Class Notes for CS6363

Design and Analysis of Computer Algorithms

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About This Note

These are lecture notes taken from *CS6363: Design and Analysis of Computer Algorithms* taught by Benjamin Raichel at University of Texas at Dallas in Fall 2016.

The first part of the note is generally based on the famous CLRS, though they are still the copy from instructor's mysterious notebook. The latter part (starting from graph algorithm) is basically the same as Jeff Erickson's Lecture Notes, which Benjamin strongly recommended as off-class additional reading material.

Taking note in Ben's class is a pleasant experience, though there might be some times in the semester you cannot read the note on blackboard. I hope this note would help you, if you are taking the same class as I did, and are bored enough to find this note. I wish you receive a good grade in this course and all other courses in the future.

Contents

1 Basic **1**

1.1 What is an algorithm? . . . 1

1.2 Input & Output . . . 1

1.2.1 Algorithm 1: Stupid way . . . 1

1.2.2 Algorithm 2: Sort & Find . . . 2

1.2.3 Algorithm 3: Dynamically store the biggest one . . . 2

1.3 Can we do better? . . . 2

2 Asymptotic Notation – big "O" notation **3**

2.1 Growth of Functions . . . 3

2.2 big "O" notation . . . 3

2.3 Asymptotic Relation's feature . . . 3

2.4 Properties of $\log(n)$ . . . 4

2.5 Something More . . . 5

3 Series **6**

3.1 Some Definition . . . 6

3.2 Some Theorem . . . 6

4 Induction **6**

4.1 When to use? . . . 6

4.2 Definition . . . 7

4.3 Example . . . 7

4.3.1 Good Induction: . . . 7

4.3.2 Bad Induction: Prove all horses are the same color . . . 8

5 Recursion (Divide & Conquer) **9**

5.1 Definition of Recursion: a Powerful type of reduction . . . 9

5.2 Tower of Hanoi . . . 9

5.2.1 Description of Problem . . . 9

5.2.2 Analysis . . . 9

5.3 Binary Search . . . 10

5.4 Maximum Sub-array Sum . . . 12

5.4.1 Description of Problem . . . 12

5.4.2 Analysis . . . 12

5.4.3 Applying Divide & Conquer on Max Sub-Array Sum Problem . . . 13

5.5 Solving Recurrence . . . 15
5.5.1 Basic Idea . . . 15
5.5.2 Typical Proving Method . . . 15
5.5.3 Generalizing: Recursion Tree Method . . . 16
5.5.4 Examples: $T(n) = \sqrt{n}T(\sqrt{n}) + n$ . . . 17
5.5.5 Examples: $T(n) = 2T(\frac{n}{4}) + n \log n$ . . . 18
5.5.6 Examples: $T(n) = 2T(\frac{n}{3}) + \sqrt{n}$ . . . 19

6 Sorting **20**
6.1 Insertion Sort & Merge Sort . . . 20
6.2 Sorting Lower Bound . . . 22
6.3 Sorting in Linear Time . . . 23
6.3.1 Counting Sort . . . 23
6.3.2 Radix Sort . . . 23
6.4 Quick Sort & Median Selection . . . 24
6.4.1 Pivoting . . . 24
6.4.2 Median Selection . . . 26

7 Dynamic Programming **29**
7.1 Rod Cutting . . . 29
7.1.1 Description of Problem . . . 29
7.1.2 Analysis . . . 29
7.1.3 Memoized Version . . . 30
7.1.4 Dynamic Programming Version to Solve RodCut . . . 30
7.2 Solving DP problem . . . 31
7.3 Longest Increasing Subsequence (LIS) . . . 32
7.3.1 Description of Problem . . . 32
7.3.2 Analysis . . . 32
7.3.3 Dynamic Programming Version to Solve LIS . . . 33
7.4 Longest Common Subsequence . . . 34
7.4.1 Description of Problem . . . 34
7.4.2 Analysis . . . 35
7.4.3 Dynamic Programming Version to Solve LCS . . . 36
7.5 Edit Distance . . . 37
7.5.1 Description of Problem . . . 37
7.5.2 Analysis . . . 38
7.5.3 Dynamic Programming Version to Solve Edit Distance Problem . . . 40

7.6 Longest Convex Subsequence . . . 40
7.6.1 Description of Problem . . . 40
7.6.2 Analysis . . . 41
7.7 Professor's note on LIS/LCS/LXS/SCS . . . 41
7.8 Summary . . . 48

8 Greedy Algorithm **49**
8.1 Greedy in a Glance . . . 49
8.2 Greedy Class Scheduling . . . 49
8.2.1 Description of Problem . . . 49
8.2.2 Analysis . . . 49
8.2.3 Proof . . . 50
8.3 General Prove Method for Greedy Algorithm: Exchange Argument . . . 51
8.4 Huffman Codes . . . 51
8.4.1 Description of Problem . . . 51
8.4.2 Definition . . . 53
8.4.3 Analysis . . . 53

9 Graph Algorithm **55**
9.1 Basic Stuff . . . 55
9.1.1 Fundamental Concept . . . 55
9.1.2 Concepts Used in this Note . . . 55
9.1.3 Graph Data Structure . . . 56
9.2 Traversing Graph . . . 57
9.2.1 Intuitive Approach . . . 57
9.2.2 Generic Traverse Algorithm . . . 58
9.2.3 Usage of Using Generic Travers Algorithm . . . 60
9.3 Standard Graph Algorithm . . . 61
9.4 Depth First Search . . . 61
9.4.1 Formal Definition . . . 61
9.4.2 Count and Label Components . . . 62
9.4.3 Pre/Post Order Traverse of Binary Tree . . . 63
9.4.4 Directed Graphs and Reachability . . . 64
9.5 Directed Acyclic Graphs . . . 65
9.5.1 Definition of DAG . . . 65
9.5.2 Algorithm to Determine a DAG . . . 65
9.5.3 Proof of Correctness . . . 66

9.6 Topological Sort . . . 67
9.7 Strong Connectivity . . . 69
9.8 Minimum Spanning Tree (MST) . . . 69
9.8.1 MST Definition . . . 69
9.8.2 MST Property . . . 70
9.8.3 MST Algorithm (General) . . . 71
9.8.4 Prim's Algorithm . . . 71
9.8.5 Borvka's Algorithm . . . 73
9.8.6 Kruskal's Algorithm . . . 75
9.9 Shortest Paths . . . 76
9.9.1 Single Source Shortest Path (SSSP) . . . 77
9.9.2 SSSP Algorithm . . . 77
9.9.3 Generic SSSP algorithm . . . 77
9.9.4 Dijkstra's Algorithm . . . 79
9.9.5 Bellman-Ford Algorithm . . . 80
9.9.6 Some Intuition on SSSP . . . 81
9.10 Max Flows and Min Cuts . . . 81
9.10.1 Flows . . . 81
9.10.2 Cuts . . . 82
9.10.3 The Maxflow MinCut Theorem . . . 83
9.10.4 Ford-Fulkerson Algorithm . . . 85
9.10.5 Application of Max Flow . . . 86

10 NP-Hardness **88**
10.1 A Game You Can't Win . . . 88
10.2 P versus NP . . . 89
10.3 NP-Hard/NP-Complete . . . 89
10.4 Reductions & SAT . . . 90
10.5 3SAT . . . 91
10.6 Independent Set . . . 92
10.7 Clique . . . 93
10.8 Vertex Cover . . . 94
10.9 Some Intuition . . . 94

List of Figures

1 Generalize Recursion Tree 16
2 Recursion Tree for $T(n) = \sqrt{n}T(\sqrt{n}) + n$ 17
3 Recursion Tree for $T(n) = 2T(\frac{n}{4}) + n \log n$ 18
4 Recursion Tree for $T(n) = 2T(\frac{n}{3}) + \sqrt{n}$ 19
5 Comparison Tree 22
6 Binary Code Tree for Fixed Length Codes 51
7 Binary Code Tree for Not Fixed Length Codes 52
8 Example Graph 56
9 Adjacency List 56
10 Example for Topological Sort 67
11 Example for MST 70
12 Example for MST 72
13 Example for No Shortest Path 77
14 An $(s,t) - flow$ with value 10. Each edge is labeled with its flow/capacity. . 82
15 An $(s,t) - cut$ with capacity 15. Each edge is labeled with its capacity. . . . 82
16 Enforcing the one-direction assumption 83
17 A flow f in a weighted graph G and the corresponding residual graph G_f. . . 84
18 A bad example for the Ford-Fulkerson algorithm 86
19 An AND, an OR, and a NOT gate 88
20 A boolean circuit 88
21 A boolean circuit with gate variables added, and an equivalent boolean formula 91
22 A graph derived from a 3CNF, and an independent set of size 4 93
23 Reduction Tree 94

List of Tables

1 Comparison between Traditional & Modern Algorithm 1
2 Function List 3
3 Definition for all Asymptotic Notation 3
4 Comparison Between Previous Two Method 14
5 Adjacency Matrix 56
6 Running Time Analysis for Adjacency Matrix and Adjacency List 57

List of Algorithms

1 Stupid Find Max Algorithm 1
2 Sort & Find Max Algorithm 2
3 Search & Find Max Algorithm 2
4 Recursive Hanoi 10
5 Binary Search Algorithm 11
6 Recursive Solution for Maximum Sub-Array Sum Problem 13
7 Divide & Conquer Solution for Maximum Sub-Array Sum Problem 14
8 Merge Two Sorted Arrays 20
9 Insertion Sort 21
10 Combine Insertion Sort with Merge 21
11 Merge Sort 22
12 Least Significant Radix Sort 24
13 Pivoting & Quick Sort 25
14 Quick Select 27
15 Median of Median Select 28
16 First Attempt of Solving Cutting Rod Problem 29
17 Memoized Version of Solving Cutting Rod Problem 30
18 Dynamic Programming Version of Solving Cutting Rod Problem 31
19 Store the Cutting Position in the Process 31
20 Original Algorithm for LIS Problem 33
21 Dynamic Programming Algorithm for LIS Problem 34
22 Original Algorithm for LCS Problem 36
23 Dynamic Programming Algorithm for LCS Problem 37
24 Original Algorithm for Edit Distance 39
25 Dynamic Programming Algorithm for Edit Distance Problem 40
26 Recursive Algorithm for Longest Convex Subsequence 41
27 Greedy Solution for Class Scheduling 50
28 Recursive Depth First Search Algorithm 57
29 Iterative Depth First Search Algorithm 58
30 Generic Traverse Algorithm 59
31 Remember Parent version of Generic Traverse Algorithm 60
32 Wrapper for Traverse 61
33 Formal Definition of DFS 62
34 DFS All the Component 62

35 Count and Label the Components 63
36 Pre/Post Order Based on DFS 64
37 Determine Whether a Graph is DAG 66
38 Topo-Sort from Source 68
39 Topo-Sort from Sink 68
40 Algorithm to Compute Strong Components 69
41 Prim's Algorithm 72
42 Borvka's Algorithm 74
43 Kruskal's Algorithm 76
44 Relax Operation in SSSP Algorithm 78
45 Generic SSSP Algorithm 79
46 Bellman-Ford Algorithm 80

1 Basic

1.1 What is an algorithm?

Unambiguous, mechanically executable sequence of elementary operations. There are certain types of algorithm:

Table 1: Comparison between Traditional & Modern Algorithm

Traditional (This courses main focus.)	Modern algorithm research
Deterministic	Randomized
Exact	Approximate
Off-line	On-line
Sequential	Parallel

1.2 Input & Output

View algorithm as a function with well defined inputs mapping to specific outputs.

Input: $A[1...n]$ // Positive real number, distinct.

Output: $MAXA[i], 1 <= i <= n.$

1.2.1 Algorithm 1: Stupid way

Algorithm 1 Stupid Find Max Algorithm

```
1: procedure FindMax
2:     for i = 1 to n do
3:         count = 0
4:         for j = 1 to n do
5:             if A[i] > A[j] then
6:                 count = count + 1
7:             end if
8:         end for
9:         if count = n then
10:            return A[i]
11:        end if
12:    end for
13: end procedure
```

Analysis: Worst Case, n^2 comparison.

1.2.2 Algorithm 2: Sort & Find

Algorithm 2 Sort & Find Max Algorithm

```
1: procedure FindMax
2:     A̅ = sort(A)
3:     return A̅[n]
4: end procedure
```

Analysis: Worst Case, sorting takes $c\ n \log n$ time.

1.2.3 Algorithm 3: Dynamically store the biggest one

Algorithm 3 Search & Find Max Algorithm

```
1: procedure FindMax
2:     current = 1
3:     for i = 2 to n do
4:         if A[i] > A[current] then
5:             current = i
6:         end if
7:     end for
8:     return A[current]
9: end procedure
```

1.3 Can we do better?

It depends on the operations allowed. For example the dropping the curtain and find the first appearing one.

2 Asymptotic Notation – big "O" notation

2.1 Growth of Functions

The growth of function in table 2 increase downwards.

Table 2: Function List

$\log_{10} n$	binary search
n	input
n^2	pairs
$10^{10}n^{10}$	
$1.000.1^n$	
2^n	Binary string of length n
$n!$	Permutation

Let $f(n)$, $g(n)$ be function.

2.2 big "O" notation

Definition 2.2.1. $f(n) = \mathcal{O}(g(n))$, if $\exists n_0 \in \mathbb{N}$, $c \in \mathbb{R}^+$, s.t. $\forall n \geq n_0$, $f(n) \leq c * g(n)$, and $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq \infty$, i.e. it is $\lim_{n\to\infty} \frac{f(n)}{g(n)} < k$, for some constant k.

Table 3 shows the basic definition of all the asymptotic notations.

Table 3: Definition for all Asymptotic Notation

$f(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)}$	relation
$\mathcal{O}(g(n))$	$\neq \infty$	$\leq$
$\Omega(g(n))$	$\neq \infty$	$\leq$
$\Theta(g(n))$	$= k > 0$	$=$
$o(g(n))$	$= 0$	$<$
$\omega(g(n))$	$= \infty$	$>$

2.3 Asymptotic Relation's feature

Theorem 2.3.1. *Multiplying by positive constant does NOT change asymptotic relations. i.e. if* $f(n) = \mathcal{O}(g(n))$, *then* $100 * f(n) = \mathcal{O}(g(n))$.

