Ordinary Differential Equations Lecture Notes

Hand written summary from lectures

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Disclaimer

These notes were written in real-time during the lectures, this is not the final version, yet. so they may contain :

- Incomplete or incorrect information
- Typos, transcription mistakes, or missing content
- Interpretations or notations that reflect my own understanding at the moment

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Notes on Contribution :

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Chapter 1

General Theory of Ordinary Differential Equations

1.1 Generalities And Physical Motivation

Lecture 1 `08:05 AM Sun, Sep 28 2025`

An nth-order ordinary differential equation (ODE for short) is a functional relationship having the form :

$$F(t, x, x', \dots, x^{(n)}) = 0$$

The variable t laying in the real interval I is commonly called the independent variable, and the $x \in C^n(I, \mathbb{R}^k)$ is the dependent variable.
An equation such as the above, is said to be in the implicit form. An ODE is said to be in explicit form if it's written in the form:

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

Unfortunately, there is not too much to say about ODEs in the implicit form. Notice that such an equation can be reduced to the explicit form above, when the implicit function theorem applys.
Radioactive Desintegration:
The law of radiocative desintegration have been formulated in **1902** by constating that the instantaneous rate of desintegration of a given radiocative element is propotional to the number of atoms

existings at the time considered, and doesnt depend on any other external factors. we write:

$$X'(t) = -ax(t) \qquad\qquad (x(t) = x(0)e^{-at})$$

where $x(t)$ is the number of non desintegrated atoms at time t, the positive constant is called desintegration constant and is related to the radiocative element and is experminetally determined.

Mathematical Pendulum:

Consider a pendulum of length l and denote by $\Delta(t)$ the length of the arc described by the free extrimity at time t. we have $s(t) = lx(t)$ is the measure in the radian of the angle between the vertical axis and the pendulum.

$\vec{P} = mg$ is the force exercised upon the pendulum. Decomposing the force $\vec{P}$ on the tangential axis and the thread axis and considering that the component of $\vec{P}$ is conunter-balanced by the resistance of the resistance of the thread, we obtain by Newtons second law:

$$mlx'' = -mg\sin(x)$$

thus we get:

$$x'' + \frac{g}{l}\sin(x) = 0$$

A Spatial Model in Ecology:

$$x' = \lambda x(1-x) - x$$

we have an infinite number of sites linked by immigration, all the sites are equally accessible $x(t)$ is the number of occupied sites and assume that the time is scalled so that the rate at which the sites become vaccout equals 1.

$x^{'}$ is propotional to the product of the occupied sites and vaccout sites.

The Prey Predator Model (Dynamic of population):

$$\begin{cases} x' = (a - hy)x \\ y' = -(b - kx)y \end{cases} \qquad \text{Lotha Volterra System}$$

- x is the population of prey species
- y is the population of predator species

R.L.C Circuit:

$$\begin{cases} \mathcal{L} \quad i'_L(t) = v_L - g(i_L) \\ \mathcal{C} \quad v'_c(t) = -i_L \end{cases}$$

$\vec{i}(t) = (i_k(t), i_L(t), i_C(t))$ is the state of the current in the circuit.

$v(t) = (v_k(t), v_L(t), v_C(t))$ is the state of the volatage in the circuit, the above is obtained by, Kirochoff, Faraday and Ohm Laws.

1.2 Initial Value Problem (or Cauchy Problem)

We start by following remark. Any explicit ODE of nth-order ($n \geq 2$) can be reduced to a first order ODE in explicit form. Indeed,

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

Put $U = (u_1, u_2, \dots, u_n)$, where:

$$\begin{cases} u_1 = x \\ u_2 = x' \\ \quad\vdots \\ u_n = x^{(n-1)} \end{cases}$$

Notice:

$$U' = (u_1', u_2', \dots, u_n') = (x', x'', \dots, x^{(n)})$$

Thus:

$$U' = (u_2, u_3, \dots, u_{n-1}, f(t, U)) = \overline{f}(t, u)$$

Hence, from now on, we consider only 1st-order ODEs.

In all what follows, we let $\Omega \subset \mathbb{R} \times \mathbb{R}^k$ is a domain, and $f : \Omega \longrightarrow \mathbb{R}^k$ a continuous function.

Definition 1.2.1 : *An initial value problem (IVP for short) or a Cauchy is given by:*

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

where $(t_0, x_0) \in \Omega$

Definition 1.2.2 : *A function* $\varphi : I \longrightarrow \mathbb{R}^k$ *of class* $\mathcal{C}^1$ *on the real interval* I *is a solution of the IVP (I.1) if* $(t, \varphi(t)) \in \Omega$, *and* $\varphi'(t) = f(t, \varphi(t))$ *for all* $t \in I$.

Definition 1.2.3 : *Let $\varphi : I \longrightarrow \mathbb{R}^k$ be a solution of the IVP (I.1). The form:*

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds$$

is called the integral form of the solution.

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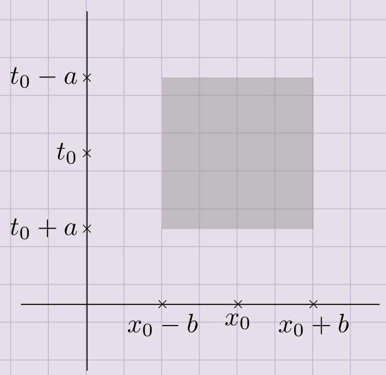
`08:06 AM Sun, Oct 05 2025`

Let a, b be a positive real number. set:

$$\mathcal{R}(t, x_0, a, b) = \{(t, x) \in \Omega : \quad |t - t_0| \leq a \text{ and } \|x - x_0\| \leq b\}$$

where $\|.\|$ is an appropriate norm in $\mathbb{R}^k$.



