Second-order partial derivatives

Example Take $f(x, y, z) = x^2y + y^2z$.

Then the first-order partial derivatives are

$$\begin{aligned}\partial_x f &= 2xy, \\ \partial_y f &= x^2 + 2yz, \\ \partial_z f &= y^2.\end{aligned}$$

We can take derivatives again to get *second-order* (and higher order, and so on) partial derivatives:

$$\begin{aligned}\partial_x^2 f = \frac{\partial^2 f}{\partial x^2} &= 2y, \\ \partial_y^2 f &= 2z, \\ \partial_z^2 f &= 0.\end{aligned}$$

Mixed partial derivatives

If some multivariable function $f \colon \mathbb{R}^n \to \mathbb{R}$ has continuous first- and second-order partial derivatives, then its mixed derivatives are the same (regardless of the order in which the partial derivatives are taken):

$$\frac{\partial^2 f}{\partial x_{i_1}\,\partial x_{i_2}} = \frac{\partial^2 f}{\partial x_{i_2}\,\partial x_{i_1}}$$

Thus the Hessian matrix, or the derivative matrix of the gradient vector field, is diagonally symmetric.

Rules of Derivatives

The rules of derivatives remain the same as in one-variable calculus.

Product rule For functions $f \colon \mathbb{R}^n \to \mathbb{R}$ and $g \colon \mathbb{R}^n \to \mathbb{R}$, the product $f(\mathbf{x})g(\mathbf{x})$ has derivative matrix

$$Df(\mathbf{x})\, g(\mathbf{x}) + g(\mathbf{x})\, Df(\mathbf{x}).$$

Potential Functions

Consider some "potential function," for example, $f(x, y, z) = xy \sin z$. It has gradient

$$\nabla f = (f_x, f_y, f_z) = (y \sin z, x \sin z, xy \cos z).$$

The potential function $f \colon \mathbb{R}^3 \to \mathbb{R}$ is a *real-valued* function (or *many-to-one*). The gradient $\nabla f \colon \mathbb{R}^3 \to \mathbb{R}^3$, however, is a vector field (a *many-to-many* function). *Whooo, that is so cool.*

Question Given a vector field $\mathbf{F}$ (for example, $\mathbb{R}^3 \to \mathbb{R}^3$). Can this function $\mathbf{F}$ come from some real-valued function f by taking the gradient ∇f? That is, is there some function $f \colon \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = \mathbf{F}$?

Good stuff. This shows up in physics.

Example Take some vector field $\mathbf{F}(x, y, z) \colon \mathbb{R}^3 \to \mathbb{R}^3 = (2x, 2y, 2z)$. We happen to notice that this function can be the gradient of

$$f \colon \mathbb{R}^3 \to \mathbb{R} = x^2 + y^2 + z^2 \quad (+C),$$

since

$$\nabla f = (2x, 2y, 2z) = \mathbf{F}.$$

Then we can call $\mathbf{F}$ a *gradient vector field*, and f is a potential function of the vector field $\mathbf{F}$.

Equipotential surfaces An *equipotential surface* of a vector field $\mathbf{F}$ is just a *level surface* of f, the (real-valued) potential function of the vector field $\mathbf{F}$, where $\nabla f = \mathbf{F}$.

How do we find the potential function of a vector field? Consider again the example $\mathbf{F} = (2x, 2y, 2z)$. Suppose there exists some function $f \colon \mathbb{R}^3 \to \mathbb{R}$, such that

$$\nabla f = \mathbf{F}.$$

Then

$$(f_x, f_y, f_z) = (2x, 2y, 2z).$$

Then

$$\begin{aligned} f_x &= 2x, \\ f_y &= 2y, \\ f_z &= 2z. \end{aligned}$$

Then we integrate with respect to x,

$$f(x, y, z) = x^2 + h_1(y, z),$$

where h_1 may be a function of y and z since, when taking partial derivatives, other variables are held constant. Likewise,

$$\begin{aligned} f(x, y, z) &= y^2 + h_2(x, z), \\ f(x, y, z) &= z^2 + h_3(x, y). \end{aligned}$$

Now, doing our best to reconcile the three results, we guess a little, piece things together, and conclude something along the lines of

$$f = x^2 + y^2 + z^2 \quad (+C).$$

Eventually, we'll do something that might be called *path integrals*, but we shouldn't spoil any secrets right now.