EXPECTED VALUE AND VARIANCE

Expected value of a random variable:

Definition: Let X be a continuous random variable with range $[a, b]$ and probability density function $f(x)$. The *expected value* of X is defined by

$$E(X) = \int_a^b x f(x)\, dx.$$

Let's see how this compares with the formula for a discrete random variable:

$$E(X) = \sum_{i=1}^{n} x_i p(x_i).$$

f(x) is often written as p(x) and it denotes the probability density function.

A **probability density function** is a function f defined on an interval (a, b) and having the following properties.

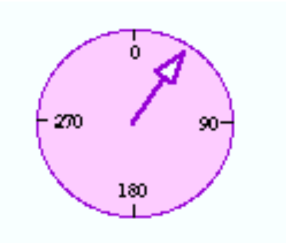
(a) $f(x) \geq 0$ for every x

(b) $\int_a^b f(x)\, dx = 1$

Expected value is also called the mean or average.

1. Let $f(x) = 2/x^2$. You would like to use this function as the pdf over a range [a, b]. You set a = 1. Find the value of b such that both the the above properties for pdf hold.

2. For the speed dial shown below:



Find a valid pdf that ensures uniform distribution and satisfies both properties. What is the probability of the following: 5 <= X <= 300 i.e. P(5 <= X <= 300)
3. Suppose X is a uniform function in range [0, 1]. What is E(X)

4. [Discrete Case]
Suppose that we toss a fair coin until a head first comes up, and let X represent the number of tosses that were made. Then the possible values of X are 1, 2, . . ., and the distribution function of X is defined by _________________. What is the expected value of X?

Some properties of E(X)

1. If X and Y are random variables on a sample space Ω then

$$E(X+Y) = E(X) + E(Y)$$

2. If a and b are constants then $E(aX+b) = aE(X) + b$.

5. The standard normal function is defined as:

$$Z = \frac{X-\mu}{\sigma} \sim \mathrm{N}(0,1).$$

The expected value of Z i.e. E(Z) is 0. Using the properties above, find E(X).

6. Find the expected value of the exponential probability density function, which is defined as:

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Variance of a random variable

Variance is defined as:

Definition: Let X be a continuous random variable with mean μ. The *variance* of X is

$$\text{Var}(X) = E((X-\mu)^2).$$

Properties of Variance:

1. If X and Y are *independent* then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.
2. For constants a and b, $\text{Var}(aX+b) = a^2\text{Var}(X)$.
3. **Theorem**: $\text{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2$.

7.

Let $X \sim \text{uniform}(0,1)$. Find $\text{Var}(X)$ and σ_X.

8.

Let $X \sim \exp(\lambda)$. Find $\text{Var}(X)$ and σ_X.