Q 1. Prove EM waves are transverse in nature.

Ans: Consider an EM wave with wave vector $\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k}$, with the electric field given by

$$\vec{E} = \overrightarrow{E_0}e^{i(\vec{k}\cdot\vec{r}-wt)}$$

where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. In a region empty of electric charge, from Maxwell's equations:

$$\nabla\cdot\vec{E} = \nabla\cdot\vec{B} = 0$$

Substitute $\vec{E} = \overrightarrow{E_0}e^{i(\vec{k}\cdot\vec{r}-wt)}$ into the $\nabla\cdot\vec{E} = 0$

$$\begin{aligned} \nabla\cdot\vec{E} &= 0 \\ \nabla\cdot\left(\overrightarrow{E_0}e^{i(\vec{k}\cdot\vec{r}-wt)}\right) &= 0 \\ i\vec{k}\cdot\overrightarrow{E_0}e^{i(\vec{k}\cdot\vec{r}-wt)} &= 0 \end{aligned}$$

It is only possible if,

$$\vec{k}\cdot\overrightarrow{E_0} = 0$$

Physically, this means that the direction of propagation is always perpendicular to the electric field. The exact same argument applies for the $\nabla\cdot\vec{B} = 0$.

In other words, the electric and magnetic field components transverse to the direction of propagation, therefore the electromagnetic wave is transverse.

Q 2. The magnetic field of plane electromagnetic wave is described as follows:

$$\boldsymbol{\vec{B} = B_0 cos[10y + (3\times10^9)t]\hat{k}}$$

where $\hat{k}$ is unit vector in z-direction, y in meters and t is in seconds. In which direction does this wave propagate? write the expression for the electric field of wave in terms of $\vec{B}$ and c.

Ans: A positive sign between the y and t terms in argument $ky + \omega t$ of the cosine function means that the direction of propagation is $-y$ (with unit vector $-\hat{\jmath}$). The direction of propagation, $-\hat{\jmath}$ of the electromagnetic waves corresponds to the direction of the cross product $\vec{E}\times\vec{B}$. With $\vec{B}$ pointing in the $+\hat{k}$ direction, the electric filed must be point in the $+\hat{\imath}$ direction, so that $\hat{\imath}\times\hat{k} = -\hat{\jmath}$. According to Maxwell's equations, amplitudes are related by $E_0 = cB_0$. Thus the electric field is,

$$\vec{E} = \hat{\imath}cB_0\cos[10y + (3\times10^9)t]$$

Q3. Compute the intensity of the standing electromagnetic wave given by

$$\boldsymbol{E_y(x,t) = 2E_0\cos kx\cos\omega t, \quad B_z(x,t) = 2B_0\sin kx\sin\omega t}$$

Ans: The Poynting vector for the standing wave is

$$\begin{aligned}
\vec{S} &= \frac{\vec{E}\times\vec{B}}{\mu_0} \\
&= \frac{1}{\mu_0}(2E_0\cos kx\cos\omega t\hat{j}) \times \left(2B_0\sin kx\sin\omega t\hat{k}\right) \\
&= \frac{4E_0B_0}{\mu_0}(\sin kx\cos kx\sin wt\cos wt)\hat{\imath} \\
&= \frac{E_0B_0}{\mu_0}(\sin 2kx\sin 2wt)\hat{\imath}
\end{aligned}$$

The time average of S is

$$\langle S\rangle = \frac{E_0B_0}{\mu_0}\sin 2kx\langle\sin 2\omega t\rangle = 0$$

The result is to be expected since the standing wave does not propagate. Alternatively, we may say that the energy carried by the two waves traveling in the opposite directions to form the standing wave exactly cancel each other, with no net energy transfer.

Q4. A plane wave is polarised with its electric vector along z. The wave propagates along the y-axis. The electric field is given by

$$\boldsymbol{E_z = E_0e^{i(ky-\omega t)}}$$

This wave is propagating through vacuum; its amplitude is $\boldsymbol{E_0 = 5V/m}$ and its wavelength is 0. 10 meters. What is the average rate at which energy is transported by this wave (per square meter)?

Ans: Electric field is in z direction and wave is propagating in y direction, therefore the magnetic field must in the x direction.