The rectangle $\mathcal{R}$ is called a security system, its projection on $\mathbb{R}$ is called security interval and its projection on $\mathbb{R}^k$ which is $D = \{\|x - x_0\| \leq b\}$ is called security domain.
Suppose now that f is continuous on $\mathcal{R}$ and set:

$$M = \sup_{(t,x)\in\mathbb{R}} \|f(t, x)\|$$

and,

$$\alpha = \begin{cases} a & \text{if } M = 0 \\ \min\left(a, \frac{b}{M}\right) & \text{if } M \neq 0 \end{cases}$$

Proposition 1.2.1 : *Let φ be a solution to the IVP (IVP1) defined on an interval $I_{\overline{\alpha}} = \{t \in \mathbb{R} : \quad |t - t_0| < \overline{\alpha}\} = (t_0 - \overline{\alpha}, t_0 + \overline{\alpha})$, with $\overline{\alpha} \leq \alpha$, then $\|\varphi(t) - x_0\| < b \quad \forall t \in I_{\overline{\alpha}}$*

Proof. Suppose that φ is a solution to (IVP1), with $(t, \varphi(t)) \notin \mathcal{R}$ for $|t - t_0| < \overline{\alpha}$, since φ is continuous, there exists $\beta \in (0, \overline{\alpha})$ such that $\|\varphi(t) - x_0\| < b$ for all $|t - t_0| < \beta$ and $\|\varphi(t_0 \pm \beta) - x_0\| = b$, hence we have:

$$\sup_{t \in I_\beta, x \in \overline{B}(x_0, b)} \|f(t, x)\| \leq M$$

Hence for all $|t - t_0| \leq \beta$, we have:

$$\begin{aligned} \|\varphi(t) - x_0\| &= \| \int_{t_0}^{t} f(s, \varphi(s)) ds \| \\ &\leq M |t - t_0| \leq M\beta \leq bM\alpha \leq b \end{aligned}$$

In particular for $t = t_0 \pm \beta$.

$$b = \|\varphi(t_0 \pm \beta) - x_0\| < b$$

contradiction. □

Proposition 1.2.2 : *If φ is a solution to (IVP1) defined on $I = \left[t_0 - \widetilde{\alpha}, t_0 + \widetilde{\alpha}\right]$ with $\widetilde{\alpha} \leq \alpha$, then:*

$$\|\varphi(t_1) - \varphi(t_2)\| \leq M |t_1 - t_2| \quad \forall t_1, t_2 \in I$$

☞ REMARK : If f is $\mathcal{C}^k$ and φ is a solution to (IVP1) then φ is $\mathcal{C}^{k+1}$.

Proof. From the integral form we have:

$$\begin{aligned} \|\varphi(t_1) - \varphi(t_2)\| &= \| \int_{t_1}^{t_2} f(s, \varphi(s)) ds \| \\ &\leq \int_{t_1}^{t_2} \|f(s, \varphi(s))\| ds = M |t_1 - t_2| \end{aligned}$$

□

1.3 Existence and Unicity

1.3.1 Local Existence and Uniqueness

We use the following Banach contraction principle.

Theorem 1.3.1 : *Let (E,d) be a complete metric space and let $T : E \longrightarrow E$ be a k-contraction (i.e. $k \in [0,1)$) and for all $x,y \in E$:*

$$d(f(x),f(y)) \leq k \cdot d(x,y)$$

then in I, T admits a unique fixed point $\overline{x}$ such that $T(\overline{x}) = \overline{x}$. Moreover, for any $x_0 \in E$ the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} = T(x_n)$ converges to $\overline{x}$ and we have:

$$d(x_n,\overline{x}) \leq \frac{k^n}{1-k} \cdot d(x_1,x_0)$$

Theorem 1.3.2 (Schauder) : *Let C be a nonempty closed convex set in a Banach space E . Any compact mapping $T : C \longrightarrow C$ (i.e. T is continuous and $\overline{T(C)}$ is compact) admits a fixed point i.e.*

$$\exists \overline{x} \in C : \quad T(\overline{x}) = \overline{x}$$

Theorem 1.3.3 : *Suppose that f is $L-$ Lipschitzian with respect to x, i.e.*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$$

for all $x,y \in \overline{B}(x_0,b)$ and for all $t \in [t_0-\alpha,t_0+\alpha]$. Then there exist $\delta \in (0,\alpha)$ such that IVP (IVP1) has a unique solution φ defined on $I_\delta = [t_0-\delta,t_0+\delta]$.

Proof. For any $\delta \in (0,\alpha)$ consider the mapping $T_\delta : X \longrightarrow X$ defined by

$$T_\delta x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds$$

for all $x \in X = \{x : I_\delta \longrightarrow \mathbb{R}^k : \quad \text{continuous}\}$ endowed by the sup-norm $\|.\|_\infty$ i.e.

