Mijn titel

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Abstract

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1 Introduction

Let me start off by saying this is an informal document meant to make my research efforts of the last three months accessible. I have made no huge effort to cite the first paper to establish a concept or even to cite well-established results at all.

The challenge at hand is, in the broadest sense, to use statistical models and computational power to improve healthcare.

2 Scattering transform

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3 Constructing the Morlet wavelet

In order to satisfy the Paley-Littlewood condition, we require that the Morlet wavelets average to zero in the spatial domain, which corresponds to $\hat{\psi}(\mathbf{0}) = 0$ in the Fourier domain. To achieve this for a discretely sampled Gabor wavelet, we substract a constant $\kappa_\sigma \ll 1$ from the plane-wave part

$$\psi(\boldsymbol{x}) = g_\sigma(\boldsymbol{x})(e^{i\boldsymbol{\xi}\boldsymbol{x}} - \kappa_\sigma) \tag{1}$$

where g_σ is a Gaussian of standard deviation σ and center frequency $\boldsymbol{\xi} = (\xi, 0, 0)$ by convention. We will specify how to find a suitable σ and ξ shortly. The Fourier transform is

$$\hat{\psi}(\boldsymbol{\omega}) = \hat{g}_\sigma(\boldsymbol{\omega} - \boldsymbol{\xi}) - \kappa_\sigma \hat{g}_\sigma(\boldsymbol{\omega}). \tag{2}$$

The requirement $\hat{\psi}(\mathbf{0}) = 0$ leads to

$$\kappa_\sigma = \frac{\hat{g}_\sigma(-\boldsymbol{\xi})}{\hat{g}_\sigma(\mathbf{0})}. \tag{3}$$

Instead of labeling a wavelet by its standard deviation in the spatial domain σ, it is in this case more insightful to label it by its bandwidth b in the Fourier domain, defined by

$$\hat{g}_\sigma\left(\pm\left(\frac{b}{2}, 0, 0\right)\right) = \exp\left(-\frac{1}{2}\sigma^2\left(\frac{b}{2}\right)^2\right) = \frac{1}{\sqrt{2}} \tag{4}$$

leading to

$$b^2 = \frac{4\ln 2}{\sigma^2}. \tag{5}$$

3.1 Rotating and scaling the mother wavelet

A wavelet transform is defined by dilating the mother wavelet by scale factors $\left\{a^j\right\}_{j\in\mathbb{Z}}$ and rotating it by rotations r in $\mathbb{R}^d$. $a=2$ is common for image analysis (at least in two dimensions). A wavelet of dilation a^j and orientation r looks like

$$\psi_{a^j,r}(\boldsymbol{x}) = a^{-dj}\psi(a^{-j}r\boldsymbol{x}) \tag{6}$$

where the normalisation a^{-dj} ($d=3$ in our case) is chosen such that the energy of the mother wavelet is conserved

$$\int_{\mathbb{R}^d} d\boldsymbol{x}|\psi_{a^j,r}(\boldsymbol{x})| = \int_{\mathbb{R}^d} d\boldsymbol{x}|\psi(\boldsymbol{x})|. \tag{7}$$

The dilated and scaled Morlet wavelet becomes in the Fourier domain

$$\hat{\psi}_{a^j,r}(\boldsymbol{\omega}) = a^{-dj}\left(\hat{g}_\sigma(a^j r^{-1}\boldsymbol{\omega}-\boldsymbol{\xi}) - \kappa_\sigma\hat{g}_\sigma(a^j r^{-1}\boldsymbol{\omega})\right) \tag{8}$$

so that it is (apart from a small corrective term) centered at frequency $\boldsymbol{\omega} = a^{-j}r\boldsymbol{\xi}$ with bandwidth $b_{a^j} = a^{-j}b$. Note that the corrective factor κ_σ is invariant under dilation and rotation.

3.2 Littlewood-Paley condition

The rotations and dilations of the mother wavelet form an overcomplete basis or frame of $L^2(\mathbb{R}^d)$ (finite-energy functions in d dimensions, including all audio signals, CT scans, etc.) But in practice we can only choose a finite amount of wavelets of dilations $\left\{a^j\right\}_{j=0,1,\ldots,J}$ and orientations $r\in R$. How can we still make sure we capture all relevant information in the signal? To capture the low frequency information (at scales a^j with $j>J$), we add a low-pass filter of length scale J to our set of wavelets, which we denote by $\phi_J(\boldsymbol{x})$ and take to be a Gaussian in practice. The rotations r must be chosen to discretize the rotation group in $\mathbb{R}^d$, which is not trivial in $\mathbb{R}^3$. To check if we have indeed covered the whole frequency space, our wavelets must satisfy the Littlewood-Paley condition

$$(1-\epsilon) \le A(\boldsymbol{\omega}) \le 1 \qquad \forall\boldsymbol{\omega}\in\mathbb{R}^d \tag{9}$$

with

$$A(\boldsymbol{\omega}) = \left|\hat{\phi}(\boldsymbol{\omega})\right|^2 + \frac{1}{2}\sum_{j\le J}\sum_{r\in R}\left(\left|\hat{\psi}_{a^j,r}(\boldsymbol{\omega})\right|^2 + \left|\hat{\psi}_{a^j,r}(-\boldsymbol{\omega})\right|^2\right) \tag{10}$$

and ϵ small.

4 Implementation details

4.1 Normalization of filter bank

4.2 Downsampling in the Fourier domain

I downsample in the Fourier domain by a factor 2^j by simply setting to zero all frequencies outside the range $[-\frac{\pi}{2^j}, \frac{\pi}{2^j}]$ in each dimension. This ideal low-pass filter corresponds to a convolution with a normalized sinc-filter in the spatial domain.

For a discrete-time signal $x[n]$, the Fourier transform is

$$X[\omega] = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}. \tag{11}$$

Downsampling the spatial signal x by a factor D, i.e.

$$x_D[n] = x[Dn] \tag{12}$$

corresponds in the Fourier domain to

$$X_D[\omega] = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega-2\pi k}{D}\right). \tag{13}$$

References

[1] J. Bruna and S. Mallat. “Invariant Scattering Convolution Networks”. In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 35.8 (Aug. 2013), pp. 1872–1886. ISSN: 0162-8828. DOI: 10.1109/TPAMI.2012.230.