$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}.$$

The numbers A_x, A_y, and A_z, are the "components" of $\mathbf{A}$; geometrically, they are the projections of $\mathbf{A}$ along the three coordinate axes ($A_x = \mathbf{A} \cdot \hat{\mathbf{x}}$, $A_y = \mathbf{A} \cdot \hat{\mathbf{y}}$, $A_z = \mathbf{A} \cdot \hat{\mathbf{z}}$). We can now reformulate each of the four vector operations as a rule for manipulating components:

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.\end{aligned} \tag{1.7}$$

Rule (i): *To add vectors, add like components.*

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}. \tag{1.8}$$

Rule (ii): *To multiply by a scalar, multiply each component.*

Because $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular unit vectors,

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1; \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0. \tag{1.9}$$

Accordingly,

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= A_xB_x + A_yB_y + A_zB_z.\end{aligned} \tag{1.10}$$

Rule (iii): *To calculate the dot product, multiply like components, and add.* In particular,

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2,$$

so

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.11}$$

(This is, if you like, the three-dimensional generalization of the Pythagorean theorem.)

Similarly,[1]

$$\begin{aligned}\hat{\mathbf{x}} \times \hat{\mathbf{x}} &= \quad \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}, \\ \hat{\mathbf{x}} \times \hat{\mathbf{y}} &= -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}}, \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}}.\end{aligned} \tag{1.12}$$

[1]These signs pertain to a *right-handed* coordinate system (x-axis out of the page, y-axis to the right, z-axis up, or any rotated version thereof). In a *left-handed* system (z-axis down), the signs would be reversed: $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{z}}$, and so on. We shall use right-handed systems exclusively.

Therefore,

$$\mathbf{A} \times \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \tag{1.13}$$
$$= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}.$$

This cumbersome expression can be written more neatly as a determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.14}$$

Rule (iv): *To calculate the cross product, form the determinant whose first row is* $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, *whose second row is* **A** *(in component form), and whose third row is* **B**.

Example 1.2. Find the angle between the face diagonals of a cube.

Solution

We might as well use a cube of side 1, and place it as shown in Fig. 1.10, with one corner at the origin. The face diagonals **A** and **B** are

$$\mathbf{A} = 1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 0\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}.$$

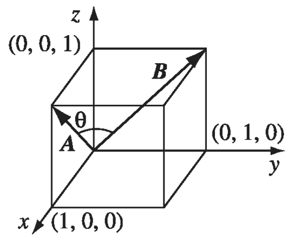

FIGURE 1.10

So, in component form,

$$\mathbf{A} \cdot \mathbf{B} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1.$$

On the other hand, in "abstract" form,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta = \sqrt{2}\sqrt{2}\cos\theta = 2\cos\theta.$$

Therefore,

$$\cos\theta = 1/2, \qquad \text{or} \qquad \theta = 60^\circ.$$

Of course, you can get the answer more easily by drawing in a diagonal across the top of the cube, completing the equilateral triangle. But in cases where the geometry is not so simple, this device of comparing the abstract and component forms of the dot product can be a very efficient means of finding angles.

Problem 1.3 Find the angle between the body diagonals of a cube.

Problem 1.4 Use the cross product to find the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the shaded plane in Fig. 1.11.

1.1.3 ■ Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a *triple* product.

(i) Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Geometrically, $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ is the volume of the parallelepiped generated by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, since $|\mathbf{B} \times \mathbf{C}|$ is the area of the base, and $|A \cos\theta|$ is the altitude (Fig. 1.12). Evidently,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \tag{1.15}$$

for they all correspond to the same figure. Note that "alphabetical" order is preserved—in view of Eq. 1.6, the "nonalphabetical" triple products,

$$\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}),$$

have the opposite sign. In component form,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{1.16}$$

Note that the dot and cross can be interchanged:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$$

(this follows immediately from Eq. 1.15); however, the placement of the parentheses is critical: $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ is a meaningless expression—you can't make a cross product from a *scalar* and a vector.

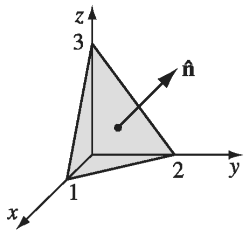

FIGURE 1.11

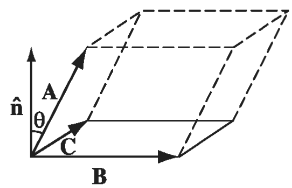

FIGURE 1.12

(ii) Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{1.17}$$

Notice that

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C})$$

is an entirely different vector (cross-products are not associative). All *higher* vector products can be similarly reduced, often by repeated application of Eq. 1.17, so it is never necessary for an expression to contain more than one cross product in any term. For instance,

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}); \\ \mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] &= \mathbf{B}[\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})] - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D}). \end{aligned} \tag{1.18}$$

Problem 1.5 Prove the **BAC-CAB** rule by writing out both sides in component form.

Problem 1.6 Prove that

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = \mathbf{0}.$$

Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

1.1.4 ■ Position, Displacement, and Separation Vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin ($\mathcal{O}$) is called the **position vector** (Fig. 1.13):

$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}. \tag{1.19}$$

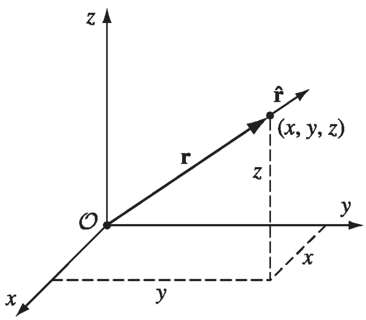

FIGURE 1.13

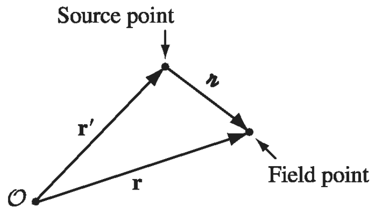

FIGURE 1.14

I will reserve the letter $\mathbf{r}$ for this purpose, throughout the book. Its magnitude,

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{1.20}$$

is the distance from the origin, and

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \tag{1.21}$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}. \tag{1.22}$$

(We could call this $d\mathbf{r}$, since that's what it *is*, but it is useful to have a special notation for infinitesimal displacements.)

In electrodynamics, one frequently encounters problems involving *two* points—typically, a **source point**, $\mathbf{r}'$, where an electric charge is located, and a **field point**, $\mathbf{r}$, at which you are calculating the electric or magnetic field (Fig. 1.14). It pays to adopt right from the start some short-hand notation for the **separation vector** from the source point to the field point. I shall use for this purpose the script letter $\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'. \tag{1.23}$$

Its magnitude is

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'|, \tag{1.24}$$

and a unit vector in the direction from $\mathbf{r}'$ to $\mathbf{r}$ is

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{1.25}$$

In Cartesian coordinates,

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}, \tag{1.26}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \tag{1.27}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{1.28}$$

(from which you can appreciate the economy of the script-$\mathscr{r}$ notation).

Problem 1.7 Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point (2,8,7) to the field point (4,6,8). Determine its magnitude ($\mathscr{r}$), and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.

1.1.5 ■ How Vectors Transform[2]

The definition of a vector as "a quantity with a magnitude and direction" is not altogether satisfactory: What precisely does "direction" *mean*? This may seem a pedantic question, but we shall soon encounter a species of derivative that *looks* rather like a vector, and we'll want to know for sure whether it *is* one.

You might be inclined to say that a vector is anything that has three components that combine properly under addition. Well, how about this: We have a barrel of fruit that contains N_x pears, N_y apples, and N_z bananas. Is $\mathbf{N} = N_x\hat{\mathbf{x}} + N_y\hat{\mathbf{y}} + N_z\hat{\mathbf{z}}$ a vector? It has three components, and when you add another barrel with M_x pears, M_y apples, and M_z bananas the result is $(N_x + M_x)$ pears, $(N_y + M_y)$ apples, $(N_z + M_z)$ bananas. So it does *add* like a vector. Yet it's obviously *not* a vector, in the physicist's sense of the word, because it doesn't really have a direction. What exactly is wrong with it?

The answer is that **N** *does not transform properly when you change coordinates.* The coordinate frame we use to describe positions in space is of course entirely arbitrary, but there is a specific geometrical transformation law for converting vector components from one frame to another. Suppose, for instance, the $\overline{x}, \overline{y}, \overline{z}$ system is rotated by angle ϕ, relative to x, y, z, about the common $x = \overline{x}$ axes. From Fig. 1.15,

$$A_y = A\cos\theta, \qquad A_z = A\sin\theta,$$

while

$$\begin{aligned}
\overline{A}_y &= A\cos\overline{\theta} = A\cos(\theta - \phi) = A(\cos\theta\cos\phi + \sin\theta\sin\phi) \\
&= \cos\phi A_y + \sin\phi A_z, \\
\overline{A}_z &= A\sin\overline{\theta} = A\sin(\theta - \phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi) \\
&= -\sin\phi A_y + \cos\phi A_z.
\end{aligned}$$

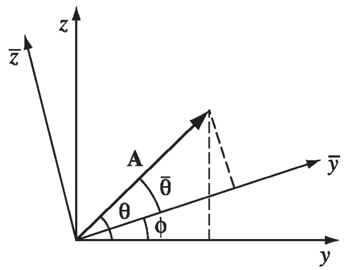

FIGURE 1.15

[2]This section can be skipped without loss of continuity.

We might express this conclusion in matrix notation:

$$\begin{pmatrix} \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}. \tag{1.29}$$

More generally, for rotation about an *arbitrary* axis in three dimensions, the transformation law takes the form

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \tag{1.30}$$

or, more compactly,

$$\overline{A}_i = \sum_{j=1}^{3} R_{ij} A_j, \tag{1.31}$$

where the index 1 stands for x, 2 for y, and 3 for z. The elements of the matrix R can be ascertained, for a given rotation, by the same sort of trigonometric arguments as we used for a rotation about the x axis.

Now: *Do* the components of **N** transform in this way? Of *course* not—it doesn't matter what coordinates you use to represent positions in space; there are still just as many apples in the barrel. You can't convert a pear into a banana by choosing a different set of axes, but you *can* turn A_x into $\overline{A}_y$. Formally, then, a *vector is any set of three components that transforms in the same manner as a displacement when you change coordinates.* As always, displacement is the *model* for the behavior of all vectors.[3]

By the way, a (second-rank) **tensor** is a quantity with *nine* components, T_{xx}, T_{xy}, T_{xz}, T_{yx}, $\ldots$, T_{zz}, which transform with *two* factors of R:

$$\begin{aligned} \overline{T}_{xx} = {} & R_{xx}(R_{xx}T_{xx} + R_{xy}T_{xy} + R_{xz}T_{xz}) \\ & + R_{xy}(R_{xx}T_{yx} + R_{xy}T_{yy} + R_{xz}T_{yz}) \\ & + R_{xz}(R_{xx}T_{zx} + R_{xy}T_{zy} + R_{xz}T_{zz}), \ldots \end{aligned}$$

or, more compactly,

$$\overline{T}_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}. \tag{1.32}$$

[3]If you're a mathematician you might want to contemplate generalized vector spaces in which the "axes" have nothing to do with direction and the basis vectors are no longer $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ (indeed, there may be more than three dimensions). This is the subject of **linear algebra**. But for our purposes all vectors live in ordinary 3-space (or, in Chapter 12, in 4-dimensional space-time.)

In general, an nth-rank tensor has n indices and 3^n components, and transforms with n factors of R. In this hierarchy, a vector is a tensor of rank 1, and a scalar is a tensor of rank zero.[4]

Problem 1.8

(a) Prove that the two-dimensional rotation matrix (Eq. 1.29) preserves dot products. (That is, show that $\overline{A}_y\overline{B}_y + \overline{A}_z\overline{B}_z = A_yB_y + A_zB_z$.)

(b) What constraints must the elements (R_{ij}) of the three-dimensional rotation matrix (Eq. 1.30) satisfy, in order to preserve the length of **A** (for all vectors **A**)?

Problem 1.9 Find the transformation matrix R that describes a rotation by 120° about an axis from the origin through the point (1, 1, 1). The rotation is clockwise as you look down the axis toward the origin.

Problem 1.10

(a) How do the components of a vector[5] transform under a **translation** of coordinates ($\overline{x} = x, \overline{y} = y - a, \overline{z} = z$, Fig. 1.16a)?

(b) How do the components of a vector transform under an **inversion** of coordinates ($\overline{x} = -x, \overline{y} = -y, \overline{z} = -z$, Fig. 1.16b)?

(c) How do the components of a cross product (Eq. 1.13) transform under inversion? [The cross-product of two vectors is properly called a **pseudovector** because of this "anomalous" behavior.] Is the cross product of two pseudovectors a vector, or a pseudovector? Name two pseudovector quantities in classical mechanics.

(d) How does the scalar triple product of three vectors transform under inversions? (Such an object is called a **pseudoscalar.**)

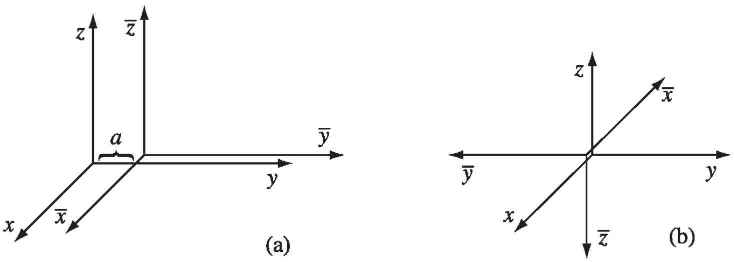

FIGURE 1.16

[4]A scalar does not change when you change coordinates. In particular, the components of a vector are *not* scalars, but the magnitude is.

[5]*Beware:* The vector **r** (Eq. 1.19) goes from a specific point in space (the origin, $\mathcal{O}$) to the point $P = (x, y, z)$. Under translations the *new* origin ($\bar{\mathcal{O}}$) is at a different location, and the arrow from $\bar{\mathcal{O}}$ to P is a completely different vector. The original vector **r** still goes from $\mathcal{O}$ to P, regardless of the coordinates used to label these points.

1.2 ■ DIFFERENTIAL CALCULUS

1.2.1 ■ "Ordinary" Derivatives

Suppose we have a function of one variable: $f(x)$. *Question:* What does the derivative, df/dx, do for us? *Answer:* It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

$$df = \left(\frac{df}{dx}\right) dx. \tag{1.33}$$

In words: If we increment x by an infinitesimal amount dx, then f changes by an amount df; the derivative is the proportionality factor. For example, in Fig. 1.17(a), the function varies slowly with x, and the derivative is correspondingly small. In Fig. 1.17(b), f increases rapidly with x, and the derivative is large, as you move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the *slope* of the graph of f versus x.

1.2.2 ■ Gradient

Suppose, now, that we have a function of *three* variables—say, the temperature $T(x, y, z)$ in this room. (Start out in one corner, and set up a system of axes; then for each point (x, y, z) in the room, T gives the temperature at that spot.) We want to generalize the notion of "derivative" to functions like T, which depend not on *one* but on *three* variables.

A derivative is supposed to tell us how fast the function varies, if we move a little distance. But this time the situation is more complicated, because it depends on what *direction* we move: If we go straight up, then the temperature will probably increase fairly rapidly, but if we move horizontally, it may not change much at all. In fact, the question "How fast does T vary?" has an infinite number of answers, one for each direction we might choose to explore.

Fortunately, the problem is not as bad as it looks. A theorem on partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz. \tag{1.34}$$

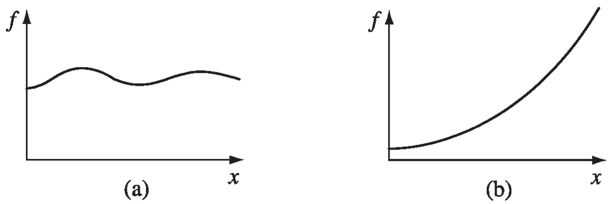

FIGURE 1.17

This tells us how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. Notice that we do *not* require an infinite number of derivatives—*three* will suffice: the *partial* derivatives along each of the three coordinate directions.

Equation 1.34 is reminiscent of a dot product:

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned} \tag{1.35}$$

where

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{1.36}$$

is the **gradient** of T. Note that ∇T is a *vector* quantity, with three components; it is the generalized derivative we have been looking for. Equation 1.35 is the three-dimensional version of Eq. 1.33.

Geometrical Interpretation of the Gradient: Like any vector, the gradient has *magnitude* and *direction.* To determine its geometrical meaning, let's rewrite the dot product (Eq. 1.35) using Eq. 1.1:

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}|\cos\theta, \tag{1.37}$$

where θ is the angle between ∇T and $d\mathbf{l}$. Now, if we *fix* the *magnitude* $|d\mathbf{l}|$ and search around in various *directions* (that is, vary θ), the *maximum* change in T evidentally occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $|d\mathbf{l}|$, dT is greatest when I move in the *same direction* as ∇T. Thus:

> *The gradient ∇T points in the direction of maximum increase of the function T.*

Moreover:

> *The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.*

Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent. That is the *direction* of the gradient. Now measure the *slope* in that direction (rise over run). That is the *magnitude* of the gradient. (Here the function we're talking about is the height of the hill, and the coordinates it depends on are positions—latitude and longitude, say. This function depends on only *two* variables, not *three*, but the geometrical meaning of the gradient is easier to grasp in two dimensions.) Notice from Eq. 1.37 that the direction of maximum *descent* is opposite to the direction of maximum *ascent*, while at right angles ($\theta = 90°$) the slope is zero (the gradient is perpendicular to the contour lines). You can conceive of surfaces that do not have these properties, but they always have "kinks" in them, and correspond to nondifferentiable functions.

What would it mean for the gradient to vanish? If $\nabla T = \mathbf{0}$ at (x, y, z), then $dT = 0$ for small displacements about the point (x, y, z). This is, then, a **stationary point** of the function $T(x, y, z)$. It could be a maximum (a summit), a minimum (a valley), a saddle point (a pass), or a "shoulder." This is analogous to the situation for functions of *one* variable, where a vanishing derivative signals a maximum, a minimum, or an inflection. In particular, if you want to locate the extrema of a function of three variables, set its gradient equal to zero.

Example 1.3. Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

Solution

$$\begin{aligned}
\nabla r &= \frac{\partial r}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\,\hat{\mathbf{z}} \\
&= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{x}} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{y}} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{z}} \\
&= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.
\end{aligned}$$

Does this make sense? Well, it says that the distance from the origin increases most rapidly in the radial direction, and that its *rate* of increase in that direction is 1...just what you'd expect.

Problem 1.11 Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

• **Problem 1.13** Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

! **Problem 1.14** Suppose that f is a function of two variables (y and z) only. Show that the gradient $\nabla f = (\partial f/\partial y)\hat{\mathbf{y}} + (\partial f/\partial z)\hat{\mathbf{z}}$ transforms as a vector under rotations, Eq. 1.29. [*Hint:* $(\partial f/\partial \overline{y}) = (\partial f/\partial y)(\partial y/\partial \overline{y}) + (\partial f/\partial z)(\partial z/\partial \overline{y})$, and the analogous formula for $\partial f/\partial \overline{z}$. We know that $\overline{y} = y\cos\phi + z\sin\phi$ and $\overline{z} = -y\sin\phi + z\cos\phi$; "solve" these equations for y and z (as functions of $\overline{y}$ and $\overline{z}$), and compute the needed derivatives $\partial y/\partial \overline{y}$, $\partial z/\partial \overline{y}$, etc.]

1.2.3 ■ The Del Operator

The gradient has the formal appearance of a vector, ∇, "multiplying" a scalar T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) T. \tag{1.38}$$

(For once, I write the unit vectors to the *left*, just so no one will think this means $\partial\hat{\mathbf{x}}/\partial x$, and so on—which would be zero, since $\hat{\mathbf{x}}$ is constant.) The term in parentheses is called **del**:

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.} \tag{1.39}$$

Of course, del is *not* a vector, in the usual sense. Indeed, it doesn't mean much until we provide it with a function to act upon. Furthermore, it does not "multiply" T; rather, it is an instruction to *differentiate* what follows. To be precise, then, we say that ∇ is a **vector operator** that *acts upon* T, not a vector that multiplies T.

With this qualification, though, ∇ mimics the behavior of an ordinary vector in virtually every way; almost anything that can be done with other vectors can also be done with ∇, if we merely translate "multiply" by "act upon." So by all means take the vector appearance of ∇ seriously: it is a marvelous piece of notational simplification, as you will appreciate if you ever consult Maxwell's original work on electromagnetism, written without the benefit of ∇.

Now, an ordinary vector $\mathbf{A}$ can multiply in three ways:

1. By a scalar a : $\mathbf{A}a$;
2. By a vector $\mathbf{B}$, via the dot product: $\mathbf{A}\cdot\mathbf{B}$;
3. By a vector $\mathbf{B}$ via the cross product: $\mathbf{A}\times\mathbf{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla\cdot\mathbf{v}$ (the **divergence**);
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla\times\mathbf{v}$ (the **curl**).

We have already discussed the gradient. In the following sections we examine the other two vector derivatives: divergence and curl.

1.2.4 ■ The Divergence

From the definition of ∇ we construct the divergence:

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{1.40}$$

Observe that the divergence of a vector function[6] $\mathbf{v}$ is itself a *scalar* $\nabla \cdot \mathbf{v}$.

Geometrical Interpretation: The name **divergence** is well chosen, for $\nabla \cdot \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ spreads out (diverges) from the point in question. For example, the vector function in Fig. 1.18a has a large (positive) divergence (if the arrows pointed *in*, it would be a *negative* divergence), the function in Fig. 1.18b has zero divergence, and the function in Fig. 1.18c again has a positive divergence. (Please understand that $\mathbf{v}$ here is a *function*—there's a different vector associated with every point in space. In the diagrams, of course, I can only draw the arrows at a few representative locations.)

Imagine standing at the edge of a pond. Sprinkle some sawdust or pine needles on the surface. If the material spreads out, then you dropped it at a point of positive divergence; if it collects together, you dropped it at a point of negative divergence. (The vector function $\mathbf{v}$ in this model is the velocity of the water at the surface—this is a *two*-dimensional example, but it helps give one a "feel" for what the divergence means. A point of positive divergence is a source, or "faucet"; a point of negative divergence is a sink, or "drain.")

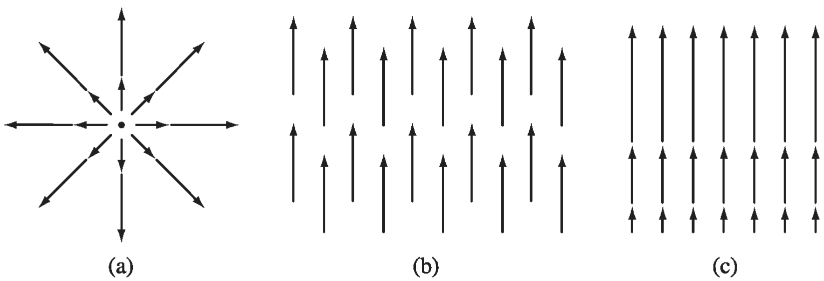

FIGURE 1.18

[6]A vector function $\mathbf{v}(x, y, z) = v_x(x, y, z)\,\hat{\mathbf{x}} + v_y(x, y, z)\,\hat{\mathbf{y}} + v_z(x, y, z)\,\hat{\mathbf{z}}$ is really *three* functions—one for each component. There's no such thing as the divergence of a scalar.

Example 1.4. Suppose the functions in Fig. 1.18 are $\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{v}_b = \hat{\mathbf{z}}$, and $\mathbf{v}_c = z\,\hat{\mathbf{z}}$. Calculate their divergences.

Solution

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

As anticipated, this function has a positive divergence.

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0,$$

as expected.

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1.$$

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

• **Problem 1.16** Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you... can you explain it?

! **Problem 1.17** In two dimensions, show that the divergence transforms as a scalar under rotations. [*Hint:* Use Eq. 1.29 to determine $\bar{v}_y$ and $\bar{v}_z$, and the method of Prob. 1.14 to calculate the derivatives. Your aim is to show that $\partial\bar{v}_y/\partial\bar{y} + \partial\bar{v}_z/\partial\bar{z} = \partial v_y/\partial y + \partial v_z/\partial z$.]

1.2.5 ■ The Curl

From the definition of ∇ we construct the curl:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \quad (1.41)$$

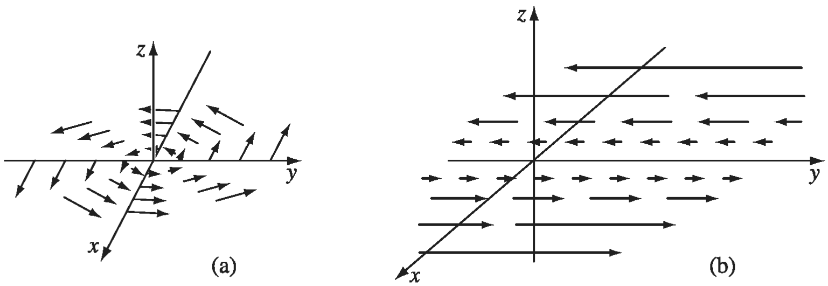

FIGURE 1.19

Notice that the curl of a vector function[7] $\mathbf{v}$ is, like any cross product, a *vector*.

Geometrical Interpretation: The name **curl** is also well chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ swirls around the point in question. Thus the three functions in Fig. 1.18 all have zero curl (as you can easily check for yourself), whereas the functions in Fig. 1.19 have a substantial curl, pointing in the z direction, as the natural right-hand rule would suggest. Imagine (again) you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero *curl.* A whirlpool would be a region of large curl.

Example 1.5. Suppose the function sketched in Fig. 1.19a is $\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, and that in Fig. 1.19b is $\mathbf{v}_b = x\hat{\mathbf{y}}$. Calculate their curls.

Solution

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{z}},$$

and

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{\mathbf{z}}.$$

As expected, these curls point in the $+z$ direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out"... it just "swirls around.")

---

[7] There's no such thing as the curl of a scalar.

Problem 1.18 Calculate the curls of the vector functions in Prob. 1.15.

Problem 1.19 Draw a circle in the xy plane. At a few representative points draw the vector $\mathbf{v}$ tangent to the circle, pointing in the clockwise direction. By comparing adjacent vectors, determine the *sign* of $\partial v_x/\partial y$ and $\partial v_y/\partial x$. According to Eq. 1.41, then, what is the direction of $\nabla \times \mathbf{v}$? Explain how this example illustrates the geometrical interpretation of the curl.

Problem 1.20 Construct a vector function that has zero divergence and zero curl everywhere. (A *constant* will do the job, of course, but make it something a little more interesting than that!)

1.2.6 ■ Product Rules

The calculation of ordinary derivatives is facilitated by a number of rules, such as the sum rule:

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx},$$

the rule for multiplying by a constant:

$$\frac{d}{dx}(kf) = k\frac{df}{dx},$$

the product rule:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx},$$

and the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}.$$

Similar relations hold for the vector derivatives. Thus,

$$\nabla(f+g) = \nabla f + \nabla g, \qquad \nabla \cdot (\mathbf{A}+\mathbf{B}) = (\nabla \cdot \mathbf{A}) + (\nabla \cdot \mathbf{B}),$$

$$\nabla \times (\mathbf{A}+\mathbf{B}) = (\nabla \times \mathbf{A}) + (\nabla \times \mathbf{B}),$$

and

$$\nabla(kf) = k\nabla f, \qquad \nabla \cdot (k\mathbf{A}) = k(\nabla \cdot \mathbf{A}), \qquad \nabla \times (k\mathbf{A}) = k(\nabla \times \mathbf{A}),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

$$
\begin{aligned}
fg \quad & \text{(product of two scalar functions)},\\
\mathbf{A}\cdot\mathbf{B} \quad & \text{(dot product of two vector functions)},
\end{aligned}
$$

and two ways to make a vector:

$$\begin{aligned} f\mathbf{A} \quad &\text{(scalar times vector)},\\ \mathbf{A}\times\mathbf{B} \quad &\text{(cross product of two vectors)}. \end{aligned}$$

Accordingly, there are *six* product rules, two for gradients:

(i) $$\nabla(fg) = f\nabla g + g\nabla f,$$

(ii) $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A},$$

two for divergences:

(iii) $$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f),$$

(iv) $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}),$$

and two for curls:

(v) $$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f),$$

(vi) $$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}).$$

You will be using these product rules so frequently that I have put them inside the front cover for easy reference. The proofs come straight from the product rule for ordinary derivatives. For instance,

$$\begin{aligned} \nabla\cdot(f\mathbf{A}) &= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)\\ &= \left(\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z}\right)\\ &= (\nabla f)\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A}). \end{aligned}$$

It is also possible to formulate three quotient rules:

$$\begin{aligned} \nabla\left(\frac{f}{g}\right) &= \frac{g\nabla f - f\nabla g}{g^2},\\ \nabla\cdot\left(\frac{\mathbf{A}}{g}\right) &= \frac{g(\nabla\cdot\mathbf{A}) - \mathbf{A}\cdot(\nabla g)}{g^2},\\ \nabla\times\left(\frac{\mathbf{A}}{g}\right) &= \frac{g(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla g)}{g^2}. \end{aligned}$$

However, since these can be obtained quickly from the corresponding product rules, there is no point in listing them separately.

Problem 1.21 Prove product rules (i), (iv), and (v).

Problem 1.22

(a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, what does the expression $(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$ mean? (That is, what are its x, y, and z components, in terms of the Cartesian components of $\mathbf{A}$, $\mathbf{B}$, and $\boldsymbol{\nabla}$?)

(b) Compute $(\hat{\mathbf{r}} \cdot \boldsymbol{\nabla})\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector defined in Eq. 1.21.

(c) For the functions in Prob. 1.15, evaluate $(\mathbf{v}_a \cdot \boldsymbol{\nabla})\mathbf{v}_b$.

Problem 1.23 (For masochists only.) Prove product rules (ii) and (vi). Refer to Prob. 1.22 for the definition of $(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$.

Problem 1.24 Derive the three quotient rules.

Problem 1.25

(a) Check product rule (iv) (by calculating each term separately) for the functions

$$\mathbf{A} = x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 3y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}}.$$

(b) Do the same for product rule (ii).

(c) Do the same for rule (vi).

1.2.7 ■ Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\boldsymbol{\nabla}$; by applying $\boldsymbol{\nabla}$ *twice*, we can construct five species of *second* derivatives. The gradient $\boldsymbol{\nabla}T$ is a *vector*, so we can take the *divergence* and *curl* of it:

(1) Divergence of gradient: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}T)$.

(2) Curl of gradient: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}T)$.

The divergence $\boldsymbol{\nabla} \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient:*

(3) Gradient of divergence: $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v})$.

The curl $\boldsymbol{\nabla} \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl:*

(4) Divergence of curl: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v})$.

(5) Curl of curl: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v})$.

This exhausts the possibilities, and in fact not all of them give anything new. Let's consider them one at a time:

(1)
$$\begin{aligned}\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}T) &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \\ &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}. \qquad (1.42)\end{aligned}$$

This object, which we write as $\nabla^2 T$ for short, is called the **Laplacian** of T; we shall be studying it in great detail later on. Notice that the Laplacian of a *scalar* T is a *scalar*. Occasionally, we shall speak of the Laplacian of a *vector,* $\nabla^2\mathbf{v}$. By this we mean a *vector* quantity whose x-component is the Laplacian of v_x, and so on:[8]

$$\nabla^2\mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}. \tag{1.43}$$

This is nothing more than a convenient extension of the meaning of ∇^2.

(2) The curl of a gradient is always zero:

$$\nabla \times (\nabla T) = \mathbf{0}. \tag{1.44}$$

This is an important fact, which we shall use repeatedly; you can easily prove it from the definition of ∇, Eq. 1.39. *Beware*: You might think Eq. 1.44 is "obviously" true—isn't it just $(\nabla \times \nabla)T$, and isn't the cross product of *any* vector (in this case, ∇) with itself always zero? This reasoning is suggestive, but not quite conclusive, since ∇ is an *operator* and does not "multiply" in the usual way. The proof of Eq. 1.44, in fact, hinges on the equality of cross derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right). \tag{1.45}$$

If you think I'm being fussy, test your intuition on this one:

$$(\nabla T) \times (\nabla S).$$

Is *that* always zero? (It *would* be, of course, if you replaced the ∇'s by an ordinary vector.)

(3) $\nabla(\nabla \cdot \mathbf{v})$ seldom occurs in physical applications, and it has not been given any special name of its own—it's just **the gradient of the divergence.** Notice that $\nabla(\nabla \cdot \mathbf{v})$ is *not* the same as the Laplacian of a vector: $\nabla^2\mathbf{v} = (\nabla \cdot \nabla)\mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$.

(4) The divergence of a curl, like the curl of a gradient, is always zero:

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0. \tag{1.46}$$

You can prove this for yourself. (Again, there is a fraudulent short-cut proof, using the vector identity $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.)

(5) As you can check from the definition of ∇:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2\mathbf{v}. \tag{1.47}$$

So curl-of-curl gives nothing new; the first term is just number (3), and the second is the Laplacian (of a vector). (In fact, Eq. 1.47 is often used to *define* the

[8] In curvilinear coordinates, where the unit vectors themselves depend on position, they too must be differentiated (see Sect. 1.4.1).

Laplacian of a vector, in preference to Eq. 1.43, which makes explicit reference to Cartesian coordinates.)

Really, then, there are just two kinds of second derivatives: the Laplacian (which is of fundamental importance) and the gradient-of-divergence (which we seldom encounter). We could go through a similar ritual to work out *third* derivatives, but fortunately second derivatives suffice for practically all physical applications.

A final word on vector differential calculus: It *all* flows from the operator ∇, and from taking seriously its vectorial character. Even if you remembered *only* the definition of ∇, you could easily reconstruct all the rest.

Problem 1.26 Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$.

(b) $T_b = \sin x \sin y \sin z$.

(c) $T_c = e^{-5x} \sin 4y \cos 3z$.

(d) $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

Problem 1.27 Prove that the divergence of a curl is always zero. *Check* it for function $\mathbf{v}_a$ in Prob. 1.15.

Problem 1.28 Prove that the curl of a gradient is always zero. *Check* it for function (b) in Prob. 1.11.

1.3 ■ INTEGRAL CALCULUS

1.3.1 ■ Line, Surface, and Volume Integrals

In electrodynamics, we encounter several different kinds of integrals, among which the most important are **line** (or **path**) **integrals**, **surface integrals** (or **flux**), and **volume integrals**.

(a) **Line Integrals.** A line integral is an expression of the form

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}, \tag{1.48}$$

where $\mathbf{v}$ is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector (Eq. 1.22), and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point $\mathbf{a}$ to point $\mathbf{b}$ (Fig. 1.20). If the path in question forms a closed loop (that is, if $\mathbf{b} = \mathbf{a}$), I shall put a circle on the integral sign:

$$\oint \mathbf{v} \cdot d\mathbf{l}. \tag{1.49}$$

At each point on the path, we take the dot product of $\mathbf{v}$ (evaluated at that point) with the displacement $d\mathbf{l}$ to the next point on the path. To a physicist, the most familiar example of a line integral is the work done by a force $\mathbf{F}$: $W = \int \mathbf{F} \cdot d\mathbf{l}$.