$\|x\|_\infty = \sup_{t\in I_\delta} \|x(t)\|$. For any $x, y \in X$ and any $t \in I_\delta$, we have

$$\begin{aligned}
\|Tx(t) - Ty(t)\| &= \| \int_{t_0}^{t} f(s, x(s))ds\| \\
&\leq \int_{\min(t,t_0)}^{\max(t,t_0)} \|f(s,x(s)) - f(s,y(s))\|ds \\
&\leq L \int_{\min(t,t_0)}^{\max(t,t_0)} \|x(s) - y(s)\|ds \\
&\leq L\|x-y\|_\infty \int_{\min(t_0,t)}^{\max(t_0,t)} ds \\
&\leq L\,|t - t_0|\,\|x-y\|_\infty \\
&\leq L\delta\|x-y\|_\infty
\end{aligned}$$

so for $\delta < \frac{1}{L}$, T_δ is a contraction. Hence, for such a δ, T_δ has a unique fixed point $\phi \in X$, i.e. $\phi : I_\delta \longrightarrow \mathbb{R}^k$ with,

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds,$$

which is the unique solution to the IVP(IVP1). □

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Example:

$$\begin{cases} x'(t) = \sin(x(t)) \\ x(0) = 0 \end{cases}$$

$\overline{B}(0, r)$ and $I_a = [-a, a]$ where $a > 0$, for all $x, y \in \mathbb{R}$,

$$|\sin(x) - \sin(y)| \leq |x - y|\,,$$

for all $\delta \in (0, \alpha)$, $\quad \alpha = \min(a, \frac{r}{1}) = \min(a, r)$, the IVP admits a unique solution defined on $I_\delta = [-\delta, \delta]$.

1.3.2 Local existence and uniqueness by successive approximation method

Picard's successive approximation method consists in using the integral form of a solution to the IVP(IVP1) to construct a sequence of apprxomiation of the solution. we do as follows,

$$\begin{cases} \varphi_0(t) = x_0 \\ \varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(n)(s))ds \end{cases}$$

Theorem 1.3.4 : *Suppose that f is L-Lipschitzian with respect to x in R. Then the sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ converges uniformally on the interval $(t_0-\alpha, t_0+\alpha)$ to the some ϕ which is the unique solution to the IVP(IVP1).*

Corollary 1.3.5 (Gronwall's Inequality) : *If $g : [t_0, t_1] \longrightarrow \mathbb{R}$ is continuous with*

$$0 \le g(t) \le k + L\int_{t_0}^{t} g(s)ds \qquad \forall t \in [t_0, t_1],$$

then,

$$0 \le g(t) \le Ke^{L(t-t_0)} \qquad \forall t \in [t_0, t_1].$$

where $K, L > 0$

Proof. Proof of *Theorem 1.3.4* .

Existence: For $t \in [t_0-\alpha, t+\alpha]$, we have,

$$\|\varphi_1(t) - \varphi_0(t)\| = \|\int_{t_0}^{t} f(s, \varphi_0(s))ds\| \le M\,|t-t_0| \le \alpha M \le b.$$

Hence, $\|\varphi_1 - \varphi_0\|_\infty \le \|\varphi_1 - x_0\|_\infty \le b$, and $\int_{t_0}^{t} f(s), \varphi_1(s)ds$ is defined for all $t \in I_\alpha$.

$$\begin{aligned}\|\varphi_2(t) - \varphi_0(t)\| &= \|\int_{t_0}^{t} f(s, \varphi_1(s))ds\| \\ &\le M\alpha \le b.\end{aligned}$$

Hence $\|\varphi_2 - \varphi_0\|_\infty \le b$ and $\int_{t_0}^{t} f(s, \varphi_2(s))ds$ is defined for all $t \in I_\alpha$. Therefore, we see that for all $n \in \mathbb{N}_0$, we have,

$$\|\varphi_n - \varphi_0\|_\infty \le b,$$

and

$$\int_{t_0}^{t} f(s, \varphi_n(s))ds,$$

is defined for all $t \in I_\alpha$. Now, for all $k \ge 1$ and for all $t \in I_\alpha$

$$\begin{aligned}\|\varphi_{k+1}(t) - \varphi_k(t)\| &= \|\int_{t_0}^{t} f(s, \varphi_k(s)) - f(s, \varphi_{k-1}(s))ds\| \\ &\le \int_{\min(t_0,t)}^{\max(t_0,t)} \|f(s, \varphi_n(s)) - f(s, \varphi_{n-1}(s))\|ds \\ &\le L\left|\int_{t_0}^{t} \|\varphi_k(s) - \varphi_{k-1}(s)\|ds\right|.\end{aligned}$$

We prove now that,

$$\|\varphi_{k+1}(t)-\varphi_k(t)\| \leq b\cdot\frac{L^k\,|t-t_0|^k}{k!} = b\cdot\frac{(L\,|t-t_0|)^k}{k!} \qquad \forall t\in I_\alpha.$$

By induction, $\square$

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Picard's Method ✎

$$\begin{cases}\varphi_0(t) & = x_0,\\ \varphi_{n+1}(t) & = x_0+\int_{t_0}^t f(s,\varphi_n(s))ds.\end{cases}$$

Theorem 1.3.6 : *$f:\mathbb{R}\longrightarrow\mathbb{R}^k$ is continuous and L-Lipschitzian on $\mathcal{R}$. $\mathcal{R}=\{(t,x):\quad |t-t_0|\leq a \text{ and } \|x-x_0\|\leq b\}$. We have;*

$$\|f(t,x)-f(t,y)\|\leq L\|x-y\|.$$

Then, IVP (IVP1) has a unique solution defined on $[t_0-\alpha,t_0+\alpha]$ where,

$$\alpha=\begin{cases} a & \text{if } M=0,\\ \min\left(a,\frac{b}{M}\right) & \text{if } M\neq 0.\end{cases}$$

