Review of one-variable Taylor stuffs

Recall, for a one-variable function f, a Taylor polynomial about a point approximates the function:

$$f(x) \sim P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots + \underbrace{\frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n}_{\text{(remainder)}}.$$

We want to extend this idea to multi-variable real-valued functions (recall that vector-valued functions can be considered component-wise as real-valued functions):

$$f(\boldsymbol{x}) = \underbrace{f(\boldsymbol{x_0}) + \nabla f(\boldsymbol{x_0})(\boldsymbol{x} - \boldsymbol{x_0}) + \frac{1}{2!}(\boldsymbol{x} - \boldsymbol{x_0})^T H(\boldsymbol{x_0})(\boldsymbol{x} - \boldsymbol{x_0})}_{P_2(\boldsymbol{x})} + \underbrace{\cdots}_{\text{remainder}}.$$

Example Consider the function

$$f(x, y) = \cos x \cos y.$$

Find, for example, $P_2(x, y)$ (about $(0, 0)$).

Solution (method 1) The polynomial is

$$P_2(x, y) = f(0, 0) + \nabla f(0, 0)(x, y) + \frac{1}{2!}(x, y)^T \mathrm{H}f(0, 0)(x, y).$$

Computing the various derivatives and values,

$$\begin{aligned} f(0, 0) &= \cos 0 \cos 0 = 1. \\ \nabla f &= (\partial_x f, \partial_y f) = (-\sin x \cos y, -\cos x \sin y) = (0, 0). \\ \mathrm{H}f &= \begin{pmatrix} \partial_x\partial_x f & \partial_y\partial_x f \\ \partial_x\partial_y f & \partial_y\partial_y f \end{pmatrix} \end{aligned}$$

Solution (method 2) We can alternatively take the Taylor polynomials of the two sub-function $(\cos x, \cos y)$ and expand. It's an alright way to check our work.

Critical points

Recall, in one-variable calculus, a critical point of a function $f(x)$ is found by looking for c such that

$$f'(c) = 0$$

(or undefined), and solving for various such possible c.

In multi-variable, the generalization is fairly straightforward. Consider the multi-variable, real-valued function $g(\boldsymbol{x})$. The critical points $\boldsymbol{c}$ are found such that

$$\nabla g(\boldsymbol{c}) = \boldsymbol{0}.$$

Example Consider

$$f(x, y) = x^2 + 6y^2.$$

Then the gradient

$$\nabla f(x, y) = (2x, 12y).$$

We set this equal to zero and thus get

$$\nabla f(x, y) = (2x, 12y) = (0, 0),$$

so that

$$x, y = 0$$

necessarily.