Ordinary Differential Equations Lecture Notes

Hand written summary from lectures

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Disclaimer

These notes were written in real-time during the lectures, this is not the final version, yet. so they may contain :

- Incomplete or incorrect information
- Typos, transcription mistakes, or missing content
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Notes on Contribution :

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Contents

Chapter 1

General Theory of Ordinary Differential Equations

1.1 Generalities And Physical Motivation

Lecture 1 08:05 AM Sun, Sep 28 2025

An nth-order ordinary differential equation (ODE for short) is a functional relationship having the form :

$$F(t, x, x', \dots, x^{(n)}) = 0$$

The variable t laying in the real interval I is commonly called the independent variable, and the $x \in C^n(I, \mathbb{R}^k)$ is the dependent variable.
An equation such as the above, is said to be in the implicit form. An ODE is said to be in explicit form if it's written in the form:

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

Unfortunately, there is not too much to say about ODEs in the implicit form. Notice that such an equation can be reduced to the explicit form above, when the implicit function theorem applys.
Radioactive Desintegration:
The law of radiocative desintegration have been formulated in **1902** by constating that the instantaneous rate of desintegration of a given radiocative element is propotional to the number of atoms

existings at the time considered, and doesnt depend on any other external factors. we write:

$$X'(t) = -ax(t) \qquad\qquad (x(t) = x(0)e^{-at})$$

where $x(t)$ is the number of non desintegrated atoms at time t, the positive constant is called desintegration constant and is related to the radiocative element and is experminetally determined.

Mathematical Pendulum:

Consider a pendulum of length l and denote by $\Delta(t)$ the length of the arc described by the free extrimity at time t. we have $s(t) = lx(t)$ is the measure in the radian of the angle between the vertical axis and the pendulum.

$\vec{P} = mg$ is the force exercised upon the pendulum. Decomposing the force $\vec{P}$ on the tangential axis and the thread axis and considering that the component of $\vec{P}$ is conunter-balanced by the resistance of the resistance of the thread, we obtain by Newtons second law:

$$mlx'' = -mg\sin(x)$$

thus we get:

$$x'' + \frac{g}{l}\sin(x) = 0$$

A Spatial Model in Ecology:

$$x' = \lambda x(1-x) - x$$

we have an infinite number of sites linked by immigration, all the sites are equally accessible $x(t)$ is the number of occupied sites and assume that the time is scalled so that the rate at which the sites become vaccout equals 1.

$x^{'}$ is propotional to the product of the occupied sites and vaccout sites.

The Prey Predator Model (Dynamic of population):

$$\begin{cases} x' = (a-hy)x \\ y' = -(b-kx)y \end{cases} \qquad \text{Lotha Volterra System}$$

- x is the population of prey species
- y is the population of predator species

R.L.C Circuit:

$$\begin{cases} \mathcal{L} \quad i'_L(t) = v_L - g(i_L) \\ \mathcal{C} \quad v'_c(t) = -i_L \end{cases}$$

$\vec{i}(t) = (i_k(t), i_L(t), i_C(t))$ is the state of the current in the circuit.

$v(t) = (v_k(t), v_L(t), v_C(t))$ is the state of the volatage in the circuit, the above is obtained by, Kirochoff, Faraday and Ohm Laws.

1.2 Initial Value Problem (or Cauchy Problem)

We start by following remark. Any explicit ODE of nth-order ($n \geq 2$) can be reduced to a first order ODE in explicit form. Indeed,

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$$

Put $U = (u_1, u_2, \ldots, u_n)$, where:

$$\begin{cases} u_1 = x \\ u_2 = x' \\ \quad \vdots \\ u_n = x^{(n-1)} \end{cases}$$

Notice:

$$U' = (u_1', u_2', \ldots, u_n') = (x', x'', \ldots, x^{(n)})$$

Thus:

$$U' = (u_2, u_3, \ldots, u_{n-1}, f(t, U)) = \overline{f}(t, u)$$

Hence, from now on, we consider only 1^{st}-order ODEs.

In all what follows, we let $\Omega \subset \mathbb{R} \times \mathbb{R}^k$ is a domain, and $f : \Omega \longrightarrow \mathbb{R}^k$ a continuous function.

Definition 1.2.1 : *An initial value problem (IVP for short) or a Cauchy is given by:*

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

where $(t_0, x_0) \in \Omega$

Definition 1.2.2 : *A function* $\varphi : I \longrightarrow \mathbb{R}^k$ *of class* $\mathcal{C}^1$ *on the real interval* I *is a solution of the IVP (I.1) if* $(t, \varphi(t)) \in \Omega$, *and* $\varphi'(t) = f(t, \varphi(t))$ *for all* $t \in I$.