Proof. We have,

$$\|\varphi_n-\varphi_0\| = \|\varphi_n-x_0\|\leq b.$$

We get,

$$\begin{aligned}\|\varphi_{k+1}(t)-\varphi_k(t)\| &= \|\int_{t_0}^t f(s,\varphi_k(s))-f(s,\varphi_{k-1}(s))ds\|\\ &\leq L\left|\int_{t_0}^t \|\varphi_k(s)-\varphi_{k-1}(s)\|ds\right|.\end{aligned}$$

Thus,

$$\|\varphi_{k+1}(t)-\varphi_k(t)\|\leq\frac{b}{k!}(L\,|t-t_0|)^k \qquad (\forall t\in I_\alpha).$$

By induction:

$$\begin{aligned}\|\varphi_2(t)-\varphi_1(t)\| &\leq L\left|\int_t^{t_0}\|\varphi_1(s)-\varphi_0(s)\|ds\right|,\\ &\leq b\,(L\,|t-t_0|)\,.\end{aligned}$$

If $\|\varphi_k(t)-\varphi_{k-1}(t)\| \leq \frac{b}{(k-1)!}(L\,|t-t_0|)^{k-1}$. Therefore,

$$\begin{aligned}
\|\varphi_{k+1}(t)-\varphi_k(t)\| &\leq L\left|\int_{t_0}^{t}\|\varphi_k(s)-\varphi_{k-1}(s)\|ds\right|,\\
&\leq L\left|\int_{t_0}^{t} b\frac{(L\,|s-s_0|)^{k-1}}{(k-1)!}ds\right|,\\
&= b\frac{(L\,|t-t_0|)^{k}}{k!}.
\end{aligned}$$

Let $N, n \in \mathbb{N}$ with $N > n$.

$$\begin{aligned}
\|\varphi_N(t)-\varphi_n(t)\| &\leq \sum_{k=n}^{N-1}\|\varphi_{k+1}(t)-\varphi_k(t)\|,\\
&\leq b\sum_{k=n}^{N-1}\frac{(L\,|t-t_0|)^{k}}{k!} \leq b\sum_{k=n}^{N-1}\frac{(L\alpha)^k}{k!},
\end{aligned}$$

then by the remainder of a convergent series, we get

$$\leq b\sum_{k\geq n}\frac{(L\alpha)^k}{k!} \to 0 \quad \text{as } n\to+\infty.$$

So (φ_n) converges uniformally to some function $\varphi : I_\alpha \longrightarrow \mathbb{R}^k$. Letting $n\to+\infty$ in $\varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s,\varphi_n(s))ds$, we get

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s,\varphi(s))ds.$$

Hence φ is a solution of IVP(IVP1).

Corollary 1.3.7 (Gronwall's Inequality) : *Let $g : [a,b] \longrightarrow \mathbb{R}^+$ be a continuous function satifying,*

$$g(t) \leq k + l\int_a^t g(s)ds, \qquad \forall t\in[a,b];$$

where $k, l \geq 0$. Then,

$$g(t) \leq ke^{l(t-a)}$$

Uniqueness: Suppose that φ and Ψ are two solutions to (IVP1). Then,

$$\begin{aligned}
\varphi(t) &= x_0 + \int_{t_0}^{t} f(s,\varphi(s))ds,\\
\Psi(t) &= x_0 + \int_{t_0}^{t} f(s,\Psi(s))ds,
\end{aligned}$$

for all $t \in I_\alpha = [t_0 - \alpha, t_0 + \alpha]$. For $t \geq t_0$,

$$\begin{aligned}\|\varphi(t) - \Psi(t)\| &= \| \int_{t_0}^{t} f(s, \varphi(s))ds - f(s, \Psi(s))ds\|, \\ &\leq \underbrace{k}_{=0\text{Gronwall's Inequality.}} + L \int_{t_0}^{t} \|\varphi(s) - \Psi(s)\|ds.\end{aligned}$$

Hence,

$$\|\varphi(t) - \Psi(s)\| \leq 0 \implies \varphi(t) = \Psi(t) \quad \forall t \geq t_0.$$

□

1.3.3 Peano's Theorem

Theorem 1.3.8 (Peano) : *Suppose that f is continuous on R, then IVP (IVP1) admits a solution defined on $[t_0 - \alpha, t_0 + \alpha]$.*

Theorem 1.3.9 (Ascoli Arzela) : *Let $\mathcal{M}$ be a nonempty set in $\mathcal{C}\left([a, b], \mathbb{R}^k\right) = E$. Then $\mathcal{M}$ is relatively compact in E ($\overline{\Omega}$ is compact) if and only if:*

① *$\mathcal{M}$ is uniformaly bounded; i.e. $\exists M > 0$ such that $\|u\| \leq M$.*

② *$\mathcal{M}$ is equicontinuous; i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that,*

$$|t_1 - t_2| \leq \delta \implies \|u(t_1) - u(t_2)\| \leq \varepsilon \qquad \forall u \in \mathcal{M}, \quad \forall t_1, t_2 \in [a, b].$$

Proof. ⇴ Proof of Peano's theorem.

