1 Decay behaviour of the quasi-particle

In this chapter we will investigate the decay behaviour shown in **??**. We will show that for large t, a $\frac{1}{t}$ decay is expected. We need to evaluate the expression:

$$\rho(x,t) = \frac{1}{N}\sum_{p,p=-N}^{N} \langle P|\hat{\rho}(0)|P'\rangle e^{i(P-P')x-i(E_p-E_{p'})t}$$

Remember that $P = \frac{2\pi p}{L}$, the dressed momentum of the state $|P\rangle$. First, we will approximate this double sum as a double integral.

$$\frac{1}{N}\sum_{p,p=-N}^{N} \langle P|\hat{\rho}(0)|P'\rangle e^{i(P-P')x-i(E_p-E_{p'})t} = \frac{1}{N}\int_{-N}^{N}\int_{-N}^{N} dpdp'\langle P|\hat{\rho}(0)|P'\rangle e^{iPx-iP'x}e^{-iE(p)t+iE(p')t} \tag{1.1}$$

Now, a change to integration variables $P = \frac{2\pi p}{L}$ and $P' = \frac{2\pi p'}{L}$ yields:

$$\frac{L^2}{4\pi^2 N}\int_{-\frac{2\pi N}{L}}^{\frac{2\pi N}{L}}\int_{-\frac{2\pi N}{L}}^{\frac{2\pi N}{L}} dPdP'\langle P|\hat{\rho}(0)|P'\rangle e^{iPx-iP'x}e^{-iE(P)t+iE(P')t} \tag{1.2}$$

Because we work at unit filling ($\frac{N}{L} = 1$), we get:

$$\frac{L}{4\pi^2}\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} dPdP'\langle P|\hat{\rho}(0)|P'\rangle e^{iPx-iP'x}e^{-iE(P)t+iE(P')t}$$

This integral can be approximated with a stationary phase approximation, which we will briefly review.

1.1 Stationary phase approximation

We can approximate oscillatory integrals of the following form:

$$\int dpg(p)e^{itf(p)}, \qquad t \gg 1 \tag{1.3}$$

We will take advantage of the fact that in regions where $\frac{\partial f}{\partial p} \neq 0$, the oscillating function $e^{itf(p)}$ kills the integral by destructive interference. So only the tiny regions around certain dominant frequencies ω_i at which $\frac{\partial f}{\partial p} = 0$ contribute to the value of the integral. In these regions, $g(p)$ is essentially constant at $g(\omega_i)$ and we Taylor-expand $f(p)$ to second order about ω_i:

$$f(p) = f(\omega_i) + \frac{f''(\omega_i)}{2}(p-\omega_i)^2$$

Here, the first order term is zero by definition of the points ω. Now, Equation 1.3 can be approximated as:

$$\sum_i g(\omega_i)e^{itf(\omega_i)} \int_{\mathbb{R}} dpe^{it\frac{f''(\omega_i)}{2}(p-\omega_i)^2}$$

When t is large, even a small difference $p-\omega_i$ leads to a highly oscillating integrand, resulting in no contribution to the total value of the integral. This allows us to freely expand the limits of the resulting integrals to $-\infty$ and ∞. They are easily evaluated (after shifting $\bar{p} = p - \omega_i$):

$$\int_{\mathbb{R}} d\bar{p}e^{it\frac{f''(\omega_i)}{2}\bar{p}^2} = \sqrt{\frac{2\pi}{-itf''(\omega_i)}}$$

1.2 Approximation of decay behaviour using the stationary phase approximation

Now we will use a stationary phase approximation to show that the notch depth decays as $\frac{1}{t}$ for large t. We start with Equation 1.2:

$$\frac{L}{4\pi^2}\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} dPdP'\langle P|\hat{\rho}(0)|P'\rangle e^{iPx-iP'x}e^{-iE(P)t+iE(P')t}$$

