

Mijn titel

Geert Kapteijns

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Abstract

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1 Introduction

Let me start off by saying this is an informal document meant to make my research efforts of the last three months accessible. I have made no huge effort to cite the first paper to establish a concept or even to cite well-established results at all.

The challenge at hand is, in the broadest sense, to use statistical models and computational power to improve healthcare.

2 Scattering transform

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fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum. [1]

3 Constructing the Morlet wavelet

In order to satisfy the Paley-Littlewood condition, we require that the Morlet wavelets average to zero in the spatial domain, which corresponds to $\hat{\psi}(\mathbf{0}) = 0$ in the Fourier domain. To achieve this for a discretely sampled Gabor wavelet, we subtract a constant $\kappa_\sigma \ll 1$ from the plane-wave part

$$\psi(\mathbf{x}) = g_\sigma(\mathbf{x})(e^{i\boldsymbol{\xi}\mathbf{x}} - \kappa_\sigma) \quad (1)$$

where g_σ is a Gaussian of standard deviation σ and center frequency $\boldsymbol{\xi} = (\xi, 0, 0)$ by convention. We will specify how to find a suitable σ and ξ shortly. The Fourier transform is

$$\hat{\psi}(\boldsymbol{\omega}) = \hat{g}_\sigma(\boldsymbol{\omega} - \boldsymbol{\xi}) - \kappa_\sigma \hat{g}_\sigma(\boldsymbol{\omega}). \quad (2)$$

The requirement $\hat{\psi}(\mathbf{0}) = 0$ leads to

$$\kappa_\sigma = \frac{\hat{g}_\sigma(-\boldsymbol{\xi})}{\hat{g}_\sigma(\mathbf{0})}. \quad (3)$$

Instead of labeling a wavelet by its standard deviation in the spatial domain σ , it is in this case more insightful to label it by its bandwidth b in the Fourier domain, defined by

$$\hat{g}_\sigma\left(\pm\left(\frac{b}{2}, 0, 0\right)\right) = \exp\left(-\frac{1}{2}\sigma^2\left(\frac{b}{2}\right)^2\right) = \frac{1}{\sqrt{2}} \quad (4)$$

leading to

$$b^2 = \frac{4 \ln 2}{\sigma^2}. \quad (5)$$

3.1 Rotating and scaling the mother wavelet

A wavelet transform is defined by dilating the mother wavelet by scale factors $\{a^j\}_{j \in \mathbb{Z}}$ and rotating it by rotations r in \mathbb{R}^d . $a = 2$ is common for image analysis (at least in two dimensions). A wavelet of dilation a^j and orientation r looks like

$$\psi_{a^j,r}(\mathbf{x}) = a^{-dj} \psi(a^{-j} r \mathbf{x}) \quad (6)$$

where the normalisation a^{-dj} ($d = 3$ in our case) is chosen such that the energy of the mother wavelet is conserved

$$\int_{\mathbb{R}^d} d\mathbf{x} |\psi_{a^j,r}(\mathbf{x})| = \int_{\mathbb{R}^d} d\mathbf{x} |\psi(\mathbf{x})|. \quad (7)$$

The dilated and scaled Morlet wavelet becomes in the Fourier domain

$$\hat{\psi}_{a^j,r}(\boldsymbol{\omega}) = a^{-dj} (\hat{g}_\sigma(a^j r^{-1} \boldsymbol{\omega} - \boldsymbol{\xi}) - \kappa_\sigma \hat{g}_\sigma(a^j r^{-1} \boldsymbol{\omega})) \quad (8)$$

so that it is (apart from a small corrective term) centered at frequency $\boldsymbol{\omega} = a^{-j} r \boldsymbol{\xi}$ with bandwidth $b_{a^j} = a^{-j} b$. Note that the corrective factor κ_σ is invariant under dilation and rotation.

3.2 Littlewood-Paley condition

The rotations and dilations of the mother wavelet form an overcomplete basis or frame of $L^2(\mathbb{R}^d)$ (finite-energy functions in d dimensions, including all audio signals, CT scans, etc.) But in practice we can only choose a finite amount of wavelets of dilations $\{a^j\}_{j=0,1,\dots,J}$ and orientations $r \in R$. How can we still make sure we capture all relevant information in the signal? To capture the low frequency information (at scales a^j with $j > J$), we add a low-pass filter of length scale J to our set of wavelets, which we denote by $\phi_J(\mathbf{x})$ and take to be a Gaussian in practice. The rotations r must be chosen to discretize the rotation group in \mathbb{R}^d , which is not trivial in \mathbb{R}^3 . To check if we have indeed covered the whole frequency space, our wavelets must satisfy the Littlewood-Paley condition

$$(1 - \epsilon) \leq A(\boldsymbol{\omega}) \leq 1 \quad \forall \boldsymbol{\omega} \in \mathbb{R}^d \quad (9)$$

with

$$A(\boldsymbol{\omega}) = \left| \hat{\phi}(\boldsymbol{\omega}) \right|^2 + \frac{1}{2} \sum_{j \leq J} \sum_{r \in R} \left(\left| \hat{\psi}_{a^j,r}(\boldsymbol{\omega}) \right|^2 + \left| \hat{\psi}_{a^j,r}(-\boldsymbol{\omega}) \right|^2 \right) \quad (10)$$

and ϵ small.

References

- [1] J. Bruna and S. Mallat. “Invariant Scattering Convolution Networks”. In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 35.8 (Aug. 2013), pp. 1872–1886. ISSN: 0162-8828. DOI: 10.1109/TPAMI.2012.230.