cdot\|_\infty$  but it's not continuous with respect to  $\|\cdot\|_1$ . - **For  $\|\cdot\|_\infty$  :**

$\forall f \in E$ , we have :

$$|\delta(f)| = |f(0)| \leq \sup_{t \in [0,1]} |f(t)| = \|f\|_\infty$$

This shows according to the above theorem, that  $\delta$  is continuous in  $(E, \|\cdot\|_\infty)$

- **For  $\|\cdot\|_1$  :**

Consider the sequence of functions  $(f_n)_{n \geq 1}$  of  $E$ , defined by  $\forall n \in \mathbb{N}$  :

$$f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0, \frac{1}{n}) \\ 0 & \text{if } x \in [\frac{1}{n}, 1] \end{cases}$$

we have for all  $n \geq 1$ :

$$\begin{aligned}|\delta(f_n)| &= |f_n(0)| = 1 \\ \|f_n\|_1 &= \int_0^1 |f_n(x)| dx = \int_0^{1/n} (1 - nx) dx + \int_{1/n}^1 0 dx \\ &= \left(x - \frac{n}{2}x^2\right)^{1/2} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}\end{aligned}$$

thus  $\forall n \in \mathbb{N}$ , we have :

$$\frac{|\delta(f_n)|}{\|f_n\|_1} = U_n \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

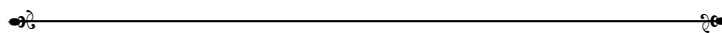
implying that  $\frac{|\delta(f)|}{\|f\|_1}$ , where  $(f \in E \setminus \{0_E\})$ , is unbounded from above, thus the Dirac operator  $\delta$  is not continuous on  $(E, \|\cdot\|_1)$ .

### Remark

If  $E$  is an infinite dimensional N.V.S, we can show that we have

$$\mathcal{L}(E, F) \subsetneq L(E, F)$$

That is there exist a linear map from  $E$  to  $F$  which is not continuous.



Let  $E$  and  $F$  be two N.V.S over  $\mathbb{K}$ , for  $f \in \mathcal{L}(E, F)$ , we define  $||| f |||$  by :

$$||| f ||| := \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E}$$

According to item (v) of the above theorem, we have that  $||| f ||| \in [0, \infty)$  i.e.,  $||| f |||$  is a non negative real number, so  $||| \cdot |||$  constitutes a map from  $\mathcal{L}(E, F)$  to  $[0, \infty)$

**Theorem 2.0.3:**

The map  $||| \cdot |||$  defined above is a norm  $\mathcal{L}(E, F)$  (seen as a  $\mathbb{K}$  vector space)

*Proof.* Let us show that  $||| \cdot |||$  satisfies the three axioms of a norm on  $\mathcal{L}(E, F)$

(i) 1<sup>st</sup> axiom :

For all  $f \in \mathcal{L}(E, F)$  we have

$$\begin{aligned} ||| f ||| = 0 &\iff \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} = 0 \\ &\iff \forall x \in E \setminus \{0_E\} : \|f(x)\|_F = 0 \\ &\iff \forall x \in E \setminus \{0_E\} : f(x) = 0_F \\ &\iff \forall x \in E : f(x) = 0_F \\ &\iff f = 0_{\mathcal{L}(E, F)} \end{aligned}$$

(ii) 2<sup>nd</sup> axiom :  $\forall f \in \mathcal{L}(E, F)$ , we have

$$\begin{aligned} ||| \lambda f ||| &= \sup_{x \in E \setminus \{0_E\}} \frac{\|(\lambda f)(x)\|_F}{\|x\|_E} \\ &= \sup_{x \in E \setminus \{0_E\}} \frac{\|\lambda f(x)\|_F}{\|x\|_E} \\ &= \sup_{x \in E \setminus \{0_E\}} \frac{|\lambda| \|f(x)\|_F}{\|x\|_E} \\ &= |\lambda| \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} = |\lambda| ||| f ||| \end{aligned}$$

As required

(iii) 3<sup>rd</sup> axiom :

let  $f, g \in \mathcal{L}(E, F)$ , we have for all  $x \in E \setminus \{0_E\}$  :

$$\begin{aligned} \|(f + g)(x)\|_F &= \|f(x) + g(x)\|_F \\ &\leq \|f(x)\|_F + \|g(x)\|_F \end{aligned}$$

Thus (by dividing by  $\|x\|_E$ ) :

$$\begin{aligned} \frac{\|(f+g)(x)\|_F}{\|x\|_E} &\leq \frac{\|f(x)\|_F}{\|x\|_E} + \frac{\|g(x)\|_F}{\|x\|_E} \\ &\leq \|f\| + \|g\| \end{aligned}$$

So all  $x \in E \setminus \{0_E\}$

$$\frac{\|(f+g)(x)\|_F}{\|x\|_E} \leq \|f\| + \|g\|$$

Hence, by taking the supremum over  $x \in E \setminus \{0_E\}$  :

$$\|f+g\| \leq \|f\| + \|g\|$$

as required, consequently,  $\|\cdot\|$  is a norm on  $\mathcal{L}(E, F)$

□

*Terminology :*

Let  $E$  and  $F$  be two N.V.S over  $\mathbb{K}$ , then the norm  $\|\cdot\|$  of  $\mathcal{L}(E, F)$  (constituted from the two norms  $\|\cdot\|_E$  of  $E$  and  $\|\cdot\|_F$  of  $F$ ), is called the subordinate norm induced by the norms  $\|\cdot\|_E$  of  $E$  and  $\|\cdot\|_F$  of  $F$ .

#### Theorem 2.0.4:

Let  $E$  and  $F$  be two N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , then for all  $f \in \mathcal{L}(E, F)$ , we have :

$$\begin{aligned} \|f\| &:= \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} \\ &= \sup_{x \in S_E(0_E, 1)} \|f(x)\|_F \\ &= \sup_{x \in B_E(0_E, 1)} \|f(x)\|_F \\ &= \sup_{x \in \overline{B_E}(0_E, 1)} \|f(x)\|_F \end{aligned}$$

*Proof.* We have to show the following multiple inequality :

$$\begin{aligned} \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} &\leq 1 \sup_{x \in S_E(0_E, 1)} \|f(x)\|_F \leq 2 \sup_{x \in B_E(0_E, 1)} \|f(x)\|_F \\ &\leq 3 \sup_{x \in \overline{B_E}(0_E, 1)} \|f(x)\|_F \\ &\leq 4 \sup_{x \in E \setminus \{0_E\}} \frac{\|f(x)\|_F}{\|x\|_E} \end{aligned}$$

Since this inequality  $\leq_3$  is obvious, because  $B_E(0_E, 1) \subset \overline{B_E}(0_E, 1)$ , we have to show the three inequalitys

$$\leq_1 \quad \leq_2 \quad \leq_4$$

Let us show  $\leq_1$  for all  $x \in E \setminus \{0_E\}$ , we have :

$$\frac{\|f(x)\|_F}{\|x\|_E} = \|f\left(\frac{x}{\|x\|_E}\right)\|_F \leq \sup_{y \in S_E(0_E, 1)} \|f(y)\|_F$$

so for all  $x \in E \setminus \{0_E\}$  :

$$\frac{\|f(x)\|_F}{\|x\|_E} \leq \sup_{y \in S_E(0_E, 1)} \|f(y)\|_F$$

Thus by taking the supremum over  $x$ , we get the required result, Now let us agains show the second inequality  $\leq_2$ , for all  $x \in S_E(0_E, 1, 1)$ , we have

$$\|f(x)\|_F = \frac{1}{r} \|f(\underbrace{rx}_{\in B_E(0_E, 1)})\|_F \leq \frac{1}{r} \sup_{y \in B_E(0_E, 1)} \|f(y)\|_F$$

so

$$\forall x \in S_E(0_E, 1), \forall r \in (0, 1) : \quad \|f(x)\|_F \leq \frac{1}{r} \sup_{y \in B_E(0_E, 1)} \|f(y)\|_F$$

So, by taking  $r \rightarrow^< 1$ , we get

$$\forall x \in S_E(0_E, 1) : \quad \|f(x)\|_F \leq \sup_{y \in B_E(0_E, 1)} \|f(y)\|_F$$

then by taking the supremum over  $x$  :

$$\sup_{x \in S_E(0_E, 1)} \|f(x)\|_F \leq \sup_{y \in B_E(0_E, 1)} \|f(y)\|_F$$

as required, now let us show the  $\leq_4$ , we have for all  $x \in \overline{B_E}(0_E, 1) \setminus \{0_E\}$ , we have :

$$0 < \|x\|_E \leq 1 \implies \frac{1}{\|x\|} \geq 1$$

so we get :

$$\begin{aligned} \|f(x)\|_F &\leq \frac{\|f(x)\|_F}{\|x\|_E} \\ &\leq \sup_{y \in E \setminus \{0_E\}} \frac{\|f(y)\|_F}{\|y\|_E} = \|f\| \end{aligned}$$

So  $\forall x \in \overline{B_E}(0_E, 1) \setminus \{0_E\}$  :

$$\|f(x)\|_F \leq \|f\|$$

which is also true for  $x = 0_E$  since  $f$  is linear, so

$$\forall x \in \overline{B_E}(0_E, 1) : \|f(x)\|_F \leq \|f\|$$

then by taking the supremum over  $x$  :

$$\sup_{x \in \overline{B_E}(0_E, 1)} \|f(x)\|_F \leq \|f\|$$

as required, this completes the proof.  $\square$

This following proposition is an immediate consequence of the definition of a subordinate norm

**Theorem 2.0.5:**

Let  $E$  and  $F$  be two N.V.S over  $\mathbb{K} = \mathbb{R}$ , or  $\mathbb{C}$  and  $f \in \mathcal{L}(E, F)$ , we have :

1.

$$\forall x \in E : \|f(x)\|_F \leq \|f\| \cdot \|x\|_E$$

2. if  $M \in [0, \infty)$  satisfies :

$$\|f(x)\|_F \leq M \|x\|_E \quad (\forall x \in E)$$

then

$$\|f\| \leq M$$

By applying theorem 5, we obtain a remarkable inequality concerning the subordinate norm of a composition of two continuous linear mappings between N.V.S

**Theorem 2.0.6:**

Let  $E, F$  and  $G$  be three N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $f : E \rightarrow F$  and  $g : F \rightarrow G$  be two continuous linear mappings then we have :

$$\|g \circ f\| \leq \|g\| \cdot \|f\|$$

*Proof.* Since  $f : E \rightarrow F$  and  $g : F \rightarrow G$  and both linear then  $g \circ f : E \rightarrow G$  is also linear, similarly, since  $f$  and  $g$  are both continuous then  $g \circ f$  is continuous therefore  $g \circ f \in \mathcal{L}(E, G)$ . Next, using twice successively the inequality of item (1), of proposition (5), we have for all  $x \in E$  :

$$\begin{aligned} \|(g \circ f)(x)\|_G &= \|g(f(x))\|_G \leq \|g\| \cdot \|f(x)\|_F \\ &\leq \|g\| \cdot \|f\| \cdot \|x\|_E \end{aligned}$$

This implies according to item (2) of proposition (5), that :

$$\|g \circ f\| \leq \|g\| \cdot \|f\|$$

as required, this completes the proof.  $\square$



## 2.1 Normed Algebra

### Definition 2.1.1:

Let  $\mathbb{K}$  be a field, an algebra over  $\mathbb{K}$  or simply a  $\mathbb{K}$ -algebra is a  $\mathbb{K}$ -vector space  $\mathcal{A}$  or  $(\mathcal{A}, +, \cdot)$  equipped with a bilinear multiplication operation,  $\times : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  such that  $(\mathcal{A}, +, \times)$  is a ring and " $\times$ " is compatible with scalar multiplication, that is

$$\forall \lambda \in \mathbb{K}, \forall x, y \in \mathcal{A} : (\lambda \cdot x) \times y = x \times (\lambda \cdot y) = \lambda \cdot (x \times y)$$

### Example

For any field  $\mathbb{K}$  and any  $n \in \mathbb{N}$ ,  $\mathcal{M}_n(\mathbb{K})$  is  $\mathbb{K}$ -algebra

### Definition 2.1.2:

let  $(\mathcal{A}, +, \times, \cdot)$  be a  $\mathbb{K}$ -algebra, an *algebra-norm* on  $\mathcal{A}$  is a norm  $||| \cdot |||$  on the  $\mathbb{K}$ -vector space  $(\mathcal{A}, +, \cdot)$  which satisfies in addition the property :

$$||| y \times x ||| \leq ||| x ||| \cdot ||| y |||$$

we say that  $||| \cdot |||$  is submultiplicative.

here are the following axioms of the algebra-norm

1.  $||| x ||| = 0 \implies x = 0_{\mathcal{A}}$
2.  $||| \lambda x ||| = |\lambda| \cdot ||| x ||| \quad \forall \lambda \in \mathbb{K}, \forall x \in \mathcal{A}$
3.  $||| x + y ||| \leq ||| x ||| + ||| y ||| \quad \forall x, y \in \mathcal{A}$
4.  $||| x \times y ||| \leq ||| x ||| \cdot ||| y ||| \quad \forall x, y \in \mathcal{A}$

### Example

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , then  $\mathcal{L}(E, E)$  with the laws  $+, \cdot, \circ$  equipped with the subordinate norm  $||| \cdot |||$  induced by  $\|\cdot\|_E$  is a normed algebra according to the above proposition

## 2.2 An important particular case (matrix norm)

### Definition 2.2.1:

Let  $n \in \mathbb{N}$ , a matrix norm on  $\mathcal{M}_n(\mathbb{K})$  where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  is a map  $||| \cdot |||: \mathcal{M}_n(\mathbb{K}) \rightarrow [0, \infty)$  which satisfies :

- (i)  $\forall A \in \mathcal{M}_n(\mathbb{K}) : ||| A ||| = 0 \implies A = 0_{\mathcal{M}_n(\mathbb{K})}$
- (ii)  $\forall A \in \mathcal{M}_n(\mathbb{K}), \forall \alpha \in \mathbb{K} : ||| \alpha A ||| = |\alpha| \cdot ||| A |||$
- (iii)  $\forall A, B \in \mathcal{M}_n(\mathbb{K}) : ||| A + B ||| \leq ||| A ||| + ||| B |||$
- (iv)  $\forall A, B \in \mathcal{M}_n(\mathbb{K}) : ||| AB ||| \leq ||| A ||| \cdot ||| B |||$

in other words, a matrix norm is an algebra norm on  $(\mathcal{M}_n(\mathbb{K}), +, \times, \cdot)$  where  $\times$  is matrix multiplication and  $\cdot$  is scalar multiplication.

### Remark

Let  $n \in \mathbb{N}$ , any norm  $\|\cdot\|$  on the  $\mathbb{K}$ -vector space  $\mathbb{K}^n$  induces a matrix norm  $||| \cdot |||$  on  $\mathcal{M}_n(\mathbb{K})$ , which is defined by :

$$||| A ||| := \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|Ax\|}{\|x\|} = \sup_{x \in \mathbb{K}, \|x\|=1} \|Ax\|$$

This particular matrix norm is called

*"The subordinate norm induced by  $\|\cdot\|$ "*

### Example

let  $n \in \mathbb{N}$ .

- the subordinate norm on  $\mathcal{M}_n(\mathbb{K})$  induced by the norm of  $\|\cdot\|_1$  on  $\mathbb{K}^n$  is given by

$$\|A\|_1 := \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| = \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|Ax\|_1}{\|x\|_1}$$

- the subordinate matrix norm on  $\mathcal{M}_n(\mathbb{K})$  induced by the norm  $\|\cdot\|_\infty$  on  $\mathbb{K}^n$  is given by :

$$||| A |||_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| = ||| A^T |||_1$$

- the subordinate norm on  $\mathcal{M}_n(\mathbb{K})$  induced by the norm  $\|\cdot\|_2$  of  $\mathbb{K}^n$  is given by :

$$\|A\|_2 = \sqrt{\rho(A^T A)} \quad (\forall A \in \mathcal{M}_n(\mathbb{K}))$$

where  $\rho$  denotes the spectral radius of a square matrix  $M$  of  $\mathcal{M}_n(\mathbb{K})$

$$(\rho(M) := \max\{|\lambda|, \lambda \in \sigma_{\mathbb{C}}(M)\})$$

the square root of the eigen values of the positive semi definite matrix  $A^T A$  are called singular values of  $A$

$$\|A\|_2 = \max S.V(A) \quad (\text{the largest singular value of } A)$$

- suppose that  $n \geq 2$ , we define

$$\begin{aligned} N : \mathcal{M}_n(\mathbb{K}) &\longrightarrow [0, \infty) \\ A &\longmapsto N(A) := \max_{1 \leq i, j \leq n} |a_{ij}| \end{aligned}$$

it's clear that  $N$  is a clear norm on  $\mathcal{M}_n(\mathbb{K})$  but it's not a matrix norm on it because we have for example

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

we have

$$A^2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = n \times A$$

so  $N(A^2) = n$  and  $N(A)^2 = 1^2 = 1$  then

$$N(A^2) \not\leq N(A)^2$$

thus  $N$  is not a matrix norm.

### Remark

let  $n \in \mathbb{N}$ , for any matrix norm  $\|\cdot\|$  on  $\mathcal{M}_n(\mathbb{K})$ , we have  $\|I_n\| \geq 1$ . Indeed,

$$\|I_n^2\| \leq \|I_n\|^2$$

that is

$$\|I_n\| \leq \|I_n\|^2$$

hence  $||| I_n ||| \geq 1$

### Definition 2.2.2:

let  $n \in \mathbb{N}$ , if a matrix norm  $||| \cdot |||$  on  $\mathcal{M}_n(\mathbb{K})$  satisfies  $||| I_n ||| = 1$  then it's said to be unital

### Example

Any subordinated matrix norm  $||| \cdot |||$  on  $\mathcal{M}_n(\mathbb{K})$  where  $(n \in \mathbb{N})$  induced by a norm  $\|\cdot\|$  on  $\mathbb{K}^n$  is unital, indeed, in such a case, we have :

$$||| I_n ||| = \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|I_n x\|}{\|x\|} = \sup_{x \in \mathbb{K}^n \setminus \{0_{\mathbb{K}^n}\}} \frac{\|x\|}{\|x\|} = 1$$

note that there exist *unital matrix norms* on  $\mathcal{M}_n(\mathbb{K})$  which are not subordinate, (i.e., not induced by any vector space norm  $\mathbb{K}^n$ )

## 2.3 The spectral radius of a complex square matrix

### Definition 2.3.1:

Let  $n \in \mathbb{N}$  and  $A \in \mathcal{M}_n(\mathbb{C})$  the spectral radius of  $A$ , denoted  $\rho(A)$ , is the maximum of the modulus of the eigen values of  $A$ , that is

$$\rho(A) := \max \{ |\lambda| : \lambda \in \sigma_{\mathbb{C}}(A) \}$$

we have the following theorem

### Theorem 2.3.1:

let  $n \in \mathbb{N}$  and let  $||| \cdot |||$  be a matrix norm on  $\mathcal{M}_n(\mathbb{C})$ , then for any  $A \in \mathcal{M}_n(\mathbb{C})$ , we have :

$$\rho(A) \leq ||| A |||$$

*Proof.* let  $A \in \mathcal{M}_n(\mathbb{C})$  and let  $\lambda \in \mathbb{C}$  be an arbitrary eigen value of  $A$ , so  $\exists x \in \mathbb{C}^n \setminus \{0_{\mathbb{C}^n}\}$  such that  $Ax = \lambda x$  consider :

$$B := (X \setminus \{0_{\mathbb{C}^n}\} \setminus \dots \setminus \{0_{\mathbb{C}^n}\}) \cap \mathcal{M}_n(\mathbb{C}) \setminus \{0_{\mathcal{M}_n(\mathbb{C})}\}$$

then we have :

$$\begin{aligned}
 AB &= (Ax \mid A0_{\mathbb{C}^n} \mid \dots \mid A0_{\mathbb{C}^n}) \\
 &= (\lambda x \mid 0_{\mathbb{C}^n} \mid \dots \mid 0_{\mathbb{C}^n}) \\
 &= \lambda (x \mid 0_{\mathbb{C}^n} \mid \dots \mid 0_{\mathbb{C}^n}) \\
 &= \lambda B
 \end{aligned}$$

thus

$$||| AB ||| = ||| \lambda B ||| = |\lambda| ||| B |||$$

so

$$\lambda ||| B ||| = ||| AB ||| \leq ||| A ||| \cdot ||| B |||$$

thus

$$|\lambda| \leq ||| A ||| \quad (\forall \lambda \in \sigma_{\mathbb{C}}(A))$$

hence

$$\max_{\lambda \in \sigma_{\mathbb{C}}(A)} |\lambda| \leq ||| A ||| \implies (\rho(A)) \leq ||| A |||$$

as required □

### Theorem 2.3.2: Gelfond's formula

Let  $n \in \mathbb{N}$  and  $||| \cdot |||$  be a matrix norm on  $\mathcal{M}_n(\mathbb{C})$  then for every  $A \in \mathcal{M}_n(\mathbb{C})$ , we have

$$\rho(A) = \lim_{k \rightarrow \infty} ||| A^k |||^{1/k}$$



# $\int$ PROPERTIES OF FINITE-DIMENSIONAL $\mathbb{K}$ -N.V.S $3$

## 3.1 Norms on a finite-dimensional $\mathbb{K}$ -vector space

Let  $n \in \mathbb{N}$  and  $E$  be an  $n$ -dimensional vector space over  $\mathbb{K}$ , let also  $\mathcal{B} = (e_1, e_2, \dots, e_n)$  be a basis of  $E$ , using  $\mathcal{B}$  we can construct on  $E$  several norms including :

$$\|\cdot\|_{1,\mathcal{B}} \quad \|\cdot\|_{2,\mathcal{B}} \quad \|\cdot\|_{p,\mathcal{B}} \quad (p \geq 1) \quad \text{and} \quad \|\cdot\|_{\infty,\mathcal{B}}$$

defined by

$$\begin{aligned} \|x\|_{1,\mathcal{B}} &:= \sum_{i=1}^n |x_i| \\ \|x\|_{2,\mathcal{B}} &:= \sqrt{\sum_{i=1}^n |x_i|^2} \\ \|x\|_{3,\mathcal{B}} &:= \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \\ \|x\|_{\infty,\mathcal{B}} &:= \max_{1 \leq i \leq n} \|x_i\| \end{aligned}$$

we easily show that these norms on  $E$  are all equivalent, lets consider in particular the norm  $\|\cdot\|_{\infty,\mathcal{B}}$ , it's immediate that the map

$$\begin{aligned} (\mathbb{K}^n, \|\cdot\|_{\infty}) &\longrightarrow (E, \|\cdot\|_{\infty,\mathcal{B}}) \\ (x_1, x_2, \dots, x_n) &\longmapsto x_1 e_1 + \dots + x_n e_n \end{aligned}$$

this map is an isometry (bijective), since the distances are conserved we call it *isomorphism isometric*, it's an homeomorphism because it's lipschitz, consequently, the  $\mathbb{K}$ -N.V.S,  $(E, \|\cdot\|_{\infty,\mathcal{B}})$  and  $(\mathbb{K}^n, \|\cdot\|_{\infty})$  have the same topological and metric properties, in particular, we derive that :

- (1) The N.V.S  $(E, \|\cdot\|_{\infty, \mathcal{B}})$  is complete (i.e., a Banach space)
- (2) The compact parts of  $(E, \|\cdot\|_{\infty, \mathcal{B}})$  are exactly bounded parts in particular

$$S_E(0_E, 1) \mid_{\|\cdot\|_{\infty, \mathcal{B}}} \text{ is compact in } (E, \|\cdot\|_{\infty, \mathcal{B}})$$

these two properties are used to prove the following fundamental theorem

**Theorem 3.1.1:**

On a finite-dimensional vector space  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , all norms are equivalent

*Proof.* let  $n \in \mathbb{N}$  and  $\mathbb{E}$  an  $n$ -dimensional vector space over  $\mathbb{K} = (\mathbb{R} \text{ or } \mathbb{C})$ , let also  $\mathcal{B} = (e_1, e_2, \dots, e_n)$  be a fixed basis of  $E$ , we are going to show that every norm on  $E$  is equivalent to the norm  $\|\cdot\|_{\infty, \mathcal{B}}$ , let  $N$  be an arbitrary norm on  $E$  and let us show that  $N \sim \|\cdot\|_{\infty, \mathcal{B}}$  on the one hand, by using the properties of  $N$  as a norm on  $E$ , we have for all  $x = x_1 e_1 + \dots + x_n e_n$  with  $(x_1, \dots, x_n \in \mathbb{K})$ , we have :

$$\begin{aligned} N(x) &= N(x_1 e_1 + \dots + x_n e_n) \\ &\leq N(x_1 e_1) + \dots + N(x_n e_n) \\ &= |x_1| N(e_1) + |x_2| N(e_2) + \dots + |x_n| N(e_n) \\ &\leq \left( \max_{1 \leq i \leq n} |x_i| \right) \sum_{i=1}^n N(e_i) = \left( \sum_{i=1}^n N(e_i) \right) \|x\|_{\infty, \mathcal{B}} \end{aligned}$$

so by setting  $\beta = \sum_{i=1}^n N(e_i) > 0$ , we have

$$N(x) \leq \beta \|x\|_{\infty, \mathcal{B}} \quad (\forall x \in E)$$

□

some recap, we have

$$\begin{aligned} (\mathbb{K}^n, \|\cdot\|_{\infty}) &\longrightarrow (E, \|\cdot\|_{\infty, \mathcal{B}}) \\ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &\longmapsto x_1 e_1 + \dots + e_n x_n \end{aligned}$$

1. we deduce that  $(\mathbb{E}, \|\cdot\|_{\infty, \mathcal{B}})$  is banach
2. the compact parts of  $(E, \|\cdot\|_{\infty})$  are exactly closed and bounded parts in particular :

$$S_E(0_E, 1) \text{ is compact}$$

**Theorem 3.1.2:**

On a finite dimensional vector space on  $\mathbb{R}$  or  $\mathbb{C}$ , all norms are equivalent.

*Proof.* Let  $N$  be an arbitrary norm on  $E$ , we want to show that

$$N \sim \|\cdot\|_{\infty, \mathcal{B}}$$

we have

$$\begin{aligned} N(x) &= N(x_1 e_1 + \dots + x_n e_n) \leq \sum_{i=1}^n N(x_i e_i) \\ &= \sum_{i=1}^n |x_i| N(e_i) \\ &\leq \left( \sum_{i=1}^n N(e_i) \right) \|x\|_{\infty, \mathcal{B}} \end{aligned}$$

On the other hand, according to a well known property of the norms on a  $\mathbb{K}$ -vector space, (See Ex 1.1), we have for all  $x, y \in E$ :

$$|N(x) - N(y)| \leq N(x - y)$$

but since  $N \leq \beta \|\cdot\|_{\infty, \mathcal{B}}$ , we derive that for all  $x, y \in E$ :

$$|N(x) - N(y)| \leq \beta \|x - y\|_{\infty, \mathcal{B}}$$

implying that the map :

$$\begin{aligned} N : (E, \|\cdot\|_{\infty, \mathcal{B}}) &\longrightarrow (\mathbb{R}, \|\cdot\|) \\ x &\longmapsto N(x) \end{aligned}$$

is  $\beta$ -Lipschitz, so continuous on  $(E, \|\cdot\|_{\infty, \mathcal{B}})$ , next, giving that the unit sphere  $S_E(0_E, 1)$ , of  $(E, \|\cdot\|_{\infty, \mathcal{B}})$ , is compact in  $(E, \|\cdot\|_{\infty, \mathcal{B}})$ , see properties of the N.V.S  $(E, \|\cdot\|_{\infty, \mathcal{B}})$  cited above, it follows according to the extreme value theorem, recall

$$\bullet \text{---} \bullet$$

Let  $X$  be a compact topological space and,  $f : X \longrightarrow \mathbb{R}$  be a continuous map, then  $f$  is bounded on  $X$  and attains its bounds, meaning there exist points  $x_{\min}, x_{\max} \in X$  such that :

$$f(x_{\min}) = \inf_{x \in X} f(x) \quad \text{and} \quad f(x_{\max}) = \sup_{x \in X} f(x)$$

$$\bullet \text{---} \bullet$$



that the map  $N$  above is bounded on the sphere  $S_E(0_E, 1) |_{\|\cdot\|_{\infty, \mathcal{B}}}$ , and attains its supremum and infimum in that sphere, so there exist  $x_0 \in S_E(0_E, 1) |_{\|\cdot\|_{\infty, \mathcal{B}}}$  such that

$$N(x) \geq N(x_0) \quad \left( \forall x \in S_E(0_E, 1) |_{\|\cdot\|_{\infty, \mathcal{B}}} \right)$$

put  $\alpha := N(x_0) \geq 0$ , if we suppose that  $\alpha = 0$ , we obtain (since  $N$  is a norm on  $E$ ) that,  $x_0 = 0_E \notin S_E(0_E, 1) |_{\|\cdot\|_{\infty, \mathcal{B}}}$ , which is a contradiction, thus  $\alpha > 0$ , and we have :

$$\forall x \in S_E(0_E, 1) |_{\|\cdot\|_{\infty, \mathcal{B}}} : \quad N(x) \geq \alpha$$

finally, giving  $x \in E \setminus \{0_E\}$ , by applying the last inequality for

$$\frac{x}{\|x\|_{\infty, \mathcal{B}}} \in S_E(0_E, 1)$$

we obtain

$$N\left(\frac{x}{\|x\|_{\infty, \mathcal{B}}}\right) \geq \alpha$$

that is

$$N(x) \geq \alpha \|x\|_{\infty, \mathcal{B}} \quad (\forall x \in E \setminus \{0_E\})$$

this inequality, is also true for  $x = 0_E$ , hence we get

$$N(x) \geq \alpha \|x\|_{\infty, \mathcal{B}} \quad (\forall x \in E)$$

hence we have show that  $N$  is equivalent to  $\|\cdot\|_{\infty, \mathcal{B}}$ , as required, this completes the proof  $\square$

## 3.2 Topological and metric properties of a finite-dimensional N.V.S

From Theorem 1, we derive several important corollaries.

### Theorem 3.2.1:

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , we have :

- (1) Every finite-dimensional N.V.S over  $\mathbb{K}$  is banach
- (2) The compact parts of a finite-dimensional N.V.S over  $\mathbb{K}$  are exactly those which are both closed and bounded.

*Proof.* Let  $(E, \|\cdot\|)$  be a finite dimensional N.V.S, over  $\mathbb{K}$ , and  $n := \dim(E)$ , since the case for  $n = 0$  is trivial, we may suppose that  $n \geq 1$ , next let  $\mathcal{B} = (e_1, e_2, \dots)$  be a basis of  $E$ , since

$$\|\cdot\| \sim \|\cdot\|_{\infty, \mathcal{B}} \quad \text{by above Theorem}$$

then  $(E, \|\cdot\|)$  has the same topological and metric properties as  $(E, \|\cdot\|_{\infty, \mathcal{B}})$  so since properties (1) and (2) of the corollary hold for  $(E, \|\cdot\|_{\infty, \mathcal{B}})$  then they also hold for  $(E, \|\cdot\|)$ , as required this achieves the proof.  $\square$

### Theorem 3.2.2:

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $E$  and  $F$  be two  $\mathbb{K}$ -N.V.S with  $E$  is finite-dimensional, then every linear mapping from  $E$  to  $F$  is continuous

$$\mathcal{L}(E, F) = L(E, F)$$

*Proof.* Put  $n = \dim(E)$  since the case  $n = 0$  is trivial, suppose that  $n \geq 1$ , fix a basis

$$\mathcal{B} = (e_1, \dots, e_n)$$

of  $E$ , let  $f : (E, \|\cdot\|_E) \rightarrow (F, \|\cdot\|_F)$  be a linear mapping and we will show that it's continuous, according to Theorem 1, all norms on  $E$  are equivalent then in particular

$$\|\cdot\|_E \sim \|\cdot\|_E$$

so there exist a positive constant  $c$  such that

$$\|\cdot\|_{E, \mathcal{B}, \infty} \leq c \|\cdot\|_E$$

using this last inequality together with the linearity of  $f$  and the properties of a norm on a vector space, we have for every

$$x = x_1 e_1 + \dots + x_n e_n \in E \quad (x_1, x_2, \dots, x_n) \in \mathbb{K}$$

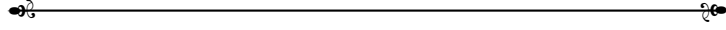
we have

$$\begin{aligned} \|f(x)\|_F &= \|f(x_1 e_1 + \dots + x_n e_n)\|_F = \|x_1 f(e_1) + \dots + x_n f(e_n)\|_F \\ &\leq \sum_{i=1}^n \|x_i f(e_i)\|_F \\ &= \sum_{i=1}^n |x_i| \|f(e_i)\|_F \\ &\leq \left( \sum_{i=1}^n \|f(e_i)\|_F \right) \|x\|_{E, \infty, \mathcal{B}} \\ &\leq \left( c \sum_{i=1}^n \|f(e_i)\|_F \right) \|x\|_E \end{aligned}$$

that is

$$\|f(x)\|_F \leq \left( c \sum_{i=1}^n (f(e_i))_F \right) \|x\|_E \quad (\forall x \in E)$$

showing that  $f$  is continuous, as required □



we have also the following important theorem

**Theorem 3.2.3:**

Let  $E$  and  $F$  be two N.V.S over  $\mathbb{K} (\{\mathbb{R}, \mathbb{C}\})$ , with  $F$  is Banach, then the  $\mathbb{K}$ -N.V.S  $\mathcal{L}(E, F)$  is Banach.

