# Fields: gradient, divergence and curl

1. Find the gradients of $f(x, y, z) = x^2 + y^3 + z^4$ and $f(x, y, z) = e^x \sin(y) \ln(z)$.

2. Show that the greatest rate of change of a scalar field $\phi(x, y, z)$, i.e., the maximum directional derivative, takes place in the direction of, and has the magnitude of, the vector $\nabla\phi$.

3. The height of certain hill in feet is given by
$$h(x, y) = 10\left(2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right) \tag{1}$$
where $y$ is the distance (in miles) north, $x$ is the distance east of South Hadley.
(a) Where is the top of the hill located?
(b) How high is the hill?

4. Let $\boldsymbol{\mathscr{r}}$ be a separation vector from a fixed point $(x', y', z')$ to the point $(x, y, z)$ and let $\mathscr{r}$ be its length. Show that
(a) $\nabla(\mathscr{r}^{\,2}) = 2\boldsymbol{\mathscr{r}}$ .
(b) $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{\,2}}$
(c) What is the general formula for $\nabla(\mathscr{r}^{\,n})$?

5. Calculate the divergence and curl of the following vector functions.
(a) $\vec{v_a} = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$
(b) $\vec{v_b} = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$
(c) $\vec{v_c} = y^2\hat{i} + (2xy + z^2)\,\hat{j} + 2yz\hat{k}$

6. The electric field due to a unit point charge is given by
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$
Sketch the function and find its divergence. interpret your answer.

7. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$, where $c$ is a constant.

8. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

9. A fluid moves so that its velocity at any point is $\vec{v}(x, y, z)$. Show that the loss of fluid per unit volume per unit time in a small parrellelepiped having center at $P(x, y, z)$ and edges parallel to the coordinate axes and having magnitude $\delta x$, $\delta y$, $\delta z$ respectively, given approximately by div $(\vec{v}) = \nabla \cdot \vec{v}$.

10. The relation between linear velocity and angular velocity is given by $\vec{v} = \vec{\omega} \times \vec{r}$. Prove that $\vec{\omega} = \frac{1}{2}\nabla \times \vec{v}$ for a constant vector $\vec{\omega}$ and interpret your answer.