Proof: $f(n) = \mathcal{O}(g(n)) \Rightarrow \exists n_0 \exists c, \forall n \geq n_0, f(n) \leq c * g(n)$,
then, $\exists n_0 \exists c'$, s.t. $\forall n \geq n_0$, $100 * f(n) \leq c' * g(n) = 100c * g(n)$. $\square$
Example:

$$C * 2^n = \Theta(2^n) \tag{1}$$
$$(C * 2)^n \neq \Theta(2^n) \tag{2}$$

Claim 2.3.2. *Show:* $2n\log(n) - 10n = \Theta(n\log(n))$

Proof: First show: $2n\log(n) - 10n = \mathcal{O}(n\log(n))$
For $n_0 = 1$, $c = 2$

$$2n\log(n) - 10n \leq 2n\log(n)$$

Now show: $2n\log(n) - 10n = \Omega(n\log(n))$
For $n_0 = 2^10$, $c = 1$,

$$\begin{aligned} 2n\log(n) - 10n &\geq n\log(n) + n\log(2^{10}) - 10n \\ &= n\log(n) + 10n - 10n \\ &= n\log(n) \end{aligned}$$

$n_0 = 1$ ($n_0 = 2^{10}$) means n is at least 1 (or 2^{10}). $\square$

Corollary 2.3.3. $\mathcal{O}(1)$ *means* ***Any Constant****.*

Attention: *Asymptotic notation has limit. It is not applicable for all scenarios.*

2.4 Properties of $\log(n)$

Definition 2.4.1. $n = C^{\log_c n}$, $c > 1$, $\lg n = \log_2 n$, $\ln n = \log_e n$.

Corollary 2.4.2. $\forall a, b > 1$

$$\begin{aligned} \log_b(n) &= \frac{\log_a(n)}{\log_a(b)} \\ \log_b(n) &= \Theta(\log_a(n)) \end{aligned} \tag{3}$$

Corollary 2.4.3. $\forall a, b \in \mathbb{R}$

$$\begin{aligned} \log(a^n) &= n * \log(a) \\ \log(a * b) &= \log(a) + \log(b)) \\ a^{\log(b)} &= b^{\log(a)} \end{aligned} \tag{4}$$

Note: $\lg(n)$ is to n as n is to 2^n.

2.5 Something More

Theorem 2.5.1. *Let $f(n)$ be a polynomial function, then* $\log(f(n)) = \Theta(\log(n))$.

Proof: The asymptotic result of n^2 and n^{10} are the same. $\square$

Definition 2.5.2. $\log^*(n) = o(\log\log\log\log\log(n)) = \alpha$.

Example: $\lg^*(2^{2^{2^{2^2}}}) = 5$.

3 Series

3.1 Some Definition

Definition 3.1.1. Harmonic Series:

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log(n))$$

Definition 3.1.2. Geometric Series:

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1} = \begin{cases} \Theta(x^n) & \text{if } \forall x > 1, \\ \Theta(1) & \text{if } \forall x < 1, \\ \Theta(n) & \text{if } \forall x = 1. \end{cases}$$

Definition 3.1.3. Arithmetic Series:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

3.2 Some Theorem

Suppose I want to know if $f(n) = o(g(n))$.

Theorem 3.2.1. *If* $\log(f(n)) = o(\log(g(n))$, *then* $f(n) = o(g(n))$.

Example: Let $f(n) = n^3$, $g(n) = 2^n$. Then $\log(f(n)) = \log(n^3) = 3\log(n)$, $\log(g(n)) = \log(2^n) = n$.

$$\text{i.e. } \log(f(n)) < \log(g(n) \Rightarrow f(n) < g(n)$$

Note that this theorem stands for 'o', NOT TRUE for 'O'.

Example: $\log(n^3) = \mathcal{O}(\log(n^2))$, but $n^3 \neq \mathcal{O}(n^2)$.

4 Induction

4.1 When to use?

Prove statement for all $n \in \mathbb{N}$, s.t. $n \geq n_0$.

4.2 Definition

Basically, induction has two parts:

1. Base case(s) – Sometimes there are more than one base cases.

   Prove statement for some n. – Often $n_0 = 0$ or 1.

2. Induction Hypothesis

   Assume statement hold true for all $m \leq n$.

   Prove the hypothesis implies that it hold true for $n + 1$.

Note that the process may be different from previous, which just hypothesize $n - 1$ is true and prove for n.

4.3 Example

4.3.1 Good Induction:

Claim: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: We are required to prove $\forall n > 0$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Base Case: $n = 1$, $\sum_{i=1}^{1} i = 1 = \frac{1 \times (1+1)}{2}$. Hence the claim holds true for $n = 1$.

Induction step: Let $k > 1$ be an arbitrary natural number.

Let us assume the induction hypothesis: for every $k < n$, assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. We will prove $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (n+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \tag{5}$$

Thus establishes the claim for $k + 1$.

Conclusion: By the principle of mathematical induction, the claim holds for all n. □

4.3.2 Bad Induction: Prove all horses are the same color

The process is omitted. The key point is that: if the base case is not true for induction hypothesis, the induction will not be solid.

5 Recursion (Divide & Conquer)

- Recursion is like Induction's twin brother, whereas induction is similar to movie filmed, and recursion is similar to movie backward.
- Recursion design may be most important course topic.
- Recursion is a type of reduction. [1]

5.1 Definition of Recursion: a Powerful type of reduction

1. if problem size very small (think $\mathcal{O}(1)$), just solve it.
2. reduce to one or more small instances of some problem.

Question: How are the smaller (but not $\mathcal{O}(1)$ size) problem solved?
Not your problem! Handled by the recursion fairy.

5.2 Tower of Hanoi

5.2.1 Description of Problem

- 3 pegs, which hold n distinct sized disks.
- initially *tmp*, *dst* empty and *src* has all disks sorted.
- 3 rules:
    1. larger cannot be placed on smaller.
    2. only one disks can move at a time.
    3. move all disks to *dst*.

Question: How long until the world end?

5.2.2 Analysis

A small hint: not consider the smallest first, but the largest first.
In order to move the largest disk:

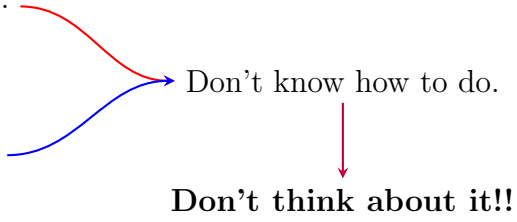
- *dst* has to be empty.

[1] Reduction is to solve problem A using a black box for B. Typically B is smaller.

- *src* has only largest one.
- *tmp* has $n-1$ disks sorted.

So we must:

1. move $n-1$ disks from *src* to *tmp* .
2. move largest from *src* to *dst* .
3. move $n-1$ disk from *tmp* to *dst* .



Don't think about how to move $n-1$ disks, recursion fairy will do it.

Algorithm 4 Recursive Hanoi

```
procedure HANOI(n, src, dst, tmp)
    if n > 0 then
        HANOI (n − 1, src, tmp, dst)
        Move disk n from src to dst.
        HANOI (n − 1, tmp, dst, src)
    end if
end procedure
```

How many moves in algorithm 4 ?

Let $T(n)$ be the total moves for n disks.

$$\begin{aligned} T(0) &= 0 \\ T(1) &= 1 \\ T(n) &= T(n-1) + 1 + T(n-1) \\ &= 2T(n-1) + 1 \end{aligned}$$

"Solve" the recurrence.

$$\sum_{l=1}^{n} 2^{l-1} = 2^n - 1$$

5.3 Binary Search

Input: Given a value *val*, and sorted aray $A[1 \ldots n]$.

Output: Is *val* contained in A?

Algorithm 5 Binary Search Algorithm

```
1: procedure BIN(val, low, high)
2:     if high < low then
3:         return Not Found
4:     end if
5:     mid = ⌊(high + low)/2⌋
6:     if val < A[mid] then
7:         return BIN (val, low, mid − 1)
8:     end if
9:     if val > A[mid] then
10:        return BIN (val, mid + 1, high)
11:    end if
12:    return mid
13: end procedure
```

To find whether val is contained in $A[1 \dots n]$, call BIN $(val, 1, n)$.
Let $m = high - low + 1$,

$$T(m) \leq \max\left\{T\left(\left\lfloor\frac{m-1}{2}\right\rfloor\right), T\left(\left\lceil\frac{m-1}{2}\right\rceil\right)\right\} + \Theta(1) \tag{6}$$

$$\leq T\left(\frac{m}{2}\right) + \Theta(1) \tag{7}$$

$$T(m) = T\left(\frac{m}{2}\right) + 1 \tag{8}$$

$$T(m) = \Theta(1) \text{ for } m = \Theta(1) \tag{9}$$

Floor and Ceilings has differences in constant time.

To think the running time in another way:

A constant size problem should has a constant solution.

$$\begin{aligned} T(m) &= T\left(\frac{m}{2}\right) + 1 \\ &= T\left(\frac{m}{2 \times 2}\right) + 1 + 1 \\ &= T\left(\frac{m}{2 \times 2 \times 2}\right) + 1 + 1 + 1 \end{aligned}$$

Counting the number of times divide by 2 to get to 1, i.e.

$$\frac{m}{2^x} = 1 \to m = 2^x \to x = \lg m$$

5.4 Maximum Sub-array Sum

5.4.1 Description of Problem

Input: Given an unsorted array $A[1 \dots n]$ of integers, including both negative and positive number.

Output: $\max_{i \le j} \left\{ \sum_{k=i}^{j} A[k], 0 \right\}$.

5.4.2 Analysis

The Naive Solution

Let $W[i][j] = \sum_{k=i}^{j} A[k]$ for $i \le j$. Return $\max\{0, \max\{W[i][j]\}\}$. The running time of this brute force solution is$T(n) = \sum_{j=1}^{n} \sum_{i=1}^{j} (j - i + 1) = \Theta(n^3)$

Think Recursively!

Let $maxSum(i, j)$ be the maximum sub-array sum in $A[1 \dots n]$ or 0. The solution is $maxSum(1, n)$.

There are two cases:

Can we express $maxSum(1, n)$ in term of $maxSum(1, n-1)$?

- $maxSum(1, n)$ does not include $A[n]$

$$maxSum(1, n) = maxSum(1, n - 1)$$

- $maxSum(1, n)$ does include $A[n]$

$$maxSum(1, n) = maxEndAt(1, n)$$

$maxEndAt(i, j)$ is the maximum sub-array sum in $A[i, j]$ restricted to include $A[j]$.

Here comes the recursive version algorithm to solve this problem, showing in .

Algorithm 6 Recursive Solution for Maximum Sub-Array Sum Problem

```
1: procedure maxSum(i, j)
2:     if j < i then
3:         return 0
4:     end if
5:     x =maxEndAt (i, j)
6:     return max{x, maxSum(i, j − 1)}
7: end procedure
8: procedure maxEndAt(i, j)
9:     if j < i then
10:        return 0
11:    end if
12:    return max{A[n], A[n] + maxEndAt(i, j − 1)}
13: end procedure
```

The running time for maxEndAt (i, j) is $T(n) = T(n-1) + \Theta(1) = \Theta(n)$. Accordingly, the running time for maxSum (i, j) is

$$\begin{aligned} T(m) &= T(m-1) + \Theta(m) \\ &= T(m-1) + m \\ &= \sum_{k=1}^{m} k \\ &= \Theta(m^2) \end{aligned}$$

Can we do better?

Yes, use Divide and Conquer.

5.4.3 Applying Divide & Conquer on Max Sub-Array Sum Problem

The key point of D&C is try to get problem size down quickly.

Algorithm 7 Divide & Conquer Solution for Maximum Sub-Array Sum Problem

1: **procedure** DCMaxSum(i, j)
2: **if** $j < i$ **then**
3: **return** 0
4: **end if**
5: $mid = \left\lfloor \frac{i+j}{2} \right\rfloor$
6: **return** $\max \left\{ \begin{array}{l} \text{maxSum}(i, mid), \\ \text{maxSum}(mid, j), \\ \text{maxGoFrom}(i, mid) + \text{maxGoFrom}(mid, j) \end{array} \right\}$
7: **end procedure**

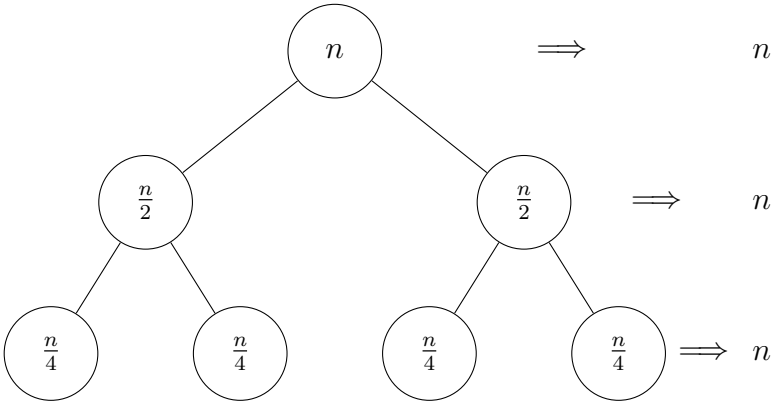
If goes cross, must include $A[mid]$ and $A[mid + 1]$.

The running time for algorithm 7 is

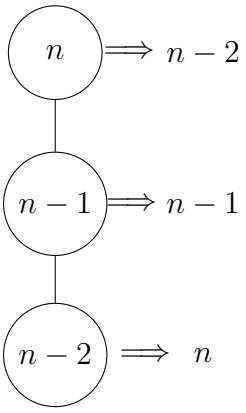
$$T(m) = 2T\left(\frac{m}{2}\right) + m$$

in which $m = j - i + 1$.

Table 4: Comparison Between Previous Two Method



At each level, the running time sum is n.
There are $\log n$ level in total.
Thus, total running time is $\Theta(n \log n)$.



$$\begin{aligned} T(m) &= T(m-1) + \Theta(m) \\ &= \sum_{k=1}^{m} k \\ &= \Theta(m^2) \end{aligned}$$

Can we do better?

The divide & conquer solution constantly calculate the MAXENDAT (i, j). Memoizing the result leads to the dynamic programming solution as shown in next section.

5.5 Solving Recurrence

5.5.1 Basic Idea

Fail-safe method for any recurrence:

Guess solution and prove correct with induction.

The subtle point is: need to make the right guess!

Example: Tower of Hanoi, $T(n) = 2T(n-1) + 1$, $T(1) = 1$.

Claim: $T(n) = 2^n - 1$

Proof:

Base Case: $T(1) = 2^1 - 1 = 1$, true.

Induction step: Assume $T(m) = 2^m - 1$, for $m < n$.

$$T(n) = 2T(n-1) + 1 = 2 \times (2^{n-1} - 1) + 1 = 2^n - 1$$

Hold true for n.

Conclusion: $T(n) = 2^n - 1$. □

5.5.2 Typical Proving Method

Example: maxSubArraySum, $T(n) = 2T(n/2) + \Theta(n)$.

Claim: $T(n) = \Theta(n \log n)$

Proof: Still using induction, only show the induction phase below.

