Contents

1

Introduction to tensor networks

1.1 Tensors, or multidimensional arrays

Make clear that there is a difference between tensors that are defined with a metric.

In the field of tensor networks, a tensor is a multidimensional table with numbers – a convenient way to organize information. It is the generalization of a vector

$$v_i = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \tag{1.1}$$

which has one index, and a matrix

$$M_{ij} = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \dots & M_{mn} \end{bmatrix} \tag{1.2}$$

which has two. A tensor of rank N has N indices:[1]

$$T_{i_1 \dots i_N} \tag{1.3}$$

A tensor of rank zero is just a scalar.

[1] The definition of rank in this context is not to be confused with the rank of a matrix, which is the number of linearly independent columns. Synonyms of tensor rank are tensor degree and tensor order.

1.2 Tensor contraction

Tensor contraction is the higher-dimensional generalization of the dot product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_i a_i b_i \tag{1.4}$$

where a lower-dimensional tensor (in this case, a scalar, which is a zero-dimensional tensor) is obtained by summing over all values of a repeated index.

Examples are matrix-vector multiplication

$$(M\boldsymbol{a})_i = \sum_j M_{ij} a_j \tag{1.5}$$

and matrix-matrix multiplication

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}, \tag{1.6}$$

but a more elaborate tensor multiplication could look like

$$w_{abc} = \sum_{d,e,f} T_{abcdef} v_{def}. \tag{1.7}$$

As with the dot product between vectors, matrix-vector multiplication and matrix-matrix multiplication, a contraction between tensors is only defined if the dimensions of the indices match.

1.3 Tensor networks

A tensor network is specified by a set of tensors, together with a set of contractions to be performed. For example:

$$M_{ab} = \sum_{i,j,k} A_{ai} B_{ij} C_{jk} D_{kb} \tag{1.8}$$

which corresponds to the matrix product $ABCD$.

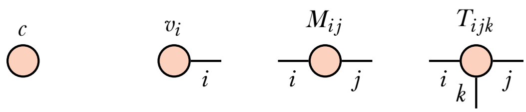

Figure 1.1: Open-ended lines, called legs, represent unsummed indices. A tensor with no open legs is a scalar.

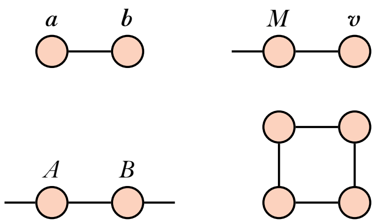

Figure 1.2: Connected legs represent contracted indices. The networks in the figure represent $\sum_i a_i b_i$ (dot product), $\sum_j M_{ij} a_j$ (matrix-vector product), $\sum_k A_{ik} B_{kj}$ (matrix-matrix product) and $\mathrm{Tr}\, ABCD$, respectively.

1.3.1 Graphical notation

It is highly convenient to introduce a graphical notation that is common in the tensor network community. It greatly simplifies expressions and makes certain properties manifest.

Each tensor is represented by a shape. Open-ended lines, called legs, represent unsummed indices. See Figure 1.1. If it clear from the context, index labels may be omitted from the open legs.

Each contracted index is represented by a connected line. See Figure 1.2.

Many tensor equations, while burdensome when written out, are readily understood in this graphical way. As an example, consider the matrix trace in Figure 1.2, where its cyclic property is manifest.

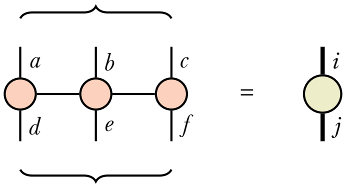

Figure 1.3: Reshaping a tensor T_{abcdef} to T_{ij}

1.3.2 Reshaping tensors

Several indices can be taken together to form a single, joint index, that runs over all combinations of the indices that fused into it. For example, an $m \times n$ matrix can be reshaped into an mn vector.

$$M_{ij} = \boldsymbol{v_a} \qquad a \in \{1, \dots, mn\} \tag{1.9}$$

Some convention has to be chosen to map the joint index i, j onto the single index a, for example

$$a = (n-1)i + j \tag{1.10}$$

which orders the indices of the matrix M row by row

$$11, 12, \dots, 1n, 21, 22, \dots, mn-1, mn \tag{1.11}$$

Graphically, this can be represented as bundling the open legs of a tensor network, as in Figure 1.3.

1.3.3 Computational complexity of contraction

Computational complexity.

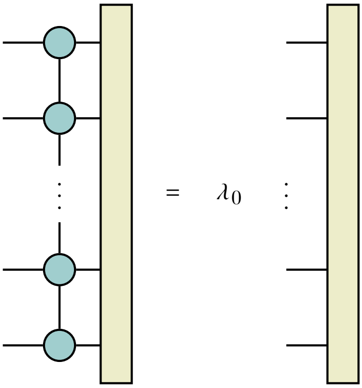

Figure 1.4: Reshaping a tensor T_{abcdef} to T_{ij}

Bibliography

[1] RJ Baxter. "Dimers on a rectangular lattice". In: *Journal of Mathematical Physics* 9.4 (1968), pp. 650–654.

[2] RJ Baxter. "Variational approximations for square lattice models in statistical mechanics". In: *Journal of Statistical Physics* 19.5 (1978), pp. 461–478.

[3] Rodney J Baxter. *Exactly solved models in statistical mechanics*. Elsevier, 1982. Chap. 13.

[4] Tomotoshi Nishino. "Density matrix renormalization group method for 2D classical models". In: *Journal of the Physical Society of Japan* 64.10 (1995), pp. 3598–3601.

[5] Tomotoshi Nishino and Kouichi Okunishi. "Corner transfer matrix renormalization group method". In: *Journal of the Physical Society of Japan* 65.4 (1996), pp. 891–894.