MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY JAIPUR

DEPARTMENT OF MATHEMATICS

Tutorial Sheet-II (Answer Key)

Subject: 22MAT102 Mathematics II

Topics: Method of variation of parameters and introduction to series solution method about an ordinary point.

Q.1 Find the solution of the following differential equations using the method of variation of parameters.

(i) $y'' + y = \operatorname{cosec}(x)$.

Ans. $y = c_1 \cos x + c_2 \sin x + \sin x \ln|\sin x| - x\cos x$

(ii) $y'' + 4y' + 8y = 16\, e^{-2x}\operatorname{cosec}^2(2x)$.

Ans. $y = e^{-2x}(c_1\cos(2x) + c_2\sin(2x)) + 4e^{-2x}\cos(2x)\ln|\operatorname{cosec}(2x) + \cot(2x)| - 4e^{-2x}$

(iii) $x^2y'' + xy' - 4y = x^2\ln|x|$.

Ans. $y = c_1x^2 + c_2\,x^{-2} + x^2[(8\ln|x|)^2 - 4\ln|x| + 1]/64$

(iv) $y'' - 3y' + 2y = \frac{e^x}{1+e^x}$.

Ans. $y = Ae^x + Be^{2x} - xe^x - xe^{2x} + e^x\ln(e^x+1) + e^{2x}\ln(e^x+1)$

(v) $(D^2+4)y = 4\sec(2x)$.

Ans. $y = (c_1 + 2x)\sin(2x) + (c_2 + \ln(|\cos(2x)|))\cos(2x)$

Q.2 Find the power series solution about the origin.

(i) $y'' - xy' + y = 0$. Ans. $y(x) = c_0\left[1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{240} - \cdots\right] + c_1x$

(ii) $(1-x^2)y'' - 2xy' + 6y = 0$. Ans. $y(x) = c_0(1-3x^2) + c_1\left(x - \frac{2x^3}{3} - \frac{x^5}{5}\ldots\right)$

Q.3 Find the solution in a series of

(i) $y'' - y = 0$ about $x = 1$.

Ans. $y(x) = c_0\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{24}(x-1)^4 + \cdots\right] + c_1\left[(x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{120}(x-1)^5 + \cdots\right]$

(ii) $y'' + xy' + y = 0$ about $x = 2$.

Ans. $y(x) = c_0\left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \cdots\right] + c_1\left[(x-2) - (x-2)^2 + \frac{1}{3}(x-2)^3 - \cdots\right]$

Q.4 Solve the following initial value problem using the power series method.

(i) $y'' + 4y = 0,\ y(2) = 2,\ y'(2) = 2$.

Ans. $y(x) = 2 + 2(x-2) - 4(x-2)^2 - \frac{4}{3}(x-2)^3 - \cdots$

(ii) $(2+x^2)y'' - 2xy' + 3y = 0,\ y(1) = 1,\ y'(1) = -1$.

Ans. $y(x) = 1 - (x-1) - \frac{5}{6}(x-1)^2 + \frac{1}{18}(x-1)^3 + \cdots$

Q.5 Write the power series expansions (about $x = 0$) of the given functions, differentiate term by term, and verify the given differentiation formulas.

(i) $[\sin(mx)]' = m\cos(mx)$.

Ans. $\sin(mx) = mx - \frac{m^3x^3}{3!} + \frac{m^5x^5}{5!} - \frac{m^7x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n m^{2n+1} x^{2n+1}}{(2n+1)!}$

$\cos(mx) = 1 - \frac{m^2x^2}{2!} + \frac{m^4x^4}{4!} - \frac{m^6x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n m^{2n} x^{2n}}{(2n)!}$

(ii) $[1/(1+x)]' = -1/(1+x)^2$.

Ans. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n$

$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

Q.6 Expand $1/(1-x^2)^{\frac{1}{2}}$ as a power series in terms of x, then integrate term by term to find a power series representation of $\sin^{-1} x$.

Ans. $(1-x^2)^{-1/2} = 1 + \frac{1}{2}(x^2) + \frac{3}{8}(x^4) + \frac{5}{16}(x^6) + \cdots$

$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$