

# Notes on “The Core Model Induction”

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## 1 The Successor Case

### 1.4 Capturing, Correctness and Genericity Iterations

**Exercise 1.4.5.** *Suppose that  $(\mathcal{M}, \Sigma)$  absorbs reals at  $\delta$  and  $\mathcal{M} \models \text{ZFC}^- \wedge \delta^+$  exists. Then  $\delta$  is either Woodin or a limit of Woodins in  $\mathcal{M}$ .*

*Proof.* By taking a countable hull

$$\sigma: \mathcal{N} \prec \mathcal{M}$$

of  $\mathcal{M}$  and considering  $(\mathcal{N}, \Sigma^\sigma)$  we may and shall assume that  $\mathcal{M}$  is countable. Let  $x \in \mathbb{R}$  code  $\mathcal{M}$  and suppose that  $\delta$  is neither Woodin nor a limit of Woodins in  $\mathcal{M}$ .

Suppose that there is some  $\eta < \delta$  such that there are no  $\mathcal{M}$ -Woodin cardinals in the interval  $[\eta, \delta]$  and let  $\delta^* \leq \delta$  be minimal such that for every real  $y$  there is some iteration tree  $\mathcal{U}$  on  $\mathcal{M}_\eta^T$  that lives on  $(\eta, \delta^*)$  and absorbs  $y$ .

Fix a real  $x$  coding  $\mathcal{M}$  and let  $\mathcal{U}$  be an iteration tree living on  $(\eta, \delta^*)$  absorbing  $x$ . Since  $\mathcal{M}$  has no Woodin cardinals in the interval  $[\eta, \delta]$ ,  $\mathcal{U}$  is guided by  $\mathcal{Q}$ -structures in  $\mathcal{M}$ , so that  $\mathcal{U}$  and  $i_\infty^\mathcal{U}$  are in fact members of  $\mathcal{M}$ .<sup>1</sup> Let  $g$  be  $\text{Coll}(\omega, i_\infty^\mathcal{U}(\delta^*))$ -generic such that  $z \in \mathcal{M}_\infty^\mathcal{U}[g]$ . Since  $x$  codes  $\mathcal{M}$  and  $x \leq_T z$ , we have  $\mathcal{M} \in \mathcal{M}_\infty^\mathcal{U}[g]$  and hence  $i_\infty^\mathcal{U} \restriction (\delta^*)^{+\mathcal{M}} \in \mathcal{M}_\infty^\mathcal{U}[g]$ . But  $i_\infty^\mathcal{U} \restriction (\delta^*)^{+\mathcal{M}}$  is cofinal in  $i_\infty^\mathcal{U}(\delta^*)^{+\mathcal{M}_\infty^\mathcal{U}}$ , so that

$$\mathcal{M}_\infty^\mathcal{U}[g] \models i_\infty^\mathcal{U}(\delta^*)^{+\mathcal{M}_\infty^\mathcal{U}} \text{ is singular.}$$

Since  $i_\infty^\mathcal{U}(\delta^*)^{+\mathcal{M}_\infty^\mathcal{U}}$  is regular in  $\mathcal{M}_\infty^\mathcal{U}$  and  $\text{Coll}(\omega, i_\infty^\mathcal{U}(\delta^*))$  has the  $i_\infty^\mathcal{U}(\delta^*)^+$ -c.c., this is a contradiction!  $\square$

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<sup>1</sup>It is here that we use that  $\mathcal{M} \models \text{ZFC}^- \wedge \delta^+$  exists.