Fine Structure Seminar
Rutgers University

Stefan Mesken

February 9, 2019

1 Fine Structure

Unless specified otherwise, M, N are acceptable $\mathcal{J}$-structures, $k, l, m, n < \omega$.

1.8 Substitution and Good Functions

Talk 1, first draft

Problem 1.8.1. *$\Sigma_1^{(n)}(M)$-relations are not necessarily closed under substitution of $\Sigma_1^{(n)}(M)$-functions.*

Exercise 1.8.1. *Find an example for a $\Sigma_1^{(n)}(M)$-relation R and a (partial) $\Sigma_1^{(n)}(M)$-function f such that $R \circ f$ is not a $\Sigma_1^{(n)}(M)$ relation.*

Hint 1.8.1. *Suppose that M is an acceptable $\mathcal{J}$-structure s.t. $\omega\rho_M^2 < \omega\rho_M^1$ and such that there is some $\Sigma_0^{(0)}$-formula ϕ and some $p \in M$ with*

$$A := \{\xi < \omega\rho_M^2 \mid M \models \exists x^1 \phi[\xi, x^{(1)}, p]\} \notin M.$$

(Such M exist. In fact, we can pick $M = J_\alpha$ for some countable α.)
Let f be the partial $\Sigma_1^{(0)}(M)$-function defined by the formula

$$\psi(y^{(0)}, z^{(1)}) :\iff y^{(0)} = z^{(1)}.$$

Consider $\phi(x^{(0)}, f(y^{(0)}), p)$.

Our goal is to identify a sufficiently large collection of $\Sigma_1^{(n)}$-functions that can be substituted into $\Sigma_1^{(n)}$-relations without increasing their complexity.

Lemma 1.8.1. *Let $n < \omega$ and $R(\vec{x}^0, \ldots, \vec{x}^n)$ be a $\Sigma_1^{(n)}(M)$-relation. Let $F^0, \ldots, F^n$ be such that for all $i \leq n$ $F^i(\vec{x}^0, \ldots, \vec{x}^{(n)})$ is a partial $\Sigma_1^{(i)}(M)$-function to H_M^i. Then*

$$R(F^0(\vec{x}^0, \ldots, \vec{x}^n), \ldots, F^n(\vec{x}^0, \ldots, \vec{x}^n))$$

is (uniformly) a $\Sigma_1^{(n)}(M)$-relation.

Proof. By induction on n. The case $n = 0$ is a simplification of the induction step $n \mapsto n+1$, hence we will only consider the latter:

Let $R(\vec{x}^0, \ldots, \vec{x}^n)$ be a $\Sigma_1^{(n+1)}(M)$ relation and let $F^0, \ldots, F^{n+1}$ be as above. Write

$$R(\vec{x}^0, \ldots, \vec{x}^{n+1}) \equiv \exists z^{n+1} B(\vec{v^{n+1}}, \vec{x}^{n+1}) R^*(\vec{x}^0, \ldots, \vec{x}^{n+1}, \vec{v}^{n+1}, z^{n+1})$$

with

1. B being a block of bounded quantifiers and
2. R^* being a Boolean combinatin of $\Sigma_1^{(n)}(M)$-relations.

By our induction hypothesis and the fact that every $\Sigma_1^{(n)}(M)$-relation is the specialization of a $\Sigma_1^{(n)}(M)$ with arguments of type $\leq n$, we obtain that

$$R(F^0(\vec{y}), \ldots, F^n(\vec{y}), \vec{x}^{n+1})$$

is $\Sigma_1^{(n+1)}(M)$. But now

$$R(F^0(\vec{y}), \ldots, F^{n+1}(\vec{y})) \equiv \exists \vec{x}^{n+1} \colon F^{n+1}(\vec{y}) \wedge R(F^0(\vec{y}), \ldots, F^n(\vec{y}), \vec{x}^{n+1})$$

is $\Sigma_1^{(n+1)}(M)$.
Note that this transformation only depends on the formulae defining $R, F^0, \ldots, F^{n+1}$ and not on M, which yields the uniformity we claimed. □

Corollary 1.8.1. *Let $n < \omega$, $1 \leq l < \omega$ and $R(\vec{x}^0, \ldots, \vec{x}^n)$ be a $\Sigma_l^{(n)}(M)$-relation. Let $F^0, \ldots, F^n$ be such that for all $i \leq n$ $F^i(\vec{x}^0, \ldots, \vec{x}^{(n)})$ is a partial $\Sigma_1^{(i)}(M)$-function to H_M^i. Then*

$$R(F^0(\vec{x}^0, \ldots, \vec{x}^n), \ldots, F^n(\vec{x}^0, \ldots, \vec{x}^n))$$

is (uniformly) a $\Sigma_l^{(n)}(M)$-relation.

Corollary 1.8.2. *Let $R(x^{i_0}, \dots, x^{i_l})$ be a $\Sigma_l^{(n)}(M)$-relation with $1 \leq l < \omega$ and $i_0, \dots, i_k \leq n < \omega$. Then there is a $\Sigma_l^{(n)}(M)$-relation $R^*(x^0, \dots, x^0)$ with the same graph as R*

Proof. Let R^{**} be the result of replacing each x^{i_0} with x^0 in R. For $j \leq k$ let F^j be the partial $\Sigma_1^{i_j}(M)$-function to $H_M^{i_j}$ defined by

$$\phi^j(x^0, y^{i_l}) \equiv x^0 = y^{i_j}.$$

Then $R^{**}(F^{i_0}(x^0), \dots, F^{i_k}(x^0))$ is a $\Sigma_l^{(n)}(M)$-relation with the same graph as R. □

Definition 1.8.1. *Let $n < \omega$. The* good $\Sigma_1^{(n)}(M)$-functions *consists of the smallest class $\mathcal{G}_1^{(n)}$ such that*

1. *Every partial $\Sigma_1^{(i)}(M)$-function $F(x^{i_0}, \dots, x^{i_k})$ to H_M^i with $i_0, \dots, i_k, i \leq n$ is in $\mathcal{G}_1^{(n)}$ and*

2. *$\mathcal{G}_1^{(n)}$ is closed under composition, i.e. if $F(x^{i_0}, \dots, x^{i_k})$ is in $\mathcal{G}_1^{(n)}$ and $G(\vec{z})$ is a function to $H_M^{i_j}$ for some $j \leq k$ in $\mathcal{G}_1^{(n)}$, then*

$$F(x^{i_0}, \dots, x^{i_{j-1}}, G(\vec{z}), x^{i_{j+1}}, \dots, x^{i_k})$$

   is in $\mathcal{G}_1^{(n)}$.