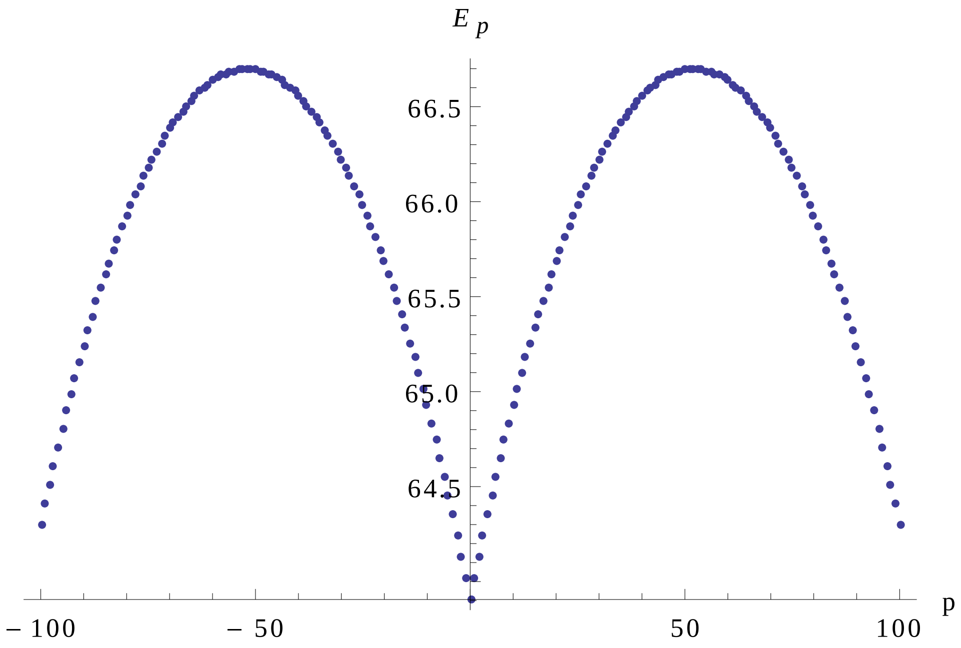

Figure 1.1: Energy spectrum of the eigenstates of the $N = L = 100$, $c = 1$ quasi particle state.

Figure 1.1 shows that $E(p)$ has two dominant frequencies at which $\frac{dE}{dp} = 0$. We label the corresponding dressed momenta P_+ and P_-.

This means the integral has 4 contributing terms, with weights $\langle P_+|\hat{\rho}(0)|P_+\rangle$, $\langle P_+|\hat{\rho}(0)|P_-\rangle$, $\langle P_-|\hat{\rho}(0)|P_+\rangle$ and $\langle P_+|\hat{\rho}(0)|P_-\rangle$. $\langle P_+|\hat{\rho}(0)|P_+\rangle = \langle P_-|\hat{\rho}(0)|P_-\rangle = 1$, since they are diagonal form factors. $\langle P_+|\hat{\rho}(0)|P_-\rangle$ and $\langle P_-|\hat{\rho}(0)|P_+\rangle$ are essentially orthogonal and are of order 10^{-8}, so they can be safely ignored. We focus on the stationary point P_+.

We expand E about $P = P_+$ to second order:

$$E(p) \approx E(P_+) + \frac{1}{2}\frac{d^2E}{dp^2}(P_+)(P - P_+)^2 \equiv E(P_+) + \frac{1}{2}\alpha(P_+ - P)^2$$

Here, we have defined $\alpha \equiv \frac{d^2E}{dp^2}(p_+)$. We approximate the original integral as:

$$\frac{L\langle P_+|\hat{\rho}(0)|P_+\rangle}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dPdP' e^{-it[E(P_+)+\frac{1}{2}\alpha(P-P_+)^2]+iPx} e^{it[E(P_+)+\frac{1}{2}\alpha(P'-P_+)^2]-iP'x}$$

We have extended the limits of both integrals from $-\infty$ to ∞, which is justified in the limit $t \to \infty$. We now have two integrals that are eachother's complex conjugate and we get:

$$\frac{L\langle P_+|\hat{\rho}(0)|P_+\rangle}{4\pi^2}\left|\int_{-\infty}^{\infty} dP e^{-it[E(P_+)+\frac{1}{2}\alpha(P-P_+)^2]+iPx}\right|^2$$

