

**Malaviya National Institute of Technology**

**Department Of Mathematics**

MAT102- Mathematics-II

Tutorial sheet

Topics: Fourier integrals, Fourier sine and cosine integrals. Fourier transform, Fourier sine and cosine transforms.

1. Find the Fourier integral representations of the following functions.

$$(a) \quad f(x) = \begin{cases} 0, & x < -1 \\ 1, & -1 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

**Answer:**  $A(\omega) = \frac{2}{\omega} \sin \omega, B(\omega) = 0$

$$(b) \quad f(x) = \begin{cases} e^{-|x|}, & |x| < 1 \\ 0, & \text{elsewhere} \end{cases}$$

**Answer:**  $A(\omega) = 2[1 + (\omega \sin \omega - \cos \omega)e^{-1}]/(1 + \omega^2), B(\omega) = 0$

$$(c) \quad f(x) = \begin{cases} -(1+x), & -1 < x < 0 \\ 1-x, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

**Answer:**  $A(\omega) = 0, B(\omega) = 2[\omega - \sin \omega]/\omega^2$

2. Find the Fourier cosine integral representations of the following functions.

$$(a) \quad f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0 & x > \pi. \end{cases}$$

**Answer:**  $A(\omega) = -2[1 + \cos(\omega\pi)]/(\omega^2 - 1), \omega \neq 1, A(1) = 0$

$$(b) \quad f(x) = e^{-2x} + e^{-3x}, x > 0$$

**Answer:**  $A(\omega) = \frac{4}{4+\omega^2} + \frac{6}{9+\omega^2}$

3. Find the Fourier sine integral representations of the following functions.

$$(a) \quad f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \\ 0 & x > 2. \end{cases}$$

**Answer:**  $B(\omega) = 2[\cos \omega - \cos(2\omega)]/\omega$

$$(b) \quad f(x) = \begin{cases} \sinh x, & 0 \leq x \leq 3 \\ 0 & x > 3. \end{cases}$$

$$\textbf{Answer: } B(\omega) = 2[\sin(3\omega) \cosh 3 - \omega \cos(3\omega) \sinh 3]/(1 + \omega^2)$$

4. Express the function  $f(x) = \begin{cases} \frac{2}{\pi} \sin x, & \text{if } 0 \leq x \leq \pi \\ 0, & \text{if } x > \pi. \end{cases}$  as a Fourier sine integral and hence evaluate the integral  $\int_0^\infty \frac{\sin(\pi\lambda) \sin(x\lambda) d\lambda}{1-\lambda^2}$ .

$$\textbf{Answer: } \begin{cases} \frac{\pi}{2} \sin x, & \text{if } 0 \leq x \leq \pi \\ 0, & \text{if } x > \pi. \end{cases}$$

5. Find the Fourier transform of the following functions.

$$(a) \quad f(t) = \begin{cases} t^2, & |t| \leq a \\ 0, & |t| > a \end{cases}$$

$$\textbf{Answer: } \frac{2}{w^3 \sqrt{2\pi}} [a^2 \omega^2 \sin a\omega + 2a\omega \cos a\omega - 2 \sin a\omega]$$

$$(b) \quad f(t) = H(t-3)e^{-4t}$$

$$\textbf{Answer: } \frac{e^{-3(4+i\omega)}}{(4+i\omega)}$$

$$(c) \quad f(t) = 1/(1+t^2)$$

$$\textbf{Answer: } \pi e^{-|\omega|}$$

$$(d) \quad f(t) = e^{-a|t+1|}, a > 0$$

$$\textbf{Answer: } \frac{2ae^{i\omega}}{(a^2+\omega^2)}$$

6. Find the inverse Fourier transform of the following functions.

$$(a) \quad \frac{e^{-i\omega}}{2(1+i\omega)}$$

$$\textbf{Answer: } \text{Use shift theorem, } e^{-(t-1)}H(t-1)/2$$

$$(b) \quad \frac{i\omega}{(i\omega+2)(i\omega+3)}$$

$$\textbf{Answer: } e^{-2t}(3e^{-t} - 2)H(t)$$

$$(c) \quad \frac{1}{(i\omega+k)^2}, k > 0$$

$$\textbf{Answer: } (f * g)(t) = e^{-kt} \int_{-\infty}^{\infty} H(\tau)H(t-\tau)d\tau = te^{-kt}H(t), \text{ since } H(\tau)H(t-$$

$\tau) = 0$  for  $t < \tau < 0$ , and 1 for  $0 < \tau < t$ .

(d)  $\frac{1}{(i\omega+k)^3}, k > 0$

**Answer:** Use the result of above problem,  $t^2 e^{-kt} H(t)/2$

7. Using Fourier transforms, find the solutions of the following differential equations.

(a)  $y' + 3y = H(t)e^{-2t}, -\infty < t < \infty$

**Answer:**  $F(\omega) = 1/[(2 + i\omega)(3 + i\omega)], y(t) = e^{-2t}(1 - e^{-t})H(t)$ .

(b)  $y'' + 3y' + 2y = \delta(t - 3)$

**Answer:**  $F(\omega) = e^{-3i\omega}/[(2 + i\omega)(1 + i\omega)], y(t) = [e^{-(t-3)} - e^{-2(t-3)}]H(t - 3)$

8. Find the Fourier cosine and sine transforms of the following functions.

(a)  $f(t) = \begin{cases} \sin t, & 0 \leq t \leq l, \\ 0, & t > l \end{cases}$

**Answer:**  $\frac{1}{1-\omega^2} + \frac{\cos(\omega-1)l}{2(\omega-1)} - \frac{\cos(\omega+1)l}{2(\omega+1)}$ ; for  $\omega = 1, \frac{1}{4}(1 - \cos(2l))$ ,

$\frac{1}{2}[\frac{\sin(\omega-1)l}{\omega-1} - \frac{\sin(\omega+1)l}{\omega+1}]$ ; for  $\omega = 1, \frac{1}{4}(2l - \sin(2l))$ .

(b)  $f(t) = \begin{cases} 1 + t, & 0 \leq t \leq l, \\ 0, & t > l \end{cases}$

**Answer:**  $[\frac{1}{\omega}(1 + l) \sin(\omega l) + \frac{1}{\omega^2}(\cos \omega l - 1)]$ ,

$[\frac{1}{\omega} + \frac{\sin(\omega l)}{\omega^2} - \frac{1}{\omega}(1 + l) \cos(\omega l)]$

(c)  $f(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq a \\ 0, & \text{if } t > a. \end{cases}$

**Answer:**  $F_s(w) = \sqrt{\frac{2}{\pi}}[\frac{1-\cos aw}{w}], F_c(w) = \sqrt{\frac{2}{\pi}}[\frac{\sin aw}{w}]$