Lemma 1.8.2. *Let $n < \omega, 1 \leq l < \omega$, $i_0, \dots, i_k \leq n$ and $R(x^{i_0}, x^{i_k})$ be a $\Sigma_l^{(n)}(M)$-relation. If, for $j \leq k$, $F_j(\vec{z})$ is a good $\Sigma_1^{(n)}(M)$-function to $H_M^{i_j}$, then*

$$R(F_0(\vec{z}), \dots, F_n(\vec{z}))$$

is $\Sigma_l^{(n)}(M)$.

Proof. Repeated application of Corollary 1.8.1. □

We are now ready to prove one of the main pillars of basic Σ^* fine structure:

Theorem 1.8.1 ($\Sigma_1^{(n)}$-Uniformization)**.** *Let $n < \omega$ and $R(\vec{x}^0, \dots, \vec{x}^n, y^n)$ be a $\Sigma_1^{(n)}(M)$-relation. Then there is a partial $\Sigma_1^{(n)}(M)$-function F to H_M^n such that*

1. $\mathrm{dom}(F) = \{\vec{x} \mid \exists y^n R(\vec{x}, y^n)\}$ *and*

2. $\forall \vec{x}(\exists y^n R(\vec{x}, y^n) \implies R(\vec{x}, F(\vec{x})))$

Moreover, F can be chosen to have a uniform definition in R's definition.

Proof. Recall that

$$R_{\vec{x}} := \{(\vec{x}^n, y^n) \mid R(\vec{x}^0, \ldots, \vec{x}^n, y^n)\}$$

is uniformly $\Sigma_1(M^{n,(\vec{x}^0,\ldots,\vec{x}^{n-1})})$. Let $i < \omega$ be such that ϕ_i defines $R_{\vec{x}}$ and let

$$F(\vec{x}^0, \ldots, \vec{x}^n) := h_{M^{n,(\vec{x}^0,\ldots,\vec{x}^{n-1})}}(i, \vec{x}^n).$$

Since $h_{M^{n,(\vec{x}^0,\ldots,\vec{x}^{n-1})}}$ is uniformly $\Sigma_1(M^{n,(\vec{x}^0,\ldots,\vec{x}^{n-1})})$, we have that F is $\Sigma_1^{(n)}(M)$ and clearly F uniformizes R. $\square$

Remark 1.8.1. *In these notes, I don't cover the definition of $Q^{(n)}$-formulae. Consult Zeman's book.*

Lemma 1.8.3. *Let $n < \omega$. There is a uniform good $\Sigma_1^{(n)}(M)$ function*

$$F\colon H_M^{n+1} \times H_M^0 \to H_M^0$$

such that for all $r \in R_M^{n+1}$

$$F_r\colon H_M^{n+1} \to M, x^{n+1} \mapsto F(x^{n+1}, p)$$

is surjective.

Proof. By induction on $n < \omega$.

$n = 0$: Let

$$F\colon H_M^1 \times H_M^0 \to H_M^0, (w, p) \mapsto h_M(w(0), (w(1), p(0))).$$

$n \mapsto n + 1$: Let

$$G\colon H_M^{n+1} \times H_M^0 \to H_M^0$$

be as above. Define

$$F\colon H_M^{n+1} \times H_M^0 \to H_M^0, (w, p) \mapsto G(h_{M^{n,p\restriction n}}(w(0), (w(1), p(n))), p \restriction n).$$

Here we let $w(k), p(k), p \restriction k$, $k < \omega$ be the usual interpretation for functions if w, p are functions whose domain contains $k + 1$ and otherwise we define them to be $\emptyset$.

$\square$

Exercise 1.8.2. *Verify that the functions defined above are as desired.*

Hint: Recall that if $r \in R_M^{n+1}$ then $r \restriction n \in R_M^n$ and $r(n) \in R_{M^{n,r\restriction n}}$.

Definition 1.8.2. *For $p \in \Gamma^n_M$ let*

1. $h^{n,p}_M := h_{M^{n,p}}$ *and*

2. $\tilde{h}^n_M$ *be uniform good $\Sigma^{(n-1)}_1(M)$-function given nesting Skolem functions of the i-th projecta as in Lemma 1.8.3, i.e.*

$$\tilde{h}^1_M(w,p) := h_M(w(0),(w(1),p(0)))$$

and

$$\tilde{h}^{n+1}_M(w,p) := h^n_M(h_{M^{n,p\restriction n}}(w(0),(w(1),p(n))),p \restriction n).$$

Remark 1.8.2. *h^{n+1}_M is a uniformly $\Sigma^{(n)}_1(M)$-function and if $r \in R^{n+1}_M$, then every $x \in M$ is of the form $\tilde{h}^{n+1}(z,r)$ for some $z \in H^{n+1}_M$. In fact, we can choose $z \in \omega\rho^{n+1}_M$.*

Corollary 1.8.3. *Let $1 \leq n < \omega$ and $r \in R^n_M$. Then every $A \subseteq H^n_M$ which is $\mathbf{\Sigma}^n_1(M)$ is $\mathbf{\Sigma}_1(M^{n,r})$.*

Proof. We already know that $\mathbf{\Sigma}_1(M^{n,r}) \subseteq \mathbf{\Sigma}^{(n)}_1(M)$.