Definition 1.2.3 : *Let $\varphi : I \longrightarrow \mathbb{R}^k$ be a solution of the IVP (I.1). The form:*

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds$$

is called the integral form of the solution.

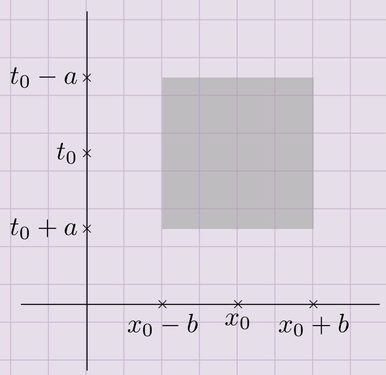
Lecture 2

08:06 AM Sun, Oct 05 2025

Let a, b be a positive real number. set:

$$\mathcal{R}(t, x_0, a, b) = \{(t, x) \in \Omega : \quad |t - t_0| \leq a \text{ and } \|x - x_0\| \leq b\}$$

where $\|.\|$ is an appropriate norm in $\mathbb{R}^k$.



The rectangle $\mathcal{R}$ is called a security system, its projection on $\mathbb{R}$ is called security interval and its projection on $\mathbb{R}^k$ which is $D = \{\|x - x_0\| \leq b\}$ is called security domain.
Suppose now that f is continuous on $\mathcal{R}$ and set:

$$M = \sup_{(t,x)\in\mathbb{R}} \|f(t, x)\|$$

and,

$$\alpha = \begin{cases} a & \text{if } M = 0 \\ \min\left(a, \frac{b}{M}\right) & \text{if } M \neq 0 \end{cases}$$

Proposition 1.2.1 : *Let φ be a solution to the IVP (IVP1) defined on an interval $I_{\overline{\alpha}} = \{t \in \mathbb{R} : \quad |t - t_0| < \overline{\alpha}\} = (t_0 - \overline{\alpha}, t_0 + \overline{\alpha})$, with $\overline{\alpha} \leq \alpha$, then $\|\varphi(t) - x_0\| < b \quad \forall t \in I_{\overline{\alpha}}$*

Proof. Suppose that φ is a solution to (IVP1), with $(t, \varphi(t)) \notin \mathcal{R}$ for $|t - t_0| < \overline{\alpha}$, since φ is continuous, there exists $\beta \in (0, \overline{\alpha})$ such that $\|\varphi(t) - x_0\| < b$ for all $|t - t_0| < \beta$ and $\|\varphi(t_0 \pm \beta) - x_0\| = b$, hence we have:

$$\sup_{t \in I_\beta, x \in \overline{B}(x_0, b)} \|f(t, x)\| \leq M$$

Hence for all $|t - t_0| \leq \beta$, we have:

$$\begin{aligned}\|\varphi(t) - x_0\| &= \|\int_{t_0}^{t} f(s, \varphi(s))ds\| \\ &\leq M\,|t - t_0| \leq M\beta \leq bM\alpha \leq b\end{aligned}$$

In particular for $t = t_0 \pm \beta$.

$$b = \|\varphi(t_0 \pm \beta) - x_0\| < b$$

contradiction. □

Proposition 1.2.2 : *If φ is a solution to (IVP1) defined on $I = \left[t_0 - \widetilde{\alpha}, t_0 + \widetilde{\alpha}\right]$ with $\widetilde{\alpha} \leq \alpha$, then:*

$$\|\varphi(t_1) - \varphi(t_2)\| \leq M\,|t_1 - t_2| \quad \forall t_1, t_2 \in I$$

☞ REMARK : If f is $\mathcal{C}^k$ and φ is a solution to (IVP1) then φ is $\mathcal{C}^{k+1}$.

Proof. From the integral form we have:

$$\begin{aligned}\|\varphi(t_1) - \varphi(t_2)\| &= \|\int_{t_1}^{t_2} f(s, \varphi(s))ds\| \\ &\leq \int_{t_1}^{t_2} \|f(s, \varphi(s))\|ds = M\,|t_1 - t_2|\end{aligned}$$

□

1.3 Existence and Unicity

1.3.1 Local Existence and Uniqueness

We use the following Banach contraction principle.