Replace original statement with:

$$T(n) = 2T(n/2) + n$$

First, prove $T(n) \leq c \times n \log n$.

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + n \\ &\leq 2 \times c \times \frac{n}{2} \times \log\left(\frac{n}{2}\right) + n \\ &= c \times n \times \log(n-1) + n \\ &= c \times n \times \log n + n(1-c) \\ &= n \log n \text{ for } c = 1 \end{aligned}$$

Then, prove $T(n) \geq c \times n \log n$. The same as previous one. □

Note: The constant in $T(n) = \mathcal{O}(n \log n)$ muse be specified. And in $T(n) = c \times n \log n - bn$, sometimes bn need to guess.

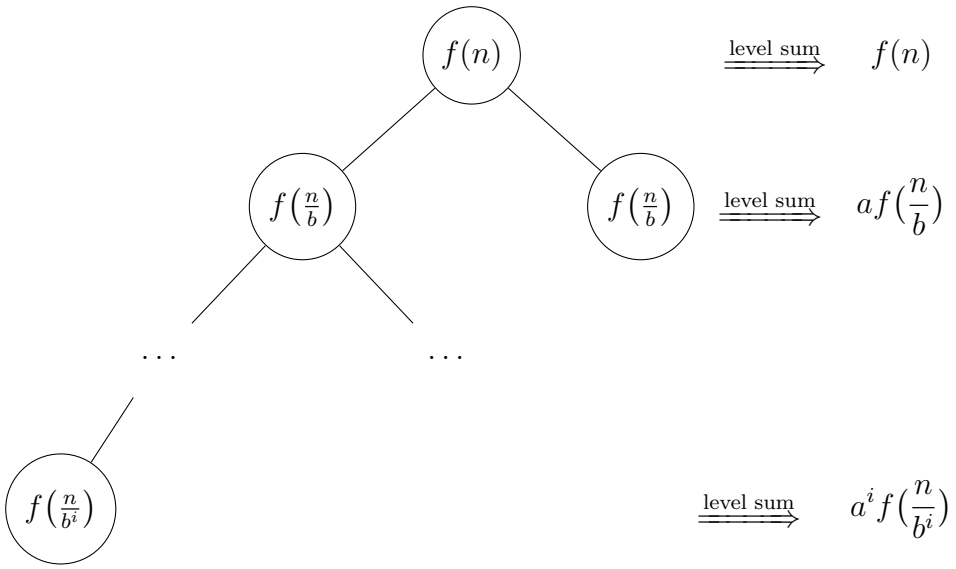
5.5.3 Generalizing: Recursion Tree Method

Generalize the equation $T(n) = T(n/2) + n$ to

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \tag{10}$$

where $a, b \geq 1$. The recursion tree for eq. (10) is shown in fig. 1.

Figure 1: Generalize Recursion Tree



Observation:

- Level sum: $a^i f(\frac{n}{b^i})$
- Depth: $\log_b n$
- Number of Leaves: $a^{\log_b n}$

In total: $T(n) = \sum_{i=1}^{\log_b n} a^i f(\frac{n}{b^i})$, which often like a geometric series.

Look at the ratio of successive level sums:

$$\frac{a^{i+1} f(\frac{n}{b^{i+1}})}{a^i f(\frac{n}{b^i})} = a \times \frac{f(\frac{n}{b^{i+1}})}{f(\frac{n}{b^{i+1}})} = \begin{cases} > 1, \text{ then sum determined by the number of leaves.} \\ < 1, \text{ then sum determined by root.} \\ = 1, \text{ then sum determined by the depth or root.} \end{cases} \tag{11}$$

A Special Case is that, $f(n) = n^c$, i.e. $f(n)$ is polynomial.

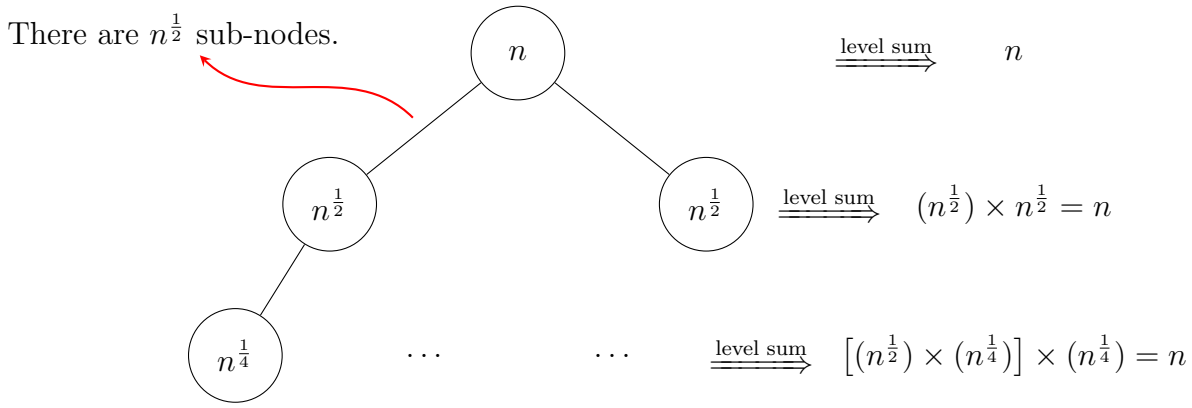
$$a \times \frac{f(\frac{n}{b^{i+1}})}{f(\frac{n}{b^{i+1}})} = a \times \frac{(\frac{n}{b^{i+1}})^c}{(\frac{n}{b^i})^c} = \frac{a}{b^c} = \begin{cases} > 1, \text{ if } a > b^c, \text{ then } T(n) = a^{\log_b n} = n^{\log_b a} \\ < 1, \text{ if } a < b^c, \text{ then } T(n) = n^c \\ = 1, \text{ if } a = b^c, \text{ then } T(n) = n^c \log_b n \end{cases} \tag{12}$$

The eq. (12) is called the Master Theorem.

5.5.4 Examples: $T(n) = \sqrt{n} T(\sqrt{n}) + n$

The recursion tree are shown in fig. 2.

Figure 2: Recursion Tree for $T(n) = \sqrt{n} T(\sqrt{n}) + n$



Observation:

Level sums: $\left(\prod_{i=1}^{l} n^{\frac{1}{2^i}}\right) \times \frac{1}{2^i} = n^{\left(\sum_{i=1}^{l} \frac{1}{2^i}\right) + \frac{1}{2^l}} = n^1$

Equal level sum, then what we need to know is the depth. So the question turns to: How many times apply $\sqrt{n}$ (root function) until get to constant size?

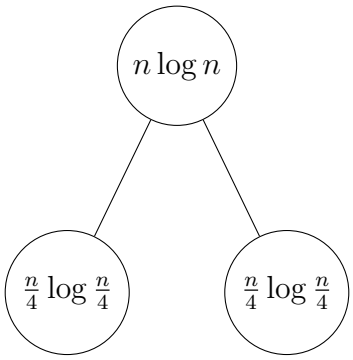
$$
\begin{aligned}
& n^{\frac{1}{2^i}} = c \\
& \Rightarrow \frac{1}{2^i} \log n = \log c \\
& \Rightarrow \log n = 2^i \log c \\
& \Rightarrow \log \log n = i \log 2 + \log \log c \\
& \Rightarrow i = \Theta(\log \log n)
\end{aligned}
$$

In total, the running time would be: $T(n) = \Theta(n \log \log n)$.

5.5.5 Examples: $T(n) = 2T(\frac{n}{4}) + n \log n$

The recursion tree are shown in fig. 3.

Figure 3: Recursion Tree for $T(n) = 2T(\frac{n}{4}) + n \log n$



Level sum: $2^i \frac{n}{4^i} \log \frac{n}{4^i} = \frac{n}{2^i}(\log(n - 2i))$ is decreasing.

Thus, total running time is dominated by root, which is $T(n) = \Theta(n \log n)$.

To prove it, first prove the upper bound:

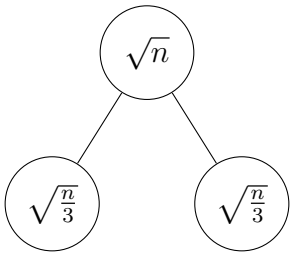
$$\begin{aligned} T(n) &\leq 2c \times \frac{n}{4} \log \frac{n}{4} + n \log n \\ &= \frac{1}{2} cn(\log n - 2) + n \log n \\ &= cn \log n \left(\frac{1}{2} + \frac{1}{c} \right) - cn \\ &\leq cn \log n \text{ for } c = 2 \end{aligned}$$

Then prove the lower bound, which is exactly the same steps as upper bound.

5.5.6 Examples: $T(n) = 2T(\frac{n}{3}) + \sqrt{n}$

The recursion tree are shown in fig. 4.

Figure 4: Recursion Tree for $T(n) = 2T(\frac{n}{3}) + \sqrt{n}$



Level sum: $2^i \sqrt{\frac{n}{3^i}} = \sqrt{n} \left(\frac{2}{\sqrt{3}} \right)$ is increasing.

Thus, total running time is dominated by number of leaves, which is $2^{\log_3 n} = n^{\log_3 2}$.

To prove it, first prove the lower bound:

$$T(n) \geq 2c \left(\frac{n}{3} \right)^{\log_3 2} + \sqrt{n} = c \times n^{\log_3 2} + \sqrt{n} \geq cn^{\log_3 2}$$

Lower bound works for any c, but upper bound fails for any c. In this case, we have to guess a upper bound:

Assume $T(n) \leq cn^{\log_3 2} - b\sqrt{n}$,

$$T(n) \leq 2 \times c \left(\frac{n}{3} \right)^{\log_3 2} - 2b\sqrt{\frac{n}{3}} + \sqrt{n} = c \times n^{\log_3 2} + \sqrt{n} \left(1 - 2 \times \frac{b}{\sqrt{3}} \right)$$

for $\sqrt{n} \times \left(1 - \frac{2b}{\sqrt{3}} \right) = -b\sqrt{n}$.

6 Sorting

Sorting elements in an array $A[1 \dots n]$, is usually a *Fundamental Task*, often first step in analysis. How do we sort? Let's begin with solving simpler problem.

Input: Given 2 sorted arrays, $B[1 \dots n]$ and $C[1 \dots m]$.

Output: Combine two arrays into sorted $A[1 \dots n+m]$.

Solution

Keep a pointer to each array starting at first element, compare values at pointers, take smaller and advance pointers. The algorithm is described in algorithm 8.

Algorithm 8 Merge Two Sorted Arrays

```
1:  procedure MERGE(A[1...n+m], B[1...n], C[1...m])
2:      Bindex = 1, Cindex = 1
3:      for Aindex = 1 to n + m do
4:          if Cindex > m then                          ▷ C exhausted. ◁
5:              A[Aindex] = B[Bindex]
6:              Bindex++
7:          else if Bindex > n then                     ▷ B exhausted. ◁
8:              A[Aindex] = C[Cindex]
9:              Cindex++
10:         else if B[Bindex] < C[Cindex] then          ▷ B smaller. ◁
11:             A[Aindex] = B[Bindex]
12:             Bindex++
13:         else                                        ▷ C smaller. ◁
14:             A[Aindex] = C[Cindex]
15:             Cindex++
16:         end if
17:     end for
18: end procedure
```

Merging takes $\mathcal{O}(n+m)$ time.

6.1 Insertion Sort & Merge Sort

Given a bag of n items (e.g playing cards).

1. Pick up one item at a time.

2. Put item into sorted set of previous items.

The algorithm is described in algorithm 9.

Algorithm 9 Insertion Sort

```
1: procedure InsertionSort(A[1...n])
2:     for j = 2 to n do
3:         key = A[j]
4:         i = j − 1
5:         while i > 0 and A[i] > key do
6:             A[i + 1] = A[i]
7:             i = i − 1
8:         end while
9:         A[i + 1] = key
10:    end for
11: end procedure
```

Loop invariant: at start of each loop iteration, $A[1 \dots n]$ consists of original $A[1 \dots j-1]$ elements, but in sorted order.

Thus, we can rewrite insertion sort using Merge () as shown in algorithm 10.

Algorithm 10 Combine Insertion Sort with Merge

```
1: procedure IS(A[1...n])
2:     for j = 2 to n do
3:         Merge (A[1...j], A[1...j − 1], A[j])
4:     end for
5: end procedure
6: procedure RIS(A[1...n])                    ▷ Recursive Rewrite. ◁
7:     if n > 1 then
8:         RIS (A[1...n − 1])
9:         Merge (A[1...n], A[1...n − 1], A[n])
10:    end if
11: end procedure
```

How long does RIS takes? $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$.

Can we do better? Yes! Divide and Conquer, cut in the middle, which leads to merge sort as described in algorithm 11.

Algorithm 11 Merge Sort

```
1: procedure MergeSort(A[1...n])
2:     if n > 1 then
3:         MergeSort (A[1...⌊n/2⌋])
4:         MergeSort (A[⌊n/2⌋ + 1...n])
5:         Merge (A[1...n], A[1...⌊n/2⌋], A[⌊n/2⌋ + 1...n])
6:     end if
7: end procedure
```

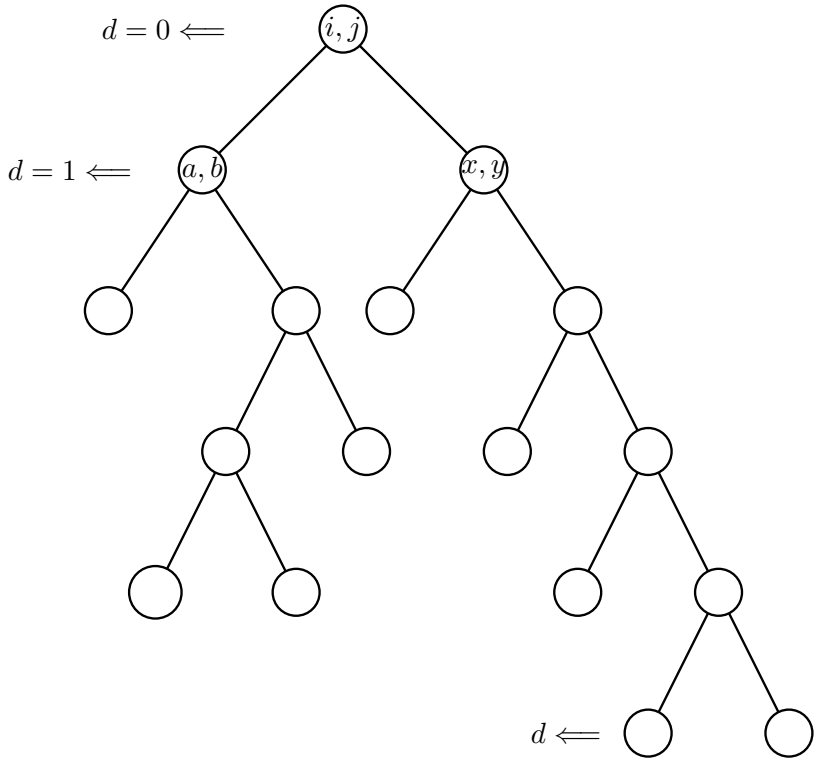
How long does Merge Sort takes? $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \log n)$.

