

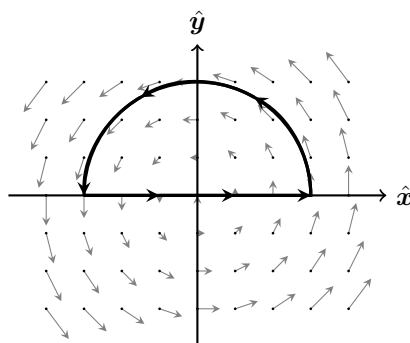
Conservative vector fields are just *gradient* vector fields. To say that some function \mathbf{F} is a gradient field is to say that there is some f such that $\mathbf{F} = \nabla f$.

Notice that the integral of a gradient vector field along a curve C is thus the *change* in the potential function across the endpoints of the curve:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C (\nabla f) \cdot d\mathbf{s} = \int_C \left(\frac{\partial f}{\partial x_i} dx_i + \dots \right) = \int_C df = f(\mathbf{b}) - f(\mathbf{a}),$$

where \mathbf{b}, \mathbf{a} are the endpoints of the curve C .

Example Not all vector fields are gradient vector fields! Consider the field $\mathbf{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 = (-y, x)$, integrated along the semicircular curve $C = C_1 \cup C_2$, where C_1 is the diameter along the x -axis and C_2 is the semicircular portion above the x -axis:



Then consider the curve $C = C_1 \cup (-C_2)$, so that C is the boundary of the semicircular region D . Then by Green's theorem

$$\oint_{\partial D=C} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA,$$

where $\mathbf{F}(x, y) = (M(x, y), N(x, y))$. Specifically

In general, these two different *path integrals* are not equal. Vector fields are typically not conservative.

But when are they equal? Notice by Green's theorem that a closed-curve path integral in a curve like this is zero iff

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Geometrically, notice then that this condition means that the vector field has no "rotation" in the plane.

But also notice that, the reverse implication

$$\mathbf{F} \text{ is conservative} \iff \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

only holds when D is "simply-connected" (there are no "holes" in D).

Fundamental Theorem of Line Integrals

If path C starts at point A , ends at point B , and $\mathbf{F} = \nabla f$, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A).$$

Notice that if F is conservative, the integral over the curve depends *only* on the endpoints, not on the path taken!

Example Consider the vector field $\mathbf{F} = (2x, 2y)$. Calculate the path integral over C , a straight line from $(1, 1)$ to $(4, 3)$.

We can compute the integral by parametrizing the curve and integrating over the parametrized curve.

We can also compute the integral by evaluating the potential function $f = x^2 + y^2$.

Flux

The flux is a double integral of a vector field over a surface:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $d\mathbf{S}$ is some sort of a differential *area* vector (a vector normal to the surface with differential area $|d\mathbf{S}|$).