Iterated Reducts and Σ^*-Theory
Exercises

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Let $M = (J^A_\alpha; \in, A, B)$ be an acceptable (which also means amenable) $\mathcal{J}$-structure.

In today's seminar, we've introduced

1. the iterated projecta $(\omega\rho^n_M \mid n < \omega)$ (section 1.5 in Zeman's book),
2. the set of n-good parameters P^n_M (section 1.5 in Zeman's book),
3. the n-th master code $A^{n,p}_M$ for $p \in P^n_M$ (section 1.5 in Zeman's book),
4. the n-th reduct $M^{n,p}$ for $p \in P^n_M$ (section 1.5 in Zeman's book),
5. $\Sigma^{(n)}_l$-formulae (section 1.6 in Zeman's book),
6. the modeling relation $M \models \phi[\vec{a}]$ for $\vec{a} \in M$ and $\phi \in \Sigma^{(n)}_l$ (section 1.6 in Zeman's book).

Here are some voluntary exercises of varying difficulty: [1]

Exercise 0.0.1. *Let κ be an M-cardinal. Then for every $a \in H^M_\kappa$*

$$\mathcal{P}(a) \cap M \subseteq H^M_\kappa.$$

(Hint: M is acceptable.)

Exercise 0.0.2. *Find a non-acceptable $N = (J^C_\gamma; \in, C, D)$ such that the claim above fails.*

[1] If you decide to tackle (some of) them, you're more than welcome to email me your solutions at stefan.m@wwu.de for feedback or to discuss them with me after next week's seminar.

(Hint: If you are already familiar with basic inner model theory, consider $L[U]$ for a normal ultrafilter $U \in V$ on κ and compute at which level of $L[U]$ $0^{\#}$ (coded as a subset of ω) appears.
Alternatively, add a "delayed Cohen real" to L via forcing, i.e. consider an isomorphic copy $\mathbb{P}$ of Cohen forcing in $L \setminus J_{\omega_1^L}$, let $g \subseteq \mathbb{P}$ be generic and consider $(J_\kappa^g; \in, g)$ for some sufficiently large L-cardinal κ.)

Exercise 0.0.3. *Let $p \in P_M^1$. Then*

1. *There is some set A^+ which is $\Sigma_1(M)$ in the parameter $p(0)$ such that*
$$A_M^{1,p} = A^+ \cap (\omega \times H^M_{\omega\rho^1_M}).$$
(And there is in fact a Σ_1-definition that defines $A_M^{1,p}$ in this fashion uniformly for all acceptable $\mathcal{J}$-structures of the same signature as M [2].)

2. *For every $p \in P_M^1$*
$$M^{1,p} := (H^M_{\omega\rho^1_M}; \in, A_M^{1,p})$$
is an acceptable $\mathcal{J}$-structure.

(Hint: Let $(\phi_i \mid i < \omega)$ be a recursive enumeration of all Σ_1-formulae in the language of M. Recall that
$$\models^M_{\Sigma_1} := \{(i, x) \in \omega \times M \mid M \models \phi_i[x]\}$$
is uniformly Σ_1-definable over M.)

Exercise 0.0.4. *Show that $M^{n,p}$ is acceptable for all $n < \omega$ and all $p \in P_M^n$.*

Exercise 0.0.5. *Suppose that $\omega\rho_M^{n+1} < o(M)$. Prove that*
$$M \models \omega\rho_M^{n+1} \text{ is a cardinal.}$$

(Hint: First prove that
$$\omega\rho_M^{n+1} < \omega\rho_M^n \implies (H^M_{\omega\rho^n_M}; \in) \models \omega\rho_M^{n+1} \text{ is a cardinal}$$
and then use acceptability.)

Exercise 0.0.6. *Let Q be a $\Sigma_1(M)$ definable in the parameter $p \in M$. Show that*
$$\Sigma_l(H^M_{\omega\rho^1_M}; \in, Q) \subseteq \Sigma_l^{(1)}(M) \text{ in parameter } p.$$

(Hint: $\Sigma_1^{(0)} \subseteq \Sigma_l^{(1)}$.)

[2] i.e. every acceptable $\mathcal{J}$-structure N of the form $N = (J_\gamma^C; \in, C, D)$, with 1-ary predicates C, D