6.2 Sorting Lower Bound

Given $A[1 \dots n]$, assume elements distinct. Algorithms we've seen so far sort only by comparisons, i.e given $A[i]$ and $A[j]$, is $A[i] < A[j]$ or $A[i] > A[j]$. How fast can comparison based sorting be?

Assume algorithms is deterministic: for any n size input, the first comparison is fixed. The comparison tree can be described as fig. 5

Figure 5: Comparison Tree



Meaning of the comparison tree:

- Pair in node is indices of elements comparing.
- Left branch means $A[i] < A[j]$.
- Right branch means $A[i] > A[j]$.
- Result of comparison uniquely determines next comparison.
- Algorithm always terminates when tree has finite depth d.
- At leaf we determined sorted order.
- Depth of leafs is number of comparisons needed.
- worst cast = deepest leaf depth.
- for binary trees, number of leaves $\leq 2^d$.

In total $n!$ (factorial) ordering of elements.

$$n! \leq \text{ number of leaves } \leq 2^d$$

which means:

$$d \geq \lg(n!) = \Theta(n \log n)$$

using Stirling's Approximation $n! \approx \sqrt{2n}\left(\frac{n}{e}\right)^n$.

6.3 Sorting in Linear Time

What if allow operations other than comparisons, and make input assumptions.

6.3.1 Counting Sort

Given array $A[1 \ldots n]$ of integers s.t $1 \leq A[i] \leq k$. If $k = \mathcal{O}(n)$, then can sort in linear time. (No comparisons)

6.3.2 Radix Sort

Given array $A[1 \ldots n]$, assume b-bit integers, where b is fixed value. Sorting based on most significant bit.

Algorithm 12 Least Significant Radix Sort

```
1: procedure LRS(A, b)
2:     for i = 1 to b do
3:         Stable sort A based on bit i                ▷ Stable is important! ◁
4:     end for
5: end procedure
```

6.4 Quick Sort & Median Selection

Quick sort is a comparison based, recursive, in place sorting (unlike radix). Recall the *Merge Sort*, using divide and conquer, each time break in half, sort each half recursively, then combine. Combining took $\Theta(n)$ time, $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \log n)$.

Quick sort reverses the order:

> spend time to carefully split elements before recursive calls.

6.4.1 Pivoting

Pick element $A[p]$, divide A into elements smaller or larger than $A[p]$. After pivoting, if we sort the "<" set and ">" set recursively, we are done when recursive calls return. The algorithm is described in algorithm 13

Algorithm 13 Pivoting & Quick Sort

```
1: procedure PIVOT(A[1...n], p)
2:     SWAP (A[p], A[n])
3:     i = 0
4:     j = n
5:     while i < j do
6:         repeat
7:             i = i + 1
8:         until i ≥ j or A[i] ≥ A[n]
9:         repeat
10:            j = j − 1
11:        until i ≥ j or A[j] ≤ A[n]
12:        if i < j then
13:            SWAP (A[i], A[j])
14:        end if
15:    end while
16:    SWAP (A[i], A[n])
17:    return i
18: end procedure
19: procedure QUICKSORT
20:    if n > 1 then
21:        Choose pivot p
22:        r = PIVOT(A[1...n], p)
23:        QUICKSORT (A[1...r − 1])
24:        QUICKSORT (A[r − 1...n])
25:    end if
26: end procedure
```

Analysis: Running time: pivot operation takes linear time ($\Theta(n)time$) after $\mathcal{O}(1)$ work either i increments or j decrements. Total running time:

$$T(n) = T(r-1) + T(n-r) + \Theta(n)$$

which depends on r.

- If $r = n$, then $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$ (worst case).

- If $r = n/2$, then $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \log n)$ (best case).

Why is quick sort popular?

- For good r, runs fast;
- Sort in place, using little memory;
- Constants in $\Theta()$ small;
- Easy to implement.

Then, what determines r? Choice of p. Can we find the median in $\mathcal{O}(n)$ [2] deterministic time?

6.4.2 Median Selection

Given $A[1 \dots n]$, let $\overline{A}$ denotes A after sorting. The "rank" of $A[i]$ is its index in $\overline{A}$, i.e finding median means find element of rank $n/2$.

We can derive the problem to a more general one:

> Given $k \in [1 \dots n]$, we want element of rank k in $A[1 \dots n]$ in $\mathcal{O}(n)$ time.

The main idea is to modify quick select.

Quick Select is described in algorithm 14.

[2] Previously, we've already use $\mathcal{O}(n)$ time to pivot

Algorithm 14 Quick Select

```
procedure QuickSelect(A[1...n], k)
    if n = 1 then
        return A[1]
    else
        Choose pivot p
        r = Pivot(A[1...n], p)
        if k < r then
            return QuickSelect (A[1...r − 1], k)
        else if k > r then
            return QuickSelect (A[r − 1...n], k − r)
        else
            return A[r]
        end if
    end if
end procedure
```

Total running time:

$$T(n) \leq \max\{T(r-1), T(n-r)\} + \Theta(n)$$

Again, depend on r, worst case is when $r = n$, $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$.

Claim: Given $T(n) = T(an) + T(bn) + n$ [3], where $a, b < 1$. Let L_i be the ith level sum in recursive tree, then $L_{i+1} = (a+b)L_i$

Proof: Suppose L_i has m sub problem of size $P_1 \dots P_m$.

Then, let $L_i = \sum_{i=1}^{m} P_i$ and $L_{i+1} = \sum_{i=1}^{m} aP_i + bP_i = (a+b)\sum_{i=1}^{m} P_i = (a+b)L_i$ $\square$

We have:

- If $a + b < 1$, then $T(n) = \Theta(n)$;
- If $a + b = 1$, then $T(n) = \Theta(n \log n)$

Let $0 < c < 1$ be some constant. Suppose our pivot always had rank cn.

The running time would be:

QuickSort $T(n) = T(cn) + T((1-c)n) + n = \Theta(n \log n)$;

[3] n means cost is linear.

QuickSelect $T(n) = \max\{T(cn), T((1-c)n)\} + n = T(bn) + n = \mathcal{O}(n)$,
where $b = \max\{c, 1-c\}$.

Note that QuickSelect has slack. It spend an extra $T(an)$ time as long as $a + b < 1$.

Now we introduce the median on median select, the algorithm is described in algorithm 15.

Algorithm 15 Median of Median Select

```
1: procedure MoMSelect(A[1...n], k)
2:     if n ≤ 25 then
3:         Use brute force
4:     else
5:         m = ⌈n/5⌉
6:         for i = 1 to n do
7:             M[i] = MedianOfFive(A[5i − 4, ..., 5i])
8:         end for
9:         mom = MoMSelect(M[1...m], ⌊m/2⌋)
10:        r = Pivot(A[1...n], mom)
11:        if k < r then
12:            return MoMSelect (A[1...r − 1], k)
13:        else if k > 0 then
14:            return MoMSelect (A[r + 1...n], k − r)
15:        else
16:            return mom
17:        end if
18:    end if
19: end procedure
```

Observation: mom is greater then $m/2$ medians, each of which is ≥ 3 items, also less then $3n/10$ elements.

7 Dynamic Programming

7.1 Rod Cutting

7.1.1 Description of Problem

- steel rod of length n, where n is some integer.
- $P[1...n]$, where $P[i]$ is market price for rod of length i.

Question:

Suppose you can cut rod to any integer length for free. How much money can you made?

7.1.2 Analysis

- Consider leftmost cut of optimal solution.

  cut can be at positions $1...n$.

  If leftmost cut at i, then you get $P[i]$ for leftmost piece and then optimally sell remaining $n - i$ length rod.

- Don't know where to make first cut, so try them all and find

$$\max(0, \max(P[i] + cutRod(n - i)))$$

So, the first attempt of the algorithm could be described as algorithm 16.

Algorithm 16 First Attempt of Solving Cutting Rod Problem

```
1: procedure CutRod(n)
2:     if n = 0 then                          ▷ If the remaining rod length is 0. ◁
3:         return 0
4:     end if
5:     q = 0
6:     for i = 1 to n do
7:         q = max(q, P[i] + CutRod(n − i))
8:     end for
9:     return q
10: end procedure
```

Running time of algorithm 16:$T(n) = n + \sum_{i=0}^{n-i} T(i)$, which is clearly **Exponential** since there are a lot of subproblem overlap!

7.1.3 Memoized Version

algorithm 17 illustrates the memoized version of the algorithm in algorithm 16

Algorithm 17 Memoized Version of Solving Cutting Rod Problem

```
1: procedure MemRodCut(n)                          ▷ Globally define R[1..n] ◁
2:     if n = 0 then
3:         return 0
4:     end if
5:     if R[n] undefined then
6:         q = 0
7:         for i = 1 to n do
8:             q = max(q, P[i] + MemRodCut(n − i))
9:         end for
10:        R[n] = q
11:    end if
12:    return R[n]
13: end procedure
```

Note that $R[1...n]$ is filled in form left to right. It means we can store the result and use it later, which brings us to the dynamic programming version of the algorithm.

7.1.4 Dynamic Programming Version to Solve RodCut

algorithm 18 illustrates the dynamic programming version of the algorithm according to the memoized version algorithm 17.

Algorithm 18 Dynamic Programming Version of Solving Cutting Rod Problem

```
1: procedure DPRodCut(n)
2:     Let R[0...n] be an array.
3:     R[0] = 0
4:     for j = 1 to n do
5:         q = 0
6:         for i = 0 to j do
7:             q = max(q, P[i] + R[j − i])
8:         end for
9:         R[i] = q
10:    end for
11:    return R[n]
12: end procedure
```

Running time of algorithm 18: $T(n) = \mathcal{O}(n^2)$.

Note that the process only computes the total number. If we are to know how to cut, we can store the cutting position during the progress.

Define $C[1...n]$, and replace the inner for loop in algorithm 18 as:

Algorithm 19 Store the Cutting Position in the Process

```
1: for i = 0 to j do
2:     q = max(q, P[i] + R[j − i])
3:     C[j] = i
4: end for
5: R[i] = q
```

The for loop does the following:

- $C[j]$ stores last leftmost cut length for rod of length j.
- $C[n]$ says where to make first

Thus $C[n - C[n]]$ tells the second cut.

7.2 Solving DP problem

According to previous examples, we can summarize the general method to solve DP problem.

1. Write recursive solution, explain why the solution is correct.

2. Identify all subproblems considered.

3. Described how to store subproblems.

4. Find order to evaluate subproblems, s.t. subproblems you depend on evaluated ***before*** current subproblem.

5. Running Time: time to fill an entry X size table.

6. Write DP/Memoized algorithm.

7.3 Longest Increasing Subsequence (LIS)

7.3.1 Description of Problem

Input: Array $A[1...n]$ of integers.

Output: Longest subsequence of indices, $1 \leq i_1 < i_2 < ... < i_k < n$, s.t. $A[i_j] < A[i_{j+1}]$ for all j.

Warning: Subarray is "contiguous". So what is a subsequence?

- if $n = 0$, the onl subsequence is empty sequence.
- otherwise, a subsequence is either
    1. a subsequence of $A[2...n] or$,
    2. $A[1]$ followed by the subsequence of $A[2...n]$.

7.3.2 Analysis

Suggest recursive strategy for any array subsequence problem.

- if empty, do nothing.
- otherwise figure out whether to take $A[1]$ and let recursion fairy handle $A[2...n]$.

However, the definition of the subsequence is not fully recursive as stated, causing handling $A[2...n]$ depends on whether take $A[1]$.

To fix it, define LIS subsequence with all elements greater than some value as follow.

- LIS(prev, start) be the LIS in $A[start, n]$, s.t. all elements greater then $A[prev]$.
- Augment A s.t $A[0] = -\infty$, then LIS of $A[1...n]$ is LIS(0, 1).

Note that the idea of adding a $A[0]$ maybe useful in many scenarios.

Algorithm 20 Original Algorithm for LIS Problem

```
1: procedure LIS(prev, start)                          ▷ prev < start ◁
2:     if start > n then
3:         return 0
4:     end if
5:     ignore = LIS(prev, start + 1)
6:     best = ignore
7:     if A[start] > A[prev] then
8:         include = 1 + LIS(start, start + 1)
9:         if include > ignore then
10:            best = include
11:        end if
12:    end if
13:    return best
14: end procedure
```

LIS $(prev, start)$ is the length of longest increasing subsequence in $A[start \ldots n]$, s.t. all elements greater than $A[prev]$.

Observation:

The procedure LIS $(prev, start)$ has the following features:

- LIS $(prev, start)$ depends on LIS $(prev, start + 1)$
- So need 2D table $B[0 \ldots n][1 \ldots n + 1]$, each entry takes $\mathcal{O}(1)$ time to fill in, and can fill in any ordeer, s.t. $B[\,][start + 1]$ filled before $B[\,][start]$.

7.3.3 Dynamic Programming Version to Solve LIS

algorithm 21 illustrates the dp version to solve LIS problem.

Algorithm 21 Dynamic Programming Algorithm for LIS Problem

```
1: procedure LISDP(A[1...n])
2:     A[0] = −∞
3:     init B[0...n][1...n+1]
4:     for i = 0 to n do
5:         B[i][n+1] = 0
6:     end for
7:     for start = n to 1 do
8:         for prev = start − 1 to 0 do
9:             if A[prev] ≥ A[start] then
10:                B[prev][start] = B[prev][start+1]
11:            else
12:                B[prev][start] = max{(B[prev][start+1], 1 + B[start][start+1]}
13:            end if
14:        end for
15:    end for
16:    return B[0][1]
17: end procedure
```

The running time for algorithm 21 is the time to fill the table, i.e. $\mathcal{O}(n^2)$.

7.4 Longest Common Subsequence

7.4.1 Description of Problem

Input: Character arrays: $A[1 \ldots n]$, $B[1 \ldots m]$.

Output: Find length of longest common subsequence, i.e. find length k of longest pair of strings of indices.

$$1 \leq i_1 < i_2 < \ldots < i_k \leq n$$

and

$$1 \leq j_1 < j_2 < \ldots < j_k \leq m$$

s.t. for all $1 \leq l \leq k$, $A[i_l] = B[j_l]$.

Example:

$$A = \{\ \texttt{C G C A A T C C A G G}\ \}$$
$$B = \{\ \texttt{G A T T A C G A}\ \}$$

LCS = G A T C A.

