Notes on “The Core Model Induction”

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1 The Successor Case

1.4 Capturing, Correctness and Genericity Iterations

Exercise 1.4.5. *Suppose that $(\mathcal{M}, \Sigma)$ absorbs reals at δ and $\mathcal{M} \models \mathrm{ZFC}^- \wedge \delta^+$ exists. Then δ is either Woodin or a limit of Woodins in $\mathcal{M}$.*

Proof. By taking a countable hull

$$\sigma \colon \mathcal{N} \prec \mathcal{M}$$

of $\mathcal{M}$ and considering $(\mathcal{N}, \Sigma^\sigma)$ we may and shall assume that $\mathcal{M}$ is countable. Let $x \in \mathbb{R}$ code $\mathcal{M}$ and suppose that δ is neither Woodin nor a limit of Woodins in $\mathcal{M}$.

Claim. *δ is a limit of measurable cardinals in $\mathcal{M}$.*

Proof. Otherwise let κ be the largest cardinal below δ, let $\eta = \kappa + 1$. Then there is no iteration above η that absorbs a real coding $o(M)$. ■

We may hence fix $\eta < \delta$ inaccessible (or measurable) in $\mathcal{M}$ such that

$$\mathcal{M} \models \forall \xi \in [\eta, \delta] \colon \xi \text{ is not Woodin.}$$

Let $\mathcal{T} \in \mathcal{M}$ be the linear iteration of $\mathcal{M}$ that applies the least measure of $\mathcal{M}$ η-many times. Let κ be the least measureable cardinal of $\mathcal{M}$. Note that $i^{\mathcal{T}}_\eta \in \mathcal{M}$ and $i^{\mathcal{T}}(\kappa) = \eta$.
Let $\delta^* \leq \delta$ be minimal such that for every real y there is some iteration tree $\mathcal{U}$ on $\mathcal{M}^{\mathcal{T}}_\eta$ that lives on $(\eta, i^{\mathcal{T}}(\delta))$ and absorbs y.
In V fix $(\xi_n \mid n < \omega)$ cofinal in δ^* and for every n fix some real x_n such that x_n cannot be absorbed by a tree living below ξ_n. Now let

$$z = x \oplus \bigoplus_{n<\omega} x_n.$$

Notice that any iteration that absorbs z must use unboundedly long extenders below δ^*. Let $\mathcal{U}$ be such an iteration tree. Since $\mathcal{M}$ has no Woodin cardinals in the interval $[\eta, \delta]$, $\mathcal{U}$ is guided by $\mathcal{Q}$-structures in $\mathcal{M}$, so that $\mathcal{U}$ and $i^{\mathcal{U}}$ are in fact members of $\mathcal{M}$. [1] Let g be $\mathrm{Coll}(\omega, i^{\mathcal{U}}(\delta^*))$-generic such that $z \in \mathcal{M}^{\mathcal{U}}_{\infty}[g]$. Since x codes $\mathcal{M}$ and $x \leq_T z$, we have $\mathcal{M} \in \mathcal{M}^{\mathcal{U}}_{\infty}[g]$ and hence $i^{\mathcal{U}} \upharpoonright (\delta^*)^{+\mathcal{M}} \in \mathcal{M}^{\mathcal{U}}_{\infty}[g]$. But $i^{\mathcal{U}} \upharpoonright (\delta^*)^{+\mathcal{M}}$ is cofinal in $i^{\mathcal{U}}(\delta^*)^{+\mathcal{M}^{\mathcal{U}}_{\infty}}$, so that

$$\mathcal{M}^{\mathcal{U}}_{\infty}[g] \models i^{\mathcal{U}}(\delta^*)^{+\mathcal{M}^{\mathcal{U}}_{\infty}} \text{ is singular.}$$

Since $i^{\mathcal{U}}(\delta^*)^{+\mathcal{M}^{\mathcal{U}}_{\infty}}$ is regular in $\mathcal{M}^{\mathcal{U}}_{\infty}$ and $\mathrm{Coll}(\omega, i^{\mathcal{U}}(\delta^*))$ has the $i^{\mathcal{U}}(\delta^*)^+$-c.c., this is a contradiction! $\square$

[1] It is here that we use that $\mathcal{M} \models \mathrm{ZFC}^- \wedge \delta^+$ exists.