Contents

1

Numerical results for the Ising model

abstract

1.1 At the critical point

1.1.1 Existence of two length scales

First, we reproduce the results presented in [1] to validate the assumption that at the critical point, the only relevant length scales are the system size N and the length scale associated to a finite dimension m of the corner transfer matrix $\xi(m)$. Here, we assume that $\xi(m)$ is given by the correlation length at the critical point, see ??.

The order parameter should obey the following scaling relation at the critical temperature

$$M(T = T_c, m) \propto \xi(T = T_c, m)^{-\beta/\nu}. \tag{1.1}$$

The left panel of Figure 1.1 shows that this scaling relation holds. The fit yields $\frac{\beta}{\nu} \approx 0.125(5)$, close to the true value of $\frac{1}{8}$.

The right panel shows the conventional finite-size scaling relation

$$M(T = T_c, N) \propto N^{-\beta/\nu}, \tag{1.2}$$

yielding $\beta/\nu \approx 0.1249(1)$.

The correlation length $\xi(m)$ shows characteristic half-moon patterns on a log-log scale, stemming from the degeneracies in the corner transfer matrix spectrum. This makes the data harder to interpret, since the effect of increasing m depends on how much of the spectrum is currently retained.

Talk about how to alleviate this partially by using entropy S as length scale.

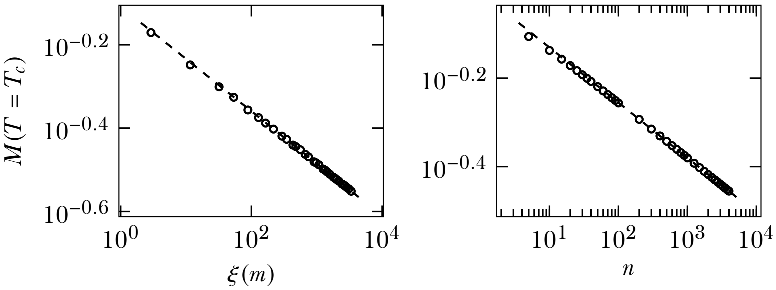

Figure 1.1: Left panel: fit to the relation in Equation 1.1, yielding $\frac{\beta}{\nu} \approx 0.125(5)$. The data points are obtained from simulations with $m = 2, 4, \ldots, 64$. The smallest 10 values of m have not been used for fitting, to diminish correction terms to the basic scaling law. Right panel: fit to conventional finite-size scaling law given in Equation 1.2.

To further test the hypothesis that N and $\xi(m)$ are the only relevant length scales, the authors of [1] propose a scaling relation for the order parameter M at the critical temperature of the form

$$M(N, m) = N^{-\beta/\nu}\mathcal{G}(\xi(m)/N) \tag{1.3}$$

with

$$\mathcal{G}(x) = \begin{cases} \text{const} & \text{if } x \to \infty, \\ x^{-\beta/\nu} & \text{if } x \to 0, \end{cases} \tag{1.4}$$

meaning that Equation 1.3 reduces to Equation 1.2 in the limit $\xi(m) \gg N$ and to Equation 1.1 in the limit $N \gg \xi(m)$. Figure 1.2 shows that the scaling relation of Equation 1.3 is justified.

Figure 1.3 shows the cross-over behaviour from the N-limiting regime, where $M(N, m) \propto N^{-\beta/\nu}$ to the $\xi(m)$-limiting regime, where $M(N, m)$ does not depend on N.

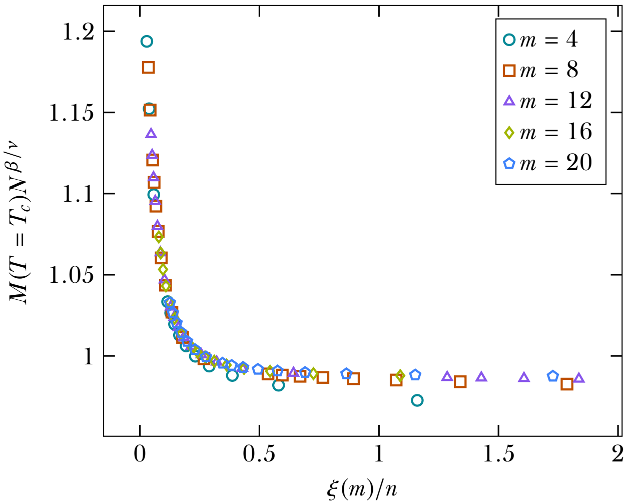

Figure 1.2: Scaling function $\mathcal{G}(\xi(m)/N)$ given in Equation 1.3.