The sequence is {2,4,6,8,9} and {1,2,3,6,8}. Note that the sequence may not be unique.

7.4.2 Analysis

Handle first element of A and B:

- if A or B is empty, return 0;
- if $A[1] \neq B[1]$, then $A[1]$ and $B[1]$ cannot both be used. So should be the best solution from throwing out $A[1]$ or $B[1]$, i.e.

$$LCS(A[1 \ldots n], B[1 \ldots m]) = \max\left\{ \begin{array}{l} LCS(A[2 \ldots n], B[1 \ldots m]), \\ LCS(A[1 \ldots n], B[2 \ldots m]) \end{array} \right\} \tag{13}$$

- if $A[1] = B[1]$, then can either match or throw both of them out, i.e

$$LCS(A[1 \ldots n], B[1 \ldots m]) = \max\left\{ \begin{array}{l} 1 + LCS(A[2 \ldots n], B[2 \ldots n]), \\ LCS(A[2 \ldots n], B[1 \ldots m]), \\ LCS(A[1 \ldots n], B[2 \ldots m]) \end{array} \right\} \tag{14}$$

Note that if $A[1] = B[1]$, why consider throw them out?

Though without proof, the option is right, but generally, the option should be considered.

The algorithm is described in algorithm 22. LCS $(curA, CurB)$ is the longest common sequence of $A[curA \ldots n]$ and $B[curB \ldots m]$.

Algorithm 22 Original Algorithm for LCS Problem

```
1: procedure LCS(curA, curB)
2:     if curA > n or curB > m then
3:         return 0
4:     end if
5:     ignore = max{LCS(curA + 1, CurB), LCS(CurA, CurB + 1)}
6:     best = ignore
7:     if A[curA] = B[curB] then
8:         include = 1 + LCS(curA + 1, curB + 1)
9:         if include > ignore then
10:            best = include
11:        end if
12:    end if
13:    return best
14: end procedure
```

To find LCS of $A[1 \ldots n]$, $B[1 \ldots m]$, call LCS $(1, 1)$.

Observation:

The procedure LCS $(curA, curB)$ has the following features:

- LCS $(curA, curB)$ depends on two parameters, need a 2D array of total size $\mathcal{O}(nm)$.
- LCS $(curA, curB)$ makes 3 recursive calls. All recursive calls have at least one parameter strictly larger, and none smaller.
- The table can be filled in two nested decreasing for loop.

7.4.3 Dynamic Programming Version to Solve LCS

The DP algorithm to solve LCS problem is showed in algorithm 23.

Algorithm 23 Dynamic Programming Algorithm for LCS Problem

```
1: procedure LCSDP(A[1...n], B[1...m])
2:     Define C[1...n+1][1...m+1]
3:     for i = 0 to n+1 do
4:         C[i][m+1] = 0
5:     end for
6:     for i = 0 to m+1 do
7:         C[n+1][i] = 0
8:     end for
9:     for curA = n to 1 do
10:        for curB = m to 1 do
11:            ignore = max{C[curA+1][CurB], C[curA][curB+1]}
12:            best = ignore
13:            if A[curA] = B[curB] then
14:                include = 1 + C[curA+1][curB+1]
15:                if include > ignore then
16:                    best = include
17:                end if
18:            end if
19:            C[curA][curB] = best
20:        end for
21:    end for
22:    return C[1][1]
23: end procedure
```

The operation to fill the table takes $\mathcal{O}(1)$ time, ignoring recursive calls.
So, total time is $\mathcal{O}(n \times m \times 1) = \mathcal{O}(nm)$.

7.5 Edit Distance

7.5.1 Description of Problem

Input: Character arrays: $A[1 \ldots m]$, $B[1 \ldots n]$

Output: Edit distance between A and B, which is the minimum number of characters insertion, deletion and substitution to turn A into B.

Example: For character arrays A and B:

$$A = \{\ \texttt{F O O D}\ \}$$
$$B = \{\ \texttt{M O N E Y}\ \}$$

The edit process is:

$$\begin{aligned}\texttt{F O O D} &\rightarrow \texttt{M O O D}\\ &\rightarrow \texttt{M O N D}\\ &\rightarrow \texttt{M O N E D}\\ &\rightarrow \texttt{M O N E Y}\end{aligned}$$

7.5.2 Analysis

Consider a better way to display edits:

Place A above B, put a gap in A for each insertion, gap in B for each deletion, i.e.

$$A = \{\ \texttt{F O O}\ \ \texttt{D}\ \}$$
$$B = \{\ \texttt{M O N E Y}\ \}$$

Then, the edit distance is the number of columns where characters don't match (in an optional alignment).

Let $edit(A, B)$ denotes the edit distance from A to B. Note that $edit(A, B) = edit(B, A)$.

Another example:

$$A = \{\texttt{A L G O R}\ \ \texttt{I}\ \ \texttt{T H M}\ \}$$
$$B = \{\texttt{A L}\ \ \texttt{T R U I S T I C}\ \}$$

If remove last column from an optimal alignment, the remaining must represent the shortest edit sequence for remaining substrings.

So now we can recursively define edit distance $edit(A, B)$:

- if $m = 0$, i.e. A is empty, then $edit(A, B) = n$;
- if $n = 0$, i.e. B is empty, then $edit(A, B) = m$;

- otherwise, $m, n \geq 1$, then look at last column in optimal alignment. There are three cases:

  a) insertion, i.e. top row empty, which means:
  - All of A remains to left;
  - All but last character of B to left;

$$edit(A[1 \ldots m], B[1 \ldots n]) = edit(A[1 \ldots m], B[1 \ldots n-1]) + 1$$

  b) deletion, i.e. bottom row empty, which means, like insertion:

$$edit(A[1 \ldots m], B[1 \ldots n]) = edit(A[1 \ldots m-1], B[1 \ldots n]) + 1$$

  c) substitution, i.e. both rows non-empty, which means:

$$edit(A[1 \ldots m], B[1 \ldots n]) = edit(A[1 \ldots m-1], B[1 \ldots n-1]) + [A[m] \neq B[n]]$$

  where the condition $[A[m] \neq B[n]]$ is 1 if true and 0 if false.

The algorithm is described in algorithm 24. $\textsc{Edit}(i, j) = edit(A[1 \ldots i], B[1 \ldots j])$.

Algorithm 24 Original Algorithm for Edit Distance

```
procedure EDIT(i, j)
    if i = 0 then
        return j
    end if
    if j = 0 then
        return i
    end if
    return min{EDIT(i − 1, j) + 1, EDIT(i, j − 1) + 1, EDIT(i − 1, j − 1) + [A[i] ≠ B[j]]}
end procedure
```

Observation:

The procedure $\textsc{Edit}\ (i, j)$ has the following features:

- i has m values.
- j has n values.
- store in table of size $\mathcal{O}(mn)$.

- EDIT (i, j) depends on three sub-problems, in each case either i or j smaller (and never larger).
- can fill in table with two nested increasing for loop.
- constant time per entry, so $\mathcal{O}(mn)$ overall.

7.5.3 Dynamic Programming Version to Solve Edit Distance Problem

The DP algorithm to solve Edit Distance problem is showed in algorithm 25.

Algorithm 25 Dynamic Programming Algorithm for Edit Distance Problem

```
1: procedure EDITDP(A[i...m], B[i...n])
2:     for j = 0 to n do
3:         E[0][j] = j
4:     end for
5:     for i = 0 to n do
6:         E[i][0] = i
7:     end for
8:     for i = 1 to m do
9:         for j = 1 to n do
10:            if A[i] = B[j] then
11:                E[i][j] = min{E[i − 1][j] + 1, E[i][j − 1] + 1, E[i − 1][j − 1]}
12:            else
13:                E[i][j] = min{E[i − 1][j] + 1, E[i][j − 1] + 1, E[i − 1][j − 1] + 1}
14:            end if
15:        end for
16:    end for
17:    return E[m][n]
18: end procedure
```

The operation to fill the table takes $\mathcal{O}(1)$ time, ignoring recursive calls.
So, total time is obviously $\mathcal{O}(mn)$.

7.6 Longest Convex Subsequence

7.6.1 Description of Problem

Input: Array $A[1 \ldots n]$ of integers.

Output: Longest subsequence of indices,

$$1 \leq i_1 < i_2 < ... < i_k < n$$

s.t. $2 \times A[i_j] < A[i_{j-1}] + A[i_{j+1}]$ for all j.

7.6.2 Analysis

If consider $A[i]$ as current element, whether choose it or throw it depends on its next element. Thus, consider $A[i+1]$ as current element.

Algorithm 26 Recursive Algorithm for Longest Convex Subsequence

```
1: procedure LXS(preprev, prev, cur)
2:     if cur > n then
3:         return 0
4:     end if
5:     ignore =LXS ( preprev, prev , cur + 1 )
6:     best = ignore
7:     if preprev = −1 or A[cur] > 2 × A[prev] − A[preprev] then
8:         include = 1 + LXS(prev, cur, cur + 1)
9:         if include > ignore then
10:            best = include
11:        end if
12:    end if
13:    return best
14: end procedure
```

Does not change.

Consider next element.

Call LXS $(-1, -1, 1)$ will get the longest convex sequence.

There are three parameter with n size, so the table is $\mathcal{O}(n^3)$ size.

Each entry require $\mathcal{O}(1)$ time to fill, so the total running time is $\mathcal{O}(n^3)$.

The third parameter *cur* is always strictly larger, so the outer for loop should be a decreasing for loop for *cur*. The inner two for loop is irrelevant.

7.7 Professor's note on LIS/LCS/LXS/SCS

Longest Increasing Subsequence

Globally defined array $A[1 \ldots n]$, augmented s.t. $A[0] = -\infty$. Assumes $0 \leq prev < start$.

```
1: procedure LIS(prev, start)
2:     if start > n then
3:         return 0
4:     end if
5:     ignore = LIS(prev, start + 1)
6:     best = ignore
7:     if A[start] > A[prev] then
8:         include = 1 + LIS(start, start + 1)
9:         if include > ignore then
10:            best = include
11:        end if
12:    end if
13:    return best
14: end procedure
```

Claim: $LIS(prev, start)$, for $prev < start$, returns the longest increasing subsequence in $A[start \ldots n]$ s.t. all elements are greater than $A[prev]$.

Proof: If $start > n$, there are no elements left in the remaining part of A, and so the algorithm correctly returns 0. Otherwise $LIS(prev, start)$ either includes $A[start]$ or not.

- if not, then $LIS(prev, start) = LIS(prev, start + 1)$.
- if so, then it must be that $A[start] > A[prev]$, and all remaining element of the LIS that come after $A[start]$ must be great than $A[start]$. Therefore, $LIS(prev, start) = LIS(start, start + 1) + 1$, where the $+1$ counts $A[start]$.

So if $A[start] \leq A[prev]$ the solution must be $LIS(prev, start + 1)$, which is what the algorithm returns, i.e. in this case the if statement is not executed. If $A[start] > A[prev]$ the solution is either $LIS(prev, start+1)$ or $LIS(start, start+1)$, whichever is bigger. Since we don't know which is bigger, our algorithm tries both and takes the max. Note in both case the problem is reduced to a subproblem on a strictly smaller array (i.e. $A[start + 1 \ldots n]$), and so can be assumed to be correctly handled by induction (where the base case is handled by the initial $start > n$ conditional). $\square$

To compute the LIS of $A[1 \ldots n]$ we call $LIS(0, 1)$ as this is the LIS in $A[1 \ldots n]$ s.t. all elements $> -\infty$, which is the same as the LIS of $A[1 \ldots n]$.

Applying DP: $LIS(prev, start)$ depends on two parameters, each ranging over $O(n)$ values, as they are both indices into $A[0 \dots n]$. Hence the above recursive algorithm can be turned into a DP algorithm using a 2D array, of total size $O(n^2)$. Note that $LIS(prev, start)$ only depends on $LIS(prev, start+1)$ and $LIS(start, start+1)$, both of which have a strictly large value of the second parameter. Therefore this table can be filled in any order such that all $LIS(\cdot, start + 1)$ values are computed before any $LIS(\cdot, start)$ value. Namely with a decreasing for loop for the second parameter, and a second inner loop going over all values of the first parameter (in any order). Ignoring the time for computing recursive calls, the above algorithm runs in $O(1)$ time. Therefore, if processed in the right order, each table entry takes $O(1)$ time to compute and so the total running time is $O(n^2)$.

Longest Common Subsequence

Input: Character arrays $A[1 \ldots n]$ and $B[1 \ldots m]$. Goal: Find the length of the longest common subsequence. Specifically, find the length k of the longest strings of indices $1 \leq i_1 \leq \ldots i_k \leq n$ and $1 \leq j_1 \leq \ldots j_k \leq m$ such that for all $1 \leq l \leq k$, $A[i_l] = B[j_l]$

If A is $[C, G, C, A, A, T, C, C, A, G, G]$ and B is $[G, A, T, T, A, C, G, A]$ an LCS is G, A, T, C, A

as witnessed by the index strings $2, 4, 6, 8, 9$ and $1, 2, 3, 6, 8$.

As this is a subsequence problem, similar to LCS, the focus is to figure out how to handle the very first element of A and B. We have the following.

- If A or B is empty, return 0.
- If $A[1] \neq B[1]$ then A[1] and B[1] cannot both be used, so it should be the best of either throwing out A[1] or B[1], i.e.
  $LCS(A[1 \ldots n], B[1 \ldots m]) = \max\{LCS(A[2 \ldots n], B[1 \ldots m]), LCS(A[1 \ldots n], B[2 \ldots m])\}$.
- If $A[1] = B[1]$ then we can either match or throw out so $LCS(A[1 \ldots n], B[1 \ldots m]) =$
  $\max\{1+LCS(A[2 \ldots n], B[2 \ldots m]), LCS(A[2 \ldots n], B[1 \ldots m]), LCS(A[1 \ldots n], B[2 \ldots m])\}$.

Note that if $A[1] = B[1]$ then one can prove $LCS(A[1 \ldots n], B[1 \ldots m]) = 1+LCS(A[2 \ldots n], B[2 \ldots m])$. While this may seem obvious, unless it is proven you cannot assume it, so we won't.

Now to turn this into a recursive algorithm, define $LCS(curA, curB)$ to be the longest common subsequence of $A[curA \ldots n]$ and $B[curB \ldots m]$ (i.e. $LCS(A[curA \ldots n], B[curB \ldots m])$) Based on the above observations we have the following.

