Contents

1

Numerical results for the Ising model

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abstract

1.1 At the critical point

1.1.1 Existence of two length scales

First, we reproduce the results presented in [1] to validate the assumption that at the critical point, the only relevant length scales are the system size N and the length scale associated to a finite dimension m of the corner transfer matrix $\xi(m)$. Here, we assume that $\xi(m)$ is given by the correlation length at the critical point, see **??**.

The order parameter should obey the following scaling relation at the critical temperature

$$M(T = T_c, m) \propto \xi(T = T_c, m)^{-\beta/\nu}. \tag{1.1}$$

The left panel of Figure 1.1 shows that this scaling relation holds. The fit yields $\frac{\beta}{\nu} \approx 0.125(5)$, close to the true value of $\frac{1}{8}$.

The right panel shows the conventional finite-size scaling relation

$$M(T = T_c, N) \propto N^{-\beta/\nu}, \tag{1.2}$$

yielding $\beta/\nu \approx 0.1249(1)$.

The correlation length $\xi(m)$ shows characteristic half-moon patterns on a log-log scale, stemming from the degeneracies in the corner transfer matrix spectrum. This makes the data harder to interpret, since the effect of increasing m depends on how much of the spectrum is currently retained.

Talk about how to alleviate this partially by using entropy S as length scale.

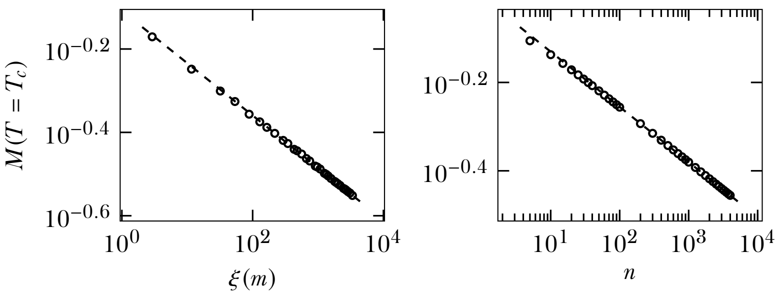

Figure 1.1: Left panel: fit to the relation in Equation 1.1, yielding $\frac{\beta}{\nu} \approx 0.125(5)$. The data points are obtained from simulations with $m = 2, 4, \ldots, 64$. The smallest 10 values of m have not been used for fitting, to diminish correction terms to the basic scaling law. Right panel: fit to conventional finite-size scaling law given in Equation 1.2.

To further test the hypothesis that N and $\xi(m)$ are the only relevant length scales, the authors of [1] propose a scaling relation for the order parameter M at the critical temperature of the form

$$M(N, m) = N^{-\beta/\nu}\mathcal{G}(\xi(m)/N) \tag{1.3}$$

with

$$\mathcal{G}(x) = \begin{cases} \text{const} & \text{if } x \to \infty, \\ x^{-\beta/\nu} & \text{if } x \to 0, \end{cases} \tag{1.4}$$

meaning that Equation 1.3 reduces to Equation 1.2 in the limit $\xi(m) \gg N$ and to Equation 1.1 in the limit $N \gg \xi(m)$. Figure 1.2 shows that the scaling relation of Equation 1.3 is justified.

Figure 1.3 shows the cross-over behaviour from the N-limiting regime, where $M(N, m) \propto N^{-\beta/\nu}$ to the $\xi(m)$-limiting regime, where $M(N, m)$ does not depend on N.

