1 Syllabus

1. Asymptotic notation, recurrence.
2. Divide and Conquer.
3. Dynamic Programming.
4. Greedy Algorithm.
5. Graph Algorithm.
6. NPC

2 Basic

2.1 What is an algorithm?

Unambiguous, mechanically executable sequence of elementary operations.

There are certain types of algorithm:

Traditional (This courses main focus.)	Modern algorithm research
Deterministic	Randomized
Exact	Approximate
Off-line	On-line
Sequential	Parallel

2.2 Input & Output

View algorithm as a function with well defined inputs mapping to specific outputs. For example:

Input: $A[1...n]$ // Positive real number, distinct.

Output: $MAX A[i], 1 <= i <= n$.

2.2.1 Algorithm 1

Stupid way.

```
procedure FindMax
    for i = 1 to n do
        count = 0
        for j = 1 to n do
            if A[i] > A[j] then
                count = count + 1
            end if
        end for
        if count = n then
            return A[i]
        end if
    end for
end procedure
```

Algorithm 1: Stupid Find Max Algorithm

Analysis: Worst Case, n^2 comparison.

2.2.2 Algorithm 2

Sort & Find.

```
procedure FindMax
    Ā = sort(A)
    return Ā[n]
end procedure
```

Algorithm 2: Sort & Find Max Algorithm

Analysis: Worst Case, sorting takes $c\ n \log n$ time.

2.2.3 Algorithm 3

Dynamically store the biggest one.

```
procedure FindMax
    current = 1
    for i = 2 to n do
        if A[i] > A[current] then
            current = i
        end if
    end for
    return A[current]
end procedure
```

Algorithm 3: Search & Find Max Algorithm

2.3 Can we do better?

It depends on the operations allowed. For example the dropping the curtain and find the first appearing one.

3 Asymptotic Notation – big "O" notation

3.1 Growth of Functions

The growth of function in Table 1 increase downwards.

Table 1: Function List

$\log_{10} n$	binary search
n	input
n^2	pairs
$10^{10}n^{10}$	
$1.000.1^n$	
2^n	Binary string of length n
$n!$	Permutation

Let $f(n)$, $g(n)$ be function.

3.2 big "O" notation

Definition 3.2.1. $f(n) = \mathcal{O}(g(n))$, if $\exists n_0 \in \mathbb{N}$, $c \in \mathbb{R}^+$, s.t. $\forall n \geq n_0$, $f(n) \leq c * g(n)$, and $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq \infty$, i.e. it is $\lim_{n\to\infty} \frac{f(n)}{g(n)} < k$, for some constant k.

Table 2 shows the basic definition of all the asymptotic notations.

Table 2: Definition for all Asymptotic Notation

$f(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)}$	relation
$\mathcal{O}(g(n))$	$\neq \infty$	$\leq$
$\Omega(g(n))$	$\neq \infty$	$\leq$
$\Theta(g(n))$	$= k > 0$	$=$
$o(g(n))$	$= 0$	$<$
$\omega(g(n))$	$= \infty$	$>$

3.3 Asymptotic Relation's feature

Theorem 3.3.1. *Multiplying by positive constant does NOT change asymptotic relations. i.e. if $f(n) = \mathcal{O}(g(n))$, then $100 * f(n) = \mathcal{O}(g(n))$.*

Proof: $f(n) = \mathcal{O}(g(n)) \Rightarrow \exists n_0 \exists c, \forall n \geq n_0, f(n) \leq c * g(n)$,

then, $\exists n_0 \exists c'$, s.t. $\forall n \geq n_0$, $100 * f(n) \leq c' * g(n) = 100c * g(n)$. $\square$

Example:

$$C * 2^n = \Theta(2^n) \tag{1}$$
$$(C * 2)^n \neq \Theta(2^n) \tag{2}$$

Claim 3.3.2. *Show:* $2n\log(n) - 10n = \Theta(n\log(n))$

Proof: First show: $2n\log(n) - 10n = \mathcal{O}(n\log(n))$

For $n_0 = 1$, $c = 2$

$$2n\log(n) - 10n \leq 2n\log(n)$$

Now show: $2n\log(n) - 10n = \Omega(n\log(n))$

For $n_0 = 2^10$, $c = 1$,

$$\begin{aligned} 2n\log(n) - 10n &\geq n\log(n) + n\log(2^{10}) - 10n \\ &= n\log(n) + 10n - 10n \\ &= n\log(n) \end{aligned}$$