*Proof.* We have to show that any Cauchy sequence of  $\mathcal{L}(E, F)$  is convergent in  $(\mathcal{L}(E, F))$  so let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence of  $\mathcal{L}(E, F)$  and let us show that it converges for some  $f \in \mathcal{L}(E, F)$ , by hypothesis, we have :

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq N \implies ||| f_p - f_q ||| \leq \varepsilon$$

it follows from the definition of the norm  $||| \cdot |||$  of  $\mathcal{L}(E, F)$  that :

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq N \implies \forall x \in E : \quad \|f_p(x) - f_q(x)\| \leq \varepsilon \|x\|_E$$

for  $x \in E \setminus \{0_E\}$  fixed, by taking instead of  $\varepsilon$  the positive real number  $\frac{\varepsilon}{\|x\|_E}$ , we desire the following

$$\forall \varepsilon > 0, \quad N(\varepsilon, x) \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq N(\varepsilon, x) \implies \|f_p(x) - f_q(x)\|_F \leq \varepsilon$$

show that, for all  $x \in E \setminus \{0_E\}$  the sequence  $(f_n(x))_{n \in \mathbb{N}}$  of  $F$  is Cauchy, since  $F$  is Banach then for all  $x \in E \setminus \{0_E\}$ , the sequence  $(f_n)_{n \in \mathbb{N}}$  of  $F$  is convergent, remark that the same sequence  $(f_n(x))_{n \in \mathbb{N}}$  of  $F$  also converge for  $x = 0_E$  to  $0_F$ , since  $f_n(0_E) = 0_F$ , then for all  $n \in \mathbb{N}$ , because the maps  $f_n$  are all linear so let us define

$$\begin{aligned} f : E &\longrightarrow F \\ x &\longmapsto f(x) := \lim_{n \rightarrow \infty} f_n(x) \end{aligned}$$

Now, we are going to show that  $f \in \mathcal{L}(E, F)$ , that is  $f$  is linear and continuous, and that  $f$  is the limit of the sequence  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{L}(E, F)$

*is  $f$  linear?*

for all  $x, y \in E$ , for all  $\lambda \in \mathbb{K}$ , we have

$$\begin{aligned} f(\lambda x + y) &:= \lim_{n \rightarrow \infty} f_n(\lambda x + y) \\ &= \lim_{n \rightarrow \infty} (\lambda f_n(x) + f_n(y)) \text{ since } f_n \text{ is linear for all } n \in \mathbb{N} \\ &= \lambda \lim_{n \rightarrow \infty} f_n(x) + \lim_{n \rightarrow \infty} f_n(y) \text{ ( by the continuity of law } + \text{ and } \cdot \text{ of } F \text{ )} \\ &= \lambda f(x) + f(y) \end{aligned}$$

implying that  $f$  is linear

*is  $f$  continuous?*

By taking in  $\varepsilon = 1$ ,  $q = N = N(1) \in \mathbb{N}$ , and by letting  $p \rightarrow \infty$ , we obtain according to the continuity of the norm  $\|\cdot\|_F$ , that

$$\begin{aligned} \|f(x) - f_N(x)\| &\leq \varepsilon \|x\|_E \quad (\forall x \in E) \\ \|(f - f_N)(x)\| &\leq \|x\|_E \quad (\forall x \in E) \end{aligned}$$

which implies that the linear map  $(f - f_N)$ , from  $E$  to  $F$  is continuous, thus  $f := f_N + (f - f_N)$  is also continuous as the sum of two continuous mappings, consequently :

$$f \in \mathcal{L}(E, F)$$

*is  $f$  the limit of  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{L}(E, F)$*

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq N \implies \forall x \in E : \|f_p(x) - f_q(x)\|_F \leq \varepsilon \|x\|_E$$

by letting  $p \rightarrow \infty$ , and taking into account the continuity of the norm  $\|\cdot\|_F$  of  $E$ , we obtain that

$$\begin{aligned} \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall q \in \mathbb{N} : \quad q \geq N \implies \forall x \in E : \|f_q(x) - f(x)\| &\leq \varepsilon \|x\|_E \\ \iff \forall x \in E : \frac{\|(f_q - f)(x)\|_F}{\|x\|_E} &\leq \varepsilon \end{aligned}$$

that is

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall q \in \mathbb{N} : \quad q \geq N \implies \sup_{x \in E \setminus \{0_E\}} \frac{\|(f_q - f)(x)\|_F}{\|x\|_E} \leq \varepsilon$$

that is

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}, \forall q \in \mathbb{N} : \quad q \geq N \implies \|f_q - f\| \leq \varepsilon$$

showing that the sequence  $(f_n)_{n \in \mathbb{N}}$  converges to  $f$  in  $\mathcal{L}(E, F)$ , this completes the proof □

**Definition 3.2.1:**

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , we call the algebraic dual space of  $E$ , denoted  $E^*$ , the  $\mathbb{K}$ -vector space of  $E$  constituting of linear forms on  $E$ , that is

$$E^* := L(E, \mathbb{K})$$

We call the continuous dual space of  $E$ , denoted  $E'$ , the  $\mathbb{K}$ -normed vector subspace of  $E$  constituted of continuous linear forms on  $E$ , that is

$$E' := \mathcal{L}(E, \mathbb{K}) \quad (||| \cdot |||)$$

note that the contrary here is relative to the subordinate norm of  $\mathcal{L}(E, \mathbb{K})$  induced by the  $\|\cdot\|_E$  of  $E$  and  $|\cdot|$  of  $\mathbb{K}$

**Example**

Let  $a, b \in \mathbb{R}$  with  $(a, b) \neq (0, 0)$ , and let  $f$  be the linear form on  $\mathbb{R}^2$  defined by :

$$f(x, y) := ax + by \quad \left( \forall (x, y) \in \mathbb{R}^2 \right)$$

- (1) Explain why  $f$  is continuous.
- (2) (a) Determine  $||| f |||$  with respect to the norm  $\|\cdot\|_1$  of  $\mathbb{R}^2$  and  $|\cdot|$  of  $\mathbb{R}$
- (b) Determine  $||| f |||$  with respect to the norm  $\|\cdot\|_2$  of  $\mathbb{R}^2$  and  $|\cdot|$  of  $\mathbb{R}$

**( Solution )**

- (1) Since  $\dim \mathbb{R}^2 = 2 < \infty$  then  $\mathcal{L}(\mathbb{R}^2, \mathbb{R}) = L(\mathbb{R}^2, \mathbb{R})$  i.e. we have :

$$(\mathbb{R}^2)' = (\mathbb{R}^2)^*$$

every linear form on  $\mathbb{R}^2$  is continuous, in particular  $f$  is continuous

- (2) (a) By definition :

$$||| f ||| := \sup_{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|f(x,y)|}{\|(x,y)\|_1} = \sup_{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|ax + by|}{|x| + |y|}$$

we have for all  $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$

$$|ax + by| \leq |ax| + |by| = \underbrace{|a|}_{\max(|a|, |b|)} |x| + \underbrace{|b|}_{\max(|a|, |b|)} |y|$$

$$\leq \max(|a|, |b|) (|x| + |y|)$$

$$\frac{|ax + by|}{|x| + |y|} \leq \max(|a|, |b|)$$

hence

$$||| f ||| \leq \max(|a|, |b|)$$

by definition, we have :

$$||| f ||| \geq \frac{|f(1,0)|}{\|(1,0)\|_1} = \frac{|a|}{1} = |a|$$

and

$$||| f ||| \geq \frac{|f(0,1)|}{\|(0,1)\|_1} = \frac{|b|}{1} = |b|$$

thus we have :

$$||| f ||| \geq \max(|a|, |b|)$$

from the above we have shown that :

$$||| f ||| = \max(|a|, |b|)$$

(b) we have

$$||| f ||| = \sup_{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|f(x,y)|}{\|(x,y)\|_2} = \sup_{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|ax + by|}{\sqrt{x^2 + y^2}}$$

According to the cauchy-schawrz in the Pre-Hilbert space  $(\mathbb{R}^2, \langle \cdot \rangle_u)$ , we have :

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$|ax + by| = \left| \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_u \right| \leq \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_2 \cdot \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 = \sqrt{a^2 + b^2} \cdot \sqrt{x^2 + y^2}$$

therefore we get

$$||| f ||| \leq \sqrt{a^2 + b^2}$$

on the other hand, we have

$$||| f ||| \geq \frac{|f(a,b)|}{\|(a,b)\|_2} = \frac{\arcsin a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}$$

hence

$$||| f ||| = \sqrt{a^2 + b^2}$$

Let us consider another example, let  $E$  be a real pre-Hilbert space and  $a$  be a fixed non zero vector of  $E$ , let also  $f$  be the linear form of  $E$  defined by

$$f(x) = \langle a, x \rangle \quad (\forall x \in E)$$

(1) Show that  $f$  is continuous and determine

## ( Solution )

According to the Cauchy-Schwarz inequality, we have for all  $x \in E$

$$|f(x)| = |\langle a, x \rangle| \leq \|a\| \|x\|$$

implying that  $f$  is continuous and

$$\|f\| \leq \|a\|$$

On the other hand, we have

$$\|f\| \geq \frac{|f(a)|}{\|a\|} = \frac{|\langle a, a \rangle|}{\|a\|} = \|a\|$$

hence

$$\|f\| = \|a\|$$

**Theorem 3.2.4:**

Let  $E$  be a N.V.S over  $\mathbb{K}$  over  $\mathbb{K}$  ( $\mathbb{R}$  or  $\mathbb{C}$ ), and let  $f$  be a linear form on  $E$ , that is  $f \in E^* = L(E, \mathbb{K})$ . Then  $f$  is continuous if and only if its kernel  $\text{Ker}(f)$  is a closed part of  $E$

*Proof.* ( $\implies$ ) Suppose that  $f : (E, \|\cdot\|) \longrightarrow (\mathbb{K}, |\cdot|)$  is continuous, then the inverse image of any closed subset of  $\mathbb{K}$  is closed in  $E$ . Next,  $\{0\}$  is a finite subset of  $(\mathbb{K}, |\cdot|)$ , which is a Hausdorff space, so  $\{0\}$  is closed in  $(\mathbb{K}, |\cdot|)$ , thus

$$f^{-1}(\{0\}) = \text{Ker}(f) \text{ is closed.}$$

( $\impliedby$ ), we shall prove the contrapositive, that is

$$f \text{ is not continuous} \implies \text{Ker}(f) \text{ is not closed}$$

Suppose that  $f$  is not continuous, so  $f \neq 0_{\mathcal{L}(E, \mathbb{K})}$ , that is there exist  $u \in E$  such that  $f(u) \neq 0$ , so by setting  $v = \frac{1}{f(u)} \cdot u$ , we have  $f(v) = 1$ . Next  $f$  is continuous which means that the quantity

$$\frac{|f|}{\|x\|_E} \quad (x \in E \setminus \{0_E\})$$

is not bounded, from above for every  $n \in \mathbb{N}$ , we can find  $x_n \in E \setminus \{0_E\}$  such that

$$\frac{|f(x_n)|}{\|x_n\|} \geq n$$

that is

$$|f(x_n)| \geq n \|x_n\| > 0$$

next, let us consider the sequence  $(y_n)_{n \in \mathbb{N}}$  of  $E$ , defined by :

$$y_n := v - \frac{1}{f(x_n)} \cdot x_n \quad \forall n \in \mathbb{N}$$

On the other hand, we have for all  $n \in \mathbb{N}$

$$f(y_n) = f(v) - \frac{1}{f(x_n)} \cdot f(x_n) = 1 - 1 = 0$$

implying that  $(y_n)_{n \in \mathbb{N}}$  is a sequence of  $\text{Ker}(f)$ , and we have for all  $n \in \mathbb{N}$  :

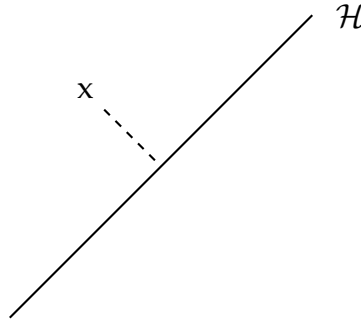
$$\|y_n - v\| = \left\| -\frac{1}{f(x_n)} x_n \right\| = \frac{\|x_n\|}{|f(x_n)|} \leq \frac{1}{n}$$

so

$$\lim_{n \rightarrow \infty} \|y_n - v\| = 0$$

implying that  $(y_n)_{n \in \mathbb{N}}$  converge to  $v$ , but we have  $f(v) = 1 \neq 0$ , so  $v \notin \text{ker}(f)$ , we can see that  $(y_n)_{n \in \mathbb{N}}$  is a sequence of  $\text{Ker}(f)$  which converges to  $v \notin \text{Ker}(f)$ , this implies that  $\text{Ker}(f)$  is not a closed set in  $E$ , as required, this completes the proof.  $\square$

### 3.3 The distance between a vector to a closed hyper plane of a N.V.S



#### Theorem 3.3.1: (Ascoli)

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $f$  be a continuous linear form on  $E$ , next let  $a \in \mathbb{K}$  and

$$\mathcal{H} := \{x \in E : f(x) = a\}$$

then for all  $u \in E$ , we have

$$d(u, H) = \frac{|f(u) - a|}{\|f\|}$$

To prove the above theorem, we use the following lemma, let  $u \in E \setminus H$  be fixed, then for any  $x \in E \setminus \text{Ker}(f)$  can be written as :

$$x = \lambda(u - h)$$



for some  $\lambda \in \mathbb{K}^*$  and some  $h \in H$

*Proof.* we will prove the lemma first, let  $x \in E \setminus \text{Ker}(f)$ , and put  $h := u - \frac{f(u)-a}{f(x)} \cdot x$ . then, we have

$$f(h) = f(u) - \frac{f(u)-a}{f(x)} \cdot f(x) = a$$

implying that  $h \in H$ , finally  $h = u - \frac{f(u)-a}{f(x)} \cdot x$  gives

$$x = \frac{f(x)}{f(u)-a} (u - h)$$

putting

$$\lambda := \frac{f(x)}{f(u)-a} \in \mathbb{K}^*$$

we get  $x = \lambda (u - h)$ , as required □

now after we warmed up, lets prove the theorem

*Proof.* The Ascoli formula is trivial when  $u \in \mathcal{H}$ , so let us prove the Ascoli formula for a fixed  $u \in E \setminus H$ , we have :

$$\begin{aligned} ||| f ||| &:= \sup_{x \in E \setminus \{0_E\}} \frac{|f(x)|}{\|x\|_E} = \sup_{x \in E \setminus \text{Ker}(f)} \frac{|f(x)|}{\|x\|} \\ &= \sup_{\lambda \in \mathbb{K}^*, h \in H} \frac{|f(\lambda(u-h))|}{\|\lambda(u-h)\|} \\ &= \sup_{\lambda \in \mathbb{K}^*, h \in H} \frac{|\lambda| |f(u-h)|}{|\lambda| \|u-h\|} \\ &= \sup_{h \in H} \frac{|f(u)-f(h)|}{\|u-h\|} \\ &= \sup_{h \in H} \frac{|f(u)-a|}{\|u-h\|} \end{aligned}$$

after factoring out the  $|f(u)-a|$  we get

$$\begin{aligned} |f(u)-a| \sup_{h \in H} \frac{1}{\|u-h\|} &= \frac{|f(u)-a|}{\inf_{h \in H} \|u-h\|} \\ &= \frac{|f(u)-a|}{\inf_{h \in H} d(u, h)} \\ &= \frac{|f(u)-a|}{d(u, H)} \end{aligned}$$

hence we get

$$||| f ||| = \frac{|f(u)-a|}{d(u, H)}$$

which gives us the result

$$d(u, H) = \frac{|f(u)-a|}{||| f |||}$$

as required. □

In the euclidean place equipped with orthonormal basis, determine a closed formula for the distance between a point  $(x_0, y_0)$  and a straight line of equation  $ax + by + c = 0$ , where  $a, b, c \in \mathbb{R}$ , where  $(a, b) \neq (0, 0)$

### Solution

we apply the Ascoli formula for  $u = (x_0, y_0) \in \mathbb{R}^2$  and  $H$  the straight line in the questio, so for the linear form  $f$  defined by

$$f(x, y) = ax + by \quad \forall (x, y) \in \mathbb{R}^2$$

doing so we get :

$$\begin{aligned} d((x_0, y_0), H) &= \frac{|f(x_0, y_0) - (-c)|}{|||f|||} \\ &= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \end{aligned}$$

### Theorem 3.3.2: F.Riesz Theorem

A N.V.S (over  $\mathbb{R}$  or  $\mathbb{C}$ ) is finite-dimensional if and only if  $\overline{B}(0_E, 1)$  is compact.

*Proof.* First

$$(\implies)$$

Suppose that  $E$  is finite-dimensional since  $\overline{B}(0_E, 1)$  is both closed and bounded then by some theorem we wrote above, then it's compact as required

$$(\impliedby)$$

Suppose that  $\overline{B}(0_E, 1)$  is a compact part of  $E$  and let us show that  $\dim E < \infty$ , obviously we have

$$\overline{B}(0_E, 1) \subset \bigcup_{x \in \overline{B}(0_E, 1)} B\left(x, \frac{1}{2}\right)$$

Since  $\overline{B}(0_E, 1)$  is compact then

$$\exists n \in \mathbb{N}, \exists x_1, \dots, x_n \in \overline{B}(0_E, 1) : \quad \overline{B}(0_E, 1) \subset \bigcup_{i=1}^n \overline{B}(x_i, 1/2)$$

we are going to show that

$$E = \langle x_1, \dots, x_n \rangle$$

implying that

$$\dim E \leq n < \infty$$

let us set

$$F := \langle x_1, \dots, x_n \rangle$$

and let us show that  $E = F$ , i.e  $E \subset F$ , let  $x \in E$  be arbitrary and let us show that  $x \in F$ , to do so we will first show that for any vector  $y \in F$ , we choose close to  $x$ , that is another  $y' \in F$  which is half closer, in other words  $x$  satisfies the property

$$\forall y \in F, \exists y' \in F : \quad \|x - y'\| \leq \frac{1}{2} \|x - y\|$$

so let  $y \in F$  be arbitrary and let us show the existence of  $y' \in F$  which satisfies the above inequality, if  $y = x$ , it suffices to take  $y' = y = x$  to have

$$\|x - y'\| \leq \frac{1}{2} \|x - y\|$$

Else if  $y \neq x$ , then we have  $\|x - y\| \neq 0$ , now we can define

$$z := \frac{x - y}{\|x - y\|}$$

since we have obviously that  $z \in \overline{B}(0_E, 1)$ , then according to the above there exist  $i \in \{1, \dots, n\}$  such that  $z \in B(x_i, \frac{1}{2})$ , next set

$$y' := \underbrace{y}_{\in F} + \|x - y\| x_i$$

since  $x_i, y \in F$  and  $F$  is a vector subspace of  $E$  then  $y' \in F$ . In addition we have

$$\begin{aligned} x - y' &= \underbrace{x - y}_{\|x - y\| z} - \|x - y\| x_i \\ &= \|x - y\| (z - x_i) \end{aligned}$$

Thus

$$\begin{aligned} \|x - y'\| &= \|x - y\| \underbrace{\|z - x_i\|}_{< 1/2} \quad (z \in B(x_i, 1/2)) \\ &\leq \frac{1}{2} \|x - y\| \end{aligned}$$

so the property is confirmed. Now by re iterating (2) several times starting from  $y = y_0 = 0_E$ , we get

$$\begin{aligned} \forall k \in \mathbb{N}, \exists y_k \in F : \quad \|x - y_k\| &\leq \frac{1}{2^k} \|x - \underbrace{y_0}_{=0_E}\| \\ \forall k \in \mathbb{N}, \exists y_k \in F : \quad \|x - y_k\| &\leq \frac{1}{2^k} \|x\| \rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

showing that the sequence  $(y_k)_{k \in \mathbb{N}}$  of  $F$  that converges to  $x$ , but since  $F$  is closed because it's finite dimensional then  $\lim_{k \rightarrow \infty} y_k = x \in F$ , consequently we have  $E = F$ , thus  $\dim E = \dim F < \infty$ , this completes the proof  $\square$

### corollary 3.3.1: F.Riesz

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , then the following properties are equivalent :

- (i)  $E$  is finite-dimensional
- (ii)  $\overline{B}(0_E, 1)$  is compact
- (iii) The compact parts of  $E$  are exactly its parts which are both closed and bounded
- (iv)  $E$  is locally compact

*Proof.* This equivalence (i)  $\iff$  (iii) is provided theorem 0,8. The implication (i)  $\implies$  (iii) is provided by corollary (2), The two implications (iii)  $\implies$  (ii) and (iii)  $\implies$  (iv) are trivial, To complete the proof it suffices to show that for example the implication

$$(iv) \implies (ii)$$

Suppose that  $E$  is locally compact and show that  $\overline{B}(0_E, 1)$  is locally compact and show that  $\overline{B}(0_E, 1)$  is compact, by hypothesis, the zero vector  $0_E$  of  $E$  has atleast a compact neighborhood  $V$ , so  $\exists r > 0$  such that  $B(0_E, r) \subset V$ , so :

$$\overline{B}(0_E, \frac{r}{2}) \subset B(0_E, r) \subset V$$

The inclusion  $\overline{B}(0_E, \frac{r}{2}) \subset V$ , implies that  $\overline{B}(0_E, \frac{r}{2})$  is compact in  $E$ , since  $\overline{B}(0_E, \frac{r}{2})$  is a closed part of  $E$ , included in the compact part  $V$ , Finally since  $\overline{B}(0_E, 1)$  is the image of closed ball  $\overline{B}(0_E, \frac{r}{2})$  by the continuous map

$$\begin{aligned} f : E &\longrightarrow E \\ x &\longmapsto \frac{2}{r}x \end{aligned}$$

we deduce that  $\overline{B}(0_E, 1)$  is compact, as required this completes the proof  $\square$

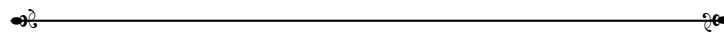


# CONTINUOUS MULTILINEAR MAPPING 4 ON N.V.S

For simplicity we only study the continuous bilinear mapping N.V.S and we give with proofs the generalization of the obtained results to the continuous multilinear mapping on N.V.S let  $\mathbb{K} = \mathbb{R}$  or  $(\mathbb{C})$  and let  $E, F$  and  $G$  be three N.V.S on  $\mathbb{K}$ . The product topology of  $E \times F$  can be induced by several norms on  $E \times F$  one of these norms is defined by

$$\begin{aligned} f : E \times F &\longrightarrow [0, \infty] \\ (x, y) &\longmapsto \max(\|x\|_E, \|y\|_E) \end{aligned}$$

For what all follows, we work with this norm which we denote  $\|\cdot\|_{E \times F}$

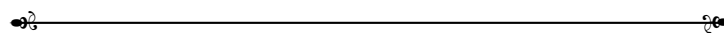


The  $\mathbb{K}$ -vector space of the bilinear mappings from  $E \times F$  to  $G$  is denoted by

$$L(E, F; G) \neq \mathcal{L}(E \times F; G)$$

and the  $\mathbb{K}$ -vector space of the continuous bilinear mappings from  $E \times F$  to  $G$  is denoted :

$$\mathcal{L}(E, F; G)$$



## Theorem 4.0.1: Fundamental

Let  $f \in L(E, F; G)$ , then the following properties are equivalent

- (i)  $f$  is continuous on  $E \times F$
- (ii)  $f$  is continuous at  $(0_E, 0_F)$
- (iii)  $f$  is bounded on  $\overline{B}_E(0_E, 1) \times \overline{B}_F(0_F, 1)$
- (iv)  $f$  is bounded on  $S_E(0_E, 1) \times S_F(0_F, 1)$

(v)  $\exists M > 0$  such that

$$\forall (x, y) \in E \times F : \|f(x, y)\|_G \leq M\|x\|_E\|y\|_F$$

*Proof.* we have to show the following implications :

$$(i) \implies (ii) \implies (iii) \implies (iv) \implies (v) \implies (i)$$

since the implication  $(i) \implies (ii)$  and  $(iii) \implies (iv)$  are obvious, we have just to show the three implications,

$$(ii) \implies (iii) \quad \text{and} \quad (iv) \implies (v) \quad \text{and} \quad (v) \implies (i)$$

$$((ii) \implies (iii))$$

Suppose that  $f$  is continuous at  $(0_E, 0_F)$ , so take  $(\varepsilon = 1)$  there exist  $\mu > 0$  such that

$$\forall (x, y) \in E \times F : \|(x, y) - (0_E, 0_F)\| \leq \mu \implies \|f(x, y) - f(0_E, 0_F)\| \leq 1$$

That is,

$$\forall (x, y) \in E \times F : (\|x\|_E \leq \mu \text{ and } \|y\|_F \leq \mu) \implies \|f(x, y)\|_G \leq 1 \quad (1)$$

Now, let  $(x, y) \in \overline{B_E}(0_E, 1) \times \overline{B_F}(0_F, 1)$  be arbitrary, then we have  $\|\mu x\|_E \leq \mu$  and  $\|\mu y\|_F \leq \mu$ , implying according to (1) that

$$\|f(\mu x, \mu y)\|_G \leq 1 \iff \|f(x, y)\|_G \leq \frac{1}{\mu^2}$$

so, we have

$$\forall (x, y) \in \overline{B_E}(0_E, 1) \times \overline{B_F}(0_F, 1) : \|f(x, y)\|_G \leq \frac{1}{\mu^2}$$

This shows that  $f$  is bounded on

$$\overline{B_E}(0_E, 1) \times \overline{B_F}(0_F, 1)$$

as required.

$$((iv) \implies (v))$$

Suppose that  $f$  is bounded on  $S_E(0_E, 1) \times S_F(0_F, 1)$  this means that there exist  $M > 0$ , such that,

$$\forall (x, y) \in S_E(0_E, 1) \times S_F(0_F, 1) : \|f(x, y)\|_G \leq M \quad (2)$$

Now, let  $(x, y) \in (E \setminus \{0_E\}) \times (F \setminus \{0_F\})$ , then we have

$$\left( \frac{x}{\|x\|_E}, \frac{y}{\|y\|_F} \right) \in S_E(0_E, 1) \times S_F(0_F, 1)$$

implying according to (2) that,

$$\left\| f \left( \frac{x}{\|x\|_E}, \frac{y}{\|y\|_F} \right) \right\|_G \leq M$$

since we have that  $f$  is bilinear we get

$$\|f(x, y)\|_G \leq M\|x\|_E\|y\|_F$$

as required.

( This inequality also holds for  $x = 0_E$  and  $y = 0_F$  )

$$(v) \implies (i)$$

Suppose that there exist  $M > 0$  such that

$$\forall (x, y) \in E \times F \quad \|f(x, y)\|_G \leq M\|x\|_E\|y\|_F$$

and let us show that  $f$  is continuous on  $E \times F$ , that is  $f$  is continuous at every  $(x_0, y_0) \in E \times F$ , so let  $(x_0, y_0) \in E \times F$  be arbitrary and let us show that  $f$  is continuous at  $(x_0, y_0)$ .

we have to show that,

$$\forall \varepsilon > 0, \exists \mu > 0 \text{ s.t. } \forall (x, y) \in E \times F : \|(x, y) - (x_0, y_0)\|_{E \times F} \leq \mu \implies \|f(x, y) - f(x_0, y_0)\|_G \leq \varepsilon \quad (2)$$

let  $\varepsilon > 0$  and take  $\mu = \min \left\{ 1, \frac{\varepsilon}{M(1 + \|x_0\|_E + \|y_0\|_F)} \right\}$ , and let  $(x, y) \in E \times F$  satisfying that,

$$\|(x, y) - (x_0, y_0)\|_{E \times F} \leq \mu$$

that is,

$$\|x - x_0\|_E \leq \mu \quad \text{and} \quad \|y - y_0\|_F \leq \mu$$

then we have,

$$\begin{aligned} \|f(x, y) - f(x_0, y_0)\|_G &= \|f(x, y) - f(x_0, y) + f(x_0, y) - f(x_0, y_0)\|_G \\ &= \text{bilinear } \|f(x - x_0, y) + f(x_0, y - y_0)\|_G \\ &\leq \underbrace{\|f(x - x_0, y)\|_G}_{\leq M\|x - x_0\|_E\|y\|_F} + \underbrace{\|f(x_0, y - y_0)\|_G}_{\leq M\|x_0\|_E\|y - y_0\|_F} \\ &\leq M \underbrace{\|x - x_0\|_E}_{\leq \mu} \|y\|_F + M\|x_0\|_E \underbrace{\|y - y_0\|_F}_{\leq \mu} \\ &\leq \mu M ( \underbrace{\|y\|_F}_{\leq \|y - y_0\|_F + \|y_0\|_F \leq \mu + \|y_0\|_F} + \|x_0\|_E ) \\ &\leq \mu M ( \underbrace{\mu}_{\leq 1} + \|x_0\|_E + \|y_0\|_F ) \\ &\leq \mu M (1 + \|x_0\|_E + \|y_0\|_F) \\ &\leq \varepsilon \end{aligned}$$

Property (3) is then confirmed. Thus  $f$  is continuous on  $E \times F$ , as required.

This completes the proof. □

### Example 01

Let  $(E, \langle \cdot \rangle)$  be a real pre-Hilbert space, prove that the inner product  $\langle \cdot \rangle : E^2 \longrightarrow \mathbb{R}$  is continuous on  $E^2$ .

### Solution 01

$\langle \cdot \rangle$  is bilinear form on  $E^2$ , we have according to the Cauchy schwarz inequality that for all  $x, y \in E$ ,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

showing that according to item (v) to the theorem, that  $\langle \cdot \rangle$  is continuous on  $E^2$ .

### Example 02

Let  $E, F$  and  $G$  be there N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let  $f : E \times F \longrightarrow G$  be a continuous bilinear mapping, show that the mappings  $f(x, \cdot)(x \in E)$  and  $f(\cdot, y)(y \in E)$  defined by,

$$\begin{aligned} f(x, \cdot) : F &\longrightarrow G \\ y &\longmapsto f(x, y) \end{aligned}$$

and

$$\begin{aligned} f(\cdot, y) : E &\longrightarrow G \\ x &\longmapsto f(x, y) \end{aligned}$$

are continuous.

### Solutions 02

Since  $f$  is bilinear then  $f(x, \cdot)(x \in E)$  and  $f(\cdot, y)(y \in F)$  are all linear, next since  $f : E \times F \longrightarrow G$  is bilinear and continuous, then there exist  $M > 0$ , such that for all  $(x, y) \in E \times F$ ,

$$\|f(x, y)\|_G \leq M \|x\|_E \|y\|_F$$

now for  $x \in E$  fixed, we have,

$$\forall y \in F, \|f(x, \cdot)(y)\|_G = \|f(x, y)\|_G \leq \underbrace{(M \cdot \|x\|_E)}_{\text{independent of } y} \|y\|_F$$

implying that  $f(x, \cdot)$  is continuous, we have,

$$\forall x \in E, \|f(\cdot, y)(x)\|_G = \|f(x, y)\|_G \leq \underbrace{(M \cdot \|y\|_F)}_{\text{independent of } x} \|x\|_E$$

implying that  $f(\cdot, y)$  is continuous on  $E$ .

### Question

Is the converse of the result of **Example 02** true?? i.e.,



The partial continuity of a bilinear map with respect to each argument.  $\implies$  ? The continuity.

### Example 03

let,

$$\ell^1 := \left\{ (x_n)_{n \in \mathbb{N}} \text{ real sequence such that } \sum_{n=1}^{\infty} |x_n| \text{ converges} \right\}$$

for  $x = (x_n)_{n \in \mathbb{N}} \in \ell^1$ , we define

$$\|x\|_{\infty} = \sup_{n \in \mathbb{N}} |x_n| \quad (\text{is a norm on } \ell^1)$$

consider,

$$\begin{aligned} f : \ell_1^2 &\longrightarrow \mathbb{R} \\ (x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}}) &\longmapsto \sum_{n=1}^{\infty} x_n y_n \end{aligned}$$

(1) Show that  $f$  is well-defined and that is symmetric and bilinear.

(2) Show that  $f(x, \cdot)$  ( $x \in \ell^1$ ) and  $f(\cdot, y)$  ( $y \in \ell^1$ ) are both continuous on  $\ell^1$ , but  $f$  is not continuous.

### Solution 03

(1) For all  $x, y \in \ell^1$ , we have,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \sum_{n=1}^{\infty} |x_n| |y_n| \leq \underbrace{\left( \sum_{n=1}^{\infty} |x_n| \right)}_{< \infty} \underbrace{\left( \sum_{n=1}^{\infty} |y_n| \right)}_{< \infty} < \infty$$

thus  $\sum_{n=1}^{\infty} |x_n y_n|$  is convergent, that  $\sum_{n=1}^{\infty} x_n y_n$  is absolutely convergent, so convergent. Hence  $f$  is well-defined.

*The symmetry and the bilinearity of  $f$  are obvious.*

(2) Let  $x \in \ell^1$  be fixed and let us show that the linear map  $f(x, \cdot)$  is continuous on  $\ell^1$ , for all  $y \in \ell^1$ , we have,

$$\begin{aligned} |f(x_i)(y)| &= |f(x, y)| = \left| \sum_{n=1}^{\infty} x_n y_n \right| \leq \sum_{n=1}^{\infty} |x_n y_n| \\ &\leq \sum_{n=1}^{\infty} |x_n| |y_n| \\ &\leq \left( \sum_{n=1}^{\infty} |x_n| \right) \|y\|_{\infty} \end{aligned}$$

i.e.

$$|f(x_i)(y)| \leq \sum_{n=1}^{\infty} \overbrace{|x_n|}^M \|y\|_{\infty}$$

Since the series  $\sum_{n=1}^{\infty} |x_n|$  converges, since  $x \in \ell^1$ , then the last inequality show that  $f(x_i)$  is continuous on  $\ell^1$  ( $\forall x \in \ell^1$ ). By the same way or by symmetry, we show that  $f(., y)$  where  $y$  is fixed in  $\ell^1$ , is continuous on  $\ell^1$ .

(3) Now Let us show that  $f$  is not continuous for  $n \in \mathbb{N}$  arbitrary, let,

$$u_n = \begin{cases} 1 & \text{if } 1 \leq n \leq N \\ 0 & \text{if } n > N \end{cases} \quad (\forall n \in \mathbb{N})$$

where

$$v_n = u_n \quad (\forall n \in \mathbb{N})$$

put  $u = (u_n)_{n \in \mathbb{N}}$ ,  $v = (v_n)_{n \in \mathbb{N}}$ .

$$u = (1, 1, \dots, 1, 0, 0, \dots)$$

$$v = (1, 1, \dots, 1, 0, 0, \dots)$$

It's clear that  $u, v \in \ell^1$ , since

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} |v_n| = N < \infty$$

On the other hand, we have,

$$\frac{|f(u, v)|}{\|u\|_{\infty} \cdot \|v\|_{\infty}} \leq \frac{N}{1 \times 1} = N$$

hence,

$$\sup_{x, y \in \ell^1 \setminus \{0_{\ell^1}\}} \frac{|f(x, y)|}{\|x\|_{\infty} \|y\|_{\infty}} = \infty$$

implying that  $f$  is not continuous.

## 4.1 A norm on $\mathcal{L}(E, F; G)$

Let  $E, F$  and  $G$  be three N.V.S over a same field,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  for  $f \in \mathcal{L}(E, F; G)$ , we define  $||| f |||$  by,

$$||| f ||| := \sup_{\substack{x \in E \setminus \{0_E\} \\ y \in F \setminus \{0_F\}}} \frac{\|f(x, y)\|_G}{\|x\|_E \|y\|_F}$$

According to item (v) of theorem 1, we have that

$$||| f ||| \in [0, \infty) \quad \text{i.e. } (||| f ||| < \infty)$$

so  $||| \cdot |||$  constitutes a map from  $\mathcal{L}(E, F; G)$  to  $[0, \infty)$

#### Theorem 4.1.1:

The map  $||| \cdot |||$  defined above is a norm on  $\mathcal{L}(E, F; G)$

*Proof.* Exercise. □

#### Terminology

The norm  $||| \cdot |||$  defined above on  $\mathcal{L}(E, F; G)$  is called the subordinate norm induced by the norm  $\|\cdot\|_E$  of  $E$  and  $\|\cdot\|_F$  of  $F$ , and  $\|\cdot\|_G$  of  $G$ .

we have several variants of the definition of a subordinate norm, including the following,  $\forall f \in \mathcal{L}(E, F; G)$ ,

$$\begin{aligned} ||| f ||| &= \sup_{\substack{x \in \overline{B_E}(0_E, 1) \\ y \in \overline{B_F}(0_F, 1)}} \|f(x, y)\|_G = \sup_{\substack{x \in \overline{B_E}(0_E, 1) \\ y \in \overline{B_F}(0_F, 1)}} \|f(x, y)\|_G \\ &= \sup_{\substack{x \in \overline{S_E}(0_E, 1) \\ y \in \overline{S_F}(0_F, 1)}} \|f(x, y)\|_G \\ &= \inf \{ M > 0 \text{ such that } \|f(x, y)\|_G \leq M \|x\|_E \|y\|_F, \forall x, y \in E, F \} \end{aligned}$$

*Proof.* Exercise! □

we have the following proposition.

#### Theorem 4.1.2:

Let  $E, F$  and  $G$  be three N.V.S over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let  $f \in \mathcal{L}(E, F; G)$  then we have,

(1) If  $f$  is continuous then

$$\forall (x, y) \in E \times F, \|f(x, y)\|_G \leq ||| f ||| \cdot \|x\|_E \cdot \|y\|_F$$

(2) if  $M > 0$  satisfies

$$\|f(x, y)\|_G \leq M \|x\|_E \|y\|_F \quad (\forall (x, y) \in E \times F)$$

then  $f$  is continuous and  $||| f ||| \leq M$

we also have the following propositions,

**Theorem 4.1.3:**

Let  $E, F$  and  $G$  be three N.V.S, over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  suppose that  $E$  and  $F$  are both dimensional, then every bilinear mapping from  $E \times F$  to  $G$  is continuous,

$$( \text{i.e. } \mathcal{L}(E, F; G) = L(E, F; G)$$

*Proof.* (Exercise) □

**Theorem 4.1.4:**

Let  $E, F$  and  $G$  be three N.V.S, over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , suppose that  $G$  is Banach, then the  $\mathbb{K}$ -N.V.S  $\mathcal{L}(E, F; G)$  is Banach.

*Proof.* Exercise □

**Corollary**

Let  $E, F$  be two N.V.S over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , then  $\mathcal{L}(E, F; \mathbb{K})$  is Banach, that space is called the space of continuous bilinear forms on  $E \times F$

## 4.2 An important isomorphism isometric

Let  $E, F$  and  $G$  be three N.V.S over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , then there exist a natural transformation from  $\mathcal{L}(E, \mathcal{L}(F, G))$  to  $\mathcal{L}(E, F; G)$ , which is defined by

$$\begin{aligned} i : \mathcal{L}(E, \mathcal{L}(F, G)) &\longrightarrow \mathcal{L}(E, F; G) \\ f &\longmapsto i(f) : \begin{array}{ccc} E \times F &\longrightarrow & G \\ (x, y) &\longmapsto & i(f)(x, y) = f(x)f(y) \end{array} \end{aligned}$$

Its easy to show that its well defined, linear and bijective with  $i^{-1}$  give :

$$\begin{aligned} i^{-1} : \mathcal{L}(E, F; G) &\longrightarrow \mathcal{L}(E, \mathcal{L}(F, G)) \\ g &\longmapsto i^{-1}(g) : \begin{array}{ccc} E &\longrightarrow & \mathcal{L}(F, G) \\ x &\longmapsto & i^{-1}(g)(x) \end{array} : \begin{array}{ccc} F &\longrightarrow & G \\ y &\longmapsto & i^{-1}(g)(x)(y) = g(x, y) \end{array} \end{aligned}$$

now let us show that  $i$  is an isometry, with respect to the natural norms defined on  $\mathcal{L}(E, F; G)$  and  $\mathcal{L}(E, \mathcal{L}(F, G))$ , for all  $f \in \mathcal{L}(E, \mathcal{L}(F, G))$ , we have

$$\begin{aligned}
 \|i(f)\|_{\mathcal{L}(E, F; G)} &= \sup_{\substack{x \in E \setminus 0_E \\ y \in F \setminus 0_F}} \frac{\|i(f)(x, y)\|_G}{\|x\|_E \|y\|_F} = \sup_{\substack{x \in E \setminus 0_E \\ y \in F \setminus 0_F}} \frac{\|f(x)(y)\|_G}{\|x\|_E \|y\|_F} \\
 &= \sup_{x \in E \setminus \{0_E\}} \frac{1}{\|x\|_E} \sup_{y \in F \setminus \{0_F\}} \frac{\|f(x, y)\|_G}{\|y\|_F} \\
 &= \sup_{x \in E \setminus \{0_E\}} \frac{1}{\|x\|_E} \|f(x)\|_{\mathcal{L}(F, G)} \\
 &= \|f\|_{\mathcal{L}(E, \mathcal{L}(F, G))}
 \end{aligned}$$

that is  $i$  is an isometry, because of the isomorphism isometric  $i$  between  $\mathcal{L}(E, \mathcal{L}(F, G))$  and  $\mathcal{L}(E, F; G)$ , we often identify  $\mathcal{L}(E, \mathcal{L}(F, G))$  to  $\mathcal{L}(E, F; G)$ , This is used in particular in differential calculus on N.V.S (for defining second derivative)

### 4.3 An introduction to differential calculus in N.V.S

Let  $E$  and  $F$  be two N.V.S over the a same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $U$  be an open subset of  $E$  and  $a \in U$ . Finally, let  $f : U \rightarrow F$  be a map

#### Definition 4.3.1:

We say that  $f$  is differentiable at  $a$  if there exist  $g \in \mathcal{L}(E, F)$  so that we have in the neighborhood of  $a$

$$\|f(x) - f(a) - g(x - a)\|_F = o(\|x - a\|_E)$$

#### Remark

- (1) If  $f$  is differentiable at  $a$  then  $f$  is continuous at  $a$ . Indeed, by letting  $x \rightarrow a$ , we obtain since ( $g$  is continuous at  $0_E$ ), that  $\lim_{x \rightarrow a} f(x) = f(a)$ , showing that  $f$  is continuous at  $a$ .
- (2) If  $f$  is idifferentiable at  $a$  then the continuous linear mapping  $g$  is unique.