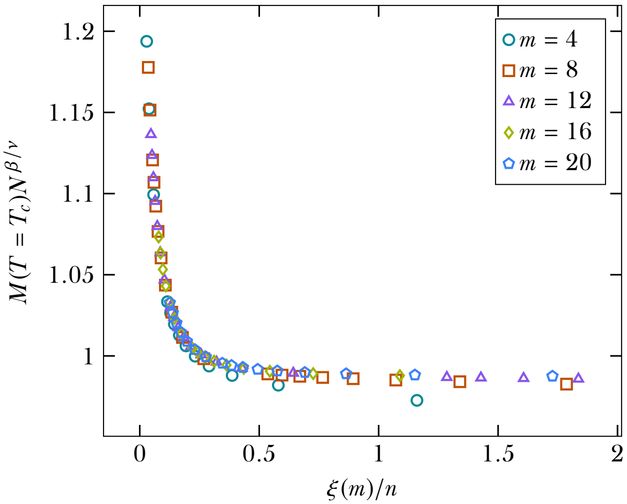

Figure 1.2: Scaling function $\mathcal{G}(\xi(m)/N)$ given in Equation 1.3.

1.1.2 Central charge

We may directly verify the value of the central charge c associated with the conformal field theory at the critical point by fitting to

$$S_{\text{classical}} \propto \frac{c}{6} \log \xi(m), \tag{1.5}$$

which yields $c = 0.501$, shown in the left panel of Figure 1.4.

The right panel of Figure 1.4 shows the fit to the scaling relation in N (or, equivalently the number of CTMRG steps n)

$$S_{\text{classical}} \propto \frac{c}{6} \log N, \tag{1.6}$$

which yields $c = 0.498$.

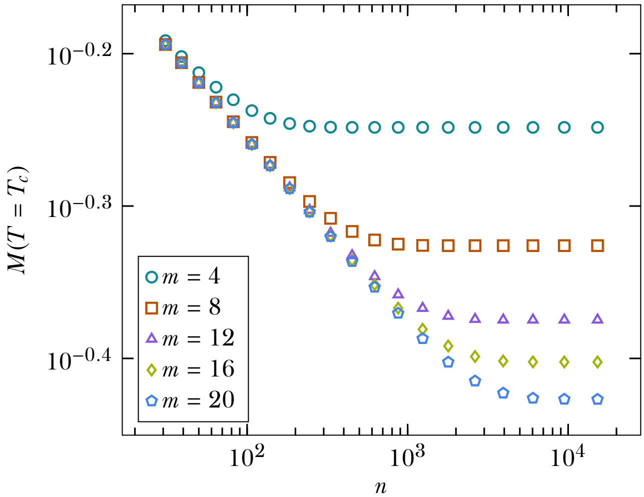

Figure 1.3: Behaviour of the order parameter at fixed m as function of the number of renormalization steps n. For small n, all curves coincide, since the system size is the only limiting length scale. For large enough n, the order parameter is only limited by the length scale $\xi(m)$. In between, there is a cross-over described by $\mathcal{G}(\xi(m)/N)$, given in Equation 1.3.

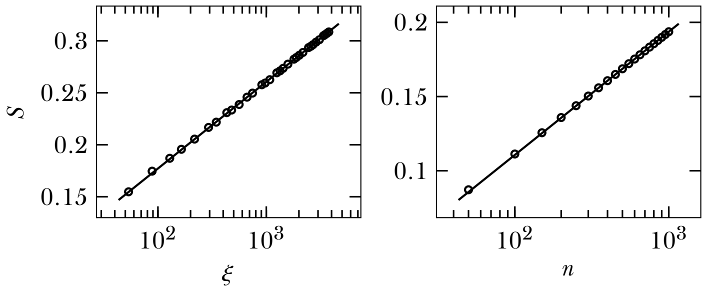

Figure 1.4: Left panel: numerical fit to Equation 1.5, yielding $c = 0.501$. Right panel: numerical fit to Equation 1.6, yielding $c = 0.498$.

