Malaviya National Institute of Technology Jaipur, India
Department of Mathematics

Tutorial Sheet: 4 (Mathematics II)

Topics: Laplace and inverse Laplace transform of some standard functions, Shifting theorems, Laplace transform of derivatives and integrals, Convolution theorem, Initial and final value theorem. Laplace transform of periodic functions, Error functions, Heaviside unit step function and Dirac delta function. Applications of Laplace transform.

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1. Use definition to find the Laplace transform of the following functions:
   (i) $e^{-t}\sin t$.
   (ii) $\sin^2(at)$.
   (iii) $f(t) = \begin{cases} \cos t, & \text{if } 0 \le t \le \pi; \\ 0, & \text{if } t \ge \pi. \end{cases}$
   (iv) $f(t) = \begin{cases} 0, & \text{if } 0 \le t < \pi; \\ \sin t, & \text{if } t \ge \pi. \end{cases}$

   Ans. (i) $\frac{1}{s^2+2s+2}$. (ii) $\frac{2a^2}{s(s^2+4a^2)}$. (iii) $\frac{s(1+e^{-s\pi})}{s^2+1}$. (iv) $-\frac{e^{-s\pi}}{s^2+1}$.

2. Find the Laplace transformation of the following functions:
   (a) $3t^2 - 5e^{-2t} + 6$. (b) $\sinh at$. (c) $e^{-t}\sinh t$.

   Ans. (a) $\frac{6}{s^3} - \frac{5}{s+2} + \frac{6}{s}$. (b) $\frac{a}{s^2-a^2}$. (c) $\frac{1}{s(s+2)}$.

3. Find the inverse Laplace transform of the following function:
   (a) $\frac{3}{s+5}$. (b) $\frac{s+3}{(s-1)(s+2)}$. (c) $\frac{s^2+2s+5}{(s-1)(s-2)(s-3)}$.

   Ans. (a) $3e^{-5t}$. (b) $\frac{4e^t - e^{-2t}}{3}$. (c) $4e^t - 13e^{2t} + 10e^{3t}$.

4. If $\mathcal{L}(f(t)) = F(s)$, then show that $\mathcal{L}(f(at)) = \frac{1}{a}F(s/a)$.

5. Show that
   (a) $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$.
   (b) $\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$.

6. Find the Laplace transform of $f(t) = t^2$ using the Laplace transform of $f''(t)$.

7. Using that $\mathcal{L}\left(\int_0^t f(u)du\right) = \frac{\mathcal{L}(f(t))}{s}$, find the inverse Laplace transform of the following functions:
   (a) $\frac{1}{s^2+5s}$. (b) $\frac{s-2}{s(s+3)}$. (c) $\frac{1}{s^4+3s^3}$.

   Ans. (a) $\frac{1-e^{-5t}}{5}$. (b) $-\frac{1}{3}(2-5e^{-3t})$. (c) $\frac{1}{18}(3t^2-2t) + \frac{1}{27}(1-e^{-3t})$.

8. Suppose $\mathcal{L}(f(t)) = F(s)$ denotes the Laplace transform of $f(t)$. Then show that $\mathcal{L}\left(\int_a^t f(u)du\right) = \frac{F(s)}{s} - \frac{1}{s}\int_0^a f(u)du$, $a > 0$.

9. Using Shifting theorems find Laplace transform of the following functions:
   (a) $e^t \sin 5t$. (b) $\sinh t \cos t$. (c) $(t^2-1)u_2(t)$, where $u_2(t)$ denotes the unit step function. **Ans.** (a) $\frac{5}{s^2-2s+26}$. (b) $\frac{s^2-2}{s^4+4}$. (c) $e^{-2s}\frac{2+4s+3s^2}{s^3}$.

10. If $L[f(t)] = F(s)$, then prove that $L[tf(t)] = -\frac{d}{ds}F(s)$. Hence find the inverse Laplace transform of $ln\left(\frac{s^2+1}{s^2}\right)$.

Ans. $\frac{2}{t}(1-\cos t)$.