1.1.2 Central charge

We may directly verify the value of the central charge c associated with the conformal field theory at the critical point by fitting to

$$S_{\text{classical}} \propto \frac{c}{6} \log \xi(m), \tag{1.5}$$

which yields $c = 0.501$, shown in the left panel of Figure 1.4.

The right panel of Figure 1.4 shows the fit to the scaling relation in N (or, equivalently the number of CTMRG steps n)

$$S_{\text{classical}} \propto \frac{c}{6} \log N, \tag{1.6}$$

which yields $c = 0.498$.

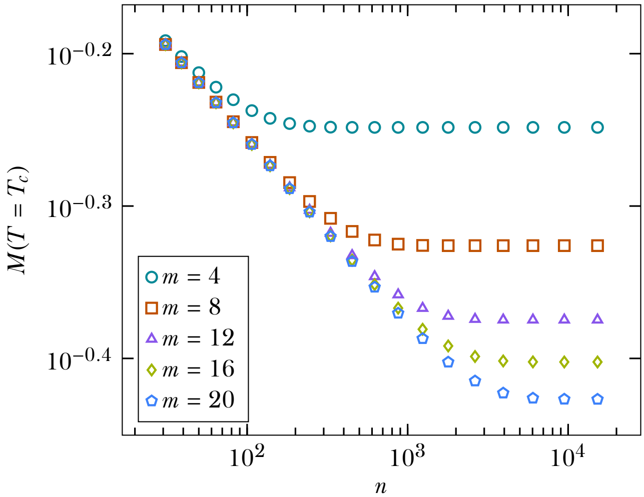

Figure 1.3: Behaviour of the order parameter at fixed m as function of the number of renormalization steps n. For small n, all curves coincide, since the system size is the only limiting length scale. For large enough n, the order parameter is only limited by the length scale $\xi(m)$. In between, there is a cross-over described by $\mathcal{G}(\xi(m)/N)$, given in Equation 1.3.

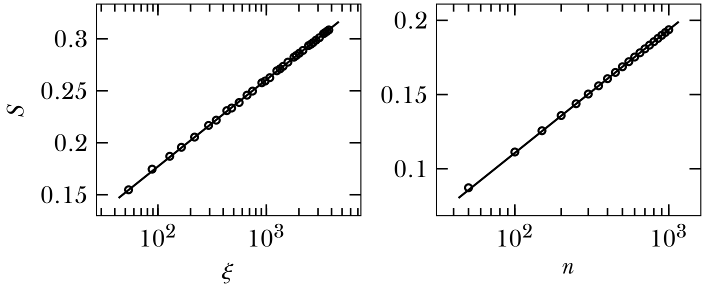

Figure 1.4: Left panel: numerical fit to Equation 1.5, yielding $c = 0.501$. Right panel: numerical fit to Equation 1.6, yielding $c = 0.498$.

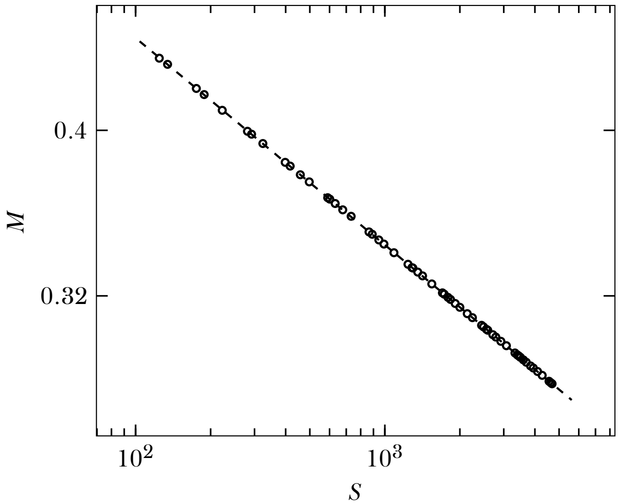

Figure 1.5

1.1.3 Using the entropy to define the correlation length

Via ??, the correlation length is expressed as

$$\xi \propto \exp(\frac{6}{c} S). \tag{1.7}$$

Figure 1.5 shows the results of fitting the relation in Equation 1.1 with this definition of the correlation length. The fit is an order of magnitude better in the least-squares sense, and the half-moon shapes have almost disappeared, yielding a much more robust exponent of $\beta / \nu = 0.12498$.

The entropy uses all eigenvalues of the corner transfer matrix, making it apparently less prone to structure in the spectrum than the correlation length as defined in ??, which uses only two eigenvalues of the row-to-row transfer matrix. Furthermore, the corner transfer matrix A is kept diagonal in the CTMRG algorithm, so S is much cheaper to compute than ξ.

1.1.4 Exponent κ

We now check the validity of the relation

$$\xi(m) \propto m^{\kappa} \tag{1.8}$$

in the context of the CTMRG method for two-dimensional classical systems. Similar checks were done for one-dimensional quantum systems in [2].

