Green's Theorem

For some *bounded* region D, Green's theorem gives a relation between a double integration over the region D and a line integration over the *boundary* ∂D of D.

Example Consider some circular disk D of radius r.

There are three ways to parametrize the integral over the circular boundary ∂D in the *counter-clockwise* direction:

- $x = R\cos\theta, y = R\sin\theta$, for $0 \le \theta < 2\pi$. This parametrization is the standard conversion to polar coordinates. It works.
- $x = R\sin\theta, y = R\cos\theta$, for $0 \le \theta < 2\pi$. While this parametrization does cover the circular boundary, the *direction* of integration is reversed; this parametrization corresponds to a *clockwise* integration along the circular boundary. Orientation matters.
- $x = R\cos 2\pi t, y = R\sin 2\pi t$, for $0 \le t < 1$. This parametrization also works, because it traverses the circle with the right orientation.

Statement of Green's theorem Consider some function in the real plane $\boldsymbol{F} \colon \mathbb{R}^2 \to \mathbb{R}^2 = (M(x,y), N(x,y))$. Green's theorem relates the integral over the boundary to the integral inside the region:

$$\oint_{\partial D} (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Stated in vector form, where $\boldsymbol{F} = (M, N)$ and $\mathrm{d}\boldsymbol{r} = (x, y)$, Green's theorem can easily be thought of as

$$\oint_{\partial D} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \iint_D (\nabla \times \boldsymbol{F})\,\mathrm{d}x\,\mathrm{d}y.$$

where $\nabla \times \boldsymbol{F}$ is thought of as a *two-dimensional curl* of the vector field (nice, Tim!).

Example Consider some function $\boldsymbol{F} = (-y, x)$. We'll still let D be the circle with radius r. The boundary integral is then

$$\oint_{\partial D} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \oint_{\partial D} (-y, x)\cdot(\mathrm{d}x, \mathrm{d}y) = \oint_{\partial D} (-y\,\mathrm{d}x + x\,\mathrm{d}y).$$

We'll use the parametrization $x = R\cos\theta, y = R\sin\theta$. Then the integral becomes

$$\oint_{\partial D} (-(R\sin\theta)(-R\cos\theta\,\mathrm{d}\theta) + -(R\cos\theta)(R\sin\theta\,\mathrm{d}\theta)),$$

since $\mathrm{d}x = R(-\cos\theta\,\mathrm{d}\theta) = -R\cos\theta\,\mathrm{d}\theta$, and $\mathrm{d}y = R\sin\theta\,\mathrm{d}\theta$. Simplify the boundary integral:

$$\oint_{\partial D} R^2(\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta = \oint_{\partial_D} R^2\,\mathrm{d}\theta = R^2(2\pi) = 2\pi R^2.$$

But notice, by not much coincidence, that the result of the integral contains a πR^2 factor, which looks very much like the *area* of the region!

Suppose we took the integral of the curl over the surface:

$$\iint_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\mathrm{d}x\,\mathrm{d}y = \iint_D 2\,\mathrm{d}x\,\mathrm{d}y = 2\pi R^2,$$

so that the two integrals match up. Wow, Green's theorem is true! What a surprise. If only we did proofs in this class...