Stokes' Theorem

Recall Green's theorem:

$$\oint_{C=\partial D} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_D \underbrace{(\nabla \times \boldsymbol{F})}_{\text{means taking some sort of derivative}} \cdot \hat{\boldsymbol{z}} \, \mathrm{d}A = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}A,$$

where $\boldsymbol{F} = (M, N)$.

Unification

Take some integration of a function ϕ:

$$\int \phi$$

If that manifold is *bounded* by some lower-dimensional manifold, then we make find some relationship between the integral over the boundary ∂M and the integral over the manifold M:

$$\int_{\partial M} \phi = \int_M \underbrace{\partial \phi}_{\text{some kind of derivative}} .$$

In essence, these relationships for various multi-dimensional manifolds form the "fundamental theorems" of calculus.

In fact, consider the one-variable fundamental theorem:

$$F(b) - F(a) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F \, \mathrm{d}t.$$

We can see this integral as an integration over the segment of the one-dimensional t curve, bounded by the endpoints a and b.

Stokes' Theorem

Consider some vector field

$$\boldsymbol{F} = (P, Q, R).$$

We want to take this integral over a curve in 3-space C. We begin by considering C as the closed boundary of some surface S. Then we relate the integrals over the boundary and over the surface:

$$\int_{C=\partial S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S},$$

where $\mathrm{d}\boldsymbol{s}$ is the curve segment, and $\mathrm{d}\boldsymbol{S}$ is the surface normal element.

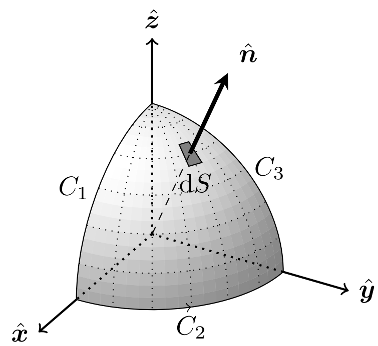
Geometric interpretations

The "circulation" of $\boldsymbol{F}$ along ∂S is the "sum" of how much field $\boldsymbol{F}$ "curls" in S.

Example Take the field $\boldsymbol{F} = (-y, 2x, z)$. We find the integral over the curve

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s},$$

where C is the outwardly-oriented (counter-clockwise, looking from outside the sphere) union of three arcs along the unit sphere (at intersection with the planes):



Then the integral is

$$\oint_{C=C_1\cup C_2\cup C_3} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s},$$

We notice that C forms a closed curve; then, we can pick some convenient spherical surface S such that C is the boundary of S.

We calculate the curl of $\boldsymbol{F}$:

$$\operatorname{curl}\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ -y & 2x & z \end{vmatrix} = 0\hat{\boldsymbol{x}} + 0\hat{\boldsymbol{y}} + (2-(-1))\hat{\boldsymbol{z}} = 3\hat{\boldsymbol{z}} = (0,0,3).$$

Then by Stokes' theorem the integral equals

$$\iint_S \underbrace{\operatorname{curl}\boldsymbol{F}}_{3\hat{\boldsymbol{z}}}\cdot \underbrace{\mathrm{d}\boldsymbol{S}}_{\hat{\boldsymbol{n}}\,\mathrm{d}S}.$$

But notice, since the curl is constant, that this integral is simply the flux through the spherical surface, given by the (magnitude of the) curl times the projection of the area onto the xy-plane:

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} = 3(\pi\cdot 1^2) = 3\pi.$$

Recall the spherical surface area element

$$\mathrm{d}S = r^2 \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta,$$

(note r is fixed here), so that the normal element is

$$\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\,\mathrm{d}S = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$