*Proof.* Let  $g_1, g_2 \in \mathcal{L}(E, F)$ , each of them satisfies

$$\|f(x) - f(a) - g_1(x - a)\|_F = o(\|x - a\|_E)$$

$$\|f(x) - f(a) - g_2(x - a)\|_F = o(\|x - a\|_E)$$

when  $x$  is in the neighborhood of  $a$ , so for all  $h \in E$  ( in the neighborhood of  $0_E$  , we have

$$\begin{aligned} \|(g_1 - g_2)(h)\|_F &= \|g_1(h) - g_2(h)\|_F \\ &= \|(f(a+h) - f(a) - g_2(h)) - (f(a+h) - f(a) - g_1(h))\| \\ &\leq \underbrace{\|f(a+h) - f(a) - g_2(h)\|_F}_{o(\|h\|_E)} + \underbrace{\|f(a+h) - f(a) - g_1(h)\|_F}_{o(\|h\|_E)} = o(\|h\|_E) \end{aligned}$$

Thus  $\|(g_1 - g_2)(h)\|_F = o(\|h\|_E)$ , in other words

$$\lim_{\|h\|_E \rightarrow 0} \frac{\|(g_1 - g_2)(h)\|_F}{\|h\|_E} = 0$$

now let  $x \in E \setminus \{0_E\}$  be arbitrary, by taking  $h = \varepsilon x$  and  $(\varepsilon \rightarrow^> 0)$ , we get

$$\lim_{\varepsilon \rightarrow^> 0} \frac{\|(g_1 - g_2)(\varepsilon x)\|_F}{\|\varepsilon x\|_E} = 0$$

thus we see

$$\frac{\|(g_1 - g_2)(x)\|_F}{\|x\|_E} = 0$$

thus we see that

$$g_1(x) = g_2(x) \quad (\forall x \in E \setminus \{0_E\})$$

which remains true for  $x = 0_E$ , hence  $g_1(x) = g_2(x)$  for all  $x \in E$ , therefore  $g_1 = g_2$ , by the uniqueness of  $g$  is then proved.  $\square$

#### Definition 4.3.2:

If  $f$  is differentiable at  $a$  then the continuous linear mapping  $g$  satisfying

$$\|f(x) - f(a) - g(x - a)\|_F = o(\|x - a\|_E)$$

is called

*The derivative of  $f$  at  $a$ , and it's denoted  $f'(a)$*

## 4.4 Relationship with the classical case $E = F = \mathbb{R}$

If  $E = F = \mathbb{R}$ , and  $U$  is an open subset of  $\mathbb{R}$ ,  $f : U \rightarrow \mathbb{R}$ , and  $a \in U$  then the classical definition of the differentiability states that

$$f \text{ is differentiable at } a \text{ if } \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \text{ exists (i.e. } \in \mathbb{R})$$

So if its the case and we let

$$l := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

we desire that

$$\lim_{x \rightarrow a} \left( \frac{f(x) - f(a)}{x - a} - l \right) = 0$$

that is

$$\lim_{x \rightarrow a} \frac{f(x) - f(a) - l(x - a)}{x - a} = 0$$

therefore we see

$$|f(x) - f(a) - l(x - a)| = o(|x - a|) \quad \text{when } x \rightarrow a$$

so hence

$$\begin{aligned} g : \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto lx \end{aligned}$$

$$\in \mathcal{L}(\mathbb{R}, \mathbb{R})$$

satisfies, so in the sense of Definition 2,  $f$  is differnetiable at  $a$  and

$$f'(a) = \left[ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & lx \end{array} \right]$$

By identifying the homothety of center 0 and ratio  $l$  to  $l$ , we obtain the equivalence between the classical case ( $E = F = \mathbb{R}$ ), and the general case on N.V.S

$$\begin{aligned} : \mathbb{R} &\longrightarrow \mathcal{L}(\mathbb{R}, \mathbb{R}) \\ l &\longmapsto \mathcal{H}(0, l) : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & lx \end{array} \end{aligned}$$

is an isomorphism isometric.

In fact, we identify  $\mathcal{L}(\mathbb{R}, \mathbb{R})$  with  $\mathbb{R}$ .

#### Definition 4.4.1:

We say that  $f$  is differentiable in  $U$ , if its differnetiable at every point of  $U$ .

- If  $f$  is differentiable in  $U$  then it's derivative is the map  $f'$  defined by :

$$\begin{aligned} f' : U &\longrightarrow \mathcal{L}(E, F) \\ a &\longmapsto f'(a) \end{aligned}$$

In the particular case  $E = \mathbb{R}$ , we can identify  $\mathcal{L}(E, F) = \mathcal{L}(\mathbb{R}, F)$  to  $F$ , so we obtain  $f' : U \longrightarrow F$  as in the classical case  $E = F = \mathbb{R}$ .

## 4.5 The Second Derivative

Let  $E$  and  $F$  be two N.V.S, and  $U$  be an open subset of  $E$ , and  $f : U \rightarrow F$  suppose that  $f$  is differentiable in  $U$  and let  $f' : U \rightarrow \mathcal{L}(E, F)$  be it's derivative so we can ask if  $f'$  is differentiable in  $U$

### Definition 4.5.1:

We say that  $f$  is twice differentiable at  $a \in U$  if  $f'$  is differentiable at  $a$ . In this case we denote  $f''(a)$  the derivative of  $f'$  at  $a$ , so

$$f''(a) \in \mathcal{L}(E, \mathcal{L}(E, F))$$

called the second derivative of  $f$  at  $a$ .

### Definition 4.5.2:

We say that  $f$  is twice differentiable in  $U$  if its twice differentiable at every point of  $U$ . In such a case, the second derivative of  $f$  is the map.

$$\begin{aligned} f'' : U &\longrightarrow \mathcal{L}(E, \mathcal{L}(E, F)) \\ a &\longmapsto f''(a) \end{aligned}$$

Then we often consider  $f''(a)(a \in U)$ , as an element of  $\mathcal{L}(E, E; F)$  that is  $f''(a)$  is a continuous bilinear map from  $E \times E$  to  $F$ .

## 4.6 Generalization of the multilinear mappings

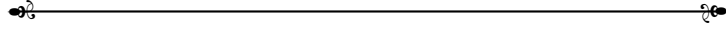
Let  $n \in \mathbb{N}$ , and  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $E_1, \dots, E_n$  and  $G$  be N.V.S over  $\mathbb{K}$ , the topological product space  $E_1 \times E_2 \times \dots \times E_n$ , can be represented by several norms, the more simple is perhaps  $\|\cdot\|_\infty$  defined by :

$$\begin{aligned} \|\cdot\|_\infty : E_1 \times E_2 \times \dots \times E_n &\longrightarrow [0, \infty) \\ (x_1, \dots, x_n) &\longmapsto \max(\|x_1\|_{E_1}, \dots, \|x_n\|_{E_n}) \end{aligned}$$





Let  $\mathbb{K}$ -Vector space of the multilinear mappings from  $E_1 \times E_2 \dots \times E_n$  to  $G$  is denoted by  $L(E_1, \dots, E_n; G)$  and the  $\mathbb{K}$ -Vector space of the continuous multilinear mappings from  $E_1, \dots, E_n$  to  $G$  is denoted by  $\mathcal{L}(E_1, \dots, E_n; G)$ .



#### Theorem 4.6.1: Fundamental

Let  $f \in \mathcal{L}(E_1, \dots, E_n)$ , Then the following properties are equivalent :

(i)  $f$  is continuous on  $E_1 \times \dots \times E_n$

(ii)  $f$  is continuous on  $(0_{E_1}, \dots, 0_{E_n})$

(iii)  $f$  is bounded on

$$\overline{B_{E_1}(0_{E_1}, 1)} \times \overline{B_{E_2}(0_{E_2}, 1)} \times \dots \times \overline{B_{E_n}(0_{E_n}, 1)}$$

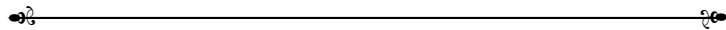
(iv)  $f$  is bounded on

$$S_{E_1}(0_{E_1}, 1) \times \dots \times S_{E_n}(0_{E_n}, 1)$$

(v)  $\exists M > 0$  such that

$$\forall (x_1, \dots, x_n) \in E_1 \times \dots \times E_n \quad \|f(x_1, \dots, x_n)\|_G \leq M \|x_1\|_{E_1} \times \dots \times \|x_n\|_{E_n}$$

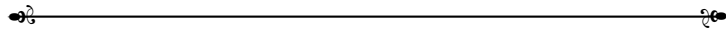
*Proof.* The same as that corresponding to the case where  $n = 2$  □



A norm on  $\overline{\mathcal{L}(E_1, \dots, E_n; G)}$  : for  $f \in \mathcal{L}(E_1, \dots, E_n; G)$ , we define  $||| f |||$  by :

$$||| f ||| := \sup_{x_1, \dots, x_n \in E_1 \setminus \{0_E\}, \dots, E_n \setminus \{0_{E_n}\}} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\|_{E_1} \dots \|x_n\|_{E_n}}$$

according to item (v) for the previous theorem, we have that  $||| f ||| \in [0, \infty)$ , i.e  $||| f |||$  is a non negative real number, so  $||| \cdot |||$  constitutes a map from  $\mathcal{L}(E_1, \dots, E_n; G)$  to  $[0, \infty)$  :



The map  $||| \cdot |||$  defined above is a norm on  $\mathcal{L}(E_1, \dots, E_n; G)$ , it's called the subordinate norm induced by the norms  $\|\cdot\|_{E_1}$  of  $E_1$ ,  $\|\cdot\|_{E_2}$  of  $E_2$ ,  $\dots$ ,  $\|\cdot\|_n$  of  $E_n$ , and  $\|\cdot\|_G$  of  $G$

*Proof.* Exercise! □

**Remark**

All the proposition of  $\mathcal{L}(E_1, \dots, E_n; G)$  seen previously for the case  $n = 2$  are easily and naturally generalizable for every  $n$

An important example, let  $n \in \mathbb{N}$  and take  $E_1 = E_2 = \dots = E_n = \mathbb{R}^n$  and  $G = \mathbb{R}$ , and we get

$$\begin{aligned} \det : \mathbb{R}^n \times \dots \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\longmapsto \det(x_1, \dots, x_n) \end{aligned}$$

It's known that for determinant is multilinear.

Next, since  $\mathbb{R}^n$  is finite-dimensional then  $\det$  is continuous let us equip  $\mathbb{R}^n$  with its euclidean norm  $\|\cdot\|_2$  and  $\mathbb{R}$  with the absolute value  $|\cdot|$ .

Then we propose to determine  $||| \det |||$ , by definition we have

$$||| \det ||| := \sup_{x_1, \dots, x_n \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}} \frac{|\det(x_1, \dots, x_n)|}{\|x_1\|_2 \dots \|x_n\|_2}$$

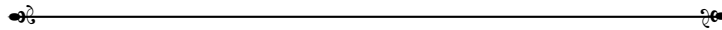
so by taking in particular  $(x_1, \dots, x_n) = (e_1, \dots, e_n)$ , the canonical basis of  $\mathbb{R}^n$ , we have that,

$$||| \det ||| \geq \frac{|\det(e_1, \dots, e_n)|}{\|e_1\|_2 \dots \|e_n\|_2} = \frac{1}{1 \times 1 \dots 1} = 1$$

so

$$||| \det ||| \geq 1$$

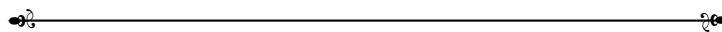
To conclude to the exact value of  $||| \det |||$ , we use the following theorem

**Theorem 4.6.2: Hadamard's inequality**

For every  $x_1, \dots, x_n \in \mathbb{R}^n$ , we have

$$|\det(x_1, \dots, x_n)| \leq \|x_1\|_2 \dots \|x_n\|_2$$

Besides, the inequality is attained if and only if  $x_1, \dots, x_n$  are pairwise orthogonal with respect to the usual inner product of  $\mathbb{R}^n$



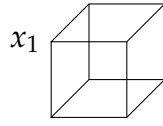
Hadamard's inequality implies immediately that  $||| \det ||| = 1$

## 4.7 The geometric sense of Hadamard's inequality

The geometric sense of Hadamard's inequality is the following

*In the Euclidean space of  $n$  dimension, the volume of the parallelepiped spanned by the  $n$  linearly independent vectors  $x_1, \dots, x_n$  of lengths  $l_1, \dots, l_n$ , is at most equal to  $l_1 \cdot l_2 \cdot \dots \cdot l_n$ .*

In addition, this volume is optimal (i.e. Equal to  $l_1 \cdot l_2 \cdot \dots \cdot l_n$ ), if and only if the vectors  $x_1, \dots, x_n$  are linearly independent



*Proof.* If  $x_1, \dots, x_n$  are linearly dependent, the Hadamard inequality is trivial, suppose for the sequel that  $x_1, \dots, x_n$  are linearly independent, in other words  $(x_1, \dots, x_n)$  constitutes a basis of  $\mathbb{R}^n$ , We use the Gram-Schmidt process to transform  $(x_1, \dots, x_n)$  to an orthogonal basis  $(y_1, \dots, y_n)$  of  $\mathbb{R}^n$ .

By The Gram-Schmidt, there exist  $\alpha_{ij} \in \mathbb{R}$  ( $1 \leq j < i \leq n$ ) such that the vectors  $y_1, \dots, y_n$  of  $\mathbb{R}^n$  defined by

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 + \alpha_{21}x_1 \\ y_3 = x_3 + \alpha_{31}x_1 + \alpha_{32}x_2 \\ \vdots \\ y_n = x_n + \alpha_{n1}x_1 + \alpha_{n2}x_2 + \dots + \alpha_{n,n-1}x_{n-1} \end{cases}$$

are pairwise orthogonal, by putting the condition in addition for  $i, j \in \{1, \dots, n\}$

$$\alpha_{i,j} = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \quad \text{and} \quad T = (\alpha_{i,j})_{1 \leq i,j \leq n} \in \mathcal{M}(\mathbb{R})$$

Which is a linear transformation with diagonal entries all equal to 1, as its non singular, specifically the system can be rewritten as

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

which gives

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = T^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$T^{-1}$  as  $(T)$  is lower triangular with diagonal entries all equal to 1, now let

$$(\beta_{i,j})_{1 \leq i,j \leq n} = T^{-1} \quad \beta_{i,j} = \begin{cases} 1 & i = j \\ 0 & j < i \end{cases}$$

and we have

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 + \beta_{21}y_1 \\ x_3 = y_3 + \beta_{31}y_1 + \beta_{32}y_2 \\ \vdots \\ x_n = y_n + \beta_{n1}y_1 + \dots + \beta_{n,n-1}y_{n-1} \end{cases}$$

Now, since the determinant is an alternating multi linear form then we desire from the above system, that

$$\det(x_1, \dots, x_n) = \det(y_1, \dots, y_n)$$

Next, by the pythagorean theorem, we have according to the system, the fact that  $y'_i$ 's are all pairwise orthogonal, we get that :

$$\begin{cases} \|x_1\|^2 = \|y_1\|^2 \\ \|x_2\|^2 = \|y_2\|^2 + \beta_{21}^2 \|y_1\|^2 \geq \|y_2\|^2 \\ \|x_3\|^2 = \|y_3\|^2 + \beta_{31}^2 \|y_1\|^2 + \beta_{32}^2 \|y_2\|^2 \geq \|y_3\|^2 \\ \vdots \\ \|x_n\|^2 = \|y_n\|^2 + \beta_{n1}^2 \|y_1\|^2 + \dots + \beta_{n,n-1}^2 \|y_{n-1}\|^2 \geq \|y_n\|^2 \end{cases}$$

hence we get

$$\|x_1\|^2 \cdot \|x_2\|^2 \cdot \dots \cdot \|x_n\|^2 \geq \|y_1\|^2 \cdot \|y_2\|^2 \cdot \dots \cdot \|y_n\|^2$$

that is

$$\|x_1\| \cdot \|x_2\| \cdot \dots \cdot \|x_n\| \geq \|y_1\| \cdot \|y_2\| \cdot \dots \cdot \|y_n\|$$

now, we are goin to show that

$$|\det(y_1, \dots, y_n)| = \|y_1\| \cdot \|y_2\| \cdot \dots \cdot \|y_n\|$$

Let  $A = (y_1 | y_2 | \dots | y_n) (\in \mathcal{M}_n(\mathbb{R}))$ , so

$$A^T = \begin{pmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{pmatrix}$$

hence

$$A^T A = \begin{pmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{pmatrix} (y_1 | y_2 | \dots | y_n)$$

which equals

$$\begin{pmatrix} \langle y_1, y_1 \rangle & \dots & \langle y_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle y_n, y_1 \rangle & \dots & \langle y_n, y_n \rangle \end{pmatrix} = \begin{pmatrix} \|y_1\|^2 & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & \|y_n\|^2 \end{pmatrix}$$

so

$$A^T A = \text{diag}(\|y_1\|^2, \dots, \|y_n\|^2)$$

then by taking the determinants

$$(\det A)^2 = \|y_1\|^2 \dots \|y_n\|^2$$

then

$$|\det(A)| = \|y_1\| \dots \|y_n\|$$

i.e

$$\det(y_1, \dots, y_n) = \|y_1\| \dots \|y_n\|$$

confirming the formula, now we have according to 1 ,2 and 3

$$\begin{aligned} |\det(x_1, \dots, x_n)| &= |\det(y_1, \dots, y_n)| \\ &= \|y_1\| \|y_2\| \dots \|y_n\| \\ &= \|x_1\| \cdot \|x_2\| \dots \|x_n\| \end{aligned}$$

as required, in addition the equality

$$|\det(x_1, \dots, x_n)| = \|x_1\| \|x_2\| \dots \|x_n\|$$

hold if and only if

$$\|y_1\| \dots \|y_n\| = \|x_1\| \dots \|x_n\|$$

but this equivalent according to 3 to  $\|x_i\| = \|y_i\|$  for all  $i$ , which is equivalent to  $\beta_{i,j} = 0$  for all  $i > j$ , that is  $T = I_n$  which is equivalent to

$$(y_1, \dots, y_n) = (x_1, \dots, x_n)$$

which holds if and only if  $x_1, \dots, x_n$  are pairwise orthogonal, the proof is complete  $\square$

## 4.8 Series in N.V.S

### Definition 4.8.1:

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $(u_n)_{n \in \mathbb{N}}$ .

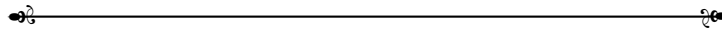
The infinite sum  $\sum_{k=0}^{\infty} u_k$ , is called the series of  $E$  with general term  $u_k$ . For  $n \in \mathbb{N}$  fixed, the finite sum  $S_n = \sum_{k=1}^n u_k$  is called the  $n^{\text{th}}$  partial sum (or the partial sum of rank  $n$ ) of the series  $\sum_{k=1}^{\infty} u_k$ , we say that the series  $\sum_{k=1}^{\infty} u_k$  converges in  $E$  if the sequence  $(S_n)_{n \in \mathbb{N}}$  converges in  $E$ . In such a case, we call the limit  $S$  of  $(S_n)_{n \in \mathbb{N}}$ , the sum of the series  $\sum_{k=1}^{\infty} u_k$ , and we write,

$$\sum_{k=1}^{\infty} u_k = S$$

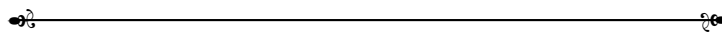
- Besides for  $n \in \mathbb{N}$ ,  $R_n := S - S_n$  is called the  $n^{\text{th}}$  remainder or the remainder of rank  $n$  of the series  $\sum_{k=1}^{\infty} u_k$ , and we often write,

$$R_n = \sum_{k=n+1}^{\infty} u_k$$

- If a series of  $E$  is not convergent, we say that it is divergent



The concept of series is rather important in a Banach space, then in an arbitrary N.V.S



### Definition 4.8.2: Cauchy Criterion

Let  $E$  be a Banach N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let  $\sum_{k=1}^{\infty} u_k$  be a series of  $E$ . Then  $\sum_{k=1}^{\infty} u_k$  is convergent if and only if it satisfies

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : \quad p > q \geq N \implies \left\| \sum_{k=q+1}^p u_k \right\| \leq \varepsilon$$

*Proof.* Let  $(S_n)_{n \in \mathbb{N}}$  be the sequence of partial sums of  $\sum_{k=1}^{\infty} u_k$ , (i.e.  $S_n = \sum_{k=1}^n u_k, \forall n \in \mathbb{N}$ ), so we have,

$$\sum_{k=1}^{\infty} u_k \text{ is convergent} \iff (S_n)_{n \in \mathbb{N}} \text{ is convergent}$$

$$\iff (S_n)_{n \in \mathbb{N}} \text{ is Cauchy ( Since } E \text{ is Banach )}$$

$$\iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N} : p > q \geq N \implies \|S_p - S_q\| < \varepsilon$$

$$\iff p > q \geq N \implies \left\| \sum_{k=q+1}^p u_k \right\| < \varepsilon$$

as required. □

**Definition 4.8.3:**

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , a series  $\sum_{k=1}^{\infty} u_k$  of  $E$  is said to be *normally convergent* if the real series (with nonnegative terms)  $\sum_{k=1}^{\infty} \|u_k\|$  converges. (in  $\mathbb{R}$ )

**Theorem 4.8.1:**

Let  $E$  be a Banach N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , if a series  $\sum_{k=1}^{\infty} u_k$  of  $E$  is *normally convergent* then its convergent and we have in this case :

$$\left\| \sum_{k=1}^{\infty} u_k \right\| \leq \sum_{k=1}^{\infty} \|u_k\|$$

*Proof.* Let  $\sum_{k=1}^{\infty} u_k$  be a series of  $E$ , suppose that  $\sum_{k=1}^{\infty} u_k$  is normally convergent (i.e. the real series  $\sum_{k=1}^{\infty} \|u_k\|$  converges), and let us prove that  $\sum_{k=1}^{\infty} u_k$  is convergent for all  $p, q \in \mathbb{N}$ , with  $p > q$  we have,

$$0 \leq \left\| \sum_{k=q+1}^p u_k \right\| \stackrel{I.I}{\leq} \sum_{k=q+1}^q \|u_k\| \quad (4.1)$$

but since  $\sum_{k=1}^{\infty} \|u_k\|$  is assumed convergent in  $\mathbb{R}$  then it satisfies the cauchy criterion i.e.,

$$\lim_{p,q \rightarrow \infty} \sum_{k=q+1}^p \|u_k\| = 0$$

Consequently by applying the squeeze theorem in (1), we get,

$$\lim_{p,q \rightarrow \infty} \left\| \sum_{k=q+1}^p u_k \right\| = 0$$

implying since  $E$  is banach, that the series  $\sum_{k=1}^{\infty} u_k$  is convergent, as required.

Now let us prove the inequality of the theorem in the case when the series  $\sum_{k=1}^{\infty} u_k$  is normally convergent then for all  $n \in \mathbb{N}$ , we have,

$$\left\| \sum_{k=1}^n u_k \right\| \leq \sum_{k=1}^n \|u_k\|$$

by letting  $n \rightarrow \infty$ , and using the continuity of  $\|\cdot\|$ , we get,

$$\left\| \sum_{k=1}^{\infty} u_k \right\| \leq \sum_{k=1}^{\infty} \|u_k\|$$

as required, This completes the proof □

**- An Important Example (Exponential of an operator of a Banach Space)**

Let  $E$  be a Banach N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $f \in \mathcal{L}(E) := \mathcal{L}(E, E)$  consider the series  $\sum_{n=0}^{\infty} \frac{f^n}{n!}$



in  $(\mathcal{L}(E))$ , then we have for all  $n \in \mathbb{N}_0$ , Note that  $f^n = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}$

$$\| \frac{f^n}{n!} \| = \frac{1}{n!} \| f^n \| \leq \frac{1}{n!} \| f \|^n$$

Since the real series  $\sum_{k=1}^{\infty} \frac{1}{k!} \| f \|^k$  converges to  $\exp(\| f \|)$  then the real series  $\sum_{k=1}^{\infty} \| \frac{f^k}{k!} \|$  is also convergent, that is the series  $\sum_{k=1}^{\infty} \frac{f^k}{k!}$  (of  $\mathcal{L}(E)$ ) is normally convergent but since  $\mathcal{L}(E)$  is Banach, (because  $E$  is Banach) then according to the theorem, The series  $\sum_{k=1}^{\infty} \frac{f^k}{k!}$  is convergent in  $\mathcal{L}(E)$ , and we have

$$\| \sum_{k=1}^{\infty} \frac{f^k}{k!} \| \leq e^{\| f \|} \quad (4.2)$$

#### Definition 4.8.4:

In the above situation (i.e. if  $E$  is a Banach space and  $f \in \mathcal{L}(E)$ ) the sum of the convergent series  $\sum_{k=1}^{\infty} \frac{f^k}{k!}$  is called the exponential of the operator  $f$  and denoted by  $e^f$  or  $\exp(f)$ , so we have according to (2),

$$\| e^f \| \leq e^{\| f \|} \quad (\forall f \in \mathcal{L}(E)) \quad (4.3)$$

#### Remark

If  $E$  is a Banach space, and  $f, g \in \mathcal{L}(E)$ , the equality of operators,

$$e^{f+g} = e^f \circ e^g$$

is in general false, but it becomes true when  $f$  and  $g$  commute.

$$\begin{aligned} e^x \cdot e^y &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} \frac{y^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \left( \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n = e^{x+y} \end{aligned}$$

In particular, we have for all  $f \in \mathcal{L}(E)$ ,

$$e^f \circ e^{-f} = e^{0_{\mathcal{L}(E)}} = id_E$$

$$e^{-f} \circ e^f = e^{0_{\mathcal{L}(E)}} = id_E$$

Consequently, for every  $f \in \mathcal{L}(E)$ , the operator  $e^f (\in \mathcal{L}(E))$  is invertible (i.e.,  $e^f \in GL(E)$ ), and  $(e^f)^{-1} = e^{-f}$ .

- **A particular case :** let  $n \in \mathbb{N}$ , we take  $E = \mathbb{K}^n$  where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and we verify identity  $\mathcal{L}(E) = L(E)$  to  $\mathcal{M}_n(\mathbb{K})$ .

Since  $E$  is finite dimensional then its Banach so, we can define the exponential of a matrix  $A$  of  $\mathcal{M}_n(\mathbb{K})$  by,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \in \mathcal{M}_n(\mathbb{K})$$

in general  $e^{A+B} \neq e^A \cdot e^B$ , for  $A, B \in \mathcal{M}_n(\mathbb{K})$ , but if  $AB = BA$ , then we have  $e^{A+B} = e^A \cdot e^B = e^B \cdot e^A$ .

### Exercise 01 :

Let  $n \in \mathbb{N}$ , and let  $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , set  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ .

(1) Show that

$$e^D = \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}) = \begin{pmatrix} e^{\lambda_1} & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & e^{\lambda_n} \end{pmatrix}$$

*Proof.*

$$\begin{aligned} e^D &= \sum_{k=0}^{\infty} \frac{D^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & \lambda_n^k \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & \dots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \dots & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \end{pmatrix} \end{aligned}$$

□

### Exercise 02 :

Let  $n \in \mathbb{N}$ , and  $P \in GL_n(\mathbb{K})$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $A \in \mathcal{M}_n(\mathbb{K})$ .

(1) Show that :

$$\exp(P^{-1}AP) = P^{-1} \exp(A)P$$

*Proof.*

$$\begin{aligned} \exp(P^{-1}AP) &= \sum_{k=0}^{\infty} \frac{(P^{-1}AP)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} (P^{-1}A^kP) \\ &= P^{-1} \left( \sum_{k=0}^{\infty} \frac{A^k}{k!} \right) P = P^{-1}e^A P \end{aligned}$$

□

#### Theorem 4.8.2:

Let  $n \in \mathbb{N}$ , and  $x_0 \in \mathbb{R}^n$ , and  $A \in \mathcal{M}_n(\mathbb{R})$  and denote by  $X$  a function of  $t$  from  $\mathbb{R}$  to  $\mathbb{R}^n$ , by

$$\begin{aligned} X : \mathbb{R} &\longrightarrow \mathbb{R}^n \\ t &\longmapsto X(t) \end{aligned}$$

then the solution of the linear differential system with initial condition

$$\begin{cases} X(0) = x_0 \\ X'(t) = A \cdot X(t) \end{cases} \quad (4.4)$$

is the following :

$$X(t) = e^{tA} x_0$$

*Proof.* Put  $Y(t) = e^{-tA}X(t)$ , then

$$Y'(t) = -Ae^{-tA}X(t) + e^{-tA}X'(t)$$

so  $X$  is a solution of (5), we have

$$\begin{cases} X(0) = x_0 \\ X'(t) = AX(t) \end{cases} \iff \begin{cases} Y(0) = x_0 \\ Y'(t) = 0_{\mathbb{R}^n} \end{cases} \iff Y(t) = x_0 \quad (\forall t \in \mathbb{R})$$

we deduce  $X(t) = e^{tA}x_0$

□

- **Problem :** (How to compute  $e^A$  in general ?)

- **The Solution :**

For  $n \in \mathbb{N}$ , and  $A \in \mathcal{M}_n(\mathbb{K})$ ,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , to compute  $e^A$ , we use the Dunford decomposition of  $A$ , we write  $A$  as,

$$A = U + N \quad (U, N \in \mathcal{M}_n(\mathbb{K}))$$

with,

- $U$  is diagonalizable in other words there exist  $P \in GL(\mathbb{K})$  and  $D \in \mathcal{M}_n(\mathbb{K})$  diagonal such that  $U = PDP^{-1}$ .
- $N$  is nilpotent i.e. there exist  $k \in \mathbb{N}$  such that.  $N^k = 0$
- $U$  commutes with  $N$  i.e.  $UN = NU$ .

So, since  $U$  and  $N$  commute with  $N$ , we have

$$e^A = e^{U+N} = e^U \cdot e^N$$

but we have

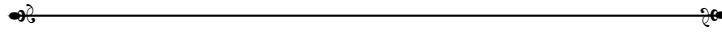
$$e^U = e^{PDP^{-1}} = Pe^DP^{-1}$$

and

$$e^N = \sum_{l=0}^{\infty} \frac{N^l}{l!} = \frac{N^l}{l!} = \sum_{l=0}^{k-1} \frac{N^l}{l!}$$

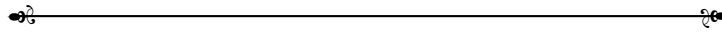
(since  $N^l = 0$  for  $l \geq k$ ), hence we obtain the closed form of  $e^A$ .

Note that the Dunford decomposition of  $A$  can be obtained by using the jordan form  $A$ .



By the same way, we can define  $\sin(f)$ ,  $\cos(f)$ ,  $\sinh(f)$ , etcetera, when  $f$  is continuous, linear operator of a Banach space

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$



- **Exercise :** (Important) Let  $E$  be a Banach space, we denote by  $\mathcal{GL}(E)$ , the set of endomorphisms of  $E$  which are continuous, invertible, and for which  $g^{-1}$  is continuous, we have

$$\mathcal{GL}(E) \subset \mathcal{L}(E)$$

(1) Let  $f \in \mathcal{L}(E)$  satisfying  $\|f\| < 1$

(a) Show that  $(id_E + f)$  and  $(id_E - f)$  are in  $\mathcal{GL}(E)$

(2) Deduce that  $\mathcal{GL}(E)$  is an open subset of  $\mathcal{L}(E)$

(3) Show that the map

$$\begin{array}{ccc} \mathcal{GL}(E) & \xrightarrow{\phi} & \mathcal{GL}(E) \\ f & \mapsto & f^{-1} \end{array}$$

is continuous

- **Solution :**

(1) First, the continuity and the linearity of  $(id_E + f)$  and  $(id_E - f)$  are obvious, are obvious next consider the series

$$\sum_{n=0}^{\infty} f^n \text{ of } \mathcal{L}(E) \text{ We have}$$

for all  $n \in \mathbb{N}_0$ ,

$$||| f^n ||| \leq ||| f |||^n$$

Since  $||| f ||| < 1$  then the real geometric series  $\sum_{n=0}^{\infty} ||| f |||^n$  is convergent, thus the real series  $\sum_{n=0}^{\infty} ||| f^n |||$  is also convergence, in other words the series  $\sum_{n=0}^{\infty} f^n$  of  $\mathcal{L}(E)$  is normally convergent, since  $\mathcal{L}(E)$  is Banach because  $E$  is banach, then  $\sum_{n=0}^{\infty} f^n$  is convergent in  $\mathcal{L}(E)$ , set

$$g = \sum_{n=0}^{\infty} f^n \in \mathcal{L}(E)$$

we have for all  $n \in \mathbb{N}_0$ ,

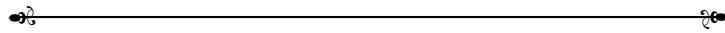
$$(id_E - f) \circ \sum_{n=0}^N f^n = \sum_{n=0}^N (f^n - f^{n+1}) = id_E - f^{N+1}$$

By letting  $N \rightarrow \infty$ , we get,

$$(id_E - f) \circ g = id_E$$

we prove by the same way that  $g \circ (id_E - f) = id_E$ , thus  $(id_E - f)$  is invertible and  $(id_E - f)^{-1} = g = \sum_{n=0}^{\infty} f^n \in \mathcal{L}(E)$ , thus,

$$(id_E - f) \in \mathcal{GL}(E)$$



(motivation  $(1 - x) \times \frac{1}{1-x} = 1$ )



by replacing  $f$  by  $-f$ , we find that  $(id_E + f)$  is also invertible is also invertible and

$$(id_E + f)^{-1} = \sum_{n=0}^{\infty} (-f)^n = \sum_{n=0}^{\infty} (-1)^n f^n \in \mathcal{L}(E)$$

Consequently  $(id_E + f) \in \mathcal{GL}(E)$

(2)  $\mathcal{GL}(E)$  is an open subset of  $\mathcal{L}(E)$  ??

we have to show that  $\mathcal{GL}(E)$  is a neighborhood of all if elements so, let  $f_0 \in \mathcal{GL}(E)$  arbitrary and let us show that  $\exists r > 0$  such that  $\mathcal{B}_{\mathcal{L}(E)}(f_0, \frac{1}{|||f_0^{-1}|||})$ .

That is  $f \in \mathcal{L}(E)$  and  $||| f - f_0 ||| < \frac{1}{||| f_0^{-1} |||}$

let us show that  $f \in \mathcal{GL}(E)$ , we have

$$||| f_0^{-1} \circ f - id_E ||| = ||| f_0^{-1} \circ (f - f_0) ||| \leq ||| f_0^{-1} ||| \cdot \underbrace{||| f - f_0 |||}_{< \frac{1}{||| f_0^{-1} |||}} < 1$$

thus according to the result of Question (1), we have

$$(f_0^{-1} \circ f - id_E) + id_E = f_0^{-1} \circ f \in \mathcal{GL}(E)$$

Thus,

$$f = f_0 \circ (f_0^{-1} \circ f) \in \mathcal{GL}(E)$$

as required, this confirms the inclusion, so  $\mathcal{GL}(E)$  is a neighborhood of any  $f_0 \in \mathcal{GL}(E)$ , so  $\mathcal{GL}(E)$  is an open subset of  $\mathcal{L}(E)$ .

$$\begin{aligned} \mathcal{GL}(\mathbb{R}^n) &= GL(\mathbb{R}^n) \simeq GL_n(\mathbb{R}) \\ &= \{A \in \mathcal{M}_n(\mathbb{R}) : \det(A) \neq 0\} \\ &= \{A \in \mathcal{M}_n(\mathbb{R}) : \det(A) \in (-\infty, 0) \cup (0, \infty)\} \\ &= \det^{-1}((-\infty, 0) \cup (0, \infty)) \end{aligned}$$

(3)

$$\begin{aligned} \mathcal{GL}(E) &\xrightarrow{\phi} \mathcal{GL}(E) \\ f &\longmapsto f^{-1} \end{aligned}$$

is continuous ??, let us show the continuity of  $\phi$  at some  $f_0 \in \mathcal{GL}(E)$  arbitrary, for all  $f \in \mathcal{GL}(E)$ , such that

$$||| f - f_0 ||| < \frac{1}{||| f_0^{-1} |||}$$

we have,

$$\begin{aligned} f^{-1} - f_0^{-1} &= f_0^{-1} \circ (f_0 \circ f^{-1} - id_E) \\ &= f_0^{-1} \circ (f_0 \circ f^{-1} - id_E) \\ &= f_0^{-1} \circ \left( (f \circ f_0^{-1})^{-1} - id_E \right) \\ &= f_0^{-1} \circ \left( (f - f_0 + f_0) \circ f_0^{-1} \right)^{-1} - id_E \\ &= f_0^{-1} \circ \left[ \left( (f - f_0) \circ f_0^{-1} + id_E \right)^{-1} - id_E \right] \end{aligned}$$

From Question (1),

$$f_0^{-1} \circ \left[ \sum_{n=0}^{\infty} (-1)^n \left( (f - f_0) \circ f_0^{-1} \right)^n - id_E \right]$$

Hence

$$||| f^{-1} - f_0^{-1} ||| \leq ||| f_0^{-1} ||| \sum_{n=0}^{\infty} ||| (-1)^n \left( (f - f_0) \circ f_0^{-1} \right)^n |||$$

Hence

$$\begin{aligned} ||| f^{-1} - f_0^{-1} ||| &\leq ||| f_0^{-1} ||| \sum_{n=1}^{\infty} ||| (-1)^n ((f - f_0) \circ f_0^{-1})^n ||| \\ &\leq ||| f_0^{-1} ||| \cdot \sum_{n=0}^{\infty} ||| f - f_0 |||^n ||| f_0^{-1} |||^n \end{aligned}$$

Thus,

$$||| f^{-1} - f_0^{-1} ||| \leq ||| f_0^{-1} ||| \cdot \left[ \frac{||| f - f_0 ||| \cdot ||| f_0^{-1} |||}{1 - ||| f - f_0 ||| \cdot ||| f_0^{-1} |||} \right]$$

This shows that,

$$\lim_{f \rightarrow f_0} ||| f^{-1} - f_0^{-1} ||| = 0$$

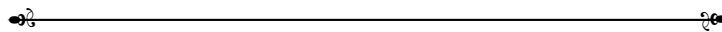
That is  $f^{-1} \rightarrow f_0^{-1}$ ,  $f \rightarrow f_0$ , hence consequently  $\phi$  is continuous

#### Definition 4.8.5:

Let  $E$  be a N.V.S, A series  $\sum_{n=1}^{\infty} x_n$  of  $E$  is said to be unconditionally convergent if for every permutation  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ , the series  $\sum_{n=0}^{\infty} x_{\sigma(n)}$  converges to the same sum (in particular, the series  $\sum_{n=0}^{\infty} x_n$  converges).



