Dynamics of quasi-particle states in a finite one-dimensional repulsive Bose gas

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Abstract

The Lieb-Liniger model for a one-dimensional ultra cold Bose gas is briefly explained. An initial state with a notch in the density profile is studied. It is demonstrated that the time evolution of this initial state resembles a classical relaxation process. Numerical calculations show that the decay of the notch depth obeys a $\frac{1}{t}$ relationship. Using the stationary phase approximation, it is shown that this behaviour is to be expected for large t.

1 Introduction

A recent article [SKD14] drew attention to the dynamics of a quasi particle initial state in a one dimensional ultra cold Bose gas, described by the Lieb-Liniger model. In this report, I have reproduced the results of this article and investigated the dynamics more thoroughly.

2 Description of the Lieb-Liniger model

We will describe an ultra cold Bose gas in one dimension with the Lieb-Liniger model. This is the Hamiltonian in the appropriate units:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j)$$

2.1 The two particle case

Let's start with the two particle case. We get:

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2c\delta(x_1 - x_2) = -\frac{1}{2}(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}) + 2c\delta(\alpha)$$

Here $\alpha = x_1 - x_2$ and $\beta = x_1 + x_2$. If we substitute this into $H\Psi = E\Psi$, and integrate both sides from $-\epsilon$ to ϵ in α, we get:

$$-\frac{1}{2}\int_{-\epsilon}^{\epsilon} d\alpha(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2})\Psi + 2c\int_{-\epsilon}^{\epsilon} d\alpha\delta(\alpha)\Psi = \int_{-\epsilon}^{\epsilon} d\alpha E\Psi$$

Now, taking the limit $\epsilon \to 0$, the right side becomes zero, and we see:

$$\lim_{\epsilon \to 0} -\frac{1}{2}\int_{-\epsilon}^{\epsilon} d\alpha(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2})\Psi + 2c\int_{-\epsilon}^{\epsilon} d\alpha\delta(\alpha)\Psi = 0$$

From which it follows that:

$$\lim_{\epsilon \to 0} 4c\int_{-\epsilon}^{\epsilon} d\alpha\delta(\alpha)\Psi = 4c\Psi(\alpha = 0) = \lim_{\epsilon \to 0}\int_{-\epsilon}^{\epsilon} d\alpha(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2})\Psi$$

$$= \lim_{\epsilon \to 0} \frac{\partial}{\partial \alpha} \Psi(\alpha = \epsilon) - \frac{\partial}{\partial \alpha} \Psi(\alpha = -\epsilon) = \Delta \frac{\partial}{\partial \alpha} \Psi(\alpha = 0)$$

So, for the discontinuity at $x_1 = x_2$ ($\alpha = 0$) we have:

$$\Delta \frac{\partial}{\partial \alpha} \Psi(\alpha = 0) = 4c\Psi(\alpha = 0) \tag{2.1}$$

Assuming the wavefunction:

$$\Psi(x_1, x_2) = \begin{cases} A_+ e^{i(k_1 x_1 + k_2 x_2)} + A_- e^{i(k_2 x_1 + k_1 x_2)} & x_1 < x_2 \\ A_- e^{i(k_1 x_1 + k_2 x_2)} + A_+ e^{i(k_2 x_1 + k_1 x_2)} & x_1 > x_2 \end{cases}$$

and solving Equation 2.1 we find:

$$A_+ = e^{-\frac{i}{2}\phi(k_1 - k_2)}, \qquad A_- = -e^{\frac{i}{2}\phi(k_1 - k_2)}$$

where

$$\phi(k) = 2 \arctan \frac{k}{c} \tag{2.2}$$

is known as the *scattering phase shift*.

Now, if we impose periodic boundary conditions $\Psi(x, 0) = \Psi(x, L)$ (this also assumes a symmetric wavefunction), where L is the system length, we find a quantization for the momenta:

$$e^{ik_1 L} = e^{-i\phi(k_1 - k_2)}, \qquad e^{ik_2 L} = -e^{-i\phi(k_2 - k_1)}$$

The above equations are the Bethe equations for two particles. Let's do a sanity check. In the limit $c \to 0$, we get:

$$e^{ik_1 L} = e^{ik_2 L} = 1$$

Which leads to the momentum quantization for particles on a ring of length L without interaction:

$$k_1L = 2\pi n_1, \qquad k_2L = 2\pi n_2$$

Where n_1 and n_2 are integers.