In the vacuum, EM wave is transverse therefore,

$$\begin{aligned}
B_0 &= \frac{E_0}{c} \\
B_0 &= \frac{5}{3\times 10^8} = 1.667\times 10^{-8}
\end{aligned}$$

The magnetic field is,

$$B_x = B_0 e^{i(ky-\omega t)}$$

The Poynting vector for the standing wave is

$$\begin{aligned}\vec{S} &= \frac{\vec{E}\times\vec{B}}{\mu_0} \\ &= \frac{E_z\hat{k}\times B_x\hat{\imath}}{\mu_0} \\ &= \frac{E_0 B_0}{\mu_0} e^{2i(ky-\omega t)}\end{aligned}$$

The time average of S is,

$$\begin{aligned}\langle S\rangle &= \frac{E_0 B_0}{\mu_0}\langle e^{2i(ky-\omega t)}\rangle \\ \langle S\rangle &= \frac{5\times 1.667\times 10^{-8}}{4\pi\times 10^{-7}}\left(\frac{1}{2}\right) \\ \langle S\rangle &= 33.16\times 10^{-3}\mathrm{watts/m^2}\end{aligned}$$

Q5. At the upper surface of the Earth's atmosphere, the time-averaged magnitude of the Poynting vector, $\langle S\rangle = 1.35\times 10^3 W/m^2$, is referred as the solar constant. Assuming that the Sun's electromagnetic radiation is plane sinusoidal wave, what are the magnitudes of the electric and magnetic fields?

Ans: The time-averaged Poynting vector is related to the amplitude of the electric field by

$$\langle S\rangle = \frac{c}{2}\epsilon_0 E_0^2$$

Thus, amplitude of the electric field is

$$E_0 = \sqrt{\frac{2\langle S\rangle}{c\epsilon_0}}$$

$$E_0 = \sqrt{\frac{2(1.35\times 10^3 W/m^2)}{(3.0\times 10^8 m/s)(8.85\times 10^{-12} C^2/N\cdot m^2)}}$$

$$E_0 = 1.01\times 10^3 V/m$$

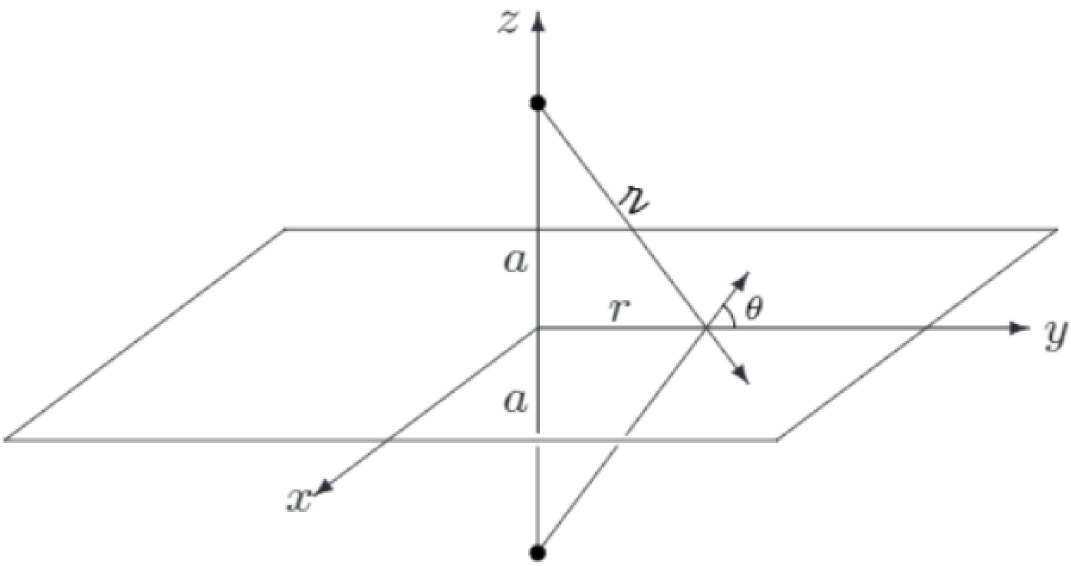
The corresponding amplitude of the magnetic field is

$$B_0 = \frac{E_0}{c}$$

$$B_0 = \frac{1.01 \times 10^3 V/m}{3.0 \times 10^8 m/s} = 3.4 \times 10^{-6} T$$

Note that the associated magnetic field is less than one-tenth the Earth's magnetic field.

