Stokes' Theorem

Recall Green's theorem:

$$\oint_{C=\partial D} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_D \underbrace{(\nabla \times \boldsymbol{F})}_{\text{means taking some sort of derivative}} \cdot \hat{\boldsymbol{z}}\, \mathrm{d}A = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}A,$$

where $\boldsymbol{F} = (M, N)$.

Unification

Take some integration of a function ϕ:

$$\int \phi$$

If that manifold is *bounded* by some lower-dimensional manifold, then we make find some relationship between the integral over the boundary ∂M and the integral over the manifold M:

$$\int_{\partial M} \phi = \int_M \underbrace{\partial \phi}_{\text{some kind of derivative}}.$$

In essence, these relationships for various multi-dimensional manifolds form the "fundamental theorems" of calculus.

In fact, consider the one-variable fundamental theorem:

$$F(b) - F(a) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F \,\mathrm{d}t.$$

We can see this integral as an integration over the segment of the one-dimensional t curve, bounded by the endpoints a and b.

Stokes' Theorem

Consider some vector field

$$\boldsymbol{F} = (P, Q, R).$$

We want to take this integral over a curve in 3-space C. We begin by considering C as the closed boundary of some surface S. Then we relate the integrals over the boundary and over the surface:

$$\int_{C=\partial S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S},$$

where $\mathrm{d}\boldsymbol{s}$ is the curve segment, and $\mathrm{d}\boldsymbol{S}$ is the surface normal element.

Geometric interpretations

The "circulation" of $\boldsymbol{F}$ along ∂S is the "sum" of how much field $\boldsymbol{F}$ "curls" in S.

Example Take the field $\boldsymbol{F} = (-y, 2x, z)$. We find the integral over the curve

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s},$$

where C is the outwardly-oriented (counter-clockwise, looking from outside the sphere) union of three arcs along the unit sphere (at intersection with the planes):

Then the integral is

$$\oint_{C=C_1\cup C_2\cup C_3} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s},$$

We notice that C forms a closed curve; then, we can pick some convenient spherical surface S such that C is the boundary of S.

We calculate the curl of $\boldsymbol{F}$:

$$\operatorname{curl}\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ -y & 2x & z \end{vmatrix} = 0\hat{\boldsymbol{x}} + 0\hat{\boldsymbol{y}} + (2-(-1))\hat{\boldsymbol{z}} = 3\hat{\boldsymbol{z}} = (0,0,3).$$

Then by Stokes' theorem the integral equals

$$\iint_S \underbrace{\operatorname{curl}\boldsymbol{F}}_{3\hat{\boldsymbol{z}}}\cdot \underbrace{\mathrm{d}\boldsymbol{S}}_{\hat{\boldsymbol{n}}\,\mathrm{d}S}.$$

Recall the spherical surface area element

$$\mathrm{d}S = r^2 \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta,$$

(note r is fixed here), so that the normal element is

$$\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\,\mathrm{d}S = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$