Ordinarily, the value of a line integral depends critically on the path taken from $\mathbf{a}$ to $\mathbf{b}$, but there is an important special class of vector functions for which the line

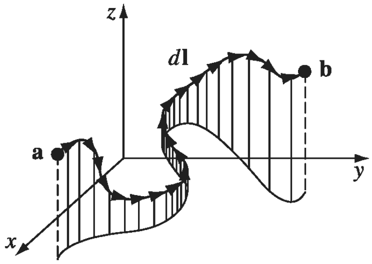

FIGURE 1.20

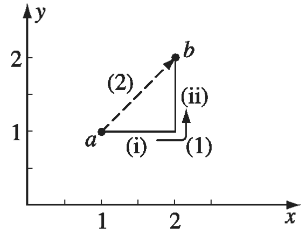

FIGURE 1.21

integral is *independent* of path and is determined entirely by the end points. It will be our business in due course to characterize this special class of vectors. (A *force* that has this property is called **conservative**.)

Example 1.6. Calculate the line integral of the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + 2x(y+1)\,\hat{\mathbf{y}}$ from the point $\mathbf{a} = (1, 1, 0)$ to the point $\mathbf{b} = (2, 2, 0)$, along the paths (1) and (2) in Fig. 1.21. What is $\oint \mathbf{v}\cdot d\mathbf{l}$ for the loop that goes from **a** to **b** along (1) and returns to **a** along (2)?

Solution

As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$. Path (1) consists of two parts. Along the "horizontal" segment, $dy = dz = 0$, so

(i) $d\mathbf{l} = dx\,\hat{\mathbf{x}},\ y = 1,\ \mathbf{v}\cdot d\mathbf{l} = y^2\,dx = dx$, so $\int \mathbf{v}\cdot d\mathbf{l} = \int_1^2 dx = 1$.

On the "vertical" stretch, $dx = dz = 0$, so

(ii) $d\mathbf{l} = dy\,\hat{\mathbf{y}},\ x = 2,\ \mathbf{v}\cdot d\mathbf{l} = 2x(y+1)\,dy = 4(y+1)\,dy$, so

$$\int \mathbf{v}\cdot d\mathbf{l} = 4\int_1^2 (y+1)\,dy = 10.$$

By path (1), then,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v}\cdot d\mathbf{l} = 1 + 10 = 11.$$

Meanwhile, on path (2) $x = y,\ dx = dy$, and $dz = 0$, so
$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dx\,\hat{\mathbf{y}},\ \mathbf{v}\cdot d\mathbf{l} = x^2\,dx + 2x(x+1)\,dx = (3x^2 + 2x)\,dx$,
and

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v}\cdot d\mathbf{l} = \int_1^2 (3x^2 + 2x)\,dx = (x^3 + x^2)\Big|_1^2 = 10.$$

(The strategy here is to get everything in terms of one variable; I could just as well have eliminated x in favor of y.)

For the loop that goes *out* (1) and *back* (2), then,

$$\oint \mathbf{v} \cdot d\mathbf{l} = 11 - 10 = 1.$$

(b) **Surface Integrals.** A surface integral is an expression of the form

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}, \tag{1.50}$$

where $\mathbf{v}$ is again some vector function, and the integral is over a specified surface $\mathcal{S}$. Here $d\mathbf{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface (Fig. 1.22). There are, of course, *two* directions perpendicular to any surface, so the *sign* of a surface integral is intrinsically ambiguous. If the surface is *closed* (forming a "balloon"), in which case I shall again put a circle on the integral sign

$$\oint \mathbf{v} \cdot d\mathbf{a},$$

then tradition dictates that "outward" is positive, but for open surfaces it's arbitrary. If $\mathbf{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \mathbf{v} \cdot d\mathbf{a}$ represents the total mass per unit time passing through the surface—hence the alternative name, "flux."

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is *independent* of the surface and is determined entirely by the boundary line. An important task will be to characterize this special class of functions.

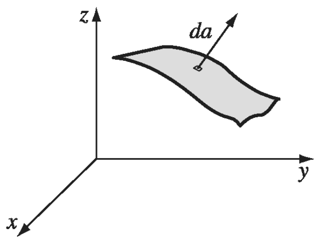

FIGURE 1.22

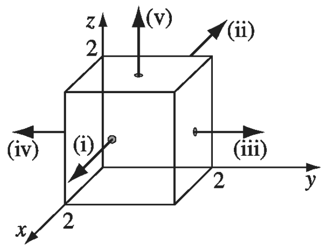

FIGURE 1.23

Example 1.7. Calculate the surface integral of $\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (x+2)\,\hat{\mathbf{y}} + y(z^2-3)\,\hat{\mathbf{z}}$ over five sides (excluding the bottom) of the cubical box (side 2) in Fig. 1.23. Let "upward and outward" be the positive direction, as indicated by the arrows.

Solution

Taking the sides one at a time:

(i) $x = 2,\ d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}},\ \mathbf{v}\cdot d\mathbf{a} = 2xz\,dy\,dz = 4z\,dy\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = 4\int_0^2 dy \int_0^2 z\,dz = 16.$$

(ii) $x = 0,\ d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}},\ \mathbf{v}\cdot d\mathbf{a} = -2xz\,dy\,dz = 0$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = 0.$$

(iii) $y = 2,\ d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}},\ \mathbf{v}\cdot d\mathbf{a} = (x+2)\,dx\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^2 (x+2)\,dx \int_0^2 dz = 12.$$

(iv) $y = 0,\ d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}},\ \mathbf{v}\cdot d\mathbf{a} = -(x+2)\,dx\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^2 (x+2)\,dx \int_0^2 dz = -12.$$

(v) $z = 2,\ d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}},\ \mathbf{v}\cdot d\mathbf{a} = y(z^2-3)\,dx\,dy = y\,dx\,dy$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^2 dx \int_0^2 y\,dy = 4.$$

The *total* flux is

$$\int_{\text{surface}} \mathbf{v}\cdot d\mathbf{a} = 16 + 0 + 12 - 12 + 4 = 20.$$

(c) **Volume Integrals.** A volume integral is an expression of the form

$$\int_{\mathcal{V}} T\,d\tau, \tag{1.51}$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$d\tau = dx\,dy\,dz. \tag{1.52}$$

For example, if T is the density of a substance (which might vary from point to point), then the volume integral would give the total mass. Occasionally we shall encounter volume integrals of *vector* functions:

$$\int \mathbf{v}\,d\tau = \int (v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}})d\tau = \hat{\mathbf{x}}\int v_x d\tau + \hat{\mathbf{y}}\int v_y d\tau + \hat{\mathbf{z}}\int v_z d\tau; \tag{1.53}$$

because the unit vectors ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$) are constants, they come outside the integral.

Example 1.8. Calculate the volume integral of $T = xyz^2$ over the prism in Fig. 1.24.

Solution

You can do the three integrals in any order. Let's do x first: it runs from 0 to $(1-y)$, then y (it goes from 0 to 1), and finally z (0 to 3):

$$\int T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\,dx \right] dy \right\} dz$$

$$= \frac{1}{2}\int_0^3 z^2\,dz \int_0^1 (1-y)^2 y\,dy = \frac{1}{2}(9)\left(\frac{1}{12}\right) = \frac{3}{8}.$$

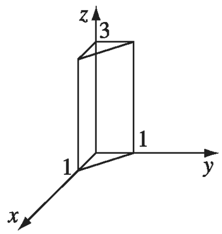

FIGURE 1.24

Problem 1.29 Calculate the line integral of the function $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + y^2\,\hat{\mathbf{z}}$ from the origin to the point (1,1,1) by three different routes:

(a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$.

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$.

(c) The direct straight line.

(d) What is the line integral around the closed loop that goes *out* along path (a) and *back* along path (b)?

Problem 1.30 Calculate the surface integral of the function in Ex. 1.7, over the *bottom* of the box. For consistency, let "upward" be the positive direction. Does the surface integral depend only on the boundary line for this function? What is the total flux over the *closed* surface of the box (*including* the bottom)? [*Note:* For the *closed* surface, the positive direction is "outward," and hence "down," for the bottom face.]

Problem 1.31 Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at (0,0,0), (1,0,0), (0,1,0), and (0,0,1).

1.3.2 ■ The Fundamental Theorem of Calculus

Suppose $f(x)$ is a function of one variable. The **fundamental theorem of calculus** says:

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a). \tag{1.54}$$

In case this doesn't look familiar, I'll write it another way:

$$\int_a^b F(x)\, dx = f(b) - f(a),$$

where $df/dx = F(x)$. The fundamental theorem tells you how to integrate $F(x)$: you think up a function $f(x)$ whose *derivative* is equal to F.

Geometrical Interpretation: According to Eq. 1.33, $df = (df/dx)dx$ is the infinitesimal change in f when you go from (x) to $(x + dx)$. The fundamental theorem (Eq. 1.54) says that if you chop the interval from a to b (Fig. 1.25) into many tiny pieces, dx, and add up the increments df from each little piece, the result is (not surprisingly) equal to the total change in f: $f(b) - f(a)$. In other words, there are two ways to determine the total change in the function: *either* subtract the values at the ends *or* go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

Notice the basic format of the fundamental theorem: the *integral* of a *derivative* over some *region* is given by the *value of the function* at the end points *(boundaries)*. In vector calculus there are three species of derivative (gradient, divergence, and curl), and each has its own "fundamental theorem," with essentially the same format. I don't plan to prove these theorems here; rather, I will explain what they *mean*, and try to make them *plausible*. Proofs are given in Appendix A.

1.3.3 ■ The Fundamental Theorem for Gradients

Suppose we have a scalar function of three variables $T(x, y, z)$. Starting at point $\mathbf{a}$, we move a small distance $d\mathbf{l}_1$ (Fig. 1.26). According to Eq. 1.37, the function T will change by an amount

$$dT = (\nabla T) \cdot d\mathbf{l}_1.$$

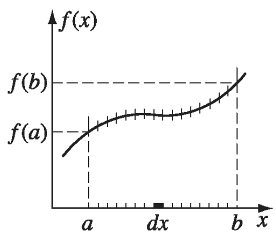

FIGURE 1.25

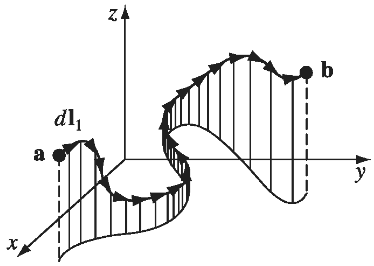

FIGURE 1.26

Now we move a little further, by an additional small displacement $d\mathbf{l}_2$; the incremental change in T will be $(\nabla T) \cdot d\mathbf{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point $\mathbf{b}$. At each step we compute the gradient of T (at that point) and dot it into the displacement $d\mathbf{l}$... this gives us the change in T. Evidently the *total* change in T in going from $\mathbf{a}$ to $\mathbf{b}$ (along the path selected) is

$$\boxed{\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).} \tag{1.55}$$

This is the **fundamental theorem for gradients**; like the "ordinary" fundamental theorem, it says that the integral (here a *line* integral) of a derivative (here the *gradient*) is given by the value of the function at the boundaries ($\mathbf{a}$ and $\mathbf{b}$).

Geometrical Interpretation: Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up (that's the left side of Eq. 1.55), or you could place altimeters at the top and the bottom, and subtract the two readings (that's the right side); you should get the same answer either way (that's the fundamental theorem).

Incidentally, as we found in Ex. 1.6, line integrals ordinarily depend on the *path* taken from $\mathbf{a}$ to $\mathbf{b}$. But the *right* side of Eq. 1.55 makes no reference to the path—only to the end points. Evidently, *gradients* have the special property that their line integrals are path independent:

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of the path taken from $\mathbf{a}$ to $\mathbf{b}$.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

Example 1.9. Let $T = xy^2$, and take point $\mathbf{a}$ to be the origin $(0, 0, 0)$ and $\mathbf{b}$ the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

Solution

Although the integral is independent of path, we must *pick* a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii) (Fig. 1.27). As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\nabla T = y^2\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}$.

(i) $y = 0$; $d\mathbf{l} = dx\,\hat{\mathbf{x}}$, $\nabla T \cdot d\mathbf{l} = y^2\,dx = 0$, so

$$\int_{\mathrm{i}} \nabla T \cdot d\mathbf{l} = 0.$$

(ii) $x = 2$; $d\mathbf{l} = dy\,\hat{\mathbf{y}}$, $\nabla T \cdot d\mathbf{l} = 2xy\,dy = 4y\,dy$, so

$$\int_{\mathrm{ii}} \nabla T \cdot d\mathbf{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2.$$

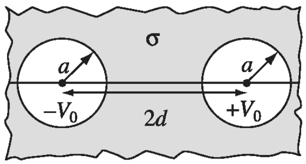

FIGURE 7.50

! **Problem 7.42** A rare case in which the electrostatic field **E** for a circuit can actually be *calculated* is the following:[28] Imagine an infinitely long cylindrical sheet, of uniform resistivity and radius a. A slot (corresponding to the battery) is maintained at $\pm V_0/2$, at $\phi = \pm\pi$, and a steady current flows over the surface, as indicated in Fig. 7.51. According to Ohm's law, then,

$$V(a, \phi) = \frac{V_0\phi}{2\pi}, \quad (-\pi < \phi < +\pi).$$

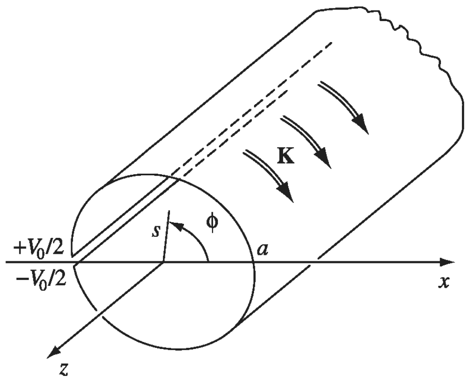

FIGURE 7.51

(a) Use separation of variables in cylindrical coordinates to determine $V(s, \phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a + s\cos\phi)]$, $(s < a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s + a\cos\phi)]$, $(s > a)$]

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

Problem 7.43 The magnetic field outside a long straight wire carrying a steady current I is

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}.$$

The *electric field inside* the wire is uniform:

$$\mathbf{E} = \frac{I\rho}{\pi a^2}\,\hat{\mathbf{z}},$$

[28] M. A. Heald, *Am. J. Phys.* **52**, 522 (1984). See also J. A. Hernandes and A. K. T. Assis, *Phys. Rev. E* **68**, 046611 (2003).

where ρ is the resistivity and a is the radius (see Exs. 7.1 and 7.3). *Question:* What is the electric field *outside* the wire?[29] The answer depends on how you complete the circuit. Suppose the current returns along a perfectly conducting grounded coaxial cylinder of radius b (Fig. 7.52). In the region $a < s < b$, the potential $V(s, z)$ satisfies Laplace's equation, with the boundary conditions

$$\text{(i)}\ \ V(a, z) = -\frac{I\rho z}{\pi a^2}; \quad \text{(ii)}\ \ V(b, z) = 0.$$

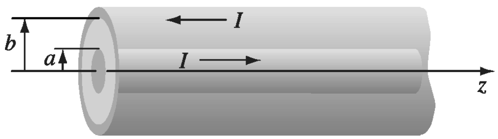

FIGURE 7.52

This does not suffice to determine the answer—we still need to specify boundary conditions at the two ends (though for a *long* wire it shouldn't matter much). In the literature, it is customary to sweep this ambiguity under the rug by simply *stipulating* that $V(s, z)$ is proportional to z: $V(s, z) = zf(s)$. On this assumption:

(a) Determine $f(s)$.

(b) Find $\mathbf{E}(s, z)$.

(c) Calculate the surface charge density $\sigma(z)$ on the wire.

[*Answer:* $V = (-Iz\rho/\pi a^2)[\ln(s/b)/\ln(a/b)]$ This is a peculiar result, since E_s and $\sigma(z)$ are *not* independent of z—as one would certainly expect for a truly *infinite* wire.]

Problem 7.44 In a **perfect conductor**, the conductivity is infinite, so $\mathbf{E} = \mathbf{0}$ (Eq. 7.3), and any net charge resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*).

(a) Show that the magnetic field is constant ($\partial\mathbf{B}/\partial t = \mathbf{0}$), inside a perfect conductor.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

A **superconductor** is a perfect conductor with the additional property that the (constant) $\mathbf{B}$ inside is in fact *zero*. (This "flux exclusion" is known as the **Meissner effect**.[30])

[29]This is a famous problem, first analyzed by Sommerfeld, and is known in its most recent incarnation as **Merzbacher's puzzle**. A. Sommerfeld, *Electrodynamics*, p. 125 (New York: Academic Press, 1952); E. Merzbacher, *Am. J. Phys.* **48**, 178 (1980); further references in R. N. Varnay and L. H. Fisher, *Am. J. Phys.* **52**, 1097 (1984).

[30]The Meissner effect is sometimes referred to as "perfect diamagnetism," in the sense that the field inside is not merely *reduced*, but canceled entirely. However, the surface currents responsible for this are *free*, not bound, so the actual *mechanism* is quite different.

(c) Show that the current in a superconductor is confined to the surface.

(d) Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{\mathbf{z}}$ while cooling it below T_c. Find the induced surface current density $\mathbf{K}$, as a function of the polar angle θ.

Problem 7.45 A familiar demonstration of superconductivity (Prob. 7.44) is the levitation of a magnet over a piece of superconducting material. This phenomenon can be analyzed using the method of images.[31] Treat the magnet as a perfect dipole $\mathbf{m}$, a height z above the origin (and constrained to point in the z direction), and pretend that the superconductor occupies the entire half-space below the xy plane. Because of the Meissner effect, $\mathbf{B} = \mathbf{0}$ for $z \leq 0$, and since $\mathbf{B}$ is divergenceless, the normal (z) component is continuous, so $B_z = 0$ just *above* the surface. This boundary condition is met by the image configuration in which an identical dipole is placed at $-z$, as a stand-in for the superconductor; the two arrangements therefore produce the same magnetic field in the region $z > 0$.

(a) Which way should the image dipole point ($+z$ or $-z$)?

(b) Find the force on the magnet due to the induced currents in the superconductor (which is to say, the force due to the image dipole). Set it equal to Mg (where M is the mass of the magnet) to determine the height h at which the magnet will "float." [*Hint:* Refer to Prob. 6.3.]

(c) The induced current on the surface of the superconductor (the xy plane) can be determined from the boundary condition on the *tangential* component of $\mathbf{B}$ (Eq. 5.76): $\mathbf{B} = \mu_0(\mathbf{K} \times \hat{\mathbf{z}})$. Using the field you get from the image configuration, show that

$$\mathbf{K} = -\frac{3mrh}{2\pi(r^2 + h^2)^{5/2}}\,\hat{\boldsymbol{\phi}},$$

where r is the distance from the origin.

! **Problem 7.46** If a magnetic dipole levitating above an infinite superconducting plane (Prob. 7.45) is free to rotate, what orientation will it adopt, and how high above the surface will it float?

Problem 7.47 A perfectly conducting spherical shell of radius a rotates about the z axis with angular velocity ω, in a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. Calculate the emf developed between the "north pole" and the equator. [*Answer:* $\frac{1}{2}B_0\omega a^2$]

! **Problem 7.48** Refer to Prob. 7.11 (and use the result of Prob. 5.42): How long does is take a falling *circular* ring (radius a, mass m, resistance R) to cross the bottom of the magnetic field B, at its (changing) terminal velocity?

[31] W. M. Saslow, *Am. J. Phys.* **59**, 16 (1991).

Problem 7.49

(a) Referring to Prob. 5.52(a) and Eq. 7.18, show that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{7.66}$$

for Faraday-induced electric fields. Check this result by taking the divergence and curl of both sides.

(b) A spherical shell of radius R carries a uniform surface charge σ. It spins about a fixed axis at an angular velocity $\omega(t)$ that changes slowly with time. Find the electric field inside and outside the sphere. [*Hint:* There are *two* contributions here: the Coulomb field due to the charge, and the Faraday field due to the changing $\mathbf{B}$. Refer to Ex. 5.11.]

Problem 7.50 Electrons undergoing cyclotron motion can be sped up by increasing the magnetic field; the accompanying electric field will impart tangential acceleration. This is the principle of the **betatron**. One would like to keep the radius of the orbit constant during the process. Show that this can be achieved by designing a magnet such that the average field over the area of the orbit is twice the field at the circumference (Fig. 7.53). Assume the electrons start from rest in zero field, and that the apparatus is symmetric about the center of the orbit. (Assume also that the electron velocity remains well below the speed of light, so that nonrelativistic mechanics applies.) [*Hint:* Differentiate Eq. 5.3 with respect to time, and use $F = ma = qE$.]

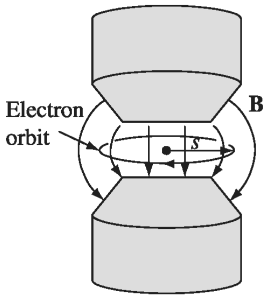

FIGURE 7.53

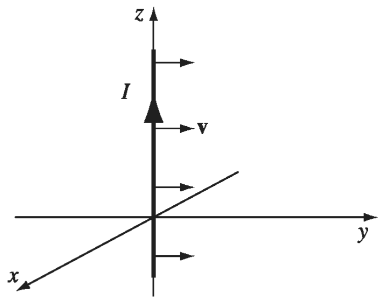

FIGURE 7.54

Problem 7.51 An infinite wire carrying a constant current I in the $\hat{\mathbf{z}}$ direction is moving in the y direction at a constant speed v. Find the electric field, in the quasistatic approximation, at the instant the wire coincides with the z axis (Fig. 7.54). [*Answer:* $-(\mu_0 I v/2\pi s)\cos\phi\,\hat{\mathbf{z}}$]

Problem 7.52 An atomic electron (charge q) circles about the nucleus (charge Q) in an orbit of radius r; the centripetal acceleration is provided, of course, by the Coulomb attraction of opposite charges. Now a small magnetic field dB is slowly turned on, perpendicular to the plane of the orbit. Show that the increase in kinetic energy, dT, imparted by the induced electric field, is just right to sustain circular motion *at the same radius* r. (That's why, in my discussion of diamagnetism, I assumed the radius is fixed. See Sect. 6.1.3 and the references cited there.)

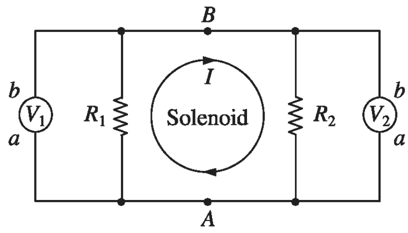

FIGURE 7.55

Problem 7.53 The current in a long solenoid is increasing linearly with time, so the flux is proportional to t: $\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in Fig. 7.55. What is the reading on each voltmeter? Assume that these are *ideal* voltmeters that draw negligible current (they have huge internal resistance), and that a voltmeter registers $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$ between the terminals and through the meter. [*Answer:* $V_1 = \alpha R_1/(R_1 + R_2)$; $V_2 = -\alpha R_2/(R_1 + R_2)$. Notice that $V_1 \neq V_2$, even though they are connected to the same points![32]]

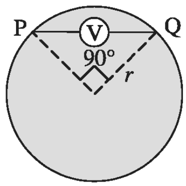

FIGURE 7.56

Problem 7.54 A circular wire loop (radius r, resistance R) encloses a region of uniform magnetic field, B, perpendicular to its plane. The field (occupying the shaded region in Fig. 7.56) increases linearly with time ($B = \alpha t$). An ideal voltmeter (infinite internal resistance) is connected between points P and Q.

(a) What is the current in the loop?

(b) What does the voltmeter read? [*Answer:* $\alpha r^2/2$]

Problem 7.55 In the discussion of motional emf (Sect. 7.1.3) I assumed that the wire loop (Fig. 7.10) has a resistance R; the current generated is then $I = vBh/R$. But what if the wire is made out of perfectly conducting material, so that R is *zero*? In that case, the current is limited only by the back emf associated with the self-inductance L of the loop (which would ordinarily be negligible in comparison with IR). Show that in this régime the loop (mass m) executes simple harmonic motion, and find its frequency.[33] [*Answer:* $\omega = Bh/\sqrt{mL}$]

[32] R. H. Romer, *Am. J. Phys.* **50**, 1089 (1982). See also H. W. Nicholson, *Am. J. Phys.* **73**, 1194 (2005); B. M. McGuyer, *Am. J. Phys.* **80**, 101 (2012).

[33] For a collection of related problems, see W. M. Saslow, *Am. J. Phys.* **55**, 986 (1987), and R. H. Romer, *Eur. J. Phys.* **11**, 103 (1990).

Problem 7.56

(a) Use the Neumann formula (Eq. 7.23) to calculate the mutual inductance of the configuration in Fig. 7.37, assuming a is very small ($a \ll b$, $a \ll z$). Compare your answer to Prob. 7.22.

(b) For the general case (*not* assuming a is small), show that

$$M = \frac{\mu_0 \pi \beta}{2} \sqrt{ab\beta} \left(1 + \frac{15}{8}\beta^2 + \dots\right),$$

where

$$\beta \equiv \frac{ab}{z^2 + a^2 + b^2}.$$

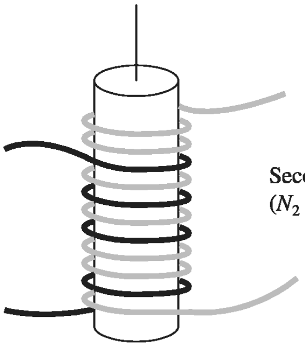

FIGURE 7.57

Problem 7.57 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils*. (In practice this is achieved by inserting an iron core through the cylinder; this has the effect of concentrating the flux.) The **primary** coil has N_1 turns and the **secondary** has N_2 (Fig. 7.57). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}, \tag{7.67}$$

where $\mathcal{E}_1$ is the (back) emf of the primary. [This is a primitive **transformer**—a device for raising or lowering the emf of an alternating current source. By choosing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, study Prob. 7.58.]

Problem 7.58 A transformer (Prob. 7.57) takes an input AC voltage of amplitude V_1, and delivers an output voltage of amplitude V_2, which is determined by the turns ratio ($V_2/V_1 = N_2/N_1$). If $N_2 > N_1$, the output voltage is greater than the input voltage. Why doesn't this violate conservation of energy? *Answer:* Power is the product of voltage and current; if the voltage goes *up*, the current must come *down*. The purpose of this problem is to see exactly how this works out, in a simplified model.

(a) In an ideal transformer, the same flux passes through all turns of the primary and of the secondary. Show that in this case $M^2 = L_1 L_2$, where M is the mutual inductance of the coils, and L_1, L_2 are their individual self-inductances.

(b) Suppose the primary is driven with AC voltage $V_{\text{in}} = V_1 \cos(\omega t)$, and the secondary is connected to a resistor, R. Show that the two currents satisfy the relations

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos(\omega t); \quad L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R.$$

(c) Using the result in (a), solve these equations for $I_1(t)$ and $I_2(t)$. (Assume I_1 has no DC component.)

(d) Show that the output voltage ($V_{\text{out}} = I_2 R$) divided by the input voltage (V_{in}) is equal to the turns ratio: $V_{\text{out}}/V_{\text{in}} = N_2/N_1$.

(e) Calculate the input power ($P_{\text{in}} = V_{\text{in}} I_1$) and the output power ($P_{\text{out}} = V_{\text{out}} I_2$), and show that their averages over a full cycle are equal.

Problem 7.59 An infinite wire runs along the z axis; it carries a current $I(z)$ that is a function of z (but not of t), and a charge density $\lambda(t)$ that is a function of t (but not of z).

(a) By examining the charge flowing into a segment dz in a time dt, show that $d\lambda/dt = -dI/dz$. If we stipulate that $\lambda(0) = 0$ and $I(0) = 0$, show that $\lambda(t) = kt$, $I(z) = -kz$, where k is a constant.

(b) Assume for a moment that the process is quasistatic, so the fields are given by Eqs. 2.9 and 5.38. Show that these are in fact the *exact* fields, by confirming that all four of Maxwell's equations are satisfied. (First do it in differential form, for the region $s > 0$, then in integral form for the appropriate Gaussian cylinder/Amperian loop straddling the axis.)

Problem 7.60 Suppose $\mathbf{J}(\mathbf{r})$ is constant in time but $\rho(\mathbf{r}, t)$ is *not*—conditions that might prevail, for instance, during the charging of a capacitor.

(a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t,$$

where $\dot{\rho}(\mathbf{r}, 0)$ is the time derivative of ρ at $t = 0$. [*Hint:* Use the continuity equation.]

This is *not* an electrostatic or magnetostatic configuration;[34] nevertheless, rather surprisingly, both Coulomb's law (Eq. 2.8) and the Biot-Savart law (Eq. 5.42) hold, as you can confirm by showing that they satisfy Maxwell's equations. In particular:

[34]Some authors *would* regard this as magnetostatic, since $\mathbf{B}$ is independent of t. For them, the Biot-Savart law is a general rule of magnetostatics, but $\nabla \cdot \mathbf{J} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply only under the *additional* assumption that ρ is constant. In such a formulation, Maxwell's displacement term can (in this very special case) be *derived* from the Biot-Savart law, by the method of part (b). See D. F. Bartlett, *Am. J. Phys.* **58**, 1168 (1990); D. J. Griffiths and M. A. Heald, *Am. J. Phys.* **59**, 111 (1991).

(b) Show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

obeys Ampère's law *with Maxwell's displacement current term.*

Problem 7.61 The magnetic field of an infinite straight wire carrying a steady current I can be obtained from the *displacement* current term in the Ampère/Maxwell law, as follows: Picture the current as consisting of a uniform line charge λ moving along the z axis at speed v (so that $I = \lambda v$), with a tiny gap of length ϵ, which reaches the origin at time $t = 0$. In the next instant (up to $t = \epsilon/v$) there is no *real* current passing through a circular Amperian loop in the xy plane, but there *is* a *displacement* current, due to the "missing" charge in the gap.

(a) Use Coulomb's law to calculate the z component of the electric field, for points in the xy plane a distance s from the origin, due to a segment of wire with uniform density $-\lambda$ extending from $z_1 = vt - \epsilon$ to $z_2 = vt$.

(b) Determine the flux of this electric field through a circle of radius a in the xy plane.

(c) Find the displacement current through this circle. Show that I_d is equal to I, in the limit as the gap width (ϵ) goes to zero.[35]

Problem 7.62 A certain transmission line is constructed from two thin metal "ribbons," of width w, a very small distance $h \ll w$ apart. The current travels down one strip and back along the other. In each case, it spreads out uniformly over the surface of the ribbon.

(a) Find the capacitance per unit length, $\mathcal{C}$.

(b) Find the inductance per unit length, $\mathcal{L}$.

(c) What is the product $\mathcal{LC}$, numerically? [$\mathcal{L}$ and $\mathcal{C}$ will, of course, vary from one kind of transmission line to another, but their *product* is a universal constant—check, for example, the cable in Ex. 7.13—provided the space between the conductors is a vacuum. In the theory of transmission lines, this product is related to the speed with which a pulse propagates down the line: $v = 1/\sqrt{\mathcal{LC}}$.]

(d) If the strips are insulated from one another by a nonconducting material of permittivity ϵ and permeability μ, what then is the product $\mathcal{LC}$? What is the propagation speed? [*Hint:* see Ex. 4.6; by what factor does L change when an inductor is immersed in linear material of permeability μ?]

Problem 7.63 Prove **Alfven's theorem**: In a perfectly conducting fluid (say, a gas of free electrons), the magnetic flux through any closed loop moving with the fluid is constant in time. (The magnetic field lines are, as it were, "frozen" into the fluid.)

(a) Use Ohm's law, in the form of Eq. 7.2, together with Faraday's law, to prove that if $\sigma = \infty$ and $\mathbf{J}$ is finite, then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

[35] For a slightly different approach to the same problem, see W. K. Terry, *Am. J. Phys.* **50**, 742 (1982).

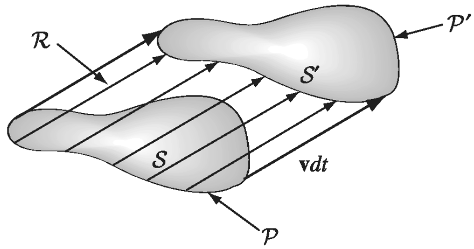

FIGURE 7.58

(b) Let $\mathcal{S}$ be the surface bounded by the loop ($\mathcal{P}$) at time t, and $\mathcal{S}'$ a surface bounded by the loop in its new position ($\mathcal{P}'$) at time $t + dt$ (see Fig. 7.58). The change in flux is

$$d\Phi = \int_{\mathcal{S}'} \mathbf{B}(t+dt) \cdot d\mathbf{a} - \int_{\mathcal{S}} \mathbf{B}(t) \cdot d\mathbf{a}.$$

Use $\nabla \cdot \mathbf{B} = 0$ to show that

$$\int_{\mathcal{S}'} \mathbf{B}(t+dt) \cdot d\mathbf{a} + \int_{\mathcal{R}} \mathbf{B}(t+dt) \cdot d\mathbf{a} = \int_{\mathcal{S}} \mathbf{B}(t+dt) \cdot d\mathbf{a}$$

(where $\mathcal{R}$ is the "ribbon" joining $\mathcal{P}$ and $\mathcal{P}'$), and hence that

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} - \int_{\mathcal{R}} \mathbf{B}(t+dt) \cdot d\mathbf{a}$$

(for infinitesimal dt). Use the method of Sect. 7.1.3 to rewrite the second integral as

$$dt \oint_{\mathcal{P}} (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l},$$

and invoke Stokes' theorem to conclude that

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{a}.$$

Together with the result in (a), this proves the theorem.

Problem 7.64

(a) Show that Maxwell's equations with magnetic charge (Eq. 7.44) are invariant under the **duality transformation**

$$\left.\begin{aligned} \mathbf{E}' &= \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha, \\ c\mathbf{B}' &= c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha, \\ cq_e' &= cq_e\cos\alpha + q_m\sin\alpha, \\ q_m' &= q_m\cos\alpha - cq_e\sin\alpha, \end{aligned}\right\} \tag{7.68}$$

where $c \equiv 1/\sqrt{\epsilon_0\mu_0}$ and α is an arbitrary rotation angle in "**E**/**B**-space." Charge and current densities transform in the same way as q_e and q_m. [This means, in particular, that if you know the fields produced by a configuration of *electric* charge, you can immediately (using $\alpha = 90°$) write down the fields produced by the corresponding arrangement of *magnetic* charge.]

(b) Show that the force law (Prob. 7.38)

$$\mathbf{F} = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) \tag{7.69}$$

is also invariant under the duality transformation.

Intermission

All of our cards are now on the table, and in a sense my job is done. In the first seven chapters we assembled electrodynamics piece by piece, and now, with Maxwell's equations in their final form, the theory is complete. There are no more laws to be learned, no further generalizations to be considered, and (with perhaps one exception) no lurking inconsistencies to be resolved. If yours is a one-semester course, this would be a reasonable place to stop.

But in another sense we have just arrived at the starting point. We are at last in possession of a full deck—it's time to deal. This is the fun part, in which one comes to appreciate the extraordinary power and richness of electrodynamics. In a full-year course there should be plenty of time to cover the remaining chapters, and perhaps to supplement them with a unit on plasma physics, say, or AC circuit theory, or even a little general relativity. But if you have room for only one topic, I'd recommend Chapter 9, on Electromagnetic Waves (you'll probably want to skim Chapter 8 as preparation). This is the segue to Optics, and is historically the most important application of Maxwell's theory.

