1 Description of the Lieb-Liniger model

I have studied the Lieb-Liniger model, a model of particles that can move in one dimension. This is the Hamiltonian in the appropriate units:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j)$$

The two particle case

Let's start with the two particle case. We get:

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2c\delta(x_1 - x_2) = -\frac{1}{2}(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}) + 2c\delta(\alpha)$$

Here $\alpha = x_1 - x_2$ and $\beta = x_1 + x_2$. If we substitute this into $H\Psi = E\Psi$, and integrate both sides from $-\epsilon$ to ϵ in α, we get:

$$-\frac{1}{2}\int_{-\epsilon}^{\epsilon} d\alpha(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2})\Psi + 2c\int_{-\epsilon}^{\epsilon} d\alpha\delta(\alpha)\Psi = \int_{-\epsilon}^{\epsilon} d\alpha E\Psi$$

Now, taking the limit $\epsilon \to 0$, the right side becomes zero, and we see:

$$\lim_{\epsilon \to 0} -\frac{1}{2}\int_{-\epsilon}^{\epsilon} d\alpha(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2})\Psi + 2c\int_{-\epsilon}^{\epsilon} d\alpha\delta(\alpha)\Psi = 0$$

From which it follows that:

$$\lim_{\epsilon \to 0} 4c\int_{-\epsilon}^{\epsilon} d\alpha\delta(\alpha)\Psi = 4c\Psi(\alpha = 0) = \lim_{\epsilon \to 0}\int_{-\epsilon}^{\epsilon} d\alpha(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2})\Psi$$

$$= \lim_{\epsilon \to 0} \frac{\partial}{\partial \alpha} \Psi(\alpha = \epsilon) - \frac{\partial}{\partial \alpha} \Psi(\alpha = -\epsilon) = \Delta \frac{\partial}{\partial \alpha} \Psi(\alpha = 0)$$

So, for the discontinuity at $x_1 = x_2$ ($\alpha = 0$) we have:

$$\Delta \frac{\partial}{\partial \alpha} \Psi(\alpha = 0) = 4c\Psi(\alpha = 0) \tag{1.1}$$

Assuming the wavefunction:

$$\Psi(x_1, x_2) = \begin{cases} A_+ e^{i(k_1 x_1 + k_2 x_2)} + A_- e^{i(k_2 x_1 + k_1 x_2)} & x_1 < x_2 \\ A_- e^{i(k_1 x_1 + k_2 x_2)} + A_+ e^{i(k_2 x_1 + k_1 x_2)} & x_1 > x_2 \end{cases}$$

and solving Equation 1.1 we find:

$$A_+ = e^{-\frac{i}{2}\phi(k_1 - k_2)}, \qquad A_- = -e^{\frac{i}{2}\phi(k_1 - k_2)}$$

where

$$\phi(k) = 2 \arctan \frac{k}{c} \tag{1.2}$$

is known as the *scattering phase shift*.

Now, if we impose periodic boundary conditions $\Psi(x, 0) = \Psi(x, L)$ (this also assumes a symmetric wavefunction), where L is the system length, we find a quantization for the momenta:

$$e^{ik_1 L} = e^{-i\phi(k_1 - k_2)}, \qquad e^{ik_2 L} = -e^{-i\phi(k_2 - k_1)}$$

The above equations are the Bethe equations for two particles. Let's do a sanity check. In the limit $c \to 0$, we get:

$$e^{ik_1 L} = e^{ik_2 L} = 1$$

Which leads to the momentum quantization for particles on a ring of length L without interaction:

$$k_1 L = 2\pi n_1, \qquad k_2 L = 2\pi n_2$$

Where n_1 and n_2 are integers.

Let's compute the energy and momentum of an eigenstate, that is labeled by a set $\{k_j\}$. For the energy, we compute $H\Psi|_{x_1<x_2}$:

$$H\Psi = (-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2})(A_+ e^{i(k_1 x_1 + k_2 x_2)} + A_- e^{i(k_2 x_1 + k_1 x_2)}) = (k_1^2 + k_2^2)\Psi = E\Psi$$

For the total momentum, we compute the eigenvalue of the momentum operator

$$\hat{P} = -i \sum_{j=1}^{N} \frac{\partial}{\partial x_j}$$

In this case:

$$\hat{P}\Psi = -i(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})(A_+ e^{i(k_1 x_1 + k_2 x_2)} + A_- e^{i(k_2 x_1 + k_1 x_2)}) = (k_1 + k_2)\Psi = P\Psi$$

Solution for general N

TODO:

- Why do we make the Bethe Ansatz?

