Department of Mathematics
Course: Mathematics I
Tutorial Sheet 5

---

1. Show that $\displaystyle\int_0^1 \frac{x^{m-1}}{(1-x)^{n-1}}(a+bx)^{m+n}dx = \frac{1}{(a+b)^m a^n}B(m,n)$, where $a \neq 0,\ a+b \neq 0$.

2. Show that $\displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = B(m,n), \quad m>0,\ n>0.$

3. Show that $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^m b^n}B(m,n)$, where $m,\ n,\ a,\ b>0$.

4. Prove that $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where $m>0,\ n>0$. Hence find the value of $\Gamma\left(\dfrac{1}{2}\right)$.

5. Prove: $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}.$

6. Show that $\displaystyle\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}\,dx = \frac{1}{4\sqrt{2}}B\left(\frac{7}{4},\frac{1}{4}\right).$

7. Evaluate $\displaystyle\iint_{x^2+y^2\leq 1} \sin\pi(x^2+y^2)dxdy.$

8. Evaluate $\displaystyle\iint \sqrt{\frac{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}}\,dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

9. Change the order of the integration of $\displaystyle\int_0^a \int_0^{\sqrt{2ay-y^2}} f(x,y)dxdy$ and verify the result by taking $f(x,y)=2x$.

10. Change the order of the integration in the followings and hence evaluate the same

    (A) $\displaystyle\int_0^a \int_y^a \frac{x}{x^2+y^2}dxdy$ (B) $\displaystyle\int_0^1 \int_{x^2}^{2-x} xydydx$

    (C) $\displaystyle\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}dydx$ (D) $\displaystyle\int_0^\infty \int_0^y ye^{-y^2/x}dydx$

11. Set up the limits of integration for evaluating the double integral of a function $f(x,y)$ over the region enclosed between the parabola $y^2=2ax$ and the circle $(x-a)^2+y^2=a^2$, and the lines $x=0$, $x=2a$ in the first quadrant by using the following orders of integration.
    (a) $dxdy$ (b) $dydx$.

12. Find the area of the region enclosed between the circles $x^2+y^2=2y$, $x^2+y^2=4y$, and the lines $y=x$, $y=-x$.

13. Using double integration, obtain the area common to the parabola $y^2=x$ and the circle $x^2+y^2=2$.

14. Find the area that lies inside the cardioids $r=a(1+\cos\theta)$ and outside the circle $r=a$, by double integration.