Conversely let

$$A = \{x \in H^n_M \mid M \models \phi[x,q]\}$$

for some $\Sigma^{(n)}_1$-formula ϕ and some $q \in M$. Fix $\xi < \omega\rho^n_M$ such that $q = \tilde{h}^n_M(\xi,r)$. Then

$$A(x^n) \iff M \models \phi[x^n, \tilde{h}^n_M(\xi^n, r^0)].$$

Since $\tilde{h}^n(y^n,z^0)$ is a good $\Sigma^{(n-1)}_1(M)$-function (and thus a good $\Sigma^{(n)}_1(M)$-function), this witnesses that A is $\Sigma^{(n)}_1(M)$ in parameters ξ, r and hence $\mathbf{\Sigma}_1(M^{n,r})$ (by the characterization of $\Sigma^{(n)}_l(M)$ subsets of H^n_M as $\Sigma_l(M^{n,r})$ relations in $\Sigma^{n-1}_1(M)$ predicates). $\square$

Corollary 1.8.4. *Let $n < \omega$ and $r \in R^n_M$. Then $\rho^{n+1}_M = \rho_{M^{n,r}}$.*

Proof. By definition $\rho^{n+1}_M \leq \rho_{M^{n,r}}$, hence it suffices to show the converse. Let $q \in P^{n+1}_M$ and A be $\Sigma^{(n)}_1(M)$ in q such that $A \cap \omega\rho^{n+1}_M \notin M$. By Corollary 1.8.3, A is $\mathbf{\Sigma_1}(M^{n,r})$, so that $\rho_{M^{n,r}} \leq \rho^{n+1}_M$. $\square$

Corollary 1.8.5. *Let $r \in R^n_M$ and $m \leq n$. Then*

- $\rho^n_M = \rho^{n-m}_{M^{m,r\restriction m}}$ *and*
- $M^{n,r} = (M^{m,r\restriction m})^{n-m,s}$*, where $s\colon n-m \to M$ is given by $s(i) := r(m+i)$.*

Proof. By induction on $n - m$ and the two preceeding corollaries. $\square$

Exercise 1.8.3. *Let $r, s \in R^n_M$. Then*

$$\boldsymbol{\Sigma}_1(M^{n,r}) = \boldsymbol{\Sigma}_1(M^{n,s}).$$

Remark 1.8.3. *In these notes, I don't cover functionally absolute definitions of good $\Sigma_1^{(n)}(M)$-functions. Consult Zeman's book.*

Lemma 1.8.4. *Let $r \in R^n_M$. Then $\boldsymbol{\Sigma}_l(M) \subseteq \Sigma_l^{(n)}(M)$ for every $l \geq 1$.*

Proof. We will prove the result for $l = 1$. The general case follows by a straightforward induction. Let ϕ be a Σ_0 formula, $q \in M$ and

$$A = \{a \in M \mid M \models \exists x \phi[x, a, q]\}.$$

Fix $\xi \in \omega\rho^n_M$ such that $q = \tilde{h}^n(\xi, r)$. Then

$$A = \{a \in M \mid M \models \exists x^{(n)} \phi[\tilde{h}^n(x^n, r^0), a^0, \tilde{h}^n(\xi^n, r^0)]\}$$

is $\Sigma_1^{(n)}(M)$. $\square$

Corollary 1.8.6. *Let $r \in R^n_M$. Then $\boldsymbol{\Sigma}_{<\omega}(M) = \boldsymbol{\Sigma}^{(n)}_{<\omega}(M)$.*

Proof. Lemma 1.8.4 yields $\boldsymbol{\Sigma}_{<\omega}(M) \subseteq \boldsymbol{\Sigma}^{(n)}_{<\omega}(M)$. For the converse just note that every $\Sigma_l^{(n)}$-formulae can be expressed as a Σ_l-formula (in parameters) by replacing each occurance of the variable x^i with $x \in H^i_M$ if $\omega\rho^i_M < o(M)$ or with x if $\omega\rho^i_M = o(M)$. $\square$

1.9 Standard Parameters

Talk 2, first draft

Lemma 1.9.1. $h_M``[o(M)]^{<\omega} = M$.

Proof. Let $X := h_M``[o(M)]^{<\omega}$. Since X is closed under pairing, we have $X \prec_1 M$. Let

$$N \overset{\pi}{\cong} X \prec_1 M$$

Be the Mostowski collapse. Since being an acceptable $\mathcal{J}$-structure is a Q-property, we have that N is an acceptable $\mathcal{J}$-structure and clearly $\pi^{-1} \restriction o(M) = \mathrm{id}$. Since there is, for acceptable $\mathcal{J}$-structures K, a uniform lightface Σ_1-definable surjection $f^K \colon [o(M)]^{<\omega} \twoheadrightarrow K$, it follows that $N = M$. $\square$

Lemma 1.9.2. *Let A be $\Sigma_1^{(n)}(M)$ in the parameter $p \in M$. Then there is some $p' \in [o(M)]^{<\omega}$ such that A is $\Sigma_1^{(n)}(M)$ in p'.*

Proof. Fix $\phi \in \Sigma_1^{(n)}(M)$ such that

$$A = \{a \in M \mid M \models \phi[a, p]\}.$$

Let $p' \in [o(M)]^{<\omega}$ be such that $p = h_M(p')$. Then

$$A = \{a \in M \mid M \models \exists x^{(0)} \colon \underbrace{h_M(p') = x^{(0)}}_{\Sigma_1^{(0)}} \wedge \phi[a, x^0]\}.$$

is $\Sigma_1^{(n)}(M)$ in p'. $\square$

Convention 1.9.1. *Let $p \in P_M^{(n)}$. If, for every $i \in \mathrm{dom}(i)$, $p(i)$ is a finite set of ordinals, we will identify p with $\bigcup \mathrm{ran}(p) \in [\mathrm{Ord}]^{<\omega}$. And we let $P_M^{(n)} \cap [o(M)]^{<\omega}$ be the collection of these good parameters.*

By the previous lemma we may restrict ourselves to good parameters which are finite sets of ordinals, i.e. to $P_M^{(n)} \cap [o(M)]^{<\omega}$ and we shall do so from now on.

Definition 1.9.1. *Let $a, b \in [\mathrm{Ord}]^{<\omega}$.*

1. $a \Delta b := (a \setminus b) \cup (b \setminus a)$ *and*
2. $a <^* b :\iff \max(a\Delta b) \in b$.

Exercise 1.9.1. *Let $a, b \in [\mathrm{Ord}]^{<\omega}$. The following are equivalent*

1. $a <^* b$,
2. $\exists \xi \in b \setminus a \colon b \setminus (\xi + 1) = a \setminus (\xi + 1)$,
3. $\vec{a} <_{\mathrm{lex}} \vec{b}$,

where $\left\{\begin{matrix} \vec{a} \\ \vec{b} \end{matrix}\right\}$ *is the strictly decreasing enumeration of* $\left\{\begin{matrix} a \\ b \end{matrix}\right\}$.

Definition 1.9.2. *A definable, binary relation R is* set-like *if for all y*

$$R_y := \{x \mid (x, y) \in R\}$$

is a set.

