Conservative vector fields are just *gradient* vector fields. To say that some function $\boldsymbol{F}$ is a gradient field is to say that there is some f such that $\boldsymbol{F} = \nabla f$.

Notice that the integral of a gradient vector field along a curve C is thus the *change* in the potential function across the endpoints of the curve:

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_C (\nabla f) \cdot \mathrm{d}\boldsymbol{s} = \int_C \left(\frac{\partial f}{\partial x_i} \,\mathrm{d}x_i + \dots \right) = \int_C \mathrm{d}f = f(\boldsymbol{b}) - f(\boldsymbol{a}),$$

where $\boldsymbol{b}, \boldsymbol{a}$ are the endpoints of the curve C.

Example Not all vector fields are gradient vector fields! Consider Then consider the curve $C = C_1 \cup (-C_2)$, so that C is the boundary of the semicircular region D. Then by Green's theorem

$$\oint_{\partial D = C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}A,$$

where $\boldsymbol{F}(x, y) = (M(x, y), N(x, y))$. Specifically

In general, these two different *path integrals* are not equal. Vector fields are typically not conservative.

But when are they equal? Notice by Green's theorem that a closed-curve path integral in a curve like this is zero iff

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Geometrically, notice then that this condition means that the vector field has no "rotation" in the plane.

But also notice that, the reverse implication

$$\boldsymbol{F} \text{ is conservative } \Longleftarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

only holds when D is "simply-connected" (there are no "holes" in D).

Fundamental Theorem of Line Integrals

If path C starts at point A, ends at point B, and $\boldsymbol{F} = \nabla f$, then

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = f(B) - f(A).$$

Notice that if F is conservative, the integral over the curve depends *only* on the endpoints, not on the path taken!

Example Consider the vector field $\boldsymbol{F} = (2x, 2y)$. Calculate the path integral over C, a straight line from $(1, 1)$ to $(4, 3)$.

We can compute the integral by parametrizing the curve and integrating over the parametrized curve.

We can also compute the integral by evaluating the potential function $f = x^2 + y^2$.

Flux

The flux is a double integral of a vector field over a surface:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

where $\mathrm{d}\boldsymbol{S}$ is some sort of a differential *area* vector (a vector normal to the surface with differential area $|\mathrm{d}\boldsymbol{S}|$).