CHAPTER 8

Conservation Laws

8.1 ■ CHARGE AND ENERGY

8.1.1 ■ The Continuity Equation

In this chapter we study conservation of energy, momentum, and angular momentum, in electrodynamics. But I want to begin by reviewing the conservation of *charge*, because it is the paradigm for all conservation laws. What precisely does conservation of charge tell us? That the total charge in the universe is constant? Well, sure—that's **global** conservation of charge. But **local** conservation of charge is a much stronger statement: If the charge in some region changes, then exactly that amount of charge must have passed in or out through the surface. The tiger can't simply rematerialize outside the cage; if it got from inside to outside it must have slipped through a hole in the fence.

Formally, the charge in a volume $\mathcal{V}$ is

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau, \tag{8.1}$$

and the current flowing out through the boundary $\mathcal{S}$ is $\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$, so local conservation of charge says

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{8.2}$$

Using Eq. 8.1 to rewrite the left side, and invoking the divergence theorem on the right, we have

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau, \tag{8.3}$$

and since this is true for *any* volume, it follows that

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.} \tag{8.4}$$

This is the continuity equation—the precise mathematical statement of local conservation of charge. It can be derived from Maxwell's equations—conservation of charge is not an *independent* assumption; it is built into the laws of electrodynamics. It serves as a constraint on the sources (ρ and $\mathbf{J}$). They can't be just *any* old functions—they have to respect conservation of charge.[1]

The purpose of this chapter is to develop the corresponding equations for local conservation of energy and momentum. In the process (and perhaps more important) we will learn how to express the energy density and the momentum density (the analogs to ρ), as well as the energy "current" and the momentum "current" (analogous to $\mathbf{J}$).

8.1.2 ■ Poynting's Theorem

In Chapter 2, we found that the work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is (Eq. 2.45)

$$W_e = \frac{\epsilon_0}{2}\int E^2\,d\tau,$$

where $\mathbf{E}$ is the resulting electric field. Likewise, the work required to get currents going (against the back emf) is (Eq. 7.35)

$$W_m = \frac{1}{2\mu_0}\int B^2\,d\tau,$$

where $\mathbf{B}$ is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields, per unit volume, is

$$\boxed{u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right).} \tag{8.5}$$

In this section I will confirm Eq. 8.5, and develop the energy conservation law for electrodynamics.

Suppose we have some charge and current configuration which, at time t, produces fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charges move around a bit. *Question*: How much work, dW, is done by the electromagnetic forces acting on these charges, in the interval dt? According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F}\cdot d\mathbf{l} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt = q\mathbf{E}\cdot\mathbf{v}\,dt.$$

In terms of the charge and current densities, $q \to \rho\,d\tau$ and $\rho\mathbf{v} \to \mathbf{J}$,[2] so the rate at which work is done on all the charges in a volume $\mathcal{V}$ is

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J})\,d\tau. \tag{8.6}$$

[1]The continuity equation is the *only* such constraint. Any functions $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ consistent with Eq. 8.4 constitute possible charge and current densities, in the sense of admitting solutions to Maxwell's equations.

[2]This is a slippery equation: after all, if charges of both signs are present, the *net* charge density can be zero even when the current is *not*—in fact, this is the case for ordinary current-carrying wires. We should really treat the positive and negative charges separately, and combine the two to get Eq. 8.6, with $\mathbf{J} = \rho_+\mathbf{v}_+ + \rho_-\mathbf{v}_-$.

Evidently $\mathbf{E} \cdot \mathbf{J}$ is the work done per unit time, per unit volume—which is to say, the *power* delivered per unit volume. We can express this quantity in terms of the fields alone, using the Ampère-Maxwell law to eliminate $\mathbf{J}$:

$$\mathbf{E} \cdot \mathbf{J} = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}.$$

From product rule 6,

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}).$$

Invoking Faraday's law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$), it follows that

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}).$$

Meanwhile,

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2), \tag{8.7}$$

so

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}). \tag{8.8}$$

Putting this into Eq. 8.6, and applying the divergence theorem to the second term, we have

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}, \tag{8.9}$$

where $\mathcal{S}$ is the surface bounding $\mathcal{V}$. This is **Poynting's theorem**; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, $\int u\, d\tau$ (Eq. 8.5). The second term evidently represents the rate at which energy is transported out of $\mathcal{V}$, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that *the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface*.

The *energy per unit time, per unit area*, transported by the fields is called the **Poynting vector**:

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).} \tag{8.10}$$

Specifically, $\mathbf{S} \cdot d\mathbf{a}$ is the energy per unit time crossing the infinitesimal surface $d\mathbf{a}$—the energy *flux* (so $\mathbf{S}$ is the **energy flux density**).[3] We will see many

[3]If you're very fastidious, you'll notice a small gap in the logic here: We know from Eq. 8.9 that $\oint \mathbf{S} \cdot d\mathbf{a}$ is the total power passing through a *closed* surface, but this does not prove that $\int \mathbf{S} \cdot d\mathbf{a}$ is the power passing through any *open* surface (there could be an extra term that integrates to zero over all closed surfaces). This is, however, the obvious and natural interpretation; as always, the precise *location* of energy is not really determined in electrodynamics (see Sect. 2.4.4).

applications of the Poynting vector in Chapters 9 and 11, but for the moment I am mainly interested in using it to express Poynting's theorem more compactly:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u\,d\tau - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}. \tag{8.11}$$

What if *no* work is done on the charges in $\mathcal{V}$—what if, for example, we are in a region of empty space, where there *is* no charge? In that case $dW/dt = 0$, so

$$\int \frac{\partial u}{\partial t}\,d\tau = -\oint \mathbf{S}\cdot d\mathbf{a} = -\int (\nabla\cdot\mathbf{S})\,d\tau,$$

and hence

$$\frac{\partial u}{\partial t} = -\nabla\cdot\mathbf{S}. \tag{8.12}$$

This is the "continuity equation" for *energy*—u (energy density) plays the role of ρ (charge density), and $\mathbf{S}$ takes the part of $\mathbf{J}$ (current density). It expresses local conservation of electromagnetic energy.

In *general*, though, electromagnetic energy by itself is *not* conserved (nor is the energy of the charges). Of course not! The fields do work on the charges, and the charges create fields—energy is tossed back and forth between them. In the overall energy economy, you must include the contributions of both the matter and the fields.

Example 8.1. When current flows down a wire, work is done, which shows up as Joule heating of the wire (Eq. 7.7). Though there are certainly *easier* ways to do it, the energy per unit time delivered to the wire can be calculated using the Poynting vector. Assuming it's uniform, the electric field parallel to the wire is

$$E = \frac{V}{L},$$

where V is the potential difference between the ends and L is the length of the wire (Fig. 8.1). The magnetic field is "circumferential"; at the surface (radius a) it has the value

$$B = \frac{\mu_0 I}{2\pi a}.$$

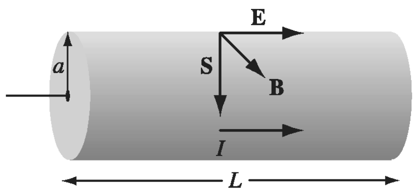

FIGURE 8.1

Accordingly, the magnitude of the Poynting vector is

$$S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L},$$

and it points radially inward. The energy per unit time passing in through the surface of the wire is therefore

$$\int \mathbf{S}\cdot d\mathbf{a} = S(2\pi a L) = VI,$$

which is exactly what we concluded, on much more direct grounds, in Sect. 7.1.1.[4]

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.62, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Problem 8.2 Consider the charging capacitor in Prob. 7.34.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the *direction* of $\mathbf{S}$. Check that Eq. 8.12 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9—in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

8.2 ■ MOMENTUM

8.2.1 ■ Newton's Third Law in Electrodynamics

Imagine a point charge q traveling in along the x axis at a constant speed v. Because it is moving, its electric field is *not* given by Coulomb's law; nevertheless, $\mathbf{E}$ still points radially outward from the instantaneous position of the charge (Fig. 8.2a), as we'll see in Chapter 10. Since, moreover, a moving point charge does not constitute a steady current, its magnetic field is *not* given by the Biot-Savart law. Nevertheless, it's a fact that $\mathbf{B}$ still circles around the axis in a manner suggested by the right-hand rule (Fig. 8.2b); again, the proof will come in Chapter 10.

[4]What about energy flow *down* the wire? For a discussion, see M. K. Harbola, *Am. J. Phys.* **78**, 1203 (2010). For a more sophisticated geometry, see B. S. Davis and L. Kaplan, *Am. J. Phys.* **79**, 1155 (2011).

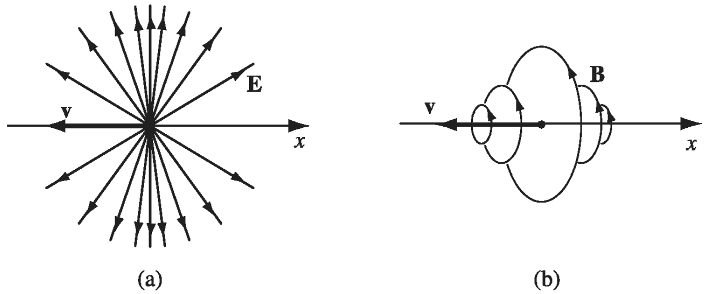

FIGURE 8.2

Now suppose this charge encounters an identical one, proceeding in at the same speed along the y axis. Of course, the electromagnetic force between them would tend to drive them off the axes, but let's assume that they're mounted on tracks, or something, so they're obliged to maintain the same direction and the same speed (Fig. 8.3). The electric force between them is repulsive, but how about the magnetic force? Well, the magnetic field of q_1 points into the page (at the position of q_2), so the magnetic force on q_2 is toward the *right*, whereas the magnetic field of q_2 is *out* of the page (at the position of q_1), and the magnetic force on q_1 is *upward*. *The net electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1, in violation of Newton's third law.* In electro*statics* and magneto*statics* the third law holds, but in electro*dynamics* it does *not*.

Well, that's an interesting curiosity, but then, how often does one actually use the third law, in practice? *Answer:* All the time! For the proof of conservation of momentum rests on the cancellation of internal forces, which follows from the third law. When you tamper with the third law, you are placing conservation of momentum in jeopardy, and there is hardly any principle in physics more sacred than *that*.

Momentum conservation is rescued, in electrodynamics, by the realization that the *fields themselves carry momentum*. This is not so surprising when you

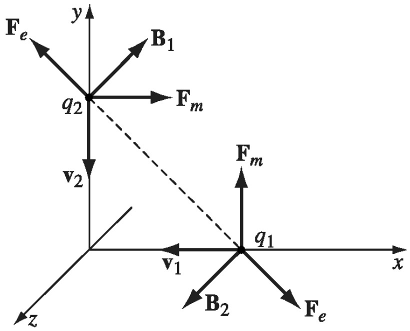

FIGURE 8.3

consider that we have already attributed *energy* to the fields. Whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum is momentum conservation restored.

8.2.2 ■ Maxwell's Stress Tensor

Let's calculate the total electromagnetic force on the charges in volume $\mathcal{V}$:

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho \, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \, d\tau. \tag{8.13}$$

The *force per unit volume* is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{8.14}$$

As before, I propose to express this in terms of fields alone, eliminating ρ and $\mathbf{J}$ by using Maxwell's equations (i) and (iv):

$$\mathbf{f} = \epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}.$$

Now

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right),$$

and Faraday's law says

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E},$$

so

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}).$$

Thus

$$\mathbf{f} = \epsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})\right] - \frac{1}{\mu_0}\left[\mathbf{B} \times (\nabla \times \mathbf{B})\right] - \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}). \tag{8.15}$$

Just to make things look more symmetrical, let's throw in a term $(\nabla \cdot \mathbf{B})\mathbf{B}$; since $\nabla \cdot \mathbf{B} = 0$, this costs us nothing. Meanwhile, product rule 4 says

$$\nabla(E^2) = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E}),$$

so

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla(E^2) - (\mathbf{E} \cdot \nabla)\mathbf{E},$$

and the same goes for $\mathbf{B}$. Therefore,

$$\mathbf{f} = \epsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}\right] + \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}\right]$$
$$-\frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}). \tag{8.16}$$

Ugly! But it can be simplified by introducing the **Maxwell stress tensor**,

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right). \tag{8.17}$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components (T_{xx}, T_{yy}, T_{xz}, T_{yx}, and so on). The **Kronecker delta**, δ_{ij}, is 1 if the indices are the same ($\delta_{xx} = \delta_{yy} = \delta_{zz} = 1$) and zero otherwise ($\delta_{xy} = \delta_{xz} = \delta_{yz} = 0$). Thus

$$T_{xx} = \frac{1}{2}\epsilon_0\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right),$$

$$T_{xy} = \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y),$$

and so on.

Because it carries *two* indices, where a vector has only one, T_{ij} is sometimes written with a double arrow: $\overleftrightarrow{\mathbf{T}}$. One can form the dot product of $\overleftrightarrow{\mathbf{T}}$ with a vector $\mathbf{a}$, in two ways—on the left, and on the right:

$$\left(\mathbf{a} \cdot \overleftrightarrow{\mathbf{T}}\right)_j = \sum_{i=x,y,z} a_i T_{ij}, \quad \left(\overleftrightarrow{\mathbf{T}} \cdot \mathbf{a}\right)_j = \sum_{i=x,y,z} T_{ji} a_i. \tag{8.18}$$

The resulting object, which has one remaining index, is itself a vector. In particular, the divergence of $\overleftrightarrow{\mathbf{T}}$ has as its jth component

$$\left(\nabla \cdot \overleftrightarrow{\mathbf{T}}\right)_j = \epsilon_0\left[(\nabla \cdot \mathbf{E})E_j + (\mathbf{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right]$$
$$+\frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})B_j + (\mathbf{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right].$$

Thus the force per unit volume (Eq. 8.16) can be written in the much tidier form

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t}, \tag{8.19}$$

where $\mathbf{S}$ is the Poynting vector (Eq. 8.10).

The *total* electromagnetic force on the charges in $\mathcal{V}$ (Eq. 8.13) is

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} \mathbf{S}\, d\tau. \tag{8.20}$$

(I used the divergence theorem to convert the first term to a surface integral.) In the *static* case the second term drops out, and the electromagnetic force on the charge configuration can be expressed entirely in terms of the stress tensor at the boundary:

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \text{(static)}. \tag{8.21}$$

Physically, $\overleftrightarrow{\mathbf{T}}$ is the force per unit area (or **stress**) acting on the surface. More precisely, T_{ij} is the force (per unit area) in the ith direction acting on an element of surface oriented in the jth direction—"diagonal" elements (T_{xx}, T_{yy}, T_{zz}) represent *pressures*, and "off-diagonal" elements (T_{xy}, T_{xz}, etc.) are *shears*.

Example 8.2. Determine the net force on the "northern" hemisphere of a uniformly charged solid sphere of radius R and charge Q (the same as Prob. 2.47, only this time we'll use the Maxwell stress tensor and Eq. 8.21).

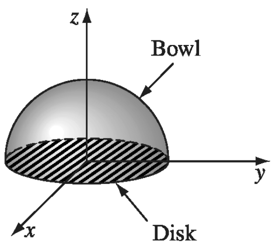

FIGURE 8.4

Solution

The boundary surface consists of two parts—a hemispherical "bowl" at radius R, and a circular disk at $\theta = \pi/2$ (Fig. 8.4). For the bowl,

$$d\mathbf{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$$

and

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{r}}.$$

In Cartesian components,

$$\hat{\mathbf{r}} = \sin\theta \cos\phi\, \hat{\mathbf{x}} + \sin\theta \sin\phi\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}},$$

so

$$T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi,$$

$$T_{zy} = \epsilon_0 E_z E_y = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi,$$

$$T_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 (\cos^2\theta - \sin^2\theta). \tag{8.22}$$

The net force is obviously in the z-direction, so it suffices to calculate

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\,d\theta\,d\phi.$$

The force on the "bowl" is therefore

$$F_{\text{bowl}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 2\pi\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}. \tag{8.23}$$

Meanwhile, for the equatorial disk,

$$d\mathbf{a} = -r\,dr\,d\phi\,\hat{\mathbf{z}}, \tag{8.24}$$

and (since we are now *inside* the sphere)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3} r(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}).$$

Thus

$$T_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2,$$

and hence

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\,dr\,d\phi.$$

The force on the disk is therefore

$$F_{\text{disk}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi\int_0^R r^3 dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}. \tag{8.25}$$

Combining Eqs. 8.23 and 8.25, I conclude that the net force on the northern hemisphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}. \tag{8.26}$$

Incidentally, in applying Eq. 8.21, *any* volume that encloses all of the charge in question (and no *other* charge) will do the job. For example, in the present case we could use the whole region $z > 0$. In that case the boundary surface consists of the entire xy plane (plus a hemisphere at $r = \infty$, but $E = 0$ out there, so it contributes nothing). In place of the "bowl," we now have the outer portion of the plane ($r > R$). Here

$$T_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}$$

(Eq. 8.22 with $\theta = \pi/2$ and $R \to r$), and $d\mathbf{a}$ is given by Eq. 8.24, so

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^3}\, dr\, d\phi,$$

and the contribution from the plane for $r > R$ is

$$\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 2\pi \int_R^\infty \frac{1}{r^3}\, dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2},$$

the same as for the bowl (Eq. 8.23).

I hope you didn't get too bogged down in the details of Ex. 8.2. If so, take a moment to appreciate what happened. We were calculating the force on a solid object, but instead of doing a *volume* integral, as you might expect, Eq. 8.21 allowed us to set it up as a *surface* integral; somehow the stress tensor sniffs out what is going on inside.

! **Problem 8.3** Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell, with radius R, angular velocity ω, and surface charge density σ. [This is the same as Prob. 5.44, but this time use the Maxwell stress tensor and Eq. 8.21.]

Problem 8.4

(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

(b) Do the same for charges that are opposite in sign.

8.2.3 ■ Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}.$$

Equation 8.20 can therefore be written in the form[5]

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\,d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}, \tag{8.27}$$

where $\mathbf{p}_{\text{mech}}$ is the (mechanical) momentum of the particles in volume $\mathcal{V}$. This expression is similar in structure to Poynting's theorem (Eq. 8.11), and it invites an analogous interpretation: The first integral represents *momentum stored in the fields*:

$$\mathbf{p} = \mu_0\epsilon_0\int_{\mathcal{V}} \mathbf{S}\,d\tau, \tag{8.28}$$

while the second integral is the *momentum per unit time flowing in through the surface*.

Equation 8.27 is the statement of *conservation of momentum* in electrodynamics: If the mechanical momentum increases, either the field momentum *de*creases, or else the fields are carrying momentum into the volume through the surface. The momentum *density* in the fields is evidently

$$\boxed{\mathbf{g} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}),} \tag{8.29}$$

and the momentum flux transported by the fields is $-\overleftrightarrow{\mathbf{T}}$ (specifically, $-\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}$ is the electromagnetic momentum per unit time passing through the area $d\mathbf{a}$).

If the mechanical momentum in $\mathcal{V}$ is not changing (for example, if we are talking about a region of empty space), then

$$\int \frac{\partial\mathbf{g}}{\partial t}\,d\tau = \oint \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} = \int \nabla\cdot\overleftrightarrow{\mathbf{T}}\,d\tau,$$

and hence

$$\frac{\partial\mathbf{g}}{\partial t} = \nabla\cdot\overleftrightarrow{\mathbf{T}}. \tag{8.30}$$

This is the "continuity equation" for electromagnetic momentum, with $\mathbf{g}$ (momentum density) in the role of ρ (charge density) and $-\overleftrightarrow{\mathbf{T}}$ playing the part of $\mathbf{J}$; it expresses the local conservation of field momentum. But in general (when there *are* charges around) the field momentum by itself, and the mechanical momentum by itself, are *not* conserved—charges and fields exchange momentum, and only the *total* is conserved.

Notice that the Poynting vector has appeared in two quite different roles: $\mathbf{S}$ itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\mu_0\epsilon_0\mathbf{S}$ is the momentum per unit volume stored in those fields.[6]

[5] Let's assume the only forces acting are electromagnetic. You can include other forces if you like—both here and in the discussion of energy conservation—but they are just a distraction from the essential story.

[6] This is no coincidence—see R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Reading, Mass.: Addison-Wesley, 1964), Vol. II, Section 27-6.

Similarly, $\overleftrightarrow{\mathbf{T}}$ plays a dual role: $\overleftrightarrow{\mathbf{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (it is the momentum current density) carried by the fields.

Example 8.3. A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other (Fig. 8.5). The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields?

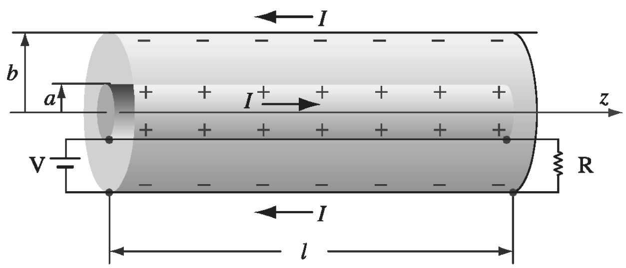

FIGURE 8.5

Solution
The fields are

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\,\hat{\mathbf{s}}, \qquad \mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}.$$

The Poynting vector is therefore

$$\mathbf{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{\mathbf{z}}.$$

So energy is flowing down the line, from the battery to the resistor. In fact, the power transported is

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{1}{s^2}2\pi s\,ds = \frac{\lambda I}{2\pi\epsilon_0}\ln(b/a) = IV,$$

as it should be.

The *momentum* in the fields is

$$\mathbf{p} = \mu_0\epsilon_0 \int \mathbf{S}\,d\tau = \frac{\mu_0\lambda I}{4\pi^2}\,\hat{\mathbf{z}}\int_a^b \frac{1}{s^2}l2\pi s\,ds = \frac{\mu_0\lambda I l}{2\pi}\ln(b/a)\,\hat{\mathbf{z}} = \frac{IVl}{c^2}\,\hat{\mathbf{z}}.$$

This is an astonishing result. The cable is not moving, **E** and **B** are static, and yet we are asked to believe that there is momentum in the fields. If something tells you this cannot be the whole story, you have sound intuitions. But the resolution of this paradox will have to await Chapter 12 (Ex. 12.12).

Suppose now that we turn up the resistance, so the current decreases. The changing magnetic field will induce an electric field (Eq. 7.20):

$$\mathbf{E} = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{\mathbf{z}}.$$

This field exerts a force on $\pm\lambda$:

$$\mathbf{F} = \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln a + K\right]\hat{\mathbf{z}} - \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln b + K\right]\hat{\mathbf{z}} = -\frac{\mu_0\lambda l}{2\pi}\frac{dI}{dt}\ln(b/a)\,\hat{\mathbf{z}}.$$

The total momentum imparted to the cable, as the current drops from I to 0, is therefore

$$\mathbf{p}_{\text{mech}} = \int \mathbf{F}\,dt = \frac{\mu_0\lambda I l}{2\pi}\ln(b/a)\,\hat{\mathbf{z}},$$

which is precisely the momentum originally stored in the fields.

Problem 8.5 Imagine two parallel infinite sheets, carrying uniform surface charge $+\sigma$ (on the sheet at $z = d$) and $-\sigma$ (at $z = 0$). They are moving in the y direction at constant speed v (as in Problem 5.17).

(a) What is the electromagnetic momentum in a region of area A?

(b) Now suppose the top sheet moves slowly down (speed u) until it reaches the bottom sheet, so the fields disappear. By calculating the (magnetic) force on the charge ($q = \sigma A$), show that the impulse delivered to the sheet is equal to the momentum originally stored in the fields.

Problem 8.6 A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\,\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{x}}$, as shown in Fig. 8.6.

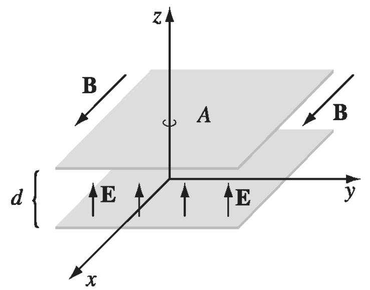

FIGURE 8.6

(a) Find the electromagnetic momentum in the space between the plates.

(b) Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?[7]

Problem 8.7 Consider an infinite parallel-plate capacitor, with the lower plate (at $z = -d/2$) carrying surface charge density $-\sigma$, and the upper plate (at $z = +d/2$) carrying charge density $+\sigma$.

(a) Determine all nine elements of the stress tensor, in the region between the plates. Display your answer as a 3×3 matrix:

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

(b) Use Eq. 8.21 to determine the electromagnetic force per unit area on the top plate. Compare Eq. 2.51.

(c) What is the electromagnetic momentum per unit area, per unit time, crossing the xy plane (or any other plane parallel to that one, between the plates)?

(d) Of course, there must be *mechanical* forces holding the plates apart—perhaps the capacitor is filled with insulating material under pressure. Suppose we suddenly *remove* the insulator; the momentum flux (c) is now absorbed by the plates, and they begin to move. Find the momentum per unit time delivered to the top plate (which is to say, the force acting on it) and compare your answer to (b). [*Note:* This is not an *additional* force, but rather an alternative way of calculating the *same* force—in (b) we got it from the force law, and in (d) we do it by conservation of momentum.]

8.2.4 ■ Angular Momentum

By now, the electromagnetic fields (which started out as mediators of forces between charges) have taken on a life of their own. They carry *energy* (Eq. 8.5)

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right), \tag{8.31}$$

and *momentum* (Eq. 8.29)

$$\mathbf{g} = \epsilon_0(\mathbf{E} \times \mathbf{B}), \tag{8.32}$$

[7]There is *much* more to be said about this problem, so don't get too excited if your answers to (a) and (b) appear to be consistent. See D. Babson, et al., *Am. J. Phys.* **77**, 826 (2009).

and, for that matter, *angular* momentum:

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} = \epsilon_0 \left[\mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \right]. \tag{8.33}$$

Even perfectly *static* fields can harbor momentum and angular momentum, as long as $\mathbf{E} \times \mathbf{B}$ is nonzero, and it is only when these field contributions are included that the conservation laws are sustained.

Example 8.4. Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical (nonconducting) shells of length l—one, *inside* the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, *outside* the solenoid at radius b, carries charge $-Q$ (see Fig. 8.7; l is supposed to be much greater than b). When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex. 7.8. *Question*: Where does the angular momentum come from?[8]

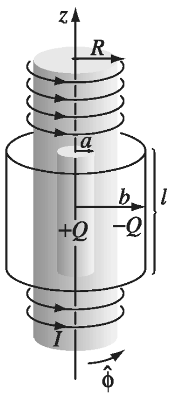

FIGURE 8.7

Solution

It was initially stored in the fields. Before the current was switched off, there was an electric field,

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 l} \frac{1}{s} \hat{\mathbf{s}} \quad (a < s < b),$$

in the region between the cylinders, and a magnetic field,

$$\mathbf{B} = \mu_0 n I \, \hat{\mathbf{z}} \quad (s < R),$$

[8]This is a variation on the "Feynman disk paradox" (R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures*, vol 2, pp. 17-5 (Reading, Mass.: Addison-Wesley, 1964) suggested by F. L. Boos, Jr. (*Am. J. Phys.* **52**, 756 (1984)). A similar model was proposed earlier by R. H. Romer (*Am. J. Phys.* **34**, 772 (1966)). For further references, see T.-C. E. Ma, *Am. J. Phys.* **54**, 949 (1986).

inside the solenoid. The momentum density (Eq. 8.29) was therefore

$$\mathbf{g} = -\frac{\mu_0 n I Q}{2\pi l s}\,\hat{\boldsymbol{\phi}},$$

in the region $a < s < R$. The z component of the *angular* momentum density was

$$(\mathbf{r} \times \mathbf{g})_z = -\frac{\mu_0 n I Q}{2\pi l},$$

which is *constant* (independent of s). To get the *total* angular momentum in the fields, we simply multiply by the volume, $\pi(R^2 - a^2)l$:[9]

$$\mathbf{L} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}. \tag{8.34}$$

When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday's law:

$$\mathbf{E} = \begin{cases} -\dfrac{1}{2}\mu_0 n \dfrac{dI}{dt}\dfrac{R^2}{s}\,\hat{\boldsymbol{\phi}}, & (s > R), \\[2ex] -\dfrac{1}{2}\mu_0 n \dfrac{dI}{dt} s\,\hat{\boldsymbol{\phi}}, & (s < R). \end{cases}$$

Thus the torque on the outer cylinder is

$$\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\hat{\mathbf{z}},$$

and it picks up an angular momentum

$$\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2\,\hat{\mathbf{z}} \int_I^0 \frac{dI}{dt}\,dt = -\frac{1}{2}\mu_0 n I Q R^2\,\hat{\mathbf{z}}.$$

Similarly, the torque on the inner cylinder is

$$\mathbf{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\,\hat{\mathbf{z}},$$

and its angular momentum increase is

$$\mathbf{L}_a = \frac{1}{2}\mu_0 n I Q a^2\,\hat{\mathbf{z}}.$$

So it all works out: $\mathbf{L}_{\text{em}} = \mathbf{L}_a + \mathbf{L}_b$. The angular momentum *lost* by the fields is precisely equal to the angular momentum *gained* by the cylinders, and the *total* angular momentum (fields plus matter) is conserved.

---

[9] The radial component integrates to zero, by symmetry.

Problem 8.8 In Ex. 8.4, suppose that instead of turning off the *magnetic* field (by reducing I) we turn off the *electric* field, by connecting a weakly[10] conducting radial spoke between the cylinders. (We'll have to cut a slot in the solenoid, so the cylinders can still rotate freely.) From the magnetic force on the current in the spoke, determine the total angular momentum delivered to the cylinders, as they discharge (they are now rigidly connected, so they rotate together). Compare the initial angular momentum stored in the fields (Eq. 8.34). (Notice that the *mechanism* by which angular momentum is transferred from the fields to the cylinders is entirely different in the two cases: in Ex. 8.4 it was Faraday's law, but here it is the Lorentz force law.)

Problem 8.9 Two concentric spherical shells carry uniformly distributed charges $+Q$ (at radius a) and $-Q$ (at radius $b > a$). They are immersed in a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$.

(a) Find the angular momentum of the fields (with respect to the center).

(b) Now the magnetic field is gradually turned off. Find the torque on each sphere, and the resulting angular momentum of the system.

! **Problem 8.10**[11] Imagine an iron sphere of radius R that carries a charge Q and a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The sphere is initially at rest.

(a) Compute the angular momentum stored in the electromagnetic fields.

(b) Suppose the sphere is gradually (and uniformly) demagnetized (perhaps by heating it up past the Curie point). Use Faraday's law to determine the induced electric field, find the torque this field exerts on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the demagnetization.

(c) Suppose instead of *demagnetizing* the sphere we *discharge* it, by connecting a grounding wire to the north pole. Assume the current flows over the surface in such a way that the charge density remains uniform. Use the Lorentz force law to determine the torque on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the discharge. (The magnetic field is discontinuous at the surface ... does this matter?) [*Answer:* $\frac{2}{9}\mu_0 M Q R^2$]

8.3 ■ MAGNETIC FORCES DO NO WORK[12]

This is perhaps a good place to revisit the old paradox that magnetic forces do no work (Eq. 5.11). What about that magnetic crane lifting the carcass of a junked car? *Somebody* is doing work on the car, and if it's not the magnetic field, who

[10] In Ex. 8.4 we turned the current off slowly, to keep things quasistatic; here we reduce the electric field slowly to keep the displacement current negligible.

[11] This version of the Feynman disk paradox was proposed by N. L. Sharma (*Am. J. Phys.* **56**, 420 (1988)); similar models were analyzed by E. M. Pugh and G. E. Pugh, *Am. J. Phys.* **35**, 153 (1967) and by R. H. Romer, *Am. J. Phys.* **35**, 445 (1967).

[12] This section can be skipped without loss of continuity. I include it for those readers who are disturbed by the notion that magnetic forces do no work.

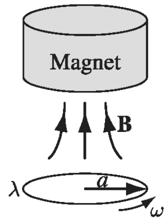

FIGURE 8.8

is it? The car is ferromagnetic; in the presence of the magnetic field, it contains a lot of microscopic magnetic dipoles (spinning electrons, actually), all lined up. The resulting magnetization is equivalent to a bound current running around the surface, so let's model the car as a circular current loop—in fact, let's make it an insulating ring of line charge λ rotating at angular velocity ω (Fig. 8.8).

The upward magnetic force on the loop is (Eq. 6.2)

$$F = 2\pi I a B_s, \tag{8.35}$$

where B_s is the radial component of the magnet's field,[13] and $I = \lambda\omega a$. If the ring rises a distance dz (while the magnet itself stays put), the work done on it is

$$dW = 2\pi a^2 \lambda\omega B_s\, dz. \tag{8.36}$$

This increases the potential energy of the ring. Who did the work? Naively, it appears that the magnetic field is responsible, but we have already learned (Ex. 5.3) that this is not the case—as the ring rises, the magnetic force is perpendicular to the *net* velocity of the charges in the ring, so it does *no* work on them.

At the same time, however, a motional emf is induced in the ring, which opposes the flow of charge, and hence reduces its angular velocity:

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

Here $d\Phi$ is the flux through the "ribbon" joining the ring at time t to the ring at time $t + dt$ (Fig. 8.9):

$$d\Phi = B_s\, 2\pi a\, dz.$$

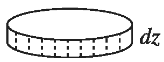

FIGURE 8.9

[13]Note that the field has to be *nonuniform*, or it won't lift the car at all.

Now

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} = f(2\pi a),$$

where **f** is the force per unit charge. So

$$f = -B_s \frac{dz}{dt}, \tag{8.37}$$

the force on a segment of length dl is $f\lambda\, dl$, the torque on the ring is

$$N = a\left(-B_s \frac{dz}{dt}\right)\lambda(2\pi a),$$

and the work done (slowing the rotation) is $N\, d\phi = N\omega\, dt$, or

$$dW = -2\pi a^2 \lambda \omega B_s\, dz. \tag{8.38}$$

The ring slows down, and the rotational energy it loses (Eq. 8.38) is precisely equal to the potential energy it gains (Eq. 8.36). All the magnetic field did was convert energy from one form to another. If you'll permit some sloppy language, the work done by the vertical component of the magnetic force (Eq. 8.35) is equal and opposite to the work done by its horizontal component (Eq. 8.37).[14]

What about the magnet? Is it completely passive in this process? Suppose we model it as a big circular loop (radius b), resting on a table and carrying a current I_b; the "junk car" is a relatively small current loop (radius a), on the floor directly below, carrying a current I_a (Fig. 8.10). This time, just for a change, let's assume both currents are constant (we'll include a regulated power supply in each loop[15]). Parallel currents attract, and we propose to lift the small loop off the floor, keeping careful track of the work done and the agency responsible.

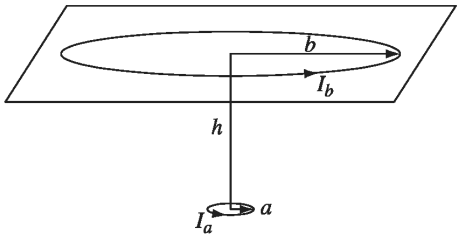

FIGURE 8.10

[14]This argument is essentially the same as the one in Ex. 5.3, except that in this case I told the story in terms of motional emf, instead of the Lorentz force law. But after all, the flux rule is a *consequence* of the Lorentz force law.

[15]The lower loop could be a single spinning electron, in which case quantum mechanics fixes its angular momentum at $\hbar/2$. It might appear that this sustains the current, with no need for a power supply. I will return to this point, but for now let's just keep quantum mechanics out of it.

