Notes on “PFA implies $\mathrm{AD}^{L(\mathbb{R})}$”

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1 Quick Reference §1

1. κ is a singular, strong limit cardinal such that $\square_\kappa$ fails.
2. $A_0 \subseteq \kappa$ codes V_κ.
3. $\lambda = \kappa^{+\operatorname{Lp}(A_0)}$.
4. $\operatorname{cf}(\lambda) < \mu < \kappa$ and $\mu^\omega = \mu$.
5. g is a $\operatorname{Col}(\omega, \mu)$-generic filter.
6. Given $U \subseteq \mathbb{R}^g$ and $k < \omega$ a coarse (k, U)-Woodin mouse (witnessed by $S, T, \Sigma, \delta_0, \dots, \delta_k$) is a countable transitive model of ZFC such that
   (a) $N \models \delta_0 < \ldots < \delta_k$ are Woodin cardinals ,
   (b) $N \models S, T$ are trees which project to complements after the collapse of δ_k to be countable and
   (c) there is a $\omega_1 + 1$-iteration strategy Σ for N such that whenever $i\colon N \to P$ is an iteration map by Σ and P is countable, then $p[i(S)] \subseteq U$ and $p[i(T)] \subseteq \mathbb{R}^g \setminus U$.
7. For a norm $\phi\colon \operatorname{dom}(\phi) \to \mathrm{Ord}$ we let $\leq_\phi$ be the associated prewellorder given by $x \leq_\phi y$ iff $\phi(x) \leq \phi(y)$ and similar for $<_\phi$.
8. For a scale $\vec{\phi}$ we let $\vec{\phi}^*$ be the associated sequence of prewellorderes.
9. (W^*_α) Let $U \subseteq \mathbb{R}^g$ and suppose there are scales $\vec{\phi}, \vec{\psi}$ on U and $\mathbb{R}^g \setminus U$ such that $\vec{\phi}^*, \vec{\psi}^* \in J_\alpha(\mathbb{R}^g)$. Then for all $k < \omega$ and all $x \in \mathbb{R}^g$ there are N, Σ such that

(a) $x \in N$ and N is a coarse (k, U)-Woodin mouse as witnessed by Σ and

(b) $\Sigma \restriction \mathrm{HC}^{V[g]} \in J_\alpha(\mathbb{R}^g)$.g

10. If W_α^* holds, then $J_\alpha(\mathbb{R}^g) \models \mathrm{AD}$.

11. $\mathcal{L}(C)$ is the language of set theory expanded by a constant symbol for each $c \in C$.

12. For any Σ_1-formula $\theta(v)$ we associate $(\theta^k(v) \mid k < \omega)$ such that θ^k is Σ_k and for any $\gamma \in \mathrm{Ord}$ and any $x \in \mathbb{R}$

$$J_{\gamma+1}(\mathbb{R}) \models \theta[x] \iff \exists k < \omega J_\gamma(\mathbb{R}) \models \theta^k[x].$$

13. Given a $\Sigma_1^{\mathcal{L}(\{\mathbb{R}\})}$ formula $\theta(v)$ and $z \in \mathbb{R}$, a $\langle \theta, z \rangle$-witness is a ω-sound, $(\omega, \omega_1, \omega_1 + 1)$-iterable z-mouse $\mathcal{N}$ for which there are $\delta_0, \ldots, \delta_9, S, T$ such that $\mathcal{N}$ models

    (a) ZFC,

    (b) $\delta_0 < \delta_1 < \ldots, \delta_9$ are Woodin,

    (c) S, T are trees on some $\omega \times \theta$, $\theta < \delta_9$ which are absolutely complementing in $V^{\mathrm{Col}(\omega, \delta_9)}$ and

    (d) For some $k < \omega$, $p[T]$ is the $\Sigma_{k+3}^{\mathbb{R}}$-theory of $J_\gamma(\mathbb{R})$ with γ minimal such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$.

