# Malaviya National Institute of Technology Jaipur

## Department of Mathematics

## 22MAT101: Mathematics I

## ODD Semester 2024-2025: Tutorial Sheet 3

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1. Show that the following limits do not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2y-1}{x-1}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y}{x-y}$ (d) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xe^y-1}{xe^y-1+y}$

   **Answer:**

   (a) Take path $y = mx^2$

   (b) Take paths $y = \frac{1}{x}$ and $y = 1$

   (c) Take path $y = mx$; $m \neq 1$

   (d) Take paths $x = 1$ and $y = 0$

2. Show that the limit $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$ does not exist.

   **Answer:**

   Along the path $x = 1$, then given limit is 1/2.

   Along the path $y = -1$, then given limit is -1/2.

3. Let $f(x,y) = \begin{cases} x+y+1, & x \neq 0 \neq y \\ 0, & x = 0 \text{ or } y = 0 \end{cases}$ Show that the limit of $f(x,y)$ at $(0,0)$ does not exist.

   **Answer:** Take paths $y = x$ and $y = 0$.

4. (a) Consider the function defined by $f(x,y) = \begin{cases} \dfrac{x+y}{x-y}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$ Find $\displaystyle\lim_{y\to 0}\lim_{x\to 0} f(x,y)$; and $\displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y)$. Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

   (b) Consider the function defined by $f(x,y) = \begin{cases} x\sin(\frac{1}{y}), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$ Discuss $\displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y)$; and $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

   **Answer:**

   (a) $\displaystyle\lim_{y\to 0}\lim_{x\to 0} f(x,y) = -1$; $\displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1$.

(b) $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist; $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

5. Define $f(0,0)$ in a way that extends $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$ to be continuous at origin.

**Answer:** Since $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, define $f(0,0) = 0$.

6. Consider the function $f(x,y) = \begin{cases} (x+y)\sin(\frac{1}{x+y}), & x+y \neq 0 \\ 0, & x+y = 0. \end{cases}$ Show that $f(x,y)$ is continuous at $(0,0)$ but the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ do not exist.

**Answer:** Use $|\sin(\frac{1}{x+y})| \leq 1$.

7. Consider the function $f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) \neq (0,0). \end{cases}$ Show that the partial derivatives $f_x(0,0) = f_y(0,0) = 0$ but $f(x,y)$ is not continuous at $(0,0)$.

**Answer:** Since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, $f$ can not be continuous at $(0,0)$.

8. Let $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

   (a) Show that $f$ is continuous at $(0,0)$.

   (b) Show that $\dfrac{\partial f}{\partial y}(x,0) = x$ for all $x$, and $\dfrac{\partial f}{\partial x}(0,y) = -y$ for all $y$.

   (c) Show that $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) \neq \dfrac{\partial^2 f}{\partial x \partial y}(0,0)$

   **Answer:**

   (a) Use definition and take $\delta = \sqrt{\frac{\epsilon}{2}}$.

   (b) Use limit definition of partial derivatives.

   (c) $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = -1$; and $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = 1$.

9. Let $f(x,y) = \begin{cases} \frac{x^2 y(x-y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

   (a) Show that $f_y(x,0) = x$ for all $x$, and $f_x(0,y) = 0$ for all $y$.

   (b) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. **(5 Marks)**

   **Answer:** (b) $f_{xy}(0,0) = 0$ and $f_{yx}(0,0) = 1$.

10. (a) If $u = f(x-y, y-z, z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

    (b) Suppose $z = f(x, y)$, $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha - v\cos\alpha$, where $\alpha$ is constant. Then show that $\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$.

11. Verify Euler's theorem for the following functions

    (a) $z = \log\left(\frac{x^2-y^2}{x^2+y^2}\right)$.

    (b) $z = (x^{1/2} + y^{1/2})(x^n + y^n)$.