☞ Recall Let  $E$  be a N.V.S  $\sum_{n=0}^{\infty} x_n$  is unconditionally convergent if and only if  $\forall \sigma : \mathbb{N} \rightarrow \mathbb{N}$  a bijective, the series  $\sum_{n=0}^{\infty} x_{\sigma(n)}$  is convergent to the same sum.



#### Example

In  $\mathbb{R}$ , the series,

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

which is convergent to  $\ln(2)$ , is conditionally convergent, consider the permutation of  $\mathbb{N}$ , that

is given by,

$$(1, 2, 3, 5, 4, 7, 9, 11, 6, \dots)$$

therefore it transforms to

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$$

transform it to a divergent series also the permutation,

$$(1, 2, 4, 3, 6, 8, \dots) = (n, 2n, 2n + 2)$$

transforms the series to,

$$\begin{aligned} \frac{1}{1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots &= \sum_{n=0}^{\infty} \left( \frac{1}{2n+1} - \frac{1}{2(2n+1)} - \frac{1}{2(2n+2)} \right) \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{2(2n+1)} - \frac{1}{2(2n+2)} \right) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= \frac{1}{2} \left[ \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots \right] \\ &= \frac{1}{2} \ln(2) \neq \ln(2) \end{aligned}$$

#### Theorem 4.8.3: The Riemann rearrangement

If a real series is conditionally convergent then its terms can be rearranged so that the new series converges to an arbitrary real number, or diverges

#### Theorem 4.8.4:

Let  $E$  be a Banach space, then any normally convergent series of  $E$  is unconditionally convergent

*Proof.* Let  $\sum_{n=0}^{\infty} x_n$  be a normally convergent series of  $E$  (i.e. the real series  $\sum_{n=0}^{\infty} \|x_n\|$  is convergent), then for the permutation  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  we have for all  $n \in \mathbb{N}$ , we will consider the series,

$$\begin{aligned} \sum_{n=0}^N \|x_{\sigma(n)}\| &= \sum_{k \in \{\sigma(0), \dots, \sigma(N)\}} \|x_k\| \leq \sum_{k=1}^{\max(\sigma(i)), 1 \leq i \leq N} \|x_k\| \\ &\leq \sum_{k=1}^{\infty} \|x_k\| \end{aligned}$$



This implies that the nonnegative real series  $\sum_{n=0}^{\infty} \|x_{\sigma(n)}\|$  is convergent, that is the series  $\sum_{n=0}^{\infty} x_{\sigma(n)}$  of  $E$  is normally convergent, since  $E$  is Banach so we conclude that the series  $\sum_{n=0}^{\infty} x_{\sigma(n)}$  is convergent, as required.

Now let us show that  $\sum_{n=0}^{\infty} x_{\sigma(n)}$  has the same sum as  $\sum_{n=0}^{\infty} x_n$  let us define for all  $n \in \mathbb{N}$ .

$$a_n = \begin{cases} \min(A = \{1, 2, \dots, n\} \Delta \{\sigma(1), \dots, \sigma(n)\}) & \text{if } A \neq \emptyset \\ n & \text{if } A = \emptyset \end{cases}$$

and let us admit for the moment that

$$\lim_{n \rightarrow \infty} a_n = \infty$$

then we have for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} \left\| \sum_{n=1}^N x_{\sigma(n)} - \sum_{n=1}^N x_n \right\| &= \left\| \sum_{i \in \{\sigma(1), \dots, \sigma(N)\}} x_i - \sum_{i \in \{1, \dots, N\}} x_i \right\| \\ &= \left\| \sum_{i \in \{\sigma(1), \dots, \sigma(N)\} \setminus \{1, 2, \dots, N\}} x_i - \sum_{i \in \{1, \dots, N\} \setminus \{\sigma(1), \dots, \sigma(N)\}} x_i \right\| \\ &\leq \sum_{i \in \{\sigma(1), \dots, \sigma(N)\} \setminus \{1, \dots, N\}} \|x_i\| + \sum_{i \in \{1, \dots, N\} \setminus \{\sigma(1), \dots, \sigma(N)\}} \|x_i\| \\ &= \sum_{i \in \{1, \dots, N\} \Delta \{\sigma(1), \dots, \sigma(N)\}} \|x_i\| \\ &\leq \sum_{i \geq a_N} \|x_i\| \end{aligned}$$

Then by letting  $N \rightarrow \infty$ , we get since  $\sum_{i=1}^{\infty} \|x_i\|$  converge and  $a_N \rightarrow \infty$  as  $N \rightarrow \infty$ , we get,

$$\sum_{n=0}^{\infty} x_{\sigma(n)} = \sum_{n=0}^{\infty} x_n$$

as required.

Now, it remains to prove that  $\lim_{n \rightarrow \infty} a_n = \infty$ , this is equivalent to show that for all  $k \in \mathbb{N}$ , there exist  $N_k$  such that  $\forall n \in \mathbb{N} : n \geq N_k \implies a_n \geq k$ , now let  $k \in \mathbb{N}$ , and take  $N_k := \max \{1, \dots, k, \sigma^{-1}(1), \dots, \sigma^{-1}(k)\}$ , then for any  $n \in \mathbb{N}$ , we have in one hand:

$$N \geq N_k \implies N \geq k \quad (\text{since } N_k \geq k) \implies \{\sigma(1), \dots, \sigma(N)\} \setminus \{1, \dots, N\} \subset \{k+1, k+2, \dots\}$$

On the other hand,

$$N \geq N_k \implies \sigma\sigma^{-1}(1), \sigma\sigma^{-1}(2), \dots, \sigma\sigma^{-1}(k) \leq N_k \leq N$$

which implies,

$$\begin{aligned} &\implies \sigma^{-1}(1), \sigma^{-1}(2), \dots, \sigma^{-1}(k) \in \{1, \dots, N\} \\ &\implies 1, \dots, k \in \{\sigma(1), \dots, \sigma(N)\} \\ &\implies \{1, \dots, N\} \setminus \{\sigma(1), \dots, \sigma(N)\} \subset \{k+1, k+2, \dots\} \end{aligned}$$

so from the two hands, we get  $\forall n \in \mathbb{N}$ ,

$$\begin{aligned} N \geq N_k &\implies \{1, \dots, N\} \Delta \{\sigma(1), \dots, \sigma(N)\} \subset \{k+1, k+2, \dots\} \\ &\implies a_N \geq k \quad (\text{also true for } a_N = N, \text{ since } N \geq N_k \geq k) \end{aligned}$$

as required. Thus  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ . which completes the proof.  $\square$

## 4.9 The summability of general series

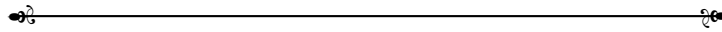
We call a general series any infinite sum of element of a N.V.S, that is a  $\sum_{i \in I} x_i$ , where  $I$  is infinite.

### Definition 4.9.1: Generalize the unconditional convergence

Let  $E$  be a N.V.S. A general series  $\sum_{i \in I} x_i$  of  $E$  is said to be summable with sum  $S \in E$ , if it satisfies the following property,

$\forall \varepsilon > 0, \exists I_\varepsilon \subset I$  finite, s.t.  $\forall J$  a finite subset of  $I$ , we have

$$I_\varepsilon \subset J \implies \left\| \sum_{i \in J} x_i - S \right\| < \varepsilon$$



Let  $E$  be a N.V.S. If a general series  $\sum_{i \in I} x_i$  is summable then it has a unique sum,

*Proof.* Let  $\sum_{i \in I} x_i$  be a general summable series with sums  $S$  and  $S'$  ( $S, S' \in E$ ), and let us prove that  $S = S'$ . Let  $\varepsilon > 0$  arbitrary, By definition  $\exists I_\varepsilon \subset I$ , with  $I_\varepsilon$  finite, such that,

$\forall J$  a finite subset of  $I$ , we have

$$I_\varepsilon \subset J \implies \left\| \sum_{i \in J} x_i - S \right\| < \frac{\varepsilon}{2}$$

Similarly,  $\exists I_\varepsilon \subset I$ , with  $I_\varepsilon$  finite, such that

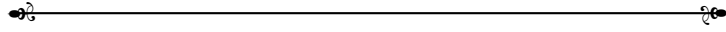
$\forall J$  a finite subset of  $I$ , we have,

$$I_\varepsilon \subset J \implies \left\| \sum_{i \in J} x_i - S' \right\| < \frac{\varepsilon}{2}$$

So, by taking  $J = I_\varepsilon \cup I'_\varepsilon$  which is a finite subset of  $I$  and contains both  $I_\varepsilon$  and  $I'_\varepsilon$ , we have,  $\left\| \sum_{i \in J} x_i - S \right\| < \frac{\varepsilon}{2}$  and  $\left\| \sum_{i \in J} x_i - S' \right\| < \frac{\varepsilon}{2}$ . Hence,

$$\begin{aligned} \|S - S'\| &= \left\| S - \sum_{i \in J} x_i + \sum_{i \in J} x_i - S' \right\| \\ &\leq \left\| S - \sum_{i \in J} x_i \right\| + \left\| \sum_{i \in J} x_i - S' \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad (= \varepsilon) \end{aligned}$$

Thus  $\|S - S'\| < \varepsilon$  for all  $\varepsilon > 0$ , implying that  $S = S'$ , as required.  $\square$

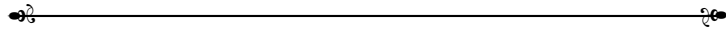


### The Cauchy Criterion

Let  $E$  be a N.V.S. We say that a general series  $\sum_{i \in I} x_i$  satisfies the Cauchy Criterion if,

$\forall \varepsilon > 0, \exists I_\varepsilon \subset I$ , with  $I_\varepsilon$  finite, s.t.  $\forall J$  a finite subset of  $I$ , disjoint with  $I_\varepsilon$ , we have

$$\left\| \sum_{i \in J} x_i \right\| < \varepsilon$$



$\sum_{i \in \mathbb{N}} x_i$  is Cauchy if and only if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ s.t. } \forall p, q \in \mathbb{N} : p > q > N_\varepsilon \implies \left\| \sum_{i=q+1}^p x_i \right\| < \varepsilon$$

which implies that

$$\forall \varepsilon > 0, \exists I_\varepsilon = \{1, \dots, N_\varepsilon\} \subset \mathbb{N} \text{ finite s.t. } \forall J = \{q+1, \dots, p\} \subset \mathbb{N} \text{ finite}$$

and

$$J \cap I_\varepsilon = \emptyset \implies \left\| \sum_{i \in J} x_i \right\| < \varepsilon$$

#### Theorem 4.9.1:

Let  $E$  be a Banach Space. Then every general series  $\sum_{i \in I} x_i$  of  $E$  which satisfies the Cauchy criterion is summable.

*Proof.* Let  $\sum_{i \in I} x_i$  be a general series of  $E$ . Which satisfies the Cauchy criterion then for all  $n \in \mathbb{N}$ , there exist  $I_n \subset I$  with  $I_n$  finite, such that  $\forall J$  a finite subset of  $I$ , with  $J \cap I_n = \emptyset$ , we have  $\left\| \sum_{i \in J} x_i \right\| <$

$\frac{1}{n}$ , let us define for all  $n \in \mathbb{N}$ ,

$$S_n := \sum_{i \in I_1 \cup I_2 \cup \dots \cup I_n} x_i \quad (\text{a finite sum})$$

$(S_n)_{n \in \mathbb{N}}$  is a sequence of  $E$

we have for any  $p, q \in \mathbb{N}$ , with  $p > q$ ,

$$\|S_p - S_q\| = \left\| \sum_{i \in \underbrace{I_1 \cup \dots \cup I_p \setminus I_1 \cup \dots \cup I_q}_{\text{disjoint } (I_p, I_q)}} x_i \right\| < \frac{1}{q} \rightarrow 0 \text{ as } q \rightarrow \infty$$

Thus  $(S_n)_{n \in \mathbb{N}}$  is Cauchy. Since  $E$  is Banach then  $(S_n)_{n \in \mathbb{N}}$  is convergent. Let  $S = \lim_{n \rightarrow \infty} S_n \in E$ , and let us show that the general series  $\sum_{i \in I} x_i$  is sommable with sum  $S$   $\square$

#### Theorem 4.9.2:

Let  $E$  be a Banach space. Then every general series  $\sum_{i \in I} x_i$  of  $E$  which satisfies Cauchy criterion is summable.

*Proof.* Let  $\sum_{i \in I} x_i$  be a general series  $E$  which satisfies the Cauchy criterion, Then for all  $n \in \mathbb{N}$ ,  $\exists I_n \subset I$ , with  $I_n$  finite, such that  $\forall J$  a finite subset of  $I$ , with  $J \cap I_n = \emptyset$ , we have,

$$\left\| \sum_{i \in J} x_i \right\| < \frac{1}{n}$$

Let us define for all  $n \in \mathbb{N}$ ,

$$S_n := \sum_{i \in I_1 \cup I_2 \cup \dots \cup I_n} x_i (\in E)$$

Clearly,  $(S_n)_{n \in \mathbb{N}}$  is a sequence of  $E$ .

we have for any  $p, q \in \mathbb{N}$ , with  $p > q$ ,

$$\|S_p - S_q\| = \left\| \sum_{i \in I_1 \cup \dots \cup I_p} x_i - \sum_{i \in I_1 \cup \dots \cup I_q} x_i \right\| = \left\| \sum_{i \in \underbrace{(I_1 \cup \dots \cup I_p) \setminus (I_1 \cup \dots \cup I_q)}_{\text{finite, disjoint with } I_q}} x_i \right\| < \frac{1}{q}$$

Hence  $\lim_{p, q \rightarrow \infty} \|S_p - S_q\| = 0$ , implying that  $(S_n)_{n \in \mathbb{N}}$  is Cauchy since  $E$  is Banach then  $(S_n)_{n \in \mathbb{N}}$  is convergent. Let  $S := \lim_{n \rightarrow \infty} S_n \in E$ , and let us show that the general series  $\sum_{i \in I} x_i$  is summable with sum  $S$   $\forall \varepsilon > 0, \exists I_\varepsilon \subset I$ , with  $I_\varepsilon$  finite,  $\forall J \subset I$ ,  $J$  finite

$$I_\varepsilon \subset J \implies \left\| \sum_{i \in J} x_i - S \right\| < \varepsilon$$

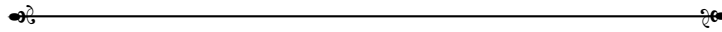
Let  $\varepsilon > 0$  arbitrary then since  $S_n \rightarrow S$  in  $E$  and  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$  in  $\mathbb{R}$ , then  $\exists n_0 \in \mathbb{N}$ , such that,

$$\|S_{n_0} - S\| < \frac{\varepsilon}{2} \text{ and } \frac{1}{n_0} < \frac{\varepsilon}{2}$$

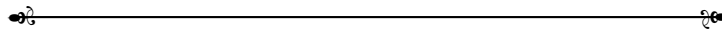
take  $I_\varepsilon = I_1 \cup \dots \cup I_{n_0}$ , For any subset  $J$  of  $I$  which is finite and contains  $I_\varepsilon$ , we have,

$$\begin{aligned} \left\| \sum_{i \in J} x_i - S \right\| &= \left\| \sum_{i \in I_1 \cup \dots \cup I_{n_0}} x_i + \sum_{i \in J \setminus (I_1 \cup \dots \cup I_{n_0})} x_i - S \right\| = \|S_{n_0} - S + \sum_{i \in J \setminus (I_1 \cup \dots \cup I_{n_0})} x_i\| \\ &\leq \underbrace{\|S_{n_0} - S\|}_{< \varepsilon/2} + \left\| \sum_{i \in J \setminus (I_1 \cup \dots \cup I_{n_0})} x_i \right\| \\ &< \varepsilon \end{aligned}$$

Thus  $\sum_{i \in I} x_i$  is summable with sum  $S$ , hence the proof is complete.  $\square$



Let  $E$  be N.V.S prove that if a general series of  $E$  is summable then it satisfies the Cauchy criterion



#### Definition 4.9.2:

Let  $E$  be a N.V.S and  $\sum_{i \in I} x_i$  be a general series of  $E$ , We say that  $\sum_{i \in I} x_i$  is normally summable if the real general series  $\sum_{i \in I} \|x_i\|$  is summable.

#### Theorem 4.9.3:

Let  $E$  be a Banach Space and  $\sum_{i \in I} x_i$  be a general series, if  $\sum_{i \in I} x_i$  is normally summable then its summable and we have

$$\left\| \sum_{i \in I} x_i \right\| \leq \sum_{i \in I} \|x_i\|$$

*Proof.* Suppose that  $\sum_{i \in I} x_i$  is normally summable, that is, the real general series  $\sum_{i \in I} \|x_i\|$  is summable, Thus  $\sum_{i \in I} \|x_i\|$  satisfies the Cauchy criterion (see Previous exercise).

It follows that  $\sum_{i \in I} x_i$  also satisfies the Cauchy criterion  $\forall \varepsilon > 0, \exists I_\varepsilon \subset I$  finite,  $\forall J \subset I$ ,  $J$  finite,  $J \cap I_\varepsilon = \emptyset$

$$\implies \sum_{i \in J} \|x_i\| < \varepsilon \implies \left\| \sum_{i \in J} x_i \right\| < \varepsilon$$

Thus according to the previous theorem, The general series  $\sum_{i \in I} x_i$  is summable as required.

Now, let us prove the inequality of the theorem, Let  $S := \sum_{i \in I} x_i$  and  $S' := \sum_{i \in I} \|x_i\| \in \mathbb{R}$ , we have to show that  $\|S\| \leq S'$ , For all  $\varepsilon > 0$ , there exist  $I_\varepsilon \subset I$ , with  $I_\varepsilon$  finite such that  $\forall J \subset I$ , such that  $J$  finite,

$$I_\varepsilon \subset J \implies \left\| \sum_{i \in J} x_i - S \right\| < \varepsilon$$

Similarly, for all  $\varepsilon > 0$ , there exist  $I'_\varepsilon \subset I$ , with  $I'_\varepsilon$  finite, such that  $\forall J \subset I$ , with  $J$  finite, with  $J$  finite,

$$I'_\varepsilon \subset J \implies \left\| \sum_{i \in J} \|x_i\| - S' \right\| < \varepsilon$$

For  $\varepsilon > 0$ , by taking  $J = I_\varepsilon \cup I'_\varepsilon$ , we have

$$\begin{aligned} \left\| \sum_{i \in J} x_i - S \right\| &< \varepsilon \\ \left\| \sum_{i \in J} \|x_i\| - S' \right\| &< \varepsilon \end{aligned}$$

Hence, using the above inequalitys, we have,

$$\begin{aligned} \|S\| &= \left\| S - \sum_{i \in J} x_i + \sum_{i \in J} x_i \right\| \\ &\leq \underbrace{\left\| S - \sum_{i \in J} x_i \right\|}_{< \varepsilon} + \underbrace{\left\| \sum_{i \in J} x_i \right\|}_{< S' + \varepsilon} \\ &< S' + 2\varepsilon \end{aligned}$$

Thus  $\|S\| < S' + 2\varepsilon$  for all  $\varepsilon > 0$ , by taking  $\varepsilon \rightarrow 0^+$  gives  $\|S\| \leq S'$ , as required. this completes the proof.  $\square$

The following theorem shows that every generla series of a N.V.S, can always be reduced to an ordinary series i.e  $I = \mathbb{N}$ .

**Theorem 4.9.4:**

Let  $E$  be a N.V.S and  $\sum_{i \in I} x_i$ , be a general series of  $E$ , Suppose that  $\sum_{i \in I} x_i$  is summable. then the set

$$I' := \{i \in I : x_i \neq 0_E\}$$

is at most countable. In addition, the general series  $\sum_{i \in I'} x_i$  is summable and we have

$$\sum_{i \in I'} x_i = \sum_{i \in I} x_i$$

*Proof.* for all  $n \in \mathbb{N}$ , put

$$I'_n := \left\{ i \in I : \|x_i\| > \frac{1}{n} \right\}$$

So, we have that

$$\begin{aligned} \bigcup_{n \in \mathbb{N}} I'_n &= \left\{ i \in I : \exists n \in \mathbb{N} \text{ such that } \|x_i\| > \frac{1}{n} \right\} \\ &= \{i \in I : x_i \neq 0_E\} = I' \end{aligned}$$

$$I = \bigcup_{n \in \mathbb{N}} I'_n$$

Next, let us prove that  $I'_n$  is finite for every  $n \in \mathbb{N}$ . So let  $n \in \mathbb{N}$ , since  $\sum_{i \in I} x_i$  is assumed to be summable then it satisfies the Cauchy criterion, So  $\exists I_n \subset I$ , with  $I_n$  finite, such that  $\forall J \subset I$ , with  $J$  finite,

$$J \cap I_n = \emptyset \implies \left\| \sum_{i \in J} x_i \right\| < \frac{1}{n}$$

(Cauchy criterion for  $\varepsilon = \frac{1}{n}$ )

In Particular, for every  $j \in I$ , we have for  $J = \{j\}$ ,

$$\forall j \in I, \{j\} \cap I_n = \emptyset \implies \|x_j\| < \frac{1}{n}$$

Equivalently,

$$\begin{aligned} \forall j \in I, j \notin I_n &\implies \|x_j\| < \frac{1}{n} \\ &\implies j \notin I'_n \end{aligned}$$

$$\forall j \in I, j \notin I_n \implies j \notin I'_n$$

By the contrapositive we have,

$$\forall j \in I, j \in I'_n \implies j \in I_n$$

Thus,

$$I'_n \subset I_n$$

Since  $I_n$  is finite, we derive that  $I'_n$  is finite.

Consequently according to the above,  $I'$  is a countable union of finite sets, implying that  $I'$  is at most countable, as required.

Now, let us prove the second part of the theorem, set  $S := \sum_{i \in I} x_i$  then  $\forall \varepsilon > 0, \exists I_\varepsilon \subset I$ , with  $I_\varepsilon$  finite,  $\forall J \subset I$ , with  $J$  finite, we have,

$$I_\varepsilon \subset J \implies \left\| \sum_{i \in J} x_i - S \right\| < \varepsilon$$

Let  $\varepsilon > 0$  be arbitrary, by putting  $I'_\varepsilon = I_\varepsilon \cap I'$ , which is finite since  $I_\varepsilon$  is finite and  $\subset I'$ , we have for any finite subset  $J'$  of  $I'$ , containing  $I'_\varepsilon$ ,

$$\begin{aligned} \sum_{i \in J'} x_i &= \sum_{i \in J' \cup I'_\varepsilon} x_i && \text{since } I'_\varepsilon \subset J' \\ &= \sum_{i \in (J' \cup I_\varepsilon) \cap I'} x_i \\ &= \sum_{i \in J' \cup I'_\varepsilon} x_i \end{aligned}$$

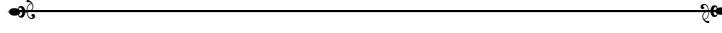
But since  $J' \cup I_\varepsilon$  is finite and contains  $I_\varepsilon$  it fololws that

$$\left\| \sum_{i \in J'} x_i - S \right\| = \left\| \sum_{i \in J' \cup I_\varepsilon} x_i - S \right\| < \varepsilon$$

This concludes that the general series  $\sum_{i \in I'} x_i$  is summable and we have

$$\sum_{i \in I'} x_i = \sum_{i \in I} x_i$$

This completes the proof. □



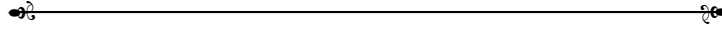
#### Theorem 4.9.5:

Let  $E$  be a N.V.S and  $\sum_{i \in I} x_i$  be a general series of  $E$ . Suppose that  $\sum_{i \in I} x_i$  is summable. then for all other set  $L$  equinumerous, with  $I$  (I forgot about 2 words here) all bijection  $\sigma : L \rightarrow I$  the general series  $\sum_{l \in L} x_{\sigma(l)}$  is summable and we have

$$\sum_{l \in L} x_{\sigma(l)} = \sum_{i \in I} x_i$$

*Proof.* Set  $S := \sum_{i \in I} x_i$  and let  $\varepsilon > 0$ , be arbitrary, then  $\exists I_\varepsilon \subset I$ , with  $I_\varepsilon$  finite, such that for all  $J \subset I$ , with  $J$  finite, and

$$I_\varepsilon \subset J \implies \left\| \sum_{i \in J} x_i - S \right\| < \varepsilon$$



Does?  $\exists L_\varepsilon \subset L$ , with  $L_\varepsilon$  finite such that  $\forall K \subset L$ , with  $K$  finite, and,

$$\underbrace{L_\varepsilon \subset K} \implies \left\| \sum_{l \in K} x_{\sigma(l)} - S \right\| < \varepsilon$$

I didnt see this clearly from the table, could be wrong

Define  $L_\varepsilon = \sigma^{-1}(I_\varepsilon)$  since  $I_\varepsilon \subset I$  then,  $L_\varepsilon \subset L$ ,  $L_\varepsilon$  is finite ( Since  $I_\varepsilon$  is finite and  $\sigma$  is bijective), Next for all  $K \subset L$ , with  $K$  is finite, and  $L_\varepsilon \subset K$ , and we have

$$\sum_{l \in K} x_{\sigma(l)} = \sum_{i \in \sigma(K)} x_i \quad (i = \sigma(l))$$

Since  $L_\varepsilon \subset K$ , then  $I_\varepsilon = \sigma(L_\varepsilon) \subset \sigma(K)$ , implying that

$$\left\| \sum_{i \in \sigma(K)} x_i - S \right\| < \varepsilon \text{ i.e. } \left\| \sum_{l \in K} x_{\sigma(l)} - S \right\| < \varepsilon$$

this shows that the general series  $\sum_{l \in L} x_{\sigma(l)}$  is summable and we have

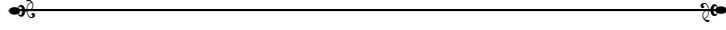
$$\sum_{l \in L} x_{\sigma(l)} = \sum_{i \in I} x_i$$

the proposition is proved. □





*Corollary* , Let  $E$  be a N.V.S. Then every summable general series can be transformed either into a finite sum or into an arbitrary series



*Proof.* Let  $\sum_{i \in I} x_i$  be a summable general series of  $E$ . Let

$$I' := \{i \in I : x_i \neq 0_E\}$$

Its proved previously that  $I'$  is at most countable and that

$$\sum_{i \in I} x_i = \sum_{i \in I'} x_i$$

We distinguish two cases.

1. If  $I'$  is finite, in this case  $\sum_{i \in I} x_i$  is transformed to the finite sum  $\sum_{i \in I'} x_i$
2. If  $I'$  is countably infinite. In this case  $\exists \sigma : \mathbb{N} \rightarrow I'$  a bijection. So, by the previous proposition, we have

$$\sum_{i \in I'} x_i = \sum_{l \in \mathbb{N}} x_{\sigma(l)} = \sum_{l=1}^{\infty} x_{\sigma(l)}$$

which is an ordinary series of  $E$ .

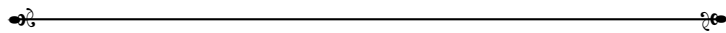
The corollary is proved. □

*Exercise : (Summation by Packet)* Let  $E$  be a Banach Space. then  $\sum_{i \in I} x_i$  be asumable general series of  $E$ , and  $(I_\alpha)_{\alpha \in A}$  be a partition of  $I$ ,

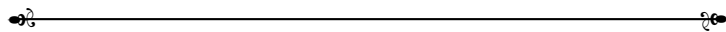
1. Show that for every  $\alpha \in A$ , the general  $\sum_{i \in I_\alpha} x_i$  is summable
2. Show that the general series

$$\sum_{\alpha \in A} \left( \sum_{i \in I_\alpha} x_i \right)$$

is summable with sum equal to  $\sum_{i \in I} x_i$ .

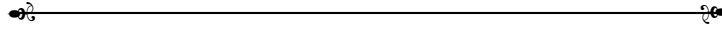


**Remainder :** (*Separable spaces*) A topological space is said to be separable if it contains a countable dense subset.



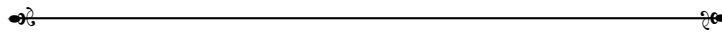
**Example**

$\mathbb{R}$  equipped with its usual topology is separable since  $Q \subset \mathbb{R}$  is countable dense subset of  $\mathbb{R}$ , is a countable dense subset of  $\mathbb{R}$ . More generally,  $\mathbb{R}^n$  is separable for all  $n \in \mathbb{N}$  (consider the subset  $Q^n$  of  $\mathbb{R}^n$ )



**Generalization** Every finite dimensional N.V.S (Over  $\mathbb{R}$  or  $\mathbb{C}$ ) is separable, since,

$$E \simeq \mathbb{K}^n \simeq \mathbb{R}^n \simeq \mathbb{C}^n$$

**An important example****Theorem 4.9.6: The Weierstrass approximation theorem**

Let  $a, b \in \mathbb{R}$  with  $a < b$ , then for every real valued continuous function on  $[a, b]$ , there exist a real polynomial sequence  $(P_n)_{n \in \mathbb{N}}$  which uniformly converges to  $f$  on  $[a, b]$ , in other words, for every  $\varepsilon > 0$ , there exist a real polynomial  $P$  such that

$$|f(x) - p(x)| < \varepsilon \quad (\forall x \in [a, b])$$

If  $[a, b] = [0, 1]$ , we can take

$$P_n(x) = \sum_{k=0}^n f(k/n) \binom{n}{k} x^k (1-x)^{n-k}$$

The Bernstein polynomials associated to  $f$

**Consequence :** let  $a, b \in \mathbb{R}$ , with  $a < b$ , then N.V.S  $(\mathcal{C}^0([a, b], \mathbb{R}), \|\cdot\|_\infty)$ , is separable. Indeed, the subset of polynomial functions with rational coefficients on  $[a, b]$  is countable and dense in  $(\mathcal{C}^0([a, b], \mathbb{R}), \|\cdot\|_\infty)$

**Definition 4.9.3:**

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$

1. A subset  $S$  of  $E$  is said to be total if its span (i.e., the set of finite linear combinations of elements of  $E$ ) is dense.
2. A Hamel basis of  $E$  is linearly independent subset of  $E$  which spans  $E$  (The concept already known in Linear Algebra-Algebra2) It follows from Zorn's lemma that every vector space has a Hamel basis and that two Hamel bases of a same vector space are necessarily

equinumerous.

3. A schauder basis of  $E$  is a sequence  $(l_n)_{n \in \mathbb{N}}$  of  $E$  such that for each vector  $x \in E$ , there exists a unique sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of scalars such that

$$x = \sum_{n=0}^{\infty} \lambda_n l_n$$

that is,

$$\|x - \sum_{n=1}^N l_n \lambda_n\| \rightarrow 0 \quad 0 \text{ as } N \rightarrow \infty$$

**Remark :**

1. Its easy to show that if a N.V.S  $E$  has a Schauder basis then its separable (Exercise)
2. A Hamel basis (if its finite or countable) of a N.V.S is always Schauder basis (obvious) but the converse is false (see below!)
3. In a finite dimensional N.V.S the concept of Hamel basis and Schauder basis coincides

**Example**

1. (In relation with Fourier series let  $p > 1$ , It's show showed that the trigonometric,

$$1, \cos(x), \sin(x), \dots$$

is a Schauder basis of the  $\mathbb{R}$ -N.V.S  $L^p([0, 2\pi])$ ,

$$L^p([0, 2\pi]) = \left\{ f : [0, 2\pi] \rightarrow \mathbb{R} \text{ s.t. } \int_0^{2\pi} |f(x)|^p d(\mu(x)) < \infty \right\}$$

with the norm  $\|\cdot\|_p$ )

2. Let  $C_0$  denote the  $\mathbb{R}$ -vector space of real sequences which converge to 0 and let

$$\begin{aligned} \|\cdot\|_{\infty} : \quad C_0 &\longrightarrow [0, \infty] \\ x = (x_n)_{n \in \mathbb{N}} &\longmapsto \|x\|_{\infty} := \sup_{n \in \mathbb{N}} |x_n| \end{aligned}$$

It's obvious that  $\|\cdot\|_{\infty}$  is a norm on  $C_0$  (In fact  $C_0$  is a normed subspace of  $(l^{\infty}, \|\cdot\|_{\infty})$ ), where,

$$l^{\infty} = \{ \text{the real bounded sequences} \}$$

for all  $n \in \mathbb{N}$ , let,

$$l^{(n)} = (l_i^{(n)})_{i \in \mathbb{N}}$$

be the real sequence of  $C_0$  defined by,

$$l_i^{(n)} := \begin{cases} 1 & i = n \\ 0 & \text{else} \end{cases} = (0, 0, \dots, 0, 0, \dots) \in C_0$$

Its clear that  $(e^{(n)})_{n \in \mathbb{N}}$  is linearly independent and is not a Hamel basis of  $C_0$ . Because

$$\langle e^{(n)}, n \in \mathbb{N} \rangle = C_{00} \neq C_0$$

where

$$C_{00} = \{ \text{real sequences } (u_n)_{n \in \mathbb{N}}, \text{ for } u_n = 0 \text{ for } n \text{ sufficiently large} \}$$

$C_{00} \neq C_0$  since we have for example  $(\frac{1}{n})_{n \in \mathbb{N}} \in C_0$ , but  $(\frac{1}{n})_{n \in \mathbb{N}} \notin C_{00}$ .

Next, for any  $x = (x_n)_{n \in \mathbb{N}} \in C_0$ , we have for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \|x - \sum_{n=1}^N x_n e^{(n)}\|_\infty &= \|(x_1, x_2, \dots) - (x_1, \dots, x_N, 0, \dots)\|_\infty \\ &= \|(0, \dots, 0, x_{N+1}, \dots)\|_\infty \\ &= \sup_{n \geq N+1} |x_n| \end{aligned}$$

hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x - \sum_{n=1}^N x_n e^{(n)}\| &= \lim_{n \geq N+1} \sup |x_n| \\ &= \overline{\lim}_{n \rightarrow \infty} |x_n| \\ &= \lim_{n \rightarrow \infty} |x_n| = 0 \end{aligned}$$

This implies that the sequence  $(\sum_{n=1}^N x_n e^{(n)})_{n \in \mathbb{N}}$  of  $C_0$  is convergent to  $x$ .

Equivalently, the series  $\sum_{n=0}^\infty x_n e^{(n)}$  of  $E$  is convergent to  $x$ , i.e.

$$x = \sum_{n=0}^\infty x_n e^{(n)} \quad (\text{in } C_0)$$

Let us show the uniqueness of a such representation of  $x \in C_0$ . Suppose that  $x \in C_0$  is representable as

$$x = \sum_{n=0}^\infty \alpha_n e^{(n)} = \sum_{n=0}^\infty \beta_n e^{(n)} \quad (\alpha_n, \beta_n \in \mathbb{R}, \forall n \in \mathbb{N})$$

we have for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \left\| \sum_{i=1}^N \alpha_i e^{(i)} - \sum_{i=1}^N \beta_i e^{(i)} \right\| \\ = \left\| \sum_{i=1}^N (\alpha_i - \beta_i) e^{(i)} \right\| = \max_{1 \leq i \leq N} |\alpha_i - \beta_i| \end{aligned}$$

So for all  $n, N \in \mathbb{N}$ , with  $n \leq N$ , we have,

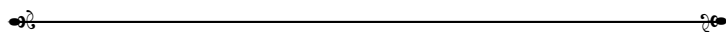
$$|\alpha_n - \beta_n| \leq \max_{1 \leq i \leq N} |\alpha_i - \beta_i| = \left\| \sum_{i=1}^N \alpha_i e^{(i)} - \sum_{i=1}^N \beta_i e^{(i)} \right\|_{\infty} \text{ By taking } N \rightarrow \infty$$

we get that,  $|\alpha_n - \beta_n| \leq 0$ , thus we have that,

$$\alpha_n = \beta_n \quad (\forall n \in \mathbb{N})$$

Thus, the representation of  $x, \sum_{n=1}^{\infty} x_n e^{(n)}$  is unique.

