Spherical coordinates

In a spherical coordinate system (ρ, θ, ϕ), the spherical volume element is given by

$$\mathrm{d}V = \rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

Example

Find the mass of a cone with density at position (x, y, z) given by

$$f(x,y,z) = \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}.$$

The mass can be found by integrating the density over the volume:

$$m = \iiint_W f(x,y,z)\,\mathrm{d}V.$$

We write f in spherical coordinates:

$$f(x,y,z) = \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}} = \frac{e^{\rho^2}}{\rho}.$$

Then the volume element is

$$\mathrm{d}V = \rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

Then integrate:

$$\begin{aligned} m &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} \int_{\rho=0}^{2} \frac{e^{\rho^2}}{\rho}\,\rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \iiint \rho e^{\rho^2} \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \end{aligned}$$

In general, it is a good idea to consider using spherical coordinates rather than cartesian coordinates if the function seems to depend on the spherical radius $\rho^2 = x^2 + y^2 + z^2$, or if the shape of the region W lends itself to spherical coordinates.

Likewise, if the integrand function depends on the plane ("cylindrical") radius $r^2 = x^2 + y^2$, or if the region has some cylindrical symmetry, it may be wise to consider using cylindrical geometry.

Line integrals

There are two kinds of line integrals.

- We may integrate a *real-valued* function $f\colon \mathbb{R}^n \to \mathbb{R}$ over a curve. We can interpret this integral as the *total mass* of the curve given the *linear density function* f.
- We may integrate a *vector-valued* function $\boldsymbol{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ over a curve. This integral shows up a little in physics, where the integral $\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$ is called the *work* done by the force $\boldsymbol{F}$.

Example

Find the mass of the wire C which is the section of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$, given the linear mass density of the wire $\rho(x, y, z) = 2x$.

The total mass m is found by the integral of the density over the curve:

$$m = \int_C \rho\,\mathrm{d}s = \int_C 2x\,\mathrm{d}s,$$

where dS is the "differential" of the arc length, or, intuitively, an infinitesimally small section of the arc length. That is,

$$\mathrm{d}s = |\mathrm{d}\boldsymbol{x}| = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = |\dot{\boldsymbol{x}}(t)|\,\mathrm{d}t,$$

for some *parametrization* of the curve $\boldsymbol{x}(t)$ (so that the velocity is $\dot{\boldsymbol{x}}(t)$, and the speed $|\dot{\boldsymbol{x}}(t)|$. Then, to parametrize the curve, let

$$\begin{aligned} x &= t, \\ \implies y &= t^2, \end{aligned}$$

so that the curve C from $(1,1)$ to $(2,4)$ under this parametrization is defined over $t \in [1,2]$, and the speed is

$$|(\dot{x}(t), \dot{y}(t))| = |(1, 2t)| = \sqrt{1+4t^2},$$

and the arc-length-differential is then

$$\mathrm{d}s = |(\dot{x}(t), \dot{y}(t))|\,\mathrm{d}t = \sqrt{1+4t^2}\,\mathrm{d}t.$$

Then we can integrate for the mass:

$$\begin{aligned} m &= \int_C 2x\,\mathrm{d}s \\ &= \int_C 2x\sqrt{1+4t^2}\,\mathrm{d}t \\ &= \int_C 2t\sqrt{1+4t^2}\,\mathrm{d}t \\ &= \frac{1}{4}\frac{2}{3}\left(1+4t^2\right)^{\frac{3}{2}}\Big|_C \\ &= \frac{1}{6}\left(1+4t^2\right)^{\frac{3}{2}}\Big|_1^2 \\ &= \frac{1}{6}\left(1+4t^2\right) \end{aligned}$$