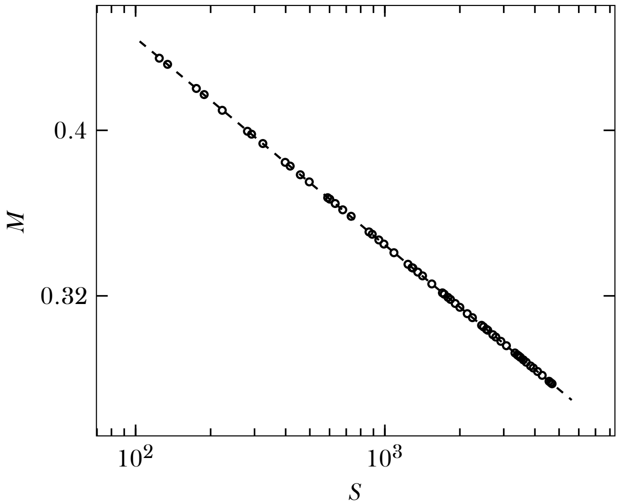

Figure 1.5

1.1.3 Using the entropy to define the correlation length

Via ??, the correlation length is expressed as

$$\xi \propto \exp(\frac{6}{c}S). \tag{1.7}$$

Figure 1.5 shows the results of fitting the relation in Equation 1.1 with this definition of the correlation length. The fit is an order of magnitude better in the least-squares sense, and the half-moon shapes have almost disappeared, yielding a much more robust exponent of $\beta/\nu = 0.12498$.

The entropy uses all eigenvalues of the corner transfer matrix, making it apparently less prone to structure in the spectrum than the correlation length as defined in ??, which uses only two eigenvalues of the row-to-row transfer matrix. Furthermore, the corner transfer matrix A is kept diagonal in the CTMRG algorithm, so S is much cheaper to compute than ξ.

1.1.4 Exponent κ

We now check the validity of the relation

$$\xi(m) \propto m^{\kappa} \tag{1.8}$$

in the context of the CTMRG method for two-dimensional classical systems. Similar checks were done for one-dimensional quantum systems in [2].

Let us first state that boundary conditions are relevant. From (*cite here!!*) we expect that for fixed boundary conditions, the entropy and therefore the correlation length is lower for a given bond dimension m.

There are various ways of extracting the exponent κ. Figure 1.6 shows the results for fixed boundary conditions and Figure 1.7 for free boundary conditions.

Directly checking Equation 1.8 yields $\kappa = 1.93$ for a fixed boundary and $\kappa = 1.96$ for a free boundary.

Under the assumption of Equation 1.8, we have the following scaling laws at the critical point

$$M(m) \propto m^{-\beta\kappa/\nu} \tag{1.9}$$

$$f(m) - f_{\text{exact}} \propto m^{(2-\alpha)\kappa/\nu} \tag{1.10}$$

for the order parameter and the singular part of the free energy, respectively. With a fixed boundary, a fit to $M(m)$ yields $\kappa = 1.93$. For a free boundary we cannot extract any exponent, since $M = 0$ for every temperature. A fit to $f(m) - f_{\text{exact}}$ yields $\kappa = 1.90$ for a fixed boundary and $\kappa = 1.93$ for a free boundary. Figure 1.6. Here, we have used $\beta = 1/8$, $\nu = 1$ and $\alpha = 0$ for the Ising model.

Tell that the κ law is indeed valid, since it is a good fit.

We may use ?? and ?? to check the relation

$$S_{\text{classical}} \propto \frac{c\kappa}{6} \log m, \tag{1.11}$$

which yields $\kappa = 1.93$ for a fixed boundary and $\kappa = 1.96$ for a free boundary, with $c = 1/2$ for the Ising model.

1.1.4.1 Comparison with exact result in asymptotic limit

The predicted value for κ [3] is $2.034\ldots$ (see also ??). With the CTMRG method, we extract the slightly lower value of 1.96 (corresponding to free

