Contents

1
Numerical results for the Ising model

abstract

1.1 At the critical point

1.1.1 Existence of two length scales

First, we reproduce the results presented in [1] to validate the assumption that at the critical point, the only relevant length scales are the system size N and the length scale associated to a finite dimension m of the corner transfer matrix $\xi(m)$. Here, we assume that $\xi(m)$ is given by the correlation length at the critical point, see **??**.

The order parameter should obey the following scaling relation at the critical temperature

$$M(T = T_c, m) \propto \xi(T = T_c, m)^{-\beta/\nu}. \tag{1.1}$$

The left panel of Figure 1.1 shows that this scaling relation holds. The fit yields $\frac{\beta}{\nu} \approx 0.125(5)$, close to the true value of $\frac{1}{8}$.

The right panel shows the conventional finite-size scaling relation

$$M(T = T_c, N) \propto N^{-\beta/\nu}, \tag{1.2}$$

yielding $\beta/\nu \approx 0.1249(1)$.

The correlation length $\xi(m)$ shows characteristic half-moon patterns on a log-log scale, stemming from the degeneracies in the corner transfer matrix spectrum. This makes the data harder to interpret, since the effect of increasing m depends on how much of the spectrum is currently retained.

Talk about how to alleviate this partially by using entropy S as length scale.

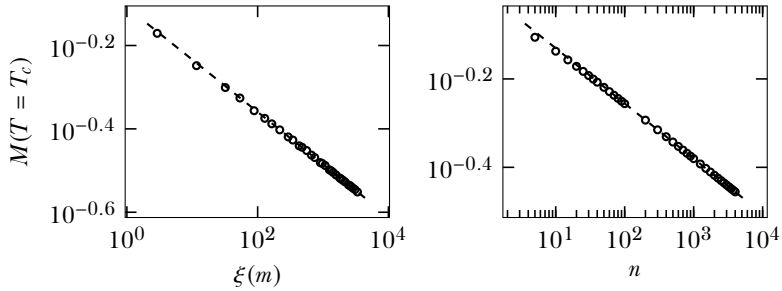

Figure 1.1: Left panel: fit to the relation in Equation 1.1, yielding $\frac{\beta}{\nu} \approx 0.125(5)$. The data points are obtained from simulations with $m = 2, 4, \ldots, 64$. The smallest 10 values of m have not been used for fitting, to diminish correction terms to the basic scaling law. Right panel: fit to conventional finite-size scaling law given in Equation 1.2.

To further test the hypothesis that N and $\xi(m)$ are the only relevant length scales, the authors of [1] propose a scaling relation for the order parameter M at the critical temperature of the form

$$M(N, m) = N^{-\beta/\nu} \mathcal{G}(\xi(m)/N) \tag{1.3}$$

with

$$\mathcal{G}(x) = \begin{cases} \text{const} & \text{if } x \to \infty, \\ x^{-\beta/\nu} & \text{if } x \to 0, \end{cases} \tag{1.4}$$

meaning that Equation 1.3 reduces to Equation 1.2 in the limit $\xi(m) \gg N$ and to Equation 1.1 in the limit $N \gg \xi(m)$. Figure 1.2 shows that the scaling relation of Equation 1.3 is justified.

Figure 1.3 shows the cross-over behaviour from the N-limiting regime, where $M(N, m) \propto N^{-\beta/\nu}$ to the $\xi(m)$-limiting regime, where $M(N, m)$ does not depend on N.

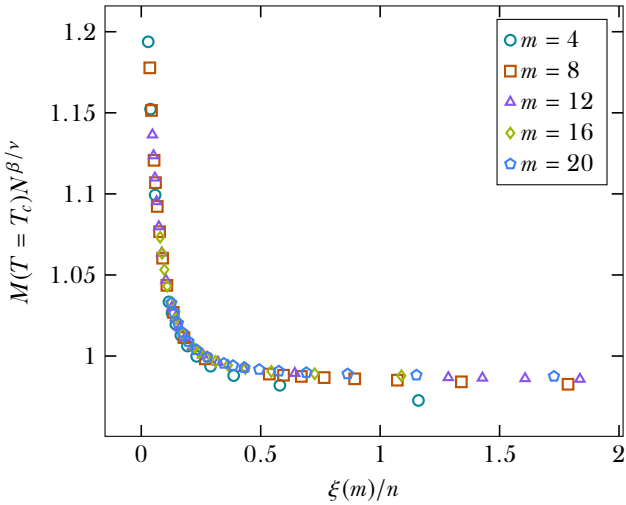

Figure 1.2: Scaling function $\mathcal{G}(\xi(m)/N)$ given in Equation 1.3.