Let's compute the energy and momentum of an eigenstate, that is labeled by a set $\{k_j\}$. For the energy, we compute $H\Psi|_{x_1<x_2}$:

$$H\Psi = (-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2})(A_+e^{i(k_1x_1+k_2x_2)} + A_-e^{i(k_2x_1+k_1x_2)}) = (k_1^2 + k_2^2)\Psi = E\Psi$$

For the total momentum, we compute the eigenvalue of the momentum operator

$$\hat{P} = -i\sum_{j=1}^{N}\frac{\partial}{\partial x_j}$$

In this case:

$$\hat{P}\Psi = -i(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})(A_+e^{i(k_1x_1+k_2x_2)} + A_-e^{i(k_2x_1+k_1x_2)}) = (k_1 + k_2)\Psi = P\Psi$$

2.2 Solution for general N

For a more complete treatment, see [Fra11]. Our assumption for the wavefunction is

$$\Psi(x_1,\ldots,x_N|k_1,\ldots,k_N)_{x_1<x_2<\cdots<x_N} = \textstyle\sum_{P_N} A_P e^{i\sum_{j=1}^{N} k_{P_j}x_j}$$

Here, P_N denotes the permutations of the set of N integers. The A_P can be derived in the same way as the two particle case (by using the discontinuity in the derivate of Ψ.) The complete expression is:

$$\Psi(x_1,\ldots,x_N) = \prod_{N\geq j>k\geq 1} sgn(x_j-x_k)\times\sum_{P_N}(-1)^{[P]}e^{i\sum_{j=1}^{N}k_{P_j}x_j+\frac{i}{2}\sum_{N\geq j>k\geq 1}sgn(x_j-x_k)\phi(k_{P_j}-k_{P_k})} \tag{2.3}$$

The periodicity conditions

$$\Psi(0, x_2, \dots, x_N | k_1, \dots, k_N) = \Psi(x_2, \dots, x_N, L | k_1, \dots, k_N)$$

Lead to the Bethe equations for general N:

$$e^{ik_jL} = (-1)^{N-1} e^{-i\sum_{l\neq j}\phi(k_j - k_l)} \qquad j = 1, \dots, N$$

Which are more conveniently expressed in the log form:

$$k_jL = 2\pi I_j - \sum_{l=1}^{N} \phi(k_j - k_l) \qquad j = 1, \dots, N \tag{2.4}$$

Where the I_j are integers when N is odd and half-integers when N is even. This is easily checked by taking the natural logarithm on both sides of the Bethe equations. From now on, we will use the (half-) integers $I_j$to label the eigenstates.

The energy and the total momentum generalize as follows:

$$E = \sum_{j=1}^{N} k_j^2, \qquad\qquad P = \sum_{j=1}^{N} k_j$$

The total momentum can be expressed nicely in terms of I_j. Equation 2.4 divided by L gives us:

$$k_j = \frac{1}{L}(2\pi I_j - \sum_{l=1}^{N} \phi(k_j - k_l))$$

Now, summing over j gives us the total momentum:

$$P = \sum_{j=1}^{N} k_j = \frac{1}{L}\sum_{j=1}^{N}(2\pi I_j - \sum_{l=1}^{N} 2\arctan\frac{k_j - k_l}{c})$$

Here, I used Equation 2.2, the definition of the scattering phase shift. But because $\arctan -x = -\arctan x$, the double sum over the scattering phase shifts gives zero.

$$\sum_{j=1}^{N}\sum_{l=1}^{N} 2\arctan\frac{k_j - k_l}{c} = 0$$

And we are left with:

$$P = \frac{2\pi}{L}\sum_{j=1}^{N} I_j \tag{2.5}$$

2.3 The ground state

TODO:

- No degeneracy means no level crossing because...?

Let's try to describe the ground state. First, note that when two I's are the same, for example $I_1 = I_2$, then also $k_1 = k_2$. This can be seen by subtracting the Bethe equations (Equation 2.4) for $k_1$and k_2.

$$(k_1 - k_2)L = 2\pi(I_1 - I_2) - \sum_{l=1}^{N} \phi(k_1 - k_l) - \phi(k_2 - k_l)$$

Setting $k_1 = k_2$ satisfies the above equation, and since it can be proved ([Cau11]) that for a given set of I_j, a unique set of k_j can be found that satisfies the Bethe equations, $k_1 = k_2$ does indeed hold.

If any two of the momentum parameters k_j are the same, Ψ is zero. This can be seen from Equation 2.3, which is asymmetric under the exchange of two momenta. When two k_j are the same, exchanging them obviously does not change the wave function. But since the asymmetry requires an extra minus sign, we have $\Psi = -\Psi$, or $\Psi = 0$.