$X = \mathcal{C}(I_\alpha, \mathbb{R}^k), I_\alpha = [t_0 - \alpha, t_0 + \alpha]$, endowed with sup-norm $\|.\|_{+\infty}$. Define the map,

$$\begin{aligned} T: \quad X &\longrightarrow X \\ x(t) &\longmapsto Tx(t) = x_0 + \textstyle\int_{t_0}^{t} f(s, x(s))ds. \end{aligned}$$

Let $\Omega = \overline{B}(x_0, b)$, Clearly Ω is nonempty closed and convex.

- $T(\Omega) \subset \Omega$. Indeed for any $x \in \Omega$,

$$\begin{aligned}\|x - x_0\|_\infty &= \sup_{t \in I_\alpha} \|x(t) - x_0\| \\ &\leq \sup_{t \in I_\alpha} \left(\| \int_{t_0}^{t} f(s, x(s))ds \| \right) \\ &\leq \sup_{t \in I_\alpha} \left(|t - t_0| \, M \right) \\ &\leq M\alpha \leq b.\end{aligned}$$

- *T is continuous.* Let $(x_n) \subset X$ convering to $\overline{x}$ in X. f being continuous on the compact $\mathcal{R}$, is uniformally continuous on $\mathcal{R}$. i.e.;

  $\forall \varepsilon > 0, \exists \delta > 0, \forall (t_1, x_1), (t_2, x_2) \in \mathcal{R} : \|(t_1, x_1) - (t_2, x_2)\| \leq \delta \implies \|f(t_1, x_1) - f(t_2, x_2)\| \leq \varepsilon.$

  In particular, for all $t \in I_\alpha, \forall x_1, x_2 \in \mathbb{R}^k$.

  $$\|x_1 - x_2\| \leq \delta \implies \|f(t, x_1) - f(t, x_2)\| \leq \varepsilon.$$

  Let $\varepsilon > 0, \exists \delta > 0$ such that for all $x, y \in \overline{B}(x_0, b)$, and for all $t \in I_\alpha$. We have,

  $$\|x - y\| \leq \delta \implies \|f(t, x) - f(t, y)\| \leq \frac{\varepsilon}{\alpha}.$$

  since $x_n \xrightarrow{\text{in X}} \overline{x}, \exists n_0 \in \mathbb{N}$ such that,

  $$\|x_n - x_0\|_\infty \leq \delta \qquad \forall n \geq n_0.$$

  Hence,

  $$\begin{aligned} \|Tx_n(t) - T\overline{x}(t)\| &= \| \int_{t_0}^{t} \left(f(s, x_n(s)) - f(s, \overline{x}(s))\right) ds \| \\ &\leq \left| \int_{t_0}^{t} \|f(s, x_n(s)) - f(s, \overline{x}(s))\| ds \right| \\ &\leq |t - t_0| \frac{\varepsilon}{\alpha} \leq \alpha \frac{\varepsilon}{\alpha} = \varepsilon, \\ &\implies \|Tx_n - T\overline{x}\|_1 \leq \varepsilon. \end{aligned}$$

- *$\overline{T(\Omega)}$ is compact in X* . $T(\Omega)$ is uniformaly bounded since for all $x \in \Omega, \|Tx - x_0\|_\infty \leq b$.

  $$\begin{aligned} \|Tx\|_\infty &\leq \|Tx - x_0\|_\infty + \|x_0\|_\infty, \\ &\leq b + \|x_0\|_\infty. \end{aligned}$$

For any $x \in \Omega$, and for any $t_1, t_2 \in I_\alpha$. We have;

$$\begin{aligned} \|Tx(t_1) - Tx(t_2)\| &= \| \int_{t_0}^{t_1} f(s, x(s)) ds - \int_{t_0}^{t_2} f(s, x(s)) ds \| \\ &= \| \int_{t_1}^{t_2} f(s, x(s)) ds \| \\ &\leq \left| \int_{t_1}^{t_2} f(s, x(s)) ds \right| \\ &\leq M \left| t_1, t_2 \right|. \end{aligned}$$

So $T(\Omega)$ is equicontinuous and $\overline{T(\Omega)}$ is compact in X.

⤳ **Conclusion:** By SCHAUDER'S THEOREM T has a fixed point φ which is a solution to (IVP1),

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds.$$

□

1.4 Continuation of solutions (or extension of solutions)

The results we have seen deal local existence. Indeed the solution was defined in a neighborhood of to and in a neighborhood of the inital data. The extension of solution (a continuous of solution) consists in studying criteria. Which allow to extend the interval of definition of the solution. Consider the IVP

$$\begin{cases} x' = f(t, x), \\ x(t_0) = x_0. \end{cases}$$

Where $f : U \subset \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is continious, U is a domain in $\mathbb{R} \times \mathbb{R}^k$ and $(t_0, x_0) \in U$.