1.1.2 Central charge

We may directly verify the value of the central charge c associated with the conformal field theory at the critical point by fitting to

$$S_{\text{classical}} \propto \frac{c}{6} \log \xi(m), \tag{1.5}$$

which yields $c = 0.501$, shown in the left panel of Figure 2.4.

The right panel of Figure 2.4 shows the fit to the scaling relation in N (or, equivalently the number of CTMRG steps n)

$$S_{\text{classical}} \propto \frac{c}{6} \log N, \tag{1.6}$$

which yields $c = 0.498$.

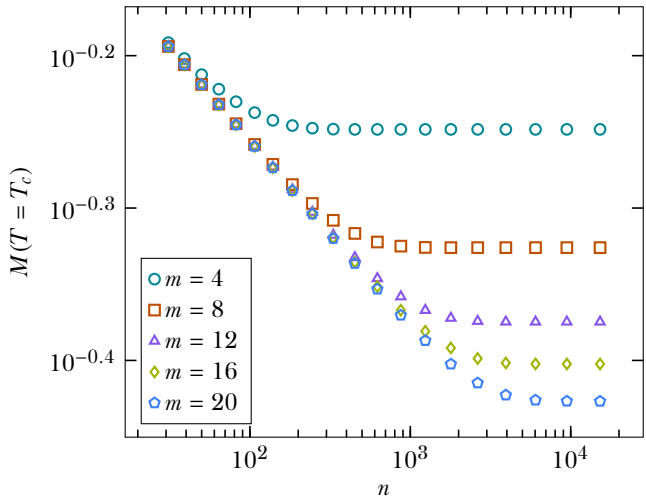

Figure 1.3: Behaviour of the order parameter at fixed m as function of the number of renormalization steps n. For small n, all curves coincide, since the system size is the only limiting length scale. For large enough n, the order parameter is only limited by the length scale $\xi(m)$. In between, there is a cross-over described by $\mathcal{G}(\xi(m)/N)$, given in Equation 1.3.

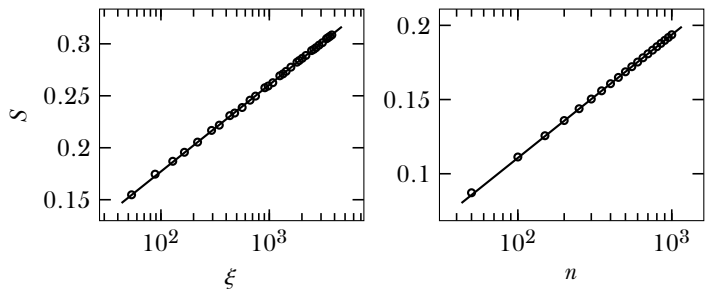

Figure 1.4: Left panel: numerical fit to Equation 2.10, yielding $c = 0.501$. Right panel: numerical fit to Equation 2.11, yielding $c = 0.498$.

