Exam tidbits

1. The final exam is *not* cumulative! Instead, it will focus on integration (multiple integrals, ..., Stokes', Green's, Gauss's).
2. Relevant homework assignments are HW7–HW12.
3. You are allowed a hand-written, double-sided cheat sheet.

Miscellaneous tidbits

Operators and kinds of functions

Recall the way different "derivative-like" operators (curl, divergence, gradient) convert between real-valued functions ($\mathbb{R}^3 \to \mathbb{R}$) and vector fields ($\mathbb{R}^3 \to \mathbb{R}^3$):

- The *gradient* converts a real-valued function into a vector field:

$$\nabla f(x, y, z)\colon \mathbb{R}^3 \to \mathbb{R} = \boldsymbol{F}(x, y, z)\colon \mathbb{R}^3 \to \mathbb{R}^3.$$

- The *divergence* converts a vector field into a real-valued function:

$$\nabla \cdot \boldsymbol{F}(x, y, z)\colon \mathbb{R}^3 \to \mathbb{R}^3 = f(x, y, z)\colon \mathbb{R}^3 \to \mathbb{R}.$$

- The *curl* converts a vector field to a vector fieled:

$$\nabla \times \boldsymbol{F}(x, y, z)\colon \mathbb{R}^3 \to \mathbb{R}^3 = \boldsymbol{G}(x, y, z)\colon \mathbb{R}^3 \to \mathbb{R}^3.$$

Conservative curls?

Is the curl of a vector field always conservative? No. Here's a counter-example:

$$\boldsymbol{F} = (0, 0, x^2 + y^2).$$

Curl and conservatism

Recall as a corollary of Green's/Stokes's theorem that a vector field $\boldsymbol{F}$ is conservative iff $\operatorname{curl} F = \nabla \times \boldsymbol{F} = \boldsymbol{0}$ on a simply-connected regions. What's a simply-connected region, you ask? Loosely, a simply-region is a region with no hole. As far as this course goes, the only simply-connected region we really care about is the real plane $\mathbb{R}^2$.

What's a boundary?

Consider a sheet of paper. Think of the paper itself as a surface; then, the boundary of the surface corresponds to the *edge* of a paper.

Consider a bowl. The "surface" of the bowl has the boundary corresponding to the *rim* of the bowl.

If a surface is closed (think of the "skin" of a basketball), it has no boundary, because the surface has no *edge*. If we poke a hole in the ball with a needle, the surface now has a boundary along the perimeter of the hole.

Homework tips

Stokes's theorem

We have to check that Stokes's theorem is true. This is a very enlightening homework problem, because mathematics is all about plugging things in and checking specific examples and definitely *not* about generality and enlightening proofs. Who needs mathematical beauty when you can do routine computations?

Say, for example, we have a surface defined by $x^2 + y^2 + z^2 = 2^2$ for $z \leq 0$, a hemisphere below the xy plane. Let the normals be oriented outward, so that the boundary of the hemisphere should be oriented "downward"—a *clockwise* semicircle in the xy plane.

We have a vector field $\boldsymbol{F} = (2y - z, x + y^2 - z, 4y - 3x)$. We might like to take the curl:

$$\operatorname{curl} F = ((4) - (-1), (-1) - (-3), (1) - (2)) = (5, 2, -1).$$

Applying theorems

Suppose we have some integral over some *horrible* surface S:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$