Corollary 1.4.1 : *Assume that $\varphi : I = [t_1, t_2] \longrightarrow \mathbb{R}^b$ satisfies $\varphi'(t) = f(t, \varphi(t))$ and $(t, \varphi(t)) \in U$ for all $t \in I$. If $\exists (\tau_n) \subset I$ such that;*

$$\lim_{n\to\infty} (\tau_n, \varphi(\tau_n)) = (t_1, r) \in U,$$

Then,

$$\lim_{z\to t_1} (z, \varphi(z)) = (t_1, r).$$

Lecture 5 `08:03 AM Sun, Oct 26 2025`

Definition 1.4.1 : *Let (φ, I) and $(\widetilde{\varphi}, \widetilde{I})$ be two solutions to the IVP above, we say that $(\widetilde{\varphi}, \widetilde{I})$ is an extension of (φ, I) if $I \subset \widetilde{I}$ and $\widetilde{\varphi}_{|I} = \emptyset$.*

Definition 1.4.2 (Maximal solution) : *A function $\varphi : I \longrightarrow \mathbb{R}^k$ is said to be a maximal solution to IVP (or the ODE $x' = f(t, x)$). If φ has no extension.*

Definition 1.4.3 (Global solution) : *Suppose that $U = I \times \Omega$ where I is an interval and Ω is a domain in $\mathbb{R}^k$. Any solution $\varphi : I \longrightarrow \mathbb{R}^k$ such that $\varphi'(t) = f(t, \varphi(t))$ and $(t, \varphi(t)) \in I \times \mathbb{R}$ is said to be a global solution to the ODE in IVP.*

Corollary 1.4.2 : *Suppose that $\varphi : (t_1, t_2) \longrightarrow \mathbb{R}^k$ satisfies $\varphi'(t) = f(t, \varphi(t))$ and $(t, \varphi(t)) \in U$ for all $t \in (t_1, t_2)$. If $\lim_{n\to\infty}(z_n, \varphi(z_n)) = (t_1, \tau) \in U$ (resp. $\lim_{n\to\infty}(z_n, \varphi(z_n)) = (t_2, \tau) \in U$), then $\lim_{z\to t_1}(z, \varphi(z)) = (t_1, \tau)$. (resp. $\lim_{z\to t_2}(z, \varphi(z)) = (t_2, \tau)$. $(z_n)_{n\in\mathbb{N}} \subset (t_1, t_2)$.*

Proof. Let W be a neighoborhood of (t_1, t_2) such that $W \subset \overline{W} \subset U$. then $(t, \varphi(t)) \in W$ for $t \in (t_1, t_2) \subset (t_1, t_2)$. Indeed, let $M = \sup_{(t,x)\in W} \|f(t, x)\|$, for every $j \in \mathbb{N}$ and any $\varepsilon > 0$ small consider the

$$\mathcal{R}(\varepsilon) = \{(t, x) : |t - t_j| \leq \varepsilon \text{ and } \|x - \varphi(t_j)\| \leq M\varepsilon\}.$$

Then there exists $\varepsilon > 0$ and $j \in \mathbb{N}$ such that

$$(t_1, \tau) \in \mathcal{R}_j(\varepsilon) \subset W.$$

From above proposition applied to φ the solution of

$$\begin{cases} x' = f(t, x), \\ x(\tau_j) = \varphi(\tau_j). \end{cases}$$

We obtain that $(\tau, \varphi(\tau)) \subset \mathcal{R}_j(\varepsilon) \subset W$ for $\tau \in (t_1, \tau_j)$, $\square$

Theorem 1.4.3 : *Let $\varphi : I = (a, b) \longrightarrow \mathbb{R}^k$ satisfying $\varphi'(t) = f(t, \varphi(t))$ and $(t, \varphi(t)) \in U$ for all $t \in I$, If the following conditions are satisfying.*

① *φ can not be extended to the left of a.*

② $\lim_{j\to\infty}(z_j, \varphi(z_j)) = (a, \tau) \qquad ((z_j)_{j\in\mathbb{N}} \subset I)$.

Then $(a, \tau) \in \partial U$.

Proof. If $(a, \tau) \in U$, then by above corollary we can extend φ at left of a. Contradiction! $\square$

Corollary 1.4.4 : *Assume that* $f : (a, b) \times \Omega \longrightarrow \mathbb{R}^k$ *is continuous, where* Ω *is a domain in* $\mathbb{R}^k$ *and there exists* $\varphi : (a, b) \longrightarrow \mathbb{R}^k$ *such that*

① φ *and* φ' *are continuous in a subinterval* $I \subset (a, b)$

② $\varphi'(t) = f(t, \varphi(t)) \quad \forall t \in I.$

Then, either

① φ *can be extended to all the interval* (a, b), *or*

② $\lim_{t\to t_0} \|\varphi(t)\| = \infty$ *for some* $t_0 \in (a, b)$.

Example: *Consider, the ODE:* $x' = -2tx^2$, *the solutions are:*

$$-\frac{x'}{x^2} = 2t \implies \frac{1}{x(t)} = t^2 - c$$

Thus

$$x(t) = \frac{1}{t^2 - c} \qquad c \in \mathbb{R}.$$

Define $f(t, x) = -2tx^2, \quad D_f = \mathbb{R} \times \mathbb{R}$. *If* $c < 0$, *in this case* $x(t) = \frac{1}{t^2-c}$ *is a global solution. If* $c \geq 0$ *then*

$$\begin{cases} D_f &= (-\infty, -\sqrt{c}) \\ D_f &= (-\sqrt{c}, \sqrt{c}) \\ D_f &= (-\sqrt{c}, +\infty) \end{cases}$$

On all the above intervals $x(t)$ *is a maximal solution.*

Definition 1.4.4 (Locally lipschitz) : $f : U \longrightarrow \mathbb{R}^k$ *is said to be locally lipschitz if its lipscitz on any compact of* U.

✎Remark:

If f has continuous partial derivatives with respect to x_i where $i = 1, \ldots, j$, then f is locally lipschitz. In particular function that are $\mathcal{C}^1$ are locally lipschitz.