$n_0 = 1$ ($n_0 = 2^{10}$) means n is at least 1 (or 2^{10}). $\square$

Corollary 3.3.3. $\mathcal{O}(1)$ *means* ***Any Constant****.*

Attention: *Asymptotic notation has limit. It is not applicable for all scenarios.*

3.4 Properties of $\log(n)$

Definition 3.4.1. $n = C^{\log_c n}$, $c > 1$, dd $\lg n = \log_2 n$, $\ln n = \log_e n$.
Corollary 3.4.2. $\forall a, b > 1$

$$\begin{aligned} \log_b(n) &= \frac{\log_a(n)}{\log_a(b)} \\ \log_b(n) &= \Theta(\log_a(n)) \end{aligned} \tag{3}$$

Corollary 3.4.3. $\forall a, b \in \mathbb{R}$

$$\begin{aligned} \log(a^n) &= n * \log(a) \\ \log(a * b) &= \log(a) + \log(b)) \\ a^{\log(b)} &= b^{\log(a)} \end{aligned} \tag{4}$$

Note: $\lg(n)$ is to n as n is to 2^n.

3.5 Something More

Theorem 3.5.1. *Let $f(n)$ be a polynomial function, then* $\log(f(n)) = \Theta(\log(n))$*.*

Proof: The asymptotic result of n^2 and n^10 are the same. □
Definition 3.5.2. $\log^*(n) = o(\log\log\log\log\log(n)) = \alpha$.

Example: $\lg^*(2^{2^{2^{2^2}}}) = 5$.

4 Series

4.1 Some Definition

Definition 4.1.1. Harmonic Series:

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log(n))$$

Definition 4.1.2. Geometric Series:

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1} = \begin{cases} \Theta(x^n) & \text{if } \forall x > 1, \\ \Theta(1) & \text{if } \forall x < 1, \\ \Theta(n) & \text{if } \forall x = 1. \end{cases}$$

Definition 4.1.3. Arithmetic Series:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

4.2 Some Theorem

Suppose I want to know if $f(n) = o(g(n))$.

Theorem 4.2.1. *If* $\log(f(n)) = o(\log(g(n))$, *then* $f(n) = o(g(n))$.

Example: Let $f(n) = n^3$, $g(n) = 2^n$. Then $\log(f(n)) = \log(n^3) = 3\log(n)$, $\log(g(n)) = \log(2^n) = n$.

$$\text{i.e. } \log(f(n)) < \log(g(n) \Rightarrow f(n) < g(n)$$

Note that this theorem stands for 'o', NOT TRUE for '$\mathcal{O}$'.

Example: $\log(n^3) = \mathcal{O}(\log(n^2))$, but $n^3 \neq \mathcal{O}(n^2)$.

5 Induction

5.1 When to use?

Prove statement for all $n \in \mathbb{N}$, s.t. $n \geq n_0$.

5.2 Definition

Basically, induction has two parts:

1. Base case(s) – Sometimes there are more than one base cases.

   Prove statement for some n. – Often $n_0 = 0$ or 1.

2. Induction Hypothesis

   Assume statement hold true for all $m \leq n$.

   Prove the hypothesis implies that it hold true for $n+1$.

Note that the process may be different from previous, which just hypothesize $n-1$ is true and prove for n.

5.3 Example

5.3.1 Good Induction: Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: We are required to prove $\forall n > 0,\ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Base Case: $n = 1$, $\sum_{i=1}^{1} i = 1 = \frac{1\times(1+1)}{2}$. Hence the claim holds true for $n = 1$.

Induction step: Let $k > 1$ be an arbitrary natural number.

Let us assume the induction hypothesis: for every $k < n$, assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. We will prove $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (n+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \tag{5}$$

Thus establishes the claim for $k+1$.

Conclusion: By the principle of mathematical induction, the claim holds for all n. $\square$

5.3.2 Bad Induction: Prove all horses are the same color

The process is omitted. The key point is that: if the base case is not true for induction hypothesis, the induction will not be solid.

