

What is a vector field?

A vector field is a “many-to-many” function (that is, $\mathbb{R}^n \rightarrow \mathbb{R}^m$, where $n, m > 1$).

Example Consider, for example, some

$$\mathbf{F}(x, y) = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} = (y, -x).$$

This function maps a plane to a plane ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$), *eating* two values and *spitting out* two values.

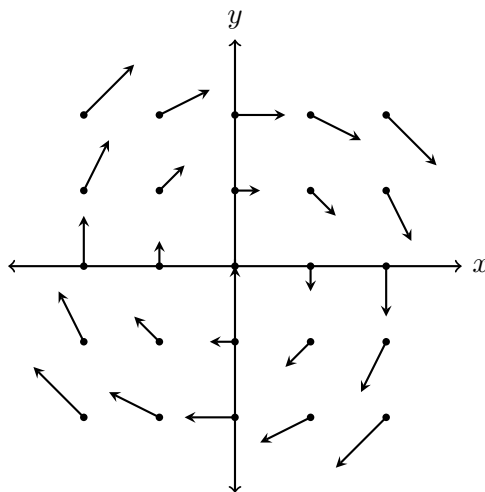
To draw such a function, we might make a table:

(x, y)	$\mathbf{F}(x, y)$
$(0, 0)$	$(0, 0)$
$(1, 0)$	$(0, -1)$
$(0, 1)$	$(1, 0)$
$(-1, 0)$	$(0, 1)$
$(0, -1)$	$(-1, 0)$

A couple observations:

- The output $\mathbf{F}(x, y)$ is always perpendicular to the input (x, y) , since $(x, y) \perp (y, -x) = \mathbf{F}(x, y)$.

Then, the “graph” of the vector field can be drawn as



Question (a very important one, most certain to appear on your test or quiz or whatever)
What are the flow lines of a given vector field \mathbf{F} ?

To visualize this concept, imagine “following” the vector field to find a curve that traces out a “path” along the vector field \mathbf{F} . Picture a curve *tangent* to the vector field \mathbf{F} everywhere along the curve.

Then the parametrized curve $\mathbf{C}(t) = (x(t), y(t))$ would have derivative equal to the vector field:

$$d_t \mathbf{C}(t) = \mathbf{F}(\mathbf{C}(t)).$$

Consider the above example $\mathbf{F}(x, y) = (y, -x)$. Then the derivative of the flow curve

$$\mathbf{C}'(t) = (x'(t), y'(t)) = \mathbf{F}(\mathbf{C}(t)) = (y(t), -x(t)),$$

so that

$$\begin{aligned}x'(t) &= y(t) \\ y'(t) &= -x(t),\end{aligned}$$

and thus the problem is reduced to that of solving an ordinary differential equation, perhaps with some initial condition (defined, for example, by some given point through which the curve passes).

How would we solve such a differential equation?

Then, say, we might try

$$\begin{aligned}x(t) &= \sin t, \\ y(t) &= x'(t) = \cos t,\end{aligned}$$

and

$$\begin{aligned}x(t) &= -\cos t, \\ y(t) &= x'(t) = \sin t.\end{aligned}$$

Then, citing a theorem from differential equations, the general solution to this set of differential equations is given by some linear combination of the two solutions, since the two solutions are *linearly independent*, since they are orthogonal: $(\sin t, \cos t) \perp (-\cos t, \sin t)$.

Velocity, speed, and acceleration The “time” derivative of some “time-parametrized” curve $\mathbf{C}(t) = (x_1(t), \dots, x_n(t))$ is called the *velocity vector*:

$$\mathbf{velocity}(t) = \mathbf{C}'(t) = d_t \mathbf{C}(t) = (C'_1(t), \dots, C'_n(t)).$$

The *acceleration vector* is the second time-derivative of the curve:

$$\mathbf{acceleration}(t) = \mathbf{C}''(t) = d_t^2 \mathbf{C}(t) = (C''_1(t), \dots, C''_n(t)).$$

The *speed*, or the absolute value (“magnitude”) of the velocity, is geometrically the length of the velocity vector:

$$\text{speed}(t) = |\mathbf{velocity}(t)| = \sqrt{(C'_1(t))^2 + \dots + (C'_n(t))^2}.$$

Example Take some parametrized curve $\mathbf{C}(t) = (\cos t, \sin t, t)$. Then the velocity vector is

$$\mathbf{C}'(t) = (-\sin t, \cos t, 1),$$

the speed is

$$|\mathbf{C}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2},$$

and the acceleration is

$$\mathbf{C}''(t) = (-\cos t, -\sin t, 0).$$

Some terms: a curve with speed 1 is said to have *unit speed*, and curves that run in *unit speed* are said to be *parametrized by arclength*.

Distinctions between the “image” and the function Consider some possible parametrization of a circle

$$\begin{aligned}\alpha(t) &= (\cos t, \sin t), & 0 \leq t < 2\pi, \\ \beta(t) &= (\sin t, \cos t), & 0 \leq t < 2\pi, \\ \gamma(t) &= (\cos 2\pi t, \sin 2\pi t), & 0 \leq t < 1.\end{aligned}$$

The “path” of the curves look something like

Notice that these curves have different starting points and/or different orientations (directions) and/or different speeds. Consequently, even though the images traced out by the curves are the same, the curves themselves are *not* the same.

Arc length The arc length is given by the integral of speed over time:

$$S = \int_t |\mathbf{C}'(t)| \, dt.$$