Physically, this can be interpreted as follows: two particles with the same momentum either never meet or, if they have the same position, always coincide, contributing infinitely to the energy due to the δ-interaction. This gives an non-physical state and has to be avoided.

So this bosonic model has a fermionic character, in the sense that the momentum quantum numbers that label the eigenstates have to be different to get a non-zero wavefunction.

The ground state is labeled by the following I_j:

$$I_j = -\frac{N+1}{2} + j \qquad j = 1, 2, \ldots, N \tag{2.6}$$

See Figure 2.1 for a visual representation.

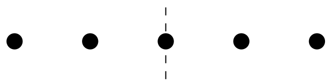

Figure 2.1: A visual representation of the ground state for $N = 5$. The dashed line denotes zero, a black dot represents a (half) integer I_j. The I_j that label this state are -2, -1, 0, 1, 2.

From Equation 2.5 it can be easily checked that this state has zero momentum. To see that this state has the lowest energy, we look first at the case $c \to \infty$. Here, the Bethe equations Equation 2.4 reduce to

$$k_j L = 2\pi I_j$$

because the scattering phase shifts are all zero. The symmetric distribution of I_j Equation 2.6 clearly gives the lowest possible energy. If we now decrease the coupling constant c, the k_j will change values, because the scattering phase shifts are no longer zero. But since the I_j are quantized they cannot change. We have already seen that the ground state cannot be degenerate (because for a set I_j, there is a unique solution for the k_j), so there can't be a level crossing upon changing c. Hence, Equation 2.6 labels the ground state for any c.

2.4 Excitations of the ground state

We can identify two fundamental excitations of the ground state: adding a particle with momentum k_p **(Type I)** and creating a hole (removing a particle) with momentum k_h **(Type II)**.

Type I excitations

Let's check what happens when we add a particle with a certain momentum $k_p > 0$ to the ground state for $N = 5$. We start with the following quantum numbers:

$$\{I_j\} = \{-2, -1, 0, 1, 2\}$$

Now we add a particle, increasing the number of particles from N to $N + 1$. It is important to realize that this excitation changes the quantum numbers I_j from integers to half integers or the other way around. The new state has quantum numbers

$$\{I_j\} = \{-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} + m\}$$

where $m > 0$ and integer. This new state is created by taking the ground state of 6 particles and increasing by m the momentum of the highest particle. We could also have created the state $\{-\frac{5}{2} - m, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$ by adding a particle with negative momentum. In general, for positive momentum, we go from Equation 2.6, the ground state, to

$$\{I_j\} = \{-\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} + m\}$$

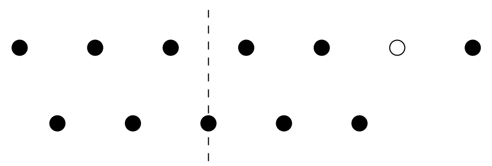

Figure 2.2: Visual representation of a type I excitation of a ground state for $N = 5$, $m = 1$. The bottom row represents the ground state and the top row, which has 6 particles, represents the excited state.

The momentum of the newly created state is

$$P = \frac{2\pi}{L} m \tag{2.7}$$

whereas the ground state had $P = 0$. This momentum change $\frac{2\pi}{L} m$ is different from k_p, because by adding one particle with momentum k_p, we changed the momentum of all the particles. The momentum k_p is called the bare momentum, and P the observed momentum.

Type II excitations

In a type II excitation, we remove a particle with momentum k_h, creating a hole. The number of particles decreases from N to $N-1$. For a hole with positive momentum, we would go from the ground state, Equation 2.6, to the following state:

$$\{I_j\} = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, \frac{N}{2}-m-1, \frac{N}{2}-m+1, \ldots, \frac{N}{2}\}$$

Again, we go from integers to half-integers or vice versa. The new state can be viewed as the ground state for $N-1$ particles, but with one momentum k_h absent, and an extra momentum one level above the highest momentum of the ground state. For our $N=5$ example, we would go from the ground state

$$\{I_j\} = \{-2, -1, 0, 1, 2\}$$

to, for example, the excited state

$$\{I_j\} = \{-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$$

Here, $m=2$, since we miss the $I=\frac{1}{2}$ quantum number. You can check that the dressed momentum is the same as Equation 2.7.