Let us first state that boundary conditions are relevant. From (*cite here!!*) we expect that for fixed boundary conditions, the entropy and therefore the correlation length is lower for a given bond dimension m.

There are various ways of extracting the exponent κ. Figure 1.6 shows the results for fixed boundary conditions and Figure 1.7 for free boundary conditions.

Directly checking Equation 1.8 yields $\kappa = 1.93$ for a fixed boundary and $\kappa = 1.96$ for a free boundary.

Under the assumption of Equation 1.8, we have the following scaling laws at the critical point

$$M(m) \propto m^{-\beta\kappa/\nu} \tag{1.9}$$

$$f(m) - f_{\text{exact}} \propto m^{(2-\alpha)\kappa/\nu} \tag{1.10}$$

for the order parameter and the singular part of the free energy, respectively. With a fixed boundary, a fit to $M(m)$ yields $\kappa = 1.93$. For a free boundary we cannot extract any exponent, since $M = 0$ for every temperature. A fit to $f(m) - f_{\text{exact}}$ yields $\kappa = 1.90$ for a fixed boundary and $\kappa = 1.93$ for a free boundary. Figure 1.6. Here, we have used $\beta = 1/8$, $\nu = 1$ and $\alpha = 0$ for the Ising model.

Tell that the κ law is indeed valid, since it is a good fit.

We may use ?? and ?? to check the relation

$$S_{\text{classical}} \propto \frac{c\kappa}{6} \log m, \tag{1.11}$$

which yields $\kappa = 1.93$ for a fixed boundary and $\kappa = 1.96$ for a free boundary, with $c = 1/2$ for the Ising model.

1.1.4.1 Comparison with exact result in asymptotic limit

The predicted value for κ [3] is $2.034\ldots$ (see also ??). With the CTMRG method, we extract the slightly lower value of 1.96 (corresponding to free

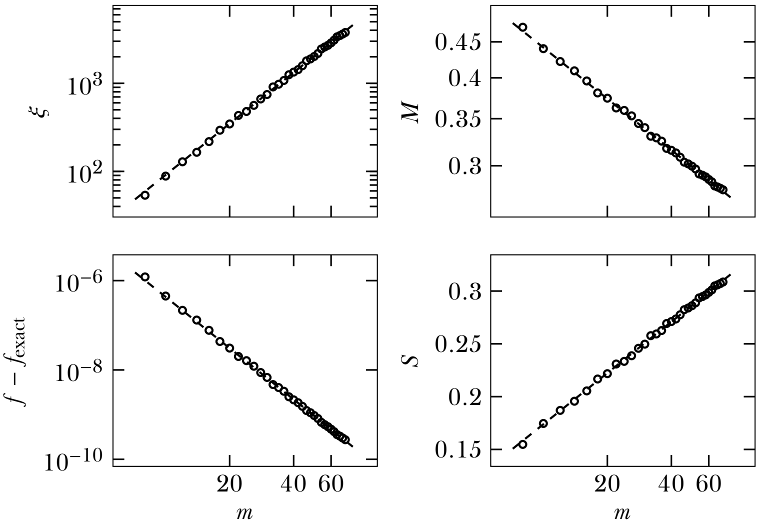

Figure 1.6: Numerical evidence for Equation 1.8, Equation 1.9, Equation 1.11 with fixed boundary, yielding, from left to right and top to bottom, $\kappa = \{1.93, 1.93, 1.90, 1.93\}$.

boundary conditions). But, the structure in the quantities as function of m makes it hard to get an accurate fit to κ.

It is interesting to note that for fixed boundary conditions, the relation in Equation 1.8 holds, but with a lower exponent κ. This is to be expected, since half the spectrum is missing.

phrase this better. Maybe subsubsection?

MUST SAY THAT I LEFT AWAY VALUES OF m in symmetric case!!

1.2 Locating the critical point

In general, the critical point is not known, but it may be located using the spectrum of the corner transfer matrix as described in ??. For a given value