Exercise 1.9.2. $<^*$ *is a set-like strict,* Σ_0*-definable well-order of* $[\mathrm{Ord}]^{<\omega}$*.*

(Hint: The only tricky bit is to show that $<^*$ is well-founded. So, suppose it isn't. Fix a set $A \subseteq [\mathrm{Ord}]^{<^*}$ with no $<^*$-minimal element. Recursively construct a strictly decreasing sequence $(\xi_n \mid n < \omega)$ via $\xi_0 := \min\{\max(a) \mid a \in A\}$ and $\xi_{n+1} := \min\{\max(a \cap \xi_n) \mid a \in A \wedge a \setminus \xi_n = \{\xi_0, \ldots, \xi_n\}\}$. Verify that this construction never breaks down (i.e. $\xi_{n+1} \neq 0$).)

Definition 1.9.3. *The* $<^*$*-least* $p \in P_M^{(n)} \cap [o(M)]^{<\omega}$*, denoted by* $p_{M,n}$*, is called the* n-th standard parameter of M*.* [1]
The $<^*$*-least* $p \in P_M^* \cap [o(M)]^{<\omega}$ *is called the* (ultimate) standard parameter *of* M*.*

Notation 1.9.1. *1. If* $a \in [o(M)]^{<\omega}$ *we set*

(a) $a^n := a \cap [\omega\rho_M^{n+1}, \omega\rho_M^n]$ *and*

(b) $a \restriction n = a \setminus \omega\rho_M^n$*.*

Exercise 1.9.3. *Let* $p \in [o(M)]^{<\omega}$*. Then*

1. $p \in P_M^n \implies p \restriction (n-1) \in P_M^{n-1}$ *and* $p^{n-1} \in P_{M^{n-1,p\restriction n-1}}$*,*
2. $p \restriction (n-1) \in P_M^{n-1}$*,* $p^{n-1} \in P_{M^{n-1,p\restriction n-1}}$ *and* $\omega\rho_M^n = \omega\rho_{M^{n-1,p\restriction n-1}} \implies p \in P_M^n$*.*
3. $r \in R_M^n \iff \forall i < n\colon r^i \in R_{M^{i,r\restriction i}}$*.*

Corollary 1.9.1. $p_{M,n} \setminus \omega\rho_M^n = \emptyset$*.*

Proof. $p_{M,n} \setminus \omega\rho_M^n \leq^* p_{M,n}$ and by Exercise 1.9.3 we have that $p_{M,n} \setminus \omega\rho_M^n \in P_M^n$. □

Corollary 1.9.2. *Let* $r \in R_M^n$*. Then* r *can be lengthended to some* $p \in P_M^{n+1}$*. If, in addtion,* $r \in [o(M)]^{<\omega}$*, then* r *can be lengthened to some* $p \in P_M^{n+1} \cap [o(M)]^{<\omega}$*.*

Proof. Let $p^- \in P_{M^{n,r}}$. Then, by Exercise 1.9.3, $p := r^\frown p^- \in P_M^{n+1}$.
If $r \in [o(M)]^{<\omega}$, pick $p^- \in P_{M^{n,r}} \cap [o(M)]^{<\omega}$ and let $p = r \cup p^-$. □

Corollary 1.9.3. *Let* M *be sound and* $p \in P_M^n$*. Then* p *can be lengthened to a* $p^* \in P_M^*$*. If, in addition,* $p \in [o(M)]^{<\omega}$*, then* p *can be lengthened to some* $p^* \in P_M^* \cap [o(M)]^{<\omega}$*.*

[1] Zeman calls $p_{M,n}$ the standard parameter above $\omega\rho_M^n$.

Proof. Let $k < \omega$ be such that $\omega\rho_M^\omega = \omega\rho_M^k$. Now apply Corollary 1.9.2 k times. $\square$

Corollary 1.9.4. *Let M be n-sound. Then $p_{M,n-1} = p_{M,n} \restriction (n-1)$.*

Proof. By Corollary 1.9.2 there is some q such that $p_{M,n-1} \cup q \in P_M^n$. Note that we may pick $q \subseteq [\omega\rho_M^n, \omega\rho_M^{n-1})$. Now $p_{M,n-1} = p_{M,n-1} \setminus \omega\rho_M^{n-1}$ and $p_{M,n-1} \cup q \leq^* p_{M,n}$. If $p_{M,n-1} \cup q = p_{M,n}$, we are done. Otherwise fix $\xi \in p_{M,n} \setminus p_{M,n-1} \cup q$ such that

$$p_{M,n} \setminus (\xi + 1) = (p_{M,n-1} \cup q) \setminus (\xi + 1).$$

Since $p_{M,n} \restriction (n-1) \in P_M^n$ and hence $p_{M,n} \restriction (n-1) \leq^* p_{M,n-1}$, we must have that $\xi \geq \omega\rho^{n-1}$ and thus $p_{M,n-1} = p_{M,n} \restriction (n-1)$ as desired. $\square$

We are now ready to prove the main result of this section

Theorem 1.9.1. *M is n-sound iff $p_{M,n} \in R_M^n$.*

Proof. If M is n-sound, then $p_{M,n} \in P_M^n = R_M^n$ and there is nothing to do. Conversely, suppose that n is minimal with $p_{M,n} \in R_M^n$ but M is not n-sound. We will derive a contradiction via the downward extension of embeddings lemma:

Let $q = \min_{<^*}(P_M^n \setminus R_M^n)$. Since $p_{M,n} = \min_{<^*} P_M^n$, we have $p_{M,n} <^* q$. Let $i < \omega$ be minimal such that $(p_{M,n} \cap [\omega\rho_M^{i+1}, \omega\rho_M^i) =)p_{M,n}^i <^* q^i$.

Claim 1.9.1. *$i = n - 1$.*

Proof. Suppose $i < n - 1$. Consider the map

$$\mathrm{id}\colon M^{n,q} \to M^{n,q}.$$

By the downward extension of embeddings lemma there are unique $\pi, \bar{M}, \bar{q}$ such that

1. $\pi\colon \bar{M} \to_{\Sigma_1^{(n)}} M$,
2. $\pi \restriction H_M^n = \mathrm{id}$,
3. $\pi(\bar{q}) = q$,
4. $\bar{q} \in R_{\bar{M}}^n$ and
5. $\bar{M}^{n,\bar{q}} = M^{n,q}$.