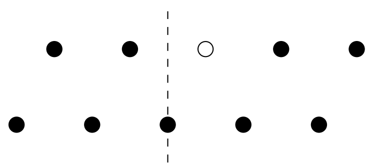

Figure 2.3: Visual representation of a type II excitation for $N=5$, $m=2$. The bottom row represents the ground state and the top row the excited state.

3 Density profile of a quasiparticle state

3.1 Definition

In this chapter we'll define a quasiparticle state, or a density notch state. This is a state that is localized in position space: it has a sharp notch in the density. To build such a state from momentum eigenstates, we have to sum over them. Let $|P\rangle$ be the type II excitation with total (dressed) momentum $\frac{2\pi p}{L}$, as in Equation 2.7. The density notch state $|\Psi\rangle$ for N particles is then defined as:

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{p=-N}^{N} e^{-2\pi i p q/N} |P\rangle \tag{3.1}$$

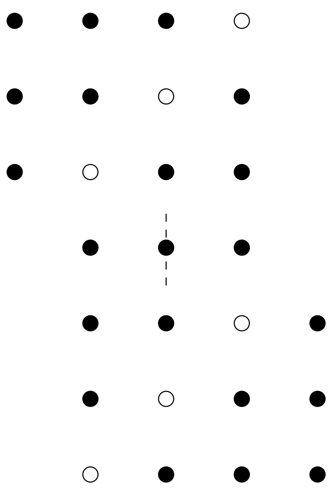

Figure 3.1: Visual representation of the density notch state for $N = 3$ particles. The state $|0\rangle$ with zero momentum is the ground state, drawn on row 4.

This gives a state with a density profile that has a notch at position $\frac{qL}{N} + \frac{L}{2}$ [SKKD12a]. The factor $e^{-2\pi i p q/N}$ in each term acts as a displacement operator. We will set $q = 0$ from now on, so the expression reduces to

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{p=-N}^{N} |P\rangle$$

To see intuitively why summing over momentum eigenstates gives a state that is localized in position, consider the expansion of a general state $|\Psi\rangle$ in eigenstates of the position operator:

$$|\psi\rangle = \int_{-\infty}^{\infty} dx |x\rangle\langle x|\psi\rangle \tag{3.2}$$

Here, $\langle x|\psi\rangle$ is the probability amplitude for the state $|\psi\rangle$. This means $|\langle x|\psi\rangle|^2$ is the probability to get a value between x and $x + dx$ when the position is measured. $\langle x|\psi\rangle$ is usually denoted by $\psi(x)$.

Now, consider a momentum eigenstate

$$\hat{p}|p\rangle = p|p\rangle$$

Here, p is the eigenvalue of the operator $\hat{p}$. The representation of this state in terms of position eigenstates,$\langle x|p\rangle$, satisfies the following relation:

$$\hat{p}\langle x|p\rangle = -i\frac{\partial}{\partial x} = p\langle x|p\rangle$$

Evidently, $\langle x|p\rangle$ is of the form $C(p)e^{ipx}$. Now, to write a general state $|\psi\rangle$ in the momentum basis, take the inner product with $\langle p|$ on both sides of Equation 3.2. $\langle p|x\rangle$, being the conjugate of $\langle x|p\rangle$, can be subsituted by $C(p)e^{-ipx}$.

$$\langle p|\psi\rangle = \int_{-\infty}^{\infty} dx\langle p|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dxC(p)e^{-ipx}\psi(x)$$

This is a fourier transform. To go from momentum space to position space, like in our case, we can do

$$\langle x|\psi\rangle = \int_{-\infty}^{\infty} dpC(x)e^{ipx}\phi(p)$$

Summing with all coefficients equal to one gives a delta peak.

$$\int_{-\infty}^{\infty} dp e^{2\pi i p x} = \delta(x)$$

Since in our case, we sum over a finite amount of momentum states, we only obtain an approximate localized position state.

3.2 Density profile

The density operator is defined as

$$\hat{\rho}(x) = \frac{1}{L} \sum_{x_j} \delta(x - x_j)$$

We evaluate its expectation value as function of time as:

$$\rho(x,t) = \langle \Psi | \hat{\rho}(x) | \Psi \rangle = \frac{1}{N} \sum_{p,p=-N}^{N} \langle P(t) | \hat{\rho}(x) | P'(t) \rangle$$

Now $|P(t)\rangle$ can be written as $e^{-iE_p t}|P\rangle$ and we apply the translation operator

$$\hat{T}(a) = e^{-ia\hat{p}}$$

to shift the eigenstates $|P\rangle$ to the origin by an amount x. We obtain:

$$\rho(x,t) = \frac{1}{N} \sum_{p,p=-N}^{N} \langle P | \hat{\rho}(0) | P' \rangle e^{i(P-P')x - i(E_p - E_{p'})t} \tag{3.3}$$

The form factors $\langle P|\hat{\rho}(0)|P\rangle$ can be calculated by Slavnov's formula [Sla89], [Sla90], and the Gaudin-Korepin norm formula [KBI93]. The energy eigenvalues of the one-hole excitations E_p can be calculated by solving the Bethe equations numerically. Both the form factors and the energy eigenvalues have been computed with the ABACUS library, written by Jean-Sébastien Caux.