Let's start by adjusting the currents so the small ring just "floats," a distance h below the table, with the magnetic force exactly balancing the weight ($m_a g$) of the little ring. I'll let you calculate the magnetic force (Prob. 8.11):

$$F_{\text{mag}} = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2b^2h}{(b^2+h^2)^{5/2}} = m_a g. \tag{8.39}$$

Now the loop rises an infinitesimal distance dz; the work done is equal to the gain in its potential energy

$$dW_g = m_a g\, dz = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2b^2h}{(b^2+h^2)^{5/2}}\, dz. \tag{8.40}$$

Who did it? The magnetic field? *No!* The work was done by the power supply that sustains the current in loop a (Ex. 5.3). As the loop rises, a motional emf is induced in it. The flux through the loop is

$$\Phi_a = M I_b,$$

where M is the mutual inductance of the two loops:

$$M = \frac{\pi\mu_0}{2}\frac{a^2b^2}{(b^2+h^2)^{3/2}}$$

(Prob. 7.22). The emf is

$$\begin{aligned} \mathcal{E}_a &= -\frac{d\Phi_a}{dt} = -I_b\frac{dM}{dt} = -I_b\frac{dM}{dh}\frac{dh}{dt} \\ &= -I_b\left(-\frac{3}{2}\right)\frac{\pi\mu_0}{2}\frac{a^2b^2}{(b^2+h^2)^{5/2}}2h\frac{(-dz)}{dt}. \end{aligned}$$

The work done by the power supply (fighting against this motional emf) is

$$dW_a = -\mathcal{E}_a I_a\, dt = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2b^2h}{(b^2+h^2)^{5/2}}\, dz \tag{8.41}$$

—same as the work done in lifting the loop (Eq. 8.40).

Meanwhile, however, a *Faraday* emf is induced in the *upper* loop, due to the changing flux from the lower loop:

$$\Phi_b = M I_a \Rightarrow \mathcal{E}_b = -I_a\frac{dM}{dt},$$

and the work done by the power supply in ring b (to sustain the current I_b) is

$$dW_b = -\mathcal{E}_b I_b\, dt = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2b^2h}{(b^2+h^2)^{5/2}}\, dz, \tag{8.42}$$

exactly the same as dW_a. That's embarrassing—the power supplies have done *twice* as much work as was necessary to lift the junk car! Where did the "wasted" energy go? *Answer:* It increased the energy stored in the fields. The energy in a system of two current-carrying loops is (see Prob. 8.12)

$$U = \frac{1}{2}L_a I_a^2 + \frac{1}{2}L_b I_b^2 + M I_a I_b, \tag{8.43}$$

so

$$dU = I_a I_b \frac{dM}{dt} dt = dW_b.$$

Remarkably, all four energy increments are the same. If we care to apportion things this way, the power supply in loop a contributes the energy necessary to lift the lower ring, while the power supply in loop b provides the extra energy for the fields. If all we're interested in is the work done to raise the ring, we can ignore the upper loop (and the energy in the fields) altogether.

In both these models, the magnet itself was stationary. That's like lifting a paper clip by holding a magnet over it. But in the case of the magnetic crane, the car stays in contact with the magnet, which is attached to a cable that lifts the whole works. As a model, we might stick the upper loop in a big box, the lower loop in a little box, and crank up the currents so the force of attraction is much greater than $m_a g$; the two boxes snap together, and we attach a string to the upper box and pull up on it (Fig. 8.11).

The same old mechanism (Ex. 5.3) prevails: as the lower loop rises, the magnetic force tilts backwards; its vertical component lifts the loop, but its horizontal component opposes the current, and no net work is done. This time, however, the motional emf is perfectly balanced by the Faraday emf fighting to keep the current going—the flux through the lower loop is not changing. (If you like, the flux is *increasing* because the loop is moving upward, into a region of higher magnetic field, but it is *decreasing* because the magnetic field of the upper loop—at any give point in space—is decreasing as that loop moves up.) No power supply is needed to sustain the current (and for that matter, no power supply is required in the upper loop either, since the energy in the fields is not changing. Who did the work to lift the car? The person pulling up on the rope, obviously. The role of the magnetic field was merely to transmit this energy to the car, via the vertical component of the magnetic force. But the magnetic field itself (as always) did no work.

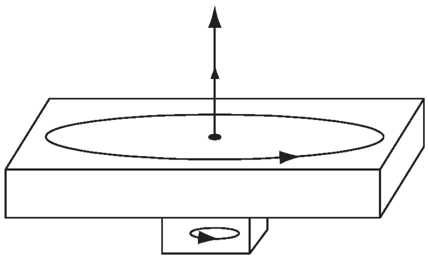

FIGURE 8.11

The fact that magnetic fields do no work follows directly from the Lorentz force law, so if you think you have discovered an exception, you're going to have to explain why that law is incorrect. For example, if magnetic monopoles exist, the force on a particle with electric charge q_e and magnetic charge q_m becomes (Prob. 7.38):

$$\mathbf{F} = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q_m\,(\mathbf{B} - \epsilon_0\mu_0\mathbf{v} \times \mathbf{E})\,. \tag{8.44}$$

In that case, magnetic fields *can* do work ... but *only on magnetic charges*. So unless your car is made of monopoles (I don't think so), this doesn't solve the problem.

A somewhat less radical possibility is that in addition to electric charges there exist permanent point magnetic dipoles (electrons?), whose dipole moment $\mathbf{m}$ is not associated with any electric current, but simply *is*. The Lorentz force law acquires an extra term

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \nabla(\mathbf{m} \cdot \mathbf{B}).$$

The magnetic field *can* do work on these "intrinsic" dipoles (which experience no motional or Faraday emf, since they enclose no flux). I don't know whether a consistent theory can be constructed in this way, but in any event it is *not* classical electrodynamics, which is predicated on Ampère's assumption that all magnetic phenomena are due to electric charges in motion, and point magnetic dipoles must be interpreted as the limits of tiny current loops.

Problem 8.11 Derive Eq. 8.39. [*Hint:* Treat the lower loop as a magnetic dipole.]

Problem 8.12 Derive Eq. 8.43. [*Hint:* Use the method of Section 7.2.4, building the two currents up from zero to their final values.]

More Problems on Chapter 8

Problem 8.13[16] A very long solenoid of radius a, with n turns per unit length, carries a current I_s. Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire, with resistance R. When the current in the solenoid is (gradually) decreased, a current I_r is induced in the ring.

(a) Calculate I_r, in terms of dI_s/dt.

(b) The power ($I_r^2 R$) delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid (the *electric* field is due to the changing flux in the solenoid; the *magnetic* field is due to the current in the ring). Integrate over the entire surface of the solenoid, and check that you recover the correct total power.

[16] For extensive discussion, see M. A. Heald, *Am. J. Phys.* **56**, 540 (1988).

Problem 8.14 An infinitely long cylindrical tube, of radius a, moves at constant speed v along its axis. It carries a net charge per unit length λ, uniformly distributed over its surface. Surrounding it, at radius b, is another cylinder, moving with the same velocity but carrying the opposite charge ($-\lambda$). Find:

(a) The energy per unit length stored in the fields.

(b) The momentum per unit length in the fields.

(c) The energy per unit time transported by the fields across a plane perpendicular to the cylinders.

Problem 8.15 A point charge q is located at the center of a toroidal coil of rectangular cross section, inner radius a, outer radius $a + w$, and height h, which carries a total of N tightly-wound turns and current I.

(a) Find the electromagnetic momentum $\mathbf{p}$ of this configuration, assuming that w and h are both much less than a (so you can ignore the variation of the fields over the cross section).

(b) Now the current in the toroid is turned off, quickly enough that the point charge does not move appreciably as the magnetic field drops to zero. Show that the impulse imparted to q is equal to the momentum originally stored in the electromagnetic fields. [*Hint:* You might want to refer to Prob. 7.19.]

Problem 8.16[17] A sphere of radius R carries a uniform polarization $\mathbf{P}$ and a uniform magnetization $\mathbf{M}$ (not necessarily in the same direction). Find the electromagnetic momentum of this configuration. [*Answer:* $(4/9)\pi\mu_0 R^3(\mathbf{M} \times \mathbf{P})$]

Problem 8.17[18] Picture the electron as a uniformly charged spherical shell, with charge e and radius R, spinning at angular velocity ω.

(a) Calculate the total energy contained in the electromagnetic fields.

(b) Calculate the total angular momentum contained in the fields.

(c) According to the Einstein formula ($E = mc^2$), the energy in the fields should contribute to the mass of the electron. Lorentz and others speculated that the *entire* mass of the electron might be accounted for in this way: $U_{\text{em}} = m_e c^2$. Suppose, moreover, that the electron's spin angular momentum is entirely attributable to the electromagnetic fields: $L_{\text{em}} = \hbar/2$. On these two assumptions, determine the radius and angular velocity of the electron. What is their product, ωR? Does this classical model make sense?

Problem 8.18 Work out the formulas for u, $\mathbf{S}$, $\mathbf{g}$, and $\overleftrightarrow{\mathbf{T}}$ in the presence of magnetic charge. [*Hint:* Start with the generalized Maxwell equations (7.44) and Lorentz force law (Eq. 8.44), and follow the derivations in Sections 8.1.2, 8.2.2, and 8.2.3.]

[17] For an interesting discussion and references, see R. H. Romer, *Am. J. Phys.* **63**, 777 (1995).
[18] See J. Higbie, *Am. J. Phys.* **56**, 378 (1988).

! **Problem 8.19**[19] Suppose you had an electric charge q_e and a magnetic monopole q_m. The field of the electric charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_e}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}$$

(of course), and the field of the magnetic monopole is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}.$$

Find the total angular momentum stored in the fields, if the two charges are separated by a distance d. [*Answer:* $(\mu_0/4\pi)q_e q_m$.][20]

Problem 8.20 Consider an ideal stationary magnetic dipole $\mathbf{m}$ in a static electric field $\mathbf{E}$. Show that the fields carry momentum

$$\mathbf{p} = -\epsilon_0\mu_0(\mathbf{m}\times\mathbf{E}). \tag{8.45}$$

[*Hint:* There are several ways to do this. The simplest method is to start with $\mathbf{p} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$, write $\mathbf{E} = -\nabla V$, and use integration by parts to show that

$$\mathbf{p} = \epsilon_0\mu_0\int V\mathbf{J}\,d\tau.$$

So far, this is valid for *any* localized static configuration. For a current confined to an infinitesimal neighborhood of the origin we can approximate $V(\mathbf{r}) \approx V(\mathbf{0}) - \mathbf{E}(\mathbf{0})\cdot\mathbf{r}$. Treat the dipole as a current loop, and use Eqs. 5.82 and 1.108.][21]

Problem 8.21 Because the cylinders in Ex. 8.4 are left rotating (at angular velocities ω_a and ω_b, say), there is actually a residual magnetic field, and hence angular momentum in the fields, even after the current in the solenoid has been extinguished. If the cylinders are heavy, this correction will be negligible, but it is interesting to do the problem *without* making that assumption.[22]

(a) Calculate (in terms of ω_a and ω_b) the final angular momentum in the fields. [Define $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$, so ω_a and ω_b could be positive or negative.]

(b) As the cylinders begin to rotate, their changing magnetic field induces an extra azimuthal electric field, which, in turn, will make an additional contribution to

[19] This system is known as **Thomson's dipole**. See I. Adawi, *Am. J. Phys.* **44**, 762 (1976) and *Phys. Rev.* **D31**, 3301 (1985), and K. R. Brownstein, *Am. J. Phys.* **57**, 420 (1989), for discussion and references.

[20] Note that this result is *independent of the separation distance d*! It points from q_e toward q_m. In quantum mechanics, angular momentum comes in half-integer multiples of $\hbar$, so this result suggests that if magnetic monopoles exist, electric and magnetic charge must be quantized: $\mu_0 q_e q_m/4\pi = n\hbar/2$, for $n = 1, 2, 3, \ldots$, an idea first proposed by Dirac in 1931. If even *one* monopole is lurking somewhere in the universe, this would "explain" why electric charge comes in discrete units. (However, see D. Singleton, *Am. J. Phys.* **66**, 697 (1998) for a cautionary note.)

[21] As it stands, Eq. 8.45 is valid only for *ideal* dipoles. But $\mathbf{g}$ is linear in $\mathbf{B}$, and therefore, if $\mathbf{E}$ is held fixed, obeys the superposition principle: For a *collection* of magnetic dipoles, the total momentum is the (vector) sum of the momenta for each one separately. In particular, if $\mathbf{E}$ is *uniform* over a localized steady current distribution, then Eq. 8.45 is valid for the whole thing, only now $\mathbf{m}$ is the *total* magnetic dipole moment.

[22] This problem was suggested by Paul DeYoung.

the torques. Find the resulting extra angular momentum, and compare it with your result in (a). [*Answer:* $-\mu_0 Q^2 \omega_b (b^2 - a^2)/4\pi l\,\hat{\mathbf{z}}$]

Problem 8.22[23] A point charge q is a distance $a > R$ from the axis of an infinite solenoid (radius R, n turns per unit length, current I). Find the linear momentum and the angular momentum (with respect to the origin) in the fields. (Put q on the x axis, with the solenoid along z; treat the solenoid as a nonconductor, so you don't need to worry about induced charges on its surface.) [*Answer:* $\mathbf{p} = (\mu_0 qnIR^2/2a)\,\hat{\mathbf{y}}$; $\mathbf{L} = \mathbf{0}$]

Problem 8.23

(a) Carry through the argument in Sect. 8.1.2, starting with Eq. 8.6, but using $\mathbf{J}_f$ in place of $\mathbf{J}$. Show that the Poynting vector becomes

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{8.46}$$

and the rate of change of the energy density in the fields is

$$\frac{\partial u}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}.$$

For *linear* media, show that[24]

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}). \tag{8.47}$$

(b) In the same spirit, reproduce the argument in Sect. 8.2.2, starting with Eq. 8.15, with ρ_f and $\mathbf{J}_f$ in place of ρ and $\mathbf{J}$. Don't bother to construct the Maxwell stress tensor, but do show that the momentum density is[25]

$$\mathbf{g} = \mathbf{D} \times \mathbf{B}. \tag{8.48}$$

Problem 8.24

A circular disk of radius R and mass M carries n point charges (q), attached at regular intervals around its rim. At time $t = 0$ the disk lies in the xy plane, with its center at the origin, and is rotating about the z axis with angular velocity ω_0, when it is released. The disk is immersed in a (time-independent) external magnetic field

$$\mathbf{B}(s, z) = k(-s\,\hat{\mathbf{s}} + 2z\,\hat{\mathbf{z}}),$$

where k is a constant.

(a) Find the position of the center if the ring, $z(t)$, and its angular velocity, $\omega(t)$, as functions of time. (Ignore gravity.)

(b) Describe the motion, and check that the total (kinetic) energy—translational plus rotational—is constant, confirming that the magnetic force does no work.[26]

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[23] See F. S. Johnson, B. L. Cragin, and R. R. Hodges, *Am. J. Phys.* **62**, 33 (1994), and B. Y.-K. Hu, *Eur. J. Phys.* **33**, 873 (2012), for discussion of this and related problems.

[24] Refer to Sect. 4.4.3 for the meaning of "energy" in this context.

[25] For over 100 years there has been a raging debate (still not completely resolved) as to whether the field momentum in polarizable/magnetizable media is Eq. 8.48 (Minkowski's candidate) or $\epsilon_0\mu_0$ ($\mathbf{E} \times \mathbf{H}$) (Abraham's). See D. J. Griffiths, *Am. J. Phys.* **80**, 7 (2012).

[26] This cute problem is due to K. T. McDonald, http://puhep1.princeton.edu/mcdonald/examles/disk.pdf (who draws a somewhat different conclusion).

CHAPTER

9 Electromagnetic Waves

9.1 ■ WAVES IN ONE DIMENSION

9.1.1 ■ The Wave Equation

What is a "wave"? I don't think I can give you an entirely satisfactory answer—the concept is intrinsically somewhat vague—but here's a start: A wave is a *disturbance of a continuous medium that propagates with a fixed shape at constant velocity.* Immediately I must add qualifiers: In the presence of absorption, the wave will diminish in size as it moves; if the medium is dispersive, different frequencies travel at different speeds; in two or three dimensions, as the wave spreads out, its amplitude will decrease; and of course *standing* waves don't propagate at all. But these are refinements; let's start with the simple case: fixed shape, constant speed (Fig. 9.1).

How would you represent such an object mathematically? In the figure, I have drawn the wave at two different times, once at $t = 0$, and again at some later time t—each point on the wave form simply shifts to the right by an amount vt, where v is the velocity. Maybe the wave is generated by shaking one end of a taut string; $f(z, t)$ represents the displacement of the string at the point z, at time t. Given the *initial* shape of the string, $g(z) \equiv f(z, 0)$, what is the subsequent form, $f(z, t)$? Well, the displacement at point z, at the later time t, is the same as the displacement a distance vt to the left (i.e. at $z - vt$), back at time $t = 0$:

$$f(z, t) = f(z - vt, 0) = g(z - vt). \tag{9.1}$$

That statement captures (mathematically) the essence of wave motion. It tells us that the function $f(z, t)$, which *might* have depended on z and t in *any* old way, in *fact* depends on them only in the very special combination $z - vt$; when that

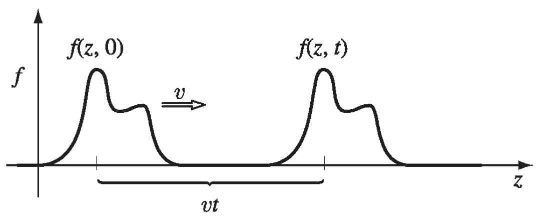

FIGURE 9.1

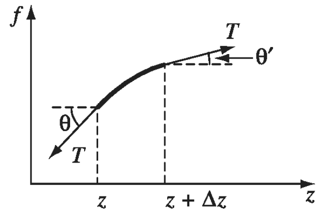

FIGURE 9.2

is true, the function $f(z, t)$ represents a wave of fixed shape traveling in the z direction at speed v. For example, if A and b are constants (with the appropriate units),

$$f_1(z, t) = Ae^{-b(z-vt)^2}, \quad f_2(z, t) = A\sin[b(z - vt)], \quad f_3(z, t) = \frac{A}{b(z - vt)^2 + 1}$$

all represent waves (with different shapes, of course), but

$$f_4(z, t) = Ae^{-b(bz^2+vt)}, \quad \text{and} \quad f_5(z, t) = A\sin(bz)\cos(bvt)^3,$$

do *not*.

Why does a stretched string support wave motion? Actually, it follows from Newton's second law. Imagine a very long string under tension T. If it is displaced from equilibrium, the net transverse force on the segment between z and $z + \Delta z$ (Fig. 9.2) is

$$\Delta F = T\sin\theta' - T\sin\theta,$$

where θ' is the angle the string makes with the z-direction at point $z + \Delta z$, and θ is the corresponding angle at point z. Provided that the distortion of the string is not too great, these angles are small (the figure is exaggerated, obviously), and we can replace the sine by the tangent:

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z.$$

If the mass per unit length is μ, Newton's second law says

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2},$$

and therefore

$$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}.$$

Evidently, small disturbances on the string satisfy

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2},} \tag{9.2}$$

where v (which, as we'll soon see, represents the speed of propagation) is

$$v = \sqrt{\frac{T}{\mu}}. \tag{9.3}$$

Equation 9.2 is known as the (classical) **wave equation**, because it admits as solutions all functions of the form

$$f(z,t) = g(z - vt), \tag{9.4}$$

(that is, all functions that depend on the variables z and t in the special combination $u \equiv z - vt$), and we have just learned that such functions represent waves propagating in the z direction with speed v. For Eq. 9.4 means

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}, \quad \frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du},$$

and

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2},$$

$$\frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2},$$

so

$$\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}. \qquad \square$$

Note that $g(u)$ can be *any* (differentiable) *function whatever.* If the disturbance propagates without changing its shape, then it satisfies the wave equation.

But functions of the form $g(z - vt)$ are not the only solutions. The wave equation involves the *square* of v, so we can generate another class of solutions by simply changing the sign of the velocity:

$$f(z,t) = h(z + vt). \tag{9.5}$$

This, of course, represents a wave propagating in the *negative* z direction, and it is certainly reasonable (on physical grounds) that such solutions would be allowed. What *is* perhaps surprising is that the *most general* solution to the wave equation is the sum of a wave to the right and a wave to the left:

$$f(z,t) = g(z - vt) + h(z + vt). \tag{9.6}$$

(Notice that the wave equation is **linear**: The sum of any two solutions is itself a solution.) *Every* solution to the wave equation can be expressed in this form.

Like the simple harmonic oscillator equation, the wave equation is ubiquitous in physics. If something is vibrating, the oscillator equation is almost certainly responsible (at least, for small amplitudes), and if something is waving (whether the context is mechanics or acoustics, optics or oceanography), the wave equation (perhaps with some decoration) is bound to be involved.

Problem 9.1 By explicit differentiation, check that the functions f_1, f_2, and f_3 in the text satisfy the wave equation. Show that f_4 and f_5 do *not*.

Problem 9.2 Show that the **standing wave** $f(z, t) = A \sin(kz) \cos(kvt)$ satisfies the wave equation, and express it as the sum of a wave traveling to the left and a wave traveling to the right (Eq. 9.6).

9.1.2 ■ Sinusoidal Waves

(i) Terminology. Of all possible wave forms, the sinusoidal one

$$f(z, t) = A \cos[k(z - vt) + \delta] \tag{9.7}$$

is (for good reason) the most familiar. Figure 9.3 shows this function at time $t = 0$. A is the **amplitude** of the wave (it is positive, and represents the maximum displacement from equilibrium). The argument of the cosine is called the **phase**, and δ is the **phase constant** (obviously, you can add any integer multiple of 2π to δ without changing $f(z, t)$; ordinarily, one uses a value in the range $0 \leq \delta < 2\pi$). Notice that at $z = vt - \delta/k$, the phase is zero; let's call this the "central maximum." If $\delta = 0$, the central maximum passes the origin at time $t = 0$; more generally, δ/k is the distance by which the central maximum (and therefore the entire wave) is "delayed." Finally, k is the **wave number**; it is related to the **wavelength** λ by the equation

$$\lambda = \frac{2\pi}{k}, \tag{9.8}$$

for when z advances by $2\pi/k$, the cosine executes one complete cycle.

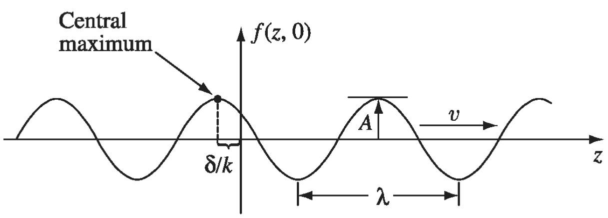

FIGURE 9.3

Now that we have taken care of the exponential factors—they cancel, given Eq. 9.94—the boundary conditions (Eq. 9.74) become:

$$\left.\begin{array}{ll} \text{(i)} & \epsilon_1\left(\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R}\right)_z=\epsilon_2\left(\tilde{\mathbf{E}}_{0_T}\right)_z \\ \text{(ii)} & \left(\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R}\right)_z=\left(\tilde{\mathbf{B}}_{0_T}\right)_z \\ \text{(iii)} & \left(\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R}\right)_{x,y}=\left(\tilde{\mathbf{E}}_{0_T}\right)_{x,y} \\ \text{(iv)} & \dfrac{1}{\mu_1}\left(\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R}\right)_{x,y}=\dfrac{1}{\mu_2}\left(\tilde{\mathbf{B}}_{0_T}\right)_{x,y} \end{array}\right\} \tag{9.101}$$

where $\tilde{\mathbf{B}}_0=(1/v)\hat{\mathbf{k}}\times\tilde{\mathbf{E}}_0$ in each case. (The last two represent *pairs* of equations, one for the x-component and one for the y-component.)

Suppose the polarization of the incident wave is *parallel* to the plane of incidence (the xz plane); it follows (see Prob. 9.15) that the reflected and transmitted waves are also polarized in this plane (Fig. 9.15). (I shall leave it for you to analyze the case of polarization *perpendicular* to the plane of incidence; see Prob. 9.17.) Then (i) reads

$$\epsilon_1\left(-\tilde{E}_{0_I}\sin\theta_I+\tilde{E}_{0_R}\sin\theta_R\right)=\epsilon_2\left(-\tilde{E}_{0_T}\sin\theta_T\right); \tag{9.102}$$

(ii) adds nothing ($0=0$), since the magnetic fields have no z components; (iii) becomes

$$\tilde{E}_{0_I}\cos\theta_I+\tilde{E}_{0_R}\cos\theta_R=\tilde{E}_{0_T}\cos\theta_T; \tag{9.103}$$

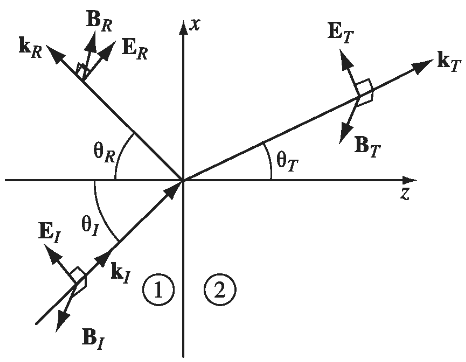

FIGURE 9.15

and (iv) says

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0_I} - \tilde{E}_{0_R}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{0_T}. \tag{9.104}$$

Given the laws of reflection and refraction, Eqs. 9.102 and 9.104 both reduce to

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T}, \tag{9.105}$$

where (as before)

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}, \tag{9.106}$$

and Eq. 9.103 says

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \alpha \tilde{E}_{0_T}, \tag{9.107}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}. \tag{9.108}$$

Solving Eqs. 9.105 and 9.107 for the reflected and transmitted amplitudes, we obtain

$$\boxed{\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0_I}.} \tag{9.109}$$

These are known as **Fresnel's equations**, for the case of polarization in the plane of incidence. (There are two other Fresnel equations, giving the reflected and transmitted amplitudes when the polarization is *perpendicular* to the plane of incidence—see Prob. 9.17.) Notice that the transmitted wave is always *in phase* with the incident one; the reflected wave is either in phase ("right side up"), if $\alpha > \beta$, or 180° out of phase ("upside down"), if $\alpha < \beta$.[15]

The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because α is a function of θ_I:

$$\alpha = \frac{\sqrt{1 - \sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1 - [(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}. \tag{9.110}$$

In the case of normal incidence ($\theta_I = 0$), $\alpha = 1$, and we recover Eq. 9.82. At grazing incidence ($\theta_I = 90°$), α diverges, and the wave is totally reflected (a fact

[15] There is an unavoidable ambiguity in the phase of the reflected wave, since (as I mentioned in the footnote to Eq. 9.36) changing the sign of the polarization vector is equivalent to a 180° phase shift. The convention I adopted in Fig. 9.15, with $\mathbf{E}_R$ positive "upward," is consistent with some, but not all, of the standard optics texts.

that is painfully familiar to anyone who has driven at night on a wet road). Interestingly, there is an intermediate angle, θ_B (called **Brewster's angle**), at which the reflected wave is completely extinguished.[16] According to Eq. 9.109, this occurs when $\alpha = \beta$, or

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2-\beta^2}. \tag{9.111}$$

For the typical case $\mu_1 \cong \mu_2$, so $\beta \cong n_2/n_1$, $\sin^2\theta_B \cong \beta^2/(1+\beta^2)$, and hence

$$\tan\theta_B \cong \frac{n_2}{n_1}. \tag{9.112}$$

Figure 9.16 shows a plot of the transmitted and reflected amplitudes as functions of θ_I, for light incident on glass ($n_2 = 1.5$) from air ($n_1 = 1$). (On the graph, a *negative* number indicates that the wave is 180° out of phase with the incident beam—the amplitude itself is the absolute value.)

The power per unit area striking the interface is $\mathbf{S}\cdot\hat{\mathbf{z}}$. Thus the incident intensity is

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0_I}^2 \cos\theta_I, \tag{9.113}$$

while the reflected and transmitted intensities are

$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0_R}^2 \cos\theta_R, \quad \text{and} \quad I_T = \frac{1}{2}\epsilon_2 v_2 E_{0_T}^2 \cos\theta_T. \tag{9.114}$$

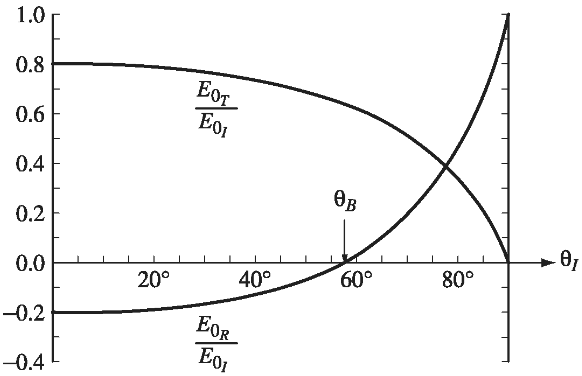

FIGURE 9.16

[16] Because waves polarized *perpendicular* to the plane of incidence exhibit no corresponding quenching of the reflected component, an arbitrary beam incident at Brewster's angle yields a reflected beam that is *totally* polarized parallel to the interface. That's why Polaroid glasses, with the transmission axis vertical, help to reduce glare off a horizontal surface.

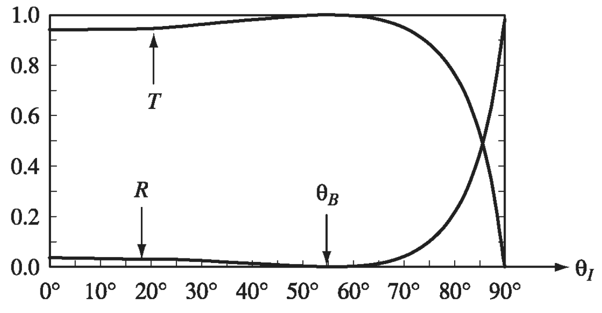

FIGURE 9.17

(The cosines are there because I am talking about the average power per unit area of *interface*, and the interface is at an angle to the wave front.) The reflection and transmission coefficients for waves polarized parallel to the plane of incidence are

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2, \tag{9.115}$$

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2. \tag{9.116}$$

They are plotted as functions of the angle of incidence in Fig. 9.17 (for the air/glass interface). R is the fraction of the incident energy that is reflected—naturally, it goes to zero at Brewster's angle; T is the fraction transmitted—it goes to 1 at θ_B. Note that $R + T = 1$, as required by conservation of energy: the energy per unit time *reaching* a particular patch of area on the surface is equal to the energy per unit time *leaving* the patch.

Problem 9.16 Suppose $Ae^{iax} + Be^{ibx} = Ce^{icx}$, for some nonzero constants A, B, C, a, b, c, and for all x. Prove that $a = b = c$ and $A + B = C$.

! **Problem 9.17** Analyze the case of polarization *perpendicular* to the plane of incidence (i.e. electric fields in the y direction, in Fig. 9.15). Impose the boundary conditions (Eq. 9.101), and obtain the Fresnel equations for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$. Sketch $(\tilde{E}_{0_R}/\tilde{E}_{0_I})$ and $(\tilde{E}_{0_T}/\tilde{E}_{0_I})$ as functions of θ_I, for the case $\beta = n_2/n_1 = 1.5$. (Note that for this β the reflected wave is *always* 180° out of phase.) Show that there is no Brewster's angle for *any* n_1 and n_2: $\tilde{E}_{0_R}$ is *never* zero (unless, of course, $n_1 = n_2$ and $\mu_1 = \mu_2$, in which case the two media are optically indistinguishable). Confirm that your Fresnel equations reduce to the proper forms at normal incidence. Compute the reflection and transmission coefficients, and check that they add up to 1.

Problem 9.18 The index of refraction of diamond is 2.42. Construct the graph analogous to Fig. 9.16 for the air/diamond interface. (Assume $\mu_1 = \mu_2 = \mu_0$.) In particular, calculate (a) the amplitudes at normal incidence, (b) Brewster's angle, and (c) the "crossover" angle, at which the reflected and transmitted amplitudes are equal.

9.4 ■ ABSORPTION AND DISPERSION

9.4.1 ■ Electromagnetic Waves in Conductors

In Sect. 9.3 I stipulated that the free charge density ρ_f and the free current density $\mathbf{J}_f$ are zero, and everything that followed was predicated on that assumption. Such a restriction is perfectly reasonable when you're talking about wave propagation through a vacuum or through insulating materials such as glass or (pure) water. But in the case of conductors we do not independently control the flow of charge, and in general $\mathbf{J}_f$ is certainly *not* zero. In fact, according to Ohm's law, the (free) current density in a conductor is proportional to the electric field:

$$\mathbf{J}_f = \sigma\mathbf{E}. \tag{9.117}$$

With this, Maxwell's equations for linear media assume the form

$$\left.\begin{array}{ll} \text{(i)}\;\; \nabla\cdot\mathbf{E} = \dfrac{1}{\epsilon}\rho_f, & \text{(iii)}\;\; \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}, \\[2ex] \text{(ii)}\;\; \nabla\cdot\mathbf{B} = 0, & \text{(iv)}\;\; \nabla\times\mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon\dfrac{\partial\mathbf{E}}{\partial t}. \end{array}\right\} \tag{9.118}$$

Now, the continuity equation for free charge,

$$\nabla\cdot\mathbf{J}_f = -\frac{\partial\rho_f}{\partial t}, \tag{9.119}$$

together with Ohm's law and Gauss's law (i), gives

$$\frac{\partial\rho_f}{\partial t} = -\sigma(\nabla\cdot\mathbf{E}) = -\frac{\sigma}{\epsilon}\rho_f$$

for a homogeneous linear medium, from which it follows that

$$\rho_f(t) = e^{-(\sigma/\epsilon)t}\rho_f(0). \tag{9.120}$$

Thus any initial free charge $\rho_f(0)$ dissipates in a characteristic time $\tau \equiv \epsilon/\sigma$. This reflects the familiar fact that if you put some free charge on a conductor, it will flow out to the edges. The time constant τ affords a measure of how "good" a conductor is: For a "perfect" conductor, $\sigma = \infty$ and $\tau = 0$; for a "good" conductor, τ is much less than the other relevant times in the problem (in oscillatory systems, that means $\tau \ll 1/\omega$); for a "poor" conductor, τ is *greater* than the characteristic times in the problem ($\tau \gg 1/\omega$).[17] But we're not interested in this transient

[17] N. Ashby, *Am. J. Phys.* **43**, 553 (1975), points out that for good conductors τ is absurdly short (10^{-19} s, for copper, whereas the time between collisions is $\tau_c = 10^{-14}$ s). The problem is that Ohm's law itself breaks down on time scales shorter than τ_c; actually, the time it takes free charge to dissipate in a good conductor is of order τ_c, not τ. Moreover, H. C. Ohanian, *Am. J. Phys.* **51**, 1020 (1983), shows that it takes even longer for the fields and currents to equilibrate. But none of this is relevant to our present purpose; the net free charge density in a conductor does quickly dissipate, and exactly how long the process takes is beside the point.

behavior—we'll wait for any accumulated free charge to disappear. From then on, $\rho_f = 0$, and we have

$$\left.\begin{array}{ll} \text{(i)}\ \nabla\cdot\mathbf{E}=0, & \text{(iii)}\ \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla\cdot\mathbf{B}=0, & \text{(iv)}\ \nabla\times\mathbf{B}=\mu\epsilon\dfrac{\partial\mathbf{E}}{\partial t}+\mu\sigma\mathbf{E}. \end{array}\right\} \tag{9.121}$$

These differ from the corresponding equations for *non*conducting media (Eq. 9.67) only in the last term in (iv)—which is absent, obviously, when $\sigma = 0$.

