Gauss's theorem

Suppose W is a three-dimensional region bounded by ∂W, consisting of *closed surfaces*, oriented so that normals point away from W, and $\boldsymbol{F}$ is a vector field defined on (at least) W, then

$$\oiint_{\partial W} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iiint_W \operatorname{div} \boldsymbol{F} \, \mathrm{d}V.$$

Example Some vector field $\boldsymbol{F}$ is called *incompressible* if $\operatorname{div} \boldsymbol{F} = 0$. By Gauss's theorem,

$$\oiint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iiint_W \operatorname{div} \boldsymbol{F} \, \mathrm{d}V = \iiint_W 0 \, \mathrm{d}V = 0.$$

Then, the flux through any surface is zero, because the divergence is zero everywhere.

Example Let S be a closed sphere of radius 3. Let $\boldsymbol{F} = (y^3, xe^z, \sin x + z)$ be a vector field. Find the flux of $\boldsymbol{F}$ through S.

Notice the divergence

$$\operatorname{div} \boldsymbol{F} = 0 + 0 + 1 = 1.$$

Then the flux is given by

$$\begin{aligned} \oiint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} &= \iiint \operatorname{div} \boldsymbol{F} \, \mathrm{d}V \\ &= \operatorname{div} \boldsymbol{F} \iiint \mathrm{d}V \\ &= \iiint \mathrm{d}V \\ &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(27) \\ &= 36\pi. \end{aligned}$$

Example An *inverse-square law* commonly appears in physics:

$$\begin{aligned} \boldsymbol{F} &= \frac{q}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \\ &= \frac{q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{r}}}{|\boldsymbol{r}|^2}. \end{aligned}$$

Suppose we want to find the flux of such a field through a surface. We might try Gauss's theorem. Take the divergence:

$$\begin{aligned} \operatorname{div} \boldsymbol{F} &= \frac{q}{4\pi\varepsilon_0} \operatorname{div} \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{3}{2} \frac{2x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \dots \right) \\ &= \frac{q}{4\pi\varepsilon_0} \left(\frac{3}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right) \\ &= \frac{q}{4\pi\varepsilon_0} \left(\frac{3}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{3}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \\ &= 0. \end{aligned}$$

Then, if we apply Gauss's theorem, this tells us that the total flux through any closed surface caused by a point charge is zero. But that can't be right! If there's a point charge sitting somewhere, it clearly generates a non-zero electric field outward and thus clearly generates a non-zero flux! What's with this?

Ah! Recall that Gauss's theorem can only be applied if the field is differentiable *everywhere inside* the region bounded by the surface. Now notice that the divergence is actually undefined at the origin $\boldsymbol{r} = (x, y, z) = \mathbf{0}$. There's a sort of "singularity" in the field. Then, we can't apply Gauss's theorem. Are we doomed?

Nope. We can do a little trick: notice that the field generated by the charge is spherically symmetric ($\boldsymbol{F}$ does not depend on the angles θ and ϕ). Then, if we consider the flux through a small sphere of radius a, we simply get

$$\varPhi_0 = \oiint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{F} \oiint_S \mathrm{d}\boldsymbol{S} = |\boldsymbol{F}|\, 4\pi a^2 = \frac{q}{4\pi\epsilon_0} \frac{1}{a^2} 4\pi a^2 = \frac{q}{\epsilon_0}.$$

Then, we can split the region into the small sphere centered on the point charge and the rest, so that the flux through the whole region is given by the flux through the sphere plus the flux through the region *excluding* the small origin sphere. But the divergence is defined-ly zero everywhere other than the origin, so the flux through the region excluding the origin sphere is zero, and thus the total flux through the region is simply equal to the total flux through the small sphere of radius a:

$$\varPhi = \frac{q}{\epsilon_0}.$$

Notice that we can always find such a sphere if the region includes the origin, because a (which we can think of as some $\epsilon > 0$) can be any arbitrary positive real.