By induction hypothesis $p_{M,n} \restriction (i+1) = p_{M,i+1} \in R_M^{i+1}$. We may thus fix a good $\Sigma_1^{(i)}(M)$-function and some $z \in [\omega\rho_M^{i+1}]$ such that $q^i = f(z, p_{M,n} \restriction i, p_{M,n}^i)$. Since $p_{M,n} \restriction i = q \restriction i$, this witnesses

$$M \models \exists z^{i+1} \exists r^i <^* q^i \colon z^{i+1}, r^i \in [\mathrm{Ord}]^{<\omega} \wedge q^i = f(z^{i+1}, q \restriction i, r^i).$$

This is a $\Sigma_1^{(i+1)}$-statement and thus preserved downwards by π. Hence there is a finite set of ordinals $\bar{z} \subseteq \omega\rho_{\bar{M}}^{i+1}$, $\bar{r} \subseteq \omega\rho_{\bar{M}}^{i+1}$ such that $\bar{q}^i = \bar{f}(\bar{z}, \bar{q} \restriction i, \bar{r})$, where $\bar{f}$ is the interpretation of f's $\Sigma_1^{(i)}$ definition over $\bar{M}$. Let $z = \pi(\bar{z})$, $r = \pi(\bar{r})$. Then $q^i = f(z, q \restriction i, r)$. Now consider

$$q^* := q \restriction i \cup r \cup z \cup (q \cap \omega\rho_M^{i+1}).$$

q and q^* can be translated into each other via a $\Sigma_1^{(i)}(M)$ function, so that $q^* \in P_M^n \restriction R_M^n$. On the other hand, sincee $r < q^i$, $q^* <^* q$. This contradicts the fact that $q = \min_{<^*} P_M^n \setminus R_M^n$. $\square$

We may now run the same proof as for the claim, but for $i = n-1$, to show that in fact $R_M^n = P_M^n$. $\square$

1.10 Two Applications to L

Theorem 1.10.1. *For every $\alpha \in \mathrm{Ord}$ J_α is acceptable and sound.*

Before we can prove this, we need one more lemma about $\Sigma_1^{(n)}$-definability:

Lemma 1.10.1. *Suppose $R_M^n \neq \emptyset$. If $A \in \boldsymbol{\Sigma}_{n+1}(M) \cap \mathcal{P}(H_M^n)$, then A is $\boldsymbol{\Sigma}_1^{(n)}(M)$.*

Proof. For $n = 0$ this is trivial. We'll provide the proof for $n = 1$ and leave the induction step as an exercise.
Let ϕ be a Σ_0-formula, $\vec{p} \in M$ such that

$$A = \{a \mid M \models \exists x \phi[a, x, \vec{p}]\}.$$

Now let $r \in R_M^1$ and fix $\xi \in \omega\rho_M^1$ such that $\vec{p} = h_M(\xi, r)$. We have

$$\begin{aligned} A &= \{a \mid M \models \exists x^1 \phi[a^1, h_M(x^1, r^0), h_M(\xi^1, r^0)]\} \\ &= \{a \mid M^{1,r} \models \exists x \phi^*[a, x, \xi]\} \end{aligned}$$

where *phi** is the natural Σ_1-formula corresponding to ϕ. [2] Since A is $\Sigma_1(M^{1,r})$, it is $\Sigma_1^{(1)}(M)$. $\square$

[2]Recall that ϕ is Σ_1 in the language $\{\in, A_M^{1,r}\}$

Exercise 1.10.1. *Provide the induction step.*

(Hint: Let $r \in R_M^{(n+1)}$. Then $r(0) \in R_M^1$ and $H_M^{n+1} = H^n_{M^{1,r(0)}}$. Furthermore recall that $M^{r(0)}$ has a very good parameter $r^* \in R^n_{M^{1,r(0)}}$ such that $r = r(0)^\frown r^*$. Use the induction hypothesis on $M^{1,r(0)}$.)

Proof of Theorem 1.10.1. We proceed by induction on α:
First let $\alpha = 1$: $J_1 = (V_\omega; \in)$ is trivially acceptable and since V_ω is the image of ω under the Ackermann coding function (without parameter), it is also sound.

Suppose J_β is acceptable and sound for all $\beta < \alpha$. We will first show that J_α is acceptable. If α is a limit ordinal, there is nothing to do. So suppose that $\alpha = \beta + 1$. It suffices to show that

$$(\exists \tau < \beta \exists a \subseteq \tau a \in J_{\beta+1} \setminus J_\beta) \implies \exists f \in J_{\beta+1} f \colon \tau \twoheadrightarrow J_\beta.$$

Fix a, τ as above with τ minimal.

Claim 1.10.1. $\tau = \omega\rho^\omega_{J_\beta}$.

Proof. Let $n < \omega$ be such that $\omega\rho^\omega_{J_\beta} = \omega\rho^n_{J_\beta}$. Then there is some $\mathbf{\Sigma}_1^{(n-1)}(M)$-subset of $\omega\rho^\omega_{J_\beta}$ not in J_β. Since $\mathbf{\Sigma}_{<\omega}(J_\beta) \subseteq J_{\beta+1}$, this new subset is in $J_{\beta+1}$ and hence witnesses that $\tau \leq \omega\rho^\omega_{J_\beta}$.
Conversely, let $a \subseteq \tau$ be $\mathbf{\Sigma}_n(J_\beta)$. Since $\tau \leq \omega\rho^n_{J_\beta}$, we have that $a \subseteq H^n_{J_\beta}$. Since J_β is sound, we have $R^n_{J_\beta} \neq \emptyset$. And hence, by Lemma 1.10.1, we have that a is $\mathbf{\Sigma}_1^{(n-1)}(J_\beta)$, witnessing that $\omega\rho^\omega_{J_\beta} \leq \tau$. □

Now, since J_β is sound, there is a $\mathbf{\Sigma}^*(J_\beta)$ function

$$f \colon \omega\rho^\omega_{J_\beta} \twoheadrightarrow J_\beta.$$

But $f \in \mathbf{\Sigma}_{<\omega}(J_\beta) \subseteq J_{\beta+1}$. Thus $J_{\beta+1}$ is acceptable.