Applying the curl to (iii) and (iv), as before, we obtain modified wave equations for $\mathbf{E}$ and $\mathbf{B}$:

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}, \quad \nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{B}}{\partial t}. \tag{9.122}$$

These equations still admit plane-wave solutions,

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}, \tag{9.123}$$

but this time the "wave number" $\tilde{k}$ is complex:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega, \tag{9.124}$$

as you can easily check by plugging Eq. 9.123 into Eq. 9.122. Taking the square root,

$$\tilde{k} = k + i\kappa, \tag{9.125}$$

where

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}. \tag{9.126}$$

The imaginary part of $\tilde{k}$ results in an attenuation of the wave (decreasing amplitude with increasing z):

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)}. \tag{9.127}$$

The distance it takes to reduce the amplitude by a factor of $1/e$ (about a third) is called the **skin depth**:

$$d \equiv \frac{1}{\kappa}; \tag{9.128}$$

it is a measure of how far the wave penetrates into the conductor. Meanwhile, the real part of $\tilde{k}$ determines the wavelength, the propagation speed, and the index of refraction, in the usual way:

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}. \tag{9.129}$$

The attenuated plane waves (Eq. 9.127) satisfy the modified wave equation (9.122) for *any* $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$. But Maxwell's equations (9.121) impose further constraints, which serve to determine the relative amplitudes, phases, and polarizations of $\mathbf{E}$ and $\mathbf{B}$. As before, (i) and (ii) rule out any z components: the fields are *transverse*. We may as well orient our axes so that $\mathbf{E}$ is polarized along the x direction:

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{x}}. \tag{9.130}$$

Then (iii) gives

$$\tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{y}}. \tag{9.131}$$

(Equation (iv) says the same thing.) Once again, the electric and magnetic fields are mutually perpendicular.

Like any complex number, $\tilde{k}$ can be expressed in terms of its modulus and phase:

$$\tilde{k} = K e^{i\phi}, \tag{9.132}$$

where

$$K \equiv |\tilde{k}| = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \tag{9.133}$$

and

$$\phi \equiv \tan^{-1}(\kappa/k). \tag{9.134}$$

According to Eq. 9.130 and 9.131, the complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ and $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related by

$$B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}. \tag{9.135}$$

Evidently the electric and magnetic fields are no longer in phase; in fact,

$$\delta_B - \delta_E = \phi; \tag{9.136}$$

the magnetic field *lags behind* the electric field. Meanwhile, the (real) amplitudes of $\mathbf{E}$ and $\mathbf{B}$ are related by

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}. \tag{9.137}$$

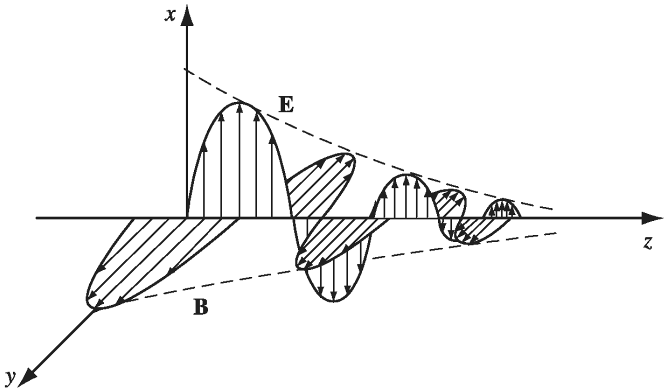

FIGURE 9.18

The (real) electric and magnetic fields are, finally,

$$\left.\begin{aligned} \mathbf{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}}, \\ \mathbf{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}. \end{aligned}\right\} \tag{9.138}$$

These fields are shown in Fig. 9.18.

Problem 9.19

(a) Suppose you imbedded some free charge in a piece of glass. About how long would it take for the charge to flow to the surface?

(b) Silver is an excellent conductor, but it's expensive. Suppose you were designing a microwave experiment to operate at a frequency of 10^{10} Hz. How thick would you make the silver coatings?

(c) Find the wavelength and propagation speed in copper for radio waves at 1 MHz. Compare the corresponding values in air (or vacuum).

Problem 9.20

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) is $(2/\sigma)\sqrt{\epsilon/\mu}$ (independent of frequency). Find the skin depth (in meters) for (pure) water. (Use the static values of ϵ, μ, and σ; your answers will be valid, then, only at relatively low frequencies.)

(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ (where λ is the wavelength *in the conductor*). Find the skin depth (in nanometers) for a typical metal ($\sigma \approx 10^7\,(\Omega\,\mathrm{m})^{-1}$) in the visible range ($\omega \approx 10^{15}$/s), assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

(c) Show that in a good conductor the magnetic field lags the electric field by 45°, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).

Problem 9.21

(a) Calculate the (time-averaged) energy density of an electromagnetic plane wave in a conducting medium (Eq. 9.138). Show that the magnetic contribution always dominates. [*Answer:* $(k^2/2\mu\omega^2)E_0^2 e^{-2\kappa z}$]

(b) Show that the intensity is $(k/2\mu\omega)E_0^2 e^{-2\kappa z}$.

9.4.2 ■ Reflection at a Conducting Surface

The boundary conditions we used to analyze reflection and refraction at an interface between two dielectrics do not hold in the presence of free charges and currents. Instead, we have the more general relations (Eq. 7.64):

$$\left.\begin{array}{ll} \text{(i)}\ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}, \end{array}\right\} \tag{9.139}$$

where σ_f (not to be confused with conductivity) is the free surface charge, $\mathbf{K}_f$ is the free surface current, and $\hat{\mathbf{n}}$ (not to be confused with the polarization of the wave) is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1). For ohmic conductors ($\mathbf{J}_f = \sigma\mathbf{E}$) there can be no free surface current, since this would require an infinite electric field at the boundary.

Suppose now that the xy plane forms the boundary between a nonconducting linear medium (1) and a conductor (2). A monochromatic plane wave, traveling in the z direction and polarized in the x direction, approaches from the left, as in Fig. 9.13:

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}}. \tag{9.140}$$

This incident wave gives rise to a reflected wave,

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}, \tag{9.141}$$

propagating back to the left in medium (1), and a transmitted wave

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{y}}, \tag{9.142}$$

which is attenuated as it penetrates into the conductor.

At $z = 0$, the combined wave in medium (1) must join the wave in medium (2), pursuant to the boundary conditions (Eq. 9.139). Since $E^{\perp} = 0$ on both sides, boundary condition (i) yields $\sigma_f = 0$. Since $B^{\perp} = 0$, (ii) is automatically satisfied. Meanwhile, (iii) gives

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}, \tag{9.143}$$

and (iv) (with $\mathbf{K}_f = 0$) says

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0_T} = 0, \tag{9.144}$$

or

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta}\tilde{E}_{0_T}, \tag{9.145}$$

where

$$\tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2. \tag{9.146}$$

It follows that

$$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}. \tag{9.147}$$

These results are formally identical to the ones that apply at the boundary between *non*conductors (Eq. 9.82), but the resemblance is deceptive since $\tilde{\beta}$ is now a complex number.

For a *perfect* conductor ($\sigma = \infty$), $k_2 = \infty$ (Eq. 9.126), so $\tilde{\beta}$ is infinite, and

$$\tilde{E}_{0_R} = -\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = 0. \tag{9.148}$$

In this case the wave is totally reflected, with a 180° phase shift. (That's why excellent conductors make good mirrors. In practice, you paint a thin coating of silver onto the back of a pane of glass—the glass has nothing to do with the *reflection*; it's just there to support the silver and to keep it from tarnishing. Since the skin depth in silver at optical frequencies is less than 100 Å, you don't need a very thick layer.)

Problem 9.22 Calculate the reflection coefficient for light at an air-to-silver interface ($\mu_1 = \mu_2 = \mu_0$, $\epsilon_1 = \epsilon_0$, $\sigma = 6 \times 10^7 (\Omega \cdot \mathrm{m})^{-1}$), at optical frequencies ($\omega = 4 \times 10^{15}/\mathrm{s}$).

9.4.3 ■ The Frequency Dependence of Permittivity

In the preceding sections, we have seen that the propagation of electromagnetic waves through matter is governed by three properties of the material: the permittivity ϵ, the permeability μ, and the conductivity σ. Actually, each of these parameters depends to some extent on the frequency of the waves you are considering. Indeed, it is well known from optics that $n \cong \sqrt{\epsilon_r}$ is a function of wavelength (Fig. 9.19 shows the graph for a typical glass). A prism or a raindrop bends blue light more sharply than red, and spreads white light out into a rainbow of colors. This phenomenon is called **dispersion**. By extension, whenever the speed of a wave depends on its frequency, the supporting medium is called **dispersive**.

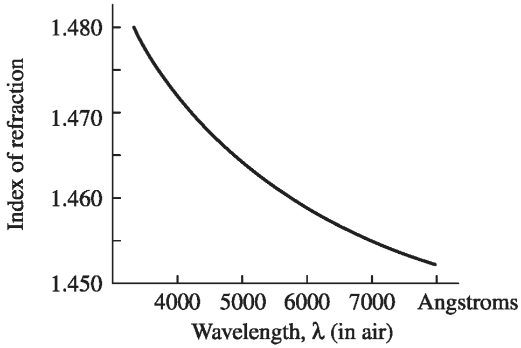

FIGURE 9.19

Because waves of different frequency travel at different speeds in a dispersive medium, a wave form that incorporates a range of frequencies will change shape as it propagates. A sharply peaked wave typically flattens out, and whereas each sinusoidal component travels at the ordinary **wave** (or **phase**) **velocity**,

$$v = \frac{\omega}{k}, \tag{9.149}$$

the packet as a whole (the "envelope") moves at the so-called **group velocity**[18]

$$v_g = \frac{d\omega}{dk}. \tag{9.150}$$

[You can demonstrate this by dropping a rock into the nearest pond and watching the waves that form: While the disturbance as a whole spreads out in a circle, moving at speed v_g, the ripples that go to make it up will be seen to travel *twice* as fast ($v = 2v_g$ in this case). They appear at the back end of the packet, growing as they move forward to the center, then shrinking again and fading away at the front (Fig. 9.20).] We shall not concern ourselves with these matters—I'll stick to monochromatic waves, for which the problem does not arise. But I should just mention that the *energy* carried by a wave packet in a dispersive medium does not travel at the phase velocity. Don't be too alarmed, therefore, if in some circumstances v comes out greater than c.[19]

[18] See A. P. French, *Vibrations and Waves* (New York: W. W. Norton & Co., 1971), p. 230, or F. S. Crawford, Jr., *Waves* (New York: McGraw-Hill, 1968), Sect. 6.2.

[19] Even the group velocity can exceed c in special cases—see P. C. Peters, *Am. J. Phys.* **56**, 129 (1988), or work Prob. 9.26. For delightful commentary, see C. F. Bohren, *Am. J. Phys.* **77**, 101 (2009). And if *two* different "speeds of light" are not enough to satisfy you, check out S. C. Bloch, *Am. J. Phys.* **45**, 538 (1977), in which no fewer than *eight* distinct velocities are identified! Indeed, it's not clear what you *mean* by the "velocity" of something that changes shape as it moves, and has no precise beginning or end. Do you mean the speed at which the *peak intensity* propagates? Or the speed at which *energy* is transported? Or *information* transmitted? In special relativity no *causal* signal can travel faster than c, but some of the other "velocities" have no such restriction.

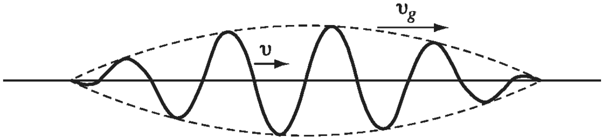

FIGURE 9.20

My purpose in this section is to account for the frequency dependence of ϵ in dielectrics, using a simplified model for the behavior of the electrons. Like all classical models of atomic-scale phenomena, it is at best an approximation to the truth; nevertheless, it does yield qualitatively satisfactory results, and it provides a plausible mechanism for dispersion in transparent media.

The electrons in a nonconductor are bound to specific molecules. The actual binding forces can be quite complicated, but we shall picture each electron as attached to the end of a spring, with force constant k_{spring} (Fig. 9.21):

$$F_{\text{binding}} = -k_{\text{spring}}x = -m\omega_0^2 x, \tag{9.151}$$

where x is displacement from equilibrium, m is the electron's mass, and ω_0 is the natural oscillation frequency, $\sqrt{k_{\text{spring}}/m}$. [If this strikes you as an implausible model, look back at Ex. 4.1, where we were led to a force of precisely this form. As a matter of fact, practically *any* binding force can be approximated this way for sufficiently small displacements from equilibrium, as you can see by expanding the potential energy in a Taylor series about the equilibrium point:

$$U(x) = U(0) + xU'(0) + \frac{1}{2}x^2U''(0) + \cdots .$$

The first term is a constant, with no dynamical significance (you can always adjust the zero of potential energy so that $U(0) = 0$). The second term automatically vanishes, since $dU/dx = -F$, and by the nature of an equilibrium, the force at that point is zero. The third term is precisely the potential energy of a spring with force constant $k_{\text{spring}} = d^2U/dx^2\big|_0$ (the second derivative is positive, for a point of stable equilibrium). As long as the displacements are small, the higher terms in the series can be neglected. Geometrically, all I am saying is that practically *any* function can be fit near a minimum by a suitable parabola.]

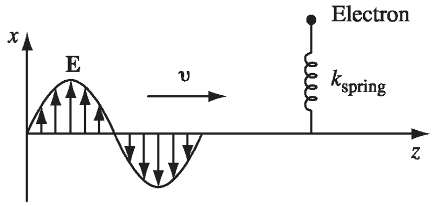

FIGURE 9.21

Meanwhile, there will presumably be some damping force on the electron:

$$F_{\text{damping}} = -m\gamma \frac{dx}{dt}. \tag{9.152}$$

[Again I have chosen the simplest possible form; the damping must be opposite in direction to the velocity, and making it *proportional* to the velocity is the easiest way to accomplish this. The *cause* of the damping does not concern us here—among other things, an oscillating charge radiates, and the radiation siphons off energy. We will calculate this "radiation damping" in Chapter 11.]

In the presence of an electromagnetic wave of frequency ω, polarized in the x direction (Fig. 9.21), the electron is subject to a driving force

$$F_{\text{driving}} = qE = qE_0 \cos(\omega t), \tag{9.153}$$

where q is the charge of the electron and E_0 is the amplitude of the wave at the point z where the electron is situated. (Since we're only interested in one point, I have reset the clock so that the maximum E occurs there at $t = 0$. For simplicity, I assume the magnetic force is negligible.) Putting all this into Newton's second law gives

$$m\frac{d^2x}{dt^2} = F_{\text{tot}} = F_{\text{binding}} + F_{\text{damping}} + F_{\text{driving}},$$

or

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos(\omega t). \tag{9.154}$$

Our model, then, describes the electron as a damped harmonic oscillator, driven at frequency ω. (The much more massive nucleus remains at rest.)

Equation 9.154 is easier to handle if we regard it as the real part of a *complex* equation:

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}. \tag{9.155}$$

In the steady state, the system oscillates at the driving frequency:

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}. \tag{9.156}$$

Inserting this into Eq. 9.155, we obtain

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0. \tag{9.157}$$

The resulting dipole moment is the real part of

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t}. \tag{9.158}$$

The imaginary term in the denominator means that p is *out of phase* with E—lagging behind by an angle $\tan^{-1}[\gamma\omega/(\omega_0^2 - \omega^2)]$ that is very small when $\omega \ll \omega_0$ and rises to π when $\omega \gg \omega_0$.

In general, differently situated electrons within a given molecule experience different natural frequencies and damping coefficients. Let's say there are f_j electrons with frequency ω_j and damping γ_j in each molecule. If there are N molecules per unit volume, the polarization $\mathbf{P}$ is given by[20] the real part of

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}}. \tag{9.159}$$

Now, I defined the electric susceptibility as the proportionality constant between $\mathbf{P}$ and $\mathbf{E}$ (specifically, $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$). In the present case, $\mathbf{P}$ is *not* proportional to $\mathbf{E}$ (this is not, strictly speaking, a linear medium) because of the difference in phase. However, the *complex* polarization $\tilde{\mathbf{P}}$ is proportional to the *complex* field $\tilde{\mathbf{E}}$, and this suggests that we introduce a **complex susceptibility**, $\tilde{\chi}_e$:

$$\tilde{\mathbf{P}} = \epsilon_0\tilde{\chi}_e\tilde{\mathbf{E}}. \tag{9.160}$$

All of the manipulations we went through before carry over, on the understanding that the physical polarization is the real part of $\tilde{\mathbf{P}}$, just as the physical field is the real part of $\tilde{\mathbf{E}}$. In particular, the proportionality between $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{E}}$ is the **complex permittivity** $\tilde{\epsilon} = \epsilon_0(1 + \tilde{\chi}_e)$, and the **complex dielectric constant** (in this model) is

$$\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}. \tag{9.161}$$

Ordinarily, the imaginary term is negligible; however, when ω is very close to one of the resonant frequencies (ω_j) it plays an important role, as we shall see.

In a dispersive medium, the wave equation for a given frequency reads

$$\nabla^2\tilde{\mathbf{E}} = \tilde{\epsilon}\mu_0\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2}; \tag{9.162}$$

it admits plane wave solutions, as before,

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)}, \tag{9.163}$$

with the complex wave number

$$\tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\,\omega. \tag{9.164}$$

[20]This applies directly to the case of a dilute gas; for denser materials the theory is modified slightly, in accordance with the Clausius-Mossotti equation (Prob. 4.41). By the way, don't confuse the "polarization" of a medium, $\mathbf{P}$, with the "polarization" of a *wave*—same *word*, but two completely unrelated meanings.

Writing $\tilde{k}$ in terms of its real and imaginary parts,

$$\tilde{k} = k + i\kappa, \tag{9.165}$$

Eq. 9.163 becomes

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}. \tag{9.166}$$

The wave is *attenuated* (this is hardly surprising, since the damping absorbs energy). Because the intensity is proportional to E^2 (and hence to $e^{-2\kappa z}$), the quantity

$$\alpha \equiv 2\kappa \tag{9.167}$$

is called the **absorption coefficient**. Meanwhile, the wave velocity is ω/k, and the index of refraction is

$$n = \frac{ck}{\omega}. \tag{9.168}$$

I have deliberately used notation reminiscent of Sect. 9.4.1. However, in the present case k and κ have nothing to do with conductivity; rather, they are determined by the parameters of our damped harmonic oscillator. For gases, the second term in Eq. 9.161 is small, and we can approximate the square root (Eq. 9.164) by the first term in the binomial expansion, $\sqrt{1+\varepsilon} \cong 1 + \frac{1}{2}\varepsilon$. Then

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right], \tag{9.169}$$

so

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2}, \tag{9.170}$$

and

$$\alpha = 2\kappa \cong \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2}. \tag{9.171}$$

In Fig. 9.22 I have plotted the index of refraction and the absorption coefficient in the vicinity of one of the resonances. *Most* of the time the index of refraction *rises* gradually with increasing frequency, consistent with our experience from optics (Fig. 9.19). However, in the immediate neighborhood of a resonance the index of refraction *drops* sharply. Because this behavior is atypical, it is called

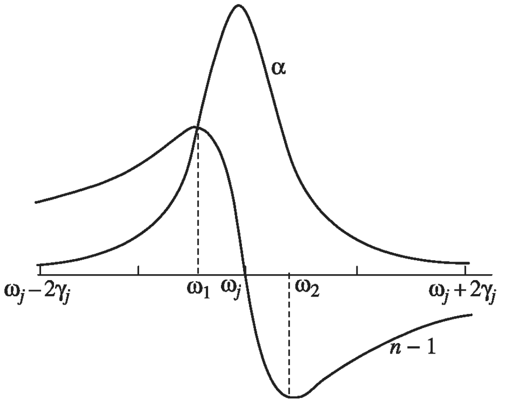

FIGURE 9.22

anomalous dispersion. Notice that the region of anomalous dispersion ($\omega_1 < \omega < \omega_2$, in the figure) coincides with the region of maximum absorption; in fact, the material may be practically opaque in this frequency range. The reason is that we are now driving the electrons at their "favorite" frequency; the amplitude of their oscillation is relatively large, and a correspondingly large amount of energy is dissipated by the damping mechanism.

In Fig. 9.22, n runs below 1 above the resonance, suggesting that the wave speed exceeds c. As I mentioned earlier, this is no immediate cause for alarm, since energy does not travel at the wave velocity. Moreover, the graph does not include the contributions of other terms in the sum, which add a relatively constant "background" that, in some cases, keeps $n > 1$ on both sides of the resonance. Incidentally, the group velocity can also exceed c in the neighborhood of a resonance, in this model (see Prob. 9.26).

If you agree to stay away from the resonances, the damping can be ignored, and the formula for the index of refraction simplifies:

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}. \tag{9.172}$$

For most substances the natural frequencies ω_j are scattered all over the spectrum in a rather chaotic fashion. But for transparent materials, the nearest significant resonances typically lie in the ultraviolet, so that $\omega < \omega_j$. In that case,

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \cong \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right),$$

and Eq. 9.172 takes the form

$$n = 1 + \left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right). \tag{9.173}$$

Or, in terms of the wavelength in vacuum ($\lambda = 2\pi c/\omega$):

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right). \tag{9.174}$$

This is known as **Cauchy's formula**; the constant A is called the **coefficient of refraction**, and B is called the **coefficient of dispersion**. Cauchy's equation applies reasonably well to most gases, in the optical region.

What I have described in this section is certainly not the complete story of dispersion in nonconducting media. Nevertheless, it does indicate how the damped harmonic motion of electrons can account for the frequency dependence of the index of refraction, and it explains why n is ordinarily a slowly increasing function of ω, with occasional "anomalous" regions where it precipitously drops.

Problem 9.23

(a) Shallow water is nondispersive; waves travel at a speed that is proportional to the square root of the depth. In deep water, however, the waves can't "feel" all the way down to the bottom—they behave as though the depth were proportional to λ. (Actually, the distinction between "shallow" and "deep" itself depends on the wavelength: If the depth is less than λ, the water is "shallow"; if it is substantially greater than λ, the water is "deep.") Show that the wave velocity of deep water waves is *twice* the group velocity.

(b) In quantum mechanics, a free particle of mass m traveling in the x direction is described by the wave function

$$\Psi(x, t) = Ae^{i(px - Et)/\hbar},$$

where p is the momentum, and $E = p^2/2m$ is the kinetic energy. Calculate the group velocity and the wave velocity. Which one corresponds to the classical speed of the particle? Note that the wave velocity is *half* the group velocity.

Problem 9.24 If you take the model in Ex. 4.1 at face value, what natural frequency do you get? Put in the actual numbers. Where, in the electromagnetic spectrum, does this lie, assuming the radius of the atom is 0.5 Å? Find the coefficients of refraction and dispersion, and compare them with the measured values for hydrogen at 0°C and atmospheric pressure: $A = 1.36 \times 10^{-4}$, $B = 7.7 \times 10^{-15}\,\mathrm{m}^2$.

Problem 9.25 Find the width of the anomalous dispersion region for the case of a single resonance at frequency ω_0. Assume $\gamma \ll \omega_0$. Show that the index of refraction assumes its maximum and minimum values at points where the absorption coefficient is at half-maximum.

Problem 9.26 Starting with Eq. 9.170, calculate the group velocity, assuming there is only one resonance, at ω_0. Use a computer to graph $y \equiv v_g/c$ as a function of $x \equiv (\omega/\omega_0)^2$, from $x = 0$ to 2, (a) for $\gamma = 0$, and (b) for $\gamma = (0.1)\omega_0$. Let $(Nq^2)/(2m\epsilon_0\omega_0^2) = 0.003$. Note that the group velocity can exceed c.

9.5 ■ GUIDED WAVES

9.5.1 ■ Wave Guides

So far, we have dealt with plane waves of infinite extent; now we consider electromagnetic waves confined to the interior of a hollow pipe, or **wave guide** (Fig. 9.23). We'll assume the wave guide is a perfect conductor, so that $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$ inside the material itself, and hence the boundary conditions at the inner wall are[21]

$$\left.\begin{array}{ll}\text{(i)} & \mathbf{E}^{\parallel} = \mathbf{0}, \\ \text{(ii)} & B^{\perp} = 0.\end{array}\right\} \tag{9.175}$$

Free charges and currents[22] will be induced on the surface in such a way as to enforce these constraints. We are interested in monochromatic waves that propagate down the tube, so $\mathbf{E}$ and $\mathbf{B}$ have the generic form

$$\left.\begin{array}{ll}\text{(i)} & \tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)}, \\ \text{(ii)} & \tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}.\end{array}\right\} \tag{9.176}$$

(For the cases of interest, k is real, so I shall dispense with its tilde.) The electric and magnetic fields must, of course, satisfy Maxwell's equations, in the interior of the wave guide:

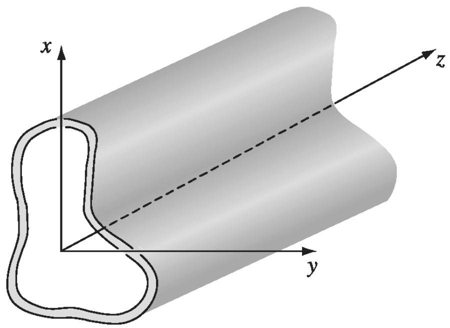

FIGURE 9.23

See Eq. 9.139 and Prob. 7.44. In a perfect conductor, $\mathbf{E} = \mathbf{0}$, and hence (by Faraday's law) $\partial\mathbf{B}/\partial t = \mathbf{0}$; assuming the magnetic field *started out* zero, then, it will *remain* so.

[22] In Section 9.4.2 I argued that there can be no surface currents in an ohmic conductor (with finite conductivity). But there *are* volume currents, extending in (roughly) to the skin depth. As the conductivity increases, they are squeezed into a thinner and thinner layer, and in the limit of a perfect conductor they become true surface currents.

$$\left.\begin{array}{ll} \text{(i)}\ \nabla\cdot\mathbf{E}=0, & \text{(iii)}\ \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla\cdot\mathbf{B}=0, & \text{(iv)}\ \nabla\times\mathbf{B}=\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}}{\partial t}. \end{array}\right\} \tag{9.177}$$

The problem, then, is to find functions $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ such that the fields (Eq. 9.176) obey the differential equations (Eq. 9.177), subject to boundary conditions (Eq. 9.175).

As we shall soon see, *confined* waves are *not* (in general) transverse; in order to fit the boundary conditions we shall have to include longitudinal components (E_z and B_z):[23]

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}}, \quad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}, \tag{9.178}$$

where each of the components is a function of x and y. Putting this into Maxwell's equations (iii) and (iv), we obtain (Prob. 9.27a)

$$\left.\begin{array}{ll} \text{(i)}\ \dfrac{\partial E_y}{\partial x}-\dfrac{\partial E_x}{\partial y}=i\omega B_z, & \text{(iv)}\ \dfrac{\partial B_y}{\partial x}-\dfrac{\partial B_x}{\partial y}=-\dfrac{i\omega}{c^2}E_z, \\ \text{(ii)}\ \dfrac{\partial E_z}{\partial y}-ikE_y=i\omega B_x, & \text{(v)}\ \dfrac{\partial B_z}{\partial y}-ikB_y=-\dfrac{i\omega}{c^2}E_x, \\ \text{(iii)}\ ikE_x-\dfrac{\partial E_z}{\partial x}=i\omega B_y, & \text{(vi)}\ ikB_x-\dfrac{\partial B_z}{\partial x}=-\dfrac{i\omega}{c^2}E_y. \end{array}\right\} \tag{9.179}$$

Equations (ii), (iii), (v), and (vi) can be solved for E_x, E_y, B_x, and B_y:

$$\left.\begin{array}{l} \text{(i)}\ E_x=\dfrac{i}{(\omega/c)^2-k^2}\left(k\dfrac{\partial E_z}{\partial x}+\omega\dfrac{\partial B_z}{\partial y}\right), \\ \text{(ii)}\ E_y=\dfrac{i}{(\omega/c)^2-k^2}\left(k\dfrac{\partial E_z}{\partial y}-\omega\dfrac{\partial B_z}{\partial x}\right), \\ \text{(iii)}\ B_x=\dfrac{i}{(\omega/c)^2-k^2}\left(k\dfrac{\partial B_z}{\partial x}-\dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right), \\ \text{(iv)}\ B_y=\dfrac{i}{(\omega/c)^2-k^2}\left(k\dfrac{\partial B_z}{\partial y}+\dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right). \end{array}\right\} \tag{9.180}$$

It suffices, then, to determine the longitudinal components E_z and B_z; if we knew those, we could quickly calculate all the others, just by differentiating. Inserting

[23] To avoid cumbersome notation, I shall leave the subscript 0 and the tilde off the individual components.

Eq. 9.180 into the remaining Maxwell equations (Prob. 9.27b) yields uncoupled equations for E_z and B_z:

$$\left.\begin{array}{ll}\text{(i)} & \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0, \\ \text{(ii)} & \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0.\end{array}\right\} \tag{9.181}$$

If $E_z = 0$, we call these **TE** ("transverse electric") **waves**; if $B_z = 0$, they are called **TM** ("transverse magnetic") **waves**; if both $E_z = 0$ and $B_z = 0$, we call them **TEM waves**.[24] It turns out that TEM waves cannot occur in a hollow wave guide.

Proof. If $E_z = 0$, Gauss's law (Eq. 9.177i) says

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0,$$

and if $B_z = 0$, Faraday's law (Eq. 9.177iii) says

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0.$$

Indeed, the vector $\tilde{\mathbf{E}}_0$ in Eq. 9.178 has zero divergence and zero curl. It can therefore be written as the gradient of a scalar potential that satisfies Laplace's equation. But the boundary condition on **E** (Eq. 9.175) requires that the surface be an equipotential, and since Laplace's equation admits no local maxima or minima (Sect. 3.1.4), this means that the potential is constant throughout, and hence the electric field is *zero*—no wave at all. □

Notice that this argument applies only to a completely *empty* pipe—if you run a separate conductor down the middle, the potential at *its* surface need not be the same as on the outer wall, and hence a nontrivial potential is possible. We'll see an example of this in Sect. 9.5.3.

! **Problem 9.27**

(a) Derive Eqs. 9.179, and from these obtain Eqs. 9.180.

(b) Put Eq. 9.180 into Maxwell's equations (i) and (ii) to obtain Eq. 9.181. Check that you get the same results using (i) and (iv) of Eq. 9.179.

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[24] In the case of TEM waves (including the unconfined plane waves of Sect. 9.2), $k = \omega/c$, Eqs. 9.180 are indeterminate, and you have to go back to Eqs. 9.179.

9.5.2 ■ TE Waves in a Rectangular Wave Guide

Suppose we have a wave guide of rectangular shape (Fig. 9.24), with height a and width b, and we are interested in the propagation of TE waves. The problem is to solve Eq. 9.181ii, subject to the boundary condition 9.175ii. We'll do it by separation of variables. Let

$$B_z(x, y) = X(x)Y(y),$$

so that

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \left[(\omega/c)^2 - k^2\right]XY = 0.$$

Divide by XY, and note that the x- and y-dependent terms must be constants:

$$\text{(i)}\ \frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2, \quad \text{(ii)}\ \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2, \tag{9.182}$$

with

$$-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0. \tag{9.183}$$

The general solution to Eq. 9.182i is

$$X(x) = A\sin(k_x x) + B\cos(k_x x).$$

But the boundary conditions require that B_x—and hence also (Eq. 9.180iii) dX/dx—vanishes at $x = 0$ and $x = a$. So $A = 0$, and

$$k_x = m\pi/a, \quad (m = 0, 1, 2, \dots). \tag{9.184}$$

The same goes for Y, with

$$k_y = n\pi/b, \quad (n = 0, 1, 2, \dots), \tag{9.185}$$

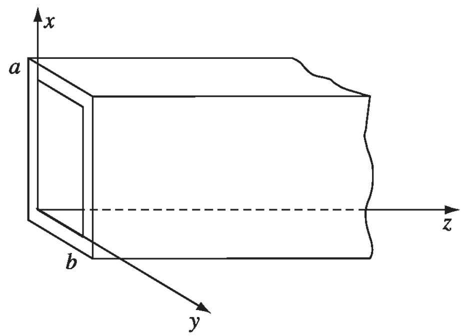

FIGURE 9.24

and we conclude that

$$B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b). \tag{9.186}$$

This solution is called the TE_{mn} mode. (The first index is conventionally associated with the *larger* dimension, so we assume $a \geq b$. By the way, at least *one* of the indices must be nonzero—see Prob. 9.28.) The wave number (k) is obtained by putting Eqs. 9.184 and 9.185 into Eq. 9.183:

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}. \tag{9.187}$$

If

$$\omega < c\pi\sqrt{(m/a)^2 + (n/b)^2} \equiv \omega_{mn}, \tag{9.188}$$

the wave number is imaginary, and instead of a traveling wave we have exponentially attenuated fields (Eq. 9.176). For this reason, ω_{mn} is called the **cutoff frequency** for the mode in question. The *lowest* cutoff frequency for a given wave guide occurs for the mode TE_{10}:

$$\omega_{10} = c\pi/a. \tag{9.189}$$

Frequencies less than this will not propagate at all.

The wave number can be written more simply in terms of the cutoff frequency:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}. \tag{9.190}$$

The wave velocity is

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}}, \tag{9.191}$$

which is greater than c. However (see Prob. 9.30), the energy carried by the wave travels at the *group* velocity (Eq. 9.150):

$$v_g = \frac{1}{dk/d\omega} = c\sqrt{1 - (\omega_{mn}/\omega)^2} < c. \tag{9.192}$$

There's another way to visualize the propagation of an electromagnetic wave in a rectangular pipe, and it serves to illuminate many of these results. Consider an ordinary *plane* wave, traveling at an angle θ to the z axis, and reflecting perfectly off each conducting surface (Fig. 9.25). In the x and y directions, the (multiply reflected) waves interfere to form standing wave patterns, of wavelength $\lambda_x = 2a/m$ and $\lambda_y = 2b/n$ (hence wave number $k_x = 2\pi/\lambda_x = \pi m/a$ and $k_y = \pi n/b$), respectively. Meanwhile, in the z direction there remains a traveling wave, with wave number $k_z = k$. The propagation vector for the "original" plane wave is therefore

$$\mathbf{k}' = \frac{\pi m}{a}\hat{\mathbf{x}} + \frac{\pi n}{b}\hat{\mathbf{y}} + k\,\hat{\mathbf{z}},$$

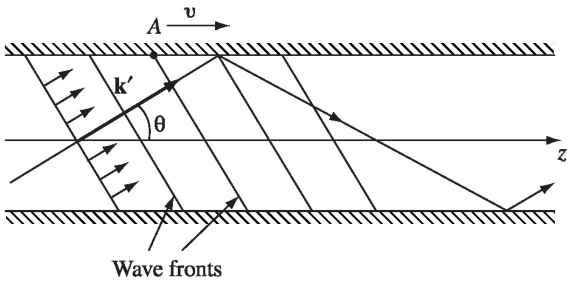

FIGURE 9.25

and the frequency is

$$\omega = c|\mathbf{k}'| = c\sqrt{k^2 + \pi^2[(m/a)^2 + (n/b)^2]} = \sqrt{(ck)^2 + (\omega_{mn})^2}.$$

Only certain angles will lead to one of the allowed standing wave patterns:

$$\cos\theta = \frac{k}{|\mathbf{k}'|} = \sqrt{1 - (\omega_{mn}/\omega)^2}.$$

The plane wave travels at speed c, but because it is going at an angle θ to the z axis, its net velocity down the wave guide is

$$v_g = c\cos\theta = c\sqrt{1 - (\omega_{mn}/\omega)^2}.$$

The *wave* velocity, on the other hand, is the speed of the wave fronts (A, say, in Fig. 9.25) down the pipe. Like the intersection of a line of breakers with the beach, they can move much faster than the waves themselves—in fact

$$v = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}}.$$

Problem 9.28 Show that the mode TE_{00} cannot occur in a rectangular wave guide. [*Hint:* In this case $\omega/c = k$, so Eqs. 9.180 are indeterminate, and you must go back to Eq. 9.179. Show that B_z is a constant, and hence—applying Faraday's law in integral form to a cross section—that $B_z = 0$, so this would be a TEM mode.]