Theorem 1.3.1 : *Let (E, d) be a complete metric space and let $T : E \longrightarrow E$ be a k-contraction (i.e. $k \in [0, 1)$) and for all $x, y \in E$:*

$$d(f(x), f(y)) \leq k \cdot d(x, y)$$

then in I, T admits a unique fixed point $\overline{x}$ such that $T(\overline{x}) = \overline{x}$. Moreover, for any $x_0 \in E$ the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} = T(x_n)$ converges to $\overline{x}$ and we have:

$$d(x_n, \overline{x}) \leq \frac{k^n}{1-k} \cdot d(x_1, x_0)$$

Theorem 1.3.2 (Schauder) : *Let C be a nonempty closed convex set in a Banach space E . Any compact mapping $T : C \longrightarrow C$ (i.e. T is continuous and $\overline{T(C)}$ is compact) admits a fixed point i.e.*

$$\exists \overline{x} \in C : \quad T(\overline{x}) = \overline{x}$$

Theorem 1.3.3 : *Suppose that f is $L-$ Lipschitzian with respect to x, i.e.*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $x, y \in \overline{B}(x_0, b)$ and for all $t \in [t_0 - \alpha, t_0 + \alpha]$. Then there exist $\delta \in (0, \alpha)$ such that IVP (IVP1) has a unique solution φ defined on $I_\delta = [t_0 - \delta, t_0 + \delta]$.

Proof. For any $\delta \in (0, \alpha)$ consider the mapping $T_\delta : X \longrightarrow X$ defined by

$$T_\delta x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

for all $x \in X = \{x : I_\delta \longrightarrow \mathbb{R}^k : \quad \text{continuous}\}$ endowed by the sup-norm $\|.\|_\infty$ i.e.

$\|x\|_\infty = \sup_{t\in I_\delta} \|x(t)\|$. For any $x, y \in X$ and any $t \in I_\delta$, we have

$$\begin{aligned}
\|Tx(t) - Ty(t)\| &= \|\int_{t_0}^{t} f(s, x(s))ds\| \\
&\leq \int_{\min(t,t_0)}^{\max(t,t_0)} \|f(s,x(s)) - f(s,y(s))\|ds \\
&\leq L \int_{\min(t,t_0)}^{\max(t,t_0)} \|x(s) - y(s)\|ds \\
&\leq L\|x-y\|_\infty \int_{\min(t_0,t)}^{\max(t_0,t)} ds \\
&\leq L\,|t-t_0|\,\|x-y\|_\infty \\
&\leq L\delta\|x-y\|_\infty
\end{aligned}$$

so for $\delta < \frac{1}{L}$, T_δ is a contraction. Hence, for such a δ, T_δ has a unique fixed point $\phi \in X$, i.e. $\phi : I_\delta \longrightarrow \mathbb{R}^k$ with,

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds,$$

which is the unique solution to the IVP(IVP1). □

Lecture 3 08:00 AM Sun, Oct 12 2025

Example:

$$\begin{cases} x'(t) = \sin(x(t)) \\ x(0) = 0 \end{cases}$$

$\overline{B}(0,r)$ and $I_a = [-a,a]$ where $a > 0$, for all $x, y \in \mathbb{R}$,

$$|\sin(x) - \sin(y)| \leq |x-y|\,,$$

for all $\delta \in (0,\alpha)$, $\quad \alpha = \min(a, \frac{r}{1}) = \min(a,r)$, the IVP admits a unique solution defined on $I_\delta = [-\delta,\delta]$.

1.3.2 Local existence and uniqueness by successive approximation method

Picard's successive approximation method consists in using the integral form of a solution to the IVP(IVP1) to construct a sequence of apprxomiation of the solution. we do as follows,

$$\begin{cases} \varphi_0(t) = x_0 \\ \varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(n)(s))ds \end{cases}$$

Theorem 1.3.4 : *Suppose that f is L-Lipschitzian with respect to x in R. Then the sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ converges uniformally on the interval $(t_0-\alpha, t_0+\alpha)$ to the some ϕ which is the unique solution to the IVP(IVP1).*

Corollary 1.3.5 (Gronwall's Inequality) : *If $g : [t_0, t_1] \longrightarrow \mathbb{R}$ is continuous with*

$$0 \leq g(t) \leq k + L\int_{t_0}^{t} g(s)ds \qquad \forall t \in [t_0, t_1],$$

then,

$$0 \leq g(t) \leq Ke^{L(t-t_0)} \qquad \forall t \in [t_0, t_1] .$$

where $K, L > 0$

Proof. Proof of *Theorem 1.3.4* .

