1
Results

1.1 Abstract

Cite more

We present finite-size scaling results using the corner transfer matrix renormalization group method on two-dimensional classical square lattices [4]. We compare the results of conventional finite-size scaling in the system size N with scaling in the number of states kept during the renormalization step of the algorithm, denoted by m. Such a comparison was first done in [3]. We highlight the areas in which method excels over the other.

Calculate critical temperature and exponents using information that is directly extractable from the corner transfer matrix.

1.2 Introduction

The first direct comparison of finite-size scaling in the system size N with scaling in the bond dimension of the corner transfer matrix renormalization group method m was done in [3]. In explaining the basic concepts, we largely follow this paper.

The error in the approximation of the partition function (and thus all thermodynamic quantities) in the thermodynamic limit with the corner transfer matrix method depends on two characteristic length scales. The first is the size of the system N. After n steps of the infinite-system algorithm, we have

$$N = 2n + 1. \tag{1.1}$$

The second length scale is related to the finite bond dimension m. Baxter [1], and later Östlund and Rommer [5] (in the context of one-dimensional quantum systems) showed that in the thermodynamic limit, CTMRG and

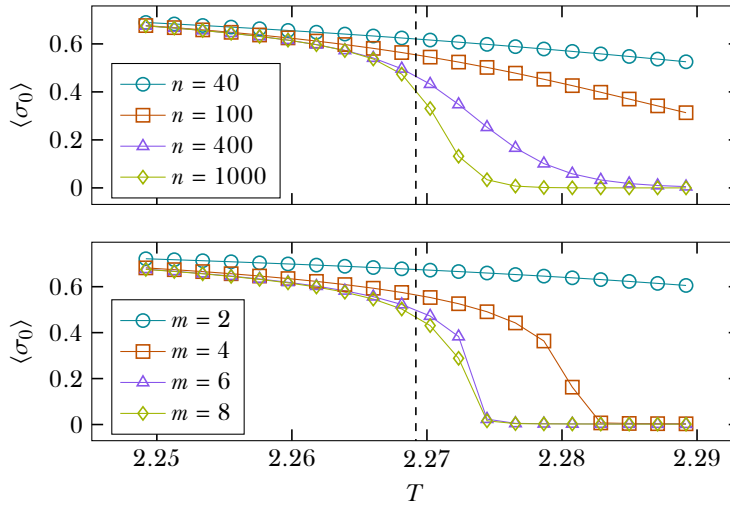

Figure 1.1: Upper panel: expectation value of the central spin $\langle\sigma_0\rangle$ after n CTMRG steps. m is chosen such that the truncation error is smaller than 10^{-6}. Lower panel: $\langle\sigma_0\rangle$ for systems with bond dimension m.

DMRG are variational optimizations in the space of matrix product states.

Can extend this idea a bit.

It is known that an MPS-ansatz with finite bond dimension inherently limits the correlation length of the system to a finite value [6]. Hence, thermodynamic quantities obtained from the CTMRG algorithm with finite m, in the limit $N \to \infty$, cannot diverge and must show finite-size effects similar to those of some effective finite system of size $N_{\text{eff}}(m)$ depending on the bond dimension m.

Figure 1.1 shows the behaviour of the order parameter of the two-dimensional Ising model for systems of finite-size, where m has been chosen such that the truncation error is smaller than 10^{-6}, and for systems of finite m, where the result is converged in the system size N. The results look very similar and support the above claim.

Order parameter is not the same as magnetization central spin. Where to explain this?

In the thermodynamic limit (corresponding to infinite m and N), we have the following expression for the correlation length of a classical system [2]

$$\xi(T) = \frac{1}{\log\left(\frac{T_0}{T_1}\right)}. \tag{1.2}$$

Here, T_0 and T_1 are the largest and second-largest eigenvalues of the row-to-row transfer matrix T, respectively. With N tending towards infinity and finite m, near the critical point $\xi(T)$ should obey a scaling law of the form

$$\xi(T, m) = N_{\text{eff}}(m)\mathcal{F}(N_{\text{eff}}(m)/\xi(T)) \tag{1.3}$$

with

$$\mathcal{F}(x) = \begin{cases} \text{const} & \text{if } x \to 0, \\ x^{-1} & \text{if } x \to \infty. \end{cases} \tag{1.4}$$

Hence, the effective length scale corresponding to the finite bond dimension m is proportional to the correlation length of the system at the critical point $t = 0$.

$$N_{\text{eff}}(m) \propto \xi(T = T_c, m). \tag{1.5}$$

Under this assumption, the order parameter should obey the following scaling relation at the critical temperature

$$M(T = T_c, m) \propto \xi(T = T_c, m)^{-\beta/\nu}. \tag{1.6}$$

The left panel of Figure 1.2 shows that this scaling relation holds. The fit yields $\frac{\beta}{\nu} \approx 0.125(5)$, close to the true value of $\frac{1}{8}$.

The right panel shows the conventional finite-size scaling relation

$$M(T = T_c, N) \propto N^{-\beta/\nu}, \tag{1.7}$$

yielding $\beta/\nu \approx 0.1249(1)$, which can be systematically improved by fitting to larger system sizes, obtained with a fixed truncation error.