Q6. Consider two equal point charges q, separeted by distance 2a. Construct equidistant from the two charges. By integrating Maxwell's stress tensor over the plane, determine the force of one charge on the other.



Ans:

$$\left(\overleftrightarrow{T} \cdot da\right)_z = T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$$

But for the xy plane $da_x = da_y = 0$, and $da_z = -r dr d\phi$.

$$\left(\overleftrightarrow{T} \cdot da\right)_z = \epsilon_0 \left(E_z E_z - \frac{1}{2} E^2\right)(-r dr d\phi)$$

Now, $E = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{(r^2+a^2)} \cos\theta \hat{r}$, and $\cos\theta = \frac{r}{\sqrt{r^2+a^2}}$, Therefore, $E_z = 0$ and $E^2 = \left(\frac{q}{2\pi\epsilon}\right)^2 \frac{r^2}{(r^2+a^2)^3}$ The stress tensor is

$$\left(\overleftrightarrow{T} \cdot da\right)_z = \frac{1}{2}\left(\frac{q}{2\pi\epsilon}\right)^2 \frac{r^2}{(r^2 + a^2)^3}(rdrd\phi)$$

Integrate the stress tensor to determine the force on other particle,

$$F_z = \int \left(\overleftrightarrow{T} \cdot da\right)_z$$

$$F_z = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2}\left(\frac{q}{2\pi\epsilon}\right)^2 \frac{r^2}{(r^2 + a^2)^3}(rdrd\phi)$$

$$= \frac{1}{2}\left(\frac{q}{2\pi\epsilon}\right)^2 2\pi \int_0^{\infty} \frac{r^3}{(r^2 + a^2)^3}dr$$

Let $u = r^2$, therefore $dr = \frac{1}{2\sqrt{u}}du$

$$F_z = \frac{q^2}{4\pi\epsilon}\frac{1}{2}\int_0^{\infty} \frac{udu}{(u + a^2)^3}$$

$$= \frac{q^2}{4\pi\epsilon}\frac{1}{2}\left[\frac{-1}{(u + a^2)} + \frac{a^2}{2(u + a^2)^3}\right]\Big|_0^{\infty} = \frac{q^2}{4\pi\epsilon}\frac{1}{(2a)^2}$$

Q7. A charged parallel-plate capacitor (with uniform electric field $\boldsymbol{E} = E\hat{\boldsymbol{k}}$) is placed in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{\imath}}$, Find the electromagnetic momentum in the space between the plates?

Ans: The momentum density in fields is given,

$$g = \mu_0\epsilon_0(E \times B)$$

Substitute E and B into g,

$$g = \mu_0\epsilon_0\left(E\hat{k} \times B\hat{\imath}\right) = \mu_0\epsilon_0 EB\hat{\jmath}$$

If the area between the plates is A, then momentum is, $p_{em} = Ag = \mu_0\epsilon_0 AEB$.

Q8. Imagine two parallel infinite sheets, carrying uniform surface charge +σ (on the sheet at z = d) and −σ (at z = 0). They are moving in the y direction at constant speed v. What is the electromagnetic momentum in a region of area A?

Ans: Electric field between the two parallel plates is give by $E = -\frac{\sigma}{\epsilon}\hat{k}$ and magnetic field is $B = -\mu_0 \sigma v \hat{\imath}$. The momentum density in fields is given,

$$g = \mu_0 \epsilon_0 (E \times B)$$

Substitute E and B into g,

$$\begin{aligned} g &= \mu_0 \epsilon_0 \left(-\frac{\sigma}{\epsilon}\hat{k}\right) \times (-\mu_0 \sigma v \hat{\imath}) \\ g &= \mu_0 \sigma^2 v \hat{\jmath} \end{aligned}$$

Integrate momentum density over the region A to find electromagnetic momentum,

$$p = \int g dA = \int \mu_0 \sigma^2 v dA \hat{\jmath}$$

Q9. An infinitely long cylindrical tube, of radius a, moves at constant speed v along its axis. It carries a net charge per unit length λ, uniformly distributed over its surface. Surrounding it, at radius b, is another cylinder, moving with the same velocity but carrying the opposite charge ($-\lambda$). Find The momentum per unit length in the fields. Given that the field between the cylinders,

$$\boldsymbol{E} = \frac{1}{2\pi\epsilon}\frac{\lambda}{s}\hat{\boldsymbol{s}}, \quad \boldsymbol{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\boldsymbol{\phi}}$$