Problem 9.29 Consider a rectangular wave guide with dimensions 2.28 cm × 1.01 cm. What TE modes will propagate in this wave guide, if the driving frequency is 1.70×10^{10} Hz? Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use? What are the corresponding wavelengths (in open space)?

Problem 9.30 Confirm that the energy in the TE_{mn} mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle\mathbf{S}\rangle$ and the energy density $\langle u\rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

Problem 9.31 Work out the theory of TM modes for a rectangular wave guide. In particular, find the longitudinal electric field, the cutoff frequencies, and the wave and group velocities. Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given wave guide. [*Caution*: What is the lowest TM mode?]

9.5.3 ■ The Coaxial Transmission Line

In Sect. 9.5.1, I showed that a *hollow* wave guide cannot support TEM waves. But a coaxial transmission line, consisting of a long straight wire of radius a, surrounded by a cylindrical conducting sheath of radius b (Fig. 9.26), *does* admit modes with $E_z = 0$ and $B_z = 0$. In this case Maxwell's equations (Eq. 9.179) yield

$$k = \omega/c \tag{9.193}$$

(so the waves travel at speed c, and are nondispersive),

$$cB_y = E_x \quad \text{and} \quad cB_x = -E_y \tag{9.194}$$

(so $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular), and (together with $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$):

$$\left.\begin{aligned} \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0, \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0, \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0. \end{aligned}\right\} \tag{9.195}$$

These are precisely the equations of *electrostatics* and *magnetostatics*, for empty space, in two dimensions; the solution with cylindrical symmetry can be borrowed directly from the case of an infinite line charge and an infinite straight current, respectively:

$$\mathbf{E}_0(s, \phi) = \frac{A}{s}\,\hat{\mathbf{s}}, \qquad \mathbf{B}_0(s, \phi) = \frac{A}{cs}\,\hat{\boldsymbol{\phi}}, \tag{9.196}$$

for some constant A. Substituting these into Eq. 9.176, and taking the real part:

$$\left.\begin{aligned} \mathbf{E}(s, \phi, z, t) &= \frac{A\cos(kz - \omega t)}{s}\,\hat{\mathbf{s}}, \\ \mathbf{B}(s, \phi, z, t) &= \frac{A\cos(kz - \omega t)}{cs}\,\hat{\boldsymbol{\phi}}. \end{aligned}\right\} \tag{9.197}$$

FIGURE 9.26

Problem 9.32

(a) Show directly that Eqs. 9.197 satisfy Maxwell's equations (Eq. 9.177) and the boundary conditions (Eq. 9.175).

(b) Find the charge density, $\lambda(z, t)$, and the current, $I(z, t)$, on the inner conductor.

More Problems on Chapter 9

! **Problem 9.33** The "inversion theorem" for Fourier transforms states that

$$\tilde{\phi}(z) = \int_{-\infty}^{\infty} \tilde{\Phi}(k) e^{ikz}\, dk \quad \Longleftrightarrow \quad \tilde{\Phi}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(z) e^{-ikz}\, dz. \qquad (9.198)$$

Use this to determine $\tilde{A}(k)$, in Eq. 9.20, in terms of $f(z, 0)$ and $\dot{f}(z, 0)$.
[*Answer:* $(1/2\pi) \int_{-\infty}^{\infty} [f(z, 0) + (i/\omega) \dot{f}(z, 0)] e^{-ikz}\, dz$]

Problem 9.34 [The naive explanation for the pressure of light offered in Section 9.2.3 has its flaws, as you discovered if you worked Problem 9.11. Here's another account, due originally to Planck.[25]] A plane wave traveling through vacuum in the z direction encounters a perfect conductor occupying the region $z \geq 0$, and reflects back:

$$\mathbf{E}(z, t) = E_0 [\cos(kz - \omega t) - \cos(kz + \omega t)]\, \hat{\mathbf{x}}, \quad (z < 0).$$

(a) Find the accompanying magnetic field (in the region $z < 0$).

(b) Assuming $\mathbf{B} = \mathbf{0}$ inside the conductor, find the current $\mathbf{K}$ on the surface $z = 0$, by invoking the appropriate boundary condition.

(c) Find the magnetic force per unit area on the surface, and compare its time average with the expected radiation pressure (Eq. 9.64).

Problem 9.35 Suppose

$$\mathbf{E}(r, \theta, \phi, t) = A \frac{\sin\theta}{r} \left[\cos(kr - \omega t) - (1/kr) \sin(kr - \omega t)\right] \hat{\boldsymbol{\phi}}, \quad \text{with } \frac{\omega}{k} = c.$$

(This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.)

(a) Show that $\mathbf{E}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\mathbf{S}$ over a full cycle to get the intensity vector $\mathbf{I}$. (Does it point in the expected direction? Does it fall off like r^{-2}, as it should?)

(c) Integrate $\mathbf{I} \cdot d\mathbf{a}$ over a spherical surface to determine the total power radiated. [*Answer:* $4\pi A^2 / 3\mu_0 c$]

[25] T. Rothman and S. Boughn, *Am. J. Phys.* **77**, 122 (2009), Section IV.

! **Problem 9.36** Light of (angular) frequency ω passes from medium 1, through a slab (thickness d) of medium 2, and into medium 3 (for instance, from water through glass into air, as in Fig. 9.27). Show that the transmission coefficient for normal incidence is given by

$$T^{-1} = \frac{1}{4n_1 n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right]. \tag{9.199}$$

[*Hint:* To the *left*, there is an incident wave and a reflected wave; to the *right*, there is a transmitted wave; inside the slab, there is a wave going to the right and a wave going to the left. Express each of these in terms of its complex amplitude, and relate the amplitudes by imposing suitable boundary conditions at the two interfaces. All three media are linear and homogeneous; assume $\mu_1 = \mu_2 = \mu_3 = \mu_0$.]

Problem 9.37 A microwave antenna radiating at 10 GHz is to be protected from the environment by a plastic shield of dielectric constant 2.5. What is the minimum thickness of this shielding that will allow perfect transmission (assuming normal incidence)? [*Hint:* Use Eq. 9.199.]

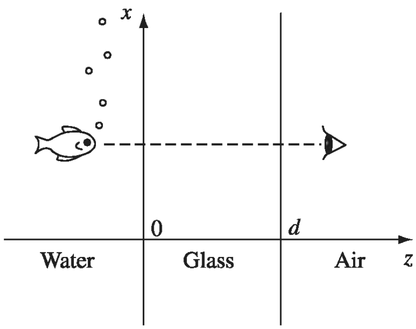

FIGURE 9.27

Problem 9.38 Light from an aquarium (Fig. 9.27) goes from water ($n = \frac{4}{3}$) through a plane of glass ($n = \frac{3}{2}$) into air ($n = 1$). Assuming it's a monochromatic plane wave and that it strikes the glass at normal incidence, find the minimum and maximum transmission coefficients (Eq. 9.199). You can see the fish clearly; how well can it see you?

! **Problem 9.39** According to Snell's law, when light passes from an optically dense medium into a less dense one ($n_1 > n_2$) the propagation vector **k** bends *away* from the normal (Fig. 9.28). In particular, if the light is incident at the **critical angle**

$$\theta_c \equiv \sin^{-1}(n_2/n_1), \tag{9.200}$$

then $\theta_T = 90°$, and the transmitted ray just grazes the surface. If θ_I *exceeds* θ_c, there is no refracted ray at all, only a reflected one (this is the phenomenon of **total internal reflection**, on which light pipes and fiber optics are based). But the *fields* are not zero in medium 2; what we get is a so-called **evanescent wave**, which is rapidly attenuated and transports no energy into medium 2.[26]

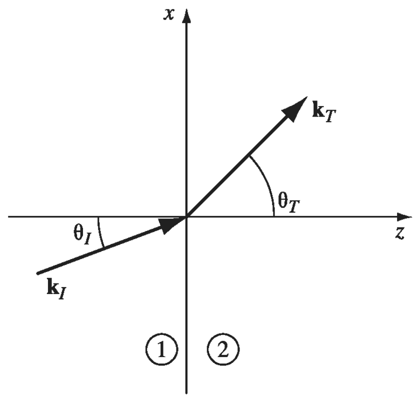

FIGURE 9.28

A quick way to construct the evanescent wave is simply to quote the results of Sect. 9.3.3, with $k_T = \omega n_2/c$ and

$$\mathbf{k}_T = k_T(\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{z}});$$

the only change is that

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$$

is now greater than 1, and

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\sqrt{\sin^2\theta_T - 1}$$

is imaginary. (Obviously, θ_T can no longer be interpreted as an *angle*!)

(a) Show that

$$\tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}, \tag{9.201}$$

where

$$\kappa \equiv \frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2^2} \quad \text{and} \quad k \equiv \frac{\omega n_1}{c}\sin\theta_I. \tag{9.202}$$

This is a wave propagating in the x direction (*parallel* to the interface!), and attenuated in the z direction.

[26] The evanescent fields can be detected by placing a second interface a short distance to the right of the first; in a close analog to quantum mechanical **tunneling**, the wave crosses the gap and reassembles to the right. See F. Albiol, S. Navas, and M. V. Andres, *Am. J. Phys.* **61**, 165 (1993).

(b) Noting that α (Eq. 9.108) is now imaginary, use Eq. 9.109 to calculate the reflection coefficient for polarization parallel to the plane of incidence. [Notice that you get 100% reflection, which is better than at a conducting surface (see, for example, Prob. 9.22).]

(c) Do the same for polarization perpendicular to the plane of incidence (use the results of Prob. 9.17).

(d) In the case of polarization perpendicular to the plane of incidence, show that the (real) evanescent fields are

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_0 e^{-\kappa z} \cos(kx - \omega t)\,\hat{\mathbf{y}}, \\ \mathbf{B}(\mathbf{r}, t) &= \frac{E_0}{\omega} e^{-\kappa z} \left[\kappa \sin(kx - \omega t)\,\hat{\mathbf{x}} + k \cos(kx - \omega t)\,\hat{\mathbf{z}}\right]. \end{aligned}\right\} \tag{9.203}$$

(e) Check that the fields in (d) satisfy all of Maxwell's equations (Eq. 9.67).

(f) For the fields in (d), construct the Poynting vector, and show that, on average, no energy is transmitted in the z direction.

! **Problem 9.40** Consider the **resonant cavity** produced by closing off the two ends of a rectangular wave guide, at $z = 0$ and at $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by

$$\omega_{lmn} = c\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}, \tag{9.204}$$

for integers l, m, and n. Find the associated electric and magnetic fields.

CHAPTER

10 Potentials and Fields

10.1 ■ THE POTENTIAL FORMULATION

10.1.1 ■ Scalar and Vector Potentials

In this chapter we seek the *general* solution to Maxwell's equations,

$$\left.\begin{array}{llll}\text{(i)} & \nabla\cdot\mathbf{E}=\dfrac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t},\\[2ex] \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\dfrac{\partial\mathbf{E}}{\partial t}.\end{array}\right\} \tag{10.1}$$

Given $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$, what are the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$? In the static case, Coulomb's law and the Biot-Savart law provide the answer. What we're looking for, then, is the generalization of those laws to time-dependent configurations.

This is not an easy problem, and it pays to begin by representing the fields in terms of potentials. In electrostatics $\nabla\times\mathbf{E}=\mathbf{0}$ allowed us to write $\mathbf{E}$ as the gradient of a scalar potential: $\mathbf{E}=-\nabla V$. In electro*dynamics* this is no longer possible, because the curl of $\mathbf{E}$ is nonzero. But $\mathbf{B}$ remains divergenceless, so we can still write

$$\boxed{\mathbf{B}=\nabla\times\mathbf{A},} \tag{10.2}$$

as in magnetostatics. Putting this into Faraday's law (iii) yields

$$\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}(\nabla\times\mathbf{A}),$$

or

$$\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=\mathbf{0}.$$

Here is a quantity, unlike $\mathbf{E}$ alone, whose curl *does* vanish; it can therefore be written as the gradient of a scalar:

$$\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla V.$$

In terms of V and $\mathbf{A}$, then,

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.} \tag{10.3}$$

This reduces to the old form, of course, when $\mathbf{A}$ is constant.

The potential representation (Eqs. 10.2 and 10.3) automatically fulfills the two homogeneous Maxwell equations, (ii) and (iii). How about Gauss's law (i) and the Ampère/Maxwell law (iv)? Putting Eq. 10.3 into (i), we find that

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho; \tag{10.4}$$

this replaces Poisson's equation (to which it reduces in the static case). Putting Eqs. 10.2 and 10.3 into (iv) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2},$$

or, using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, and rearranging the terms a bit:

$$\left(\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{10.5}$$

Equations 10.4 and 10.5 contain all the information in Maxwell's equations.

Example 10.1. Find the charge and current distributions that would give rise to the potentials

$$V = 0, \quad \mathbf{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\mathbf{z}}, & \text{for } |x| < ct, \\ \mathbf{0}, & \text{for } |x| > ct, \end{cases}$$

where k is a constant, and (of course) $c = 1/\sqrt{\epsilon_0\mu_0}$.

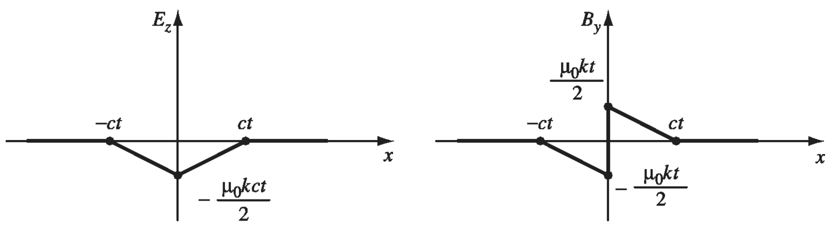

FIGURE 10.1

Solution

First we'll determine the electric and magnetic fields, using Eqs. 10.2 and 10.3:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\,\hat{\mathbf{z}},$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\,\hat{\mathbf{y}} = \pm\frac{\mu_0 k}{2c}(ct - |x|)\,\hat{\mathbf{y}},$$

(plus, for $x > 0$; minus, for $x < 0$). These are for $|x| < ct$; when $|x| > ct$, $\mathbf{E} = \mathbf{B} = \mathbf{0}$ (Fig. 10.1). Calculating every derivative in sight, I find

$$\nabla \cdot \mathbf{E} = 0; \quad \nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = \mp\frac{\mu_0 k}{2}\hat{\mathbf{y}}; \quad \nabla \times \mathbf{B} = -\frac{\mu_0 k}{2c}\,\hat{\mathbf{z}};$$

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\mu_0 k c}{2}\,\hat{\mathbf{z}}; \quad \frac{\partial \mathbf{B}}{\partial t} = \pm\frac{\mu_0 k}{2}\,\hat{\mathbf{y}}.$$

As you can easily check, Maxwell's equations are all satisfied, with ρ and $\mathbf{J}$ both *zero*. Notice, however, that $\mathbf{B}$ has a discontinuity at $x = 0$, and this signals the presence of a surface current $\mathbf{K}$ in the yz plane; boundary condition (iv) in Eq. 7.64 gives

$$kt\,\hat{\mathbf{y}} = \mathbf{K} \times \hat{\mathbf{x}},$$

and hence

$$\mathbf{K} = kt\,\hat{\mathbf{z}}.$$

Evidently we have here a uniform surface current flowing in the z direction over the plane $x = 0$, which starts up at $t = 0$, and increases in proportion to t. Notice that the news travels out (in both directions) at the speed of light: for points $|x| > ct$ the message ("current is now flowing") has not yet arrived, so the fields are zero.

Problem 10.1 Show that the differential equations for V and $\mathbf{A}$ (Eqs. 10.4 and 10.5) can be written in the more symmetrical form

$$\left.\begin{aligned} \Box^2 V + \frac{\partial L}{\partial t} &= -\frac{1}{\epsilon_0}\rho, \\ \Box^2 \mathbf{A} - \nabla L &= -\mu_0 \mathbf{J}, \end{aligned}\right\} \tag{10.6}$$

where

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \quad \text{and} \quad L \equiv \nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}.$$

Problem 10.2 For the configuration in Ex. 10.1, consider a rectangular box of length l, width w, and height h, situated a distance d above the yz plane (Fig. 10.2).

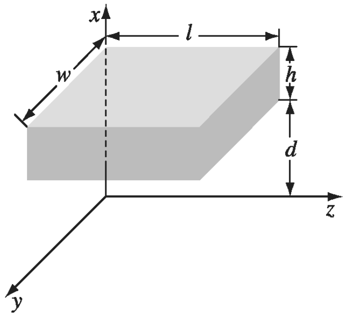

FIGURE 10.2

(a) Find the energy in the box at time $t_1 = d/c$, and at $t_2 = (d+h)/c$.

(b) Find the Poynting vector, and determine the energy per unit time flowing into the box during the interval $t_1 < t < t_2$.

(c) Integrate the result in (b) from t_1 to t_2, and confirm that the increase in energy (part (a)) equals the net influx.

10.1.2 ■ Gauge Transformations

Equations 10.4 and 10.5 are *ugly*, and you might be inclined to abandon the potential formulation altogether. However, we *have* succeeded in reducing six problems—finding **E** and **B** (three components each)—down to four: V (one component) and **A** (three more). Moreover, Eqs. 10.2 and 10.3 do not uniquely define the potentials; we are free to impose extra conditions on V and **A**, as long as nothing happens to **E** and **B**. Let's work out precisely what this **gauge freedom** entails.

Suppose we have two sets of potentials, $(V, \mathbf{A})$ and $(V', \mathbf{A}')$, which correspond to the *same* electric and magnetic fields. By how much can they differ? Write

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \quad \text{and} \quad V' = V + \beta.$$

Since the two **A**'s give the same **B**, their curls must be equal, and hence

$$\nabla \times \boldsymbol{\alpha} = \mathbf{0}.$$

We can therefore write $\boldsymbol{\alpha}$ as the gradient of some scalar:

$$\boldsymbol{\alpha} = \nabla\lambda.$$

The two potentials also give the same **E**, so

$$\nabla\beta + \frac{\partial\boldsymbol{\alpha}}{\partial t} = \mathbf{0},$$

or

$$\nabla\left(\beta+\frac{\partial\lambda}{\partial t}\right)=\mathbf{0}.$$

The term in parentheses is therefore independent of position (it could, however, depend on time); call it $k(t)$:

$$\beta=-\frac{\partial\lambda}{\partial t}+k(t).$$

Actually, we might as well absorb $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')dt'$ to the old one. This will not affect the gradient of λ; it just adds $k(t)$ to $\partial\lambda/\partial t$. It follows that

$$\left.\begin{aligned}\mathbf{A}'&=\mathbf{A}+\nabla\lambda,\\ V'&=V-\frac{\partial\lambda}{\partial t}.\end{aligned}\right\}\tag{10.7}$$

Conclusion: For any old scalar function $\lambda(\mathbf{r},t)$, we can with impunity add $\nabla\lambda$ to $\mathbf{A}$, provided we simultaneously subtract $\partial\lambda/\partial t$ from V. This will not affect the physical quantities $\mathbf{E}$ and $\mathbf{B}$. Such changes in V and $\mathbf{A}$ are called **gauge transformations**. They can be exploited to adjust the divergence of $\mathbf{A}$, with a view to simplifying the "ugly" equations 10.4 and 10.5. In magnetostatics, it was best to choose $\nabla\cdot\mathbf{A}=0$ (Eq. 5.63); in electrodynamics, the situation is not so clear cut, and the most convenient gauge depends to some extent on the problem at hand. There are many famous gauges in the literature; I'll show you the two most popular ones.

Problem 10.3

(a) Find the fields, and the charge and current distributions, corresponding to

$$V(\mathbf{r},t)=0,\quad \mathbf{A}(\mathbf{r},t)=-\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}}.$$

(b) Use the gauge function $\lambda=-(1/4\pi\epsilon_0)(qt/r)$ to transform the potentials, and comment on the result.

Problem 10.4 Suppose $V=0$ and $\mathbf{A}=A_0\sin(kx-\omega t)\,\hat{\mathbf{y}}$, where A_0, ω, and k are constants. Find $\mathbf{E}$ and $\mathbf{B}$, and check that they satisfy Maxwell's equations in vacuum. What condition must you impose on ω and k?

10.1.3 ■ Coulomb Gauge and Lorenz Gauge

The Coulomb Gauge. As in magnetostatics, we pick

$$\nabla\cdot\mathbf{A}=0.\tag{10.8}$$

With this, Eq. 10.4 becomes

$$\nabla^2V=-\frac{1}{\epsilon_0}\rho.\tag{10.9}$$

This is Poisson's equation, and we already know how to solve it: setting $V = 0$ at infinity,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}}\,d\tau'. \tag{10.10}$$

There is a very peculiar thing about the scalar potential in the Coulomb gauge: it is determined by the distribution of charge *right now*. If I move an electron in my laboratory, the potential V on the moon immediately records this change. That sounds particularly odd in the light of special relativity, which allows no message to travel faster than c. The point is that V *by itself* is not a physically measurable quantity—all the man in the moon can measure is $\mathbf{E}$, and that involves $\mathbf{A}$ as well (Eq. 10.3). Somehow it is built into the vector potential (in the Coulomb gauge) that whereas V instantaneously reflects all changes in ρ, the combination $-\nabla V - (\partial \mathbf{A}/\partial t)$ does *not*; $\mathbf{E}$ will change only after sufficient time has elapsed for the "news" to arrive.[1]

The *advantage* of the Coulomb gauge is that the scalar potential is particularly simple to calculate; the *disadvantage* (apart from the acausal appearance of V) is that $\mathbf{A}$ is particularly *difficult* to calculate. The differential equation for $\mathbf{A}$ (Eq. 10.5) in the Coulomb gauge reads

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \mu_0\epsilon_0\nabla\left(\frac{\partial V}{\partial t}\right). \tag{10.11}$$

The Lorenz gauge. In the Lorenz[2] gauge, we pick

$$\boxed{\nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}.} \tag{10.12}$$

This is designed to eliminate the middle term in Eq. 10.5 (in the language of Prob. 10.1, it sets $L = 0$). With this,

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \tag{10.13}$$

Meanwhile, the differential equation for V, (Eq. 10.4), becomes

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho. \tag{10.14}$$

The virtue of the Lorenz gauge is that it treats V and $\mathbf{A}$ on an equal footing: the same differential operator

$$\boxed{\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \equiv \Box^2,} \tag{10.15}$$

[1]See O. L. Brill and B. Goodman. *Am. J. Phys.* **35**, 832 (1967) and J. D. Jackson, *Am. J. Phys.* **70**, 917 (2001).

[2]Until recently, it was spelled "Lorentz," in honor of the Dutch physicist H. A. Lorentz, but it is now attributed to L. V. Lorenz, the Dane. See J. Van Bladel, *IEEE Antennas and Propagation Magazine* **33**(2), 69 (1991); J. D. Jackson and L. B. Okun, *Rev. Mod. Phys.* **73**, 663 (2001).

(called the **d'Alembertian**) occurs in both equations:

$$\boxed{\begin{array}{ll}\text{(i)} & \Box^2 V = -\dfrac{1}{\epsilon_0}\rho, \\[2ex] \text{(ii)} & \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}.\end{array}} \tag{10.16}$$

This democratic treatment of V and $\mathbf{A}$ is especially nice in the context of special relativity, where the d'Alembertian is the natural generalization of the Laplacian, and Eqs. 10.16 can be regarded as four-dimensional versions of Poisson's equation. In this same spirit, the wave equation $\Box^2 f = 0$, might be regarded as the four-dimensional version of Laplace's equation. In the Lorenz gauge, V and $\mathbf{A}$ satisfy the **inhomogeneous wave equation**, with a "source" term (in place of zero) on the right. From now on, I shall use the Lorenz gauge exclusively, and the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for a specified source*.

Problem 10.5 Which of the potentials in Ex. 10.1, Prob. 10.3, and Prob. 10.4 are in the Coulomb gauge? Which are in the Lorenz gauge? (Notice that these gauges are not mutually exclusive.)

Problem 10.6 In Chapter 5, I showed that it is always possible to pick a vector potential whose divergence is zero (the Coulomb gauge). Show that it is always possible to choose $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0(\partial V/\partial t)$, as required for the Lorenz gauge, assuming you know how to solve the inhomogeneous wave equation (Eq. 10.16). Is it always possible to pick $V = 0$? How about $\mathbf{A} = \mathbf{0}$?

Problem 10.7 A time-dependent point charge $q(t)$ at the origin, $\rho(\mathbf{r}, t) = q(t)\delta^3(\mathbf{r})$, is fed by a current $\mathbf{J}(\mathbf{r}, t) = -(1/4\pi)(\dot{q}/r^2)\,\hat{\mathbf{r}}$, where $\dot{q} \equiv dq/dt$.

(a) Check that charge is conserved, by confirming that the continuity equation is obeyed.

(b) Find the scalar and vector potentials in the Coulomb gauge. If you get stuck, try working on (c) first.

(c) Find the fields, and check that they satisfy all of Maxwell's equations.[3]

10.1.4 ■ Lorentz Force Law in Potential Form[4]

It is illuminating to express the Lorentz force law in terms of potentials:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\left[-\nabla V - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right], \tag{10.17}$$

[3] P. R. Berman, *Am. J. Phys.* **76** 48 (2008).
[4] This section can be skipped without loss of continuity.

where $\mathbf{p} = m\mathbf{v}$ is the momentum of the particle. Now, product rule 4 says

$$\nabla(\mathbf{v}\cdot\mathbf{A}) = \mathbf{v}\times(\nabla\times\mathbf{A}) + (\mathbf{v}\cdot\nabla)\mathbf{A}$$

($\mathbf{v}$, the velocity of the particle, is a function of time, but not of position). Thus

$$\frac{d\mathbf{p}}{dt} = -q\left[\frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} + \nabla(V - \mathbf{v}\cdot\mathbf{A})\right]. \tag{10.18}$$

The combination

$$\left[\frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}\right]$$

is called the **convective derivative** of $\mathbf{A}$, and written $d\mathbf{A}/dt$ (*total* derivative). It represents the time rate of change of $\mathbf{A}$ *at the (moving) location of the particle*. For suppose that at time t the particle is at point $\mathbf{r}$, where the potential is $\mathbf{A}(\mathbf{r}, t)$; a moment dt later it is at $\mathbf{r} + \mathbf{v}\,dt$, where the potential is $\mathbf{A}(\mathbf{r} + \mathbf{v}\,dt, t + dt)$. The *change* in $\mathbf{A}$, then, is

$$\begin{aligned} d\mathbf{A} &= \mathbf{A}(\mathbf{r} + \mathbf{v}\,dt, t + dt) - \mathbf{A}(\mathbf{r}, t) \\ &= \left(\frac{\partial\mathbf{A}}{\partial x}\right)(v_x\,dt) + \left(\frac{\partial\mathbf{A}}{\partial y}\right)(v_y\,dt) + \left(\frac{\partial\mathbf{A}}{\partial z}\right)(v_z\,dt) + \left(\frac{\partial\mathbf{A}}{\partial t}\right)dt, \end{aligned}$$

so

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}. \tag{10.19}$$

As the particle moves, the potential it "feels" changes for two distinct reasons: first, because the potential varies with *time*, and second, because it is now in a new location, where $\mathbf{A}$ is different because of its variation in *space*. Hence the two terms in Eq. 10.19.

With the aid of the convective derivative, the Lorentz force law reads:

$$\frac{d}{dt}(\mathbf{p} + q\mathbf{A}) = -\nabla\left[q(V - \mathbf{v}\cdot\mathbf{A})\right]. \tag{10.20}$$

This is reminiscent of the standard formula from mechanics, for the motion of a particle whose potential energy U is a specified function of position:

$$\frac{d\mathbf{p}}{dt} = -\nabla U.$$

Playing the role of $\mathbf{p}$ is the so-called **canonical momentum**,

$$\mathbf{p}_{\text{can}} = \mathbf{p} + q\mathbf{A}, \tag{10.21}$$

while the part of U is taken by the velocity-dependent quantity

$$U_{\text{vel}} = q(V - \mathbf{v}\cdot\mathbf{A}). \tag{10.22}$$

A similar argument (Prob. 10.9) gives the rate of change of the particle's *energy*:

$$\frac{d}{dt}(T + qV) = \frac{\partial}{\partial t}[q(V - \mathbf{v}\cdot\mathbf{A})], \tag{10.23}$$

where $T = \frac{1}{2}mv^2$ is its kinetic energy and qV is its potential energy (The derivative on the right acts only on V and $\mathbf{A}$, not on $\mathbf{v}$). Curiously, the same quantity[5] U_{vel} appears on the right side of both equations. The parallel between Eq. 10.20 and Eq. 10.23 invites us to interpret $\mathbf{A}$ as a kind of "potential momentum" per unit charge, just as V is potential *energy* per unit charge.[6]

Problem 10.8 The vector potential for a uniform magnetostatic field is $\mathbf{A} = -\frac{1}{2}(\mathbf{r}\times\mathbf{B})$ (Prob. 5.25). Show that $d\mathbf{A}/dt = -\frac{1}{2}(\mathbf{v}\times\mathbf{B})$, in this case, and confirm that Eq. 10.20 yields the correct equation of motion.

Problem 10.9 Derive Eq. 10.23. [*Hint:* Start by dotting $\mathbf{v}$ into Eq. 10.17.]

10.2 ■ CONTINUOUS DISTRIBUTIONS

10.2.1 ■ Retarded Potentials

In the static case, Eq. 10.16 reduces to (four copies of) Poisson's equation,

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \quad \nabla^2\mathbf{A} = -\mu_0\mathbf{J},$$

with the familiar solutions

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau', \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau', \tag{10.24}$$

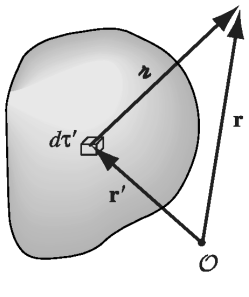

FIGURE 10.3

[5]I don't know what to call U_{vel}—it's not potential energy, exactly (that would be qV).
[6]There are other arguments for this interpretation, which Maxwell himself favored, and many modern authors advocate. For a fascinating discussion, see M. D. Semon and J. R. Taylor, *Am. J. Phys.* **64**, 1361 (1996). Incidentally, it is the *canonical* angular momentum (derived from $\mathbf{p}_{\text{can}}$), not the mechanical portion alone, that is quantized—see R. H. Young, *Am. J. Phys.* **66**, 1043 (1998).

where $\mathscr{r}$, as always, is the distance from the source point $\mathbf{r}'$ to the field point $\mathbf{r}$ (Fig. 10.3). Now, electromagnetic "news" travels at the speed of light. In the *nonstatic* case, therefore, it's not the status of the source *right now* that matters, but rather its condition at some earlier time t_r (called the **retarded time**) when the "message" left. Since this message must travel a distance $\mathscr{r}$, the delay is $\mathscr{r}/c$:

$$\boxed{t_r \equiv t - \frac{\mathscr{r}}{c}.} \tag{10.25}$$

The natural generalization of Eq. 10.24 for *nonstatic* sources is therefore

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau', \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau'.} \tag{10.26}$$

Here $\rho(\mathbf{r}', t_r)$ is the charge density that prevailed at point $\mathbf{r}'$ at the retarded time t_r. Because the integrands are evaluated at the retarded time, these are called **retarded potentials**. (I speak of "the" retarded time, but of course the more distant parts of the charge distribution have earlier retarded times than nearby ones. It's just like the night sky: The light we see now left each star at the retarded time corresponding to that star's distance from the earth.) Note that the retarded potentials reduce properly to Eq. 10.24 in the static case, for which ρ and $\mathbf{J}$ are independent of time.

Well, that all sounds *reasonable*—and surprisingly simple. But are we sure it's *right*? I didn't actually *derive* the formulas for V and $\mathbf{A}$ (Eq. 10.26); all I did was invoke a heuristic argument ("electromagnetic news travels at the speed of light") to make them seem *plausible*. To *prove* them, I must show that they satisfy the inhomogeneous wave equation (Eq. 10.16) and meet the Lorenz condition (Eq. 10.12). In case you think I'm being fussy, let me warn you that if you apply the same logic to the *fields* you'll get entirely the *wrong* answer:

$$\mathbf{E}(\mathbf{r}, t) \neq \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau', \quad \mathbf{B}(\mathbf{r}, t) \neq \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'.$$

Let's stop and check, then, that the retarded scalar potential satisfies Eq. 10.16; essentially the same argument would serve for the vector potential.[7] I shall leave it for you (Prob. 10.10) to show that the retarded potentials obey the Lorenz condition.

In calculating the Laplacian of $V(\mathbf{r}, t)$, the crucial point to notice is that the integrand (in Eq. 10.26) depends on $\mathbf{r}$ in *two* places: *explicitly*, in the denominator ($\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$), and *implicitly*, through $t_r = t - \mathscr{r}/c$, in the numerator. Thus

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int \left[(\nabla\rho)\frac{1}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right] d\tau', \tag{10.27}$$

[7] I'll give you the straightforward but cumbersome proof; for a clever indirect argument see M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed., Sect. 8.1 (Orlando, FL: Saunders (1995)).

and

$$\nabla\rho = \dot{\rho}\,\nabla t_r = -\frac{1}{c}\dot{\rho}\,\nabla\boldsymbol{\mathscr{r}} \tag{10.28}$$

(The dot denotes differentiation with respect to time).[8] Now $\nabla\mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$ and $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$ (Prob. 1.13), so

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} - \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right]d\tau'. \tag{10.29}$$

Taking the divergence,

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\cdot(\nabla\dot{\rho}) + \dot{\rho}\,\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right)\right]\right.$$
$$\left. - \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot(\nabla\rho) + \rho\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\right\}d\tau'.$$

But

$$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\,\nabla\mathscr{r} = -\frac{1}{c}\ddot{\rho}\,\hat{\boldsymbol{\mathscr{r}}},$$

as in Eq. 10.28, and

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$$

(Prob. 1.63), whereas

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

(Eq. 1.100). So

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}})\right]d\tau' = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\mathbf{r}, t),$$

confirming that the retarded potential (Eq. 10.26) satisfies the inhomogeneous wave equation (Eq. 10.16). □

Incidentally, this proof applies equally well to the **advanced potentials,**

$$V_a(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t_a)}{\mathscr{r}}\,d\tau', \quad \mathbf{A}_a(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}', t_a)}{\mathscr{r}}\,d\tau', \tag{10.30}$$

in which the charge and the current densities are evaluated at the **advanced time**

$$t_a \equiv t + \frac{\mathscr{r}}{c}. \tag{10.31}$$

A few signs are changed, but the final result is unaffected. Although the advanced potentials are entirely consistent with Maxwell's equations, they violate the most sacred tenet in all of physics: the principle of **causality**. They suggest that the potentials *now* depend on what the charge and the current distribution *will be* at some

[8]Note that $\partial/\partial t_r = \partial/\partial t$, since $t_r = t - \mathscr{r}/c$ and $\mathscr{r}$ is independent of t.

time in the future—the effect, in other words, precedes the cause. Although the advanced potentials are of some theoretical interest, they have no direct physical significance.[9]

Example 10.2. An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ I_0, & \text{for } t > 0. \end{cases}$$

That is, a constant current I_0 is turned on abruptly at $t = 0$. Find the resulting electric and magnetic fields.