Theorem 1.4.5 : *Suppose that $\varphi_1, \varphi_2 :\longrightarrow \mathbb{R}^k$ are two solutions of $x' = f(t,x)$, where f is locally lipschitz on U (with respect to variable x). If $\varphi_1(t_0) = \varphi_2(t_0)$ for some $t_0 \in I$ then $\varphi_1(t) = \varphi_2(t)$ for all $t \in I$.*

Proof. By contradiction, suppose that there is $t_1 \in I$ such that $\varphi_1(t_1) \neq \varphi_2(t_2)$. and without loss of generality suppose suppose that $t_1 > t_0$. By uniqueness, there exists $\beta > 0$ such that

$$\varphi_1(t) = \varphi_2(t) \qquad \forall t \in (t_0 - \beta, t_0 + \beta).$$

Let

$$E = \{t \in [t_0, t_1] : \quad \varphi_1(t) \neq \varphi_2(t)\}$$

we have $E \neq \emptyset$ since $t_1 \in E$. Let $\alpha = \inf E \in (t_0, t_1)$ and for all $t \in [t_0, \alpha)$ $\varphi_1(t) = \varphi_2(t)$. By continuity, $\varphi_1(\alpha) = \varphi_2(\alpha)$ similarly, there eixsts a neighborhood W of α such that

$$\varphi_1(t) = \varphi_2(t) \quad \forall t \in W$$

This contradicts the definition of α. □

Corollary 1.4.6 (Global Uniqueness) : *suppose that f is locally lipschitz with respect to the variable x on U. then by any initial data $(t_0, x_0) \in U$, then passes a unique maximal solution. If there is a global solution there its unique.*

Example:

$$u'(t) + a(t)u(t) = f(t) \quad a, f \in \mathcal{C}([a,b])$$

$a, b \in \mathcal{C}(I)$ interval in $\mathbb{R}$, $f(t,u) = a(t)u$ for any $(t_0, u_0) \in I \times \mathbb{R}$ there exists $[a,b] \times [c,d] \subset I \times \mathbb{R}$ such that $(t_0, x_0) \in [a,b] \times [c,d]$. we have

$$|f(t,u) - f(t,v)| \leq \sup_{t \in [a,b]} |a(t)| \, |u - v|$$

since the solution has the form

$$u(t) = e^{-A(t)} \int e^{A(t)} b(t) dt (A' = a)$$

are defined on I, by any $(t_0, u_0) \in I \times \mathbb{R}$ passes a unique global solution.

1.5 Continuous Dependence on data and parameters

Lecture 6 08:06 AM Sun, Nov 02 2025

Theorem 1.5.1 : *Let $f : \mathbb{R} \longrightarrow \mathbb{R}^k$ be a continuous and $L-$ Lipschitz function in x, where $\mathcal{R} = \{(t, x) : |t - t_0| < x$ and $\|x - x_0\| \leq b\}$, Let φ and Ψ be respectively unique solutions to*

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0, \end{cases} \quad \textit{and} \quad \begin{cases} x' = f(t,x) \\ x(t_1) = x_1, \end{cases}$$

where $(t_1, x_1) \in \mathcal{R}$. Suppose that φ and Ψ are well defined on an interval $I \subset [t_0 - a, t_0 + a]$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$|t_1 - t_0| < \delta, \|x_1 - x_0\| \leq \delta \implies \|\varphi(t) - \Psi(t)\| \leq \varepsilon$$

Proof. For all t in I, we have

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds \text{ and } \Psi(t) = x_1 + \int_{t_1}^{t} f(s, \Psi(s))ds.$$

Hence, we have for all $t \in I$,

$$\varphi(t) - \Psi(t) = (x_0 - x_1) + \int_{t_1}^{t} f(s, \Psi(s))ds + \int_{t_0}^{t} f(s, \varphi(s))ds,$$

then,

$$\|\varphi(t) - \Psi(t)\| \leq \|x_0 - x_1\| + M|t_1 - t_0| + L\left|\int_{t_1}^{t} \|\varphi(s) - \Psi(s)\| ds\right|,$$

By Gronwall's inequality, we get

$$\|\varphi(t) - \Psi(t)\| \leq (\|x_0 - x_1\| + M|t_1 - t_0|) \exp(L|t - t_1|).$$

This leads to the claim. □

Theorem 1.5.2 : *Let $f, g : \mathcal{R} \longrightarrow \mathbb{R}^k$ (where $\mathcal{R}$ is the rectangle from the theorem above), be two continuous and k-Lipschitzian in x, let φ and Ψ be respectively the unique solutions to*

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0, \end{cases} \quad \text{and} \quad \begin{cases} x' = g(t,x) \\ x(t_0) = x_1. \end{cases}$$

where $(t_0, x_1) \in \mathcal{R}$, defined on $I = (\alpha, \beta)$. Suppose that $\|f(t,x) - g(t,x)\| \leq \varepsilon$ for all $(t,x) \in \mathcal{R}$. Then,

$$\|\varphi(t) - \Psi(t)\| \leq \|x_1 - x_0\| e^{K|t-t_0|} + \varepsilon\,(\beta - \alpha)\, e^{K|t-t_0|} \quad \forall t \in I$$

Proof. We have,

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s)) ds \text{ and } \Psi(t) = x_1 + \int_{t_0}^{t} g(s, \Psi(s)) ds.$$