In the case of scaling in correlation length $\xi(m)$, the exponent does not improve when taking bigger values of m, while keeping the termination criterion (relative change of singular values) fixed. This points to a flaw in the termination criterion of the algorithm.

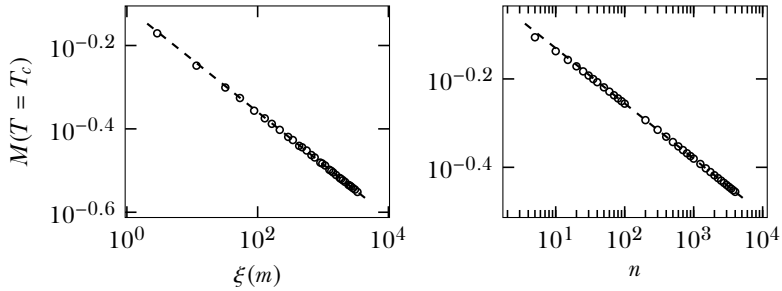

Figure 1.2: Left panel: fit to the relation in Equation 1.6, yielding $\frac{\beta}{\nu} \approx 0.125(5)$. The data points are obtained from simulations with $m = 2, 4, \ldots, 64$. The smallest 10 values of m have not been used for fitting, to diminish correction terms to the basic scaling law. Right panel: fit to conventional finite-size scaling law given in Equation 1.7.

Furthermore, the correlation length $\xi(m)$ shows characteristic half-moon patterns on a log-log scale, stemming from the degeneracies in the corner transfer matrix spectrum. This makes the data harder to interpret, since the effect of increasing m depends on how much of the spectrum is currently retained.

Talk about how to alleviate this partially by using entropy S as length scale.

To further test the hypothesis that N and $\xi(m)$ are the only relevant length scales, the authors of [3] propose a scaling relation for the order parameter M at the critical temperature of the form

$$M(N, m) = N^{-\beta/\nu} \mathcal{G}(\xi(m)/N) \tag{1.8}$$

with

$$\mathcal{G}(x) = \begin{cases} \text{const} & \text{if } x \to \infty, \\ x^{-\beta/\nu} & \text{if } x \to 0, \end{cases} \tag{1.9}$$

meaning that ?? reduces to Equation 1.7 in the limit $\xi(m) \gg N$ and to Equation 1.6 in the limit $N \gg \xi(m)$. Figure 1.3 shows that the scaling relation of Equation 1.8 is justified.

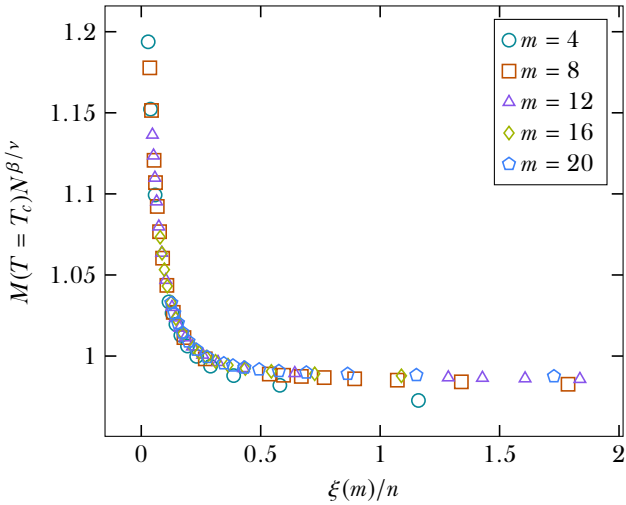

Figure 1.3: Scaling function $\mathcal{G}(\xi(m)/N)$ given in Equation 1.8.

Figure 1.4 shows the cross-over behaviour from the N-limiting regime, where $M(N, m) \propto N^{-\beta/\nu}$ to the $\xi(m)$-limiting regime, where $M(N, m)$ does not depend on N.

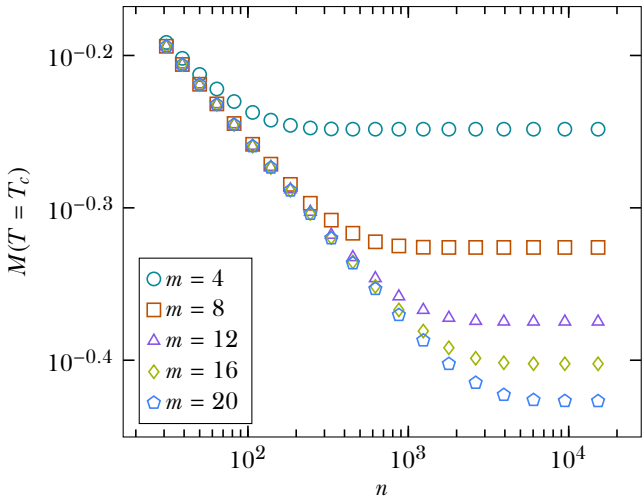

Figure 1.4: Behaviour of the order parameter at fixed m as function of the number of renormalization steps n. For small n, all curves coincide, since the system size is the only limiting length scale. For large enough n, the order parameter is only limited by the length scale $xi(m)$. In between, there is a cross-over described by $\mathcal{G}(\xi(m)/N)$, given in Equation 1.8.

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