Let us now verify that J_α is sound:

By Theorem 1.9.1, it suffices to show that $p_{n,J_\alpha} \in R^n_{J_\alpha}$ for all $n < \omega$. Suppose this is false and let $n < \omega$ be minimal such that $p := p_{n,J_\alpha} \notin R^n_{J_\alpha}$. Let a be $\Sigma_1^{(n-1)}(M)$ in p such that $a \cap \omega\rho^n_{J_\alpha} \notin J_\alpha$. Once again, consider

$$\mathrm{id} \colon (H^n_{J_\alpha}; \in, A^{n,p}_{J_\alpha}) \to (H^n_{J_\alpha}; \in, A^{n,p}_{J_\alpha}).$$

By the downward extension of embeddings lemma there are unique $\pi, \bar{M}, \bar{p}$ such that

1. $\pi\colon \bar{M} \to_{\Sigma_1^{(n)}} J_\alpha$,
2. $\pi \restriction H^n_M = \mathrm{id}$,
3. $\pi(\bar{p}) = p$,
4. $\bar{p} \in R^n_{\bar{M}}$ and
5. $\bar{M}^{n,\bar{q}} = (H^n_{J_\alpha}; \in, A^{n,p}_{J_\alpha})$.

By condensation we have that $\bar{M} = J_{\bar{\alpha}}$ for some $\bar{\alpha} \le \alpha$.

Claim 1.10.2. *$\bar{\alpha} = \alpha$.*

Proof. Let $\bar{a}$ be the $\Sigma_1^{(n-1)}(J_{\bar{\alpha}})$ set in q via the definition for a, call it ϕ. Let $\xi < a \cap \omega\rho^n_{J_\alpha} = \omega\rho^n_{J_{\bar{\alpha}}}$. We have

$$\begin{aligned} \xi \in \bar{a} &\iff J_{\bar{\alpha}} \models \phi[\xi, q] \\ &\iff J_\alpha \models \phi[\underbrace{\pi(\xi)}_{=\xi}, \underbrace{\pi(q)}_{=p}] \\ &\iff \xi \in a, \end{aligned}$$

i.e. $a \cap \omega\rho^n_{J_\alpha} = \bar{a} \cap \omega\rho^n_{J_\alpha}$. If $\bar{\alpha} < \alpha$, then $\bar{a} \cap \omega\rho^n_{J_\alpha} \in J_{\bar{\alpha}+1} \subseteq J_\alpha$, which is absurd. Hence $\bar{\alpha} = \alpha$. ■

Since $\pi(q) = p$, we have $q \le^* p$. On the other hand p is the $<^*$-least good nth parameter and thus $p = q$. By $q \in R^n_{J_{\bar{\alpha}}} = R^n_{J_\alpha}$. Contradiction! □

2 More on Downward Extensions of Embeddings

Talk 3, first draft

Definition 2.0.1. *Let $M = (|M|; A_1, \ldots, A_n)$ and $X \subseteq |M|$. Then*

$$M|X := (|M| \cap X; A_1 \cap X, \ldots, A_n \cap X).$$

Lemma 2.0.1. *Let $M = (J^A_\alpha; \in, A, B)$ be an acceptable structure and let $X \subseteq M$ be closed under good $\Sigma_1^{(n)}(M)$-functions with $X \cap P^n_M \neq \emptyset$. Let $\bar{M}$ be the transitive collapse of X and let σ be the inverse of the collapsing map. Then*

$$\sigma\colon \bar{M} \to_{\Sigma_1^{(n)}} M.$$

Proof. $\vec{x} \in X$ and let ϕ be a $\Sigma_0^{(n)}$-formula such that

$$M \models \exists y^n \phi[y^n, \vec{x}].$$

By $\Sigma_1^{(n)}(M)$-uniformization, leveraging that X is closed under good $\Sigma_1^{(n)}(M)$-functions, we obtain that

$$M|X \models \exists y^n \phi[y^n, \vec{x}]$$

and thus that $M|X \prec_{\Sigma_1^{(n)}} M$. Now, for $i \leq n$, define

1. $H^i := \sigma" H^i_M$ and
2. $\omega\rho^i := H^i \cap \mathrm{Ord}$.

We establish a pseudo-interpretation of $\Sigma_1^{(n)}$-formulae over $\bar{M}$ by declaring that variables of type i ($i \leq n$) range over H^i. For this pseudo-interpretation we obtain

$$\sigma \colon \bar{M} \to_{\Sigma_1^{(n)}} M$$

as above. It now suffices to show

Exercise 2.0.1. *$\omega\rho^i_{\bar{M}} = \omega\rho^i$ for all $i \leq n$.*

$\square$

Lemma 2.0.2. *Let*

$$\sigma \colon M \to_{\Sigma_1^{(n)}} N$$

be such that $\sigma \upharpoonright \omega\rho_N^{n+1} = id$ and $\mathrm{ran}(\sigma) \cap P_N^* \neq \emptyset$. Then σ is $\Sigma^*(= \bigcup_{k<\omega} \Sigma_1^{(k)})$-elementary.

Proof. provide proof

By induction on $m > n$ we prove that

1. $\omega\rho_M^m = \omega\rho_N^m$ and
2. $\sigma \colon M \to_{\Sigma_1^{(m)}} N$.

For this it suffices to prove the following two claims.

Claim 2.0.1. *Suppose $\sigma \colon M \to_{\Sigma_1^{(m)}} N$, $m \geq n$. Then $\omega\rho_M^{m+1} = \omega\rho_N^{m+1}$.*

Proof. Let $\bar{p} \in P^*_M$ be such that $\sigma(\bar{p}) \in P^*_N$. Let A be $\Sigma_1^{(m)}(N)$ in $\sigma(\bar{p})$ such that $A \cap \omega\rho_N^{m+1} \notin N$. Let $\bar{A}$ be $\Sigma_1^{(m)}(M)$ in $\bar{p}$ by the same definition. We have $\bar{A} \cap \omega\rho_N^{m+1} \notin M$ as otherwise

$$\sigma(\bar{A} \cap \omega\rho_N^{m+1}) \cap \omega\rho_N^{m+1} = A \cap \omega\rho_N^{m+1} \in N.$$

