Malaviya National Institute of Technology, Jaipur
Department of Mathematics
B.Tech. First Semester | 22MAT101: Mathematics - I
Tutorial Sheet: 7
Topics covered: *Differentiation of vectors, gradient, directional derivative, divergence, curl, and Line integrals.*

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1. Find the directional derivative of the given scalar functions at the given points in the indicated directions.
   (a) $f(x,y,z) = xyz$ at $(1,4,3)$ in the direction of the line from $(1,2,3)$ to $(1,-1,-3)$.
   (b) $f(x,y,z) = x^2 + y^2 + 2z^2$ at $(1,1,2)$ in the direction of grad(f).

Ans: (a) $-\dfrac{11}{\sqrt{5}}$. (b) $6\sqrt{2}$.

2. If the maximum value of the directional derivative of a scalar field $\phi = ax^2y + by^2z + cz^2x$ at the point $(1,1,1)$ is in the direction parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$ and has magnitude 15, then find the values of the parameters a, b, and c.

Ans: $a = 5$, $b = -5$, $c = 5/2$.

3. Find the angle between the two surfaces at the indicated point of intersection.
   (a) $z = x^2 + y^2$ and $z = 2x^2 - 3y^2$ at $(2,1,5)$.
   (b) $x^2 + y^2 + z^2 = 9$ and $z + 3 = x^2 + y^2$ at $(-2,1,2)$.

Ans: (a) $\cos^{-1}\left(\dfrac{21}{101}\right)$. (b) $\cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$.

4. Show that the following vector fields are irrotational and hence find their scalar potentials.
   (a) $\mathbf{f}(x,y,z) = \cos(x^2+y^2+z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.
   (b) $\mathbf{f}(x,y,z) = 3x^2y^2z^4\mathbf{i} + 2x^3yz^4\mathbf{j} + 4x^3y^2z^3\mathbf{k}$.

Ans: Let $\mathbf{f} = \nabla\phi$ then (a) $\phi = \sin(x^2+y^2+z^2)/2$. (b) $\phi = x^3y^2z^4$.

5. Show that the following vector fields $\mathbf{f}$ are solenoidal.
   (a) $(2x+3y)\mathbf{i} + (x-y)\mathbf{j} - (x+y+z)\mathbf{k}$.
   (b) $e^{x+y-2z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Ans: Since $\nabla \cdot \mathbf{f} = 0$, the fields are solenoidal.

6. Prove the following identities:
   (a) curl$(f\mathbf{v})$ = (grad f) $\times\, \mathbf{v}$ + curl $\mathbf{v}$.
   (b) div(grad$f$) $= \nabla^2 f$, where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplacian operator.
   (c) curl(curl $\mathbf{v}$) $= \nabla(\nabla \cdot \mathbf{v}) - \nabla^2\mathbf{v}$.

Ans: Refer to Example 15.24 on page 15.23 of the textbook.

7. Find the area of one side of the *wall* standing orthogonally on the curve $y = x^2,\ 0 \le x \le 2$, and beneath the curve on the surface $f(x, y) = x + \sqrt{y}$.

Ans: $A = \displaystyle\int_C f(x,y)ds = \int_0^2 (t + \sqrt{t^2})\frac{ds}{dt}dt = \int_0^2 (2t)\left\|\frac{d\mathbf{r}}{dt}\right\| dt = \frac{1}{6}\left(17^{3/2} - 1\right).$

8. Find the mass of a wire that has (linear) density $\rho = \rho_0(1 + z)$ and is in the shape of the curve

$$\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + (2\sqrt{2}/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le 1.$$

Ans: $M = \displaystyle\int_C \rho ds = \rho_0 \int_0^1 \left(1 + (2\sqrt{2}/3)t^{3/2}\right)(1+t)dt = \left(\frac{3}{2} + \frac{16\sqrt{2}}{35}\right)\rho_0.$

9. Find the work done by the force $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ in moving a particle from the origin $(0, 0, 0)$ to the point $(2\pi, 0, 2\pi)$ along the conic-helical curve: $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + t\mathbf{k}$ with t between 0 and 2π.

Ans: $W = \displaystyle\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_0^{2\pi} 2tdt = 4\pi^2.$

10. Find the circulation of the vector field $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t:

$$\begin{aligned} C_1 &: \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, && 0 \le t \le \pi/2. \\ C_2 &: \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1-t)\mathbf{k}, && 0 \le t \le 1. \\ C_3 &: \mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}, && 0 \le t \le 1. \end{aligned}$$

Ans: $\displaystyle\int_{C_1} \mathbf{f}\cdot d\mathbf{r} = \int_0^{\pi/2}(2t)dt = \frac{\pi^2}{4}$, $\displaystyle\int_{C_2} \mathbf{f}\cdot d\mathbf{r} = \int_0^1 1/2\pi^2(-1+t)dt = -\frac{\pi^2}{4}$, and $\displaystyle\int_{C_3} \mathbf{f}\cdot d\mathbf{r} = \int_0^1 (-2+4t)dt = 0$. Thus, by path-additivity property, the circulation is 0.

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