Again assume $A[1 \ldots n]$ and $B[1 \ldots m]$ defined globally, and $curA, curB > 0$

```
1: procedure LCS(curA, curB)
2:     if curA > n or curB > m then
3:         return 0
4:     end if
5:     ignore = max{LCS(curA + 1, curB), LCS(curA, curB + 1)}
6:     best = ignore
7:     if A[curA] = A[curB] then
8:         include = 1 + LIS(curA + 1, curB + 1)
9:         if include > ignore then
10:            best = include
11:        end if
12:    end if
13:    return best
14: end procedure
```

To find the longest common subsequence of $A[1 \ldots n]$ and $B[1 \ldots m]$ we then call $LCS(1, 1)$.

The correctness follows immediately from the above (arguing the same way as for LIS).

Applying DP. $LCS(curA, curB)$ depends on two parameters, the first ranging over $O(n)$ values and the second over $O(m)$ values, since they are indices into $A[1 \ldots n]$ and $B[1 \ldots m]$, respectively . Hence the above recursive algorithm can be turned into a DP algorithm using a 2D array, of total size $O(mn)$. $LCS(curA, curB)$ makes at most three recursive call to $LCS(curA + 1, curB)$, $LCS(curA, curB + 1)$, and $LCS(curA + 1, curB + 1)$. In each case at least one of the two parameters increases and the other does not decrease. Therefore, the 2D array can be filled in using a pair of nested for the loops, the outer one ranging over the first parameter and starting at n and going to down 1, and the inner one ranging over the second parameter and starting at m and going down to 1. Ignoring the time for computing recursive calls, the above algorithm runs in $O(1)$ time. Therefore, if processed in the right order, each table entry takes $O(1)$ time to compute and so the total running time is $O(n^2)$.

```
1:  procedure LCSDP(A[1...n], B[1...m])
2:      Define C[1...n+1][1...m+1]
3:      for i = 1 to n+1 do
4:          C[i][m+1] = 0
5:      end for
6:      for i = 1 to m+1 do
7:          C[n+1][i] = 0
8:      end for
9:      for curA = n to 1 do
10:         for curB = m to 1 do
11:             ignore = max{C[curA+1][curB], C[curA][curB+1]}
12:             best = ignore
13:             if A[curA] = A[curB] then
14:                 include = 1 + C[curA+1][curB+1]
15:                 if include > ignore then
16:                     best = include
17:                 end if
18:             end if
19:             C[curA][curB] = best
20:         end for
21:     end for
22:     return C[1][1]
23: end procedure
```

Longest Convex Subsequence

Call a sequence $x_1 \dots x_m$ of numbers convex if $2x_i < x_{i-1} + x_{i+1}$ for all i. Describe an efficient algorithm to compute the length of the longest convex subsequence of an arbitrary array $A[1 \dots n]$ of integers.

In the following *preprev* may get set to the dummy value -1, which removes the convexity requirement.

```
1: procedure LXS(preprev, prev, cur)
2:     if cur > n then
3:         return 0
4:     ignore = LXS(preprev, prev, cur + 1)
5:     best = ignore
6:     if (preprev = −1) ∨ (A[cur] > 2A[prev] − A[preprev]) then
7:         include = 1 + LXS(prev, cur, cur + 1)
8:         if include > ignore then
9:             best = include
10:    return best
```

$LXS(preprev, prev, cur)$ computes the length of the longest convex subsequence in the array $A[cur \dots n]$, but with the requirement that the sequence is still convex after appending $A[preprev]A[prev]$ to the front (unless $preprev = -1$, in which case there is no additional requirement involving $A[preprev]A[prev]$). Thus the longest convex subsequence of A is given by $LXS(-1, -1, 1)$. Note that the optimal such subsequence either "includes" or "ignores" the element $A[cur]$ and the algorithm just returns the best of these two options (where including is only allowed if convexity is satisfied).

Each parameter in the above algorithm ranges over $O(n)$ values, and hence the DP table has size $O(n^3)$. Each entry takes $O(1)$ to compute so the total run time is $O(n^3)$. Every subproblem has a strictly larger last parameter hence the table can be filled with a decreasing outermost for loop for this parameter, and two nested inner for loops with any order for the other parameters.

Shortest Common Supersequence

Let $A[1 \ldots m]$ and $B[1 \ldots n]$ be two arbitrary arrays. A common supersequence of A and B is another sequence that contains both A and B as subsequences. Describe an efficient algorithm to compute the length of the shortest common supersequence of A and B.

```
1: procedure SCS(curA, curB)
2:     if curA > m then
3:         return n − curB + 1
4:     if curB > n then
5:         return m − curA + 1
6:     different = min{1 + SCS(curA + 1, curB), 1 + SCS(curA, curB + 1)}
7:     best = different
8:     if A[curA] = B[curB] then
9:         match = 1 + SCS(curA + 1, curB + 1)
10:        if match < different then
11:            best = match
12:    return best
```

Assuming we argued you can always take a match, this simplifies to:

```
1: procedure SCS(curA, curB)
2:     if curA > m then
3:         return n − curB + 1
4:     if curB > n then
5:         return m − curA + 1
6:     if A[curA] = B[curB] then
7:         return 1 + SCS(curA + 1, curB + 1)
8:     return min{1 + SCS(curA + 1, curB), 1 + SCS(curA, curB + 1)}
```

$SCS(curA, curB)$ is the length of the shortest common supersequence of $A[curA \ldots m]$ and $B[curB \ldots n]$, hence the shortest common supersequence of $A[1 \ldots m]$ and $B[1 \ldots n]$ is $SCS(1, 1)$. Specifically the next element in the shortest common supersequence is either $A[curA]$ or $B[curB]$, and the algorithm tries both options (where if $A[curA] = B[curB]$ then the supersequence only needs one character to cover both $A[curA]$ and $B[curB]$).

Each of the two parameters range over $O(n)$ values so use a DP table of size $O(n^2)$. Each entry takes $O(1)$ time to compute, so the total time is $O(n^2)$. At least one of the two parameters is larger (and the other not smaller), hence we can fill in the table with two nested decreasing for loops going over the two parameters.

7.8 Summary

Quote from professor:

> When solving the DP problem on the exam ask yourself:
>
> > "What is the minimal amount of information needed to determine whether I can take the current element as the next element in the subsequence?"
>
> For the longest convex subsequence I need to know what the previous two elements in the subsequence were. Shortest common super-sequence was even easier, as there is no requirement imposed by earlier elements. In both problems we then try all options of taking or not taking the current element(s) and return the best result.

For increasing/decreasing in the For-Loop, look for the parameter(s) that is strictly increasing or decreasing.

- If only one is increasing/decreasing, then the outer For-Loop should be decreasing/increasing about that parameter. The other For-Loop is irrelevant (for example the inner two For-Loop in LXS).
- If both parameter is increasing/decreasing, then two For-Loop should both be decreasing/increasing about its parameter. Which For-Loop is the outer one is irrelevant (for example the two For-Loop in SCS).

8 Greedy Algorithm

8.1 Greedy in a Glance

Greedy Algorithm is repeatedly taking locally optimal step in hopes reaching global optimal solution.

Often Greedy is WRONG!

So we should assume greedy is wrong, unless proven otherwise.

However, if we can prove greedy works in some circumstance, it would be a simple and fast solution in most cases.

8.2 Greedy Class Scheduling

8.2.1 Description of Problem

Target: On given day of week, we want to take as many classes as possible. Constraint: We cannot take two classes whose times overlap.

Input: Given $S[1 \dots n]$ and $F[1 \dots n]$, which

$$S[i] = \text{start time of class} i$$

$$F[i] = \text{finish time of class} i$$

Output: Select largest subset $X \subseteq \{1 \dots n\}$, s.t. for $i \neq j \in X$ either $S[i] > F[j]$ or $S[j] > F[i]$.

8.2.2 Analysis

Recursive approach: consider class 1.

$$B_4 = \{i | 2 \leq i \leq n \text{ and } F[i] < S[1]\}$$

$$L_8 = \{i | 2 \leq i \leq n \text{ and } S[i] > F[1]\}$$

DP approach: $\mathcal{O}(n)$ running time.

But we can do better with greedy: Pick class which finishes first.

In other word, the algorithm is:

> Scan classes in increasing order of finishing time. Each time encounter non-conflicting class, select it.

as described in algorithm 27.

Algorithm 27 Greedy Solution for Class Scheduling

```
1: procedure GreedySchedule(S[1...n], F[1...n])
2:     Sort F and permute S to match.
3:     count = 1
4:     X[count] = 1                                   ▷ X[1...n] ◁
5:     for i = 2 to n do
6:         if S[i] > F[X[count]] then
7:             count = count + 1
8:             X[count] = i
9:         end if
10:    end for
11:    return X[1...count]
12: end procedure
```

Running time: $\Theta(n \log n) \rightarrow$ sorting running time.

8.2.3 Proof

Note that there may be many optimal solutions. And Greedy produces unique solution. So we cannot argue any optimal solution looks like greedy.

Lemma 8.2.1. *At least one optimal solution include class that finishes first.*

Proof: Let f be class that finishes first, X be an optimal set of classes. If $f \in X$ then lemma proven. Otherwise, $f \notin X$. Let g be class in X that finishes first. Since f finishes before g, f cannot conflict with any class in $X \setminus \{g\}$

Thus $X' = X \cup \{f\} \setminus \{g\}$ is a valid solution with max size and contain f. □

Theorem 8.2.2. *Greedy schedule is optimal.*[4]

Proof: Let f be class that finishes first. L is set that starts after f finishes. Best schedule containing f is optimal. This best schedule must be optimal on L. L is strictly smaller set of classes, so can apply induction. □

[4]Can be proved using induction

8.3 General Prove Method for Greedy Algorithm: Exchange Argument

1. Assume there is a optimal solution other than greedy algorithm.
2. Fine the "first" difference between it and greedy algorithm.
3. Argue can exchange optimal choice for greedy one, without degrading solution value.

8.4 Huffman Codes

8.4.1 Description of Problem

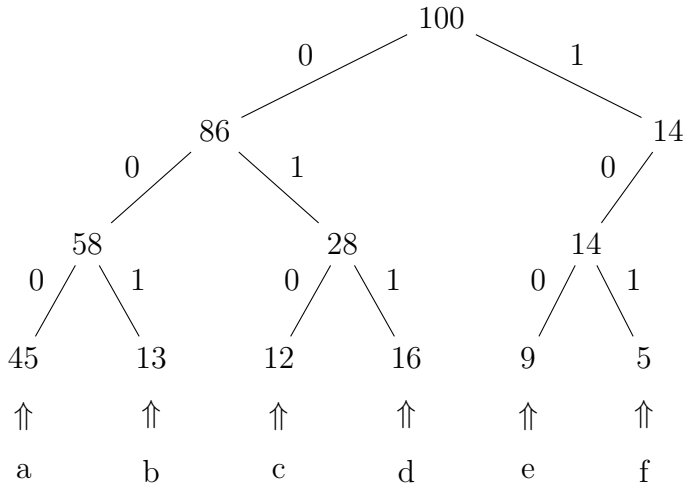
Input: Given an alphabet Σ.

Output: A binary code for Σ maps every $x \in \Sigma$ to a unique binary string.

Example: Fixed length codes: Map Σ to strings all having same length.
Given Alphabet: $\Sigma = \{a, b, c, d, e, f\}$
Frequency for each character $F = \{45, 13, 12, 16, 9, 5\}$.

$a = 000$	$b = 001$	$c = 010$
$d = 011$	$e = 100$	$f = 101$

Figure 6: Binary Code Tree for Fixed Length Codes



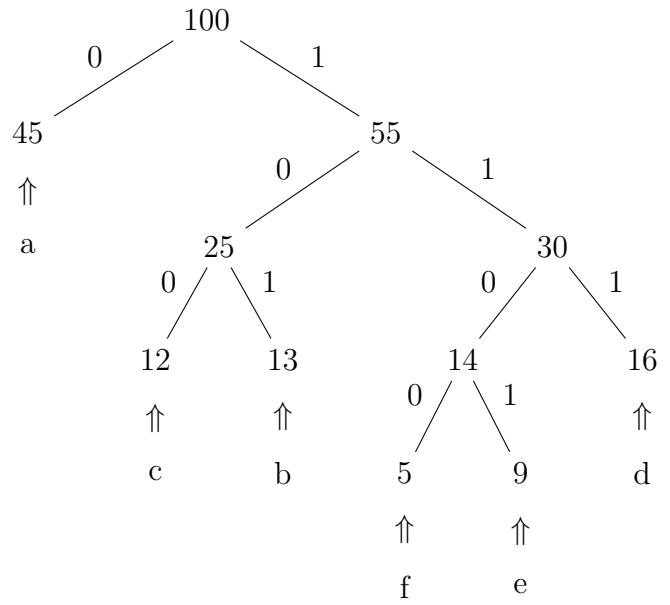
The length of coded files:

$$3 \times 45 + 3 \times 13 + 3 \times 12 + 3 \times 14 + 3 \times 16 + 3 \times 9 + 3 \times 5$$
$$= \sum F[i] \times depth(i)$$

What if the codes is not fixed length?

$a = 0$	$b = 101$	$c = 100$
$d = 111$	$e = 1101$	$f = 1100$

Figure 7: Binary Code Tree for Not Fixed Length Codes



The length of coded files:

$$1 \times 45 + 3 \times 13 + 3 \times 12 + 3 \times 16 + 4 \times 9 + 4 \times 5$$
$$= 224$$

Note that, in order to decode, the codes need to be prefix-free, i.e. no code word is a prefix of another, which means characters must at leaves.

So here comes the greedy algorithm: Huffman Codes.

8.4.2 Definition

Input: Alphabet Σ of size n and the frequency array $F[1 \dots n]$, $F[i]$ is the frequency of character $\Sigma[i]$.

Output: Find a prefix free tree, s.t. $\sum F[i] \times depth(i)$ minimized.

Huffman Algorithm: Recursively merge two least frequent.

8.4.3 Analysis

Claim 8.4.1. *Huffman's Algorithm is optimal.*

Observation: Optimal code trees are full binary tree (gbt[5]). In other word, leaf at maximum depth has a sibling.

Lemma 8.4.2. *Let x and y be two least frequent characters. There is an optimal tree where x, y are siblings.*

Proof: Let T be an optimal tree of depth d.

By observation, there are siblings a, b at depth d. If $a = x$ and $b = y$, then proved.

Let T' be the tree obtained by swapping x and a.

Then the depth of x increased by $d - depth(x, T) = z$, and the depth of a decreased by z, so,

$$Cost(T') = Cost(T) - (F[a] - F[x]) \times z$$

where x was least frequent. Thus

$$F[a] - F[x] \geq 0 \Rightarrow Cost(T') \leq Cost(T)$$

Based on assumption, T is a optimal tree, so $Cost(T) \leq Cost(T')$.

Hence, T equals to T' $\square$

Theorem 8.4.3. *Huffman codes is optimal.*

Proof: Assume $F[1]$, $F[2]$ is least frequent. Define $F[n+1] = F[1] + F[2]$.