Consequently,  $(e^{(n)})_{n \in \mathbb{N}}$  is a Schauder basis of  $C_0$





# 5

## ∫ FUNDAMENTAL THEOREMS ON BANACH SPACES :

- The open mapping theorem.
- The closed graph theorem.
- The Banach-Steinhaus Theorem
- The Hahn-Banach

### 5.1 The open mapping theorem

**Reminders :** A mapping  $f$  from a topological space  $X$  into a topological space  $Y$  is said to be an open mapping. if the image by  $f$  of every open subset of  $X$  is an open subset of  $Y$

#### Theorem 5.1.1: (The open mapping theorem-Schauder

Let  $f$  be a continuous linear mapping from a Banach space  $E$  to a Banach space  $F$ . Then the two following properties are equivalent,

- i  $f$  is surjective
- ii  $f$  is an open mapping

*Proof.* (ii)  $\implies$  (i)

We argue by contradiction. Suppose that  $f$  is an open mapping that  $f$  is not surjective ( i.e.  $f(E) \neq F$ ), so  $f(E)$ , is a proper subspace of  $F$ , implying (see the tutorial worksheet number 1 ), that

$$\text{int}(f(E)) = \emptyset$$

On the other hand, since  $f$  is an open mapping and  $E$  is open in  $E$  then  $f(E)$  is open in  $F$ , thus  $\text{int}(f(E)) = f(E)$ , Hence  $f(E) = \emptyset$ , which is a contradiction.

$$(i) \implies (ii)$$

we need preliminary results.

### Theorem 5.1.2:

Let  $E$  and  $F$  be two N.V.S and  $f : E \longrightarrow F$  be a linear mapping then the two following properties are equivalent,

i  $f$  is an open mapping

ii  $\exists r > 0$  such that

$$B_F(0_F, r) \subset f(B_E(0_E, 1))$$

$$\text{Proof. } (i) \implies (ii)$$

Suppose that  $f$  is an open mapping. Since  $B_E(0_E, 1)$  is an open subset of  $E$ , then  $f(B_E(0_E, 1))$  is an open subset of  $F$ . So since,

$$0_F = f(0_E) \in f(B_E(0_E, 1))$$

then  $f(B_E(0_E, 1))$  is a neighborhood of  $0_F$ , that is  $\exists r > 0$  such that

$$B_F(0_F, r) \subset f(B_E(0_E, 1))$$

as required. □

□

### Theorem 5.1.3: (The open mapping theorem)

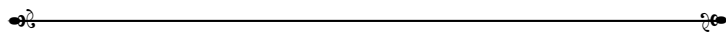
Let  $E, F$  be two Banach spaces. and let  $f \in \mathcal{L}(E, F)$ , then the following assertions are equivalent,

(i)  $f$  is surjective

(ii)  $f$  is an open mapping

*Proof.* Last time we have proved that  $(ii) \implies (i)$ , now

$$(i) \implies (ii)$$



*Proposition 01:* let  $E, F$  be two N.V.S. and  $f : E \longrightarrow F$  be a linear map, then,

(a)  $f$  is an open mapping

(b)  $\exists r > 0$  such that  $f(B_E(0_E, 1)) \supset B_F(0_F, r)$

*Proof.*  $(\alpha) \implies (\beta)$

Suppose that  $f$  is an open mapping  $B_E(0_E, 1)$  is open in  $E$ , then  $f(B_E(0_E, 1))$  is open in  $F$ .

$$0_F = f(0_E) \in f(B_E(0_E, 1))$$

Thus there exist  $r > 0$  such that

$$f(B_E(0_E, 1)) \in \mathcal{V}(0_F)$$

Therefore

$$B_F(0_F, r) \subset f(B_E(0_E, 1))$$

$$(\beta) \implies (\alpha)$$

**Notation :** For a given non empty subsets  $A$  and  $B$  of a N.V.S  $V$ , then  $x_0 \in V$ , and a given scalar  $\lambda$ , we let  $(A + B)$ ,  $A + x_0$ , and  $\lambda A$ , respectively, denote the following subsets of  $V$  :

$$A + B := \{a + b : a \in A, b \in B\}$$

$$A + x_0 := A + \{x_0\} = \{a + x_0 : a \in A\}$$

$$\lambda A := \{\lambda a, a \in A\}$$

Note that  $2A \neq A + A$  because,

$$\{2a : a \in A\} \subset \{a + b : a, b \in A\}$$

Suppose that  $\exists r > 0$  such that

$$B_F(0_F, r) \subset f(B_E(0_E, 1))$$

Let  $\mathcal{O}$  be an open subset of  $E$ , and let us show that  $f(\mathcal{O})$  is an open subset of  $F$ , we have to show that  $f(\mathcal{O})$  is a neighborhood of every element of  $f(\mathcal{O})$ .

Let  $y \in f(\mathcal{O})$  arbitrary and show that  $f(\mathcal{O})$  is a neighborhood of  $y$ .

$y \in f(\mathcal{O})$ , which means that  $\exists x \in \mathcal{O}$  such that  $y = f(x)$ . But since  $\mathcal{O}$  is an open set in  $E$ , and  $x \in \mathcal{O}$ , then  $\exists \varepsilon > 0$  such that

$$B_E(x, \varepsilon) \subset \mathcal{O}$$

Hence

$$f(B_E(x, \varepsilon)) \subset f(\mathcal{O})$$



Since  $f$  is linear, then we have

$$\begin{aligned}
 f(B_E(x, \varepsilon)) &= f(\varepsilon B_E(0_E, 1) + x) \\
 &= \varepsilon \underbrace{f(B_E(0_E, 1))}_{\supset B_F(0_F, r)} + f(x) \supset \varepsilon B_F(0_F, r) + f(x) \\
 &= B_F(f(x), \varepsilon r) \\
 &= B_F(y, \varepsilon r)
 \end{aligned}$$

Hence  $f(\mathcal{O}) \supset B_F(y, \varepsilon r)$  implying that  $f(\mathcal{O})$  is a neighborhood of  $y$ . Thus since  $y$  is arbitrary in  $f(\mathcal{O})$ , then  $f(\mathcal{O})$  is open in  $F$ . Consequently,  $f$  is an open mapping, as required, this completes the proof.  $\square$

#### Theorem 5.1.4:

Let  $E$  be a Banach space, and  $F$  be an arbitrary N.V.S. And  $f \in \mathcal{L}(E, F)$  let  $\varepsilon \in (0, 1)$  and  $A$  be a bounded subset of  $F$ , satisfying

$$A \subset f(B_E(0_E, 1)) + \varepsilon A$$

Then we have

$$A \subset \frac{1}{1-\varepsilon} f(B_E(0_E, 1))$$

*Proof.* Let  $a_0 \in A$  and let us show that  $a_0 \in \frac{1}{1-\varepsilon} f(B_E(0_E, 1))$  and let us show that  $a_0 \in \frac{1}{1-\varepsilon} f(B_E(0_E, 1))$ . Since  $a_0 \in A$  and  $A \subset f(B_E(0_E, 1)) + \varepsilon A$ , then  $a_0 \in f(B_E(0_E, 1)) + \varepsilon A$ , this  $\exists x_0 \in B_E(0_E, 1)$  and  $\exists a_1 \in A$  such that,

$$a_0 = f(x_0) + \varepsilon a_1$$

Similarly, since  $a_1 \in A$  and

$$A \subset f(B_E(0_E, 1)) + \varepsilon A$$

then  $a_1 \in f(B_E(0_E, 1)) + \varepsilon A$ . Thus there exist  $x_1 \in B_E(0_E, 1)$  and there exist  $a_2 \in A$ , such that

$$a_1 = f(x_1) + \varepsilon a_2$$

By iterating the process, we get a sequence  $(x_n)_{n \in \mathbb{N}_0}$  of  $B_E(0_E, 1)$  and a sequence  $(a_n)_{n \in \mathbb{N}_0}$  of  $A$  such that

$$a_n = f(x_n) + \varepsilon a_{n+1} \quad (\forall n \in \mathbb{N}_0)$$

Thus,

$$\begin{aligned}
 a_0 &= f(x_0) + \varepsilon a_1 \\
 &= f(x_0) + \varepsilon(f(x_1) + \varepsilon a_2) \\
 &= f(x_0 + \varepsilon x_1) + \varepsilon^2 a_2 \\
 &= f(x_0 + \varepsilon x_1) + \varepsilon^2(f(x_2) + \varepsilon a_3) \\
 &= f(x_0 + \varepsilon x_1 + \varepsilon^2 x_2) + \varepsilon^3 a_3 \\
 &= f(x_0 + \varepsilon x_1 + \dots + \varepsilon^n x_n) + \varepsilon^{n+1} a_{n+1}
 \end{aligned}$$

Since the series  $\sum_{n=0}^{\infty} \varepsilon^n x_n$  of  $E$  is normally convergent (because for every  $n \in \mathbb{N}_0$ ), we have

$$\|\varepsilon^n x_n\|_E = \varepsilon^n \|x_n\|_E < \varepsilon^n$$

and the real geometric series  $\sum_{n=0}^{\infty} \varepsilon^n$  converges since its ratio  $\varepsilon \in (0, 1)$ , then we derive that  $\sum_{n=0}^{\infty} \varepsilon^n x_n$  is convergent in  $E$ , and since  $E$  is Banach. So setting

$$x := \sum_{n=0}^{\infty} \varepsilon^n x_n \in E$$

and letting  $n \rightarrow \infty$ , we get,

$$a_0 = f(x) \quad (\text{since } f \text{ is continuous and } \varepsilon^{n+1} a_{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ because } A \text{ is bounded and } 0 < \varepsilon < 1)$$

finally, we observe that,

$$\begin{aligned}
 \|x\|_E &= \left\| \sum_{n=0}^{\infty} \varepsilon^n x_n \right\|_E \leq \sum_{n=0}^{\infty} \|\varepsilon^n x_n\|_E \\
 &= \sum_{n=0}^{\infty} \varepsilon^n \|x_n\|_E < 1
 \end{aligned}$$

Thus,

$$\|x\|_E < \sum_{n=0}^{\infty} \varepsilon^n = \frac{1}{1-\varepsilon}$$

by setting  $u = (1 - \varepsilon)x$ , we get,

$$\|u\|_E < 1 \quad \text{i.e.} \quad u \in B_E(0_E, 1)$$

Hence,

$$\begin{aligned}
 a_0 &= f(x) = f\left(\frac{1}{1-\varepsilon}u\right) \\
 &= \frac{1}{1-\varepsilon}f(u) \\
 &\in \frac{1}{1-\varepsilon}f(B_E(0_E, 1))
 \end{aligned}$$

consequently  $A \subset \frac{1}{1-\varepsilon}f(B_E(0_E, 1))$ , as required. □

**Theorem 5.1.5:**

Let  $E$  be a Banach space, and  $F$  be an arbitrary N.V.S. Next, let  $f \in \mathcal{L}(E, F)$  and  $r, s > 0$ , suppose that,

$$\overline{f(B_E(0_E, r))} \supset B_F(0_F, s)$$

then,

$$f(B_E(0_E, r)) \supset B_F(0_F, s)$$

**Remark :** In the context of Proposition 3 (i.e. above theorem), we have,

$$\begin{aligned} f(B_E(0_E, r)) \supset B_F(0_F, s) &\iff rf(B_E(0_E, 1)) \supset sB_F(0_F, 1) \\ &\iff \frac{r}{s}f(B_E(0_E, 1)) \supset B_F(0_F, 1) \end{aligned}$$

similarly,

$$\begin{aligned} f(B_E(0_E, r)) \supset B_F(0_F, s) &\iff r\overline{f(B_E(0_E, 1))} \supset sB_F(0_F, 1) \\ &\iff \frac{r}{s}\overline{f(B_E(0_E, 1))} \supset B_F(0_F, 1) \end{aligned}$$

if we put  $g = \frac{r}{s}f \in \mathcal{L}(E, F)$ , the proposition becomes,

$$\overline{g(B_E(0_E, 1))} \supset B_F(0_F, 1) \implies g(B_E(0_E, 1)) \supset B_F(0_F, 1)''$$

*Proof.* By replacing if necessary  $f$  by  $\frac{r}{s}f$ , we may suppose that  $r = s = 1$ . So, we have to show the implication,

$$B_F(0_F, 1) \subset \overline{f(B_E(0_E, 1))} \implies B_F(0_F, 1) \subset f(B_E(0_E, 1))$$

suppose that

$$B_F(0_F, 1) \subset \overline{f(B_E(0_E, 1))}$$

and let us show that  $B_F(0_F, 1) \subset f(B_E(0_E, 1))$  for all  $\varepsilon \in (0, 1)$ , we have,

$$\overline{f(B_E(0_E, 1))} \subset f(B_E(0_E, 1)) + \varepsilon B_F(0_F, 1)$$

Indeed, for any  $y \in \overline{f(B_E(0_E, 1))}$ , we have  $B_F(y, \varepsilon) \cap f(B_E(0_E, 1)) \neq \emptyset$ , so, by considering  $u \in B_F(y, \varepsilon) \cap f(B_E(0_E, 1))$ , we have

$$y = u + \underbrace{(y - u)}_{\in B_F(0_F, \varepsilon) = \varepsilon B_F(0_F, 1)} \in f(B_E(0_E, 1)) + \varepsilon B_F(0_F, 1)$$

Thus the claimed inclusion is proved.

From  $B_F(0_F, 1) \subset \overline{f(B_E(0_E, 1))}$  and

$$\overline{f(B_E(0_E, 1))} \subset f(B_E(0_E, 1)) + \varepsilon B_F(0_F, 1)$$

we deduce the inclusion

$$B_F(0_F, 1) \subset f(B_E(0_E, 1)) + \varepsilon B_F(0_F, 1)$$

so, by applying one of the above theorems (find it!) for  $A = B_F(0_F, 1)$ , we desire,

$$B_F(0_F, 1) \subset \frac{1}{1-\varepsilon} f(B_E(0_E, 1))$$

Now let  $y \in B_F(0_F, 1)$  arbitrary, so  $\|y\|_F < 1$ , thus

$$\exists \varepsilon \in (0, 1) \text{ s.t. } \|y\|_F < 1 - \varepsilon < 1$$

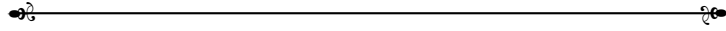
implying that  $\frac{1}{1-\varepsilon}y \in B_F(0_F, 1)$ , so by the above inclusion,

$$\frac{1}{1-\varepsilon}y \in \frac{1}{1-\varepsilon} f(B_E(0_E, 1))$$

thus  $y \in f(B_E(0_E, 1))$ . Hence the inclusion

$$B_F(0_F, 1) \subset f(B_E(0_E, 1))$$

as required. □



Lets finish the proof that we initially started, suppose that  $f$  is surjective and let us show that  $f$  is an open mapping. According to Theorem 1, it suffices to show that  $\exists r > 0$ , such that

$$f(B_E(0_E, 1)) \supset B_F(0_F, 1)$$

Next, according to Proposition 03, it suffices to show  $\exists r > 0$ , such that

$$\overline{f(B_E(0_E, 1))} \supset B_F(0_F, r)$$

we have obviously

$$E = \bigcup_{n=1}^{\infty} B_E(0_E, n)$$

thus,

$$F = f(E) = \bigcup_{n=1}^{\infty} f(B_E(0_E, n)) \quad (\text{since } f \text{ is surjective})$$

in other words,

$$F = \bigcup_{n=1}^{\infty} f(B_E(0_E, n))$$

by inserting the closure on both sides,

$$F = \bigcup_{n=1}^{\infty} \overline{f(B_E(0_E, n))}$$

we get

$$\text{int}(F) = F \neq \emptyset \text{ so } \bigcup_{n=1}^{\infty} \overline{f(B_E(0_E, 1))} \neq \emptyset$$

It follows according to the Baire theorem, that there exist  $n_0 \in \mathbb{N}$  such that

$$\text{int}(\overline{f(B_E(0_E, n_0))}) \neq \emptyset$$

But

$$\overline{f(B_E(0_E, n_0))} = n_0 \overline{f(B_E(0_E, 1))}$$

Hence

$$\overline{f(B_E(0_E, 1))} \neq \emptyset$$

Consequently, there exist  $y \in \overline{f(B_E(0_E, 1))}$ , and there exist  $r > 0$  such that

$$B_F(y, r) \subset \overline{f(B_E(0_E, 1))}$$

Now by using the above inclusion, and the immediate fact that the set  $\overline{f(B_E(0_E, 1))}$  is convex and symmetric, since

$$\begin{aligned} B_E(0_E, 1) \text{ is convex} &\implies f \text{ is linear therefore } f(B_E(0_E, 1)) \text{ is convex} \\ &\implies \overline{f(B_E(0_E, 1))} \text{ is convex} \end{aligned}$$

$\overline{f(B_E(0_E, 1))}$  is symmetric ( $\forall a \in A, -a \in A$ ), since  $B_E(0_E, 1)$  is symmetric.

$$\begin{aligned} &\implies f \text{ is linear therefore } f(B_E(0_E, 1)) \text{ is symmetric} \\ &\implies \overline{f(B_E(0_E, 1))} \end{aligned}$$

we have for all  $z \in B_F(0_F, r)$ ,

$$z + y, -z + y \in B_F(y, r)$$

thus we get,

$$z + y, -z + y \in \overline{f(B_E(0_E, 1))}$$

thus (since  $\overline{f(B_E(0_E, 1))}$  is symmetric),

$$z + y, z - y \in \overline{f(B_E(0_E, 1))}$$

thus (since  $\overline{f(B_E(0_E, 1))}$  is convex),

$$\frac{1}{2}((z + y) + (z - y)) = z \in \overline{f(B_E(0_E, 1))}$$

hence the required inclusion,

$$B_F(0_F, r) \subset \overline{f(B_E(0_E, 1))}$$

This completes the proof. □

We can derive a bunch of theorems from the latter.

### Theorem 5.1.6: (The Banach Isomorphism Theorem)

Let  $E$  and  $F$  be two Banach spaces, and let  $f \in \mathcal{L}(E, F)$  bijective, then  $f$  is an isomorphism of N.V.S (i.e.  $f^{-1}$  is continuous)

*Proof.* Since  $f$  is surjective, then (according to the open mapping theorem)  $f$  is open; that is the image (by  $f$ ) of an open subset of  $E$  is an open subset of  $F$ . Equivalently, the preimage by  $f^{-1}$  of any open subset of  $E$  is open in  $F$ . this shows that  $f^{-1}$  is continuous thus  $f$  is an isomorphism of N.V.S.  $\square$

### Theorem 5.1.7:

Let  $N_1$  and  $N_2$  be two norms on  $\mathbb{K}$ -vector space  $E$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , such that the two N.V.S  $(E, N_1)$  and  $(E, N_2)$  are both Banach. Then for  $N_1$  and  $N_2$  to be equivalent, it suffices to have  $N_2 \leq \alpha N_1$  or the converse for some  $\alpha > 0$

*Proof.* Suppose that  $\exists \alpha > 0$ , such that  $N_2 \leq \alpha N_1$ . So the identity map of  $E$ ,

$$\begin{aligned} Id_E : (E, N_1) &\longrightarrow (E, N_2) \\ x &\longmapsto x \end{aligned}$$

$$\begin{aligned} N_2 \leq \alpha N_1 &\implies id_E \text{ is } \alpha\text{-Lipschitz} \\ &\implies id_E \text{ is continuous} \end{aligned}$$

$id_E$  is linear, bijective, and continuous this implies (according to the above theorem), that  $id_E$  is an isomorphism of N.V.S, i.e., so  $id_E^{-1}$  is continuous, so Lipschitz continuous, so  $\exists \beta > 0$  such that  $N_1 \leq \beta N_2$ , Hence  $N_1$  and  $N_2$  are equivalent.  $\square$

### Theorem 5.1.8: (The closed graph theorem)

Let  $E$  and  $F$  be two Banach spaces over some field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $f : E \longrightarrow F$  be a linear mapping, then  $f$  is continuous if and only if its graph  $G(f)$  is closed in the Banach space  $E \times F$ , Recall that

$$G(f) := \{(x, f(x)) : x \in E\}$$

*Proof.*

$$(\implies)$$

Suppose that  $f$  is continuous and show that  $G(f)$  is closed in  $E \times F$ . So, let  $\{(x_n, f(x_n))\}_{n \in \mathbb{N}}$ , be an

arbitrary sequence of  $G(f)$ , converging in  $E \times F$  to some  $(x, y) \in E \times F$  and let show that

$$(x, y) \in G(f) \quad y = f(x)$$

since the projections are continuous

$$\begin{aligned} \pi_1 : E \times F &\longrightarrow E \\ (u, v) &\longmapsto u \end{aligned}$$

and

$$\begin{aligned} \pi_2 : E \times F &\longrightarrow F \\ (u, v) &\longmapsto v \end{aligned}$$

are both continuous, then the fact

$$(x_n, f(x_n)) \rightarrow (x, y) \quad \text{as } n \rightarrow \infty$$

implies

$$x_n \rightarrow x \quad f(x_n) \rightarrow y \quad \text{as } n \rightarrow \infty$$

But on the other hand, we have since  $f$  is continuous, we have

$$x_n \rightarrow x \text{ (in } E) \implies f(x_n) \rightarrow f(x) \text{ (in } F) \quad \text{as } n \rightarrow \infty$$

It follows according to the uniqueness of the limit that  $y = f(x)$ , as required.

$$(\Leftarrow)$$

Conversly, suppose that  $G(f)$  is closed in  $E \times F$ . This implies that the vector subspace  $G(f)$  of  $E \times F$  is Banach (a closed subset of complete space is complete). Next, consider the two maps,

$$p_1 = \pi_1|_{G(f)} \quad p_2 = \pi_2|_{G(f)}$$

where

$$\begin{aligned} p_1 : G(f) &\longrightarrow E \\ (u, f(u)) &\longmapsto u \end{aligned}$$

and

$$\begin{aligned} p_2 : G(f) &\longrightarrow F \\ (u, f(u)) &\longmapsto f(u) \end{aligned}$$

Since  $\pi_1$  and  $\pi_2$  are linear and continuous then  $p_1$  and  $p_2$  are also linear and continuous, Besides  $p_1$  is clearly bejective. So according to the Banach Isomorphism theorem we get that  $p_1^{-1}$  is continuous, then,

$$\begin{aligned} f : E &\longrightarrow G(f) \longrightarrow F \\ u &\longmapsto (u, f(u)) \longrightarrow f(u) \end{aligned}$$

clearly

$$f = p_2 \circ p_1^{-1}$$

is continuous, since its a composition of two continuous maps, as required. this completes the proof of the theorem.  $\square$

### The Banach-Steinhans Theorem

#### Definition 5.1.1: Meager Sets

Let  $E$  be a topological space and  $X$  be a subset of  $E$ . Then  $X$  is said to be meager if it can be included in a countable union of closed subsets of  $E$  of empty interior.

Equivalently,  $X$  is meager if its a countable union of subsets whose closure has empty interior.

A set that is not meager is said to be nonmeager

#### Example

1.  $\mathbb{Q}$  is meager in  $\mathbb{R}$  equipped with its usual topology. Indeed we can write,

$$\mathbb{Q} = \bigcup_{n \in \mathbb{Q}} \{n\}$$

$\{x\}$  is closed in  $\mathbb{R}$ , and  $\overline{\{x\}}^\circ = \emptyset$ , Other method is,

$$\mathbb{Q} = \mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z} \cup \dots$$

for all  $n \in \mathbb{N}$ , we have

$$\overline{\frac{1}{n}\mathbb{Z}}^\circ = \frac{1}{n}\overline{\mathbb{Z}}^\circ = \emptyset$$

since  $\overline{\mathbb{Z}} = \mathbb{Z}$  and  $\mathbb{Z}^\circ = \emptyset$

2. Let  $E$  be Baire space (i.e., a topological space that satisfies the Baire property).

- $E$  is nonmeager in  $E$ .

*Proof.* Indeed if  $E = \bigcup_{n=0}^{\infty} F_n$ , where  $F_n = \emptyset, \forall n \in \mathbb{N}$ , then since  $E$  is Baire we get  $\mathring{E} = \emptyset$ , which is a contradiction.  $\square$

- More generally, if  $A$  is a meager subset of  $E$ , then  $E \setminus A$  is dense in  $E$

*Proof.* Since  $A$  is meager then we have

$$A \subset \bigcup_{n=1}^{\infty} F_n \quad \mathring{F}_n = \emptyset \quad \forall n \in \mathbb{N}$$



Since  $E$  is Biare then  $\bigcup_{n=1}^{\infty} F_n = \emptyset$ . Thus  $\mathring{A} \subset \bigcup_{n=1}^{\infty} F_n = \emptyset$ , thus  $\mathring{A} = \emptyset$ , hence

$$\overline{E \setminus A} = E \setminus \mathring{A} = E \setminus \emptyset = E$$

that is  $X \setminus A$  is dense in  $E$

□

### Theorem 5.1.9: Banach-Steinhaus 1927

Let  $E$  and  $F$  be two N.V.S for a family of continuous mappings from  $E$  to  $F$  to be uniformly bounded on the unit ball of  $E$ , it suffices that it be pointwise bounded on a noneager subset of  $E$ .

### Definition 5.1.2: (Uniformly bounded in Unit ball)

$(f_i)_{i \in I}$  linear continuous.

$$\exists M > 0, \forall x \in B_E(0_E, 1) \|f_i(x)\| \leq M$$

### Definition 5.1.3: (Pointwise bounded on A)

Pointwise bounded on  $A$ , for all  $x \in A$ ,  $\exists M_x$  such that,

$$\forall i \in I : \|f_i(x)\| \leq M_x$$

More explicitly, let  $A \subset \mathcal{L}(E, F)$ , and for all  $x \in E$ , let

$$A_x := \{f(x), f \in A\}$$

Finally, let

$$B := \{x \in E, A_x \text{ is bounded in } F\}$$

Suppose that  $B$  is nonmeager in  $E$ , then  $A$  is bounded in  $\mathcal{L}(E, F)$ , In particular  $B = E$

*Proof.* We can write  $B$  as,

$$\begin{aligned} B &= \bigcup_{n=1}^{\infty} \{x \in E, A_x \text{ is bounded by } n \text{ in } F\} \\ &= \bigcup_{n=1}^{\infty} \{x \in E : \|f(x)\|_F \leq n, \forall f \in A\} \end{aligned}$$

next for all  $n \in \mathbb{N}$ , we have

$$B_n = \bigcap_{f \in A} \underbrace{\{x \in E : \|f(x)\|_F \leq n\}}_{B_{n,f}}$$

since for any  $n \in \mathbb{N}$  and any  $f \in A$ ,  $B_{n,f}$  is the preimage of the closed subset  $(-\infty, n]$  of  $\mathbb{R}$  by the continuous map

$$\begin{aligned} E &\longrightarrow \mathbb{R} \\ x &\longmapsto \|f(x)\| = \|\cdot\| \circ f \end{aligned}$$

then  $B_{n,f}$  is closed in  $E$  for all  $n \in \mathbb{N}, \forall f \in A$ , thus  $B_n(n \in \mathbb{N})$  is closed in  $E$  as its the intersction of closed subsets of  $E$ , but since  $B$  is non meager and  $B = \bigcup_{n=1}^{\infty} B_n$ , where  $B_n$  is closed for all  $n$ , there exist  $N \in \mathbb{N}$  such that

$$B_N^\circ \neq \emptyset$$

therefore  $\exists x_0 \in E, \exists r > 0$  such that

$$B_E(x_0, r) \subset B_N$$

Now, for all  $f \in A$  and for all  $x \in B_E(0_E, 1)$ , we have that

$$x_0(+/-)rx \in B_E(x_0, r) \subset B_N$$

implying that

$$\|f(x_0(+/-)rx)\|_F \leq N$$

consequently, we have

$$\forall f \in A, \forall x \in B_E(0_E, 1) \quad f(x) = f\left(\frac{1}{2r}[(x_0 + rx) - (x_0 - rx)]\right)$$

since  $f$  is linear we get

$$f(x) = \frac{1}{2r} [f(x_0 + rx) - f(x_0 - rx)]$$

thus

$$\forall f \in A, \forall x \in B_E(0_E, 1)$$

we get

$$\begin{aligned} \|f(x)\|_F &\leq \frac{1}{2r} [\|f(x_0 + rx)\|_F + \|f(x_0 - rx)\|_F] \\ &\leq \frac{N}{r} \end{aligned}$$

implying that

$$\|f\| \leq \frac{N}{r} \quad (f \in A)$$

showing that  $A$  is bounded in  $\mathcal{L}(E, F)$ , as required. □

before we continue the main proof, we will add some small theorems

**Theorem 5.1.10: 1**

Let  $E$  be a Banach space and  $F$  be an arbitrary N.V.S. Let also  $A$  be a subset of  $\mathcal{L}(E, F)$ . Then the two following properties are equivalent,

- (i)  $A$  is bounded in  $\mathcal{L}(E, F)$
- (ii) for all  $x \in E$ , the subset

$$\{f(x), f \in A\} \text{ of } F \text{ is bounded.}$$

*Proof.* Since  $E$  is Banach then its Baire, hence  $E$  is nonmeager in it self the result of the corollary then follows from the previous proof.  $\square$

**Theorem 5.1.11: 2**

Let  $E$  be a Banach space, and  $F$  be an arbitrary N.V.S. Let also  $(f_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathcal{L}(E, F)$ , suppose that for all  $x \in E$ , the sequence  $(f_n(x))_{n \in \mathbb{N}}$  converges in  $F$  and denote by  $f(x)$  its limit, then

- $(f_n)_{n \in \mathbb{N}}$  is bounded in  $\mathcal{L}(E, F)$
- $f \in \mathcal{L}(E, F)$
- $\|f\| \leq \lim_{n \rightarrow \infty} \inf \|f_n\|$

*Proof.* The Boundedness of  $f$

for all  $x \in E$ , since the sequence  $(f_n)_{n \in \mathbb{N}}$  of  $F$  is assumed convergent, then its bounded. this implies according to the Theorem 1, that the sequence  $(f_n)_{n \in \mathbb{N}}$  of  $\mathcal{L}(E, F)$  is bounded.

The Linearity of  $f$  (obvious)

for all  $\lambda \in \mathbb{K}, \forall x, y \in E$ , we have,

$$\begin{aligned} f(\lambda x + y) &= \lim_{n \rightarrow \infty} f_n(\lambda x + y) \\ &= \lim_{n \rightarrow \infty} (\lambda f_n(x) + f_n(y)) \\ &= \lambda \lim_{n \rightarrow \infty} f_n(x) + \lim_{n \rightarrow \infty} f_n(y) \\ &= \lambda f(x) + f(y) \end{aligned}$$

showing that  $f$  is linear.

The continuity of  $f$  and the estimate of  $\|f\|$ ,

$\forall x \in E$ , we have

$$\begin{aligned}
 \|f(x)\|_F &= \left\| \lim_{n \rightarrow \infty} f_n(x) \right\|_F \\
 &= \lim_{n \rightarrow \infty} \|f_n(x)\|_F \\
 &= \lim_{n \rightarrow \infty} \inf \|f_n(x)\|_F \\
 &\leq \lim_{n \rightarrow \infty} \inf (\|f_n\| \|x\|_E) = \left( \lim_{n \rightarrow \infty} \inf \|f_n\| \right) \|x\|_E
 \end{aligned}$$

implying that  $f$  is continuous and that

$$\|f\| \leq \lim_{n \rightarrow \infty} \inf \|f_n\|$$

This completes the proof. □

*Corollary :* Let  $E$  be a Banach space and  $F$  and  $G$  be two arbitrary N.V.S. let also  $h : E \times F \longrightarrow G$  be a bilinear mapping that is separately continuous, that is  $h$  satisfies the following properties,

(1) for all  $y \in F$ , the linear mapping

$$\begin{aligned}
 h(., y) : E &\longrightarrow G \\
 x &\longmapsto h(x, y)
 \end{aligned}$$

is continuous

(2) for all  $x \in E$ , the linear mapping

$$\begin{aligned}
 h(x, .) : F &\longrightarrow G \\
 y &\longmapsto h(x, y)
 \end{aligned}$$

is continuous

Then  $h$  is continuous

*Proof.* Define

$$A = \{h(., y) : y \in \overline{B_F}(0_F, 1)\} \subset \mathcal{L}(E, G)$$

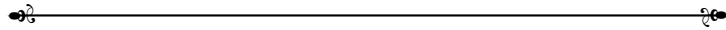
and for all  $x \in E$ ,

$$\begin{aligned}
 A_x &:= \{f(x), f \in A\} \\
 &= \{h(x, y), y \in \overline{B_F}(0_F, 1)\} \\
 &= \{h(x, .)(y), y \in \overline{B_F}(0_F, 1)\}
 \end{aligned}$$

Giving  $x \in E$ , since the linear mapping  $h(x, \cdot)$  is continuous by hypothesis then the last inequality shows that the subset  $A_x$  of  $G$  is bounded. Thus (by Banach Steinhaus theorem), the subset  $A$  of  $\mathcal{L}(E, G)$  is bounded (say by a pointwise constant  $M$ ). Hence, we have for all  $x \in \overline{B_F}(0_E, 1)$  and  $y \in \overline{B_F}(0_F, 1)$ ,

$$\begin{aligned} \|h(x, y)\|_G &= \|h(\cdot, y)(x)\|_G \\ &\leq \underbrace{\|h(\cdot, y)\|_{\mathcal{L}(E, G)}}_{\in A} \cdot \|x\|_E \leq M \end{aligned}$$

implying that  $h$  is continuous, hence the corollary is proved. □





# $\int$ QUOTIENT VECTOR NORMED SPACES 6

Let  $E$  be a N.V.S. and  $H$  be a vector subspace of  $E$ . Recall that the quotient vector space of  $E$  on  $H$  is given by,

$$E_{\setminus H} = \{x + H, x \in E\}$$

Consider the map

$$\begin{aligned} \|\cdot\|_{E_{\setminus H}} : E_{\setminus H} &\longrightarrow [0, \infty) \\ C &\longmapsto \inf_{x \in C} \|x\|_E \end{aligned}$$

the map  $\|\cdot\|_{E_{\setminus H}}$  defines a seminorm on  $E_{\setminus H}$ . In addition,  $\|\cdot\|_{E_{\setminus H}}$  becomes a norm on  $E_{\setminus H}$  if and only if  $H$  is closed in  $E$ .

*Proof.* Let us show that the map  $\|\cdot\|_{E_{\setminus H}}$  satisfies the three properties of a seminorm on the quotient vector space  $E_{\setminus H}$ .

1. The zero vector of the quotient vector space  $E_{\setminus H}$  is  $C(0_E) = 0_E + H = H$ , and we have,

$$\|H\|_{E_{\setminus H}} = \inf_{x \in H} \|x\|_E \leq \|0_E\|_E$$

Thus,  $\|H\|_{E_{\setminus H}} = 0$ , as required.

2. Let  $\lambda \in \mathbb{K}$  and  $C \in E_{\setminus H}$ , since  $\lambda C = \{\lambda x, x \in C\}$  then we have,

$$\begin{aligned} \|\lambda C\|_{E_{\setminus H}} &= \inf_{x \in C} \|\lambda x\|_E \\ &= \inf_{x \in C} (|\lambda| \|x\|_E) \\ &= |\lambda| \left( \inf_{x \in C} \|x\|_E \right) = |\lambda| \|C\|_{E_{\setminus H}} \end{aligned}$$

as required.

3. Let  $C_1, C_2 \in E_{\setminus H}$  which we can write as

$$C_1 = x_1 + H \quad C_2 = x_2 + H$$

where  $x_1, x_2 \in E$ ,

$$\|C_1 + C_2\|_{E \setminus H} \stackrel{?}{\leq} \|C_1\|_{E \setminus H} + \|C_2\|_{E \setminus H}$$

then  $C_1 + C_2 = x_1 + x_2 + H$ , By the triangle inequality in  $E$ , we have for all  $h_1, h_2 \in H$ ,

$$(x_1 + h_1) + (x_2 + h_2) \leq \|x_1 + h_1\|_E + \|x_2 + h_2\|_E$$

taking in the two sides of this inequality the infimum where  $h_1, h_2 \in H$ , we obtain since  $(\{h_1 + h_2, h_1, h_2 \in H\} = H)$

$$\inf_{h \in H} \|x_1 + x_2 + h\|_E \leq \inf_{h_1 \in H} \|x_1 + h_1\|_E + \inf_{h_2 \in H} \|x_2 + h_2\|_E$$

That is

$$\|C_1 + C_2\|_{E \setminus H} \leq \|C_1\|_{E \setminus H} + \|C_2\|_{E \setminus H}$$

as required. Consequently,  $\|\cdot\|_{E \setminus H}$  defines a seminorm on  $E \setminus H$ .