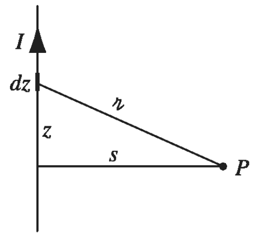

FIGURE 10.4

Solution

The wire is presumably electrically neutral, so the scalar potential is zero. Let the wire lie along the z axis (Fig. 10.4); the retarded vector potential at point P is

$$\mathbf{A}(s, t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}} \int_{-\infty}^{\infty} \frac{I(t_r)}{\mathscr{r}}\,dz.$$

For $t < s/c$, the "news" has not yet reached P, and the potential is zero. For $t > s/c$, only the segment

$$|z| \leq \sqrt{(ct)^2 - s^2} \tag{10.32}$$

contributes (outside this range t_r is negative, so $I(t_r) = 0$); thus

$$\mathbf{A}(s, t) = \left(\frac{\mu_0 I_0}{4\pi}\,\hat{\mathbf{z}}\right) 2 \int_0^{\sqrt{(ct)^2 - s^2}} \frac{dz}{\sqrt{s^2 + z^2}}$$

$$= \frac{\mu_0 I_0}{2\pi}\,\hat{\mathbf{z}}\,\ln\left(\sqrt{s^2 + z^2} + z\right)\Bigg|_0^{\sqrt{(ct)^2 - s^2}} = \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right)\hat{\mathbf{z}}.$$

[9]Because the d'Alembertian involves t^2 (as opposed to t), the theory itself is **time-reversal invariant**, and does not distinguish "past" from "future." Time asymmetry is introduced when we select the retarded potentials in preference to the advanced ones, reflecting the (not unreasonable!) belief that electromagnetic influences propagate forward, not backward, in time.

The electric field is

$$\mathbf{E}(s,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2 - s^2}}\,\hat{\mathbf{z}},$$

and the magnetic field is

$$\mathbf{B}(s,t) = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2 - s^2}}\,\hat{\boldsymbol{\phi}}.$$

Notice that as $t \to \infty$ we recover the static case: $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = (\mu_0 I_0 / 2\pi s)\,\hat{\boldsymbol{\phi}}$.

! **Problem 10.10** Confirm that the retarded potentials satisfy the Lorenz gauge condition. [*Hint:* First show that

$$\nabla \cdot \left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla \cdot \mathbf{J}) + \frac{1}{\mathscr{r}}(\nabla' \cdot \mathbf{J}) - \nabla' \cdot \left(\frac{\mathbf{J}}{\mathscr{r}}\right),$$

where ∇ denotes derivatives with respect to $\mathbf{r}$, and ∇' denotes derivatives with respect to $\mathbf{r}'$. Next, noting that $\mathbf{J}(\mathbf{r}', t - \mathscr{r}/c)$ depends on $\mathbf{r}'$ both explicitly and through $\mathscr{r}$, whereas it depends on $\mathbf{r}$ only through $\mathscr{r}$, confirm that

$$\nabla \cdot \mathbf{J} = -\frac{1}{c}\dot{\mathbf{J}} \cdot (\nabla \mathscr{r}), \quad \nabla' \cdot \mathbf{J} = -\dot{\rho} - \frac{1}{c}\dot{\mathbf{J}} \cdot (\nabla' \mathscr{r}).$$

Use this to calculate the divergence of $\mathbf{A}$ (Eq. 10.26).]

! **Problem 10.11**

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt,$$

for $t > 0$. Find the electric and magnetic fields generated.

(b) Do the same for the case of a sudden burst of current:

$$I(t) = q_0 \delta(t).$$

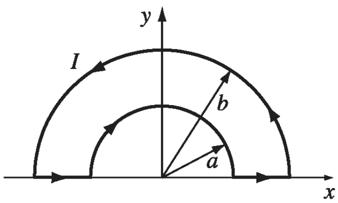

FIGURE 10.5

Problem 10.12 A piece of wire bent into a loop, as shown in Fig. 10.5, carries a current that increases linearly with time:

$$I(t) = kt \quad (-\infty < t < \infty).$$

Calculate the retarded vector potential $\mathbf{A}$ at the center. Find the electric field at the center. Why does this (neutral) wire produce an *electric* field? (Why can't you determine the *magnetic* field from this expression for $\mathbf{A}$?)

10.2.2 ■ Jefimenko's Equations

Given the retarded potentials

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau', \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau', \tag{10.33}$$

it is, in principle, a straightforward matter to determine the fields:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{10.34}$$

But the details are not entirely trivial because, as I mentioned earlier, the integrands depend on $\mathbf{r}$ both explicitly, through $\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$ in the denominator, and implicitly, through the retarded time $t_r = t - \mathscr{r}/c$ in the argument of the numerator.

I already calculated the gradient of V (Eq. 10.29); the time derivative of $\mathbf{A}$ is easy:

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\dot{\mathbf{J}}}{\mathscr{r}}\,d\tau'. \tag{10.35}$$

Putting them together (and using $c^2 = 1/\mu_0\epsilon_0$):

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{c\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2\mathscr{r}}\right] d\tau'.} \tag{10.36}$$

This is the time-dependent generalization of Coulomb's law, to which it reduces in the static case (where the second and third terms drop out and the first term loses its dependence on t_r).

As for $\mathbf{B}$, the curl of $\mathbf{A}$ contains two terms:

$$\nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\int \left[\frac{1}{\mathscr{r}}(\nabla \times \mathbf{J}) - \mathbf{J} \times \nabla\left(\frac{1}{\mathscr{r}}\right)\right] d\tau'.$$

Now

$$(\nabla \times \mathbf{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z},$$

and

$$\frac{\partial J_z}{\partial y} = \dot{J}_z\frac{\partial t_r}{\partial y} = -\frac{1}{c}\dot{J}_z\frac{\partial \mathscr{r}}{\partial y},$$

so

$$(\nabla \times \mathbf{J})_x = -\frac{1}{c}\left(\dot{J}_z\frac{\partial \mathscr{r}}{\partial y} - \dot{J}_y\frac{\partial \mathscr{r}}{\partial z}\right) = \frac{1}{c}\left[\dot{\mathbf{J}} \times (\nabla \mathscr{r})\right]_x.$$

But $\nabla \eta = \hat{\boldsymbol{\eta}}$ (Prob. 1.13), so

$$\nabla \times \mathbf{J} = \frac{1}{c}\dot{\mathbf{J}} \times \hat{\boldsymbol{\eta}}. \tag{10.37}$$

Meanwhile $\nabla(1/\eta) = -\hat{\boldsymbol{\eta}}/\eta^2$ (again, Prob. 1.13), and hence

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\eta^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\eta} \right] \times \hat{\boldsymbol{\eta}}\, d\tau'.} \tag{10.38}$$

This is the time-dependent generalization of the Biot-Savart law, to which it reduces in the static case.

Equations 10.36 and 10.38 are the (causal) solutions to Maxwell's equations. For some reason, they do not seem to have been published until quite recently—the earliest explicit statement of which I am aware was by Oleg Jefimenko, in 1966.[10] In practice **Jefimenko's equations** are of limited utility, since it is typically easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields. Nevertheless, they provide a satisfying sense of closure to the theory. They also help to clarify an observation I made in the previous section: To get to the retarded *potentials*, all you do is replace t by t_r in the electrostatic and magnetostatic formulas, but in the case of the *fields* not only is time replaced by retarded time, but completely new terms (involving derivatives of ρ and $\mathbf{J}$) appear. And they provide surprisingly strong support for the quasistatic approximation (see Prob. 10.14).

Problem 10.13 Suppose $\mathbf{J}(\mathbf{r})$ is constant in time, so (Prob. 7.60) $\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t$. Show that

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\eta^2} \hat{\boldsymbol{\eta}}\, d\tau';$$

that is, Coulomb's law holds, with the charge density evaluated at the *non-retarded* time.

Problem 10.14 Suppose the current density changes slowly enough that we can (to good approximation) ignore all higher derivatives in the Taylor expansion

$$\mathbf{J}(t_r) = \mathbf{J}(t) + (t_r - t)\dot{\mathbf{J}}(t) + \cdots$$

(for clarity, I suppress the **r**-dependence, which is not at issue). Show that a fortuitous cancellation in Eq. 10.38 yields

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t) \times \hat{\boldsymbol{\eta}}}{\eta^2} d\tau'.$$

[10] O. D. Jefimenko, *Electricity and Magnetism* (New York: Appleton-Century-Crofts, 1966), Sect. 15.7. Related expressions appear in G. A. Schott, *Electromagnetic Radiation* (Cambridge, UK: Cambridge University Press, 1912), Chapter 2, W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Reading, MA: Addison-Wesley, 1962), Sect. 14.3, and elsewhere. See K. T. McDonald, *Am. J. Phys.* **65**, 1074 (1997) for illuminating commentary and references.

That is: the Biot-Savart law holds, with $\mathbf{J}$ evaluated at the *non-retarded* time. This means that the quasistatic approximation is actually much *better* than we had any right to expect: the *two* errors involved (neglecting retardation and dropping the second term in Eq. 10.38) *cancel* one another, to first order.

10.3 ■ POINT CHARGES

10.3.1 ■ Liénard-Wiechert Potentials

My next project is to calculate the (retarded) potentials, $V(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$, of a point charge q that is moving on a specified trajectory

$$\mathbf{w}(t) \equiv \text{position of } q \text{ at time } t. \tag{10.39}$$

A naïve reading of the formula (Eq. 10.26)

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}} \, d\tau' \tag{10.40}$$

might suggest to you that the potential is simply

$$\frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}$$

(the same as in the static case, with the understanding that $\mathscr{r}$ is the distance to the *retarded* position of the charge). But this is wrong, for a very subtle reason: It is true that for a point source the denominator $\mathscr{r}$ comes outside the integral,[11] but what remains,

$$\int \rho(\mathbf{r}', t_r) \, d\tau', \tag{10.41}$$

is *not* equal to the charge of the particle (and depends, through t_r, on the location of the point $\mathbf{r}$). To calculate the total charge of a configuration, you must integrate ρ over the entire distribution at *one instant of time*, but here the retardation, $t_r = t - \mathscr{r}/c$, obliges us to evaluate ρ at *different times* for different parts of the configuration. If the source is moving, this will give a distorted picture of the total charge. You might think that this problem would disappear for *point* charges, but it doesn't. In Maxwell's electrodynamics, formulated as it is in terms of charge and current *densities*, a point charge must be regarded as the limit of an extended charge, when the size goes to zero. And for an extended particle, no matter how small, the retardation in Eq. 10.41 throws in a factor $(1 - \hat{\mathscr{r}} \cdot \mathbf{v}/c)^{-1}$, where $\mathbf{v}$ is the velocity of the charge at the retarded time:

$$\int \rho(\mathbf{r}', t_r) \, d\tau' = \frac{q}{1 - \hat{\mathscr{r}} \cdot \mathbf{v}/c}. \tag{10.42}$$

[11] There is, however, an implicit change in its functional dependence: *Before* the integration, $\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$ is a function of $\mathbf{r}$ and $\mathbf{r}'$; *after* the integration, which fixes $\mathbf{r}' = \mathbf{w}(t_r)$, $\mathscr{r} = |\mathbf{r} - \mathbf{w}(t_r)|$ is (like t_r) a function of $\mathbf{r}$ and t.

Proof. This is a purely *geometrical* effect, and it may help to tell the story in a less abstract context. You will not have noticed it, for obvious reasons, but the fact is that a train coming towards you looks a little longer than it really is, because the light you receive from the caboose left earlier than the light you receive simultaneously from the engine, and at that earlier time the train was farther away (Fig. 10.6). In the interval it takes light from the caboose to travel the extra distance L', the train itself moves a distance $L' - L$:

$$\frac{L'}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v/c}.$$

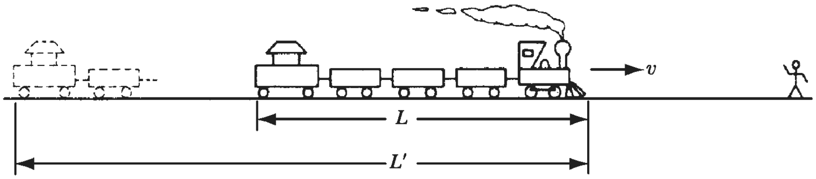

FIGURE 10.6

So approaching trains appear *longer*, by a factor $(1 - v/c)^{-1}$. By contrast, a train going *away* from you looks *shorter*,[12] by a factor $(1 + v/c)^{-1}$. In general, if the train's velocity makes an angle θ with your line of sight,[13] the extra distance light from the caboose must cover is $L' \cos\theta$ (Fig. 10.7). In the time $L' \cos\theta / c$, then, the train moves a distance $(L' - L)$:

$$\frac{L' \cos\theta}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v\cos\theta/c}.$$

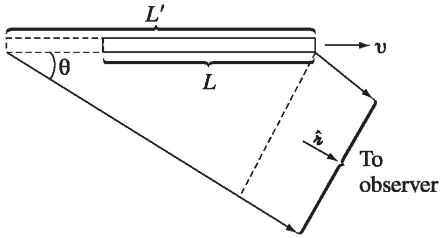

FIGURE 10.7

Notice that this effect does *not* distort the dimensions perpendicular to the motion (the height and width of the train). Never mind that the light from the far

[12]Please note that this has nothing whatever to do with special relativity or Lorentz contraction—L is the length of the *moving* train, and its *rest* length is not at issue. The argument *is* somewhat reminiscent of the Doppler effect.

[13]I assume the train is far enough away or (more to the point) *short* enough so that rays from the caboose and engine can be considered parallel.

side is delayed in reaching you (relative to light from the near side)—since there's no *motion* in that direction, they'll still look the same distance apart. The apparent *volume* τ' of the train, then, is related to the *actual* volume τ by

$$\tau' = \frac{\tau}{1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}/c}, \tag{10.43}$$

where $\hat{\boldsymbol{\mathscr{r}}}$ is a unit vector from the train to the observer.

In case the connection between moving trains and retarded potentials eludes you, the point is this: Whenever you do an integral of the type in Eq. 10.41, in which the integrand is evaluated at the retarded time, the effective volume is modified by the factor in Eq. 10.43, just as the apparent volume of the train was. Because this correction factor makes no reference to the *size* of the particle, it is every bit as significant for a point charge as for an extended charge. □

Meanwhile, for a point charge the retarded time is determined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r). \tag{10.44}$$

The left side is the distance the "news" must travel, and $(t - t_r)$ is the time it takes to make the trip (Fig. 10.8); $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point $\mathbf{r}$:

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r). \tag{10.45}$$

It is important to note that at most *one* point on the trajectory is "in communication" with $\mathbf{r}$ at any particular time t. For suppose there were *two* such points, with retarded times t_1 and t_2:

$$\mathscr{r}_1 = c(t - t_1) \quad \text{and} \quad \mathscr{r}_2 = c(t - t_2).$$

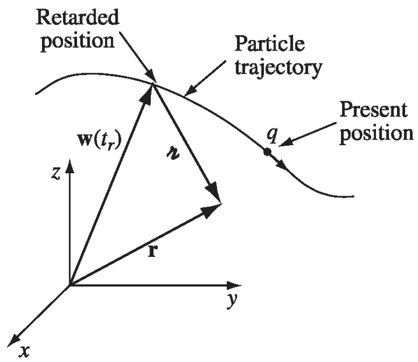

FIGURE 10.8

Then $\mathscr{r}_1 - \mathscr{r}_2 = c(t_2 - t_1)$, so the average speed of the particle in the direction of the point $\mathbf{r}$ would have to be c—and that's not counting whatever velocity the charge might have in *other* directions. Since no charged particle can travel at the speed of light, it follows that only *one retarded point contributes to the potentials, at any given moment.*[14]

It follows, then, that

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})},} \tag{10.46}$$

where $\mathbf{v}$ is the velocity of the charge at the retarded time, and $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point $\mathbf{r}$. Moreover, since the current density is $\rho\mathbf{v}$ (Eq. 5.26), the vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\mathbf{r}', t_r)\mathbf{v}(t_r)}{\mathscr{r}}\, d\tau' = \frac{\mu_0}{4\pi} \frac{\mathbf{v}}{\mathscr{r}} \int \rho(\mathbf{r}', t_r)\, d\tau',$$

or

$$\boxed{\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t).} \tag{10.47}$$

Equations 10.46 and 10.47 are the famous **Liénard-Wiechert potentials** for a moving point charge.[15]

Example 10.3. Find the potentials of a point charge moving with constant velocity.

Solution

For convenience, let's say the particle passes through the origin at time $t = 0$, so that

$$\mathbf{w}(t) = \mathbf{v}t.$$

We first compute the retarded time, using Eq. 10.44:

$$|\mathbf{r} - \mathbf{v}t_r| = c(t - t_r),$$

[14] For the same reason, an observer at $\mathbf{r}$ *sees* the particle in only one place at a time. By contrast, it is possible to *hear* an object in two places at once. Consider a bear who growls at you and then runs toward you at the speed of sound and growls again; you hear both growls at the same time, coming from two different locations, but there's only one bear.

[15] There are many ways to obtain the Liénard-Wiechert potentials. I have tried to emphasize the *geometrical* origin of the factor $(1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}/c)^{-1}$; for illuminating commentary, see W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2d ed. (Reading, MA: Addison-Wesley, 1962), pp. 342-3. A more rigorous derivation is provided by J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, 3d ed. (Reading, MA: Addison-Wesley, 1979), Sect. 21.1, or M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed. (Orlando, FL: Saunders, 1995), Sect. 8.3.

or, squaring:

$$r^2 - 2\mathbf{r}\cdot\mathbf{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2).$$

Solving for t_r by the quadratic formula, I find that

$$t_r = \frac{(c^2t - \mathbf{r}\cdot\mathbf{v}) \pm \sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}. \tag{10.48}$$

To fix the sign, consider the limit $v = 0$:

$$t_r = t \pm \frac{r}{c}.$$

In this case the charge is at rest at the origin, and the retarded time should be $(t - r/c)$; evidently we want the *minus* sign.

Now, from Eqs. 10.44 and 10.45,

$$\mathscr{r} = c(t - t_r), \quad \text{and} \quad \hat{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{v}t_r}{c(t - t_r)},$$

so

$$\begin{aligned}
\mathscr{r}(1 - \hat{\mathscr{r}}\cdot\mathbf{v}/c) &= c(t - t_r)\left[1 - \frac{\mathbf{v}}{c}\cdot\frac{(\mathbf{r} - \mathbf{v}t_r)}{c(t - t_r)}\right] = c(t - t_r) - \frac{\mathbf{v}\cdot\mathbf{r}}{c} + \frac{v^2}{c}t_r \\
&= \frac{1}{c}\left[(c^2t - \mathbf{r}\cdot\mathbf{v}) - (c^2 - v^2)t_r\right] \\
&= \frac{1}{c}\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}
\end{aligned}$$

(I used Eq. 10.48, with the minus sign, in the last step). Therefore,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}, \tag{10.49}$$

and (Eq. 10.47)

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}. \tag{10.50}$$

Problem 10.15 A particle of charge q moves in a circle of radius a at constant angular velocity ω. (Assume that the circle lies in the xy plane, centered at the origin, and at time $t = 0$ the charge is at $(a, 0)$, on the positive x axis.) Find the Liénard-Wiechert potentials for points on the z axis.

• **Problem 10.16** Show that the scalar potential of a point charge moving with constant velocity (Eq. 10.49) can be written more simply as

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1 - v^2\sin^2\theta/c^2}}, \tag{10.51}$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ is the vector from the *present (!)* position of the particle to the field point $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Note that for nonrelativistic velocities ($v^2 \ll c^2$),

$$V(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0} \frac{q}{R}.$$

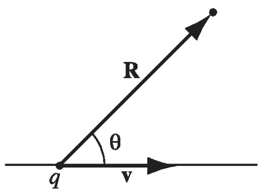

FIGURE 10.9

Problem 10.17 I showed that at *most* one point on the particle trajectory communicates with $\mathbf{r}$ at any given time. In some cases there may be *no* such point (an observer at $\mathbf{r}$ would not see the particle—in the colorful language of general relativity, it is "over the **horizon**"). As an example, consider a particle in **hyperbolic motion** along the x axis:

$$\mathbf{w}(t) = \sqrt{b^2 + (ct)^2}\,\hat{\mathbf{x}} \qquad (-\infty < t < \infty). \tag{10.52}$$

(In special relativity, this is the trajectory of a particle subject to a constant force $F = mc^2/b$.) Sketch the graph of w versus t. At four or five representative points on the curve, draw the trajectory of a light signal emitted by the particle at that point—both in the plus x direction and in the minus x direction. What region on your graph corresponds to points and times (x, t) from which the particle cannot be seen? At what time does someone at point x first see the particle? (Prior to this the potential at x is zero.) Is it possible for a particle, once seen, to *disappear* from view?

! **Problem 10.18** Determine the Liénard-Wiechert potentials for a charge in hyperbolic motion (Eq. 10.52). Assume the point $\mathbf{r}$ is on the x axis and to the right of the charge.[16]

10.3.2 ■ The Fields of a Moving Point Charge

We are now in a position to calculate the electric and magnetic fields of a point charge in arbitrary motion, using the Liénard-Wiechert potentials:[17]

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t), \tag{10.53}$$

[16]The fields of a point charge in hyperbolic motion are notoriously tricky. Indeed, a straightforward application of the Liénard-Wiechert potentials yields an electric field in violation of Gauss's law. This paradox was resolved by Bondi and Gold in 1955. For a history of the problem, see E. Eriksen and Ø. Grøn, *Ann. Phys.* **286**, 320 (2000).

[17]You can get the fields directly from Jefimenko's equations, but it's not easy. See, for example, M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed. (Orlando, FL: Saunders, 1995), Sect. 8.4.

and the equations for **E** and **B**:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The differentiation is tricky, however, because

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \quad \text{and} \quad \mathbf{v} = \dot{\mathbf{w}}(t_r) \tag{10.54}$$

are both evaluated at the retarded time, and t_r—defined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r) \tag{10.55}$$

—is *itself* a function of **r** and t.[18] So hang on: the next two pages are rough going ... but the answer is worth the effort.

Let's begin with the gradient of V:

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \nabla(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}). \tag{10.56}$$

Since $\mathscr{r} = c(t - t_r)$,[19]

$$\nabla \mathscr{r} = -c \nabla t_r. \tag{10.57}$$

As for the second term, product rule 4 gives

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}}). \tag{10.58}$$

Evaluating these terms one at a time:

$$\begin{aligned}
(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} &= \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right) \mathbf{v}(t_r) \\
&= \mathscr{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z} \\
&= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r),
\end{aligned} \tag{10.59}$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the *acceleration* of the particle at the retarded time. Now

$$(\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} = (\mathbf{v} \cdot \nabla)\mathbf{r} - (\mathbf{v} \cdot \nabla)\mathbf{w}, \tag{10.60}$$

[18]The following calculation is done by the most direct, "brute force" method. For a more clever and efficient approach, see J. D. Jackson, *Classical Electrodynamics*, 3d ed. (New York: John Wiley, 1999), Sect. 14.1.

[19]Remember that $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$ (Fig. 10.8), and t_r is itself a function of **r**. Contrast Prob. 1.13 (and Section 10.2), where $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$ (Fig. 10.3), and $\mathbf{r}'$ was an *independent* variable. In that case $\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$, but here we have a more complicated problem on our hands.

and

$$(\mathbf{v}\cdot\nabla)\mathbf{r} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$$
$$= v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} = \mathbf{v}, \tag{10.61}$$

while

$$(\mathbf{v}\cdot\nabla)\mathbf{w} = \mathbf{v}(\mathbf{v}\cdot\nabla t_r)$$

(same reasoning as Eq. 10.59). Moving on to the third term in Eq. 10.58,

$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$
$$= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\mathbf{y}}$$
$$+ \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{\mathbf{z}}$$
$$= -\mathbf{a}\times\nabla t_r. \tag{10.62}$$

Finally,

$$\nabla\times\boldsymbol{\mathscr{r}} = \nabla\times\mathbf{r} - \nabla\times\mathbf{w}, \tag{10.63}$$

but $\nabla\times\mathbf{r} = \mathbf{0}$, while, by the same argument as Eq. 10.62,

$$\nabla\times\mathbf{w} = -\mathbf{v}\times\nabla t_r. \tag{10.64}$$

Putting all this back into Eq. 10.58, and using the "BAC-CAB" rule to reduce the triple cross products,

$$\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = \mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r) + \mathbf{v} - \mathbf{v}(\mathbf{v}\cdot\nabla t_r) - \boldsymbol{\mathscr{r}}\times(\mathbf{a}\times\nabla t_r) + \mathbf{v}\times(\mathbf{v}\times\nabla t_r)$$
$$= \mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a} - v^2)\nabla t_r. \tag{10.65}$$

Collecting Eqs. 10.57 and 10.65, we have

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\nabla t_r\right]. \tag{10.66}$$

To complete the calculation, we need to know ∇t_r. This can be found by taking the gradient of the defining equation (Eq. 10.55)—which we have already done in Eq. 10.57—and expanding $\nabla\mathscr{r}$:

$$-c\nabla t_r = \nabla\mathscr{r} = \nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}} = \frac{1}{2\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}})$$
$$= \frac{1}{\mathscr{r}}\left[(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}})\right]. \tag{10.67}$$

But

$$(\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}} = \boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)$$

(same idea as Eq. 10.60), while (from Eqs. 10.63 and 10.64)

$$\nabla \times \boldsymbol{\mathscr{r}} = (\mathbf{v} \times \nabla t_r).$$

Thus

$$-c\nabla t_r = \frac{1}{\mathscr{r}}\left[\boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) + \boldsymbol{\mathscr{r}} \times (\mathbf{v} \times \nabla t_r)\right] = \frac{1}{\mathscr{r}}\left[\boldsymbol{\mathscr{r}} - (\boldsymbol{\mathscr{r}} \cdot \mathbf{v})\nabla t_r\right],$$

and hence

$$\nabla t_r = \frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}. \tag{10.68}$$

Incorporating this result into Eq. 10.66, I conclude that

$$\nabla V = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3}\left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\boldsymbol{\mathscr{r}}\right]. \tag{10.69}$$

A similar calculation, which I shall leave for you (Prob. 10.19), yields

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3}\Big[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{v}\Big]. \tag{10.70}$$

Combining these results, and introducing the vector

$$\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}, \tag{10.71}$$

I find

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})\right].} \tag{10.72}$$

Meanwhile,

$$\nabla \times \mathbf{A} = \frac{1}{c^2}\nabla \times (V\mathbf{v}) = \frac{1}{c^2}\left[V(\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla V)\right].$$

We have already calculated $\nabla \times \mathbf{v}$ (Eq. 10.62) and ∇V (Eq. 10.69). Putting these together,

$$\nabla \times \mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u} \cdot \boldsymbol{\mathscr{r}})^3}\boldsymbol{\mathscr{r}} \times \left[(c^2 - v^2)\mathbf{v} + (\boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{v} + (\boldsymbol{\mathscr{r}} \cdot \mathbf{u})\mathbf{a}\right].$$

The quantity in brackets is strikingly similar to the one in Eq. 10.72, which can be written, using the BAC-CAB rule, as $[(c^2 - v^2)\mathbf{u} + (\boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}} \cdot \mathbf{u})\mathbf{a}]$; the main difference is that we have $\mathbf{v}$'s instead of $\mathbf{u}$'s in the first two terms. In fact, since it's all crossed into $\boldsymbol{\mathscr{r}}$ *anyway*, we can with impunity *change* these $\mathbf{v}$'s into $-\mathbf{u}$'s; the extra term proportional to $\boldsymbol{\mathscr{r}}$ disappears in the cross product. It follows that

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t).} \tag{10.73}$$

Evidently *the magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.*

The first term in $\mathbf{E}$ (the one involving $(c^2 - v^2)\mathbf{u}$) falls off as the inverse *square* of the distance from the particle. If the velocity and acceleration are both zero, this term alone survives and reduces to the old electrostatic result

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.$$

For this reason, the first term in $\mathbf{E}$ is sometimes called the **generalized Coulomb field**. (Because it does not depend on the acceleration, it is also known as the **velocity field**.) The second term (the one involving $\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})$) falls off as the inverse *first* power of $\mathscr{r}$ and is therefore dominant at large distances. As we shall see in Chapter 11, it is this term that is responsible for electromagnetic radiation; accordingly, it is called the **radiation field**—or, since it is proportional to a, the **acceleration field**. The same terminology applies to the magnetic field.

Back in Chapter 2, I commented that if we could write down the formula for the force one charge exerts on another, we would be done with electrodynamics, in principle. That, together with the superposition principle, would tell us the force exerted on a test charge Q by any configuration whatsoever. Well ... here we are: Eqs. 10.72 and 10.73 give us the fields, and the Lorentz force law determines the resulting force:

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}\Big\{[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] + \frac{\mathbf{V}}{c} \times \big[\hat{\boldsymbol{\mathscr{r}}} \times [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]\big]\Big\}, \tag{10.74}$$

where $\mathbf{V}$ is the velocity of Q, and $\boldsymbol{\mathscr{r}}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{a}$ are all evaluated at the retarded time. The entire theory of classical electrodynamics is contained in that equation ... but you see why I preferred to start out with Coulomb's law.

Example 10.4. Calculate the electric and magnetic fields of a point charge moving with constant velocity.

Solution

Putting $\mathbf{a} = \mathbf{0}$ in Eq. 10.72,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{(c^2 - v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}\mathbf{u}.$$

In this case, using $\mathbf{w} = \mathbf{v}t$,

$$\mathscr{r}\mathbf{u} = c\boldsymbol{\mathscr{r}} - \mathscr{r}\mathbf{v} = c(\mathbf{r} - \mathbf{v}t_r) - c(t - t_r)\mathbf{v} = c(\mathbf{r} - \mathbf{v}t).$$

In Ex. 10.3 we found that

$$\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v} = \boldsymbol{\mathscr{r}} \cdot \mathbf{u} = \sqrt{(c^2 t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}.$$

In Prob. 10.16 you showed that this radical could be written as

$$Rc\sqrt{1 - v^2 \sin^2\theta / c^2},$$

where

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$$

is the vector from the *present* location of the particle to $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Thus

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta / c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}.} \tag{10.75}$$

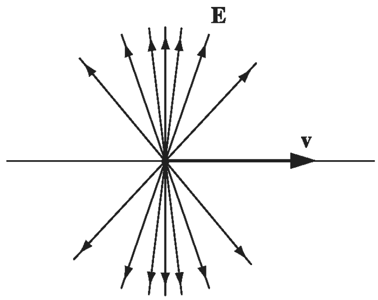

FIGURE 10.10

Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an extraordinary coincidence, since the "message" came from the *retarded* position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is *reduced* by a factor $(1 - v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$.

As for $\mathbf{B}$, we have

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} = \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c},$$

and therefore

$$\boxed{\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).} \tag{10.76}$$

Lines of **B** *circle around* the charge, as shown in Fig. 10.11.

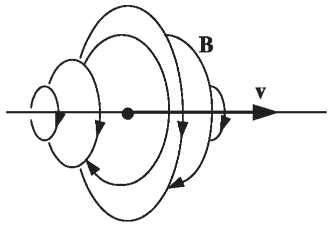

FIGURE 10.11

The fields of a point charge moving at constant velocity (Eqs. 10.75 and 10.76) were first obtained by Oliver Heaviside in 1888.[20] When $v^2 \ll c^2$ they reduce to

$$\mathbf{E}(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}}; \qquad \mathbf{B}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi}\frac{q}{R^2}(\mathbf{v} \times \hat{\mathbf{R}}). \tag{10.77}$$

The first is essentially Coulomb's law, and the second is the "Biot-Savart law for a point charge" I warned you about in Chapter 5 (Eq. 5.43).

Problem 10.19 Derive Eq. 10.70. First show that

$$\frac{\partial t_r}{\partial t} = \frac{\mathscr{r} c}{\boldsymbol{\mathscr{r}} \cdot \mathbf{u}}. \tag{10.78}$$

Problem 10.20 Suppose a point charge q is constrained to move along the x axis. Show that the fields at points on the axis to the *right* of the charge are given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}}, \quad \mathbf{B} = \mathbf{0}.$$

(Do not assume v is constant!) What are the fields on the axis to the *left* of the charge?

Problem 10.21 For a point charge moving at constant velocity, calculate the flux integral $\oint \mathbf{E} \cdot d\mathbf{a}$ (using Eq. 10.75), over the surface of a sphere centered at the present location of the charge.[21]

[20]For history and references, see O. J. Jefimenko, *Am. J. Phys.* **62**, 79 (1994).

[21]Feynman was fond of saying you should never begin a calculation before you know the answer. It doesn't always work, but this is a good problem to try it on.

Problem 10.22

(a) Use Eq. 10.75 to calculate the electric field a distance d from an infinite straight wire carrying a uniform line charge λ, moving at a constant speed v down the wire.

(b) Use Eq. 10.76 to find the *magnetic* field of this wire.

Problem 10.23 For the configuration in Prob. 10.15, find the electric and magnetic fields at the center. From your formula for **B**, determine the magnetic field at the center of a circular loop carrying a steady current I, and compare your answer with the result of Ex. 5.6

More Problems on Chapter 10

Problem 10.24 Suppose you take a plastic ring of radius a and glue charge on it, so that the line charge density is $\lambda_0|\sin(\theta/2)|$. Then you spin the loop about its axis at an angular velocity ω. Find the (exact) scalar and vector potentials at the center of the ring. $\left[\textit{Answer: } \mathbf{A} = (\mu_0\lambda_0\omega a/3\pi)\left\{\sin[\omega(t-a/c)]\,\hat{\mathbf{x}} - \cos[\omega(t-a/c)]\,\hat{\mathbf{y}}\right\}\right]$

Problem 10.25 Figure 2.35 summarizes the laws of electrostatics in a "triangle diagram" relating the source (ρ), the field (**E**), and the potential (V). Figure 5.48 does the same for magnetostatics, where the source is **J**, the field is **B**, and the potential is **A**. Construct the analogous diagram for *electrodynamics*, with sources ρ and **J** (constrained by the continuity equation), fields **E** and **B**, and potentials V and **A** (constrained by the Lorenz gauge condition). Do not include formulas for V and **A** in terms of **E** and **B**.