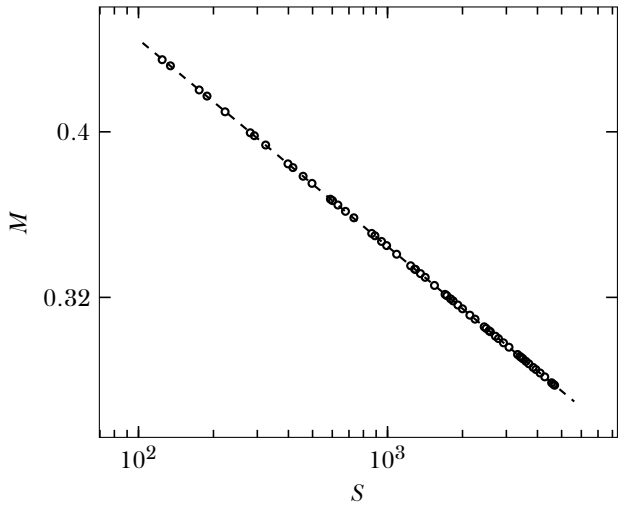

Figure 1.5

1.1.3 Using the entropy to define the correlation length

Via ??, the correlation length is expressed as

$$\xi \propto \exp(\frac{6}{c}S). \tag{1.7}$$

Figure 1.5 shows the results of fitting the relation in Equation 1.1 with this definition of the correlation length. The fit is an order of magnitude better in the least-squares sense, and the half-moon shapes have almost disappeared, yielding a much more robust exponent of $\beta/\nu = 0.12498$.

The entropy uses all eigenvalues of the corner transfer matrix, making it apparently less prone to structure in the spectrum than the correlation length as defined in ??, which uses only two eigenvalues of the row-to-row transfer matrix. Furthermore, the corner transfer matrix A is kept diagonal in the CTMRG algorithm, so S is much cheaper to compute than ξ.

1.1.4 Exponent κ

We now check the validity of the relation

$$\xi(m) \propto m^{\kappa} \tag{1.8}$$

in the context of the CTMRG method for two-dimensional classical systems. Similar checks were done for one-dimensional quantum systems in [2].

Let us first state that boundary conditions are relevant. From (*cite here!!*) we expect that for fixed boundary conditions, the entropy and therefore the correlation length is lower for a given bond dimension m.

There are various ways of extracting the exponent κ. Figure 2.3 shows the results for fixed boundary conditions and Figure 1.7 for free boundary conditions.

Directly checking Equation 1.8 yields $\kappa = 1.93$ for a fixed boundary and $\kappa = 1.96$ for a free boundary.

Under the assumption of Equation 1.8, we have the following scaling laws at the critical point

$$M(m) \propto m^{-\beta\kappa/\nu} \tag{1.9}$$

$$f(m) - f_{\text{exact}} \propto m^{(2-\alpha)\kappa/\nu} \tag{1.10}$$

for the order parameter and the singular part of the free energy, respectively. With a fixed boundary, a fit to $M(m)$ yields $\kappa = 1.93$. For a free boundary we cannot extract any exponent, since $M = 0$ for every temperature. A fit to $f(m) - f_{\text{exact}}$ yields $\kappa = 1.90$ for a fixed boundary and $\kappa = 1.93$ for a free boundary. Figure 2.3. Here, we have used $\beta = 1/8$, $\nu = 1$ and $\alpha = 0$ for the Ising model.

Tell that the κ law is indeed valid, since it is a good fit.

We may use ?? and ?? to check the relation

$$S_{\text{classical}} \propto \frac{c\kappa}{6} \log m, \tag{1.11}$$

which yields $\kappa = 1.93$ for a fixed boundary and $\kappa = 1.96$ for a free boundary, with $c = 1/2$ for the Ising model.

The predicted value for κ [3] is $2.034\ldots$ (see also ??). With the CTMRG method, we extract the slightly lower value of 1.96 (corresponding to free boundary conditions). But, the structure in the quantities as function of m makes it hard to get an accurate fit to κ.

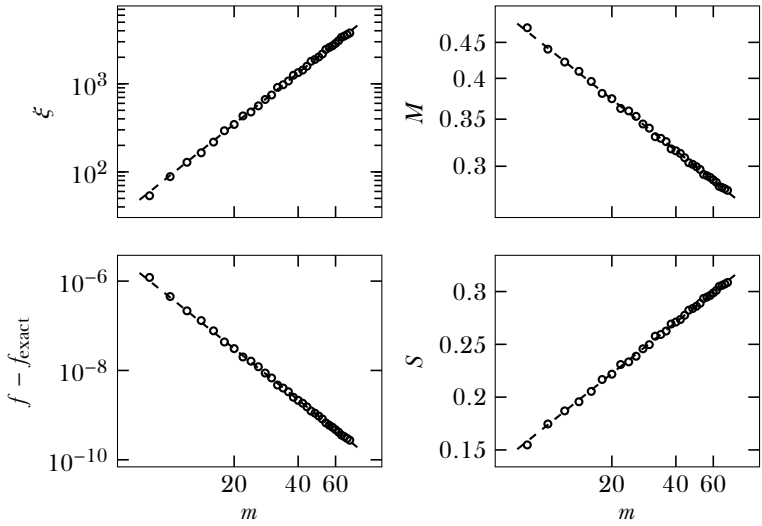

Figure 1.6: Numerical evidence for Equation 1.8, Equation 2.7, Equation 2.9 with fixed boundary, yielding, from left to right and top to bottom, $\kappa = \{1.93, 1.93, 1.90, 1.93\}$.