Next, denoting by  $d$  the metric associated to the norm of  $E$ , we have for all  $x \in E$ ,

$$\begin{aligned} \|x + H\|_{E \setminus H} &= \inf_{h \in H} \|x + h\|_E \\ &= \inf_{h \in H} \|x - h\|_E \\ &= \inf_{h \in H} d(x, H) \\ &= d(x; H) \end{aligned}$$

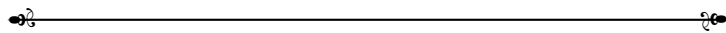
It follows according to the well-known results on metric spaces, that for all  $x \in E$ ,

$$\begin{aligned} \|x + H\|_{E \setminus H} = 0 &\iff d(x, H) = 0 \\ &\iff x \in \overline{H} \end{aligned}$$

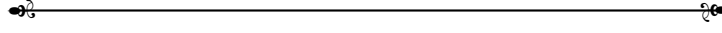
Therefore,  $\|\cdot\|_{E \setminus H}$  defines a norm on  $E \setminus H$  if and only if  $\overline{H} = 0_{E \setminus H} = H$ , that is if and only if  $H$  is closed in  $E$ , the proof is complete. □

### Terminology :

The map  $\|\cdot\|_{E \setminus H}$  defined above is called the quotient seminorm of  $E \setminus H$ , if  $H$  is closed in  $E$ , its called the quotient norm of  $E \setminus H$ .



NB : whenever the quotient space  $E \setminus H$  is mentioned (where  $E$  is N.V.S. and  $H$  is closed vector subspace of  $E$ ) its completely assumed that  $E \setminus H$  is equipped with the quotient norm  $\|\cdot\|_{E \setminus H}$  defined previously.



### Theorem 6.0.1:

Let  $E$  be a N.V.S. and  $H$  be a closed *proper* subspace of  $E$ . then the quotient map

$$\begin{aligned} \Pi : E &\longrightarrow E \setminus H \\ x &\longmapsto x + H \end{aligned}$$

is continuous, and satisfies  $||| \pi ||| = 1$

*Proof.* Recall that  $\pi$  is linear. Next, for all  $x \in E$ , we have,

$$\begin{aligned} \|\pi(x)\|_{E \setminus H} &= \|x + H\|_{E \setminus H} := \inf_{h \in H} \|x + h\|_E \\ &\leq \|x + 0_E\|_E = \|x\|_E \end{aligned}$$

implying that  $\pi$  is continuous and that

$$||| \pi ||| \leq 1$$

Now, let us show that

$$||| \pi ||| \geq 1$$

To do so, fix  $a \in E \setminus H$ , thus  $\pi(a) \neq H = 0_{E \setminus H}$ , implying that  $\|\pi(a)\|_{E \setminus H} > 0$ , by definition of  $\|\pi(a)\|_{E \setminus H}$  and the characterization of the infimum of a subset of  $\mathbb{R}$ ,

$$\|\pi(a)\|_{E \setminus H} = \inf_{x \in \pi(a)} \|x\|_E$$

for all  $\varepsilon > 0$ , there exist  $x_\varepsilon \in \pi(a)$  such that,

$$\begin{aligned} \|\pi(a)\|_{E \setminus H} &\leq \|x_\varepsilon\|_E \\ &\leq \|\pi(a)\|_{E \setminus H} + \varepsilon \end{aligned}$$

implying that,

$$\frac{\|\pi(x_\varepsilon)\|_{E \setminus H}}{\|x_\varepsilon\|_E} \geq 1 - \frac{\varepsilon}{\|\pi(a)\|_{E \setminus H}}$$

Thus,

$$\begin{aligned} ||| \pi ||| &= \sup_{x \in E \setminus \{0_E\}} \frac{\|\pi(x)\|_{E \setminus H}}{\|x\|_E} \geq \frac{\|\pi(x_\varepsilon)\|_{E \setminus H}}{\|x_\varepsilon\|_E} \\ &\geq 1 - \frac{\varepsilon}{\|\pi(a)\|_{E \setminus H}} \end{aligned}$$



hence

$$||| \pi ||| \geq 1 - \frac{\varepsilon}{\|\pi(a)\|_{E \setminus H}}$$

by taking  $\varepsilon \rightarrow 0^+$  gives  $||| \pi ||| \geq 1$ , as required here  $||| \pi ||| = 1$ , completing this proof.  $\square$

### Theorem 6.0.2:

Let  $E$  be a Banach N.V.S. and  $H$  be a closed vector subspace of  $E$ , then  $E \setminus H$  is Banach.

*Proof.* To show that  $E \setminus H$  is Banach, we will prove that every normally convergent series in  $E \setminus H$  is convergent, Let  $\sum_{n=1}^{\infty} C_n$  be a normally convergent series in  $E \setminus H$ , This means that the real series  $\sum_{n=1}^{\infty} \|C_n\|_{E \setminus H}$  is convergent, by the definition of  $\|C_n\|_{E \setminus H} (= \inf_{x \in C_n} \|x\|_E)$ , and the chracterzation of the infimum of a subset of  $\mathbb{R}$ , for all  $n \in \mathbb{N}$ , there exist  $x_n \in C_n$  such that

$$\|x_n\|_E \leq \|C_n\|_{E \setminus H} + \frac{1}{2^n}$$

This implies that the real series

$$\sum_{n=1}^{\infty} \|x_n\|_E$$

converges, namely the series  $\sum_{n=1}^{\infty} x_n$  is normally convergent in  $E$ , but since  $E$  is Banach, it follows that the series  $\sum_{n=1}^{\infty} x_n$  is convergent in  $E$ . Finally, since  $\pi$  is continuous (according to proposition 2), we conclude that the series  $\sum_{n=1}^{\infty} \pi(x_n) = \sum_{n=1}^{\infty} C_n$  is convergent in  $E \setminus H$ , as required therefore  $E \setminus H$  is Banach, completing the proof.  $\square$

### The Hahn-Banach theorem

#### PreLiminaries :

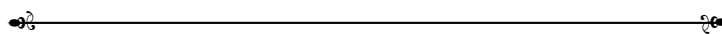
### Theorem 6.0.3: Zorn's Lemma

Let  $X$  be partially ordered suppose that every *chain*  $\mathcal{C}$  in  $X$ , (That is, every totally ordered subset of  $X$ ), has an upper bound in  $X$ . Then  $X$  contains atleast one maximal element



Note :  $m$  is upper-bound

$$\forall x \in A, x \leq m$$



## Example

**Theorem 6.0.4:**

Every vector space has a basis. (Teacher provided a Skrtch proof, we may prove it next time)

**Theorem 6.0.5: Zorn's Lemma**

Let  $X$  be a partially ordered set, suppose that every chain in  $\mathcal{C}$  in  $X$ , that is every totally ordered subset of  $X$ , has an upper bound in  $X$ . Then  $X$  contains atleast one maximal element.

**Theorem 6.0.6:**

Every vector space has (atleast) a basis.

*Proof.* Let  $E$  be a vector space over some field  $\mathbb{K}$ , (not necessarily  $\mathbb{R}$  or  $\mathbb{C}$ ), if  $E = \{0_E\}$  then  $\emptyset$  is the basis of  $E$ . Now suppose that  $E \neq \{0_E\}$ , Consider  $X$  the set of all linearly independent subsets of  $E$ , we have  $X \neq \emptyset$  because every nonzero vector of  $E$  is a linearly independent subset of  $E$ . we equip  $X$  with the partial order of set inclusion

$$(X, \subset)$$

for every chain  $\mathcal{C}$  of  $X$  we claim that the set  $\bigcup_{S \in \mathcal{C}} S$  is linearly independent. (i.e.  $\in X$ ), so  $\bigcup_{S \in \mathcal{C}} S$  constitutes an upper bound of  $\mathcal{C}$  in  $X$ , let  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{K}$  and  $x_1, \dots, x_n \in \bigcup_{S \in \mathcal{C}} S$  such that,

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = 0_E$$

and show that

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0_{\mathbb{K}}$$

by hypothesis, for all  $i \in \{1, 2, \dots, n\}$  there exists  $S_i \in \mathcal{C}$  such that  $x_i \in S_i$ . Next, since  $\mathcal{C}$  is totally ordered, there exists a bijection from  $\{1, \dots, n\}$  to  $\{1, \dots, n\}$  such that

$$S_{\sigma(1)} \subset S_{\sigma(2)} \subset \dots \subset S_{\sigma(n)}$$

consequently, we have

$$x_1, \dots, x_n \in S_{\sigma(n)}$$

But since  $S_{\sigma(n)}$  is linearly independent, then the equality

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = 0_E$$

implies that

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0_{\mathbb{K}}$$

as required, our claim is confirmed.

So we can apply the zorn lemma which ensures that  $X$  contains atleast one maximal element. Let  $B$  be a maximal element of  $X$  so  $B$  is a linearly indepdent subset of  $E$ . Next, for every vector  $x \in E$ , we have either  $x \in B$ , thus ( $x \in \langle B \rangle$ ) or  $x \notin B$ , that is  $B \subsetneq B \cup \{x\}$ , (implying according to the maximality of  $B$  in  $X$ ) that

$$B \cup \{x\} \notin X$$

that is,  $B \cup \{x\}$  is linearly dependent, hence  $x \in \langle B \rangle$ . So, we have for all  $x \in E$ ,  $x \in \langle B \rangle$ . Thus  $\langle B \rangle = E$ , Consequently,  $B$  is both linearly independent and spans  $E$ ; that is,  $B$  is a a basis of  $E$ .

Hence the proof is complete.  $\square$

## 6.1 The problem of the extension of continuous linear forms on N.V.S

**Problem 01:** Let  $E$  and  $F$  be two vector spaces over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let  $H$  be a proper subspace of  $E$ , If  $f : H \longrightarrow F$  is a linear mapping from  $H$  to  $F$  can we extend it to a linear mapping  $f^\sim : E \longrightarrow F$ .

$$\begin{array}{ccc} f^\sim : E & \xrightarrow{\pi} & H \longrightarrow F \\ x & \longmapsto & f^\sim(x) \end{array}$$

**Answer: Yes!**

It sufficies to consider a complementary subspace  $G$  of  $H$  in  $E$ , i.e.

$$G \oplus H = E$$

$$\begin{array}{ccc} f^\sim : & E & \longrightarrow F \\ x = h + g (h \in H, g \in G) & \longmapsto & f(h) \end{array}$$

In other words, we have  $f^\sim = f \circ \pi$ , where  $\pi$  is the projection of  $E$  into  $H$  parallel to  $G$

$$\begin{array}{ccccc} f : & E & \xrightarrow{\pi} & H & \xrightarrow{f} F \\ x = h + g & \longmapsto & h & \longmapsto & f(h) \end{array}$$

since  $\pi$  is linear then  $f^\sim = f \circ \pi$  is linear and since  $\pi(h) = h (\forall h \in H)$ , then

$$f^\sim|_H = f$$

that is  $f^\sim$  extends  $f$

**Problem 02:**

Now, suppose that  $E$  and  $F$  are two N.V.S over the same field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let  $H$  be a proper normed vector subspace of  $E$  and  $f : H \rightarrow F$  linear and continuous . Is't possible to extend  $f$  to some linear and continuous mapping  $f^\sim : E \rightarrow F$

**Answer : No, in general !**

Note that the method used to solve **Problem 01** fails because the considered projection  $\pi$  is in general not continuous.

#### Definition 6.1.1:

Let  $E$  be an  $\mathbb{R}$ -N.V.S, and  $p : E \rightarrow \mathbb{R}$  be a map, we say that  $p$  is sublinear if it satisfies :

- (i)  $p(x + y) \leq p(x) + p(y) \quad (\forall x, y \in \mathbb{R})$
- (ii)  $p(\lambda x) = \lambda p(x) \quad (\forall \lambda \geq 0, \forall x \in E)$

#### Theorem 6.1.1: The Hahn-Banach Theorem

Let  $E$  be an  $\mathbb{R}$ -vector space and  $p : E \rightarrow \mathbb{R}$  be a *sublinear* function. Then any linear form  $f$  on a vector subspace  $H$  of  $E$  that is dominated above by  $p$  has at least one linear extension to all  $E$  that is also dominated above by  $p$ . More explicitly, for every linear form  $f : H \rightarrow \mathbb{R}$  satisfying

$$f(x) \leq p(x) \quad (\forall x \in H)$$

there exists a linear form  $f^\sim : E \rightarrow \mathbb{R}$  such that

$$f^\sim|_H = f \text{ and } f^\sim(x) \leq p(x) \quad (\forall x \in E)$$

*Proof.* Let  $H$  be a vector subspace of  $E$  and  $f : H \rightarrow \mathbb{R}$  be a linear form on  $H$  that is dominated above by  $p$  since the result of the theorem is trivial for  $H = E$  suppose for the sequel that  $H \neq E$ .

#### 1<sup>st</sup> Step

let  $u \in E \setminus H$  be fixed we are going to show that there exist a linear form  $g : H \oplus \mathbb{R}u \rightarrow \mathbb{R}$ , extending  $f$  and satisfying  $g(x) \leq p(x)$  for all  $x \in H + \mathbb{R}u$ , the determination of such a  $g$  is clearly equivalent to the determination of its value at  $u$ , that is the determination of  $\lambda := g(u) \in \mathbb{R}$  so that we have for all  $h \in H$  and all  $t \in \mathbb{R}$ ,

$$g(h + tu) \leq p(h + tu)$$

that is, since  $g$  should be linear and extend  $f$ ,

$$g(h) + tg(u) \leq p(h + tu)$$

i.e.,

$$f(h) + t\lambda \leq p(h + tu) \quad (\forall h \in H, \forall t \in \mathbb{R}) \quad (1)$$

since (1) is obviously satisfied for  $t = 0$ , then we have

$$(1) \iff \begin{cases} f(\frac{1}{t}h) + \lambda \leq p(\frac{1}{t}h + u) & \text{if } t > 0 \\ f(\frac{1}{t}h)L + \lambda \leq -p(-\frac{1}{t}h - u) & \text{if } t < 0 \end{cases} \quad (2)$$

and we have

$$(2) \iff \lambda \leq p(x + u) - f(x) \quad (\forall x \in H)$$

$$(3) \iff \lambda \geq f(y) - p(y - u) \quad (\forall y \in H)$$

thus

$$(1) \iff f(y) - p(y - u) \leq \lambda \leq p(x + u) - f(x) \quad (\forall x, y \in H)$$

$$\iff \sup_{y \in H} \{f(y) - p(y - u)\} \leq \lambda \leq \inf_{x \in H} \{p(x + u) - f(x)\} \quad (4)$$

the existence of  $\lambda$  is then equivalent to

$$\sup_{y \in H} \{f(y) - p(y - u)\} \leq \inf_{x \in H} \{p(x + u) - f(x)\} \quad (*)$$

Let us show (\*), for all  $x, y \in H$ , we have according to the assumption made on  $f$  and  $p$ ,

$$\begin{aligned} f(x) + f(y) &= f(x + y) \leq p(x + y) = p((y - u) + (x + u)) \\ &\leq p(y - u) + p(x + u) \end{aligned}$$

hence

$$f(y) - p(y - u) \leq p(x + u) - f(x) \quad (\forall x, y \in H)$$

thus,

$$\sup_{y \in H} \{f(y) - p(y - u)\} \leq \inf_{x \in H} \{p(x + u) - f(x)\}$$

confirming (\*), Hence the existence of  $\lambda$  as required and then the existence of  $g$  as required.

## 2<sup>nd</sup> Step

Consider the set  $X$  of the pairs  $(F, \varphi)$ , where  $F$  is a subspace of  $E$  containing  $H$  and  $\varphi$  is a linear form on  $F$  extending  $f$  and satisfying

$$\varphi(x) \leq p(x) \quad (\forall x \in F)$$

Since  $(H, f) \in X$  then  $X \neq \emptyset$ , we equip  $X$  with the binary relation  $\mathcal{R}$  defined by

$$(F_1, \varphi_1) \mathcal{R} (F_2, \varphi_2) \iff F_1 \subset F_2 \text{ and } \varphi_2|_{F_1} = \varphi_1$$

we easily check that  $\mathcal{R}$  is a partial order on  $X$ .

Next for every chain  $((F_i, \varphi_i))_{i \in I}$  of  $X$ , the pair  $(F, \varphi)$  given by

$$F = \bigcup_{i \in I} F_i \quad \varphi(x) = \varphi_i(x) \quad (\forall i \in I, \forall x \in F_i)$$

Clearly

The zorn lemma to desire that  $(X, \mathcal{R})$  has at least 1 maximal element  $(F^\sim, \varphi^\sim)$  but if  $F^\sim \neq E$  and  $u \in E \setminus F^\sim$ , by the 1<sup>st</sup> step, we can construct a pair

$$(F^\sim \oplus \mathbb{R}u, \quad \Psi) \in X$$

which we strictly greater

Thus  $F^\sim = E$ . So it suffices to take  $f^\sim = \varphi^\sim$  to conclude to the result of the theorem:  $\square$

### Theorem 6.1.2: (Hahn-Banach)

Let  $E$  be a  $\mathbb{K}$ -vector space ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) and  $H$  be a vector subspace of  $E$  let also  $N : E \rightarrow [0, \infty)$  be a seminorm on  $E$  and  $f : H \rightarrow \mathbb{K}$  be a linear form on  $H$ , satisfying

$$|f(x)| \leq N(x) \quad (\forall x \in H)$$

then there exist a  $\mathbb{K}$ -linear form  $\tilde{f} : E \rightarrow \mathbb{K}$ , extending  $f$  and satisfying

$$\tilde{f} \leq N(x) \quad (\forall x \in E)$$

*Proof. Case 01 :*

If  $\mathbb{K} = \mathbb{R}$  since we have for all  $x \in H$ ,

$$f(x) \leq |f(x)| \leq N(x)$$

then by applying Theorem 1 for  $p = N$ , we find that there exist a linear form  $\tilde{f} : E \rightarrow \mathbb{R}$  extending  $f$  and satisfying

$$\forall x \in E : \tilde{f}(x) \leq N(x) \quad (1)$$

By applying (1) for  $(-x)$  instead of  $x$ , we get,

$$\begin{aligned}\tilde{f}(-x) &\leq N(-x) = N(x) \\ -\tilde{f}(x) &\leq N(x) \\ \tilde{f}(x) &\geq -N(x)\end{aligned}\quad (2)$$

from (1) and (2), we have

$$\begin{aligned}\iff -N(x) &\leq \tilde{f}(x) \\ \iff \left| \tilde{f}(x) \right| &\leq N(x)\end{aligned}$$

### Case 02:

Define

$$\begin{aligned}g : H &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) := \operatorname{Re} f(x) = \frac{f(x) + \overline{f(x)}}{2}\end{aligned}$$

Its clear that  $g$  is an  $\mathbb{R}$ -linear form on  $H$ , next we have for all  $x \in H$ ,

$$\begin{aligned}|g(x)| &= |\operatorname{Re}(f(x))| \leq |f(x)| \\ &\leq N(x)\end{aligned}$$

for all  $x \in H$ ,

$$|g(x)| \leq N(x)$$

so we can apply the result of the first case, for the linear form  $g$  on  $H$ , we find that  $\exists \tilde{g} : E \longrightarrow \mathbb{R}$  an  $\mathbb{R}$ -linear extending  $g$ , and satisfying,

$$\forall x \in E : \quad \left| \tilde{g}(x) \right| \leq N(x)$$

Furthermore, we have, for all  $x \in H$ ,

$$\begin{aligned}g(ix) &= \operatorname{Re}(\overline{f}(ix)) \\ &= \operatorname{Re}(if(x)) \\ &= -\operatorname{Im}f(x) \\ \implies \operatorname{Im}f(x) &= -g(ix)\end{aligned}$$

Then for all  $x \in H$ ,

$$\begin{aligned}f(x) &= \operatorname{Re}f(x) + i\operatorname{Im}f(x) \\ &= g(x) - ig(ix)\end{aligned}$$

Thus, we have for all  $x \in H$ ,

$$f(x) = g(x) - ig(ix) \quad (1)$$

therefore define,  $\tilde{f} : E \longrightarrow \mathbb{C}$ , by,

$$\tilde{f}(x) = \tilde{g}(x) - i\tilde{g}(ix)$$

**We will prove that it's an extension**

(1) Show that  $\tilde{f}$  extends  $f$ .

(2) Show that  $\tilde{f}$  is  $\mathbb{C}$ -linear.

*Proof.* Since  $\tilde{g}$  is  $\mathbb{R}$ -linear then  $\tilde{f}$  is obviously  $\mathbb{R}$ -linear. So, to show that  $\tilde{f}$  is  $\mathbb{C}$ -linear it suffices to show that

$$\tilde{f}(ix) = i\tilde{f}(x) \quad (\forall x \in E)$$

for all  $x \in E$ , we have,

$$\begin{aligned} \tilde{f}(ix) &= \tilde{g}(ix) - i\tilde{g}(-x) \\ &= \tilde{g}(ix) + i\tilde{g}(x) \\ &= i(\tilde{g}(x) - i\tilde{g}(ix)) = i\tilde{f}(x) \end{aligned}$$

as required, then  $\tilde{f}$  is  $\mathbb{C}$ -linear. □

Now we have to show that

$$\left| \tilde{f}(x) \right| \leq N(x) \quad (\forall x \in E)$$

Finally, for all  $x \in E$ , by writting the complex number  $\tilde{f}(x)$  in it exponential form, say,

$$\tilde{f}(x) = \left| \tilde{f}(x) \right| e^{i\theta} \quad (\theta \in \mathbb{R})$$

we have,

$$\begin{aligned} \left| \tilde{f}(x) \right| &= \tilde{f}(x) e^{-i\theta} \\ &= \tilde{f}(xe^{-i\theta}) \\ &= \operatorname{Re} \tilde{f}(xe^{-i\theta}) \\ &= \tilde{g}(xe^{-i\theta}) \\ &\leq N(xe^{-i\theta}) = \left| e^{-i\theta} \right| N(x) = N(x) \end{aligned}$$

Thus

$$\left| \tilde{f}(x) \right| \leq N(x) \quad (\forall x \in E)$$

as required, thus this completes the proof. □



**Theorem 6.1.3: Hahn-Banach**

Let  $E$  be a N.V.S over a field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $H$  be a non zero subspace of  $E$ , then for all  $f \in H' = \mathcal{L}(H, \mathbb{K})$  there exists

$$\tilde{f} \in E' = \mathcal{L}(E, \mathbb{K})$$

extending  $f$  and satisfying,

$$||| \tilde{f} |||_{E'} = ||| f |||_{H'}$$

*Proof.* let  $f \in H'$ . By applying Theorem 2 for  $N(x) = ||| f ||| \cdot \|x\|$ , let us verify that  $f$  is dominated by  $N$  on  $H$  we have for all  $x \in H$ ,

$$|f(x)| \leq ||| f ||| \|x\| = N(x)$$

we find that there exist  $\tilde{f} : E \rightarrow \mathbb{K}$  linear and extending  $f$  and satisfying for all  $x \in E$ ,

$$|\tilde{f}(x)| \leq N(x) = ||| f |||_{H'} \cdot \|x\|$$

implying that  $\tilde{f}$  is continuous, thus  $\tilde{f} \in E'$  and that

$$||| \tilde{f} ||| \leq ||| f |||$$

On the other hand, we have,

$$\begin{aligned} ||| \tilde{f} |||_E &= \sup_{x \in E \setminus \{0_E\}} \frac{|\tilde{f}(x)|}{\|x\|} \geq \sup_{x \in H \setminus \{0_E\}} \frac{|\tilde{f}(x)|}{\|x\|} \\ &= \sup_{x \in H \setminus \{0_E\}} \frac{|f(x)|}{\|x\|} \\ &= ||| f |||_{H'} \end{aligned}$$

Hence

$$||| \tilde{f} |||_{E'} = ||| f |||_{H'}$$

completing this proof. □

**Some Theorems****Theorem 6.1.4:**

Let  $E$  be a nonzero N.V.S over a field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , Then

(1) for all  $x \in E \setminus \{0_E\}$ , there exist a continuous linear form  $f$  on  $E$  such that

$$f(x) = \|x\|_E \quad \text{and} \quad ||| f ||| = 1$$

In particular

$$E' \neq \{0_{E'}\}$$

(2) Let  $x, y \in E$  such that,

$$f(x) = f(y) \quad (\forall f \in E') \implies x = y$$

*Proof.* (1) Consider  $H := \langle x \rangle$ ,  $H$  is a subspace of  $E$ , and

$$\begin{aligned} h : H &\longrightarrow \mathbb{K} \\ \lambda x &\longmapsto \lambda \|x\| \end{aligned} \quad (\forall \lambda \in \mathbb{K})$$

It's clear that  $h$  is linear,  $h(x) = \|x\|$  by taking  $\lambda = 1$ ,  $h$  is continuous because ( $\dim(H) = 1 < \infty$ ), by Theorem 3, there exists  $f : E \longrightarrow \mathbb{K}$ , linear continuous and satisfies

$$\begin{aligned} \|f\|_{E'} &= \|h\|_{H'} := \sup_{\lambda \in \mathbb{K}^*} \frac{|h(\lambda x)|}{\|\lambda x\|} \\ &= \frac{|h(x)|}{\|x\|} = 1 \end{aligned}$$

so  $f$  extends  $h$  and  $x \in H$ , we have

$$f(x) = h(x) = \|x\|$$

this completes the proof of (1).

(2) Let us show the contrapositive, i.e.

$$\forall x, y \in E : (x \neq y \implies \exists f \in E' : f(x) \neq f(y))$$

let  $x, y \in E$  such that  $x \neq y$ , and set  $z := x - y \in E \setminus \{0_E\}$ , by applying the result of (1) for  $z$ , we find that there exist  $f \in E'$  such that,

$$f(z) = \|z\| \neq 0$$

hence we have,

$$f(x - y) = f(x) - f(y)$$

thus there exist  $f \in E'$  such that

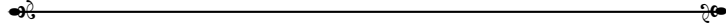
$$f(x) \neq f(y)$$

as required. Hence this completes the proof. □

**Remark :**

The property of item 2 of Theorem 1 is expressed literally by saying that,

*"The continuous linear forms on  $E$  separate the vectors of  $E$  "*



*Remark by the Writer : Sometimes when i write  $E \setminus H$  i mean quotient space not minus, understand from context.*

**Theorem 6.1.5: Theorem 2**

Let  $E$  be a N.V.S over a field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and let  $H$  be a subspace of  $E$ , and  $x \in E \setminus \overline{H}$  then there exists a continuous, linear form  $f$  on  $E$  such that

$$||| f ||| \leq 1$$

and

$$f(x) = d(x, H) \neq 0 \quad f(H) = \{0\}$$

*Proof.* We apply Item 1 of Theorem 1 for the N.V.S Quotient space  $E_{\setminus \overline{H}}$  and the non zero vector  $cl(x) = x + \overline{H}$ , where

$$(cl(x) \neq 0_{E_{\setminus \overline{H}}} \neq 0_{E_{\setminus \overline{H}}} \text{ since } x \notin \overline{H})$$

let,  $\pi : E \longrightarrow E_{\setminus \overline{H}}$  be the quotient map. It's known that  $\pi$  is continuous and that  $||| \pi ||| = 1$ , By Item 1 from Theorem 1, there exists a continuous linear form  $\bar{f}$  on  $E_{\setminus \overline{H}}$  if

$$\bar{f}(\pi(x)) = \|\pi(x)\|_{E_{\setminus \overline{H}}}$$

and

$$||| \bar{f} ||| = 1$$

consider,  $f : E \xrightarrow{\pi} E_{\setminus \overline{H}} \xrightarrow{\bar{f}} \mathbb{K}$ , i.e.

$$f = \bar{f} \circ \pi$$

$f$  is linear and continuous because its a composition of two linear and continuous maps. Then  $f \in E'$ , Next, we have

$$||| f ||| = ||| \bar{f} \circ \pi ||| \leq \underbrace{||| \bar{f} |||}_{=1} \cdot \underbrace{||| \pi |||}_{=1}$$

thus,

$$||| f ||| \leq 1$$

Next, we have,

$$\begin{aligned}
 f(x) &= (\bar{f} \circ \pi)(x) = \bar{f}(\pi(x)) = \|\pi(x)\|_{E \setminus \bar{H}} \\
 &= \inf_{y \in \pi(x)} \|y\|_E \\
 &= \inf_{h \in \bar{H}} \|x + h\|_E \\
 &= \inf_{h \in \bar{H}} \|x - h\|_E \\
 &= \inf_{h \in \bar{H}} d(x, h) \\
 &= d(x, \bar{H}) = d(x, H)
 \end{aligned}$$

Finally, we have,

$$\begin{aligned}
 f(H) &= (\bar{f} \circ \pi)(H) = \bar{f} \left( \underbrace{\pi(H)}_{= \{0_{E \setminus \bar{H}} \text{ since } H \subset \bar{H}\}} \right) \\
 &= \bar{f}(\{0_{E \setminus \bar{H}}\}) = \{0\}
 \end{aligned}$$

This completes the proof. □

### Theorem 6.1.6: Theorem 3

Let  $E$  be a N.V.S over a field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $H$  be a subspace of  $E$ , Then the two following properties are equivalent,

- (i)  $H$  is dense in  $E$
- (ii) for all  $f \in E'$ , we have,

$$f|_H \text{ is zero} \implies f \text{ is zero}$$

*Proof.* Let's start proving!

$$(i) \implies (ii)$$

Already known !.

Suppose that  $H$  is dense in  $E$  (i.e.  $\bar{H} = E$ ) and let  $f \in E'$  such that  $f|_H = 0$ , that is  $f(h) = 0$  for all  $h \in H$ .

Then, giving  $x \in E$  since  $H$  is dense in  $E$ , then there exist a sequence  $(h_n)_{n \in \mathbb{N}}$  in  $H$  converging to  $x$ ,

thus we have,

$$\begin{aligned} f(x) &= f(\lim_{n \rightarrow \infty} h_n) \\ &= \lim_{n \rightarrow \infty} f(h_n) \\ &= \lim_{n \rightarrow \infty} 0 = 0 \quad (\text{since } h_n \in H \text{ and } f|_H \text{ is zero}) \end{aligned}$$

Thus  $f = 0_E$ , as required.

$$(ii) \implies (i)$$

let us show the contrapositive

$$\overline{(i)} \implies \overline{(ii)}$$

suppose that  $\overline{(i)}$  i.e.  $\overline{H} \neq E$ , thus there exists  $x \in E \setminus \overline{H}$ . By Theorem 2, there exists  $f \in E'$  such that  $f(H) = \{0\}$ , and  $f(x) = d(x, H) \neq 0$ , in other words  $d(x, H) \neq 0$  since  $x \notin \overline{H}$  so  $f \in E'$ , and  $f|_H = 0_{H'}$  and  $f \neq 0_{E'}$  since  $f(x) \neq 0$ .

This completes the proof. □

#### Theorem 6.1.7: Theorem 4

Let  $E$  be a N.V.S,  $n$  be a positive integer,  $x_1, \dots, x_n$  be  $n$  vector linearly independent of  $E$ , and  $c_1, \dots, c_n$  be  $n$  scalars then there exists a continuous linear form on  $f$  on  $E$  such that

$$f(x_i) = c_i \quad \forall i \in \{1, \dots, n\}$$

#### Theorem 6.1.8:

Let  $E$  be a N.V.S,  $n$  be a positive integer,  $x_1, \dots, x_n$  be  $n$  linearly independent vectors of  $E$ , and  $c_1, \dots, c_n$  be  $n$  scalars. Then there exist a continuous linear form  $f$  on  $E$  such that  $f(x_i) = c_i$  for all  $i \in \{1, \dots, n\}$ .

*Proof.* Let

$$H := \langle x_1, \dots, x_n \rangle$$

and  $h : H \longrightarrow \mathbb{K}$  be the linear form on  $H$  defined by

$$h\left(\sum_{i=1}^n \lambda_i x_i\right) = \sum_{i=1}^n \lambda_i c_i \quad (\forall \lambda_i \in \mathbb{K} \forall i = 1, \dots, n)$$

so for all  $i \in \{1, \dots, n\}$ , we have  $h(x_i) = c_i$ , since  $\dim(H) = n < \infty$ , then  $h$  is continuous, so by the Hahn-Banach theorem, there exist  $f \in E'$  extending  $h$ , so for all  $i \in \{1, \dots, n\}$ , we have that

$$f(x_i) = h(x_i) = c_i$$

hence the proof is complete. □

## 6.2 The Geometric form of the Hahn-Banach Theorem

The geometric form of the Hahn-Banach Theorem deals with the separation of disjoint convex sets using affine hyperplanes.

### Reminders :

Let  $E$  be a N.V.S over  $\mathbb{K}$  or  $\mathbb{C}$ . An affine hyperplane of  $E$  is a subset  $H$  of  $E$ , of the form,

$$H := \{x \in E : f(x) = \alpha\}$$

for some  $f \in E^* \setminus \{0_{E^*}\}$  and  $\alpha \in \mathbb{K}$ , Its known that  $H$  is closed if and only if  $f$  is continuous.

### Theorem 6.2.1:

Let  $E$  be a N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $C$  be an open and convex subset of  $E$ , containing  $0_E$ , for all  $x \in E$ , define,

$$p(x) := \inf \left\{ \alpha > 0, \alpha^{-1}x \in C \right\}$$

then,

(i)  $p$  is sublinear i.e.

$$\begin{cases} \text{Sub additive} \rightarrow p(x+y) \leq p(x) + p(y) & \forall x, y \in E \\ \text{Positively homogenous} \rightarrow p(\lambda x) = \lambda(x) & \forall \lambda \geq 0 \end{cases}$$

(ii)  $\exists M > 0$  such that for all  $x \in E$ , we have,

$$p(x) \leq M\|x\|$$

(iii) and we have,

$$C = \{x \in E : p(x) < 1\}$$

we have that  $p$  is called the Minkowski functional of  $C$ .

*Proof.* Let us first prove item (ii), Since  $C$  is open and contains  $0_E$ , then there exist  $r > 0$ , such that  $B(0_E, r)$ , Now for all  $x \in E \setminus \{0_E\}$ , we have

$$\frac{r}{2} \frac{x}{\|x\|} \in B(0_E, r) \subset C$$

implying that the positive real number,  $\alpha = \frac{2}{r}\|x\|$  satisfies

$$\alpha^{-1}x \in C$$

thus, by definition of  $p$ ,

$$p(x) \leq \frac{2}{r}\|x\|$$

This proves then the positive constant  $M = \frac{2}{r}$ .

Now let us prove then (iii)

$$C \subset \{x \in E : p(x) < 1\}$$

let  $x \in C$ , for  $x = 0_E$ , then we have clearly that

$$p(x) = p(0_E) = 0 < 1$$

suppose that  $x \neq 0_E$  and let us show that  $p(x) < 1$ , since  $C$  is open and  $x \in C$ , then  $\exists \varepsilon > 0$  such that

$$B_E(x, \varepsilon) \subset C$$

so from,

$$\left(1 + \frac{\varepsilon}{2\|x\|}\right)x \in B_E(x, \varepsilon) \subset C$$

we desire that  $\alpha_0 = \left(1 + \frac{\varepsilon}{2\|x\|}\right)^{-1}$ , satisfies that  $\alpha_0^{-1}x \in C$ , thus,

$$p(x) \leq \alpha_0 < 1$$

hence  $p(x) < 1$  as required.

$$\{x \in E : p(x) < 1\} \subset C$$

let  $x \in E$  such that  $p(x) < 1$  and let us prove that  $x \in C$ . So by definition of  $p(x)$  there exist  $t \in (0, 1)$  such that  $t^{-1}x \in C$  now since  $C$  is convex and  $0_E, t^{-1}x \in C$ , then we have

$$t(t^{-1}x) + (1-t)0_E \in C$$

in other words,

$$x \in C$$

as required, Hence we have the equality,

$$C = \{x \in E : p(x) < 1\}$$

Finally let us prove (i).