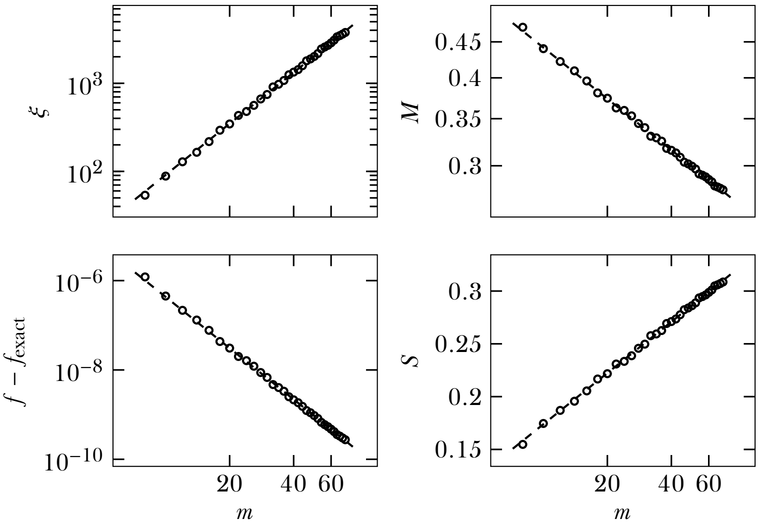

Figure 1.6: Numerical evidence for Equation 1.8, Equation 1.9, Equation 1.11 with fixed boundary, yielding, from left to right and top to bottom, $\kappa = \{1.93, 1.93, 1.90, 1.93\}$.

boundary conditions). But, the structure in the quantities as function of m makes it hard to get an accurate fit to κ.

It is interesting to note that for fixed boundary conditions, the relation in Equation 1.8 holds, but with a lower exponent κ. This is to be expected, since half the spectrum is missing.

phrase this better. Maybe subsubsection?

MUST SAY THAT I LEFT AWAY VALUES OF m in symmetric case!!

1.2 Locating the critical point

In general, the critical point is not known, but it may be located using the spectrum of the corner transfer matrix as described in ??. For a given value

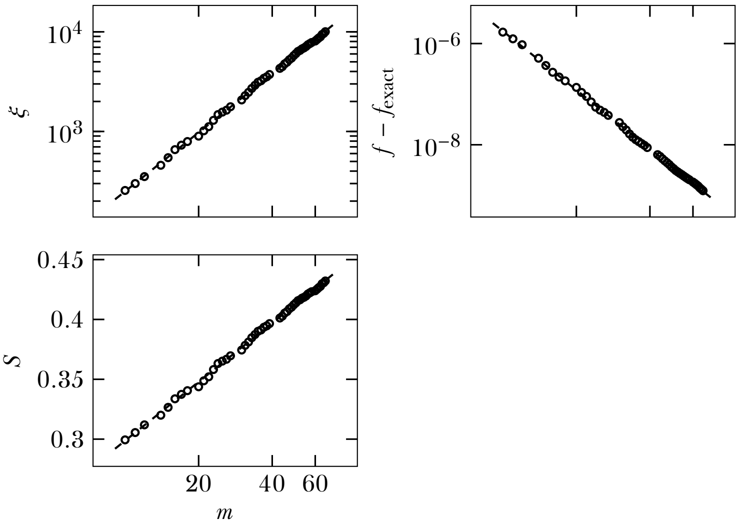

Figure 1.7: Numerical evidence for Equation 1.8 with free boundary, yielding from left to right and then bottom $\kappa = \{1.96, 1.93, 1.96\}$.

of m (or N, in a finite-size approximation), the pseudocritical temperature is defined as the point of maximum entropy.

Figure 1.8 shows the classical analogue to the entanglement entropy as a function of temperature for different values of m.

The critical point is located by fitting the scaling law in ??.

1.2.1 Finite m

For approximations with finite bond dimension m, it is not clear what length scale should be used to fit the scaling behaviour of $T^\star(m)$. Figure 1.9 shows the results for different choices of this length scale. To obtain $T^\star$, a convergence threshold of 10^{-8} and a temperature tolerance of 10^{-6} are used. The boundaries are fixed to +1.

Maybe I've used a too low TolX??

Explain results here

We denote the estimated value of the critical temperature as $\widetilde{T}_c$. Recall that the exact value is

$$T_c = 2.2691853\ldots \tag{1.12}$$

and $\nu = 1$.

When using $\xi(T_c, m)$, the correlation length at the exact critical point, the result shows a lot of structure, yielding $\widetilde{T}_c = 2.269172$ and $\nu = 1.057$.