- Why do we assume a superposition of plane waves? - Why do we only consider one ordering of the particles? Explain the symmetry.

Our assumption for the wavefunction is

$$\Psi(x_1, \ldots, x_N | k_1, \ldots, k_N)_{x_1 < x_2 < \cdots < x_N} = \textstyle\sum_{P_N} A_P e^{i \sum_{j=1}^{N} k_{P_j} x_j}$$

Here, P_N denotes the permutations of the set of N integers. The A_P can be derived in the same way as the two particle case (by using the discontinuity in the derivate of Ψ.) The complete expression is:

$$\Psi(x_1, \ldots, x_N) = \prod_{N \geq j > k \geq 1} sgn(x_j - x_k) \times \sum_{P_N} (-1)^{[P]} e^{i \sum_{j=1}^{N} k_{P_j} x_j + \frac{i}{2} \sum_{N \geq j > k \geq 1} sgn(x_j - x_k) \phi(k_{P_j} - k_{P_k})}$$

$$(1.3)$$

The periodicity conditions

$$\Psi(0, x_2, \ldots, x_N | k_1, \ldots, k_N) = \Psi(x_2, \ldots, x_N, L | k_1, \ldots, k_N)$$

Lead to the Bethe equations for general N:

$$e^{ik_jL} = (-1)^{N-1} e^{-i\sum_{l\neq j}\phi(k_j - k_l)} \qquad j = 1, \ldots, N$$

Which are more conveniently expressed in the log form:

$$k_j L = 2\pi I_j - \sum_{l=1}^{N} \phi(k_j - k_l) \qquad j = 1, \ldots, N \tag{1.4}$$

Where the I_j are integers when N is odd and half-integers when N is even. This is easily checked by taking the natural logarithm on both sides of the Bethe equations. From now on, we will use the (half-) integers $I_j$to label the eigenstates.

The energy and the total momentum generalize as follows:

$$E = \sum_{j=1}^{N} k_j^2, \qquad P = \sum_{j=1}^{N} k_j$$

The total momentum can be expressed nicely in terms of I_j. Equation 1.4 divided by L gives us:

$$k_j = \frac{1}{L}(2\pi I_j - \sum_{l=1}^{N} \phi(k_j - k_l))$$

Now, summing over j gives us the total momentum:

$$P = \sum_{j=1}^{N} k_j = \frac{1}{L}\sum_{j=1}^{N}(2\pi I_j - \sum_{l=1}^{N} 2\arctan\frac{k_j - k_l}{c})$$

Here, I used Equation 1.2, the definition of the scattering phase shift. But because $\arctan -x = -\arctan x$, the double sum over the scattering phase shifts gives zero.

$$\sum_{j=1}^{N}\sum_{l=1}^{N} 2\arctan\frac{k_j - k_l}{c} = 0$$

And we are left with:

$$P = \frac{2\pi}{L}\sum_{j=1}^{N} I_j \tag{1.5}$$

The ground state

TODO:

- Refer to notes by Fabio Franchini

- Refer to proof of uniqueness

- No degeneracy means no level crossing because...?

Let's try to describe the ground state. First, note that when two I's are the same, for example $I_1 = I_2$, then also $k_1 = k_2$. This can be seen by subtracting the Bethe equations (Equation 1.4) for $k_1$and k_2.

$$(k_1 - k_2)L = 2\pi(I_1 - I_2) - \sum_{l=1}^{N}\phi(k_1 - k_l) - \phi(k_2 - k_l)$$

Setting $k_1 = k_2$ satisfies the above equation, and since it can be proved (REF) that for a given set of I_j, a unique set of k_j can be found that satisfies the Bethe equations, $k_1 = k_2$ does indeed hold.

If any two of the momentum parameters k_j are the same, Ψ is zero. This can be seen from Equation 1.3, which is asymmetric under the exchange of two momenta. When two k_j are the same, exchanging them obviously does not change the wave function. But since the asymmetry requires an extra minus sign, we have $\Psi = -\Psi$, or $\Psi = 0$.

Physically, this can be interpreted as follows: two particles with the same momentum either never meet or, if they have the same position, always coincide, contributing infinitely to the energy due to the δ-interaction. This gives an non-physical state and has to be avoided.

So this bosonic model has a fermionic character, in the sense that the momentum quantum numbers that label the eigenstates have to be different to get a non-zero wavefunction.