It is interesting to note that for fixed boundary conditions, the relation in Equation 1.8 holds, but with a lower exponent κ. This is to be expected, since half the spectrum is missing.

1.2 Locating the critical point

1.3 Away from T_c

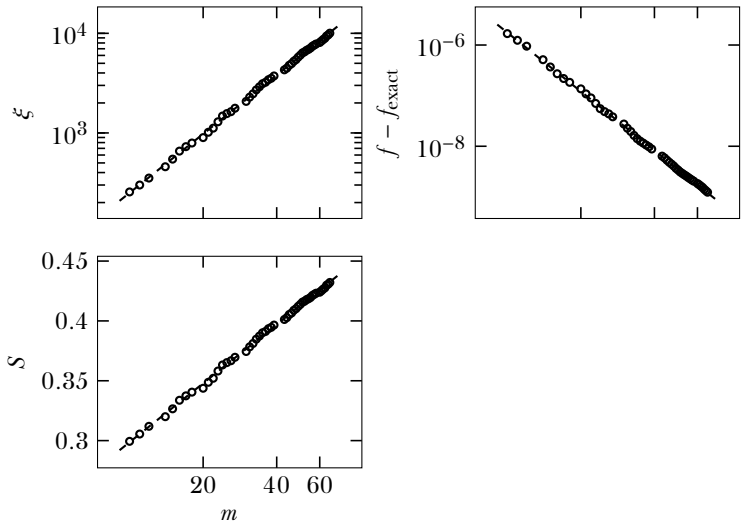

Figure 1.7: Numerical evidence for Equation 1.8 with free boundary, yielding from left to right and then bottom $\kappa = \{1.96, 1.93, 1.96\}$.

2
Results old

2.1 Abstract

Cite more

We present finite-size scaling results using the corner transfer matrix renormalization group method on two-dimensional classical square lattices [4]. We compare the results of conventional finite-size scaling in the system size N with scaling in the number of states kept during the renormalization step of the algorithm, denoted by m. Such a comparison was first done in [1]. We highlight the areas in which method excels over the other.

Calculate critical temperature and exponents using information that is directly extractable from the corner transfer matrix.

2.2 Introduction

The first direct comparison of finite-size scaling in the system size N with scaling in the bond dimension of the corner transfer matrix renormalization group method m was done in [1]. In explaining the basic concepts, we largely follow this paper.

The error in the approximation of the partition function (and thus all thermodynamic quantities) in the thermodynamic limit with the corner transfer matrix method depends on two characteristic length scales. The first is the size of the system N. After n steps of the infinite-system algorithm, we have

$$N = 2n + 1. \tag{2.1}$$

The second length scale is related to the finite bond dimension m. Baxter [5], and later Östlund and Rommer [6] (in the context of one-dimensional quantum systems) showed that in the thermodynamic limit, CTMRG and