Let T' be the Huffman tree on $F[3 \dots n+1]$, T be the Huffman tree on $F[1 \dots n]$ obtained from T' by replacing $n+1$ with 1 and 2.

[5] Definition is in homework 1.

For any prefix tree $\overline{T}$ obtained from any $\overline{T'}$ on $F[3 \dots n+1]$ by replacing $n+1$ with 1 and 2 can prove

$$Cost(\overline{T}) = Cost(\overline{T'}) + F[1] + F[2]$$

Use Induction:

Base Case: For one or two characters, Huffman is optimal. Now want to prove Huffman tree T is optimal.

Induction step: Assume T' is an optimal tree, then $Cost(\overline{T}) = Cost(\overline{T'}) + F[1] + F[2]$, which means minimizing $Cost(\overline{T})$ is equivalent to minimizing $Cost(\overline{T'})$.

T is optimal among trees on $\Sigma[1 \dots n]$ obtained by an expanding $n+1$ node to 1 and 2 from an optimal tree on $F[3 \dots n+1]$.

In conclusion, Huffman is optimal. $\square$

Greedy Ends Here.

9 Graph Algorithm

9.1 Basic Stuff

9.1.1 Fundamental Concept

The following is the basic concept of graph.

1. A **graph** is a pair $G = (V, E)$.
2. V is set of vertices or nodes.
3. E is set of edges. An edge e in E is a pair of vertices $e = \{u, v\}$.
4. If G is undirected, e is unordered, i.e. $e = uv = vu$.
   If G is directed, e is ordered, i.e. $e = u \to v$ or (u, v).
5. Graphs represent relations between pairs of objects.

In this course, we mainly consider simple graphs, no multi-edges and no self-loop.

9.1.2 Concepts Used in this Note

The following conclude the terminology and notations used in this notes.

1. The **degree** of v is the number of adjacent edges.
2. n denotes $|v|$, m denotes $|E|$.
3. For undirected graph: $m \leq \binom{n}{2}$. For directed graph: $m \leq 2\binom{n}{2}$.
4. Sub-graph of $G = (V, E)$ is a graph $G' = (V', E')$, s.t. $V' \subseteq V$, $E' \subseteq E$.
5. A **walk** is a sequence $v_1 \dots v_l$, s.t. $v_i \in V$ and $v_i v_{i+1} \in E$.
6. A **path** is a walk where v_i distinct.
7. A walk is close, if $v_i = V_k$.
8. A cycle is a "closed path".
9. Undirected graph connected if path between every pair $u, v \in V$.
10. If not connected, a component is a maximal connect sub-graph.

11. A **tree** is a connected "acyclic" graph.
12. A **forest** is a graph where each component is a tree.
13. A **Spanning Tree** of G is a sub-graph that is a tree and contains all vertices of G.
14. For directed graps: directed path/cycles.
15. A graph is **Strongly Connected** if directed path from any vertex to another exists.
16. Directed acyclic graph is called a **DAG**.

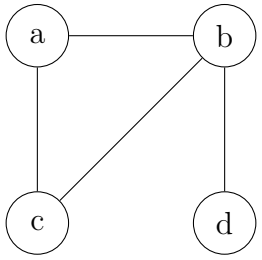
9.1.3 Graph Data Structure

There are two widely used data structure of graph:

- Adjacency Matrix: $|V| \times |V|$ matrix, where $A[i, j] = 1$ if $(i, j) \in E$.
- Adjacency List: an array of length $|V|$, where every entry in the array stores a list of neighbors.

For example:

Figure 8: Example Graph



The adjacency matrix and adjacency list for fig. 8 would be:

	A	B	C	D
A	0	1	1	0
B	1	0	1	1
C	1	1	0	0
D	0	1	0	0

Table 5: Adjacency Matrix

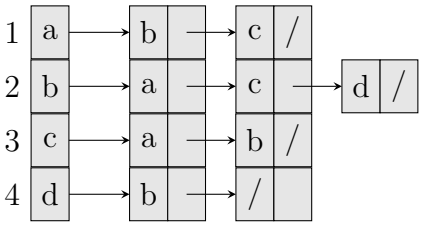
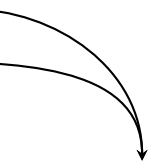


Figure 9: Adjacency List

The running time for these two data structures are shown in table 6.

Table 6: Running Time Analysis for Adjacency Matrix and Adjacency List

Time	Adjacency Matrix	Adjacency List
visit neighbor	$\mathcal{O}(1)$	$\mathcal{O}(1)$
visit all neighbors	$\mathcal{O}(n)$	$\mathcal{O}(degree(v))$
test if $uv \in E$	$\mathcal{O}(1)$	$\mathcal{O}(degree(v))$
add/delete edge	$\mathcal{O}(1)$	$\mathcal{O}(n)$
size	$\mathcal{O}(mn)$	$\mathcal{O}(m+n)$



$\mathcal{O}(1)$ with hashing.

9.2 Traversing Graph

9.2.1 Intuitive Approach

Assume G is a connected, undirected graph. Suppose we want to visit every vertex in G, we can use Depth First Search (DFS) is a recursive manner, as shown in algorithm 28.

Algorithm 28 Recursive Depth First Search Algorithm

```
1: procedure RecursiveDFS(v)
2:     if v unmarked then
3:         Mark v                                  ▷ has v been visited. ◁
4:         for each edge vw. do
5:             RecursiveDFS (w)
6:         end for
7:     end if
8: end procedure
```

Initially, call RecursiveDFS (s), where $s \in V$ is some start vertex.

Note that the functions was called recursively, forming an implicit stack. We can re-write as an iteration using an explicit stack, as shown in algorithm 29.

Algorithm 29 Iterative Depth First Search Algorithm

```
1: procedure IterativeDFS(s)
2:     Push (s)
3:     while stack not empty do
4:         s = Pop()
5:         if v is unmarked. then
6:             Mark v
7:             for each edge vw do
8:                 if w is unmarked then        ▷ This if is optional in some cases. ◁
9:                     Push (w)
10:                end if
11:            end for
12:        end if
13:    end while
14: end procedure
```

Observation: Traversal algorithms store candidate vertices in a "bag", which can be any structure allowing insertion/deletion.

9.2.2 Generic Traverse Algorithm

So we can define this generic traverse algorithm as shown in algorithm 30.

Algorithm 30 Generic Traverse Algorithm

```
1:  procedure TRAVERSE(s)
2:      Put s in bag
3:      while bag is not empty do
4:          Take v from bag.
5:          if v is unmarked then
6:              Mark v
7:              for each edge vw do
8:                  if w is unmarked then
9:                      Put w in bag.
10:                 end if
11:             end for
12:         end if
13:     end while
14: end procedure
```

This step is ambiguous.

▷ This *if* is optional in some cases. ◁

In other application, this line may bot be needed, as for traversing, it is needed.

Observation: This traverse algorithm clearly marks each vertex at most once. Now we want to prove it marks each vertex at least once, i.e. "visit" every vertex. To do this, we can remember each vertex's parent in the process. The modified algorithm is shown in algorithm 31

Algorithm 31 Remember Parent version of Generic Traverse Algorithm

```
1: procedure TRAVERSE(s)
2:     Put (∅, s) in bag
3:     while bag is not empty do
4:         Take (p, v) from bag                          ▷ O(1) per round. ◁
5:         if v is unmarked then          ▷ O(m) overall, regardless of inner for ◁
6:             Mark v
7:             Parent(v) = p
8:             for each edge vw do                       ▷ O(m) overall. ◁
9:                 if w is unmarked then
10:                    Put (v, w) in bag
11:                end if
12:            end for
13:        end if
14:    end while
15: end procedure
```

Lemma 9.2.1. TRAVERSE (s) *marks every vertex in connected graph exactly once.*

Set of pair $(parent(v), v)$ *with* $parent \neq \emptyset$ *defines spanning tree.*

Proof: s is marked, so let $v \neq s$.

Let $(s, \ldots, u, v)$ be shortest path from s to v, use induction on path length. By induction, u is visited, which implies that when u is marked, (u, v) is put in the bag.

Call a pair $(parent(v), v)$, s.t. $parent(v) \neq \emptyset$ a parent edge. Consider the path $(v, parent(v), parent(parent(v)), \ldots)$, this path eventually lead to s. If every vertex has path to s, then parent edge define connected spanning graph. □

Running Time for Generic Traverse Algorithm

Each vertex marked exactly once. Each edge (v, w) added to bag at most once by v and at most once by w.

Inner loop executed at most $\mathcal{O}(m)$ time. Outer loop $\mathcal{O}(m)$ time. In total, $\mathcal{O}(m)$ time.

Note that, this running time analysis ignore bag operation time (consider $\mathcal{O}(1)$ for all bag operations). If bag operation takes b per operation, $\mathcal{O}(mb)$ total time is required.

For connected graph, $n = \mathcal{O}(m)$.

9.2.3 Usage of Using Generic Travers Algorithm

1. Bag is stack $\Longrightarrow$ DFS (results in a long skinny tree).

2. Bag is queue $\Longrightarrow$ BFS (results in a short fat tree).

3. If edges have weight, and bag is min priority queue on edge weights.
   $\Longrightarrow$ MST, Prim's Algorithm.

4. Bag is min priority queue on vertex distance.
   $\Longrightarrow$ Dijkstra's Algorithm, shortest path tree.

If G is disconnected, we can use a wrap functions to traverse, as shown in algorithm 32.

Algorithm 32 Wrapper for Traverse

```
1: procedure TraverseAll
2:     for all vertices v do
3:         if v is unmarked then
4:             Traverse (v)
5:         end if
6:     end for
7: end procedure
```

Lemma 9.2.2. TraverseAll () *marks every vertex in disconnected graph exactly once. Set of pair* $(parent(v), v)$ *with* $parent \neq \emptyset$ *defines spanning forest.*

9.3 Standard Graph Algorithm

Step 1 Maintain set S of visited vertices, and $T \setminus S$ of unvisited.

Step 2 Maintain tree over S.

Step 3 Repeat:

(a) Pick vertex w in T, which is adjacent to S.

(b) Add to S, remove from T, update tree on S.

9.4 Depth First Search

9.4.1 Formal Definition

The Formal definition of DFS is shown in algorithm 33.

Algorithm 33 Formal Definition of DFS

```
1: procedure DFS(v)
2:     Mark v
3:     for each edge vw do
4:         if w is unmarked then
5:             DFS (w)
6:         end if
7:     end for
8: end procedure
```

Observation: The procedure create a similar stack of that in algorithm 30, but it's the program's stack. When DFS (w) makes recursive call, it stores current program on program stack, i.e. when DFS (v) calls DFS (w), DFS (v) is on stack.

Given v in undirected G, DFS (v) visits all vertices in v's component.

Algorithm 34 DFS All the Component

```
1: procedure DFSAll(G)
2:     for all v ∈ V do
3:         if v is unmarked then
4:             DFS (v)
5:         end if
6:     end for
7: end procedure
```

9.4.2 Count and Label Components

DFS Algorithm can be modified to count and label all the components.

Algorithm 35 Count and Label the Components

```
1: procedure CountAndLabel(G)
2:     count = 0
3:     for all v ∈ V do
4:         if v is unmarked. then
5:             count = count + 1
6:             LabelComponent (v, count)
7:         end if
8:     end for
9:     return count
10: end procedure
11: procedure LabelComponent(v, count)
12:     Mark v
13:     component(v) = count
14:     for each vw do
15:         if w is unmarked then
16:             LabelComponent (w, count)
17:         end if
18:     end for
19: end procedure
```

9.4.3 Pre/Post Order Traverse of Binary Tree

Originally, the definition are:

- Pre-order: visit, recursive call on left, recursive call on right.
- Post-order: recursive call on left, recursive call on right, visit.

Consider the problem in the view of DFS:

- Pre-order: vertices put on stack;
- Post-order: vertices taken off from stack.

A modified algorithm is shown in algorithm 36

Algorithm 36 Pre/Post Order Based on DFS

```
1: procedure DFSAll(G)
2:     clock = 0
3:     for all v ∈ V do
4:         if v is unmarked then
5:             DFS (v)
6:         end if
7:     end for
8: end procedure
9: procedure DFS(v)
10:    Mark v
11:    Previsit (v)
12:    for each edge vw do
13:        if w is unmarked then
14:            PreVisit (w)
15:        end if
16:    end for
17:    PostVisit (v)
18: end procedure
19: procedure PreVisit(v)
20:    pre(v) = clock
21:    clock = clock + 1
22: end procedure
23: procedure PostVisit(v)
24:    post(v) = clock
25:    clock = clock − 1
26: end procedure
```

9.4.4 Directed Graphs and Reachability

- For undirected graph, DFS (v) explores the component of v.
- For directed graph, it is trickier.
  - For v, u in directed graph, u is reachable from v, if directed path from v to u.
  - Let Reach (v) be set of vertices reachable from v.
  - If $v \in$ Reach(v), v may or may not be in Reach (u).

9.5 Directed Acyclic Graphs

9.5.1 Definition of DAG

A **DAG** is a directed graph with no directed cycles.

- For a DAG, if $u \in \textsc{Reach}(v)$, then $v \notin \textsc{Reach}(u)$.
- A source is a vertex with no incoming edges.
- A sink is a vertex with no outgoing edges.
- Every DAG has at least one source and at least one sink.
- Checking if graph is a DAG need $\mathcal{O}(n+m)$ time using DFS.

To make concept simpler:

- Let G be a directed graph.
- G' obtained by adding source s, a directed edge from s to all vertices of G.
- G is a DAG $\iff$ G' is a DAG (if and only if).

9.5.2 Algorithm to Determine a DAG

To illustrate is a program manner,consider 3 status during the traversal.

- New: never been of program stack;
- Active: is currently on the stack;
- Done: is off stack, but was previously on.

Thus, the DFS algorithm can be modified as shown in algorithm 37

Algorithm 37 Determine Whether a Graph is DAG

```
1: procedure IsAcyclic(G)
2:     Add vertex s
3:     for all v ≠ s in V do
4:         Add edge s → v
5:         Status (v) = New
6:     end for
7:     return IsAcyclicDFS (s)
8: end procedure
9: procedure IsAcyclicDFS(v)
10:    Status (v) = Active
11:    for each edge v → w do
12:        if Status (w) = Active then
13:            return False
14:        else if Status (w) = New then
15:            if IsAcyclicDFS (w) = False then
16:                return False
17:            end if
18:        end if
19:    end for
20:    Status (v) = Done
21:    return True
22: end procedure
```

9.5.3 Proof of Correctness

Prove "False" implies cycle.

Suppose "False" is returned, then find edge $v \to w$, s.t. Status (w) = Active.

