Notes on "The Core Model Induction"

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February 27, 2019

1 The Successor Case

1.4 Capturing, Correctness and Genericity Iterations

Exercise 1.4.5. *Suppose that* $(\mathcal{M}, \Sigma)$ *absorbs reals at* δ *and* $\mathcal{M} \models \mathrm{ZFC}^- \wedge \delta^+$ *exists. Then* δ *is either Woodin or a limit of Woodins in* $\mathcal{M}$*.*

Proof. By taking a countable hull

$$\sigma \colon \mathcal{N} \prec \mathcal{M}$$

of $\mathcal{M}$ and considering $(\mathcal{N}, \Sigma^\sigma)$ we may and shall assume that $\mathcal{M}$ is countable. Let $x \in \mathbb{R}$ code $\mathcal{M}$ and suppose that δ is neither Woodin nor a limit of Woodins in $\mathcal{M}$.

Fix $\eta < \delta$ and let $\delta^* \leq \delta$ be minimal such that for every real y there is some iteration tree $\mathcal{U}$ on $\mathcal{M}_\eta^\mathcal{T}$ that lives on (η, δ^*) and absorbs y.
In V, fix $(\xi_n \mid n < \omega)$ cofinal in δ^* and for every n fix some real x_n such that x_n cannot be absorbed by a tree living below ξ_n. Now let

$$z = x \oplus \bigoplus_{n<\omega} x_n,$$

where x is a real coding $\mathcal{M}$.
Notice that any iteration that absorbs z must use unboundedly long extenders below δ^*. Let $\mathcal{U}$ be such an iteration tree. Since $\mathcal{M}$ has no Woodin cardinals in the interval $[\eta, \delta]$, $\mathcal{U}$ is guided by $\mathcal{Q}$-structures in $\mathcal{M}$, so that $\mathcal{U}$ and $i^\mathcal{U}_\infty$ are in fact members of $\mathcal{M}$. [1] Let g be $\mathrm{Coll}(\omega, i^\mathcal{U}_\infty(\delta^*))$-generic such that $z \in \mathcal{M}^\mathcal{U}_\infty[g]$. Since x codes $\mathcal{M}$ and $x \leq_T z$, we have $\mathcal{M} \in \mathcal{M}^\mathcal{U}_\infty[g]$ and hence $i^\mathcal{U}_\infty \restriction (\delta^*)^{+\mathcal{M}} \in \mathcal{M}^\mathcal{U}_\infty[g]$. But $i^\mathcal{U}_\infty \restriction (\delta^*)^{+\mathcal{M}}$ is cofinal in $i^\mathcal{U}(\delta^*)^{+\mathcal{M}^\mathcal{U}_\infty}$, so that

$$\mathcal{M}^\mathcal{U}_\infty[g] \models i^\mathcal{U}_\infty(\delta^*)^{+\mathcal{M}^\mathcal{U}_\infty} \text{ is singular.}$$

[1] It is here that we use that $\mathcal{M} \models \mathrm{ZFC}^- \wedge \delta^+$ exists.

Since $i^{\mathcal{U}}_\infty(\delta^*)^{+\mathcal{M}^{\mathcal{U}}_\infty}$ is regular in $\mathcal{M}^{\mathcal{U}}_\infty$ and $\mathrm{Coll}(\omega, i^{\mathcal{U}}_\infty(\delta^*))$ has the $i^{\mathcal{U}}_\infty(\delta^*)^+$-c.c., this is a contradiction! $\square$