22MAT101- Mathematics-I

Department of Mathematics

Tutorial Sheet-6 Hints

1. Suppose $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$, where $\rho > 0$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. Then find the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$.
   Hint: $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 \sin\phi$.

2. Suppose $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$, $w = \frac{z}{3}$. Verify $\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)\left(\frac{\partial(u,v,w)}{\partial(x,y,z)}\right) = 1$.

3. Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.
   Hint: $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy = 9/2$

4. Using polar integration, find the area of the region R in the xy-plane enclose by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.
   Hint: $\int_{\pi/6}^{\pi/3} \int_{csc\theta}^{2} rdrd\theta = \pi - \sqrt{3}/3$

5. Find the area of the region enclosed between the circles $x^2 + y^2 = 2y$, $x^2 + y^2 = 4y$, and the lines $y = x$, $y = -x$.
   Hint: Area$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{2\sin\theta}^{4\sin\theta} rdrd\theta = \frac{3(\pi+2)}{2}$.

6. Find the total mass of the circular plate $x^2 + y^2 \le 1$ whose density at any point is equal to the square of the distance of the point from origin.
   Hint: $M = \int\int_{x^2+y^2\le 1} (x^2 + y^2)dxdy = \frac{\pi}{2}$.

7. Find the average value of $f(x, y) = x \cos xy$ over the rectaingle R: $0 \le x \le \pi, 0 \le y \le 1$.
   Hint: $\pi \int_0^\pi \int_0^1 dydx = 2$

8. Find the centre of gravity of a plate whose density $\rho(x, y)$ is constant and is bounded by the curve $y = x^2$ and $y = x + 2$. Also, find the moments of inertia about the axes.
   Hint: Take $\rho(x, y) = k$, then
   $M = k \int_{-1}^2 \int_{x^2}^{x+2} dydx = \frac{9k}{2}$
   $\overline{x} = \frac{1}{M} \int_{-1}^2 \int_{x^2}^{x+2} kxdydx = \frac{1}{2}$, $\overline{y} = \frac{1}{M} \int_{-1}^2 \int_{x^2}^{x+2} kydydx = \frac{8}{5}$
   $I_x = \int_1^2 \int_{x^2}^{x+2} y^2 dydx = \frac{423}{28}k$, $I_y = \int_{-1}^2 \int_{x^2}^{x+2} x^2 dydx = \frac{63}{20}k$

9. Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.
   Hint: $\int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)dxdy = 12.4$ approx

10. Find the volume of solid region bounded above by paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.
    Hint: $\int_0^{2\pi} \int_0^1 (9 - r^2)rdrd\theta = 17\pi/2$

11. Setup the triple integration to find the volume of tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ anc the following orders: (a) $dzdydx$ (b) $dydzdx$ (c) $dxdydz$ Show that all these integrals yield same volume.

    Hint:
    (a)$\int_0^1 \int_x^1 \int_0^{y-x} dzdydx = 1/6$
    (b)$\int_0^1 \int_0^{1-x} \int_{x+z}^1 dydzdx = 1/6$
    (c)$\int_0^1 \int_z^1 \int_0^{y-z} dxdydz = 1/6$.

12. (a) Find the volume of the region D enclosed by the surface $z = x^2+3y^2$ and $z = 8-x^2-y^2$.
    Hint: $V = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dzdydx = 8\pi\sqrt{2}$
    (b) Find the volume of the solid enclosed between the surfaces $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.
    Hint: $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dzdydx = 16a^3/3$

13. Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.
    Hint: $\iiint\limits_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, rdr\, d\theta$

14. Find the volume of the "ice cream cone "D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$ (i.e Cone opening up from the origin,making an angle of $\pi/3$ radians with the positive z-axis).
    Hint: $\int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 sin\phi\, d\rho\, d\phi\, d\theta = \pi/3$

15. Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the plane $x = 2$, $y = 2$ and $z = 2$ in the first octant. **Hint:** $\frac{1}{\text{Volume of D}} \iiint\limits_D xyzdxdydz = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyzdxdydz = 1$

16. A solid fills the region between two concentric spheres of radii a and b,$0 < a < b$. The density at each point is inversely proportional to its square of distance from the origin. Find the total mass.
    Hint: Take $\rho(x, y, z) = \frac{k}{x^2+y^2+z^2}$, then $M = k \int_0^{2\pi} \int_0^{\pi} \int_a^b \frac{r^2 sin\phi}{r^2} drd\phi d\theta = 4\pi k(b-a)$.

17. A cylindrical hole with radius b is drilled through the sphere $x^2 + y^2 + z^2 = 1$ keeping z-axis as the axis of the cylindrical hole. Determine b such that the remaining volume of the sphere (after the drill) is one-eighth of the original volume of the sphere.
    Hint: The volume outside the drilled cylinder is given by

$$V = \int_0^{2\pi} \int_b^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} rdzdrd\theta = \frac{4}{3}(1-b^2)^{3/2}\pi.$$

    For V to be one-eighth of the original volume of the sphere: $\frac{4}{3}(1-b^2)^{3/2}\pi = \frac{1}{8} \times \frac{4\pi}{3} \implies b = \sqrt{\frac{3}{2}}$.