Hence $\omega\rho_M^{n+1} \leq \omega\rho_N^{n+1}$.
Conversely let $\bar{A}$ be $\Sigma_1^{(m)}(M)$ in parameter $\bar{p}$ and let $\alpha < \omega\rho_N^{m+1}$. Let A be $\Sigma_1^{(m)}(N)$ in parameter $\sigma(\bar{p})$ via the same definition. Then

$$\bar{A} \cap \alpha = A \cap \alpha \in \mathcal{J}^N_{\omega\rho_N^{m+1}}.$$

Since $\sigma \restriction \omega\rho_N^{m+1} = \mathrm{id}$, we have $\mathcal{J}^M_{\omega\rho_M^{m+1}} = \mathcal{J}^N_{\omega\rho_N^{m+1}}$, so that $\bar{A} \cap \alpha \in M$ and hence $\omega\rho_N^{m+1} \leq \omega\rho_M^{m+1}$. ■

Claim 2.0.2. *Suppose $\omega\rho_M^m = \omega\rho_N^m$ and $\sigma\colon M \rightarrow_{\Sigma_1^{(m-1)}} N$, $m > n$. Then*

$$\sigma\colon M \rightarrow_{\Sigma_1^{(m)}} N.$$

Proof. By induction on the complexity of $\Sigma_1^{(m)}$-formulae. The only nontrivial step is the introduction of an existential quantifier of type m. So, suppose $\vec{x} \in M$, ϕ is preserved by σ and

$$N \models \exists x^m \phi[x^m, \sigma(\vec{x})].$$

Pick some such $x \in H_N^m$. Since $\sigma \restriction H_N^m = \mathrm{id}$, we have $\sigma(x) = x$ and

$$M \models \phi[x, \vec{x}].$$

Since $\omega\rho_M^m = \omega\rho_N^m$, it follows that

$$M \models \exists x^m \phi[x^m, \vec{x}].$$

■

□

3 Witnesses and Solidity

Problem 3.0.1. *Let*

$$\sigma\colon \bar{M} \rightarrow_{\Sigma_1^{(n)}} M^3$$

[3] In a typical application, σ will be the canonical embedding associated to a fine structural ultrapower of $\bar{M}$.

In general we might have that

$$p_{M,n+1} <^* \sigma(p_{\bar{M},n+1}),$$

i.e. that standard parameters are moved incorrectly by fine structural maps. In this section we introduce the concept of solidity that, provided it holds for $\bar{M}$, implies that standard parameters will be moved correctly.

Remark 3.0.1. *Fine structure theory and its later refinement $\Sigma^{(*)}$-theory was developed by Ronald Jensen. The concept of solidity, however, is due to Bill Mitchell and later, independently, Sy Friedman.*

Definition 3.0.1. *Let $p \in [o(M)]^{<\omega}$. (Q, r) is a* generalized witness *for $\nu \in o(M)$ w.r.t. M, p iff*

1. *Q is acceptable and $\nu \subseteq Q$,*
2. *$r \in [o(Q)]^{<\omega}$ and*
3. *Let n be such that $\omega\rho_M^{n+1} \leq \nu < \omega\rho_M^n$. Then for every $\Sigma_1^{(n)}$-formula ϕ and $\xi_0, \ldots, \xi_l < \nu$*

$$M \models \phi[p \setminus (\nu+1), \xi_0, \ldots, \xi_l] \implies Q \models \phi[r, \xi_0, \ldots, \xi_l].$$

Lemma 3.0.1. *There is a $\Sigma_1^{(n)}$-formula ψ such that for every M, p, n, ν as above (Q, r) is a generalized witness for ν w.r.t. M, p if for all $\xi_0, \ldots, \xi_l < \nu$*

$$M \models \psi[p \setminus (\nu+1), \xi_0, \ldots, \xi_l] \implies Q \models \psi[r, \xi_0, \ldots, \xi_l].$$

Proof. Let

$$\psi(i, q, \langle l, \langle \xi_0, \ldots, \xi_l \rangle \rangle) \equiv M^{n,q} \models \phi_i(q \cap \omega\rho_M^n, \xi_0, \ldots, \xi_l),$$

where $(\phi_i \mid i < \omega)$ is the fixed recursive enumeration of all Σ_1-formulae in the language of acceptable $\mathcal{J}$-structures. □

Exercise 3.0.1. *Verify that ψ is as desired.*

Theorem 3.0.1. *Let $p \in P_M^n$ be such that for every $\nu \in p$ there is a generalized witness (Q, r) for ν w.r.t. M, p such that $Q \in M$. Then $p = p_{M,n}$.*

Proof. Suppose not. Then $p_{M,n} <^* p$. Hence there is some $\nu \in p \setminus p_{M,n}$ with

$$q := p \setminus (\nu+1) = p_{M,n} \setminus (\nu+1).$$

Fix $i < n$ such that $\omega\rho_M^{i+1} \leq \nu < \omega\rho_M^i$ and let A be a $\Sigma_1^{(i)}(M)$ set in $p_{M,n}$ such that $A \cap \omega\rho_M^{i+1} \notin M$. Now fix a generalized witness (Q, r) for ν w.r.t. M, p such that $Q \in M$.
Let X be the closure of $\nu \cup q$ under good $\Sigma_1^{(i)}(M)$ functions and let

$$\sigma\colon W \to X \prec_{\Sigma_1^{(i)}} M$$

be the transitive collapse. Let $\bar{q} = \sigma^{-1}(q)$. W is the closure of $\nu \cup \bar{q}$ under good $\Sigma_1^{(i)}(W)$ functions, so that

$$W = \bar{h}_W^{i+1}(\nu \cup \{\bar{q}\}).$$

Claim 3.0.1. $p_{M,n} \in \operatorname{ran}(\sigma)$.

Proof.

$$p_{M,n} = \underbrace{p_{M,n} \setminus (\nu+1)}_{=q} \cup (p_{m,n} \cap [0, \nu)).$$

Since $q \in X$, $\nu \subseteq X$ and X is closed under pairing, we have $p_{M,n} \in X = \operatorname{ran}(\sigma)$. ■

Now let $\bar{A}$ be the $\Sigma_1^{(i)}(W)$ set in $\bar{p} = \sigma^{-1}(p_{M,n})$ by the same definition as A.
Define

$$\sigma^*\colon W \to Q, f_W(\xi_0, \ldots, \xi_l, q) \mapsto f_Q(\xi_0, \ldots, \xi_l, r),$$

where $\xi_0, \ldots, \xi_l \in \nu$ and f_W, f_Q are the corresponding interpretation of the same functionally absolute definition of a good $\Sigma_1^{(i)}$-function f in W, Q.