Existence: For $t \in [t_0-\alpha, t+\alpha]$, we have,

$$\|\varphi_1(t) - \varphi_0(t)\| = \|\int_{t_0}^{t} f(s, \varphi_0(s))ds\| \leq M\,|t-t_0| \leq \alpha M \leq b.$$

Hence, $\|\varphi_1 - \varphi_0\|_\infty \leq \|\varphi_1 - x_0\|_\infty \leq b$, and $\int_{t_0}^{t} f(s), \varphi_1(s)ds$ is defined for all $t \in I_\alpha$.

$$\begin{aligned}\|\varphi_2(t) - \varphi_0(t)\| &= \|\int_{t_0}^{t} f(s, \varphi_1(s))ds\| \\ &\leq M\alpha \leq b.\end{aligned}$$

Hence $\|\varphi_2 - \varphi_0\|_\infty \leq b$ and $\int_{t_0}^{t} f(s, \varphi_2(s))ds$ is defined for all $t \in I_\alpha$. Therefore, we see that for all $n \in \mathbb{N}_0$, we have,

$$\|\varphi_n - \varphi_0\|_\infty \leq b,$$

and

$$\int_{t_0}^{t} f(s, \varphi_n(s))ds,$$

is defined for all $t \in I_\alpha$. Now, for all $k \geq 1$ and for all $t \in I_\alpha$

$$\begin{aligned}\|\varphi_{k+1}(t) - \varphi_k(t)\| &= \|\int_{t_0}^{t} f(s, \varphi_k(s)) - f(s, \varphi_{k-1}(s))ds\| \\ &\leq \int_{\min(t_0,t)}^{\max(t_0,t)} \|f(s, \varphi_n(s)) - f(s, \varphi_{n-1}(s))\|ds \\ &\leq L\left|\int_{t_0}^{t} \|\varphi_k(s) - \varphi_{k-1}(s)\|ds\right| .\end{aligned}$$

We prove now that,

$$\|\varphi_{k+1}(t)-\varphi_k(t)\| \leq b\cdot\frac{L^k\,|t-t_0|^k}{k!} = b\cdot\frac{(L\,|t-t_0|)^k}{k!} \qquad \forall t\in I_\alpha.$$

By induction, □

Lecture 4 `08:04 AM Sun, Oct 19 2025`

Picard's Method ✎

$$\begin{cases}\varphi_0(t) &= x_0,\\ \varphi_{n+1}(t) &= x_0+\int_{t_0}^t f(s,\varphi_n(s))ds.\end{cases}$$

Theorem 1.3.6 : *$f:\mathbb{R}\longrightarrow\mathbb{R}^k$ is continuous and L-Lipschitzian on $\mathcal{R}$. $\mathcal{R}=\{(t,x):\ \ |t-t_0|\leq a \text{ and } \|x-x_0\|\leq b\}$. We have;*

$$\|f(t,x)-f(t,y)\|\leq L\|x-y\|.$$

Then, IVP (IVP1) has a unique solution defined on $[t_0-\alpha,t_0+\alpha]$ where,

$$\alpha=\begin{cases}a & \text{if } M=0,\\ \min\left(a,\frac{b}{M}\right) & \text{if } M\neq 0.\end{cases}$$

Proof. We have,

$$\|\varphi_n-\varphi_0\|=\|\varphi_n-x_0\|\leq b.$$

We get,

$$\begin{aligned}\|\varphi_{k+1}(t)-\varphi_k(t)\| &= \|\int_{t_0}^t f(s,\varphi_k(s))-f(s,\varphi_{k-1}(s))ds\|\\ &\leq L\left|\int_{t_0}^t \|\varphi_k(s)-\varphi_{k-1}(s)\|ds\right|.\end{aligned}$$

Thus,

$$\|\varphi_{k+1}(t)-\varphi_k(t)\|\leq\frac{b}{k!}(L\,|t-t_0|)^k \qquad (\forall t\in I_\alpha).$$

By induction:

$$\begin{aligned}\|\varphi_2(t)-\varphi_1(t)\| &\leq L\left|\int_t^{t_0}\|\varphi_1(s)-\varphi_0(s)\|ds\right|,\\ &\leq b\,(L\,|t-t_0|)\,.\end{aligned}$$