6 Recursion (Divide & Conquer)

- Recursion is like Induction's twin brother, whereas induction is similar to movie filmed, and recursion is similar to movie backward.
- Recursion design may be most important course topic.

- Recursion is a type of reduction. [1]

6.1 Definition of Recursion: a Powerful type of reduction

1. if problem size very small (think $\mathcal{O}(1)$), just solve it.
2. reduce to one or more small instances of some problem.

Question: How are the smaller (but not $\mathcal{O}(1)$ size) problem solved?

Not your problem! Handled by the recursion fairy.

6.2 Tower of Hanoi

- 3 pegs, which hold n distinct sized disks.
- initially *tmp*, *dst* empty and *src* has all disks sorted.
- 3 rules:
    1. larger cannot be placed on smaller.
    2. only one disks can move at a time.
    3. move all disks to *dst*.

Question: How long until the world end?

Solution

7 Dynamic Programming

7.1 Rod Cutting

- steel rod of length n, where n is some integer.
- $P[1...n]$, where $P[i]$ is market price for rod of length i.

[1]Reduction is to solve problem A using a black box for B. Typically B is smaller.

Question:

Suppose you can cut rod to any integer length for free. How much money can you made?

Analysis:

- Consider leftmost cut of optimal solution.

  cut can be at positions $1...n$.

  If leftmost cut at i, then you get $P[i]$ for leftmost piece and then optimally sell remaining $n-i$ length rod.

- Don't know where to make first cut, so try them all and find

$$\max(0, \max(P[i] + cutRod(n-i)))$$

So, the first attempt of the algorithm could be described as algorithm 4.

```
procedure CutRod(n)
    if n = 0 then                          ▷ If the remaining rod length is 0.
        return 0
    end if
    q = 0
    for i = 1 to n do
        q = max (q, P[i] + CutRod(n − i))
    end for
    return q
end procedure
```

Algorithm 4: First Attempt of Solving Cutting Rod Problem

Running time of algorithm 4:$T(n) = n + \sum_{i=0}^{n-i} T(i)$, which is clearly **Exponential** since there are <u>a lot of subproblem overlap</u>!

7.1.1 Memoized Version

algorithm 5 illustrates the memoized version of the algorithm in algorithm 4

```
procedure MemRodCut(n)                          ▷ Globally define R[1..n]
    if n = 0 then
        return 0
    end if
    if R[n] undefined then
        q = 0
        for i = 1 to n do
            q = max (q, P[i] + MemRodCut(n − i))
        end for
        R[n] = q
    end if
    return R[n]
end procedure
```

Algorithm 5: Memoized Version of Solving Cutting Rod Problem

Note that $R[1...n]$ *is filled in form left to right.* It means we can store the result and use it later, which brings us to the dynamic programming version of the algorithm.

7.1.2 Dynamic Programming Version

algorithm 6 illustrates the dynamic programming version of the algorithm according to the memoized version algorithm 5.

```
procedure DPRodCut(n)
    Let R[0...n] be an array.
    R[0] = 0
    for j = 1 to n do
        q = 0
        for i = 0 to j do
            q = max(q, P[i] + R[j − i])
        end for
        R[i] = q
    end for
    return R[n]
end procedure
```

Algorithm 6: Dynamic Programming Version of Solving Cutting Rod Problem

Running time of algorithm 6: $T(n) = \mathcal{O}(n^2)$.

Note that the process only computes the total number. If we are to know how to cut, we can store the cutting position during the progress.

Define $C[1...n]$, and replace the inner for loop in algorithm 6 as:

for $i = 0$ to j **do**
    $q = \max(q, P[i] + R[j-i])$
    $C[j] = i$
end for
$R[i] = q$

The for loop does the following:

- $C[j]$ stores last leftmost cut length for rod of length j.
- $C[n]$ says where to make first

Thus $C[n - C[n]]$ tells the second cut.

7.2 Solving DP problem

According to previous examples, we can summarize the general method to solve DP problem.

1. Write recursive solution, explain why the solution is correct.
2. Identify all subproblems considered.
3. Described how to store subproblems.
4. Find order to evaluate subproblems, s.t. subproblems you depend on evaluated ***before*** current subproblem.
5. Running Time: time to fill an entry X size table.
6. Write DP/Memoized algorithm.