By numerically evaluating Equation 3.3, a movie of the density profile can be created.

The density notch collapses and vanishes into the sea of particles. With this data, the findings of [SKD14] have been reproduced, but with a symmetric summation of momentum eigenstates (Equation 3.1), instead of a summation of only one-hole excitations with positive momentum, i.e.

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{p=1}^{N} |P\rangle \tag{3.4}$$

3.3 Notch depth as function of time

We define the notch depth d as the equilibrium density minus lowest density. The density notch is initially located at $\frac{L}{2}$ (since we have set $q = 0$), but splits into two notches that move into opposite directions with speed approximately $\frac{2\pi}{L}$. This approximation holds as long as the notch is has not collapsed, i.e. the quasi-particle is intact. By taking the limit $c \to 0$, a non-collapsing notch can be created that shows soliton like behaviour, such as constant speed [SKKD12b]. The initial state in the referenced article is of the form Equation 3.4, so its density profile features a notch that travels in the positive direction.

Quasi-particle states with a higher interaction parameter c show a faster decay.

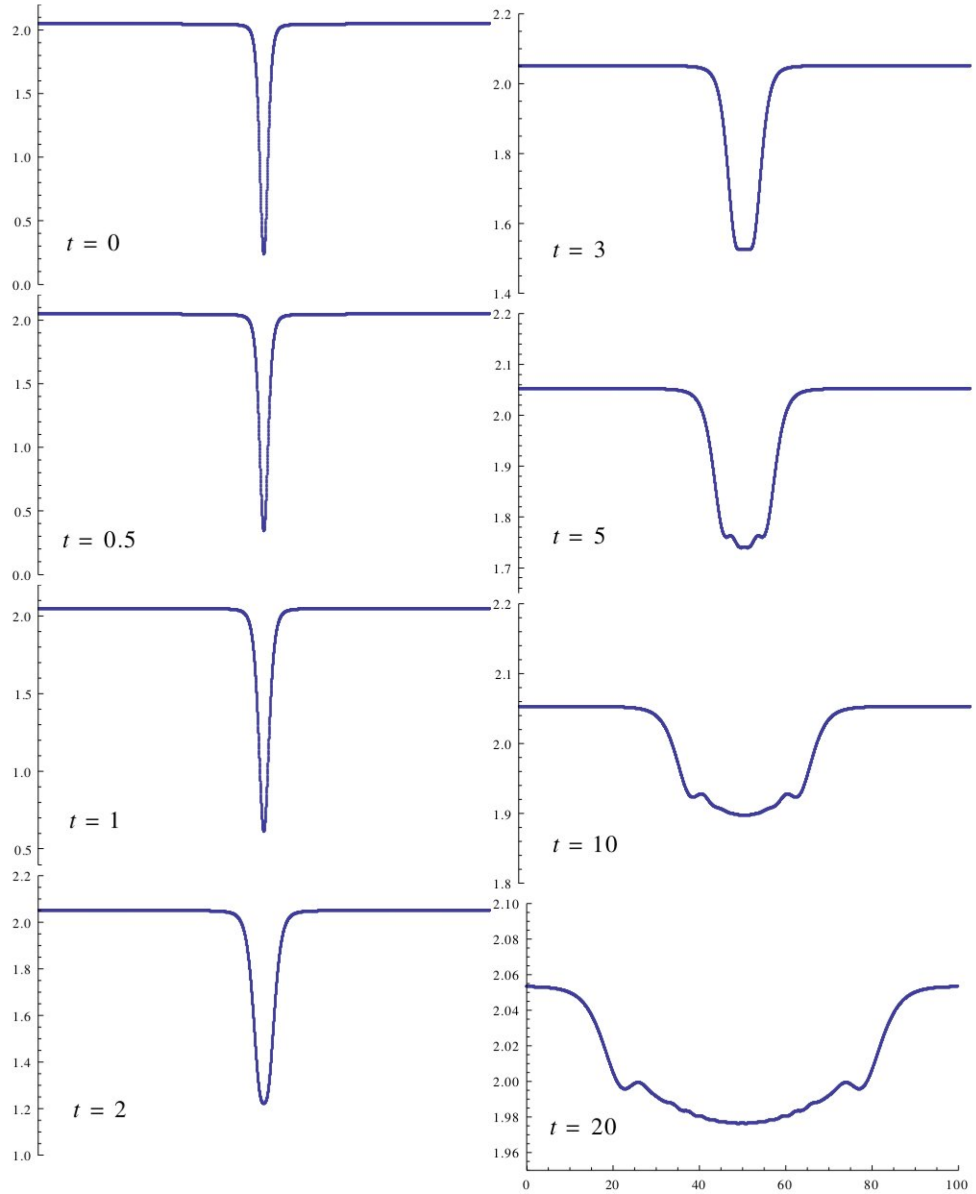

Figure 3.2: Scatter plot of the density profile for $N = L = 100$, $c = 1$.

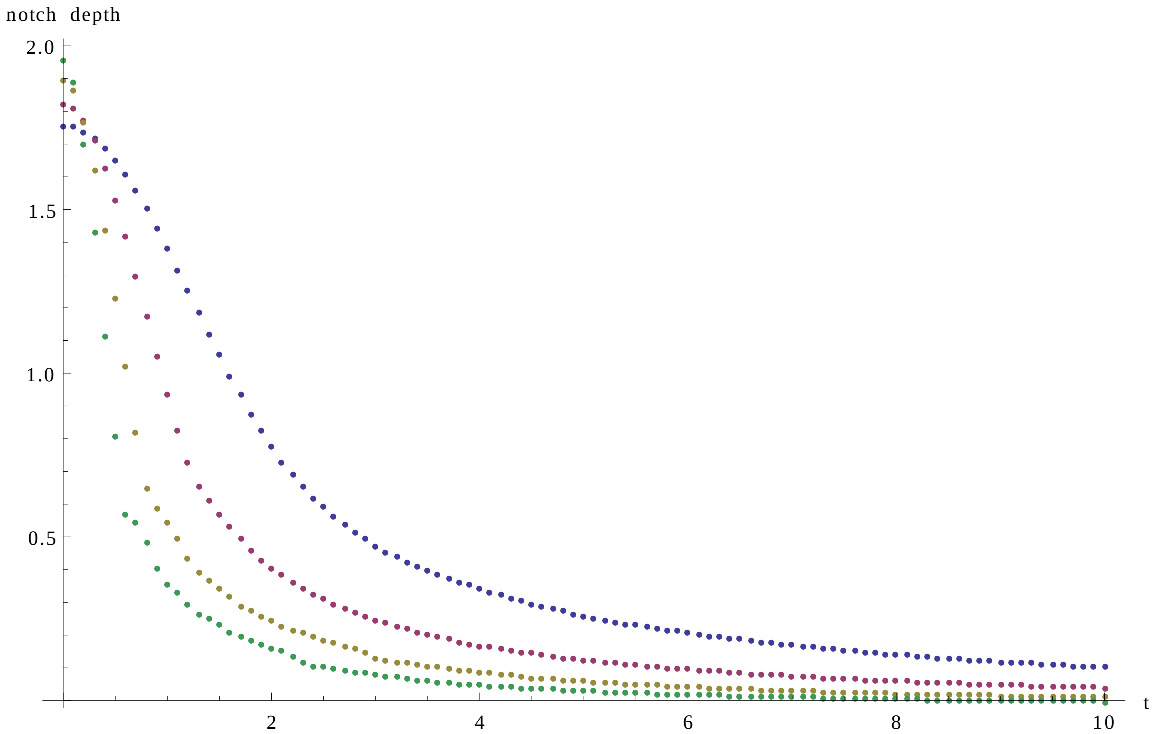

Figure 3.3: Notch depth decay for $N = L = 100$, $c = 1$ (blue dots), $c = 2$ (red dots), $c = 4$ (yellow dots), $c = 8$ (green dots).