If $\|\varphi_k(t) - \varphi_{k-1}(t)\| \leq \frac{b}{(k-1)!}(L\,|t-t_0|)^{k-1}$. Therefore,

$$\begin{aligned}
\|\varphi_{k+1}(t) - \varphi_k(t)\| &\leq L\left|\int_{t_0}^{t} \|\varphi_k(s) - \varphi_{k-1}(s)\| ds\right|, \\
&\leq L\left|\int_{t_0}^{t} b\frac{(L\,|s-s_0|)^{k-1}}{(k-1)!} ds\right|, \\
&= b\frac{(L\,|t-t_0|)^{k}}{k!}.
\end{aligned}$$

Let $N, n \in \mathbb{N}$ with $N > n$.

$$\begin{aligned}
\|\varphi_N(t) - \varphi_n(t)\| &\leq \sum_{k=n}^{N-1} \|\varphi_{k+1}(t) - \varphi_k(t)\|, \\
&\leq b\sum_{k=n}^{N-1} \frac{(L\,|t-t_0|)^k}{k!} \leq b\sum_{k=n}^{N-1} \frac{(L\alpha)^k}{k!},
\end{aligned}$$

then by the remainder of a convergent series, we get

$$\leq b\sum_{k\geq n} \frac{(L\alpha)^k}{k!} \to 0 \quad \text{as } n \to +\infty.$$

So (φ_n) converges uniformally to some function $\varphi : I_\alpha \longrightarrow \mathbb{R}^k$. Letting $n \to +\infty$ in $\varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))ds$, we get

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds.$$

Hence φ is a solution of IVP(IVP1).

Corollary 1.3.7 (Gronwall's Inequality) : *Let $g : [a, b] \longrightarrow \mathbb{R}^+$ be a continuous function satifying,*

$$g(t) \leq k + l\int_{a}^{t} g(s)ds, \qquad \forall t \in [a, b];$$

where $k, l \geq 0$. Then,

$$g(t) \leq ke^{l(t-a)}$$

Uniqueness: Suppose that φ and Ψ are two solutions to (IVP1). Then,

$$\begin{aligned}
\varphi(t) &= x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds, \\
\Psi(t) &= x_0 + \int_{t_0}^{t} f(s, \Psi(s))ds,
\end{aligned}$$

for all $t \in I_\alpha = [t_0-\alpha, t_0+\alpha]$. For $t \geq t_0$,

$$\begin{aligned}\|\varphi(t)-\Psi(t)\| &= \|\int_{t_0}^{t} f(s,\varphi(s))ds - f(s,\Psi(s))ds\|,\\ &\leq \underbrace{k}_{=0\text{Gronwall's Inequality.}} +L\int_{t_0}^{t}\|\varphi(s)-\Psi(s)\|ds.\end{aligned}$$

Hence,

$$\|\varphi(t)-\Psi(s)\| \leq 0 \implies \varphi(t)=\Psi(t) \quad \forall t \geq t_0.$$

□

1.3.3 Peano's Theorem

Theorem 1.3.8 (Peano) : *Suppose that f is continuous on R, then IVP (IVP1) admits a solution defined on $[t_0-\alpha, t_0+\alpha]$.*

Theorem 1.3.9 (Ascoli Arzela) : *Let $\mathcal{M}$ be a nonempty set in $\mathcal{C}\left([a,b],\mathbb{R}^k\right)=E$. Then $\mathcal{M}$ is relatively compact in E ($\overline{\Omega}$ is compact) if and only if:*

① *$\mathcal{M}$ is uniformaly bounded; i.e. $\exists M>0$ such that $\|u\| \leq M$.*

② *$\mathcal{M}$ is equicontinuous; i.e. $\forall \varepsilon>0, \exists \delta>0$ such that,*

$$|t_1-t_2| \leq \delta \implies \|u(t_1)-u(t_2)\| \leq \varepsilon \qquad \forall u \in \mathcal{M}, \quad \forall t_1,t_2 \in [a,b].$$

Proof. ⤞ Proof of Peano's theorem.

$X=\mathcal{C}(I_\alpha,\mathbb{R}^k), I_\alpha=[t_0-\alpha,t_0+\alpha]$, endowed with sup-norm $\|.\|_{+\infty}$. Define the map,

$$\begin{array}{rccl} T: & X & \longrightarrow & X\\ & x(t) & \longmapsto & Tx(t)=x_0+\int_{t_0}^{t} f(s,x(s))ds.\end{array}$$

Let $\Omega=\overline{B}(x_0,b)$, Clearly Ω is nonempty closed and convex.