If, instead, the correlation length at the estimated pseudocritical temperature $\xi(T^\star(m))$ is used, the data shows less structure and we obtain the much more precise results $\widetilde{T}_c = 2.269183$ and $\nu = 1.002$.

Another option is to use the entropy to define the correlation length, via Equation 1.7, which gave more accurate results than using the transfer matrix definition in subsection 1.1.3. In this case, the results are slightly worse than the transfer matrix definition: $T_c = 2.269183$ and $\nu = 1.02$.

Finally, we may directly fit the law

$$|T_c - T^\star(m)| \propto m^{-\kappa/\nu}, \tag{1.13}$$

yielding $T_c = 2.269181$ and $\kappa/\nu = 1.91$. Incidentally, this is another way to confirm $\kappa \approx 1.9$ for systems with a fixed boundary.

Why do length scales defined at $T^\star$ work better?? It is fortunate that we don't need the length scales at T_c, since we don't know it.

1.2.2 Finite N

As a cross check, we can instead use systems of finite size to extract T_c and ν. This yields *values*. See the *figure*.

1.3 Away from the critical point

We may also verify the validity of the different length scales by asserting that the data for different values of m should collapse on a single curve

$$\mathcal{G}(tN_{\text{eff}}(m)^{1/\nu}) = M(T, m)N_{\text{eff}}(m)^{\beta/\nu}. \tag{1.14}$$

All data points were calculated with a convergence threshold of 10^{-7}. The values of the pseudocritical temperatures are taken from the results in sec-

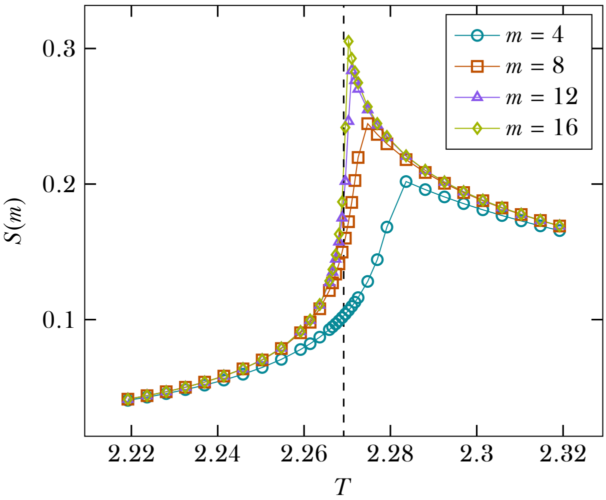

Figure 1.8: Classical analogue to the entanglement entropy, as in ??, near the critical point (shown as dashed line).

tion 1.2. No temperatures beyond T_c is considered because the order parameter drops off sharply, causing the curve $\mathcal{G}(x)$ to tend to zero almost vertically, making the fitness P unreliable.

Figure 1.10 shows that for all length scales, the results more or less fall on one curve. Table 1.1 shows the fitness of the data collapse [4] (given by ??) for all length scales used.

Say which length scales apparently don't work so well

Using m^κ as a length scale for optimized fitness $P(\kappa)$ yields $\kappa \approx 1.98$, substantially higher than found previously for fixed boundary conditions.

As a cross-check, the bottom-right panel of Figure 1.10 shows data points for finite-N simulations. Here, the bond dimension is chosen such that the truncation error is smaller than 10^{-6}.