11. If $L[f(t)] = F(s)$, then prove that $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty f(u)du$. Hence find the Laplace transform of $\frac{\cos t - \cos 2t}{t}$. **Ans.** $\frac{1}{2}ln\left(\frac{s^2+4}{s^2+1}\right)$.

12. Find the Laplace transform of the function $f(t) = \frac{kt}{T}$, $0 < t < T$; $f(t+T) = f(t)$.

Ans. $\frac{-ke^{-sT}}{s(1-e^{-sT})} + \frac{k}{s^2T}$.

13. Express the following functions in terms of Heaviside unit step functions and find the Laplace transforms:

(a) $f(t) = \begin{cases} 8, & \text{if} \quad t < 2, \\ 6, & \text{if} \quad t \geq 2. \end{cases}$ (b) $f(t) = \begin{cases} t-1, & \text{if} \quad 1 < t < 2, \\ 3-t, & \text{if} \quad 2 < t < 3. \end{cases}$

Ans. (a) $\frac{8}{s} - 2\frac{e^{-2s}}{s}$, (b) $\frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$.

14. If $\delta_k(t-a)$ represents the Dirac delta function, then obtain its Laplace transforms.

15. Find the Laplace transform of $\frac{\sin at}{t}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Ans. $\frac{\cot^{-1}s}{a}$ & Laplace transform of $\frac{\cos at}{t}$ does not exist.

16. Using Shifting theorems find inverse Laplace transform of the following functions:

(a) $\frac{s}{(s-2)^3}$. (b) $\frac{s}{s^2+4s+8}$. **Ans.** (a) $(t+t^2)e^{2t}$. (b) $(\cos 2t - \sin 2t)e^{-2t}$.

17. State and prove convolution theorem, and hence obtain the Laplace inverse transform for the following functions:

(a) $\frac{s}{(s^2+a^2)^2}$. (b) $\frac{1}{s^2(s^2+a^2)^2}$. (c) $\frac{1}{s(s+1)^3}$. (d) $\frac{s^2}{(s^2+1)(s^2+4)}$.

Ans. (a) $\frac{t\sin at}{2a}$. (b) $\frac{at - \sin at}{a^3}$. (c) $1 - e^{-t}\left(\frac{t^2}{2}+t+1\right)$. (d) $\frac{1}{3}(\sin 2t - \sin t)$.

18. Obtain the solution for the following differential equations:

(i) $\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} - 6y = 2$, $y(0) = 0$ $y'(0) = 0$. **Ans.** $y(t) = t^2$.

(ii) $t\frac{d^2y}{dt^2} - t\frac{dy}{dt} + y = 2$, $y(0) = 2$ $y'(0) = -4$. **Ans.** $y(t) = 2 - 4t$.

(iii) $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 2y = te^{-2t}$, $y(0) = 0$, $y'(0) = -2$.

Ans. $y(t) = \frac{1}{125}\left(-96e^{\frac{t}{2}} + 96e^{-2t} - 10te^{-2t} - \frac{25}{2}t^2e^{-2t}\right)$.

(iv) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$.

Ans. $y(t) = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$.

(v) $\dfrac{d^2y}{dt^2} + 4y = 2\sin t + 3\cos t, \qquad y(0) = 2, \qquad y'(0) = 2.$

Ans. $y(t) = \dfrac{5}{3}\sin t - \dfrac{1}{3}\sin(2t) + 3\cos(2t) - \cos t.$

19. Solve the integro-differential equation: $\dfrac{dy}{dt} + 2y + \int_0^t y\,dt = 2\cos t,\ y(0) = 1.$
Ans. $y(t) = \cos t.$

20. Solve the integral equation $f(t) = 1 + \int_0^t f(u).\sin(t-u)du$ and verify your solution.
Ans. $f(t) = 1 + \dfrac{t^2}{2}$

21. The co-ordinate (x, y) of a particle moving along a plane curve at any time t is given by

$$\frac{dy}{dt} + 2x = \sin 2t$$

$$\frac{dx}{dt} - 2y = \cos 2t.$$

If at $t = 0$, $x = 1$ and $y = 0$, show by using transforms that particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

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