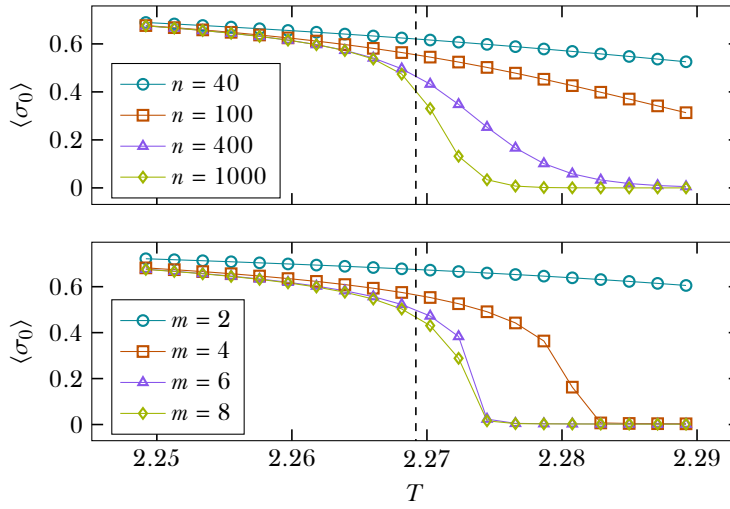

Figure 2.1: Upper panel: expectation value of the central spin $\langle\sigma_0\rangle$ after n CTMRG steps. m is chosen such that the truncation error is smaller than 10^{-6}. Lower panel: $\langle\sigma_0\rangle$ for systems with bond dimension m.

DMRG are variational optimizations in the space of matrix product states.

Can extend this idea a bit.

It is known that an MPS-ansatz with finite bond dimension inherently limits the correlation length of the system to a finite value [7]. Hence, thermodynamic quantities obtained from the CTMRG algorithm with finite m, in the limit $N \to \infty$, cannot diverge and must show finite-size effects similar to those of some effective finite system of size $N_{\text{eff}}(m)$ depending on the bond dimension m.

Figure 2.1 shows the behaviour of the order parameter of the two-dimensional Ising model for systems of finite-size, where m has been chosen such that the truncation error is smaller than 10^{-6}, and for systems of finite m, where the result is converged in the system size N. The results look very similar and support the above claim.

Order parameter is not the same as magnetization central spin. Where to explain this?

2.2.1 Definition of in terms of the correlation length at T_c

In the thermodynamic limit (corresponding to infinite m and N), we have the following expression for the correlation length of a classical system [8]

$$\xi(T) = \frac{1}{\log\left(\frac{T_0}{T_1}\right)}. \tag{2.2}$$

Here, T_0 and T_1 are the largest and second-largest eigenvalues of the row-to-row transfer matrix T, respectively. With N tending towards infinity and finite m, near the critical point $\xi(T)$ should obey a scaling law of the form

$$\xi(T, m) = N_{\text{eff}}(m)\mathcal{F}(N_{\text{eff}}(m)/\xi(T)) \tag{2.3}$$

with

$$\mathcal{F}(x) = \begin{cases} \text{const} & \text{if } x \to 0, \\ x^{-1} & \text{if } x \to \infty. \end{cases} \tag{2.4}$$

Hence, the effective length scale corresponding to the finite bond dimension m is proportional to the correlation length of the system at the critical point $t = 0$.

$$N_{\text{eff}}(m) \propto \xi(T = T_c, m). \tag{2.5}$$

2.2.2 Scaling relations away from the critical point

In general, the position of the critical point is not known. In that situation, the scaling relation in Equation 1.3 cannot be used to calculate thermodynamic information. Instead, in the limit $N \to \infty$, we should have

$$M(t, m) \propto \xi(m)^{-\beta/\nu}\mathcal{P}(t\xi(m)^{1/\nu}), \tag{2.6}$$

which is confirmed in Figure 2.2.

However, in practice this is still problematic, since $\xi(m)$ is defined at the critical point. Thus, we must find a way to define the length scale corresponding to a finite bond dimension m without making use of the position of the critical point.

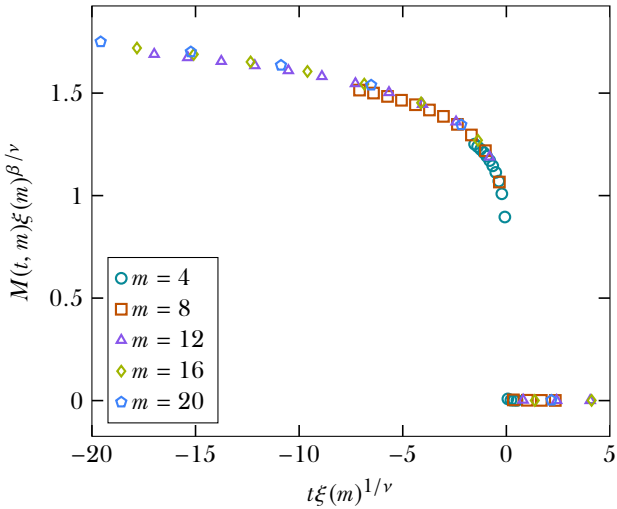

Figure 2.2: Scaling function $\mathcal{P}(t\xi(m)^{1/\nu})$ in Equation 2.6.