14. If there is a $\langle \theta, z \rangle$-witness, then $L(\mathbb{R}) \models \theta[z]$.

15. (W_α) If $\theta(v)$ is a Σ_1-formula, $z \in \mathbb{R}^g$ and $J_\alpha(\mathbb{R}^g) \models \theta[z]$, then there is a $\langle \theta, z \rangle$-witness $\mathcal{N}$ whose iteration strategy Σ satisfies $\Sigma \restriction \mathrm{HC} \in J_\alpha(\mathbb{R}^g)$.

16. If α is a limit ordinal, then $W_\alpha^* \implies W_\alpha$.

17. An ordinal β is critical iff there is some $U \subseteq \mathbb{R}^g$ such that U and $\mathbb{R}^g \setminus U$ admit scales in $J_{\beta+1}(\mathbb{R}^g)$ but U admits no scale in $J_\beta(\mathbb{R}^g)$.

18. If β is critical, then $\beta + 1$ is critical. If β is a limit of critical ordinals, then β is critical iff $J_\beta(\mathbb{R}^g)$ is not admissible.

19. Let $\alpha \leq \beta \in \mathrm{Ord}$. $[\alpha, \beta]$ is a Σ_1-gap iff

    (a) $J_\alpha(\mathbb{R}) \prec_1^{\mathcal{L}(\mathbb{R} \cup \{V_{\omega+1}\})} J_\beta(\mathbb{R})$,

    (b) $\forall \alpha' < \alpha \colon J_{\alpha'}(\mathbb{R}) \not\prec_1^{\mathcal{L}(\mathbb{R} \cup \{V_{\omega+1}\})} J_\alpha(\mathbb{R})$ and

(c) $\forall \beta' > \beta J_\beta(\mathbb{R}) \not\prec_1^{\mathcal{L}(\mathbb{R}\cup\{V_{\omega+1}\})} J_{\beta'}(\mathbb{R})$.

20. Let β be critial. Then one of the following cases holds true:
    (1) $\beta = \eta + 1$ for some critical η,
    (2) β is a limit of critical ordinals and either
        (a) $\mathrm{cf}(\beta) = \omega$ or
        (b) The "inadmissible case": $\mathrm{cf}(\beta) > \omega$ but $J_\beta(\mathbb{R}^g)$ is not admissible.
    (3) The "end-of-gap case": $\alpha = \sup(\{\eta < \beta \mid \eta \text{ is critical}\}) < \beta$ and either
        (a) $[\alpha, \beta]$ is a Σ_1 gap or
        (b) $\beta - 1$ exists and $[\alpha, \beta - 1]$ is a Σ_1 gap.

21. A self-justifying system is a countable set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ which is closed under complements in $\mathbb{R}$ and such that every $A \in \mathcal{A}$ admits a scale $\vec{\psi} = (\psi_i \mid i < \omega)$ such that $\leq_\phi \in \mathcal{A}$ for all $i < \omega$. Here we identify $\leq_\phi \subseteq \mathbb{R} \times \mathbb{R}$ with its code $\{x \oplus y \mid x \leq_{\psi_i} y\} \subseteq \mathbb{R}$.

1.1 The inadmissible case.

1. κ is a singular strong limit cardinal such that $\kappa^{+\mathrm{Lp}(A)} < \kappa^+$ for all bounded $A \subseteq \kappa^+$ and $\mu = \mathrm{cf}(\kappa^{+\mathrm{Lp}(A_0)})^\omega$ where $A_0 \subseteq \kappa$ codes V_κ.

2. to be continued...

1.2 The end-of-gap case.

1. Let β be critical $\alpha = \sup(\{\eta < \beta \mid \eta \text{ is critical}\}) < \beta$ and
    (a) $[\alpha, \beta]$ is a Σ_1-gap or
    (b) $\beta - 1$ exists and $[\alpha, \beta - 1]$ is a Σ_1-gap.

2. to be continued...

1.3 Details in the inadmissible case

1. to be continued...

1.4 Details in the end-of-gap case

1. Let β be critical $\alpha = \sup(\{\eta < \beta \mid \eta \text{ is critical}\}) < \beta$ and
   (a) $[\alpha, \beta]$ is a Σ_1-gap or
   (b) $\beta - 1$ exists and $[\alpha, \beta - 1]$ is a Σ_1-gap.