*Is  $p$  positively homogenous ?*

for all  $\lambda > 0$ , and  $x \in E$ , we have,

$$\begin{aligned} p(\lambda x) &:= \inf \left\{ \alpha > 0 : \alpha^{-1} \lambda x \in C \right\} \\ &= \lambda \left\{ \lambda^{-1} \alpha : \left( (\lambda^{-1} \alpha)^{-1} x \in C \right) \right\} \\ &= \lambda \left\{ \beta > 0, \beta^{-1} x \in C \right\} \end{aligned}$$

thus,

$$\begin{aligned} p(\lambda x) &:= \inf \lambda \left\{ \beta > 0, \beta^{-1} x \in C \right\} \\ &= \lambda \underbrace{\inf \left\{ \beta > 0, \beta^{-1} x \in C \right\}}_{p(x)} \\ &= \lambda p(x) \end{aligned}$$

*Is  $p$  sub additive ?*

Let  $x, y \in E$  be arbitrary, and show that,

$$p(x + y) \leq p(x) + p(y)$$

For  $\varepsilon > 0$ , we have from the positive homogeneity of  $p$  that,

$$p\left(\frac{1}{p(x) + \varepsilon} x\right) = \frac{1}{p(x) + \varepsilon} p(x) < 1$$

implying then (iii) already proved that,

$$\frac{1}{p(x) + \varepsilon} x \in C$$

similarly

$$\frac{1}{p(y) + \varepsilon} y \in C$$

so setting,

$$t = \frac{p(x) + \varepsilon}{p(x) + p(y) + 2\varepsilon} \in (0, 1)$$

we have from the convexity of  $C$ ,

$$t \left( \frac{1}{p(x) + \varepsilon} x \right) + (1 - t) \left( \frac{1}{p(y) + \varepsilon} y \right) \in C$$

hence,

$$\frac{1}{p(x) + p(y) + 2\varepsilon} x + \frac{1}{p(x) + p(y) + 2\varepsilon} y \in C$$



we get then,

$$\frac{1}{p(x) + p(y) + 2\varepsilon}(x + y) \in C$$

hence

$$p\left(\frac{1}{p(x) + p(y) + 2\varepsilon}(x + y)\right) < 1$$

by the positive homogeneity of  $p$ , it follows that,

$$\frac{1}{p(x) + p(y) + 2\varepsilon}p(x + y) < 1$$

i.e.

$$p(x + y) < p(x) + p(y) + 2\varepsilon$$

by taking  $\varepsilon \rightarrow 0^+$  it gives us, the inequality,

$$p(x + y) \leq p(x) + p(y)$$

as required. This completes the proof. □

### The geometric versions of the Hahn-Banach Theorem;

#### Theorem 6.2.2: The first geometric version of the Hahn-Banach Theorem

Let  $E$  be an  $\mathbb{R}$  N.V.S,  $A$  and  $B$  be two *nonempty disjoint convex* subsets of  $E$ , Suppose that  $A$  is open then. There exists affine hyperplane of  $E$  which separates  $A$  and  $B$ , that is there exists a non-zero continuous linear form  $f$  on  $E$  and a real number  $\alpha$  such that,

$$f(x) \leq \alpha \leq f(y) \quad (\forall x \in A, \forall y \in B)$$

#### Theorem 6.2.3: The second geometric version of the Hahn-Banach Theorem

let  $E$  be on  $\mathbb{R}$ -N.V.S and  $A$  and  $B$  be two nonempty disjoint convex subsets of  $E$ , suppose that  $A$  is closed and  $B$  is compact, then there exists closed affine hyperplane of  $E$  which separates strictly  $A$  and  $B$ , that is, there exists a nonzero continuous linear form  $f$  on  $E$  and a real number  $\alpha$  such that

$$f(x) < \alpha < f(y) \quad (\forall x \in A, \forall y \in B)$$

To prove these theorems, we need the propositions

**corollary 6.2.1:**

Let  $E$  be an  $\mathbb{R}$ -N.V.S,  $C$  be a non empty open convex subset of  $E$  and  $x_0 \in E \setminus C$ , then there exists a non zero continuous linear form  $f$  on  $E$  such that,

$$f(x) < f(x_0) \quad (\forall x \in C)$$

In other words, the closed affine hyper plane of  $E$  of equation

$$f(x) = f(x_0)$$

separates  $\{x_0\}$  and  $C$

*Proof.* By translating if necessary  $C$  and  $x$  by  $a$  some vector of  $(-C)$ , suppose that  $0_E \in C$ , and let  $p$  denote the Minkowski functional of  $C$ , introduce

$$H := \langle x_0 \rangle$$

and  $h : H \rightarrow \mathbb{R}$  and  $h(\lambda x_0) = \lambda$  for all  $\lambda \in \mathbb{R}$ , clearly  $h$  is a linear form on  $H$ , Next since

$$C = \{x \in E, p(x) < 1\}$$

By item (3) of the previous proposition, and  $x_0 \notin C$  then  $p(x_0) \geq 1$ , then

$$h(x_0) = 1 \leq p(x_0)$$

it follows by distinguishing the cases  $\lambda > 0$  and  $\lambda \leq 0$  that,

if  $\lambda > 0$ , then we have,

$$h(\lambda x_0) = \lambda h(x_0) = \lambda$$

$$p(\lambda x_0) = \lambda p(x_0) \geq \lambda$$

so  $h(\lambda x_0) \leq p(\lambda x_0)$ .

if  $\lambda \leq 0$ , then we have

$$h(\lambda x_0) = \lambda h(x_0) = \lambda \leq 0$$

and

$$p(\lambda x_0) \geq 0$$

then

$$h(\lambda x_0) \leq p(\lambda x_0)$$

so for all  $\lambda \in \mathbb{R}$ , we have

$$h(\lambda x_0) \leq p(\lambda x_0)$$

i.e.,

$$\forall x \in H, h(x) \leq p(x)$$

(according to the Hahn Banach Theorem) there exists a linear form  $f$  on  $E$ , extending  $h$  such that,

$$f(x) \leq p(x) \quad (\forall x \in E)$$

Let us show that  $f$  is continuous, by item (ii) of the previous propositions, there exists  $M > 0$  constant such that

$$p(x) \leq M\|x\|$$

for all  $x \in E$ , thus

$$f(x) \leq p(x) \leq M\|x\| \quad (\forall x \in E)$$

therefore

$$f(x) \leq M\|x\| \quad (\forall x \in E)$$

so by taking  $(-x)$  instead of  $x$ , we get

$$f(-x) \leq M\|-x\| \quad (\forall x \in E)$$

therefore

$$f(x) \geq -M\|x\|$$

thus,

$$-M\|x\| \leq f(x) \leq M\|x\| \quad (\forall x \in E)$$

that is,

$$|f(x)| \leq M\|x\| \quad (\forall x \in E)$$

Implying that  $f$  is continuous.

1. Since  $f$  extends  $h$  and  $x_0 \in H$ , then,

$$f(x_0) = h(x_0) = 1 \neq 0$$

thus  $f$  is non-zero,

2. for all  $x \in C$ , we have  $p(x) < 1$ , thus

$$f(x) \leq p(x) < 1 = f(x_0)$$

thus

$$\forall x \in C : f(x) < f(x_0)$$

This completes the proof.

□

## 6.3 The Geometric versions of the Hahn-Banach Theorem

**Lemma 02** Let  $(E, d)$  be a metric space and let  $A$  and  $B$  be two nonempty disjoint subsets of  $E$  such that  $A$  is closed and  $B$  is compact then we have  $d(A, B) > 0$

*Proof.* Suppose for contradiction that  $d(A, B) = 0$ . Then by the definition of  $d(A, B) := \inf_{a \in A, b \in B} d(a, b)$ , for all  $n \in \mathbb{N}$ : there exists  $a_n \in A$  and  $b_n \in B$  such that,

$$d(a_n, b_n) < \frac{1}{n}$$

since  $B$  is compact, then we can extract a subsequence  $(b_{n_k})_{k \in \mathbb{N}}$  of  $B$  a convergent subsequence  $(b_{n_k})_{k \in \mathbb{N}}$ . let  $b = \lim_{k \rightarrow \infty} b_{n_k}$ , then for all  $k \in \mathbb{N}$ ,

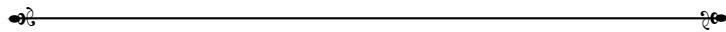
$$\begin{aligned} d(a_{n_k}, b) &\leq d(a_{n_k}, b_{n_k}) + d(b_{n_k}, b) \\ &\leq \frac{1}{n_k} + d(b_{n_k}, b) \end{aligned}$$

hence,  $\lim_{k \rightarrow \infty} d(a_{n_k}, b) = 0$ , implying that the sequence  $(a_{n_k})_{k \in \mathbb{N}}$  of  $A$  converges to  $b$ , then  $b \in \overline{A} = A$  since  $A$  is closed, thus  $b \in A \cap B = \emptyset$ , Contradiction hence  $d(A, B) > 0$ , as required.  $\square$

*Now we will prove the 1<sup>st</sup> Geometric version of the Hahn-Banach theorem.*

$$\begin{aligned} C &:= A - B \\ &= \{a - b, \in A, b \in B\} \end{aligned}$$

Since  $A$  and  $B$  are convex, then  $C$  is convex, quick scratch proof,



Let  $c_1, c_2 \in C$  and  $E \in [0, 1]$ , does there exists  $tc_1 + (1 - t)c_2 \in C$ ,

$$\exists a_1 \in A, b_1 \in B \text{ such that } c_1 = a_1 - b_1$$

$$\exists a_2 \in A, b_2 \in B \text{ such that } c_2 = a_2 - b_2$$

thus

$$\begin{aligned} tc_1 + (1 - t)c_2 &= t(a_1 - b_1) + (1 - t)(a_2 - b_2) \\ &= \left( \underbrace{ta_1 + (1 - t)a_2}_{\in A \text{ convex}} \right) - \left( \underbrace{tb_1 + (1 - t)b_2}_{\in B \text{ convex}} \right) \in A - B = C \end{aligned}$$

Thus  $C$  is convex.



Since  $A \neq \emptyset$  and  $B \neq \emptyset$  then  $C \neq \emptyset$ , since  $A$  and  $B$  are disjoint then  $0_E \notin C$ , Next we remark that,

$$C = \bigcup_{b \in B} (A - b)$$

$$\begin{aligned} \tau_b : E &\longrightarrow E \\ x &\longmapsto x + b \end{aligned}$$

For all  $b \in B$ , since  $\tau_b$  is continuous and  $A$  is open then  $\tau_b^{-1}(A) = A - b$  is open, thus  $C$  is a union of open subsets of  $E$  implying that  $C$  is open in  $E$ .

By applying **Lemma 01** for the convex subset  $C$  of  $E$  and for  $x_0 = 0_E \notin C$ , we find that there exist  $f \in E^* \setminus \{0_E\}$  such that

$$f(x) < f(0_E) = 0 \quad (\forall x \in C)$$

writing  $x = a - b$  ( $a \in A, b \in B$ ), we conclude that

$$f(a - b) < 0$$

hence

$$f(a) < f(b) \quad (\forall a \in A, \forall b \in B)$$

thus we get

$$\sup_{a \in A} f(a) \leq f(b) \quad (\forall b \in B)$$

thus we get

$$\sup_{a \in A} f(a) \leq \inf_{b \in B} f(b)$$

now consider  $\alpha \in [\sup_{a \in A} f(a), \inf_{b \in B} f(b)]$  then we get for all  $a \in A$  and for all  $b \in B$ ,

$$f(a) \leq \alpha \leq f(b)$$

this completes the proof.

*Proof.* Now we will prove the second geometric version of the Hahn- Banach Theorem □

Since  $A \cap B = \emptyset$ , and  $A$  is closed and  $B$  is compact then there exist by **Lemma 02**  $d(A, B) > 0$ , fix  $\varepsilon > 0$  such that

$$\varepsilon < \frac{1}{2}d(A, B)$$

now consider

$$A_\varepsilon = A + B_E(0_E, \varepsilon)$$

$$B_\varepsilon = B + B_E(0_E, \varepsilon)$$

It's clear that  $A_\varepsilon$  and  $B_\varepsilon$  are non empty since  $A, B$  and  $B_E(0_E, \varepsilon) \neq \emptyset$ .

Next,  $A_\varepsilon$  and  $B_\varepsilon$  are both convex since they are sums of convex subsets, next  $A_\varepsilon$  and  $B_\varepsilon$  are disjoint, Indeed, suppose for contradiction that  $A_\varepsilon \cap B_\varepsilon \neq \emptyset$ , then there exist  $x \in E$  such that  $x \in A_\varepsilon \cap B_\varepsilon$ , then we can write  $x$  as,

$$x = a + u = b + v \quad (u, v \in B_E(0_E, 1) \quad a, b \in A, B)$$

Thus,

$$\begin{aligned} \|a - b\| &= \|v - u\| \\ &\leq \underbrace{\|v\|}_{< \varepsilon} + \underbrace{\|u\|}_{< \varepsilon} \\ &< 2\varepsilon < d(A, B) < \|a - b\| \end{aligned}$$

hence  $\|a - b\| < \|a - b\|$ , which is a contradiction thus  $A_\varepsilon \cap B_\varepsilon = \emptyset$ , as claimed let us show that  $A_\varepsilon$  is open, then we can write

$$A_\varepsilon = \bigcup_{a \in A} (a + B_E(0_E, \varepsilon)) = \bigcup_{a \in A} B_E(a, \varepsilon)$$

which is a union of open subsets of  $E$ , thus  $A_\varepsilon$  is open, now by applying the first geometric version of the Hahn-Banach for  $A_\varepsilon$  and  $B_\varepsilon$ , we can find that there exists a function in  $E' \setminus \{0_{E'}\}$  and there exists  $\alpha \in \mathbb{R}$  such that,

$$f(x) \leq \alpha \leq f(y) \quad (\forall x \in A_\varepsilon, \forall y \in B_\varepsilon)$$

now we can write  $x, y \in A_\varepsilon, B_\varepsilon$  like this,

$$\begin{aligned} x &= a + \varepsilon u, & a \in A, u \in B_E(0_E, 1) \\ y &= b + \varepsilon v, & b \in B, v \in B_E(0_E, 1) \end{aligned}$$

we set,

$$f(a) + \varepsilon f(u) \leq \alpha \leq f(b) + \varepsilon f(v) \quad (\forall a, b \in A, B, \forall u, v \in B_E(0_E, 1))$$

Hence,

$$f(a) + \varepsilon \sup_{u \in B_E(0_E, 1)} f(u) \leq \alpha \leq f(b) + \varepsilon \inf_{v \in B_E(0_E, 1)} f(v)$$

But,

$$\begin{aligned} \sup_{u \in B_E(0_E, 1)} f(u) &= \sup_{u \in B_E(0_E, 1)} \|f(u)\| &= f((+/-)u) \\ &= ||| f ||| \end{aligned}$$

and we have,

$$\begin{aligned}
 \inf_{u \in B_E(0_E, 1)} f(u) &= - \sup_{u \in B_E(0_E, 1)} (-f(u)) \\
 &= - \sup_{u \in B_E(0_E, 1)} f(-u) \\
 &= - ||| f |||
 \end{aligned}$$

hence,

$$f(a) + \varepsilon ||| f ||| \leq \alpha \leq f(b) - \varepsilon ||| f ||| \quad (\forall a, b \in A, B)$$

hence

$$f(a) < \alpha < f(b) \quad (\forall a \in A, b \in B)$$

This completes the proof.



# 7

# THE HILBERT-SPACES

## 7.1 Generalities

### Definition 7.1.1: (Real inner Product

Let  $E$  be an  $\mathbb{R}$ -vector space, we call inner product on  $E$  any *Positive Definite Symmetric Bilinear Form* on  $E$ , that is any map,  $f : E^2 \longrightarrow \mathbb{R}$ , Satisfying the following properties:

$\forall x, y, x_1, x_2, y_1, y_2 \in E$ , and for all  $\lambda_1, \lambda_2 \in \mathbb{R}$ :

(i)

$$f(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 f(x_1, y) + \lambda_2 f(x_2, y)$$

(Linearity with respect to 1<sup>st</sup> argument)

(ii)

$$f(x, \lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 f(x, y_1) + \lambda_2 f(x, y_2)$$

(Linearity with respect to 2<sup>nd</sup> argument)

(iii) The symmetry:

$$\forall x, y \in E : f(x, y) = f(y, x)$$

(iv) Positive definiteness:

$$\begin{cases} \forall x \in E : f(x, x) \geq 0 & \text{Positive semi-definiteness} \\ \forall x \in E : f(x, x) = 0 \implies x = 0_E & \text{Definite} \end{cases}$$

which is equivalent to,

$$\forall x \in E \setminus \{0_E\} : f(x, x) > 0$$



**Definition 7.1.2: (Complex inner product)**

Let  $E$  be a  $\mathbb{C}$ -vector space, we call inner product on  $E$  any positive definit hermitian form on  $E$ , that is any map  $f : E^2 \rightarrow \mathbb{C}$ , satisfying the following properties:

(i) Semi-linearity in the first argument: (*sesqui* = 1.5 linear)

$$\forall x_1, x_2, y \in E, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

we have,

$$f(\lambda_1 x_1 + \lambda_2 x_2, y) = \overline{\lambda_1} f(x_1, y) + \overline{\lambda_2} f(x_2, y)$$

(ii) Linearity in the second argument:

$$\forall x, y_1, y_2 \in E, \forall \lambda_1, \lambda_2 \in \mathbb{C} :$$

in other words,

$$f(x, \lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 f(x, y_1) + \lambda_2 f(x, y_2)$$

(iii) Hermitian Symmetry:

$$\forall x, y \in E : \overline{f(x, y)} = f(y, x)$$

which implies,

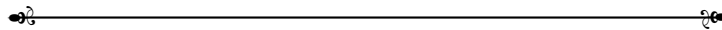
$$\forall x \in E : \overline{f(x, x)} = f(x, x) \implies \forall x \in E : f(x, x) \in \mathbb{R}$$

(iv) Positive definiteness:

$$\begin{cases} \forall x \in E : f(x, x) \geq 0 & \text{Positive semi-definiteness} \\ \forall x \in E : f(x, x) = 0 \implies x = 0_E & \text{Definiteness} \end{cases}$$

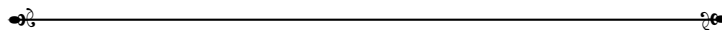
which is equivalent to,


$$\forall x \in E \setminus \{0_E\} : f(x, x) > 0$$



**NB :** The standard notation of an inner product (real or complex) is:

$$\langle \cdot, \cdot \rangle$$



**Examples**  in finite-dimensional vector spaces

Let  $n \in \mathbb{N}$  be fixed,

1. The standard inner product  $\mathbb{R}^n$  (seen as an  $\mathbb{R}$ -vector space) is defined by:


$$\forall x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n : \quad \langle x, y \rangle_{us} := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

2. The standard inner product of  $\mathbb{C}$  (seen as  $\mathbb{C}$ -vector space) is defined by,

$$\forall z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n : \langle z, w \rangle_{us} = \overline{z_1} w_1 + \overline{z_2} w_2 + \dots + \overline{z_n} w_n$$

note that,

$$\overline{\langle z, w \rangle} = \langle z, w \rangle$$

**Examples**  in infinite-dimensional vector spaces

(1) Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Consider the  $\mathbb{R}$ -vector space  $\mathcal{C}^0([a, b], \mathbb{R})$  of continuous real-valued functions on  $[a, b]$ .

The Properties of the [Riemann](#) Integrals easily show that the map defined by:

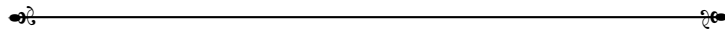
$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{C}^0([a, b], \mathbb{R}) \mathcal{C}^0([a, b], \mathbb{R}) &\longrightarrow \mathbb{R} \\ (f, g) &\longmapsto \int_a^b f(t)g(t)dt \end{aligned}$$

is an inner product on  $\mathcal{C}^0([a, b], \mathbb{R})$

(2) Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Consider the  $\mathbb{C}$ -vector space  $\mathcal{C}^0([a, b], \mathbb{C})$  of continuous complex-valued functions on  $[a, b]$ , Using the properties of the [Riemann](#) integrals

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{C}^0([a, b], \mathbb{C}) \mathcal{C}^0([a, b], \mathbb{C}) &\longrightarrow \mathbb{C} \\ (f, g) &\longmapsto \langle f, g \rangle \end{aligned}$$

is an inner product on  $\mathcal{C}^0([a, b], \mathbb{C})$



$$\begin{aligned}\overline{\langle f, g \rangle} &= \overline{\int_a^b \overline{f(t)} g(t) dt} \\ &= \int_a^b \overline{\overline{f(t)} g(t)} dt \\ &= \int_a^b \overline{g(t)} f(t) dt = \langle g, f \rangle\end{aligned}$$

## 7.2 The norm associated to an inner product

### Definition 7.2.1:

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let  $E$  be a  $\mathbb{K}$ -vector space, equipped with an inner product  $\langle \cdot, \cdot \rangle$ .

We define the norm associated to  $\langle \cdot, \cdot \rangle$  as the map:

$$\begin{aligned} \|\cdot\| : E &\longrightarrow [0, \infty) \\ x &\longmapsto \|x\| = \sqrt{\langle x, x \rangle} \end{aligned}$$

since  $\langle \cdot, \cdot \rangle$  is positive definite then  $\|\cdot\|$  is well-defined and

$$(\forall x \in E : \|x\| = 0 \implies x = 0_E)$$

Next, we have that for all  $x \in E$ , and for all  $\lambda \in \mathbb{K}$ :

$$\begin{aligned} \|\lambda x\| &= \sqrt{\langle \lambda x, \lambda x \rangle} \\ &= \sqrt{\bar{\lambda} \lambda \langle x, x \rangle} \quad \mathbb{K} = \mathbb{C} \text{ for example} \\ &= \sqrt{|\lambda|^2 \|x\|^2} = |\lambda| \|x\| \end{aligned}$$

We will see later that  $\|\cdot\|$ , also satisfies the triangle inequality:

$$\|x + y\| \leq \|x\| + \|y\| \quad (\forall x, y \in E)$$

So  $\|\cdot\|$  is really a norm on  $E$

## 7.3 The Cauchy-Schwarz Inequality

### Theorem 7.3.1:

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and be a  $\mathbb{K}$ -vector space equipped with an inner product  $\langle \cdot, \cdot \rangle$ . Let also  $\|\cdot\|$  denote the norm associated to  $\langle \cdot, \cdot \rangle$ , then we have for all  $x, y \in E$ :

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Its the **Cauchy Schwarz**, In addition this inequality becomes an equality if and only if  $x$  and  $y$

are collinear.

*Proof.* Take  $\mathbb{K} = \mathbb{C}$  (the general case), the statement of the proposition is trivial for  $x = 0_E$  or  $y = 0_E$ . Suppose for the sequel that  $x \neq 0_E$  and  $y \neq 0_E$ . Consider the unitary vector of  $E$  :

$$u := \frac{x}{\|x\|_E} \quad \text{and} \quad v := \frac{y}{\|y\|_E} \quad \text{so } \|u\|_E = \|v\|_E = 1$$

Since  $\langle, \rangle$  is Positive Definite, we have :

$$\langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle \geq 0$$

and

$$\begin{aligned} \langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle &= 0 \\ \iff u - \overline{\langle u, v \rangle} v &= 0_E \end{aligned} \quad (*)$$

which implies that  $u, v$  are collinear.

On the other hand, by expanding the inner product  $\langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle$  using the sesquilinearity and the hermitian symmetry of  $\langle \cdot, \cdot \rangle$ , we get :

$$\begin{aligned} \langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle &= \underbrace{\langle u, u \rangle}_{\|u\|^2=1} - \overline{\langle u, v \rangle} \langle u, v \rangle - \langle u, v \rangle \overline{\langle v, u \rangle} + \underbrace{\langle u, v \rangle \overline{\langle u, v \rangle} \langle v, v \rangle}_{\|v\|^2=1} \\ &= 1 - |\langle u, v \rangle|^2 \end{aligned}$$

By inserting this in  $(*)$ , we derive that :

$$\begin{cases} |\langle u, v \rangle| \leq 1 \\ |\langle u, v \rangle| = 1 \iff u \text{ and } v \text{ are collinear} \\ \left| \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \right| \leq 1 \iff |\langle x, y \rangle| \leq \|x\| \|y\| \\ \left| \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \right| = 1 \iff x \text{ and } y \text{ are collinear} \end{cases}$$

That is :

$$\begin{cases} |\langle x, y \rangle| \leq \|x\| \cdot \|y\| \\ \text{and} \\ |\langle x, y \rangle| = \|x\| \cdot \|y\| \iff x \text{ and } y \text{ are collinear} \end{cases}$$

The proposition is proved. □

**corollary 7.3.1: (The Triangle Inequality)**

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $E$  be a  $\mathbb{K}$ -vector space equipped with an inner product  $\langle \cdot, \cdot \rangle$ . Let also  $\|\cdot\|$  be the norm associated to  $\langle \cdot, \cdot \rangle$ . Then we have for all  $x, y \in E$  :

$$\|x + y\| \leq \|x\| + \|y\|$$

It's The Triangular inequality !

*Proof.* for  $\mathbb{K} = \mathbb{C}$  let  $x, y \in E$ , we have :

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 + 2 \underbrace{\operatorname{Re} \langle x, y \rangle}_{\substack{\leq |\langle x, y \rangle| \leq \|x\| \|y\| \\ \text{Cauchy-Schawrtz}}} \\ &= \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2 \end{aligned}$$

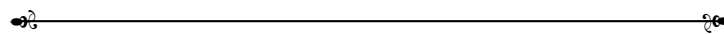
Hence :

$$\|x + y\| \leq \|x\| + \|y\|$$

as required. □

CONSEQUENCE (IMMEDIATE) : ⚠

A norm associated to an inner product of a  $\mathbb{K}$ -vector space  $E$ , where ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) is really a norm on  $E$ .

**Definition 7.3.1:**

We call a pre-Hilbert space any vector space over  $\mathbb{R}$  or  $\mathbb{C}$ , equipped with an inner product. To clarify, we sometimes use the terminology of **real pre-Hilbert space** and **complex pre-Hilbert space**

**Definition 7.3.2:**

We call **Hilbert-space** any pre-Hilbert which is **Banach** with respect to the norm associated to its inner product.

ORTHOGONALITY IN A PRE-HILBERT SPACE : ⚠

Let  $E$  be a pre-Hilbert space and let  $x, y \in E$ , we say that  $x$  and  $y$  are orthogonal (and we write  $x \perp y$

) if :

$$\langle x, y \rangle = 0$$

### SOME IMPORTANT IDENTITIES IN A PRE-HILBERT SPACE : $\triangle$

Let  $E$  be a pre-Hilbert space.

#### (1) THE PYTHAGOREAN THEOREM : $\Leftrightarrow$

For any  $x, y \in E$  with  $x \perp y$ , we have :

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

#### $\Leftrightarrow$ Generalization :

Let  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in E$ , **pairwise** orthogonal (i.e.  $x_k \perp x_l$  for  $k \neq l$ ), then we have :

$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2$$

#### (2) THE POLARIZATION FORMULA : $\Leftrightarrow$

The polarisation formula expresses the inner product in terms of it's associated norm. In the **real** case, for all  $x, y \in E$ , we have :

$$\begin{aligned} \langle x, y \rangle &= \frac{1}{2} (\|x + y\|^2 - \|x\|^2 - \|y\|^2) \\ \langle x, y \rangle &= \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \end{aligned}$$

Some additional notes, for the **imaginary** case

$$\langle x, y \rangle = \mathcal{R} \langle x, y \rangle + i \mathcal{I} \langle x, y \rangle$$

$$\begin{aligned} \langle x, y \rangle &= \mathcal{R} \langle x, y \rangle + i \mathcal{I} \langle x, y \rangle \\ &= \mathcal{R} \langle x, y \rangle - i \mathcal{R} \langle x, iy \rangle \quad (\mathcal{I} \omega = -\mathcal{R} i \omega) \\ &= \mathcal{R} \langle x, y \rangle - i \mathcal{R} \langle x, iy \rangle \\ &= \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) - \frac{1}{4} i (\|x + iy\|^2 - \|x - iy\|^2) \end{aligned}$$

To formally put it, for all  $x, y \in E$ , we have :

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) - \frac{1}{4} i (\|x + iy\|^2 - \|x - iy\|^2)$$

#### (3) THE PARALLELOGRAM IDENTITY : $\Leftrightarrow$

For every  $x, y \in E$ , we have :

$$\|x + y\|^2 + \|x - y\|^2 = 2 (\|x\|^2 + \|y\|^2)$$

"In any parallelogram, the sum of the squares of the lengths of the two diagonals is equal to the sum of the squares of the lengths of the four sides"

**Example**

Consider  $\|\cdot\|_1$  in  $\mathbb{R}^2$ :

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\|x + y\|_1 = \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 2$$

$$\|x - y\|_1 = \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\| = 2$$

and  $\|x\|_1 = \|y\|_1 = 1$ ,

$$\|x + y\|_1^2 + \|x - y\|_1^2 = 8 \neq 4 = 2(\|x\|_1^2 + \|y\|_1^2)$$

**Theorem 7.3.2: (p.Jordan & j.Von Neumann)**

A N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  is pre-Hilbert if and only if it's norm satisfies the **Parallelogram Identity**

*Proof.* It's already shown that the **Parallelogram identity** is necessary for a N.V.S over ( $\mathbb{R}$  or  $\mathbb{C}$ ) to be pre-Hilbert. Let us show that it is even sufficient we only deal with the case  $\mathbb{K} = \mathbb{R}$  and we describe how to handle the complex case.

Let  $E$  be an  $\mathbb{R}$ -N.V.S. Suppose that the norm  $\|\cdot\|$  of  $E$  satisfies the **Parallelogram identity**; that is, it satisfies :

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (\forall x, y \in E)$$

We refer to this identity by the abbreviation **P.I.** Let us define

$$\begin{aligned} f: E^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f(x, y) := \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \end{aligned}$$

we remark that for all  $x \in E$ , we have :

$$f(x, x) = \frac{1}{4}(\|2x\|^2 - \|0_E\|^2) = \|x\|^2 \geq 0$$

we also remark that  $\forall x, y \in E$ :

$$\begin{aligned} f(x, y) &= \frac{1}{4} \left( \|x + y\|^2 - \underbrace{\|x - y\|^2}_{=\|y - x\|^2} \right) \\ &= f(y, x) \end{aligned}$$



That is,  $f$  is **Symmetric**.

☞ So if we show that  $f$  is linear with respect to it's 1<sup>st</sup> argument, we are done.

1<sup>st</sup> STEP  $\triangle$ :

We show that

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y) \quad (\forall x_1, x_2, y \in E)$$

for all  $x_1, x_2, y \in E$ , we have that :

$$\begin{aligned} 4f(x_1 + x_2, y) &:= \|x_1 + x_2 + y\|^2 - \|x_1 + x_2 - y\|^2 \\ &= \|x_1 + (x_2 + y)\|^2 - \|x_1 + (x_2 - y)\|^2 \\ &\stackrel{P.I}{=} 2 \left( \cancel{\|x_1\|^2} + \|x_2 + y\|^2 \right) - \|x_1 - (x_2 + y)\|^2 - 2 \left( \cancel{\|x_1\|^2} \|x_2 - y\|^2 \right) + \|x_1 - (x_2 - y)\|^2 \\ &= 2 \left( \|x_2 + y\|^2 - \|x_2 - y\|^2 \right) + \|x_1 + y - x_2\|^2 - \|x_1 - y - x_2\|^2 \\ &\stackrel{P.I}{=} 2 \left( \|x_2 + y\|^2 - \|x_2 - y\|^2 \right) + \left( \|x_1 + y\|^2 + \cancel{\|x_2\|^2} \right) - \|x_1 + y + x_2\|^2 - 2 \left( \|x_1 - y\|^2 + \cancel{\|x_2\|^2} \right) \\ &= 8f(x_1, y) + 8f(x_2, y) - 4f(x_1 + x_2, y) \end{aligned}$$

Thus :

$$4f(x_1 + x_2, y) = 8f(x_1, y) + 8f(x_2, y) - 4f(x_1 + x_2, y)$$

Hence

$$\boxed{f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)} \quad (1)$$

as required. So it remains to show that

$$f(\lambda x, y) = \lambda f(x, y) \quad (\forall \lambda \in \mathbb{R}, \forall x, y \in E)$$

1<sup>st</sup> STEP  $\triangle$ :

Let us show by induction, and by relying on (1) that for all  $x, y \in E$ , we have :

$$f(nx, y) = nf(x, y) \quad (\forall n \in \mathbb{N}_0) \quad (2)$$

For  $n = 0$ , we have :

$$\begin{aligned} f(0x, y) &= f(0_E, y) \\ &= \frac{1}{4} \left( \underbrace{\|0_E + y\|^2}_{=\|y\|^2} - \underbrace{\|0_E - y\|^2}_{=\|y\|^2} \right) \\ &= 0 = 0f(x, y) \end{aligned}$$

So (2) is true for  $n = 0$ .

☞ Let  $n \in \mathbb{N}_0$ , Suppose that (2) is true for  $n$  and show that it remains true for  $(n + 1)$ . for all

$x, y \in E$ , we have according to (1) :

$$f((n+1)x, y) = f(nx + x, y) \stackrel{(1)}{=} \underbrace{f(nx, y)}_{=nf(x,y)} + f(x, y) = nf(x, y) + f(x, y) = (n+1)f(x, y)$$

Showing that (2) is true for  $(n+1)$ .

hence (2) is true for all  $n \in \mathbb{N}_0$ .

2<sup>nd</sup> STEP  $\triangle$ :

Let us show that for all  $x, y \in E$ , we have :

$$\boxed{f(nx, y) = nf(x, y) \quad (\forall n \in \mathbb{Z})} \quad (3)$$

We already shown that 3 holds for  $n \in \mathbb{N}_0$ . To prove (3) for  $n \in \mathbb{Z}$ , we remark first that for all  $u, v \in E$ , we have by definition :

$$\begin{aligned} f(-u, v) &= \frac{1}{4} \left( \underbrace{\| -u + v \|^2}_{=\|u-v\|^2} - \underbrace{\| -u - v \|^2}_{=\|u+v\|^2} \right) \\ &= \frac{1}{4} \left( \|u - v\|^2 - \|u + v\|^2 \right) \\ &= -f(u, v) \end{aligned}$$

That is :

$$f(-u, v) = -f(u, v) \quad (\forall u, v \in E) \quad (4)$$

Now let  $n \in \mathbb{Z}$ , that is  $n = -m$  for some  $m \in \mathbb{N}$ , so we have :

$$\begin{aligned} f(nx, y) &= f(-mx, y) \\ &\stackrel{(4)}{=} -f(\underbrace{mx}_{\in \mathbb{N}}, y) \\ &\stackrel{(2)}{=} \underbrace{-m}_{=n} f(x, y) = nf(x, y) \end{aligned}$$

as required. Consequently (3) is true for all  $n \in \mathbb{Z}$ .

3<sup>rd</sup> STEP  $\triangle$ :

Let us show that for  $x, y \in E$ , we have :

$$\boxed{f(rx, y) = rf(x, y) \quad (\forall r \in \mathbb{Q})} \quad (5)$$

let  $x, y \in E$ , and  $r \in \mathbb{Q}$ . So, we can write  $r$  as  $r = \frac{p}{q}$ , where  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . So we have :

$$\begin{aligned} qf(rx, y) &\stackrel{(3)}{=} f(qrx, y) \\ &= f(px, y) \\ &\stackrel{(3)}{=} pf(x, y) \end{aligned}$$

Thus

$$f(rx, y) = \frac{p}{q} f(x, y) = rf(x, y)$$

As required, so (5) is proved.

4<sup>th</sup> STEP  $\triangle$ :

Let us conclude that for  $x, y \in E$  we have :

$$f(\lambda x, y) = \lambda f(x, y) \quad (\forall \lambda \in \mathbb{R}) \quad (6)$$

This is simply derived to the continuity of  $f$  with respect to it's first variable, (since  $\|\cdot\|$  is continuous), and the continuity of the map  $(\lambda, x) \rightarrow \lambda x$  on  $\mathbb{R} \times E$ . Let  $\lambda \in \mathbb{R}$  and let  $x, y \in E$ . By the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , there exists  $\{\lambda_n\}_{n \in \mathbb{N}}$  a rational sequence convergin to  $\lambda$ , by (5), we have for all  $n \in \mathbb{N}$  :

$$f(\lambda_n x, y) = \lambda_n f(x, y)$$

Getting  $n \rightarrow \infty$  gives :

$$f(\lambda x, y) = \lambda f(x, y)$$

as required. Consequently,  $f$  is bilinear this completes the proof.  $\square$

*Proof.* Take  $\mathbb{K} = \mathbb{C}$  (the general case), the statement of the proposition is trivial for  $x = 0_E$  or  $y = 0_E$ . Suppose for the sequel that  $x \neq 0_E$  and  $y \neq 0_E$ . Consider the unitary vector of  $E$  :

$$u := \frac{x}{\|x\|_E} \quad \text{and} \quad v := \frac{y}{\|y\|_E} \quad \text{so } \|u\|_E = \|v\|_E = 1$$

Since  $\langle, \rangle$  is Positive Definite, we have :

$$\langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle \geq 0$$

and

$$\begin{aligned} \langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle &= 0 \\ \iff u - \overline{\langle u, v \rangle} v &= 0_E \quad (*) \end{aligned}$$

which implies that  $u, v$  are collinear.

On the other hand, by expanding the inner product  $\langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle$  using the sesquilinearity and the hermitian symmetry of  $\langle \cdot, \cdot \rangle$ , we get :

$$\begin{aligned} \langle u - \overline{\langle u, v \rangle} v, u - \overline{\langle u, v \rangle} v \rangle &= \underbrace{\langle u, u \rangle}_{\|u\|^2=1} - \overline{\langle u, v \rangle} \langle u, v \rangle - \langle u, v \rangle \overline{\langle v, u \rangle} + \underbrace{\langle u, v \rangle \overline{\langle u, v \rangle} \langle v, v \rangle}_{\|v\|^2=1} \\ &= 1 - |\langle u, v \rangle|^2 \end{aligned}$$

By inserting this in (\*), we derive that :

$$\begin{cases} |\langle u, v \rangle| \leq 1 \\ |\langle u, v \rangle| = 1 \iff u \text{ and } v \text{ are collinear} \\ \left| \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \right| \leq 1 \iff |\langle x, y \rangle| \leq \|x\| \|y\| \\ \left| \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \right| = 1 \iff x \text{ and } y \text{ are collinear} \end{cases}$$

That is :

$$\begin{cases} |\langle x, y \rangle| \leq \|x\| \cdot \|y\| \\ \text{and} \\ |\langle x, y \rangle| = \|x\| \cdot \|y\| \iff x \text{ and } y \text{ are collinear} \end{cases}$$

The proposition is proved. □

#### corollary 7.3.2: (The Triangle Inequality)

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $E$  be a  $\mathbb{K}$ -vector space equipped with an inner product  $\langle \cdot, \cdot \rangle$ . Let also  $\|\cdot\|$  be the norm associated to  $\langle \cdot, \cdot \rangle$ . Then we have for all  $x, y \in E$  :

$$\|x + y\| \leq \|x\| + \|y\|$$

It's The Triangular inequality !