Ans: Angular momentum density, g will be,

$$g = \epsilon_0 (E \times B) = \frac{1}{2\pi\epsilon}\frac{\lambda}{s}\hat{s} \times \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi}$$

Since $\hat{s} \times \hat{\phi} = \hat{z}$, therefore,

$$g = \frac{\mu_0 \lambda^2 v}{4\pi^2 s^2}\hat{z}$$

To find momentum per length integrate g along the radius of the cylinders.

$$\begin{aligned} \frac{p}{l} &= \int_a^b g 2\pi s ds \\ &= g = \frac{\mu_0 \lambda^2 v}{4\pi^2}\hat{z}\int_a^b \frac{1}{s^2} 2\pi s ds \\ \frac{p}{l} &= \frac{\mu_0 \lambda^2 v}{2\pi^2}\log(b/a)\hat{z} \end{aligned}$$

Q10. Imagine an iron sphere of radius R that carries a charge Q and a uniform magnetization $\boldsymbol{M} = \boldsymbol{M}\vec{\boldsymbol{k}}$. The sphere is initially at rest. Given that

$$\boldsymbol{E} = \begin{cases} \boldsymbol{0}, & \boldsymbol{r} < \boldsymbol{R} \\ \dfrac{\boldsymbol{1}}{\boldsymbol{4\pi\epsilon_0}}\dfrac{\boldsymbol{Q}}{\boldsymbol{r}}\hat{\boldsymbol{r}}, & \boldsymbol{r} > \boldsymbol{R} \end{cases}$$

$$\boldsymbol{B} = \begin{cases} \dfrac{\boldsymbol{2}}{\boldsymbol{3}}\boldsymbol{\mu_0 M\hat{k}} & \boldsymbol{r} < \boldsymbol{R} \\ \dfrac{\boldsymbol{\mu_0 m}}{\boldsymbol{4\pi r^3}}\left(\boldsymbol{2\cos\theta\hat{r}} + \boldsymbol{\sin\theta\hat{\theta}}\right) & \boldsymbol{r} \geq \boldsymbol{R} \end{cases}$$

Compute the angular momentum stored in the electromagnetic fields.

Ans: Substitute E and B for r > R into g,

$$\begin{aligned} g &= \epsilon_0(E \times B) \\ g &= \epsilon\frac{1}{4\pi\epsilon_0}\frac{Q}{r}\hat{r} \times \left(\frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)\right) \\ &= \frac{\mu_0}{(4\pi)^2}\frac{Q}{r^5}\left(\hat{r} \times \hat{\theta}\right)\sin\theta \end{aligned}$$

Since $\hat{r} \times \hat{\theta} = \hat{\phi}$, therefore

$$g = \frac{\mu_0}{(4\pi)^2}\frac{Q}{r^5}(\hat{\phi})\sin\theta$$

The angular momentum density is,

$$l = r \times g = \frac{\mu_0}{(4\pi)^2}\frac{Q}{r^4}\left(\hat{r} \times \hat{\phi}\right)\sin\theta$$

But $\hat{r} \times \hat{\phi} = -\theta$, and only the z component will survive integration, so (since $(\theta)_z = -\sin\theta$). The integrate l over the sphere to calculate angular momentum:

$$\begin{aligned} L &= \frac{\mu_0 mQ}{(4\pi)^2}\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{\sin^2\theta}{r^4}\left(r^2\sin\theta dr d\theta d\phi\right) \\ &= \frac{\mu_0 mQ}{(4\pi)^2}\int_R^{\infty}\frac{1}{r^2}dr\int_0^{\pi}\left(\sin^3\theta d\theta\int_0^{2\pi}d\,\phi\right) \\ &= \frac{\mu_0 mQ}{(4\pi)^2}\frac{1}{R}\cdot\frac{4}{3}\cdot 2\pi \\ L &= \frac{\mu_0 mQ}{6\pi R}\hat{z} \end{aligned}$$