- $T(\Omega)\subset\Omega$. Indeed for any $x\in\Omega$,

$$\begin{aligned}\|x-x_0\|_\infty &= \sup_{t\in I_\alpha}\|x(t)-x_0\|\\ &\leq \sup_{t\in I_\alpha}\left(\|\int_{t_0}^{t} f(s,x(s))ds\|\right)\\ &\leq \sup_{t\in I_\alpha}\left(|t-t_0|\,M\right)\\ &\leq M\alpha \leq b.\end{aligned}$$

- *T is continuous.* Let $(x_n) \subset X$ convering to $\overline{x}$ in X. f being continuous on the compact $\mathcal{R}$, is uniformally continuous on $\mathcal{R}$. i.e.;

  $$\forall \varepsilon > 0, \exists \delta > 0, \forall (t_1, x_1), (t_2, x_2) \in \mathcal{R} : \|(t_1, x_1) - (t_2, x_2)\| \leq \delta \implies \|f(t_1, x_1) - f(t_2, x_2)\| \leq \varepsilon.$$

  In particular, for all $t \in I_\alpha, \forall x_1, x_2 \in \mathbb{R}^k$.

  $$\|x_1 - x_2\| \leq \delta \implies \|f(t, x_1) - f(t, x_2)\| \leq \varepsilon.$$

  Let $\varepsilon > 0, \exists \delta > 0$ such that for all $x, y \in \overline{B}(x_0, b)$, and for all $t \in I_\alpha$. We have,

  $$\|x - y\| \leq \delta \implies \|f(t, x) - f(t, y)\| \leq \frac{\varepsilon}{\alpha}.$$

  since $x_n \xrightarrow{\text{in X}} \overline{x}, \exists n_0 \in \mathbb{N}$ such that,

  $$\|x_n - x_0\|_\infty \leq \delta \qquad \forall n \geq n_0.$$

  Hence,

  $$\begin{aligned}
  \|Tx_n(t) - T\overline{x}(t)\| &= \| \int_{t_0}^{t} \left(f(s, x_n(s)) - f(s, \overline{x}(s))\right) ds \| \\
  &\leq \left| \int_{t_0}^{t} \|f(s, x_n(s)) - f(s, \overline{x}(s))\| ds \right| \\
  &\leq |t - t_0| \frac{\varepsilon}{\alpha} \leq \alpha \frac{\varepsilon}{\alpha} = \varepsilon, \\
  &\implies \|Tx_n - T\overline{x}\|_1 \leq \varepsilon.
  \end{aligned}$$

- *$\overline{T(\Omega)}$ is compact in X* . $T(\Omega)$ is uniformaly bounded since for all $x \in \Omega, \|Tx - x_0\|_\infty \leq b$.

  $$\begin{aligned}
  \|Tx\|_\infty &\leq \|Tx - x_0\|_\infty + \|x_0\|_\infty, \\
  &\leq b + \|x_0\|_\infty.
  \end{aligned}$$

For any $x \in \Omega$, and for any $t_1, t_2 \in I_\alpha$. We have;

$$\begin{aligned}
\|Tx(t_1) - Tx(t_2)\| &= \| \int_{t_0}^{t_1} f(s, x(s)) ds - \int_{t_0}^{t_2} f(s, x(s)) ds \| \\
&= \| \int_{t_1}^{t_2} f(s, x(s)) ds \| \\
&\leq \left| \int_{t_1}^{t_2} f(s, x(s)) ds \right| \\
&\leq M |t_1, t_2| .
\end{aligned}$$

So $T(\Omega)$ is equicontinuous and $\overline{T(\Omega)}$ is compact in X.

⇝ **Conclusion:** By Schauder's Theorem T has a fixed point φ which is a solution to (IVP1),

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds.$$

$\square$

1.4 Extension of solutions

The results we have seen deal local existence. Indeed the solution was defined in a neighborhood of to and in a neighborhood of the inital data. The extension of solution (a continuous of solution) consists in studying criteria. Which allow to extend the interval of definition of the solution. Here, we let $f : U \subset \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ continuous, U is open.

Corollary 1.4.1 : *Assume that $\varphi : I = [t_1, t_2] \longrightarrow \mathbb{R}^b$ satisfies $\varphi'(t) = f(t, \varphi(t))$ and $(t, \varphi(t)) \in U$ for all $t \in I$. If $\exists(\tau_n) \subset I$ such that;*

$$\lim_{n\to\infty} (\tau_n, \varphi(\tau_n)) = (t_1, r) \in U,$$

Then,

$$\lim_{z\to t_1} (z, \varphi(z)) = (t_1, r).$$