

Stokes' Theorem

Recall Green's theorem:

$$\oint_{C=\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \underbrace{(\nabla \times \mathbf{F})}_{\text{means taking some sort of derivative}} \cdot \hat{\mathbf{z}} \, dA = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA,$$

where $\mathbf{F} = (M, N)$.

Unification

Take some integration of a function ϕ :

$$\int \phi$$

If that manifold is *bounded* by some lower-dimensional manifold, then we make find some relationship between the integral over the boundary ∂M and the integral over the manifold M :

$$\int_{\partial M} \phi = \int_M \underbrace{\partial \phi}_{\text{some kind of derivative}}.$$

In essence, these relationships for various multi-dimensional manifolds form the “fundamental theorems” of calculus.

In fact, consider the one-variable fundamental theorem:

$$F(b) - F(a) = \int_a^b \frac{d}{dt} F \, dt.$$

We can see this integral as an integration over the segment of the one-dimensional t curve, bounded by the endpoints a and b .

Stokes' Theorem

Consider some vector field

$$\mathbf{F} = (P, Q, R).$$

We want to take this integral over a curve in 3-space C . We begin by considering C as the closed boundary of some surface S . Then we relate the integrals over the boundary and over the surface:

$$\int_{C=\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S},$$

where $d\mathbf{s}$ is the curve segment, and $d\mathbf{S}$ is the surface normal element.

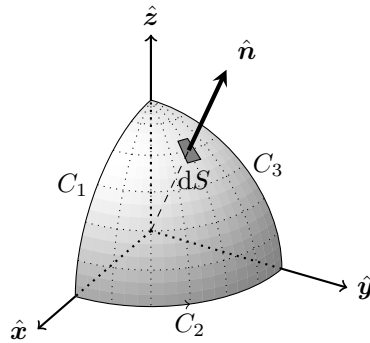
Geometric interpretations

The “circulation” of \mathbf{F} along ∂S is the “sum” of how much field \mathbf{F} “curls” in S .

Example Take the field $\mathbf{F} = (-y, 2x, z)$. We find the integral over the curve

$$\oint_C \mathbf{F} \cdot d\mathbf{s},$$

where C is the outwardly-oriented (counter-clockwise, looking from outside the sphere) union of three arcs along the unit sphere (at intersection with the planes):



Then the integral is

$$\oint_{C=C_1 \cup C_2 \cup C_3} \mathbf{F} \cdot d\mathbf{s},$$

We notice that C forms a closed curve; then, we can pick some convenient spherical surface S such that C is the boundary of S .

We calculate the curl of \mathbf{F} :

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ -y & 2x & z \end{vmatrix} = 0\hat{x} + 0\hat{y} + (2 - (-1))\hat{z} = 3\hat{z} = (0, 0, 3).$$

Then by Stokes' theorem the integral equals

$$\iint_S \underbrace{\text{curl } \mathbf{F}}_{3\hat{z}} \cdot \underbrace{d\mathbf{S}}_{\hat{n} dS}.$$

Recall the spherical surface area element

$$dS = r^2 \sin \phi \, d\phi \, d\theta,$$

(note r is fixed here), so that the normal element is

$$d\mathbf{S} = \hat{n} dS = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi).$$