2.2.3 Numerical results

We now check the validity of ?? in the context of the CTMRG method for two-dimensional classical systems. Similar checks were done for one-dimensional quantum systems in [2].

Directly checking ?? yields $\kappa = 1.93$, see top left panel of Figure 2.3. Under the assumption of ??, we have the following scaling laws at the critical point

$$M(m) \propto m^{-\beta\kappa/\nu} \tag{2.7}$$

$$f(m) - f_{\text{exact}} \propto m^{(2-\alpha)\kappa/\nu} \tag{2.8}$$

for the order parameter and the singular part of the free energy, respectively. A fit to $M(m)$ yields $\kappa = 1.93$ and a fit to $f(m) - f_{\text{exact}}$ yields $\kappa = 1.90$. See the top right and bottom left panels of Figure 2.3. Here, we have used $\beta = 1/8$, $\nu = 1$ and $\alpha = 0$ for the Ising model.

Tell that the κ law is indeed valid, since it is a good fit.

We may use ?? and ?? to check the relation

$$S_{\text{classical}} \propto \frac{c\kappa}{6} \log m, \tag{2.9}$$

which also yields $\kappa = 1.93$, where $c = 1/2$ for the Ising model. See bottom right panel of Figure 2.3.

We may directly verify the value of the central charge c associated with the conformal field theory at the critical point by fitting to

$$S_{\text{classical}} \propto \frac{c}{6} \log \xi(m), \tag{2.10}$$

which yields $c = 0.501$, shown in the left panel of Figure 2.4.

The right panel of Figure 2.4 shows the fit to the scaling relation in N (or, equivalently the number of CTMRG steps n)

$$S_{\text{classical}} \propto \frac{c}{6} \log N, \tag{2.11}$$

which yields $c = 0.498$.

To verify if the point of maximum entropy

$$T^\star(m) = \max_T S(T, m) \tag{2.12}$$

is a good definition of the pseudocritical point, we fit the relation

$$T^\star - T_c \propto \xi(m)^{-1/\nu}. \tag{2.13}$$

which yields $\widetilde{T}_c = 2.2692$ and $\nu = 0.997$ when the length scale $\xi(T^\star, m)$ is used, shown in the left panel of Figure 2.5. Here, $\widetilde{T}_c$ denotes the critical temperature found by minimising the norm of squares of a fit of the form given in Equation 2.13. In finding the position of the pseudoccritical temperature $T^\star$, a tolerance of 10^{-6} was used.

If, however, the length scale $\xi(m, T_c)$ at the actual critical point is used, a much worse fit is obtained, yielding $\widetilde{T}_c = 2.2691$ and $\nu = 0.90$, shown in the right panel of Figure 2.5.

This signifies the value of $\xi(T^\star, m)$ is heavily dependent on $T^\star$, and using the length scale at the actual pseudocritical temperature found somehow offsets the error on its position.

this is unclear.

Assuming ??, Equation 2.13 becomes

$$T^\star - T_c \propto m^{-\kappa/\nu}, \tag{2.14}$$

which yields *values*, shown in the bottom left panel of Figure 2.5.

As a cross check, we can fit instead to scaling relation of the pseudocritical temperature for finite N

$$T^\star - T_c \propto N^{-1/\nu}, \tag{2.15}$$

yielding *values*. See the bottom right panel of Figure 2.5.

- xi vs m: 1.96
- entropy vs m: 1.96
- free energy diff vs m: 1.93

- validate pseudocritical point by matching it to pseudocritical point given by correlation length and magnetization (how?)
- scaling of pseudocritical point $T^\star - T_c \propto m^{-\kappa/\nu}$.

