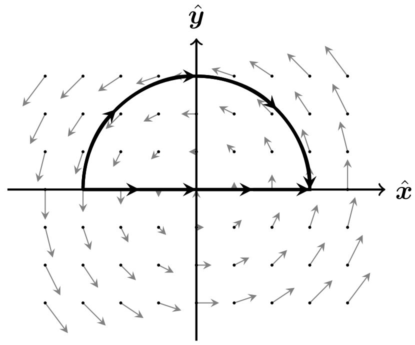
Conservative vector fields are just *gradient* vector fields. To say that some function $\boldsymbol{F}$ is a gradient field is to say that there is some f such that $\boldsymbol{F} = \nabla f$.

Notice that the integral of a gradient vector field along a curve C is thus the *change* in the potential function across the endpoints of the curve:

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_C (\nabla f) \cdot \mathrm{d}\boldsymbol{s} = \int_C \left(\frac{\partial f}{\partial x_i} \, \mathrm{d}x_i + \dots \right) = \int_C \mathrm{d}f = f(\boldsymbol{b}) - f(\boldsymbol{a}),$$

where $\boldsymbol{b}, \boldsymbol{a}$ are the endpoints of the curve C.

Example Not all vector fields are gradient vector fields! Consider the field $\boldsymbol{F} \colon \mathbb{R}^2 \to \mathbb{R}^2 = (-y, x)$, integrated from point P to point Q along two different paths C_1 and C_2:



Based on the geometry, we notice that the field is perpendicular to the x-axis everywhere along the x-axis, so that the integral

Then consider the curve $C = C_1 \cup (-C_2)$, so that C is the boundary of the semicircular region D. Then by Green's theorem

$$\oint_{\partial D = C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}A,$$

where $\boldsymbol{F}(x, y) = (M(x, y), N(x, y))$. Specifically

In general, these two different *path integrals* are not equal. Vector fields are typically not conservative.

But when are they equal? Notice by Green's theorem that a closed-curve path integral in a curve like this is zero iff

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Geometrically, notice then that this condition means that the vector field has no "rotation" in the plane.

But also notice that, the reverse implication

$$\boldsymbol{F} \text{ is conservative } \Longleftarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

only holds when D is "simply-connected" (there are no "holes" in D).

Fundamental Theorem of Line Integrals

If path C starts at point A, ends at point B, and $\boldsymbol{F} = \nabla f$, then

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = f(B) - f(A).$$

Notice that if F is conservative, the integral over the curve depends *only* on the endpoints, not on the path taken!

Example Consider the vector field $\boldsymbol{F} = (2x, 2y)$. Calculate the path integral over C, a straight line from $(1, 1)$ to $(4, 3)$.

We can compute the integral by parametrizing the curve and integrating over the parametrized curve. Let $(x, y) = (1 + 3t, 1 + 2t), 0 \leq t \leq 1$, so that the integral becomes

$$\begin{aligned}\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} &= \int_0^1 2(1 + 3t, 1 + 2t) \cdot (3, 2)\,\mathrm{d}t \\ &= \int_0^1 2(5 + 13t)\,\mathrm{d}t = \frac{1}{13}(5 + 13t)^2\Big|_0^1 = \frac{1}{13}(18^2 - 5^2) = \frac{1}{13}(18 - 5)(18 + 5) = 23.\end{aligned}$$

We can also compute the integral by evaluating the "change" in the potential function $f = x^2 + y^2$:

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = f(4, 3) - f(1, 1) = (4^2 + 3^2) - (1^2 + 1^2) = 23.$$

Wow, can't believe that the theorem is true! It essentially comes down to this rough simplification—recall the *total differential* of a function:

$$\mathrm{d}f(x_1, \ldots, x_n) = \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\,\mathrm{d}x_n.$$

Then the line integral of a conservative field $\boldsymbol{F} = \nabla f$ is just

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_C (\nabla f) \cdot \mathrm{d}\boldsymbol{s} = \int_C \left(\frac{\partial f}{\partial x_i}, \ldots\right) \cdot (\mathrm{d}x_i, \ldots) = \int_C \left(\frac{\partial f}{\partial x_i}\,\mathrm{d}x_i + \ldots\right) = \int_C \mathrm{d}f.$$

Flux

The flux is a double integral of a vector field over a surface:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

where $\mathrm{d}\boldsymbol{S}$ is some sort of a differential *area* vector (a vector normal to the surface with differential area $|\mathrm{d}\boldsymbol{S}|$).