MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY JAIPUR

DEPARTMENT OF MATHEMATICS

Tutorial Sheet-I (Answer Key)

Subject: 22MAT102 Mathematics II

Topics: Solution of linear differential equations with constant coefficients using operator method, Euler-Cauchy equations, Solution of second order differential equations by method of reduction of order.

Q.1 Find the general solution of the following differential equations.

(i) $y''' - 16y' = 0$.
Ans. $y = c_1 + c_2 e^{4x} + c_3 e^{-4x}$

(ii) $y''' - 4y'' + y' + 6y = 0$.
Ans. $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$

Q.2 Solve the following differential equations:

(i) $4\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.
Ans. $y = c_1 + (c_2 + c_3 x)e^{-\frac{1}{2}x}$

(ii) $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$.
Ans. $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)e^{-x}$

Q.3 Solve the following:

(i) $(D^2 + 9)y = 0$.
Ans. $y = c_1 \cos 3x + c_2 \sin 3x$

(ii) $(D^4 - a^4)y = 0$.
Ans. $y = c_1 e^{-ax} + c_2 e^{ax} + c_3 \cos ax + c_4 \sin ax$

(iii) $(D^4 + 6D^2 + 9)y = 0$.
Ans. $y = (c_1 + c_2 x)\cos\sqrt{3}x + (c_3 + c_4 x)\sin\sqrt{3}x$

(iv) $(D^2 - 2D + 10)y = 0$.
Ans. $y = e^{x}(c_1 \cos 3x + c_2 \sin 3x)$

Q.4 Solve the differential equation $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y = 0$ under the conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 2$.
Ans. $y = x^2 e^{-2x}$

Q.5 Find the general solution of the following differential equations.

(i) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$.
Ans. $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{12}e^{5x}$

(ii) $(D^2 + D - 2)y = e^x$.

Ans. $y = c_1 e^x + c_2 e^{-2x} + \frac{1}{3} x e^x$

(iii) $(D^3 + 1)y = (e^x + 1)^2$.

Ans. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{9} e^{2x} + e^x + 1$

(iv) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$.

Ans. $y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$

(v) $\frac{d^2y}{dx^2} + a^2 y = \sin ax$.

Ans. $y = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a}\cos ax$

(vi) $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$.

Ans. $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{8}\sin 2x$

(vii) $(D^3 + D^2 - D - 1)y = \cos 2x$.

Ans. $y = c_1 e^x + (c_2 + c_3 x)e^{-x} - \frac{2}{25}\sin 2x - \frac{1}{25}\cos 2x$

Q.6 Solve the initial value problem $(D^2 + 4D + 3)y = e^{2x}\cos x$, $y(0) = 0$, $y'(0) = 0$.

Ans. $y = -\frac{3}{20} e^{-x} + \frac{5}{52} e^{-3x} + \frac{e^{2x}}{130}(4\sin x + 7\cos x)$

Q.7 Solve the following differential equations:

(i) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$.

Ans. $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$

(ii) $(D^2 - 2D + 4)y = e^x \cos x$.

Ans. $y = e^x\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) + \frac{1}{2} e^x \cos x$

(iii) $(D^2 + 1)^2 y = x^2 \cos x$.

Ans. $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - \frac{1}{48}x^4 \cos x + \frac{3}{16}x^2 \cos x + \frac{1}{12}x^3 \sin x$

(iv) $(D - 2)^2 y = 8x^2 e^{2x} \sin 2x$.

Ans. $y = (c_1 + c_2 x)e^{2x} + e^{2x}(3\sin 2x - 4x\cos 2x - 2x^2 \sin 2x)$

(v) $(D^3 + 2D^2 + D)y = e^{-x} + \cos x + x^2$.

Ans. $y = c_1 + (c_2 + c_3 x)e^{-x} - \frac{x^2}{2} e^{-x} + 6x - 2x^2 + \frac{x^3}{3} - \frac{1}{2}\cos x$

(vi) $(D^2 + 2D + 1)y = x\cos x$.

Ans. $y = (c_1 + c_2 x)e^{-x} + \frac{x}{2}\sin x + \frac{1}{2}(-\sin x + \cos x)$

Q.8 Reduce the Euler-Cauchy equation $a_0 x^2 y'' + a_1 x y' + a_2 y = 0$ to a differential equation with constant coefficients.

Q.9 Find the general solution of the following homogeneous differential equations (Euler-Cauchy Equations):

(i) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x.$

Ans. $y = x[c_1\cos(\log x) + c_2\sin(\log x)] + x\log x$

(ii) $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$

Ans. $y = \frac{c_1}{x} + x[c_2\cos(\log x) + c_3\sin(\log x)] + 5x + \frac{2}{x}\log x$

(iii) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{\sin(\log x)}{x}.$

Ans. $y = (c_1 + c_2\log x)x^{-1} - \frac{1}{x}\sin(\log x)$

(iv) $(x^2D^2 - 3xD + 5)y = x^2\sin(\log x).$

Ans. $y = x^2[c_1\cos(\log \mathrm{x}) + c_2\sin(\log \mathrm{x})] - \frac{1}{2}x^2\log x\cos(\log x)$

(v) $(x^2D^2 - xD - 3)y = x^2\log x.$

Ans. $y = c_1x^3 + c_2x^{-1} - \frac{1}{3}x^2\log x - \frac{2}{9}x^2$

Q.10 Solve $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Ans. $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2) + 1]$

Q.11 Show that the given set of functions $\{y_1(x), y_2(x)\}$ forms a basis of the equation and hence solve the initial value problem by the method of reduction of order.

(i) $e^x, e^{4x},\ y'' - 5y' + 4y = 0,\ y(0) = 2,\ y'(0) = 1.$

Ans. $y = \frac{7}{3}e^x - \frac{1}{3}e^{4x}$

(ii) $x, x\ln x,\ x^2y'' - xy' + y = 0,\ y(1) = 3,\ y'(1) = 4.$

Ans. $y = 3x + x\,ln(x)$

(iii) $e^{-3x}, xe^{-3x},\ y'' + 6y' + 9y = 0,\ y(0) = 1,\ y'(0) = 2.$

Ans. $y = 5xe^{-3x} + e^{-3x}$

(iv) $x^2,\ \frac{1}{x^2},\ x^2y'' + xy' - 4y = 0,\ y(1) = 2,\ y'(1) = 6.$

Ans. $y = \frac{5}{2}x^2 - \frac{1}{2x^2}$