Thus,

$$\begin{aligned} \varphi(t) - \Psi(t) &= x_0 - x_1 + \int_{t_0}^{t} \left[f(s, \varphi(s)) - g(s, \Psi(s)) \right] ds. \\ &= (x_0 - x_1) + \int_{t_0}^{t} f(s, \varphi(s)) - f(s, \Psi(s)) ds + \int_{t_0}^{t} f(s, \Psi(s)) - g(s, \Psi(s)) ds. \end{aligned}$$

By using norms,

$$\|\varphi(t) - \Psi(t)\| \leq \|x_0 - x_1\| + K \left| \int_{t_0}^{t} \|\varphi(s) - \Psi(s)\| ds \right| + \varepsilon \left| t - t_0 \right|.$$

or,

$$\|\varphi(t) - \Psi(t)\| \leq \|x_0 - x_1\| + K \left| \int_{t_0}^{t} \|\varphi(s) - \Psi(s)\| \right| ds + \varepsilon(\beta - \alpha)$$

By Grownwall's inequality, we obtain,

$$\|\varphi(t) - \Psi(t)\| \leq \left(\|x_0 - x_1\| + \varepsilon(\beta - \alpha) \right) e^{K|t-t_0|}$$

□

Theorem 1.5.3 : *Let Ω be a domain in $\mathbb{R} \times \mathbb{R}^k$, and $B_\mu = B(\mu_0, c) = \{\mu \in \mathbb{R}^m : \|\mu - \mu_0\| \leq c\}$, and $\Omega_\mu = \{(t, x, \mu) : (t, x) \in \Omega$ and $\mu \in B(\mu_0, c)\}$. Let $f : \Omega_\mu \longrightarrow \mathbb{R}^k$ be continuous bounded by a constant M. Suppose that for $\mu = \mu_0$, the IVP*

$$\begin{cases} x' = f(t, x, \mu) \\ x(t_0) = x_0 \end{cases} \qquad \text{(IVP *)}$$

has a unique solution defined on $[a, b]$. Then, $\exists \delta > 0$ such that for any μ with $|\mu - \mu_0| < \varepsilon$, every solution φ_μ to the IVP (IVP) is defined on $[a, b]$ and $\varphi_\mu \to \varphi_{\mu_0}$ uniformally on $[a, b]$ as $\mu \to \mu_0$.*

Proof. Let $\alpha > 0$ be small so that $\mathcal{R} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^k : |t - t_0| \leq \alpha$ and $\|x - x_0\| \leq \mu\} \subset \Omega$. Then all solutions of the IVP. (IVP*) are defined on $[t_0 - \alpha, t_0 + \alpha]$. This is due to *Proposition 1.4*. Let φ_μ denote a solution. We claim that the set $\{\varphi_\mu : \mu \in \mathcal{B}_\mu\}$ is uniformally bounded and equicontinuous on the interval $[t_0 - \alpha, t_0 + \alpha]$. (Use the integral form of the solution). By the contrary, suppose that for some $\bar{t} \in [t_0 - \alpha, t_0 + \alpha]$, $\varphi_\mu(\bar{t}) \not\to \varphi_{\mu_0}(\bar{t})$. Then there exists a sequence $\{\mu_k\}$ such that $\mu_k \to \mu_0$ and $\varphi_{\mu_k} \to \Psi$ uniformally on $[t_0 - \alpha, t_0 + \alpha]$. (Ascoli Arzela). Then Ψ is a solution to

$$\begin{cases} x' = f(t, x, \mu_0) \\ x(t_0) = x_0. \end{cases}$$

on the interval $[t_0 - \alpha, t_0 + \alpha]$. In one hand, we have $\Psi(\bar{t}) \neq \varphi_{\mu_0}(\bar{t})$. In the other, by uniqueness we have $\varphi_{\mu_0} = \Psi$ in $[t_0 - \alpha, t_0 + \alpha]$. Now, we need to prove that the uniform convergence occurs in $[a, b]$. Let $\tau \in [t_0, b)$, and suppose that the uniform convergence holds on $[t_0, \tau - h]$; but not on $[t_0, \tau + h]$ for $h > 0$ small, its clear that $\tau \geq t_0 + \alpha$. For all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\|\varphi_\mu(\tau - \varepsilon) - \varphi_0(\tau - \varepsilon)\| \leq \varepsilon \quad \forall \mu, |\mu - \mu| \leq \delta_\varepsilon \qquad (**)$$

Let $H \subset \Omega$ the region given by

$$H = \{(t, x) : |t - \tau| \leq \gamma \quad \|x - \varphi_0(\tau - \gamma)\| \leq \gamma + M\,|t_0 - \tau + \gamma|\}$$

where γ is small. Any solution Ψ satisfying by $\varphi(\tau - \gamma) = \zeta_0$ with,

$$|\zeta_0 - \varphi_0(\tau - \gamma)| \leq \gamma$$

will remain in H, as t increases. Thus all solutions can be seen extended to $\tau + \gamma_0$. By choosing $\varepsilon = \gamma$ in $(**)$, it follows that for $|\mu - \mu_0| \leq \delta_\varepsilon$ the solution φ_μ can be extended to $\tau + \varepsilon$. Thus over

$[\tau_0, \tau_0 + \varepsilon]$ these solutions are in Ω so the argument $\varphi_k \to \varphi_0$ applies to $[\tau_0, \tau + \varepsilon]$, this contradicts the definition of τ. the case $\tau = b$ is treated similarly. $\square$