$e^{-itE(P_+)}$ simply disappears as a phase. In order to shift variables $\tilde{P} = P - P_+$, we multiply with $e^{iP_+x}e^{-iP_+x}$, yielding:

$$\frac{L\langle P_+|\hat{\rho}(0)|P_+\rangle}{4\pi^2}\left|e^{iP_+x}\int_{-\infty}^{\infty} dP e^{-it\frac{1}{2}\alpha(P-P_+)^2+i(P-P_+)x}\right|^2$$

e^{iP_+x} also disappears and we shift variables (renaming $\tilde{P}$ back to P):

$$\frac{L\langle P_+|\hat{\rho}(0)|P_+\rangle}{4\pi^2}\left|\int_{-\infty}^{\infty} dP e^{-it\frac{1}{2}\alpha P^2+iPx}\right|^2$$

This integrates to:

$$\frac{L\langle P_+|\hat{\rho}(0)|P_+\rangle}{4\pi^2}\left|\frac{e^{\frac{ix^2}{4t\alpha}}\sqrt{2\pi}}{\sqrt{it\alpha}}\right|^2 = \frac{L\langle P_+|\hat{\rho}(0)|P_+\rangle}{2\pi\alpha}\frac{1}{t}$$

Since the contribution for $\langle P_+|\hat{\rho}(0)|P_+\rangle$ is completely analogous, we can double this value and set $\langle P_+|\hat{\rho}(0)|P_+\rangle = \langle P_-|\hat{\rho}(0)|P_-\rangle = 1$ to obtain:

$$\frac{L}{\pi\alpha t}$$

By fitting to Figure 1.1, we find $\alpha \equiv \frac{d^2E}{dP^2}(P_+) = \frac{d^2E}{dP^2}(P_-) = -0.490782$, which gives us:

$$\rho(x,t) = -64.8577$$

Obviously something is wrong here. In the above expression, I inserted $L = 100$, but since we took the thermodynamic limit, this isn't right. What does that L still do there?

1.3 $\frac{1}{t}$ fit to numerically calculated notch depth

Figure 1.2 shows a fit to the model $\frac{A}{t+B}$, characterized by the parameters A and B.

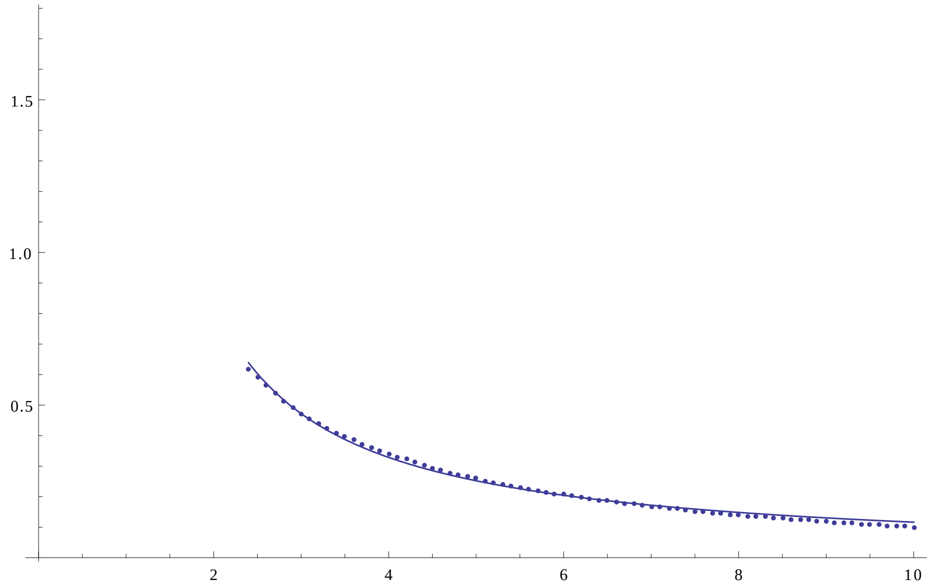

Figure 1.2: Fit to numerically calculated values of the notch depth for $N = L = 100$, $c = 1$. Only the values for t for which the peak is smaller than $\frac{1}{e}$ times the original depth have been taken into account.

c	A	B
1	2.12261	0.958587
2	1.18083	0.517246
4	0.743117	0.32157
8	0.524958	0.227496

Table 1.1: Values for fit of a $\frac{A}{t+B}$ model for different values of the interaction parameter c. $N = L = 100$ for all values.