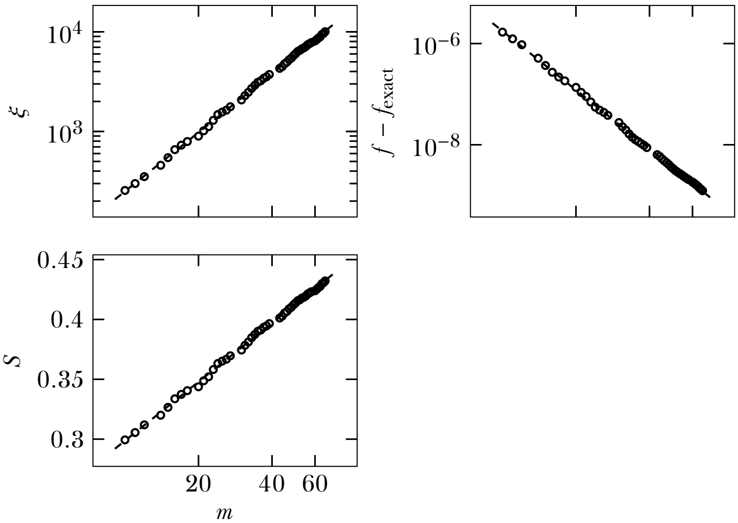

Figure 1.7: Numerical evidence for Equation 1.8 with free boundary, yielding from left to right and then bottom $\kappa = \{1.96, 1.93, 1.96\}$.

of m (or N, in a finite-size approximation), the pseudocritical temperature is defined as the point of maximum entropy.

Figure 1.8 shows the classical analogue to the entanglement entropy as a function of temperature for different values of m.

The critical point is located by fitting the scaling law in ??.

- corr length $T^\star$: $T_c = 2.269183$, $\nu = 1.002$
- corr length at T_c: $T_c = 2.269173$, $\nu = 1.057$
- entropy at $T^\star$: $T_c = 2.269183$, $\nu = 1.02$
- chi: $T_c = 2.269181$, $\kappa/\nu = 1.91$

1.3 Away from T_c

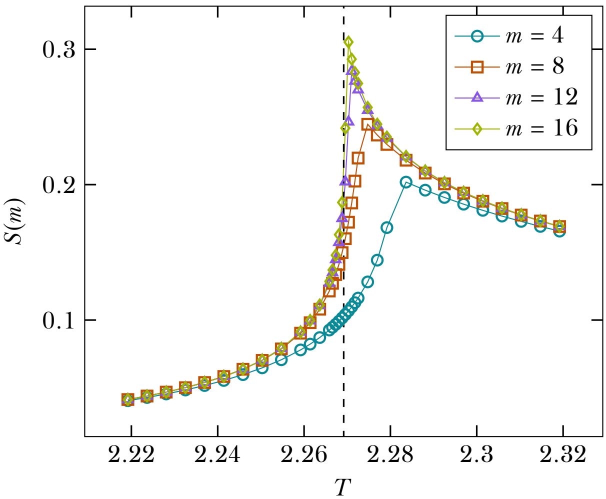

Figure 1.8: Classical analogue to the entanglement entropy, as in ??, near the critical point (shown as dashed line).

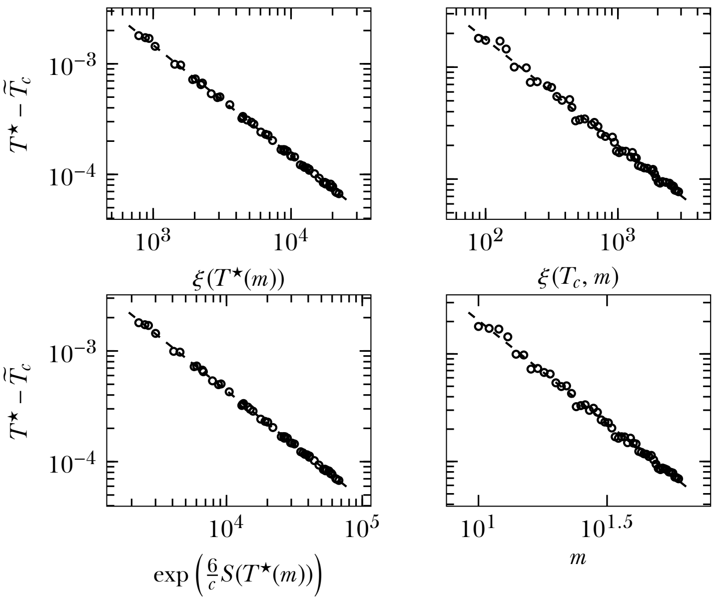

Figure 1.9: Left panel: numerical fit to ?? with $\xi(T^\star(m), m)$ used as relevant length scale. Right panel: same fit but using $\xi(T_c, m)$, the correlation length at the exact critical point.

Bibliography

[1] T Nishino, K Okunishi, and M Kikuchi. “Numerical renormalization group at criticality”. In: *Physics Letters A* 213.1-2 (1996), pp. 69–72.

[2] L Tagliacozzo et al. “Scaling of entanglement support for matrix product states”. In: *Physical review b* 78.2 (2008), p. 024410.

[3] Frank Pollmann et al. “Theory of finite-entanglement scaling at one-dimensional quantum critical points”. In: *Physical review letters* 102.25 (2009), p. 255701.