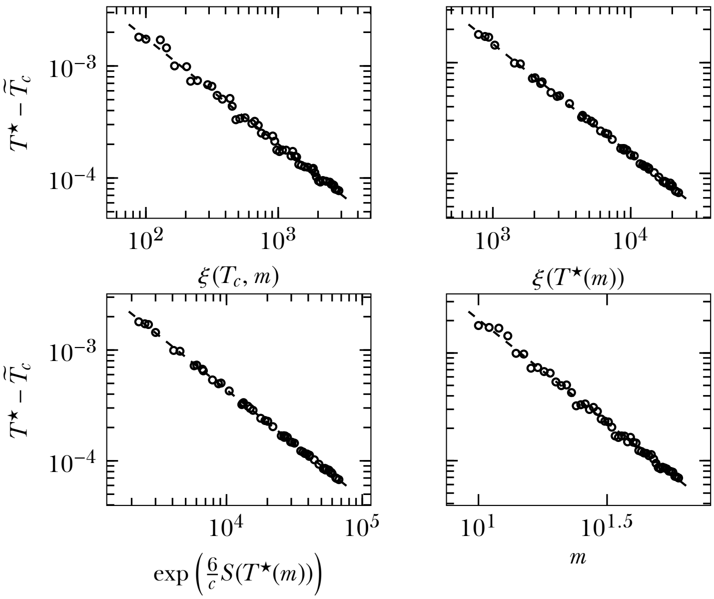

Figure 1.9: haloalaoalalalalaalkk

N_{eff}	fitness P
$\xi(T_c, m)$	0.0075
$\xi(T^\star(m))$	0.066
$\exp((6/c)S(T_c, m))$	0.057
$\exp((6/c)S(T^\star(m))$	0.087
m^κ	0.0080
N	0.0075

Table 1.1: Fitness of data collapse (??) for different length scales. $\kappa \approx 1.98$ was found to be optimal for the length scale m^κ.

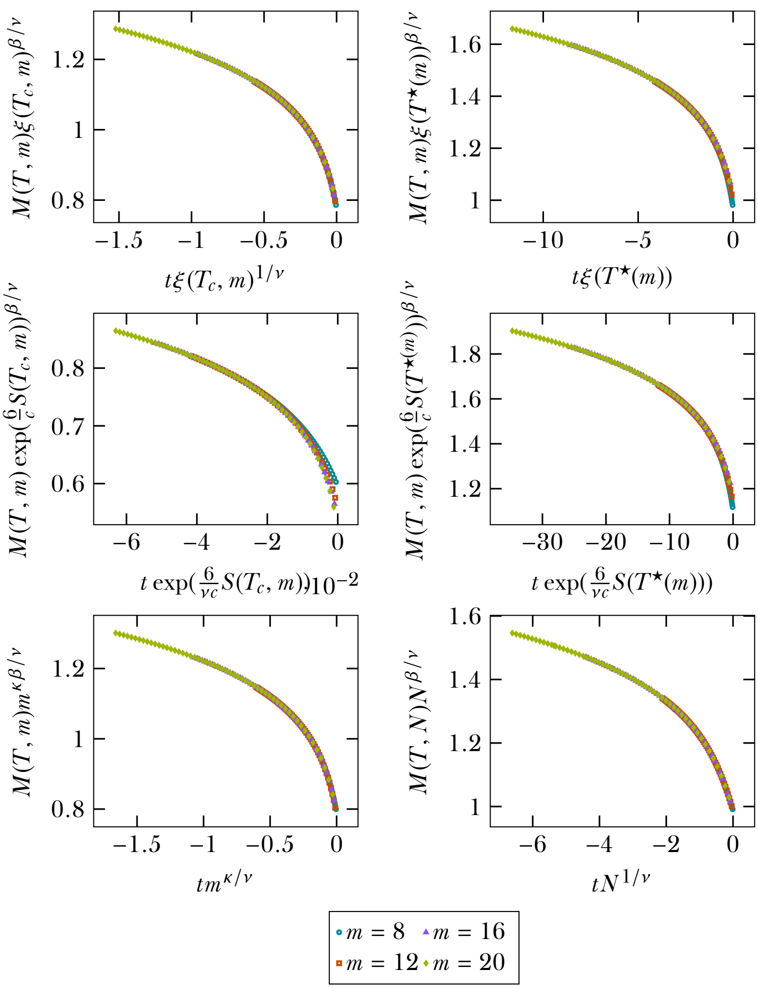

Figure 1.10: Data collapses using different length scales. For the bottom-right plot, approximations with finite N instead of finite m have been used, with $n = \{160, 480, 1000, 1500\}$ ($n = \frac{N-1}{2}$ is the number of algorithm steps).

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