Malaviya National Institute of Technology, Jaipur

Department of Mathematics

MAT 101 Tutorial 2- Eigenvalues and Eigenvectors of a matrix, Cayley- Hamilton theorem, Diagonalization of a matrix.

1. Find the eigenvalues of a matrix A. Hence find the eigenvalues of matrix $A^2 = AA$?
(i) $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, (ii) $A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{bmatrix}$.
Ans. (i) (1,5), (1, 25), (ii) (6, 3, 7), (36, 9, 49)

2. (i) Let λ be an eigenvalue of matrix A. Identify the corresponding eigenvalue of $adj(A)$.
(ii) Eigenvalues of a matrix $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ are λ_1, λ_2, then determine the characteristic roots of $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$.
Ans. (i) $\frac{det(A)}{\lambda}$ **(ii)** $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$

3. If 1 and 2 are the eigenvalues of 2×2 matrix A , what are the eigenvalues of A^2 and A^{-1}. **Ans. (i)1, 4, 1,** $\frac{1}{2}$

4. Find the eigenvalues (characteristic roots) and eigenvectors (characteristic vectors) of the following matrices:
(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.
Ans. (i) 0,3,15, $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, **(ii) -3,-3,5,** $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$,

5. Using Cayley-Hamilton theorem, find A^{-1} and A^4 , if $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$. **Ans.** $\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}, \begin{pmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{pmatrix}$

6. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. Hence, find A^{50}. **Ans.** $\begin{pmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix}$

7. Find the characteristic equation of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. Hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$. **Ans.** $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$, $\begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$

8. State Cayley-Hamilton theorem. Verify it for the matrix $A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}$. Hence find A^{-1}. **Ans.**
$\frac{1}{81}\begin{pmatrix} 17 & 16 & -20 \\ 16 & 13 & 4 \\ 20 & 4 & -5 \end{pmatrix}$

9. Find a matrix P that diagonalizes the matrix $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. Verify $P^{-1}AP = D$ where D is the diagonal matrix. **Ans.** $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$.

10. Diagonalize the matrix $A = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$ and hence find the matrix representation of e^A . **Ans.** $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\frac{1}{2}\begin{pmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{pmatrix}$.

11. Show that the matrix $A = \begin{pmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ is diagonalizable. Hence, find P such that $P^{-1}AP$ is a diagonal matrix.

    Ans. $\lambda = 1, 2, 3$; $P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

12. Show that
    (i) The eigenvalues of a square matrix and its transpose are the same.
    (ii) If A and B be two square invertible matrices, then AB and BA have the same characteristic roots.

13. (a) Let X_1 and X_2 be eigenvectors of a matrix A corresponding to distinct eigenvalues λ_1 and λ_2 respectively. Then show that X_1 and X_2 are linearly independent.
    (b) Let λ be an eigenvalue of an invertible matrix A. Then show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

    Solution:

    (a) Consider

$$c_1X_1 + c_2X_2 = \mathbf{0}, \tag{1}$$
$$\implies c_1AX_1 + c_2AX_2 = \mathbf{0}$$
$$\implies c_1\lambda_1X_1 + c_2\lambda_2X_2 = \mathbf{0}. \tag{2}$$

Solving above equations we get

$$c_1(\lambda_1 - \lambda_2)X_1 = \mathbf{0} \implies c_1 = 0,$$
$$\text{and } c_2(\lambda_1 - \lambda_2)X_2 = \mathbf{0} \implies c_2 = 0.$$

as $\lambda_1 \neq \lambda_2$ and X_1 and X_2 are non-zero vectors. Thus, whenever $c_1X_1 + c_2X_2 = \mathbf{0}$, we must have $c_1 = 0 = c_2$.

    (b) Since λ is an eigenvalue of A, there exists a nonzero vector X such that

$$AX = \lambda X \tag{3}$$
$$\implies X = \lambda A^{-1}X$$
$$\implies A^{-1}X = \frac{1}{\lambda}X. \tag{4}$$

As $det(A) \neq 0$, therefore $\lambda \neq 0$, therefore $\frac{1}{\lambda}$ exists. Hence $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

14. The eigenvectors of 3 X 3 matrix A corresponding to the eigenvalues 1, -1, 2 are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ respectively. Find matrix A and hence find A^5.

    Solution:

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{5}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{6}$$

$$A = PDP^{-1} \tag{7}$$

$$A = \begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix}. \tag{8}$$

$$A^5 = PD^5P^{-1} \tag{9}$$

$$A^5 = \begin{bmatrix} 96 & -95 & -97 \\ 31 & -30 & -31 \\ 33 & -33 & -34 \end{bmatrix}. \tag{10}$$

15. (a) Let A be a square matrix such that $A^k = 0$ for some positive integer k. Then show that all the eigenvalues of A are zero.

(b) Let A be a square matrix such that $A^2 = A$. Then show that the eigenvalues of A are either 0 or 1.

Solution:

(a) Let λ be an eigenvalue of A, then there exists a nonzero vector X such that

$$\begin{aligned} AX &= \lambda X \\ \implies A^k X &= \lambda^k X \\ \implies \lambda^k X &= 0. \end{aligned}$$

As $X \neq 0$, therefore $\lambda = 0$.

(b) Let λ be an eigenvalue of A, then there exists a nonzero vector X such that

$$\begin{aligned} AX &= \lambda X \\ \implies A^2 X &= \lambda^2 X \\ \implies \lambda^2 X &= \lambda X. \end{aligned}$$

As $X \neq 0$, therefore $\lambda(\lambda - 1) = 0$. Hence either $\lambda = 0$ or $\lambda = 1$.