Geometric intuition

Integrating a real-valued function over some line/curve/surface/volume/region can be thought of as integrating (*summing*) a density function to obtain a total mass.

Vector fields are trickier.

For most problems we might care about, surfaces are either spheres, cylinders, or general graphs.

Area elements

Over a cylindrical surface (with a cylindrical-coordinate parametrization), the area element is

$$\mathrm{d}S = r\,\mathrm{d}\theta\,\mathrm{d}z,$$

(Notice here that $r = |\boldsymbol{r}|$ is fixed across the cylinder's surface!)

Fluid flow and flux

We can imagine surface integrals of vector fields as an integral over *flux density*, or "flow" density, to get the total *flux*.

Then the integrand of the surface integral

$$\boldsymbol{F} \cdot (\hat{\boldsymbol{n}}\,\mathrm{d}S)$$

is seen as the "volume flowing out of the area element $\mathrm{d}S$ per unit time." We rewrite the area differential vector (with magnitude equal to the area, pointing in the outward normal direction) as

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

How to compute surface integrals?

1. Parametrize the surface in terms of parameters s, t.
2. Find the area element $\mathrm{d}\boldsymbol{S}$ in terms of $\mathrm{d}s, \mathrm{d}t$.
3. Integrate.

Example Take the surface parametrized by

$$(x, y, z) = (2\cos\theta, 2\sin\theta, z) = \boldsymbol{X}(\theta, z).$$

Then we find the normal area element

$$\mathrm{d}\boldsymbol{S} = (\boldsymbol{X}_\theta\,\mathrm{d}\theta) \times (\boldsymbol{X}_z\,\mathrm{d}z) = (\boldsymbol{X}_\theta \times \boldsymbol{X}_z)\,\mathrm{d}\theta\,\mathrm{d}z.$$

Take the partial derivatives

$$\begin{aligned}\boldsymbol{X}_\theta &= (-2\sin\theta, 2\cos\theta, 0),\\ \boldsymbol{X}_z &= (0, 0, 1).\end{aligned}$$

Then the cross product is

$$\boldsymbol{X}_\theta \times \boldsymbol{X}_z = (2\cos\theta, 2\sin\theta, 0),$$

and the area element is then

$$\mathrm{d}\boldsymbol{S} = (\boldsymbol{X}_\theta \times \boldsymbol{X}_z) = (2\cos\theta, 2\sin\theta, 0)\,\mathrm{d}\theta\,\mathrm{d}z.$$

Notice that this can be written as

$$\mathrm{d}S = r\,\mathrm{d}\theta\,\mathrm{d}z.$$

If, in general, we get some surface defined by

$$\boldsymbol{X}(x, y, z) = (x, y, f(x, y)),$$

we can do some vector stuff.

$$\mathrm{d}\boldsymbol{S}$$