Malaviya National Institute of Technology Jaipur
Department of Mathematics
Tutorial Sheet 5 (Mathematics II)

Topic: Fourier Series

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1. Prove that $f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$, $x \in [-\pi,\ \pi]$. Hence show that

   (a) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. (b) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. (c) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$.

   Hint. Use **Parseval's identity** for proving the part (c).

2. Find the Fourier series expansion for the function $f(x)$, if $f(x) = \begin{cases} -\pi & \text{if } -\pi \le x < 0 \\ x & \text{if } 0 < x \le \pi \end{cases}$.

   Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \cdots$.

   Ans. $f(x) = -\frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{\pi n^2} \cos nx + \sum_{n=1}^{\infty} \frac{1 - 2\cos n\pi}{n} \sin nx$

3. If $f(x) = |\cos x|$, expand as a Fourier series in the interval $[-\pi,\ \pi]$.

   Ans. $f(x) = |\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots.\right\}$

4. Find the Fourier series expansion of $f(x) = |x|$, $-2 \le x \le 2$ and hence find the Fourier series of $f'(x)$.

   Ans. $f(x) = 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(1 - (-1)^n\right) \cos\left(\frac{n\pi x}{2}\right)$ and $f'(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - (-1)^n\right) \sin\left(\frac{n\pi x}{2}\right)$.

5. Find the Fourier series of the following functions:

   (i) $f(x) = \begin{cases} x^2, & \text{if } -\pi < x < 0, \\ -x^2, & \text{if } 0 \le x < \pi. \end{cases}$ with period 2π.

   Ans. $\frac{2}{\pi}\sum \left[\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^3}\left(1 - (-1)^n\right)\right] \sin(nx)$.

   (ii) $f(x) = \begin{cases} 1 + x, & \text{if } -1 \le x < 0, \\ 0, & \text{if } 0 \le x < 1. \end{cases}$ with period 2.

   Ans. $\frac{1}{4} + \sum \left[\frac{1}{n^2\pi^2}\left(1 - (-1)^n\right)\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)\right]$.

6. Find the Fourier series of the following periodic function of period 2π

$$f(x) = \begin{cases} 0, & \text{if } -\pi \le x < 0, \\ x^2, & \text{if } 0 \le x \le \pi. \end{cases}$$

   Hence, show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \ldots$ and $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$

   Ans. $\frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^n \cos(nx) + \left\{\frac{\pi}{n}(-1)^{n+1} - \frac{2}{\pi n^3}\left(1 - (-1)^n\right)\right\} \sin(nx)\right]$.

7. Find the Fourier series of the following periodic function of period 2π

$$f(x) = \begin{cases} 0, & \text{if } \; -\pi < x < 0, \\ \sin x & \text{if } \; 0 < x < \pi. \end{cases}$$

Hence deduce that $\dfrac{\pi - 2}{4} = \dfrac{1}{1.3} - \dfrac{1}{1.5} + \dfrac{1}{5.7} - \dots$.

Hint. $a_n = \begin{cases} \dfrac{2}{\pi} & \text{if } n = 0 \\ 0 & \text{if } n \text{ is odd} \\ -\dfrac{2}{\pi\,(n^2 - 1)} & \text{if } n \text{ is even} \end{cases}$ and $b_n = \begin{cases} \dfrac{1}{2} & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$

and put $x = \dfrac{\pi}{2}$ in the Fourier series expansion of the given function.

8. Show that following function is even function and then find the Fourier series

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & \text{if } \; -\pi \le x \le 0, \\ 1 - \dfrac{2x}{\pi}, & \text{if } \; 0 < x < \pi. \end{cases}$$

Ans. $f(x) = \displaystyle\sum_{n=1}^{\infty} \left\{ \frac{4}{n^2\pi^2}[1 - (-1)^n] \right\} \cos nx$

9. Sketch the waveform of the periodic function defined by $f(x) = x, -\pi \le x \le \pi$ and $f(x) = f(x + 2\pi)$ for all x, and also find the Fourier series of $f(x)$.
Hint. $a_n = 0$ and $b_n = (-1)^{n+1}\dfrac{2}{n}$.

10. Determine the half-range Fourier sine series for $f(x) = x(\pi - x)$ in $0 \le x \le \pi$ and hence deduce that

$$(i) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}. \qquad (ii) \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

Hint. $b_n = \begin{cases} \dfrac{8}{n^3\pi}, & \text{if n is odd} \\ 0 & \text{if n is even.} \end{cases}$; and put $x = \dfrac{\pi}{2}$ in the Fourier sine series expansion to prove part (i) and use **Parseval's identity** for proving the part (ii).

11. Find the Fourier cosine series of $f(x) = x \sin x$ in $0 \le x \le \pi$.
Ans. $1 - \dfrac{1}{2}\cos x - 2\left\{ \dfrac{\cos 2x}{1.3} - \dfrac{\cos 3x}{3.5} + \dfrac{\cos 4x}{5.7} - \dots \right\}$.

12. Expand the following function as the Fourier series of sine terms:

$$f(x) = \begin{cases} \dfrac{1}{4} - x, & \text{if } 0 \le x \le \dfrac{1}{2}, \\ x - \dfrac{3}{4}, & \text{if } \; \dfrac{1}{2} \le x \le 1. \end{cases}$$

Ans. $f(x) = \displaystyle\sum_{n=1}^{\infty} \left\{ \frac{1}{2n\pi}[1 - (-1)^n] - \frac{4\sin(n\pi/2)}{n^2\pi^2} \right\} \sin n\pi x$

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