The ground state is labeled by the following I_j:

$$I_j = -\frac{N+1}{2} + j \qquad j = 1, 2, \ldots, N \tag{1.6}$$

See Figure 1.1 for a visual representation.

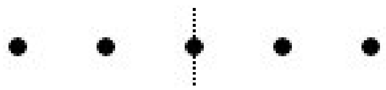

Figure 1.1: A visual representation of the ground state for $N = 5$. The dashed line denotes zero, a black dot represents a (half) integer I_j. The I_j that label this state are -2, -1, 0, 1, 2.

From Equation 1.5 it can be easily checked that this state has zero momentum. To see that this state has the lowest energy, we look first at the case $c \to \infty$. Here, the Bethe equations Equation 1.4 reduce to

$$k_j L = 2\pi I_j$$

because the scattering phase shifts are all zero. The symmetric distribution of I_j Equation 1.6 clearly gives the lowest possible energy. If we now decrease the coupling constant c, the k_j will change values, because the scattering phase shifts are no longer zero. But since the I_j are quantized they cannot change. We have already seen that the ground state cannot be degenerate (because for a set I_j, there is a unique solution for the k_j), so there can't be a level crossing upon changing c. Hence, Equation 1.6 labels the ground state for any c.

Excitations of the ground state

We can identify two fundamental excitations of the ground state: adding a particle with momentum k_p (**Type I**) and creating a hole (removing a particle) with momentum k_h (**Type II**).

Type I excitations

Let's check what happens when we add a particle with a certain momentum $k_p > 0$ to the ground state for $N = 5$. We start with the following quantum numbers:

$$\{I_j\} = \{-2, -1, 0, 1, 2\}$$

Now we add a particle, increasing the number of particles from N to $N + 1$. It is important to realize that this excitation changes the quantum numbers I_j from integers to half integers or the other way around. The new state has quantum numbers

$$\{I_j\} = \{-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} + m\}$$

where $m > 0$ and integer. This new state is created by taking the ground state of 6 particles and increasing by m the momentum of the highest particle. We could also have created the state $\{-\frac{5}{2} - m, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$ by adding a particle with negative momentum. In general, for positive momentum, we go from Equation 1.6, the ground state, to

$$\{I_j\} = \{-\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} + m\}$$

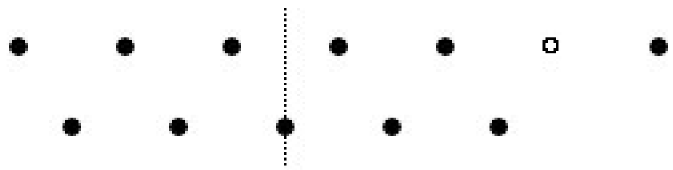

Figure 1.2: Visual representation of a type I excitation of a ground state for $N = 5$, $m = 1$. The bottom row represents the ground state and the top row, which has 6 particles, represents the excited state.

The momentum of the newly created state is

$$P = \frac{2\pi}{L} m \tag{1.7}$$

whereas the ground state had $P = 0$. This momentum change $\frac{2\pi}{L}m$ is different from k_p, because by adding one particle with momentum k_p, we changed the momentum of all the particles. The momentum k_p is called the bare momentum, and P the observed momentum.

Type II excitations

In a type II excitation, we remove a particle with momentum k_h, creating a hole. The number of particles decreases from N to $N-1$. For a hole with positive momentum, we would go from the ground state, Equation 1.6, to the following state:

$$\{I_j\} = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, \frac{N}{2}-m-1, \frac{N}{2}-m+1, \ldots, \frac{N}{2}\}$$

Again, we go from integers to half-integers or vice versa. The new state can be viewed as the ground state for $N-1$ particles, but with one momentum k_h absent, and an extra momentum one level above the highest momentum of the ground state. For our $N=5$ example, we would go from the ground state

$$\{I_j\} = \{-2, -1, 0, 1, 2\}$$

to, for example, the excited state

$$\{I_j\} = \{-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$$

Here, $m=2$, since we miss the $I=\frac{1}{2}$ quantum number. You can check that the dressed momentum is the same as Equation 1.7.

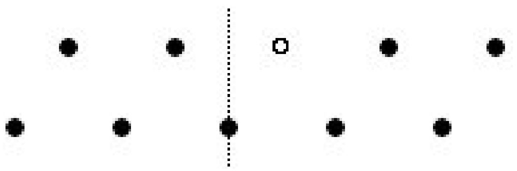

Figure 1.3: Visual representation of a type II excitation for $N=5$, $m=2$. The bottom row represents the ground state and the top row the excited state.