Problem 10.26 An expanding sphere, radius $R(t) = vt$ ($t > 0$, constant v) carries a charge Q, uniformly distributed over its volume. Evaluate the integral

$$Q_{\text{eff}} = \int \rho(\mathbf{r}, t_r)\,d\tau$$

with respect to the center. Show that $Q_{\text{eff}} \approx Q(1 - \frac{3v}{4c})$, if $v \ll c$.

Problem 10.27 Check that the potentials of a point charge moving at constant velocity (Eqs. 10.49 and 10.50) satisfy the Lorenz gauge condition (Eq. 10.12).

Problem 10.28 One particle, of charge q_1, is held at rest at the origin. Another particle, of charge q_2, approaches along the x axis, in hyperbolic motion:

$$x(t) = \sqrt{b^2 + (ct)^2};$$

it reaches the closest point, b, at time $t = 0$, and then returns out to infinity.

(a) What is the force F_2 on q_2 (due to q_1) at time t?

(b) What total impulse $\left(I_2 = \int_{-\infty}^{\infty} F_2 dt\right)$ is delivered to q_2 by q_1?

(c) What is the force F_1 on q_1 (due to q_2) at time t?

(d) What total impulse $\left(I_1 = \int_{-\infty}^{\infty} F_1 dt\right)$ is delivered to q_1 by q_2? [*Hint:* It might help to review Prob. 10.17 before doing this integral. *Answer:* $I_2 = -I_1 = q_1 q_2/4\epsilon_0 bc$]

Problem 10.29 We are now in a position to treat the example in Sect. 8.2.1 quantitatively. Suppose q_1 is at $x_1 = -vt$ and q_2 is at $y = -vt$ (Fig. 8.3, with $t < 0$). Find the electric and magnetic forces on q_1 and q_2. Is Newton's third law obeyed?

Problem 10.30 A uniformly charged rod (length L, charge density λ) slides out the x axis at constant speed v. At time $t = 0$ the back end passes the origin (so its position as a function of time is $x = vt$, while the front end is at $x = vt + L$). Find the retarded scalar potential at the origin, as a function of time, for $t > 0$. [First determine the retarded time t_1 for the back end, the retarded time t_2 for the front end, and the corresponding retarded positions x_1 and x_2.] Is your answer consistent with the Liénard-Wiechert potential, in the point charge limit ($L \ll vt$, with $\lambda L = q$)? Do not assume $v \ll c$.

Problem 10.31 A particle of charge q is traveling at constant speed v along the x axis. Calculate the total power passing through the plane $x = a$, at the moment the particle itself is at the origin. $\left[\textit{Answer: } q^2 v/32\pi\epsilon_0 a^2\right]$

Problem 10.32[22] A particle of charge q_1 is at rest at the origin. A second particle, of charge q_2, moves along the z axis at constant velocity v.

(a) Find the force $\mathbf{F}_{12}(t)$ of q_1 on q_2, at time t (when q_2 is at $z = vt$).

(b) Find the force $\mathbf{F}_{21}(t)$ of q_2 on q_1, at time t. Does Newton's third law hold, in this case?

! (c) Calculate the linear momentum $\mathbf{p}(t)$ in the electromagnetic fields, at time t. (Don't bother with any terms that are constant in time, since you won't need them in part (d)). [*Answer:* $(\mu_0 q_1 q_2/4\pi t)\,\hat{\mathbf{z}}$]

(d) Show that the sum of the forces is equal to minus the rate of change of the momentum in the fields, and interpret this result physically.

Problem 10.33 Develop the potential formulation for electrodynamics with magnetic charge (Eq. 7.44). [*Hint:* You'll need two scalar potentials and two vector potentials. Use the Lorenz gauge. Find the retarded potentials (generalizing Eqs. 10.26), and give the formulas for **E** and **B** in terms of the potentials (generalizing Eqs. 10.2 and 10.3).]

! **Problem 10.34** Find the (Lorenz gauge) potentials and fields of a time-dependent ideal electric dipole $\mathbf{p}(t)$ at the origin.[23] (It is stationary, but its magnitude and/or direction are changing with time.) Don't bother with the contact term. [*Answer:*

[22] See J. J. G. Scanio, *Am. J. Phys.* **43**, 258 (1975).
[23] W. J. M. Kort-Kamp and C. Farina, *Am. J. Phys.* **79**, 111 (2011); D. J. Griffiths, *Am. J. Phys.* **79**, 867 (2011).

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\cdot[\mathbf{p} + (r/c)\dot{\mathbf{p}}]$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\left[\frac{\dot{\mathbf{p}}}{r}\right]$$

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi}\left\{\frac{\ddot{\mathbf{p}} - \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})}{r} + c^2\frac{[\mathbf{p} + (r/c)\dot{\mathbf{p}}] - 3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot[\mathbf{p} + (r/c)\dot{\mathbf{p}}])}{r^3}\right\}$$

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi}\left\{\frac{\hat{\mathbf{r}}\times[\dot{\mathbf{p}} + (r/c)\ddot{\mathbf{p}}]}{r^2}\right\} \tag{10.79}$$

where all the derivatives of $\mathbf{p}$ are evaluated at the retarded time.]

CHAPTER 11

Radiation

11.1 ■ DIPOLE RADIATION

11.1.1 ■ What is Radiation?

When charges *accelerate*, their fields can transport energy irreversibly out to infinity—a process we call **radiation**.[1] Let us assume the source is localized[2] near the origin; we would like to calculate the energy it is radiating at time t_0. Imagine a gigantic sphere, out at radius r (Fig. 11.1) The power passing through its surface is the integral of the Poynting vector:

$$P(r, t) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}. \tag{11.1}$$

Because electromagnetic "news" travels at the speed of light,[3] this energy actually left the source at the earlier time $t_0 = t - r/c$, so the power radiated is

$$P_{\text{rad}}(t_0) = \lim_{r \to \infty} P\left(r, t_0 + \frac{r}{c}\right) \tag{11.2}$$

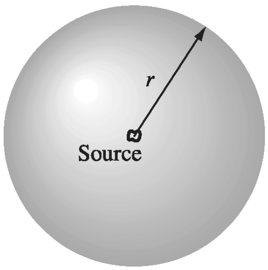

FIGURE 11.1

[1]In this chapter, the word "radiation" is used in a restricted technical sense—it might better be called "radiation to infinity." In everyday language the word has a broader connotation. We speak, for example, of radiation from a heat lamp or an x-ray machine. In this more general sense, electromagnetic "radiation" applies to any fields that transport energy—which is to say, fields whose Poynting vector is non-zero. There is nothing wrong with that language, but it is not how I am using the term here.

[2]For *non*localized configurations, such as infinite planes, wires, or solenoids, the concept of "radiation" must be reformulated (Prob. 11.28).

[3]More precisely, the fields depend on the status of the source at the retarded time.

(with t_0 held constant). This is energy (per unit time) that is carried away and never comes back.

Now, the area of the sphere is $4\pi r^2$, so for radiation to occur the Poynting vector must decrease (at large r) no faster than $1/r^2$ (if it went like $1/r^3$, for example, then P would go like $1/r$, and P_{rad} would be zero). According to Coulomb's law, electro*static* fields fall off like $1/r^2$ (or even faster, if the total charge is zero), and the Biot-Savart law says that magneto*static* fields go like $1/r^2$ (or faster), which means that $S \sim 1/r^4$, for static configurations. So *static* sources do not radiate. But Jefimenko's equations (Eqs. 10.36 and 10.38) indicate that *time-dependent* fields include terms (involving $\dot{\rho}$ and $\dot{\mathbf{J}}$) that go like $1/r$; these are the terms that are responsible for electromagnetic radiation.

The study of radiation, then, involves picking out the parts of $\mathbf{E}$ and $\mathbf{B}$ that go like $1/r$ at large distances from the source, constructing from them the $1/r^2$ term in $\mathbf{S}$, integrating over a large spherical[4] surface, and taking the limit as $r \to \infty$. I'll carry through this procedure first for oscillating electric and magnetic dipoles; then, in Sect. 11.2, we'll consider the more difficult case of radiation from an accelerating point charge.

11.1.2 ■ Electric Dipole Radiation

Picture two tiny metal spheres separated by a distance d and connected by a fine wire (Fig. 11.2); at time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose that we drive the charge back and forth through the wire, from one end to the other, at an angular frequency ω:

$$q(t) = q_0 \cos(\omega t). \tag{11.3}$$

The result is an oscillating electric dipole:[5]

$$\mathbf{p}(t) = p_0 \cos(\omega t)\,\hat{\mathbf{z}}, \tag{11.4}$$

where

$$p_0 \equiv q_0 d$$

is the maximum value of the dipole moment.

The retarded potential (Eq. 10.26) is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}, \tag{11.5}$$

where, by the law of cosines,

$$\mathscr{r}_\pm = \sqrt{r^2 \mp r d\cos\theta + (d/2)^2}. \tag{11.6}$$

[4]It doesn't have to be a sphere, of course, but this makes the calculations a lot easier.

[5]It might occur to you that a more natural model would consist of equal and opposite charges mounted on a spring, say, so that q is constant while d oscillates, instead of the other way around. Such a model would lead to the same result, but moving point charges are hard to work with, and this formulation is much simpler.

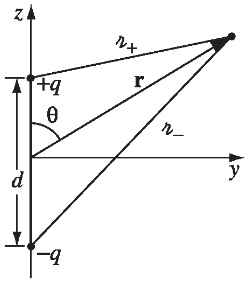

FIGURE 11.2

Now, to make this *physical* dipole into a *perfect* dipole, we want the separation distance to be extremely small:

$$\textbf{approximation 1}: \quad d \ll r. \tag{11.7}$$

Of course, if d is *zero* we get no potential at all; what we want is an expansion carried to *first order* in d. Thus

$$\mathscr{r}_\pm \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right). \tag{11.8}$$

It follows that

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right), \tag{11.9}$$

and

$$\begin{aligned}\cos[\omega(t - \mathscr{r}_\pm/c)] &\cong \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right]\\ &= \cos[\omega(t - r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right)\\ &\quad \mp \sin[\omega(t - r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right).\end{aligned}$$

In the perfect dipole limit we have, further,

$$\textbf{approximation 2}: \quad d \ll \frac{c}{\omega}. \tag{11.10}$$

(Since waves of frequency ω have a wavelength $\lambda = 2\pi c/\omega$, this amounts to the requirement $d \ll \lambda$.) Under these conditions,

$$\cos[\omega(t - \mathscr{r}_\pm/c)] \cong \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta\,\sin[\omega(t - r/c)]. \tag{11.11}$$

Putting Eqs. 11.9 and 11.11 into Eq. 11.5, we obtain the potential of an oscillating perfect dipole:

$$V(r,\theta,t) = \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)] + \frac{1}{r}\cos[\omega(t-r/c)]\right\}. \tag{11.12}$$

In the static limit ($\omega \to 0$) the second term reproduces the old formula for the potential of a stationary dipole (Eq. 3.102):

$$V = \frac{p_0\cos\theta}{4\pi\epsilon_0 r^2}.$$

This is not, however, the term that concerns us now; we are interested in the fields that survive at *large distances from the source*, in the so-called **radiation zone**:[6]

$$\textbf{approximation 3}: \quad r \gg \frac{c}{\omega} \tag{11.13}$$

(or, in terms of the wavelength, $r \gg \lambda$). In this region the potential reduces to

$$\boxed{V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)].} \tag{11.14}$$

Meanwhile, the *vector* potential is determined by the current flowing in the wire:

$$\mathbf{I}(t) = \frac{dq}{dt}\,\hat{\mathbf{z}} = -q_0\omega\sin(\omega t)\,\hat{\mathbf{z}}. \tag{11.15}$$

Referring to Fig. 11.3,

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q_0\omega\sin[\omega(t-\mathscr{r}/c)]\,\hat{\mathbf{z}}}{\mathscr{r}}\,dz. \tag{11.16}$$

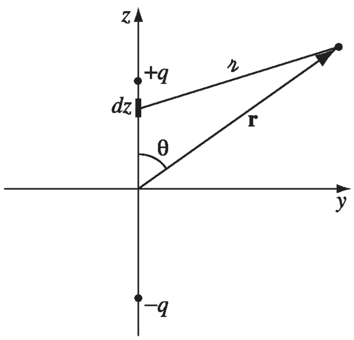

FIGURE 11.3

[6]Note that approximations 2 and 3 subsume approximation 1; all together, we have $d \ll \lambda \ll r$.

Because the integration itself introduces a factor of d, we can, to first order, replace the integrand by its value at the center:

$$\boxed{\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)]\,\hat{\mathbf{z}}.} \tag{11.17}$$

(Notice that whereas I implicitly used approximations 1 and 2, in keeping only the first order in d, Eq. 11.17 is not subject to approximation 3.)

From the potentials, it is a straightforward matter to compute the fields.

$$\begin{aligned}
\nabla V &= \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} \\
&= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\hat{\mathbf{r}}\right. \\
&\qquad\qquad \left. -\frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}\right\} \\
&\cong \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\mathbf{r}}.
\end{aligned}$$

(I dropped the first and last terms, in accordance with approximation 3.) Likewise,

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}),$$

and therefore

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}.} \tag{11.18}$$

Meanwhile

$$\begin{aligned}
\nabla \times \mathbf{A} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \\
&= -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.
\end{aligned}$$

The second term is again eliminated by approximation 3, so

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}}.} \tag{11.19}$$

Equations 11.18 and 11.19 represent monochromatic waves of frequency ω traveling in the radial direction at the speed of light. The fields are in phase,

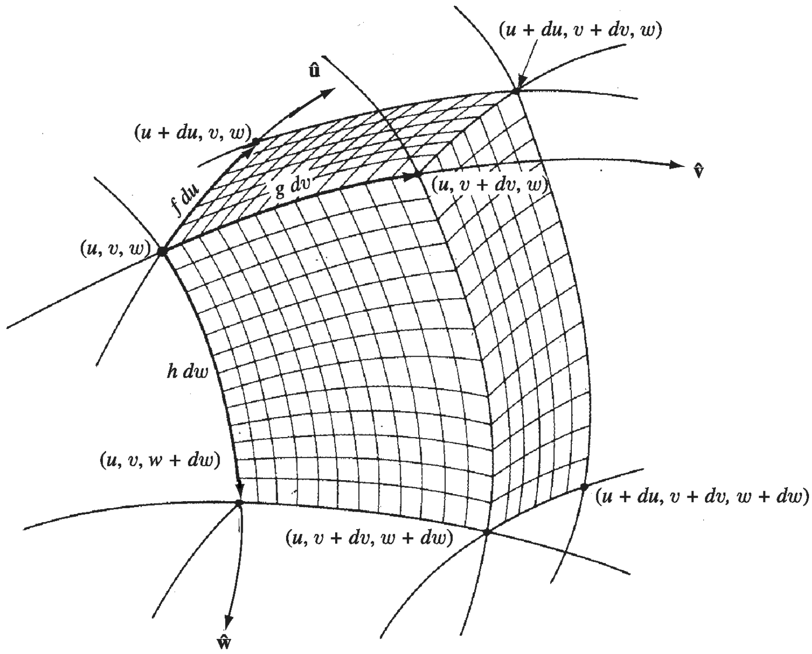

FIGURE A.2

and the top and bottom give

$$\frac{1}{fgh}\frac{\partial}{\partial w}(fgA_w)\,d\tau.$$

All told, then,

$$\oint \mathbf{A}\cdot d\mathbf{a} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right] d\tau. \tag{A.7}$$

The coefficient of $d\tau$ serves to define the **divergence** of **A** in curvilinear coordinates:

$$\boxed{\nabla\cdot\mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right],} \tag{A.8}$$

and Eq. A.7 becomes

$$\oint \mathbf{A}\cdot d\mathbf{a} = (\nabla\cdot\mathbf{A})\,d\tau. \tag{A.9}$$

Using Table A.1, you can now derive the formulas for the divergence in Cartesian, spherical, and cylindrical coordinates, which appear in the front cover of the book.

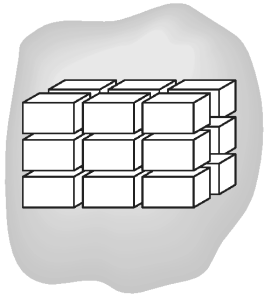

FIGURE A.3

As it stands, Eq. A.9 does not prove the divergence theorem, for it pertains only to *infinitesimal* volumes, and rather special infinitesimal volumes at that. Of course, a finite volume can be broken up into infinitesimal pieces, and Eq. A.9 can be applied to each one. The trouble is, when you then add up all the bits, the left-hand side is not just an integral over the *outer* surface, but over all those tiny *internal* surfaces as well. Luckily, however, these contributions cancel in pairs, for each internal surface occurs as the boundary of *two* adjacent infinitesimal volumes, and since $d\mathbf{a}$ always points *outward*, $\mathbf{A} \cdot d\mathbf{a}$ has the opposite sign for the two members of each pair (Fig. A.3). Only those surfaces that bound a *single* chunk—which is to say, only those at the outer boundary—survive when everything is added up. For *finite* regions, then,

$$\oint \mathbf{A} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{A})\, d\tau, \tag{A.10}$$

and you need integrate only over the *external* surface.[2] This establishes the **divergence theorem**.

A.5 ■ CURL

To obtain the curl in curvilinear coordinates, we calculate the line integral,

$$\oint \mathbf{A} \cdot d\mathbf{l},$$

around the infinitesimal loop generated by starting at (u, v, w) and successively increasing u and v by infinitesimal amounts, holding w constant (Fig. A.4). The surface is a rectangle (at least, in the infinitesimal limit), of length $dl_u = f\,du$, width $dl_v = g\,dv$, and area

$$d\mathbf{a} = (fg)du\,dv\,\hat{\mathbf{w}}. \tag{A.11}$$

[2]What about regions that cannot be fit perfectly by rectangular solids no matter *how* tiny they are—such as planes cut at an angle to the coordinate lines? It's not hard to dispose of this case; try thinking it out for yourself, or look at H. M. Schey's *Div, Grad, Curl and All That* (New York: W. W. Norton, 1973), starting with Prob. II-15.

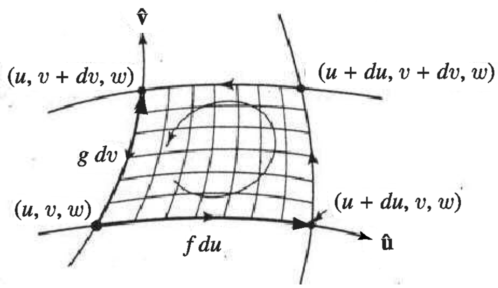

FIGURE A.4

Assuming the coordinate system is right-handed, $\hat{\mathbf{w}}$ points out of the page in Fig. A.4. Having chosen this as the positive direction for $d\mathbf{a}$, we are obliged by the right-hand rule to run the line integral counterclockwise, as shown.

Along the bottom segment,

$$d\mathbf{l} = f\,du\,\hat{\mathbf{u}},$$

so

$$\mathbf{A} \cdot d\mathbf{l} = (f A_u)\,du.$$

Along the top leg, the sign is reversed, and $f A_u$ is evaluated at $(v + dv)$ rather than v. Taken together, these two edges give

$$\left[-(f A_u)\big|_{v+dv} + (f A_u)\big|_v\right] du = -\left[\frac{\partial}{\partial v}(f A_u)\right] du\,dv.$$

Similarly, the right and left sides yield

$$\left[\frac{\partial}{\partial u}(g A_v)\right] du\,dv,$$

so the total is

$$\begin{aligned} \oint \mathbf{A} \cdot d\mathbf{l} &= \left[\frac{\partial}{\partial u}(g A_v) - \frac{\partial}{\partial v}(f A_u)\right] du\,dv \\ &= \frac{1}{fg}\left[\frac{\partial}{\partial u}(g A_v) - \frac{\partial}{\partial v}(f A_u)\right] \hat{\mathbf{w}} \cdot d\mathbf{a}. \end{aligned} \tag{A.12}$$

The coefficient of $d\mathbf{a}$ on the right serves to define the w-component of the **curl**. Constructing the u and v components in the same way, we have

$$\boxed{\begin{aligned} \nabla \times \mathbf{A} \equiv \frac{1}{gh}\left[\frac{\partial}{\partial v}(h A_w) - \frac{\partial}{\partial w}(g A_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(f A_u) - \frac{\partial}{\partial u}(h A_w)\right]\hat{\mathbf{v}} \\ + \frac{1}{fg}\left[\frac{\partial}{\partial u}(g A_v) - \frac{\partial}{\partial v}(f A_u)\right]\hat{\mathbf{w}}, \end{aligned}} \tag{A.13}$$

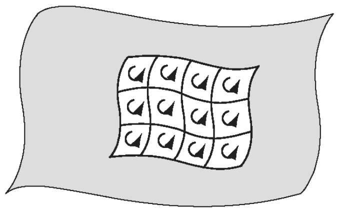

FIGURE A.5

and Eq. A.11 generalizes to

$$\oint \mathbf{A} \cdot d\mathbf{l} = (\nabla \times \mathbf{A}) \cdot d\mathbf{a}. \tag{A.14}$$

Using Table A.1, you can now derive the formulas for the curl in Cartesian, spherical, and cylindrical coordinates.

Equation A.14 does not by itself prove Stokes' theorem, however, because at this point it pertains only to very special infinitesimal surfaces. Again, we can chop any *finite* surface into infinitesimal pieces and apply Eq. A.14 to each one (Fig. A.5). When we add them up, though, we obtain (on the left) not only a line integral around the outer boundary, but a lot of tiny line integrals around the internal loops as well. Fortunately, as before, the internal contributions cancel in pairs, because every internal line is the edge of *two* adjacent loops running in opposite directions. Consequently, Eq. A.14 can be extended to finite surfaces,

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a}, \tag{A.15}$$

and the line integral is to be taken over the external boundary only.[3] This establishes **Stokes' theorem**.

A.6 ■ LAPLACIAN

Since the **Laplacian** of a scalar is by definition the divergence of the gradient, we can read off from Eqs. A.4 and A.8 the general formula

$$\boxed{\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right].} \tag{A.16}$$

Once again, you are invited to use Table A.1 to derive the Laplacian in Cartesian, spherical, and cylindrical coordinates, and thus to confirm the formulas inside the front cover.

[3]What about surfaces that cannot be fit perfectly by tiny rectangles, no matter how small they are (such as triangles) or surfaces that do not correspond to holding one coordinate fixed? If such cases trouble you, and you cannot resolve them for yourself, look at H. M. Schey's *Div, Grad, Curl, and All That*, Prob. III-2 (New York: W. W. Norton, 1973).

APPENDIX

B The Helmholtz Theorem

Suppose we are told that the divergence of a vector function $\mathbf{F}(\mathbf{r})$ is a specified scalar function $D(\mathbf{r})$:

$$\nabla \cdot \mathbf{F} = D, \tag{B.1}$$

and the curl of $\mathbf{F}(\mathbf{r})$ is a specified vector function $\mathbf{C}(\mathbf{r})$:

$$\nabla \times \mathbf{F} = \mathbf{C}. \tag{B.2}$$

For consistency, $\mathbf{C}$ must be divergenceless,

$$\nabla \cdot \mathbf{C} = 0, \tag{B.3}$$

because the divergence of a curl is always zero. *Question:* can we, on the basis of this information, determine the function $\mathbf{F}$? If $D(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ go to zero sufficiently rapidly at infinity, the answer is *yes*, as I will show by explicit construction.

I claim that

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}, \tag{B.4}$$

where

$$U(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{D(\mathbf{r}')}{\mathscr{r}} \, d\tau' \tag{B.5}$$

and

$$\mathbf{W}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r}')}{\mathscr{r}} \, d\tau'; \tag{B.6}$$

the integrals are over all of space, and, as always, $\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$. For if $\mathbf{F}$ is given by Eq. B.4, then its divergence (using Eq. 1.102) is

$$\nabla \cdot \mathbf{F} = -\nabla^2 U = -\frac{1}{4\pi} \int D \, \nabla^2 \left(\frac{1}{\mathscr{r}} \right) d\tau' = \int D(\mathbf{r}') \, \delta^3(\mathbf{r} - \mathbf{r}') \, d\tau' = D(\mathbf{r}).$$

(Remember that the divergence of a curl is zero, so the $\mathbf{W}$ term drops out, and note that the differentiation is with respect to $\mathbf{r}$, which is contained in $\mathscr{r}$.)

So the divergence is right; how about the curl?

$$\nabla \times \mathbf{F} = \nabla \times (\nabla \times \mathbf{W}) = -\nabla^2 \mathbf{W} + \nabla(\nabla \cdot \mathbf{W}). \tag{B.7}$$

(Since the curl of a gradient is zero, the U term drops out.) Now

$$-\nabla^2 \mathbf{W} = -\frac{1}{4\pi}\int \mathbf{C}\,\nabla^2\left(\frac{1}{\scriptstyle{\mathscr{r}}}\right)d\tau' = \int \mathbf{C}(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau' = \mathbf{C}(\mathbf{r}),$$

which is perfect—I'll be done if I can just persuade you that the *second* term on the right side of Eq. B.7 vanishes. Using integration by parts (Eq. 1.59), and noting that derivatives of $\scriptstyle{\mathscr{r}}$ with respect to *primed* coordinates differ by a sign from those with respect to *unprimed* coordinates, we have

$$\begin{aligned} 4\pi\nabla\cdot\mathbf{W} &= \int \mathbf{C}\cdot\nabla\left(\frac{1}{\scriptstyle{\mathscr{r}}}\right)d\tau' = -\int \mathbf{C}\cdot\nabla'\left(\frac{1}{\scriptstyle{\mathscr{r}}}\right)d\tau' \\ &= \int \frac{1}{\scriptstyle{\mathscr{r}}}\nabla'\cdot\mathbf{C}\,d\tau - \oint \frac{1}{\scriptstyle{\mathscr{r}}}\mathbf{C}\cdot d\mathbf{a}. \end{aligned} \tag{B.8}$$

But the divergence of $\mathbf{C}$ is zero, by assumption (Eq. B.3), and the surface integral (way out at infinity) will vanish, as long as $\mathbf{C}$ goes to zero sufficiently rapidly.

Of course, that proof tacitly assumes that the integrals in Eqs. B.5 and B.6 *converge*—otherwise U and $\mathbf{W}$ don't exist at all. At the large r' limit, where $\scriptstyle{\mathscr{r}} \approx r'$, the integrals have the form

$$\int^{\infty}\frac{X(r')}{r'}r'^2\,dr' = \int^{\infty} r'X(r')\,dr'. \tag{B.9}$$

(Here X stands for D or $\mathbf{C}$, as the case may be). Obviously, $X(r')$ must go to zero at large r'—but that's not enough: if $X \sim 1/r'$, the integrand is constant, so the integral blows up, and even if $X \sim 1/r'^2$, the integral is a logarithm, which is still no good at $r' \to \infty$. Evidently the divergence and curl of $\mathbf{F}$ must go to zero *more rapidly than* $1/r^2$ for the proof to hold. (Incidentally, this is *more* than enough to ensure that the surface integral in Eq. B.8 vanishes.)

Now, assuming these conditions on $D(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ are met, is the solution in Eq. B.4 *unique?* The answer is clearly *no,* for we can add to $\mathbf{F}$ any vector function whose divergence and curl both vanish, and the result still has divergence D and curl $\mathbf{C}$. However, it so happens that there is *no* function that has zero divergence and zero curl everywhere *and* goes to zero at infinity (see Sect. 3.1.5). So if we include a requirement that $\mathbf{F}(\mathbf{r})$ goes to zero as $r \to \infty$, then solution B.4 is unique.[1]

Now that all the cards are on the table, I can state the **Helmholtz theorem** more rigorously:

> If the divergence $D(\mathbf{r})$ and the curl $\mathbf{C}(\mathbf{r})$ of a vector function $\mathbf{F}(\mathbf{r})$ are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$, and if $\mathbf{F}(\mathbf{r})$ goes to zero as $r \to \infty$, then $\mathbf{F}$ is given uniquely by Eq. B.4.

[1]Typically we *do* expect the electric and magnetic fields to go to zero at large distances from the charges and currents that produce them, so this is not an unreasonable stipulation. Occasionally one encounters artificial problems in which the charge or current distribution itself extends to infinity—infinite wires, for instance, or infinite planes. In such cases, other means must be found to establish the existence and uniqueness of solutions to Maxwell's equations.

The Helmholtz theorem has an interesting **corollary**:

Any (differentiable) vector function $\mathbf{F}(\mathbf{r})$ that goes to zero faster than $1/r$ as $r \to \infty$ can be expressed as the gradient of a scalar plus the curl of a vector:[2]

$$\mathbf{F}(\mathbf{r}) = \nabla \left(\frac{-1}{4\pi} \int \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right) + \nabla \times \left(\frac{1}{4\pi} \int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right). \tag{B.10}$$

For example, in electrostatics $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ and $\nabla \times \mathbf{E} = \mathbf{0}$, so

$$\mathbf{E}(\mathbf{r}) = -\nabla \left(\frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right) = -\nabla V \tag{B.11}$$

(where V is the scalar potential), while in magnetostatics $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, so

$$\mathbf{B}(\mathbf{r}) = \nabla \times \left(\frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right) = \nabla \times \mathbf{A} \tag{B.12}$$

(where $\mathbf{A}$ is the vector potential).

[2]As a matter of fact, any differentiable vector function *whatever* (regardless of its behavior at infinity) can be written as a gradient plus a curl, but this more general result does not follow directly from the Helmholtz theorem, nor does Eq. B.10 supply the explicit construction, since the integrals, in general, diverge.

APPENDIX C

Units

In our units (the **Système International**), Coulomb's law reads

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathscr{r}^2}\hat{\mathscr{r}} \quad \text{(SI)}. \tag{C.1}$$

Mechanical quantities are measured in meters, kilograms, seconds, and charge is in **coulombs** (Table C.1). In the **Gaussian system**, the constant in front is, in effect, absorbed into the unit of charge, so that

$$\mathbf{F} = \frac{q_1 q_2}{\mathscr{r}^2}\hat{\mathscr{r}} \quad \text{(Gaussian)}. \tag{C.2}$$

Mechanical quantities are measured in centimeters, grams, seconds, and charge is in **electrostatic units** (or **esu**). For what it's worth, an esu is a (dyne)$^{1/2}$-centimeter.

Quantity	SI	Factor	Gaussian
Length	meter (m)	10^2	centimeter
Mass	kilogram (kg)	10^3	gram
Time	second (s)	1	second
Force	newton (N)	10^5	dyne
Energy	joule (J)	10^7	erg
Power	watt (W)	10^7	erg/second
Charge	coulomb (C)	3×10^9	esu (statcoulomb)
Current	ampere (A)	3×10^9	esu/second (statampere)
Electric field	volt/meter	$(1/3) \times 10^{-4}$	statvolt/centimeter
Potential	volt (V)	$1/300$	statvolt
Displacement	coulomb/meter^2	$12\pi \times 10^5$	statcoulomb/centimeter^2
Resistance	ohm (Ω)	$(1/9) \times 10^{-11}$	second/centimeter
Capacitance	farad (F)	9×10^{11}	centimeter
Magnetic field	tesla (T)	10^4	gauss
Magnetic flux	weber (Wb)	10^8	maxwell
H	ampere/meter	$4\pi \times 10^{-3}$	oersted
Inductance	henry (H)	$(1/9) \times 10^{-11}$	second^2/centimeter

TABLE C.1 Conversion Factors. [*Note:* Except in exponents, every "3" is short for $\alpha \equiv 2.99792458$ (the numerical value of the speed of light), "9" means α^2, and "12" is 4α.]

Converting electrostatic equations from SI to Gaussian units is not difficult: just set

$$\epsilon_0 \to \frac{1}{4\pi}.$$

For example, the energy stored in an electric field (Eq. 2.45),

$$U = \frac{\epsilon_0}{2} \int E^2 \, d\tau \quad \text{(SI)},$$

becomes

$$U = \frac{1}{8\pi} \int E^2 \, d\tau \quad \text{(Gaussian)}.$$

(Formulas pertaining to fields inside dielectrics are not so easy to translate, because of differing definitions of displacement, susceptibility, and so on; see Table C.2.)

The Biot-Savart law, which for us reads

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \quad \text{(SI)}, \tag{C.3}$$

becomes, in the Gaussian system,

$$\mathbf{B} = \frac{I}{c} \int \frac{d\mathbf{l} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \quad \text{(Gaussian)}, \tag{C.4}$$

where c is the speed of light, and current is measured in esu/s. The Gaussian unit of magnetic field (the **gauss**) is the one quantity from this system in everyday use: people speak of volts, amperes, henries, and so on (all SI units), but for some reason they tend to measure magnetic fields in gauss (the Gaussian unit); the correct SI unit is the **tesla** (10^4 gauss).

One major virtue of the Gaussian system is that electric and magnetic fields have the same dimensions (in principle, one could measure the electric fields in gauss too, though no one uses the term in this context). Thus the Lorentz force law, which we have written

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \text{(SI)}, \tag{C.5}$$

(indicating that E/B has the dimensions of velocity), takes the form

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \quad \text{(Gaussian)}. \tag{C.6}$$

In effect, the magnetic field is "scaled up" by a factor of c. This reveals more starkly the parallel structure of electricity and magnetism. For instance, the total energy stored in electromagnetic fields is

$$U = \frac{1}{8\pi} \int (E^2 + B^2) \, d\tau \quad \text{(Gaussian)}, \tag{C.7}$$

eliminating the ϵ_0 and μ_0 that spoil the symmetry in the SI formula,

$$U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau \quad \text{(SI)}. \tag{C.8}$$

Table C.2 lists some of the basic formulas of electrodynamics in both systems. For equations not found here, and for Heaviside-Lorentz units, I refer you to the appendix of J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (New York: John Wiley, 1999), where a more complete listing is to be found.[1]

[1] For an interesting "primer" on electrical SI units, see N. M. Zimmerman, *Am. J. Phys.* **66**, 324 (1998); the history is discussed in L. Kowalski, *Phys. Teach.* **24**, 97 (1986).

	SI	Gaussian
Maxwell's equations		
In general:	$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \partial \mathbf{E}/\partial t \end{cases}$	$\nabla \cdot \mathbf{E} = 4\pi\rho$ $\nabla \times \mathbf{E} = -\frac{1}{c}\partial \mathbf{B}/\partial t$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial \mathbf{E}/\partial t$
In matter:	$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \partial \mathbf{D}/\partial t \end{cases}$	$\nabla \cdot \mathbf{D} = 4\pi\rho_f$ $\nabla \times \mathbf{E} = -\frac{1}{c}\partial \mathbf{B}/\partial t$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_f + \frac{1}{c}\partial \mathbf{D}/\partial t$
D and H		
Definitions:	$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$
Linear media:	$\begin{cases} \mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \quad \mathbf{D} = \epsilon\mathbf{E} \\ \mathbf{M} = \chi_m\mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$	$\mathbf{P} = \chi_e\mathbf{E}, \quad \mathbf{D} = \epsilon\mathbf{E}$ $\mathbf{M} = \chi_m\mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$
Lorentz force law	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$
Energy and power		
Energy:	$U = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$	$U = \frac{1}{8\pi}\int\left(E^2 + B^2\right) d\tau$
Poynting vector:	$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$	$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B})$
Larmor formula:	$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}$	$P = \frac{2}{3}\frac{q^2a^2}{c^3}$

TABLE C.2 Fundamental Equations in SI and Gaussian Units.

Index

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Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \dfrac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\,\mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$