2.3 To do

Articles to cite:

- [9]: assumes existence of κ and compares finite-size scaling with finite-m scaling for 1D quantum systems with periodic boundary conditions.
- [10]: proves relation for classical eight vertex model

Things to check

- does $S(T^\star, m) \propto \log \xi(T^\star, m)$ hold?
- does $S(T^\star, N) \propto \log N$ hold better than $S(T_c, N) \propto \log N$?
- does fitting $T_c - T^\star(N)$ to N give better results than fitting against $S(T^\star, N)$?
- how does $T^\star(m)$ from entropy compare against $T^\star$ found from max correlation length, or vanishing magnetization?

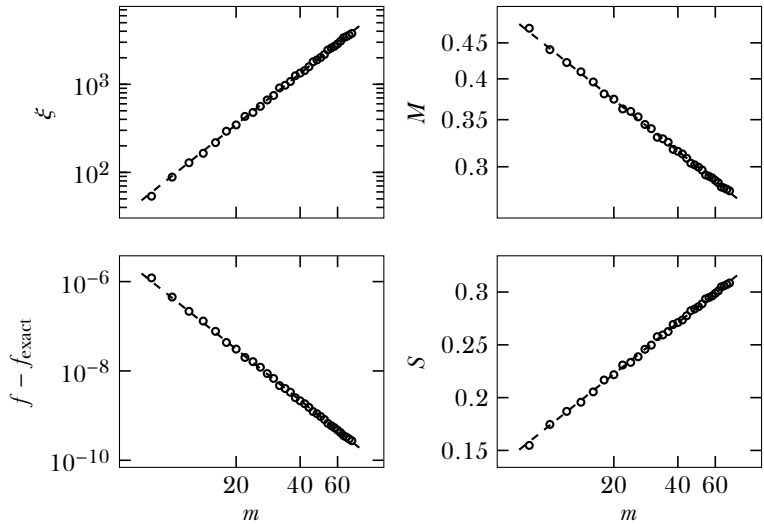

Figure 2.3: Numerical evidence for ??, Equation 2.7, Equation 2.9, yielding, from left to right and top to bottom, $\kappa = \{1.93, 1.93, 1.90, 1.93\}$.

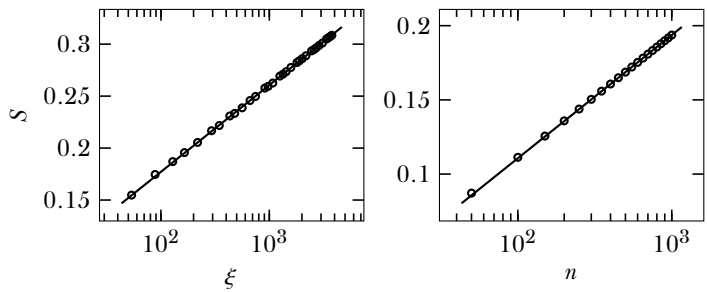

Figure 2.4: Left panel: numerical fit to Equation 2.10, yielding $c = 0.501$. Right panel: numerical fit to Equation 2.11, yielding $c = 0.498$.

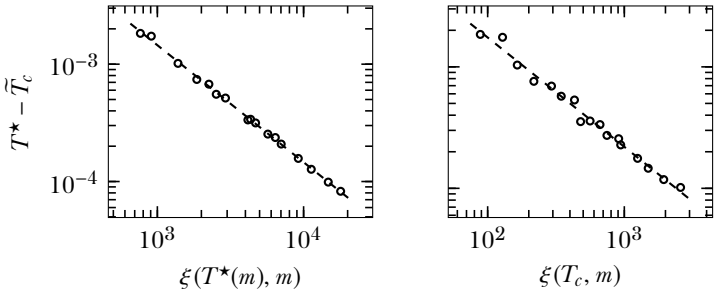

Figure 2.5: Left panel: numerical fit to Equation 2.13 with $\xi(T^\star(m), m)$ used as relevant length scale. Right panel: same fit but using $\xi(T_c, m)$, the correlation length at the exact critical point.

- optimize κ for scaling of pseudo critical point?
- find $T^{\star,N}$ for larger N for ising model.

Plots to make:

- $T^\star - T_c$ vs N
- $S(T_c, N) \propto \log N$
- generally, why should there be a difference between using entropy at critical point vs using entropy at pseudocritical point? How does each one scale?

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