*Proof.* for  $\mathbb{K} = \mathbb{C}$  let  $x, y \in E$ , we have :

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 + 2 \underbrace{\Re \langle x, y \rangle}_{\substack{\leq |\langle x, y \rangle| \leq \|x\| \|y\| \\ \text{Cauchy-Schawrtz}}} \\ &= \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2 \end{aligned}$$

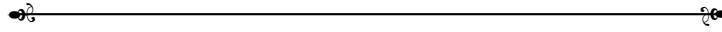
Hence :

$$\|x + y\| \leq \|x\| + \|y\|$$

as required. □

#### CONSEQUENCE (IMMEDIATE) : $\triangle$

A norm associated to an inner product of a  $\mathbb{K}$ -vector space  $E$ , where ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) is really a norm on  $E$ .

**Definition 7.3.3:**

We call a pre-Hilbert space any vector space over  $\mathbb{R}$  or  $\mathbb{C}$ , equipped with an inner product. To clarify, we sometimes use the terminology of **real pre-Hilbert space** and **complex pre-Hilbert space**

**Definition 7.3.4:**

We call **Hilbert-space** any pre-Hilbert which is **Banach** with respect to the norm associated to its inner product.

ORTHOGONALITY IN A PRE-HILBERT SPACE :  $\triangle$ 

Let  $E$  be a pre-Hilbert space and let  $x, y \in E$ , we say that  $x$  and  $y$  are orthogonal (and we write  $x \perp y$ ) if :

$$\langle x, y \rangle = 0$$

SOME IMPORTANT IDENTITIES IN A PRE-HILBERT SPACE :  $\triangle$ 

Let  $E$  be a pre-Hilbert space.

(1) THE PYTHAGOREAN THEOREM :  $\Leftrightarrow$ 

For any  $x, y \in E$  with  $x \perp y$ , we have :

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

 $\Leftrightarrow$  Generalization :

Let  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in E$ , **pairwise** orthogonal (i.e.  $x_k \perp x_l$  for  $k \neq l$ ), then we have :

$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2$$

(2) THE POLARIZATION FORMULA :  $\Leftrightarrow$ 

The polarisation formula expresses the inner product in terms of it's associated norm. In the **real** case, for all  $x, y \in E$ , we have :

$$\begin{aligned} \langle x, y \rangle &= \frac{1}{2} \left( \|x + y\|^2 - \|x\|^2 - \|y\|^2 \right) \\ \langle x, y \rangle &= \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right) \end{aligned}$$

Some additional notes, for the **imaginary** case

$$\langle x, y \rangle = \mathcal{R} \langle x, y \rangle + i \mathcal{I} \langle x, y \rangle$$

$$\begin{aligned}
\langle x, y \rangle &= \mathcal{R} \langle x, y \rangle + i\mathcal{I} \langle x, y \rangle \\
&= \mathcal{R} \langle x, y \rangle - i\mathcal{R} \langle x, iy \rangle & (\mathcal{I}\omega = -\mathcal{R}i\omega) \\
&= \mathcal{R} \langle x, y \rangle - i\mathcal{R} \langle x, iy \rangle \\
&= \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right) - \frac{1}{4} \left( \|x + iy\|^2 - \|x - iy\|^2 \right)
\end{aligned}$$

To formally put it, for all  $x, y \in E$ , we have :

$$\langle x, y \rangle = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right) - \frac{1}{4}i \left( \|x + iy\|^2 - \|x - iy\|^2 \right)$$

### (3) THE PARALLELOGRAM IDENTITY :

For every  $x, y \in E$ , we have :

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

"In any parallelogram, the sum of the squares of the lengths of the two diagonals is equal to the sum of the squares of the lengths of the four sides"

#### Example

Consider  $\|\cdot\|_1$  in  $\mathbb{R}^2$  :

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\|x + y\|_1 = \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 2$$

$$\|x - y\|_1 = \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\| = 2$$

and  $\|x\|_1 = \|y\|_1 = 1$ ,

$$\|x + y\|_1^2 + \|x - y\|_1^2 = 8 \neq 4 = 2(\|x\|_1^2 + \|y\|_1^2)$$

### Theorem 7.3.3: (p.Jordan & j.Von Neumann)

A N.V.S over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  is pre-Hilbert if and only if it's norm satisfies the **Parallelogram Identity**

*Proof.* It's already shown that the **Parallelogram identity** is necessary for a N.V.S over ( $\mathbb{R}$  or  $\mathbb{C}$ ) to be pre-Hilbert. Let us show that it is even sufficient we only deal with the case  $\mathbb{K} = \mathbb{R}$  and we describe how to handle the complex case.

Let  $E$  be an  $\mathbb{R}$ -N.V.S. Suppose that the norm  $\|\cdot\|$  of  $E$  satisfies the **Parallelogram** identity; that is, it satisfies :

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (\forall x, y \in E)$$

We refer to this identity by the abbreviation **P.I.** Let us define

$$\begin{aligned} f : E^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f(x, y) := \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \end{aligned}$$

we remark that for all  $x \in E$ , we have :

$$f(x, x) = \frac{1}{4} (\|2x\|^2 - \|0_E\|^2) = \|x\|^2 \geq 0$$

we also remark that  $\forall x, y \in E$  :

$$\begin{aligned} f(x, y) &= \frac{1}{4} \left( \|x + y\|^2 - \underbrace{\|x - y\|^2}_{=\|y - x\|^2} \right) \\ &= f(y, x) \end{aligned}$$

That is,  $f$  is **Symmetric**.

☞ So if we show that  $f$  is linear with respect to its 1<sup>st</sup> argument, we are done.

1<sup>st</sup> STEP  $\triangle$ :

We show that

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y) \quad (\forall x_1, x_2, y \in E)$$

for all  $x_1, x_2, y \in E$ , we have that :

$$\begin{aligned} 4f(x_1 + x_2, y) &:= \|x_1 + x_2 + y\|^2 - \|x_1 + x_2 - y\|^2 \\ &= \|x_1 + (x_2 + y)\|^2 - \|x_1 + (x_2 - y)\|^2 \\ &\stackrel{P.I.}{=} 2(\|x_1\|^2 + \|x_2 + y\|^2) - \|x_1 - (x_2 + y)\|^2 - 2(\|x_1\|^2 + \|x_2 - y\|^2) + \|x_1 - (x_2 - y)\|^2 \\ &= 2(\|x_2 + y\|^2 - \|x_2 - y\|^2) + \|x_1 + y - x_2\|^2 - \|x_1 - y - x_2\|^2 \\ &\stackrel{P.I.}{=} 2(\|x_2 + y\|^2 - \|x_2 - y\|^2) + (\|x_1 + y\|^2 + \|x_2\|^2) - \|x_1 + y + x_2\|^2 - 2(\|x_1 - y\|^2 + \|x_2\|^2) \\ &= 8f(x_1, y) + 8f(x_2, y) - 4f(x_1 + x_2, y) \end{aligned}$$

Thus :

$$4f(x_1 + x_2, y) = 8f(x_1, y) + 8f(x_2, y) - 4f(x_1 + x_2, y)$$

Hence

$$\boxed{f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)} \quad (1)$$

as required. So it remains to show that

$$f(\lambda x, y) = \lambda f(x, y) \quad (\forall \lambda \in \mathbb{R}, \forall x, y \in E)$$

1<sup>st</sup> STEP  $\triangle$ :

Let us show by induction, and by relying on (1) that for all  $x, y \in E$ , we have :

$$f(nx, y) = nf(x, y) \quad (\forall n \in \mathbb{N}_0) \quad (2)$$

For  $n = 0$ , we have :

$$\begin{aligned} f(0x, y) &= f(0_E, y) \\ &= \frac{1}{4} \left( \underbrace{\|0_E + y\|^2}_{=\|y\|^2} - \underbrace{\|0_E - y\|^2}_{=\|y\|^2} \right) \\ &= 0 = 0f(x, y) \end{aligned}$$

So (2) is true for  $n = 0$ .

☞ Let  $n \in \mathbb{N}_0$ , Suppose that (2) is true for  $n$  and show that it remains true for  $(n + 1)$ . for all  $x, y \in E$ , we have according to (1) :

$$f((n + 1)x, y) = f(nx + x, y) \stackrel{(1)}{=} \underbrace{f(nx, y)}_{=nf(x, y)} + f(x, y) = nf(x, y) + f(x, y) = (n + 1)f(x, y)$$

Showing that (2) is true for  $(n + 1)$ .

hence (2) is true for all  $n \in \mathbb{N}_0$ .

2<sup>nd</sup> STEP  $\triangle$ :

Let us show that for all  $x, y \in E$ , we have :

$$\boxed{f(nx, y) = nf(x, y) \quad (\forall n \in \mathbb{Z})} \quad (3)$$

We already shown that 3 holds for  $n \in \mathbb{N}_0$ . To prove (3) for  $n \in \mathbb{Z}$ , we remark first that for all  $u, v \in E$ , we have by definition :

$$\begin{aligned} f(-u, v) &= \frac{1}{4} \left( \underbrace{\| -u + v \|^2}_{=\|u-v\|^2} - \underbrace{\| -u - v \|^2}_{=\|u+v\|^2} \right) \\ &= \frac{1}{4} \left( \|u - v\|^2 - \|u + v\|^2 \right) \\ &= -f(u, v) \end{aligned}$$

That is :

$$f(-u, v) = -f(u, v) \quad (\forall u, v \in E) \quad (4)$$



Now let  $n \in \mathbb{Z}$ , that is  $n = -m$  for some  $m \in \mathbb{N}$ , so we have :

$$\begin{aligned} f(nx, y) &= f(-mx, y) \\ &\stackrel{(4)}{=} -f(\underbrace{mx}_{\in \mathbb{N}}, y) \\ &\stackrel{(2)}{=} \underbrace{-m}_{=n} f(x, y) = nf(x, y) \end{aligned}$$

as required. Consequently (3) is true for all  $n \in \mathbb{Z}$ .

3<sup>rd</sup> STEP  $\underline{\Delta}$ :

Let us show that for  $x, y \in E$ , we have :

$$\boxed{f(rx, y) = rf(x, y) \quad (\forall r \in \mathbb{Q})} \quad (5)$$

let  $x, y \in E$ , and  $r \in \mathbb{Q}$ . So, we can write  $r$  as  $r = \frac{p}{q}$ , where  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . So we have :

$$\begin{aligned} qf(rx, y) &\stackrel{(3)}{=} f(qrx, y) \\ &= f(px, y) \\ &\stackrel{(3)}{=} pf(x, y) \end{aligned}$$

Thus

$$\boxed{f(rx, y) = \frac{p}{q}f(x, y) = rf(x, y)}$$

As required, so (5) is proved.

4<sup>th</sup> STEP  $\underline{\Delta}$ :

Let us conclude that for  $x, y \in E$  we have :

$$\boxed{f(\lambda x, y) = \lambda f(x, y) \quad (\forall \lambda \in \mathbb{R})} \quad (6)$$

This is simply derived to the continuity of  $f$  with respect to it's first variable, (since  $\|\cdot\|$  is continuous), and the continuity of the map  $(\lambda, x) \rightarrow \lambda x$  on  $\mathbb{R} \times E$ . Let  $\lambda \in \mathbb{R}$  and let  $x, y \in E$ . By the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , there exists  $\{\lambda_n\}_{n \in \mathbb{N}}$  a rational sequence convergin to  $\lambda$ , by (5), we have for all  $n \in \mathbb{N}$  :

$$f(\lambda_n x, y) = \lambda_n f(x, y)$$

Getting  $n \rightarrow \infty$  gives :

$$f(\lambda x, y) = \lambda f(x, y)$$

as required. Consequently,  $f$  is bilinear this completes the proof. □

## 7.4 The Hilbert Projection Theorem

### Theorem 7.4.1: (Hilbert)

Let  $H$  be a Hilbert space and  $C$  be a non empty subset of  $H$  which is closed and convex, Then forevery  $x \in H$ , there exists a unique  $u \in C$ , such that :

$$\|x - u\| = d(x, C) \quad (1)$$

i.e.

$$\forall x \in H, \quad \exists! u \in C, \quad \|x - u\| = d(x, C)$$

in addition,  $u$  is characterized by the property :

$$\begin{cases} u \in C \\ \Re \langle x - u, v - u \rangle \leq 0 \quad \forall v \in C \end{cases} \quad (2)$$

*Proof.* Let  $x \in H$  be fixed.

By definition of  $d(x, C)$  which is :

$$d(x, C) := \inf_{u \in C} \|x - u\|$$

there exists for all  $n \in \mathbb{N}$ , a vector  $u_n \in C$  such that :

$$d(x, C) \leq \|x - u_n\| < d(x, C) + \frac{1}{n} \quad (*)$$

Let us show that  $(u_n)_{n \in \mathbb{N}}$  is a **Cauchy Sequence**.

Then for all  $p, q \in \mathbb{N}$ , we have :

$$\begin{aligned} \|u_p - u_q\|^2 &= \|(x - u_p) - (x - u_q)\|^2 \\ &\stackrel{P.I}{=} 2 \left( \|x - u_p\|^2 + \|x - u_q\|^2 \right) - 4 \left\| x - \frac{u_p + u_q}{2} \right\|^2 \end{aligned}$$

Next by  $(*)$ , we have :

$$\begin{aligned} \|x - u_p\| &< d(x, C) + \frac{1}{p} \\ \|x - u_q\| &< d(x, C) + \frac{1}{q} \end{aligned}$$

On the other hand, since  $C$  is convex and  $u_p, u_q \in C$ , then  $\frac{u_p + u_q}{2} \in C$ , and thus :

$$\left\| x - \frac{u_p + u_q}{2} \right\| \geq d(x, C)$$

hence :

$$\|u_p - u_q\|^2 \leq 2 \left( \left( d(x, C) + \frac{1}{p} \right)^2 + \left( d(x, C) + \frac{1}{q} \right)^2 \right) - 4d(x, C)^2 \longrightarrow 0 \quad p, q \longrightarrow +\infty$$

hence we have :

$$\|u_p - u_q\| \rightarrow 0 \quad p, q \rightarrow +\infty$$

Showing that  $(u_n)_{n \in \mathbb{N}}$  is a **Cauchy Sequence** of  $H$ . since  $H$  is complete then  $(u_n)_{n \in \mathbb{N}}$  is convergent to some  $u \in H$ . Next, since  $(u_n)_{n \in \mathbb{N}} \in C$  and  $C$  is closed then  $u \in \overline{C} = C$ . Therefore  $u \in C$ , thus by setting  $n \rightarrow \infty$  in  $(*)$ . we get :

$$\|x - u\| = d(x, C)$$

The existence of  $u$  is proved.

### THE UNIQUENESS : $\triangle$

i.e., The uniqueness of  $u$ . Let  $u, u' \in C$  such that :

$$\|x - u\| = \|x - u'\| = d(x, C)$$

then we have :

$$\begin{aligned} \|u - u'\|^2 &= \|(x - u) - (x - u')\|^2 \\ &\stackrel{P.I}{=} 2 \left( \|x - u\|^2 + \|x - u'\|^2 \right) - 4 \underbrace{\left\| x - \frac{u + u'}{2} \right\|^2}_{\in C \text{ (convex)}} \quad (*) \end{aligned}$$

Thus :

$$\left\| x - \frac{u + u'}{2} \right\| \geq d(x, C)$$

from  $(*)$  we continue :

$$(*) \leq 2 \left( d(x, C)^2 + d(x, C)^2 \right) - 4d(x, C)^2 = 0$$

Thus  $\|u - u'\| = 0$ , implying that  $u = u'$ . Hence the uniqueness of  $u$  is proved.

### THE EQUIVALENCE BETWEEN (1) AND (2) : $\triangle$

$$(1) \implies (2)$$

Let  $u \in C$ , satisfyin (1) i.e.  $(\|x - u\| = d(x, C))$  and show that (2) i.e.  $(\forall v \in C : \Re \langle x - u, v - u \rangle \leq 0)$ . for every  $v \in C$ , consider the vectors  $w_t (t \in [0, 1])$ , defined by :

$$w_t := (1 - t)u + tv$$

Since  $C$  is convex then  $w_t \in C$ , then we have

$$\|x - w_t\| \geq d(x, C) = \|x - u\| \quad (\forall t \in [0, 1])$$

That is :

$$\|x - (1 - t)u - tv\| \geq \|x - u\| \quad (\forall t \in [0, 1])$$

That is :

$$\|(x - u) - t(v - u)\| \geq \|x - u\| \quad (\forall t \in [0, 1])$$

By squaring, we get : <sup>1</sup>

$$\cancel{\|x - u\|^2} + \|v - u\|^2 t^2 - 2t\mathcal{R}\langle x - u, v - u \rangle \geq \cancel{\|x - u\|^2} \quad (\forall t \in [0, 1])$$

hence :

$$\|v - u\|^2 t - 2\mathcal{R}\langle x - u, v - u \rangle \geq 0 \quad (\forall t \in (0, 1])$$

i.e.

$$\mathcal{R}\langle x - u, v - u \rangle \leq \frac{t}{2} \|v - u\|^2 \quad (\forall t \in (0, 1])$$

now setting  $t \rightarrow^> 0$ , we get finally :

$$\mathcal{R}\langle x - u, v - u \rangle \leq 0$$

as required.

$$(2) \implies (1)$$

Conversely, let  $u \in C$  satisfying (2)

$$\mathcal{R}\langle x - u, v - u \rangle \leq 0 \quad (\forall v \in C)$$

and let us show that  $u$  satisfies (1) i.e.

$$\|x - u\| = d(x, C)$$

for all  $v \in C$ , we have :

$$\begin{aligned} \|x - u\|^2 - \|x - v\|^2 &= \|x - u\|^2 - \|(x - u) - (v - u)\|^2 \\ &= \|x - u\|^2 - \left( \|x - u\|^2 + \|v - u\|^2 - 2\mathcal{R}\langle x - u, v - u \rangle \right) \\ &= 2\mathcal{R}\langle x - u, v - u \rangle - \|v - u\|^2 \leq 0 \end{aligned}$$

that is :

$$\|x - u\| \leq \|x - v\| \quad (\forall v \in C)$$

---

<sup>1</sup>  $\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2\mathcal{R}\langle a, b \rangle$

By taking the infimum on  $v \in C$ , we get :

$$\|x - u\| \leq d(x, C)$$

Hence.

$$\|x - u\| = d(x, C)$$

As required. ✕

This completes the proof. □

#### Definition 7.4.1:

Let  $H$  be a hilbert space and  $C$  be a non empty subset of  $H$ . Which is closed and convex. The map associating to each  $x \in H$  the unique  $u \in C$  such that :

$$\|x - u\| = d(x, C)$$

is called " The **Projection** on  $C$  " and we denote it by  $\pi_C$  <sup>a</sup>

---

<sup>a</sup>Notice  $d(\pi(x), \pi(y)) \leq d(x, y)$

#### Theorem 7.4.2:

Let  $H$  be a Hilbert Space and  $C$  be a non empty subset of  $H$  which is closed and convex. Then the map :

$\pi_C$  is **1-Lipschitz** (so continuous).

*Proof.* For  $x, y \in H$ , let  $u := \pi_C(x)$  and  $v := \pi_C(y)$ . So we have to show that :

$$\|u - v\| \leq \|x - y\|$$

Since this inequality is trivial when  $u = v$ . we may suppose that  $u \neq v$ . According to the second part of the theorem 1, we have that :

$$\mathcal{R} \langle x - u, v - u \rangle \leq 0 \quad (A)$$

and

$$\mathcal{R} \langle y - v, u - v \rangle \leq 0$$

that is :

$$\mathcal{R} \langle v - y, v - u \rangle \leq 0 \quad (B)$$

Summing sides to side (A) and (B), we get :

$$\mathcal{R} \langle (x - u) + (v - y), v - u \rangle \leq 0$$

i.e.

$$\mathcal{R} \langle (x - y) + (v - u), v - u \rangle \leq 0$$

that is :

$$\mathcal{R} \langle x - y, v - u \rangle + \|v - u\|^2 \leq 0$$

Thus :

$$\begin{aligned} \|v - u\|^2 &\leq -\mathcal{R} \langle x - y, v - u \rangle \\ &\leq |\langle x - y, v - u \rangle| \\ &\leq^{C.S} \|x - y\| \cdot \|v - u\| \end{aligned}$$

Hence :

$$\|v - u\| \leq \|x - y\|$$

as required. □

#### corollary 7.4.1:

Let  $H$  be a Hilbert space and  $K$  be a closed vector subspace<sup>a</sup> of  $H$ . For all  $x \in H$ , the projection  $u = \pi_K(x)$  of  $x$  on  $K$  is characterized by :

$$u \in K \langle x - u, v \rangle = 0, \quad \forall v \in K$$

In addition  $\pi_K \in \mathcal{L}(H)$  (i.e.  $\pi_K$  is linear and continuous).

<sup>a</sup> "Notice that the other theorem talks about Convex sets, and this one says that dot product is zero not  $\leq 0$ ." so it's a more particular case - Author.

*Proof.* ( $\mathbb{K} = \mathbb{C}$ ) i.e. in the general case.

let  $x \in H$  and  $u := \pi_K x$  by the second part of Theorem 1,  $u$  is characterized by :

$$\begin{cases} u \in K \\ \mathcal{R} \langle x - u, w - u \rangle \leq 0 \quad (\forall w \in K) \end{cases} \quad (I)$$

since  $K$  is vector subspace of  $H$  then any vector  $w \in K$ . Can be written as

$$w = zv + u \quad (z \in \mathbb{C}, \quad v \in K)$$

and conversly any vector fof the form  $zv + u$ , ( $z \in \mathbb{C}$ ,  $v \in K$ ) belong to  $K$ . Thus (I) is equivalent to :

$$\mathcal{R} \langle x - u, zv \rangle \leq 0 \quad (\forall z \in \mathbb{C}, \quad \forall v \in K)$$

That is :

$$\mathcal{R} z \langle x - u, v \rangle \leq 0 \quad (\forall z \in \mathbb{C}, \quad \forall v \in K) \quad (II)$$

for (II) to be satisfied for all  $z \in \mathbb{C}$ , it suffices that it to be satisfied for all  $z$  real and all  $z$  pure imaginary.

For  $z$  real say  $z = t \in \mathbb{R}$ , we get :

$$t\mathcal{R} \langle x - u, v \rangle \leq 0 \quad (\forall t \in \mathbb{R}, \forall v \in K)$$

which is equivalent to :

$$\mathcal{R} \langle x - u, v \rangle = 0 \quad (\forall v \in K)$$

for  $z$  pure imaginary, say  $z = it$ ,  $t \in \mathbb{R}$ . We get :

$$\mathcal{R}it \langle x - u, v \rangle \leq 0 \quad (\forall t \in \mathbb{R}, v \in K)$$

<sup>2</sup> which is equivalent to

$$\mathcal{I} \langle x - u, v \rangle = 0 \quad (\forall v \in K) \quad (2)$$

So :

$$(II) \iff (1) \ \& \ (2)$$

$$\iff \langle x - u, v \rangle = 0 \quad (\forall v \in K)$$

This proves the first point of the corollary . Further, the continuity of  $\pi_K$  is proved in Proposition 2.

So, it remains to show the linearity of  $\pi_K$ . let  $x, y \in H$  and  $\lambda, \mu \in \mathbb{C}$  and let us show that

$$\pi_K(\lambda x + \mu y) = \lambda \pi_K(x) + \mu \pi_K(y)$$

we have for all  $v \in K$  :

$$\begin{aligned} \langle \lambda x + \mu y - (\lambda \pi_K(x) + \mu \pi_K(y)), v \rangle &= \langle \lambda(x - \pi_K(x)) + \mu(y - \pi_K(y)), v \rangle \\ &= \bar{\lambda} \langle x - \pi_K(x), v \rangle + \bar{\mu} \langle y - \pi_K(y), v \rangle \\ &= 0 \end{aligned}$$

Thus.

$$\langle \lambda x + \mu y - (\lambda \pi_K(x) + \mu \pi_K(y)), v \rangle = 0 \quad (\forall v \in K)$$

implying by the result of the first part, that:

$$\pi_K(\lambda x + \mu y) = \lambda \pi_K(x) + \mu \pi_K(y)$$

Show that  $\pi_K$  is linear. This completes the proof. □

## corollary 7.4.2:

Let  $H$  be a hilbert space and  $K$  be a closed vector subspace of  $H$ .

Then  $K^\perp$ , closed vector subspace of  $H$  and it s a complement subspace of  $K$  in  $H$  i.e.

$$K \oplus K^\perp = H$$

*Proof.* It's known that  $K^\perp$  is a vector subspace of  $H$ , of course ( [Algebra 3](#) ). Let us prove that  $K^\perp$  is closed.

let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of  $K^\perp$ , which converges in  $H$  to some  $x \in H$ . and let us show that we have necessary  $x \in K^\perp$ , using the continuity of the inner product

$$\langle \cdot, \cdot \rangle$$

(with respect to it's 1<sup>st</sup> variable). we have for all  $u \in K$  :

$$\begin{aligned} \langle x, u \rangle &= \left\langle \lim_{n \rightarrow \infty} x_n, u \right\rangle \\ &= \lim_{n \rightarrow \infty} \langle x_n, u \rangle \\ &= \lim_{n \rightarrow \infty} 0 = 0 \end{aligned}$$

implying that :

$$x \perp u \quad \forall u \in K$$

Thus consequently,  $K^\perp$  is closed in  $H$ .

$$K \cap K^\perp = \{0_H\} \quad ? : \underline{\Delta}$$

for all  $x \in K \cap K^\perp$ , we have

$$x \perp x; \quad \text{i.e. } \langle x, x \rangle = 0 \quad \text{i.e. } \|x\|^2 = 0 \quad \text{i.e. } \|x\| = 0 \implies x = 0_H$$

hence  $K \cap K^\perp \subset \{0_H\}$ ; hence

$$K \cap K^\perp = \{0_H\}$$

Thus the sum :

$$K + K^\perp \quad \text{is direct}$$

$$K + K^\perp \stackrel{?}{=} H : \underline{\Delta}$$

Let  $x \in H$  be arbitrary. Consider  $u = \pi_K(x) \in K$ . By corollary 3, we have :

$$\langle x - \pi_K(x), v \rangle = 0 \quad \forall v \in K$$

That is :

$$(x - \pi_K(x)) \in K^\perp$$



hence :

$$x = \underbrace{\pi_K(x)}_{\in K} + \underbrace{(x - \pi_K(x))}_{\in K^\perp} \in K + K^\perp$$

Thus :

$$H \subset K + K^\perp \subset H$$

Hence :

$$H = K \oplus K^\perp$$

This completes the proof. □

#### Definition 7.4.2:

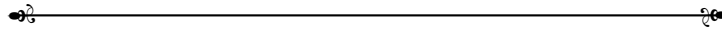
Let  $H$  be a Hilbert space, and  $K$  be a closed vector subspace of  $H$ . Then  $K^\perp$  is called the orthogonal complement of  $K$  (in  $H$ ).



#### EXERCISE :

Let  $H$  be a Hilbert and  $K$  be a vector subspace of  $H$  (not necessary closed). Show that we have :

$$K^{\perp\perp} = \overline{K}$$



The orthogonal projection on a finite-dimensional vector subspace of a Hilbert space :

Let  $H$  be a Hilbert space and  $F$  be a finite-dimensional vector subspace of  $H$ , so  $F$  is closed in  $H$  and we can speak about the (orthogonal) projection of a vector of  $H$  onto  $F$  and ask about an expression of it with respect to the appropriate basis of  $F$ .

In what follows, we will find such an expression relative to an orthonormal basis of  $F$ .

Let  $\mathcal{B} = (e_1, e_2, \dots, e_n)$  where  $(n = \dim F \in \mathbb{N}_0)$ , be an orthonormal basis of  $F$  ( $\mathcal{B}$  exists and it can be formed by the Gram-Schmidt algorithm for example). we have the following proposition :

#### Theorem 7.4.3:

For every  $x \in H$ , we have :

$$\pi_F(x) = \sum_{k=1}^n \langle e_k, x \rangle e_k$$

*Proof.* Let  $x \in H$  be fixed.

since  $\pi_F(x) \in F = \langle e_1, e_2, \dots, e_n \rangle$ , then we can write  $\pi_F(x)$  as :

$$\pi_F(x) = \sum_{k=1}^n \lambda_k e_k \quad (\lambda_k \in \mathbb{C}, k = 1, \dots, n) \quad (1)$$

Next, by Corollary 3, we have :

$$\begin{aligned}(x - \pi_F(x)) &\in F^\perp = \langle e_1, e_2, \dots, e_n \rangle^\perp \\ &= \{e_1, \dots, e_n\}^\perp\end{aligned}$$

That is :

$$\langle e_k, x - \pi_F(x) \rangle = 0 \quad \forall k \in \{1, 2, \dots, n\}$$

For all  $k \in \{1, 2, \dots, n\}$ , we have :

$$\begin{aligned}\iff \langle e_k, x - \pi_K(x) \rangle &= 0 \\ \iff \left\langle e_k, x - \sum_{\ell=1}^n \lambda_\ell e_\ell \right\rangle &= 0 \\ \iff \langle e_k, x \rangle - \sum_{\ell=1}^n \lambda_\ell \underbrace{\langle e_k, e_\ell \rangle}_{\delta_{k,\ell}} &= 0 \\ \iff \langle e_k, x \rangle - \lambda_k &= 0 \\ \iff \lambda_k = \langle e_k, x \rangle \quad \forall k \in \{1, 2, \dots, n\} &\quad (2)\end{aligned}$$

By substituting into (1), we get :

$$\pi_F(x) = \sum_{k=1}^n \langle e_k, x \rangle e_k$$

As required. □

#### corollary 7.4.3:

For all  $x \in H$ , we have :

$$\|\pi_F(x)\|^2 = \sum_{k=1}^n |\langle e_k, x \rangle|^2$$

*Proof.* Since the vectors  $\langle e_k, x \rangle e_k$  are pairwise orthogonal, we have by the [PYTHAGOREAN THEOREM](#) :

$$\begin{aligned}\|\pi_F(x)\| &= \left\| \sum_{k=1}^n \langle e_k, x \rangle e_k \right\|^2 \\ &= \sum_{k=1}^n \underbrace{\|\langle e_k, x \rangle e_k\|^2}_{=|\langle e_k, x \rangle|^2} = \sum_{k=1}^n |\langle e_k, x \rangle|^2\end{aligned}$$

as required. □

THE RIESZ REPRESENTATION THEOREM :

**Theorem 7.4.4: (F. Riesz)**

Let  $H$  be a Hilbert space. Then for every  $f \in H'$  there exists a unique  $a \in H$  such that :

$$f(x) = \langle a, x \rangle \quad (\forall x \in H)$$

*Proof.* Let  $f \in H'$  (i.e.  $f$  is continuous linear form on  $H$ ).

THE EXISTENCE OF  $a$  :  $\triangle$

If  $f = 0_{H'}$ , it suffices to take  $a = 0_H$ . Suppose for the sequel that  $f \neq 0_{H'}$ .

So  $\text{Ker}(f)$  is a closed (linear) hyperplane of  $H$ . it follows (according to the corollary that  $\text{Ker}(f) \oplus \text{Ker}(f)^\perp = H$ ), and  $\dim(\text{Ker}(f))^\perp = 1$ <sup>3</sup>, let  $a_0 \in (\text{Ker}(f))^\perp$  such that  $a_0 \neq 0_H$ , so :

$$(\text{Ker}(f))^\perp = \langle a_0 \rangle$$

we have :

$$\begin{aligned} f(a_0) &= \frac{f(a_0)}{\|a_0\|^2} \|a_0\|^2 \\ &= \frac{a_0}{\|a_0\|^2} \langle a_0, a_0 \rangle \\ &= \left\langle \frac{\overline{f(a_0)}}{\|a_0\|^2} a_0, a_0 \right\rangle \end{aligned}$$

set  $a := \frac{\overline{f(a_0)}}{\|a_0\|^2} a_0$ , so we have :

$$f(a_0) = \langle a, a_0 \rangle$$

now, for all  $x \in H$ , for  $\lambda \in \mathbb{C}, y \in \text{Ker}(f)$ , we can write  $x$  as :

$$x = y + \lambda a_0 \quad \left( \text{since } H = \text{Ker}(f) \oplus (\text{Ker}(f))^\perp = \text{Ker}(f) \oplus \langle a_0 \rangle \right)$$

---

<sup>3</sup>A note the teacher wrote,  $u, v \in (\text{Ker}(f))^\perp$

$$\begin{aligned} w &= f(v)u - f(u)v \in \text{Ker}(f)^\perp \\ f(w) &= f(v)f(u) - f(u)f(v) = 0 \\ \implies w &\in \text{Ker}(f) \\ \implies w &\in \text{Ker}(f) \cap \text{Ker}(f)^\perp = \{0_H\} \\ \implies w &= 0_H \end{aligned}$$

by inserting  $f$  on both sides :

$$\begin{aligned}
 f(x) &= f(y + \lambda a_0) \\
 &= f(y) + \lambda f(a_0) \\
 &= \lambda \langle a, a_0 \rangle \\
 &= \underbrace{\langle a, y \rangle}_{=0, \quad y \in \text{Ker}(f), a \in \text{Ker}(f)^\perp} + \lambda \langle a, a_0 \rangle \\
 &= \langle a, y \rangle + \langle a, \lambda a_0 \rangle \\
 &= \langle a, y + \lambda a_0 \rangle = \langle a, x \rangle
 \end{aligned}$$

Hence :

$$f(x) = \langle a, x \rangle \quad (\forall x \in H)$$

☞ As required.

THE UNIQUENESS OF  $a$  :  $\triangle$

Let  $a, b \in H$  such that for all  $x \in H$  :

$$f(x) = \langle a, x \rangle = \langle b, x \rangle$$

thus  $\forall x \in H$  :

$$\langle a - b, x \rangle = 0$$

☞ Hence  $(a - b) \in H^\perp = \{0_H\}$ , thus  $a - b = 0_H$ ; thus  $a = b$ , hence the uniqueness of  $a$ . This completes the proof.  $\square$

corollary 7.4.4:

Any real Hilbert space  $H$  is Isometrically isomorphic to it's continuous dual  $H'$ .

*Proof.* Let  $H$  be a Hilbert space. Consider :

$$\begin{aligned}
 \varphi : H &\longrightarrow H' \\
 a &\longmapsto \varphi_a
 \end{aligned}$$

where  $\varphi_a$  is :

$$\begin{aligned}
 \varphi_a : H &\longrightarrow \mathbb{R} \\
 x &\longmapsto \langle a, x \rangle
 \end{aligned}$$

The **RIESZ REPRESENTATION THEOREM** ensures that  $\varphi$  is bijective. Next, it's clear that  $\varphi$  is linear and that for all  $a \in H$ , we have :

$$\|\varphi(a)\|_{H'} = \| \varphi_a \|_{H'} \stackrel{\text{Cauchy Schwartz}}{=} \|a\|_H$$

Implying that  $\varphi$  is an isomorphism isometric of N.V.S □

### FURTHER APPLICATIONS OF THE RIESZ REPRESENTATION THEOREM

#### corollary 7.4.5:

Let  $H$  be a Hilbert space and  $G$  be a closed vector subspace of  $H$ .

Then, every continuous linear form on  $G$  has a unique **Hahn-Banach** extension on  $H$ . Precisely :

$$\forall g \in G', \quad \exists! h \in H', \quad f|_G = g \quad \text{and} \quad ||| h |||_{H'} = ||| g |||_{G'}$$

*Proof.* Let  $g \in G'$  and show that  $g$  has a unique **Hahn-Banach** extension in  $H$ . Since  $G$  is closed in  $H$  which is a complete space then  $G$  is complete; That is,  $G$  is **Hilbert**. By the **RIESZ REPRESENTATION THEOREM** ,

$$\exists! a \in G, \quad g(x) = \langle a, x \rangle \quad (\forall x \in G)$$

An obvious **Hahn-Banach** extension of  $g$  on  $H$  is  $h_1 \in H'$ , given by :

$$h_1(x) = \langle a, x \rangle \quad (\forall x \in H)$$

Let us show that this is the unique **Hahn-Banach** extension of  $g$ . Let  $h_2 \in H'$  be another **Hahn-Banach** extension of  $g$  on  $H$ . By the **RIESZ REPRESENTATION THEOREM** :

$$\exists! b \in H, \quad h_2(x) = \langle b, x \rangle \quad (\forall x \in H)$$

since  $h_2$  extends  $g$ , we have :

$$\begin{aligned} h_2(x) &= g(x) & (\forall x \in G) \\ \langle b, x \rangle &= \langle a, x \rangle & (\forall x \in G) \\ \langle b - a, x \rangle &= 0 & (\forall x \in G) \end{aligned}$$

In particular since  $a \in G$  :

$$\langle b - a, a \rangle = 0$$

thus :

$$(b - a) \perp a$$

so, by the **PYTHAGOREAN THEOREM** , we have :

$$\underbrace{\| (b - a) + a \|^2}_{= \| b \|^2} = \| b - a \|^2 + \| a \|^2$$

Thus :

$$\|b - a\|^2 = \|b\|^2 - \|a\|^2$$

But since  $h_2$  is a **Hahn-Banach** extension of  $g$ , we have :

$$\|h_2\|_{H'} = \|g\|_{G'}$$

i.e.

$$\|b\| = \|a\|$$

Hence :

$$\|b - a\|^2 = 0$$

implying that

$$b = a$$

Hence  $h_2 = h_1$ , as required.

This completes the proof

□