Exercise 3.0.2. *$\sigma*$ is a well-defined $\Sigma_0^{(i)}$-elementary embedding such that*

1. $\sigma^* \restriction \nu = \mathrm{id}$ *and*

2. $\sigma^*(\bar{q}) = r$.

Let $\omega\rho^* := \sup \sigma^{*}{}''\omega\rho_W^i$ and $A^* := A_Q^{i, r\restriction i} \cap \mathcal{J}_{\rho^*}^Q$. We have that $Q^* := (\mathcal{J}_{\rho^*}^Q, A^*) \in M$ and

$$\sigma^* \restriction H_W^i\colon W^{i, \bar{q}\restriction i} \to_{\Sigma_0} Q^*$$

cofinally.
Note that $\bar{A} \cap \omega\rho_W^i$ is $\Sigma_1(W^{i,\bar{q}\restriction i})$ in the parameter $\bar{p} \cap \omega\rho_W^i$. Let $\tilde{A}$ be $\Sigma_1(Q^*)$ in $\sigma^*(\bar{p} \cap \omega\rho_W^i)$ by the same definition. Since $\bar{Q}^* \in M$, we have that $\tilde{A} \in M$.
On the other hand, since $\sigma \restriction \nu = \sigma^* \restriction \nu = \mathrm{id}$, we have that

$$\tilde{A} \cap \omega\rho_M^{i+1} = \bar{A} \cap \omega\rho_M^{i+1} = A \cap \omega\rho_M^{i+1} \notin M.$$

Contradiction! □

As with good parameters, we can restrict ourselves to a natural, representative subclass of solidity witnesses.

Definition 3.0.2. *Let M be acceptable, $p \in [o(M)]^{<\omega}$ and $\nu \in o(M)$. Let*

$$W_M^{\nu,p}$$

be the transitive collapse of $M|X$, where X is the closure of $\nu \cup (p \setminus (\nu+1))$ under good $\Sigma_1^{(n)}(M)$-functions where $n \in \omega$ is such that $\omega\rho_M^{n+1} \leq \nu < \omega\rho_M^n$.
$W_M^{\nu,p}$ is the standard witness for ν w.r.t. *M, p and the corresponing inverse of the collapsing map*

$$\sigma\colon W_M^{\nu,p} \to M$$

is the canonical witness map of $W_M^{\nu,p}$.

Lemma 3.0.2. *Let $p \in P_M^n$. Then for every $\nu \in p$*

$$W_M^{\nu,p} \in M \iff \textit{there is a generalized witness } Q \in M \textit{ for } \nu \textit{ w.r.t. } M, p.$$

Proof. Zeman, Lemma 1.12.3. □

Corollary 3.0.1. *Let M be acceptable, $p \in P_M^n$ and suppose that every $\nu \in p$ has a generalized witness w.r.t. M, p in M. Then $p \upharpoonright i = p_{M,i}$ for all $i \leq n$.*

Definition 3.0.3. *An acceptable $\mathcal{J}$-structure M is* solid *iff $W_M^{\nu,p_M} \in M$ for all $\nu \in p_M$.*
M is solid above α *iff $W_M^{\nu,p_M} \in M$ whenever $\nu \in p_M \setminus \alpha$.*

Exercise 3.0.3. *J_α is solid for all α.*

The main interest in solid $\mathcal{J}$-structures is captured in the following

Theorem 3.0.2. *Let $\sigma\colon M \to N$ be $\Sigma_1^{(n)}$-preserving, $\bar{p} \in M$, $p = \sigma(\bar{p})$, $\bar{\nu} \in o(M)$, $\nu = \sigma(\bar{\nu})$ with $\omega\rho_M^{n+1} \leq \bar{\nu}$ and $\omega\rho_N^{n+1} \leq \nu$.*
Let $(\bar{Q}, \bar{r})$ be a generalized witness for $\bar{\nu}$ w.r.t. $M, \bar{p}$ such that $\bar{Q} \in M$. Then $(Q, r) := \sigma((\bar{Q}, \bar{r}))$ is a generalized witness for ν w.r.t. N, p.

Proof. Fix $i < n$ such that $\omega\rho_M^{i+1} \leq \bar{\nu} < \omega\rho_M^{i+1}$ and $\omega\rho_N^{i+1} \leq \nu < \omega\rho_N^i$. [4] Let ϕ be a $\Sigma_1^{(i)}$-formula. Then

$$M \models \forall \xi_0^i, \ldots, \xi_l^i < \bar{\nu} \left(\phi(\xi_0^i, \ldots, \xi_l^i, \bar{p} \setminus (\bar{\nu}+1)) \implies \bar{Q} \models \phi(\xi_0^i, \ldots, \xi_l^i, \bar{r}) \right).$$

This is a $\Pi_1^{(i)}$ statement and hence preserved upwards. □

Corollary 3.0.2. *Let $\sigma\colon M \to N$ be Σ^*-elementary and $p = \sigma(p_M) \in P_N^*$. If M is solid, then $p = p_N$ and N is solid.*

[4] such an i exists by Exercise 3.0.4.

Bonus Exercises

Exercise 3.0.4. *Let* $\pi\colon M \to N$ *be* $\Sigma_1^{(n)}$*-elementary. Then either*

1. $\omega\rho_M^n = o(M)$ *and* $\omega\rho_N^n = o(N)$ *or*
2. $\omega\rho_M^n < o(M)$ *and* $\pi(\omega\rho_M^n) = \omega\rho_N^n$.

Corollary 3.0.3. *Let* $\pi\colon M \to N$ *be* $\Sigma_1^{(n)}$*-elementary. Then for all* $k \leq n$ *either*

1. $\omega\rho_M^k = o(M)$ *and* $\omega\rho_N^k = o(N)$ *or*
2. $\omega\rho_M^k < o(M)$ *and* $\pi(\omega\rho_M^k) = \omega\rho_N^k$.

In particular this holds if $\pi\colon M \to N$ *is* Σ^**-elementary.*

Proof of Corollary. If $\pi\colon M \to N$ is $\Sigma_1^{(n)}$-elementary, then π is also $\Sigma_1^{(l)}$-elementary for all $l \leq n$. $\square$