4 Decay behaviour of the quasi-particle

In this chapter we will investigate the decay behaviour shown in Figure 3.3. We will show that for large t, a $\frac{1}{t}$ decay is expected. We need to evaluate the expression:

$$\rho(x,t) = \frac{1}{N}\sum_{p,p=-N}^{N} \langle P|\hat{\rho}(0)|P'\rangle e^{i(P-P')x-i(E_p-E_{p'})t}$$

First, we will approximate this double sum as a double integral.

$$\frac{1}{N}\sum_{p,p=-N}^{N} \langle P|\hat{\rho}(0)|P'\rangle e^{i(P-P')x-i(E_p-E_{p'})t} = \frac{1}{N}\int_{\mathbb{R}^2} dpdp' A_p A_{p'} e^{-iE(p)t} e^{iE(p')t} \tag{4.1}$$

This integral can be approximated with a stationary phase approximation, which we will briefly review.

4.1 Stationary phase approximation

We can approximate oscillatory integrals of the following form:

$$\int dpg(p)e^{itf(p)}, \qquad t \gg 1 \tag{4.2}$$

We will take advantage of the fact that in regions where $\frac{\partial f}{\partial p} \neq 0$, the oscillating function $e^{itf(p)}$ kills the integral by destructive interference. So only the tiny regions around certain dominant frequencies ω_i at which $\frac{\partial f}{\partial p} = 0$ contribute to the value of the integral. In these regions, $g(p)$ is essentially constant at $g(\omega_i)$ and we Taylor-expand $f(p)$ to second order about ω_i:

$$f(p) = f(\omega_i) + \frac{f''(\omega_i)}{2}(p - \omega_i)^2$$

Here, the first order term is zero by definition of the points ω. Now, Equation 4.2 can be approximated as:

$$\sum_i g(\omega_i) e^{itf(\omega_i)} \int_{\mathbb{R}} dp e^{it\frac{f''(\omega_i)}{2}(p-\omega_i)^2}$$

When t is large, even a small difference $p - \omega_i$ leads to a highly oscillating integrand, resulting in no contribution to the total value of the integral. This allows us to freely expand the limits of the resulting integrals to $-\infty$ and ∞. They are easily evaluated (after shifting $\bar{p} = p - \omega_i$):

$$\int_{\mathbb{R}} d\bar{p} e^{it\frac{f''(\omega_i)}{2}\bar{p}^2} = \sqrt{\frac{2\pi}{-itf''(\omega_i)}}$$

4.2 Approximation of decay behaviour using the stationary phase approximation

Now we will use a stationary phase approximation to show that the notch depth decays as $\frac{1}{t}$ for large t. Equation 4.1 consists of two integrals of the form:

$$\int_{\mathbb{R}} dp A(p) e^{iE(p)t} \tag{4.3}$$

Figure 4.1 shows that our oscillatory term $E(p)$ has two dominant frequencies at which $\frac{dE}{dp} = 0$. We label them p_+ and p_-. We will focus on the frequency p_+ and keep in mind that the reasoning for p_- is analogous.

We expand E about $p = p_+$ to second order:

$$E(p) \approx E(p_+) + \frac{1}{2}\frac{d^2E}{dp^2}(p_+)(p - p_+)^2 \equiv E(p_+) + \frac{1}{2}\alpha(p_L - p)^2$$

Here, we have defined $\alpha \equiv \frac{d^2E}{dp^2}(p_+)$. We now have the integral:

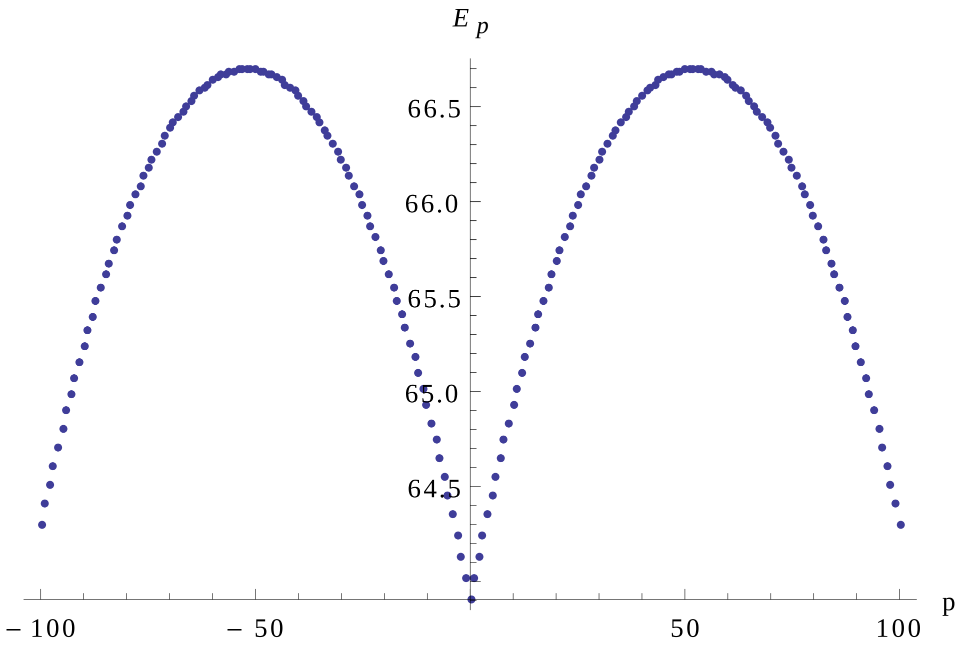

Figure 4.1: Energy spectrum of the eigenstates of the $N = L = 100$, $c = 1$ quasi particle state.

$$\int_{\mathbb{R}} dp A(p_+) e^{it[E(p_+)+\frac{1}{2}\alpha(p-p_+)^2]} = \# \int_{\mathbb{R}} dp e^{\frac{it\alpha}{2}p^2}$$

$\#$ denotes different real numbers that are not important in our reasoning. In the second step, we shifted the integral from $p - p_L$ to p. Now a rescaling $\bar{p} = p\sqrt{\alpha t}$ yields:

$$\frac{\#}{\sqrt{t}} \int_{\mathbb{R}} d\bar{p} e^{\frac{i}{2}\bar{p}^2}$$

Since we are left with an integral that does not depend on t, we can conclude that our original expression, Equation 4.3, has a $\frac{1}{\sqrt{t}}$ dependence. The expression for the density,Equation 4.1 consists of two such integrals, so we expect the density to decay as $\frac{1}{t}$ in the large t limit.

4.3 $\frac{1}{t}$ fit to numerically calculated notch depth

Figure 4.2 shows a fit to the model $\frac{A}{t+B}$, characterized by the parameters A and B.

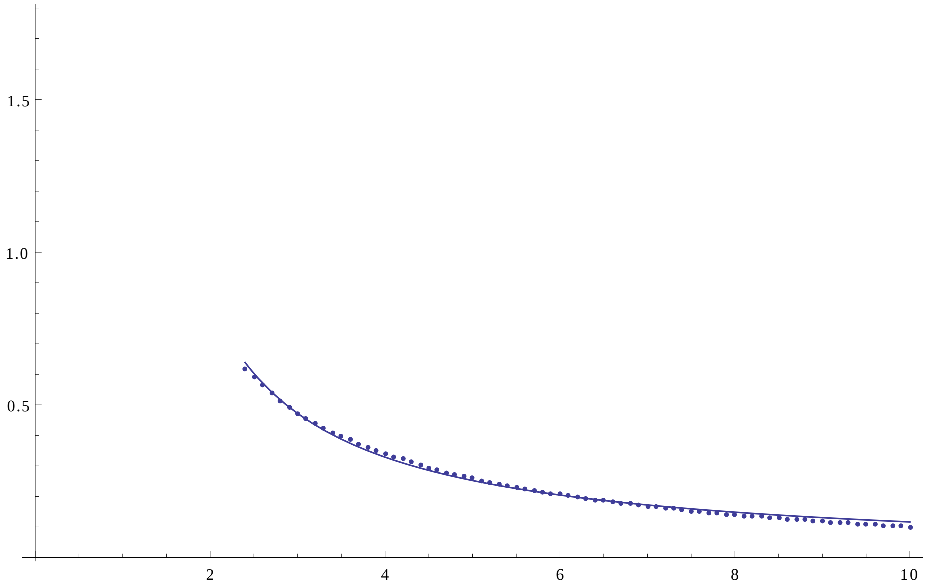

Figure 4.2: Fit to numerically calculated values of the notch depth for $N = L = 100$, $c = 1$. Only the values for t for which the peak is smaller than $\frac{1}{e}$ times the original depth have been taken into account.

c	A	B
1	2.12261	0.958587
2	1.18083	0.517246
4	0.743117	0.32157
8	0.524958	0.227496

Table 4.1: Values for fit of a $\frac{A}{t+B}$ model for different values of the interaction parameter c. $N = L = 100$ for all values.

5 Conclusion

An initial state of a one dimensional repulsive Bose gas with notch in its density profile is studied. With a stationary phase approximation, it is shown that a $\frac{1}{t}$ relationship for the relaxation process is expected. Numerically calculated values show a good fit to this approximation.

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