- If w is Active, then it's on the stack and v is currently on top of stack.
- If vertex b directly on top of c on stack, then there must be an edge $c \to b$.
- Thus there must be a direct path $w \to \ldots \to v$, appending $v \to w$ gives a cycle.

Prove cycle leads to "False".

Let w be first vertex in cycle that we visit, v is vertex before w in cycle. Then, there exist a path $w \to \ldots \to v \to w$.

Since w visited first. IsAcyclicDFS (w) called first in the cycle.

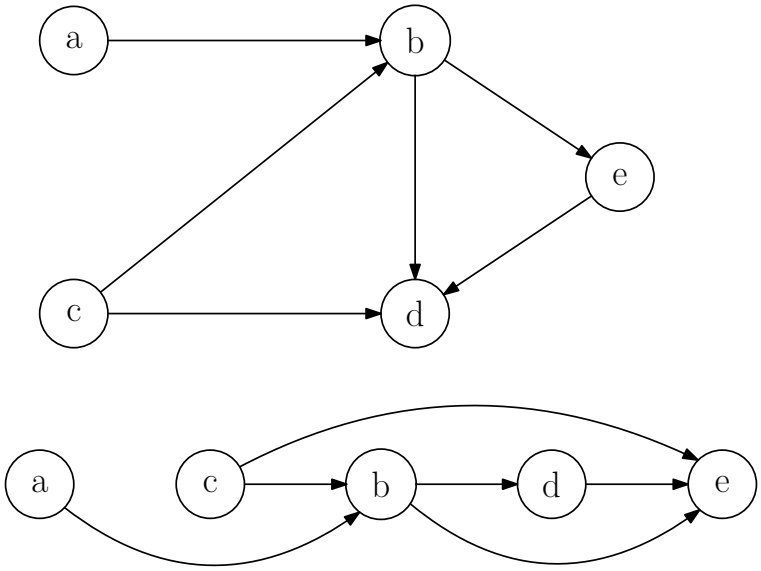
There is a directed path from w to v, so IsAcyclicDFS (v) is called while w on stack. During IsAcyclicDFS (v), $v \to w$ is looked at. Thus, result in False.

9.6 Topological Sort

Given directed G, topological ordering is a total order $\prec$ on the vertices s.t. $u \prec v$ for every edge $u \to v$.

Informally, a topological sort is the order arranges vertices from left to right, s.t. all edges go from left to right. An example is shown in fig. 10

Figure 10: Example for Topological Sort



Observation:

The most important fact:

> G has a topological ordering if and only if no cycles exist, i.e. G is a DAG.

A DAG has at least one source/sink, thus the topological sort has two implementation, one starts from sink, the other starts from source. The algorithm is shown in algorithm 39 and algorithm 38.

Algorithm 38 Topo-Sort from Source

```
1: procedure TopologicalSort(G)
2:     n = |V|
3:     for i = 1 to n do
4:         v = any Source
5:         S[i] = v
6:         Delete v
7:     end for
8:     return S[1...n]
9: end procedure
```

Algorithm 39 Topo-Sort from Sink

```
1: procedure TopologicalSort(G)
2:     n = |V|
3:     for i = n to 1 do
4:         v = any Sink
5:         S[i] = v
6:         Delete v
7:     end for
8:     return S[1...n]
9: end procedure
```

In the algorithm, $v =$ any Sink/Source is ambiguous. Naively, the algorithm takes $\mathcal{O}(nm)$ time.

Lemma 9.6.1. *For any DAG, first vertex marked* Done *by* IsAcyclic () *is a sink.*

Proof: Let v be first vertex marked Done. Suppose $v \to w$ exists.

- If w marked Done, results in contradiction.
- If w marked Active, then there is a cycle.
- If w marked New, then recursively call on w. Thus w marked Done before v.

In all three cases, v must be a sink. □

Observation:

- IsAcyclic (G) is modified DFS.
- First vertex marked Done is first off stack.
- First off stack is the first post order.
- This behavior repeats.

Thus, Topological order is reverse post order traverse.

Topological ordering computed in $\mathcal{O}(n+m)$ time with DFS.

Why does linear time matter?

- Scheduling: two jobs $a \to b$ means a depends on b.
  Reverse topological order is valid job scheduling.

- Dynamic Programming.
  Recursive sub-problem is a vertex, edges to other sub-problems you depend on.

9.7 Strong Connectivity

u and v are in same strong component if $u \in \text{REACH}(v)$ and $v \in \text{REACH}(u)$.

Strong Component defines an eequivalent relation.

- If u, v in some SC, and v, w in some SC, then u, w in same SC.
- Define a vertex partition: given two strong components S_1, S_2, $S_1 \cap S_2 = \emptyset$, and every vertex in same SC.

Denote SSC (G) as Strong Connected Component Graph, obtained by collapsing each component to vertex, each edge from S_1 to S_2, if and only if $\exists u \in S_1, v \in S_2$ s.t. $u \to v$ in G.

SSC (G) is DAG. In a DAG, every vertex is its own SC.

To compute SSC (G), call a component sink if sink in SSC (G). Observe that for sink components, DFS (v) for $v \in S$ explores all vertices in S. Thus, the algorithm can be defined as algorithm 40.

Algorithm 40 Algorithm to Compute Strong Components

```
1: procedure StrongComponents(G)
2:     count = 0
3:     while G is not empty do
4:         count = count + 1
5:         v = any vertex in sink component.
6:         c = DFSLabel(v, count)
7:         Remove c from G.
8:     end while
9: end procedure
```

9.8 Minimum Spanning Tree (MST)

9.8.1 MST Definition

Input: Undirected, weighted, connected graph G. Assume there is a function $w : E \to R$

Output: Find the minimum spanning tree of G, i.e. the spanning tree T which minmimizes $w(T) = \sum_{e \in T} w(e)$.

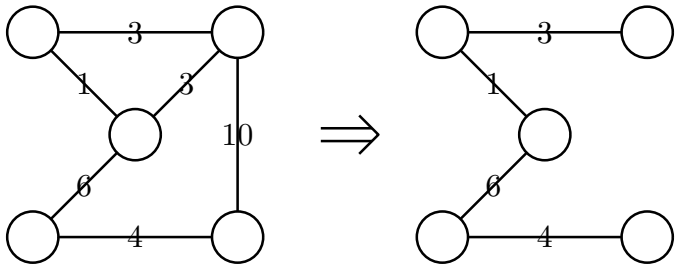
9.8.2 MST Property

Why use MST?

If vertices represent nodes in a network, and weights represent connection cost. MST is cheapest way to connect all nodes.

A example of MST is shown in fig. 11.

Figure 11: Example for MST



Observation:

Some properties are obvious:

- If all weights distinct, can prove MST is unique.
- If all weights equal, any spanning tree is a MST.

Main MST Property

A partition of V is a pair $S, R \subseteq V$, s.t. $S \cap R = \emptyset$ and $S \cup R = V$. Given a partition $S, R \subseteq V$, MWE (S, R) denotes minimum weight edge $u \to v$, s.t. $u \in S$, $v \in R$.

- For any partition S, R, MWE$(S, R) \in MST$.
- If e is not minimum weight edge of any partition, then $e \notin$ MST(G).

Proof: First, prove that for any partition S, R, MWE$(S, R) \in MST$.

Let S, R be a partition and let $e =$ MWE(S, R).

Suppose $e = u \to v$, where $u \in S$, $v \in R$. Consider any spanning tree T, which does not contain e. T contains unique path from u to v. Some edges in this path has endpoint in S and another in R. Let e' be such an edge.

Remove e' from T, gives a spanning forest, with two trees. u and v must be in different trees.

Hence adding e connects two trees, producing single spanning tree $T' = T - e' + e$. $weight(e) < weight(e')$, so $weight(T') < weight(T)$. T is not MST, result in contradiction.

Then, prove that if e is not minimum weight edge of some partition, then it's not in MST.

Let T be the MST. Consider any edge $e' \in T$. e' define partition of V.

Just proved minimum weight edge across this partition must be in MST. Since e' is the only edge in T crossing this partition, it must be min weight. □

9.8.3 MST Algorithm (General)

General idea of MST algorithm is:

> Maintain a spanning forest, initially is n isolated vertices. Add an edge of MST, update forest, and repeat.

All the algorithm follows is the variation of this general idea.

9.8.4 Prim's Algorithm

Key point

- Pick vertex s. Grow MST from s, call it T.
- In each round, find the edge $e = \text{MWE}(T, V \setminus T)$. Add e to T and repeat.
- To find e, store edges adjacent to T in min priority queue.

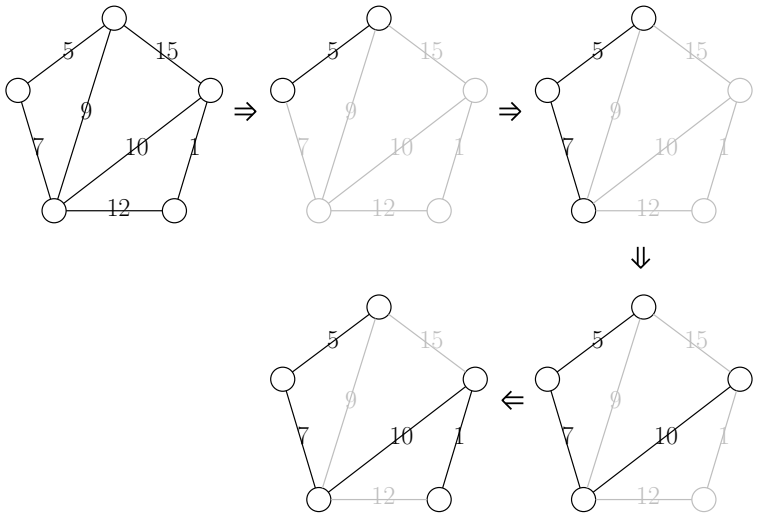
The algorithm is shown in algorithm 41

Algorithm 41 Prim's Algorithm

```
1: procedure PRIM(s)
2:     Initial an empty minimum priority queue Q
3:     Mark s
4:     for all edges s → w do
5:         Put (s, w) in Q
6:     end for
7:     while Q is not empty do
8:         (p, v) = Q.pop
9:         if v is unmarked then
10:            Mark v
11:            PARENT(v) = p
12:            for each edge v → w do
13:                Put (v, w) in Q
14:            end for
15:        end if
16:    end while
17: end procedure
```

Observation: This traverse with bag = min queue. Marked mean in tree, not in queue. An example is shown in fig. 12.

Figure 12: Example for MST



Analysis:

Each time add MWE $(T, V \setminus T)$, which we know is in MST, so correct.

Recall that for traverse, running time depend on bag operation cost. Specifically each edge is added/renewed from Q once, and this dominates runtime.

If Q implemented in binary min heap, all operations take

$$\mathcal{O}(\lg heapsize) = \mathcal{O}(\log(m)) = \mathcal{O}(\log(n))$$

time. Thus total time is $\mathcal{O}(m \log n)$

Note that the running time can be improved to $\mathcal{O}(m + n \log n)$ time using Fibonacci Heap, see Jeff's note.

9.8.5 Borvka's Algorithm

Key point

- Start with forest of singletons, $F = (v, \emptyset)$.
- Repeat until F is a single tree: For every $T \in F$, add MWE $(T, V \setminus T)$.

The algorithm is shown in algorithm 42

Algorithm 42 Borvka's Algorithm

```
1: procedure BORVKA(V, E)
2:     F = (V, ∅)
3:     count = COUNTANDLABEL(F)
4:     while count > 1 do
5:         ADDALLMWE (E, F, count)
6:         count = COUNTANDLABEL(F)
7:     end while
8:     return F
9: end procedure
10:
11: procedure ADDALLMWE(E, F, count)
12:     for i = 1 to count do
13:         S[i] = NULL                          ▷ w(null) = ∞ ◁
14:     end for
15:     for each edge uv ∈ E do
16:         if label(u) ≠ label(v) then
17:             if weight(uv) < w(S[label(u)]) then
18:                 S[label(u)] = uv
19:             end if
20:             if weight(uv) < w(S[label(v)]) then
21:                 S[label(v)] = uv
22:             end if
23:         end if
24:     end for
25:     for i = 1 to count do
26:         if S[i] ≠ NULL then
27:             add S[i] to F
28:         end if
29:     end for
30: end procedure
```

$S[i]$ stores minimum weight edge with exactly one endpoint in tree i.

Analysis:

The algorithm is correct since add MWE $(T, V \setminus T)$ for each $T \in F$.

The running time:

- CountAndLabel (F) takes $\mathcal{O}(n)$ time, since F has $\mathcal{O}(n)$ edge.
- AddAllMWE () takes $\mathcal{O}(m)$ time, since $\mathcal{O}(m+n) = \mathcal{O}(m)$.
- While-loop takes $\mathcal{O}(m+n) = \mathcal{O}(m)$ per iteration, in total performs $\mathcal{O}(\lg n)$ iterations.

Thus, total running is $\mathcal{O}(m \log n)$.

9.8.6 Kruskal's Algorithm

Key point

- Sort edges in increasing weight order.
- Go through edges in increasing order, add edge e if its endpoints in different trees.

Correctness 1 Suppose we add e joing $T_1, T_2 \in F$, then e must be MWE $(T_1, V \setminus T_1)$. If not, then there was cheaper edge e', but in which case, e' should be add first.

The difficulty is maintaining F. For Borvka, spend $\mathcal{O}(m)$ time to update F. Hence, now consider edges one at a time leads to $\mathcal{O}(m \times m) = \mathcal{O}(m^2)$ time.

Union Find

Union Find is a data structure which can do the following:

- MakeSet (V): make set (i.e. tree in forest) with just V.
- Find (u): return set id (some vertex in tree).
- Union (u, v): merges sets containing u and v.

All operation of Union Find take $\mathcal{O}(\propto(n))$ time.

The algorithm is shown in algorithm 43

Algorithm 43 Kruskal's Algorithm

```
1: procedure KRUSKAL(s)
2:     sort E by increasing weight order
3:     F = (V, ∅)
4:     for each vertex v ∈ V do MAKESET (v)
5:     end for
6:     for i = 1 to |E| do
7:         uv = ith heaviest edge in E
8:         if FIND(u) ≠ FIND(v) then
9:             UNION (u, v)
10:            add uv to F
11:        end if
12:    end for
13:    return F
14: end procedure
```

Analysis:

The running time:

- $\mathcal{O}(m)$ find operations.
- $\mathcal{O}(n)$ union operations.
- $\mathcal{O}(n)$ makeset operations.

So other than sorting, total time is $\mathcal{O}(m \propto (n))$.

Sorting takes $\mathcal{O}(m \log m) = \mathcal{O}(m \log n)$ time, dominating the total running.

Thus, total running time would be $\mathcal{O}(m \log n)$.

9.9 Shortest Paths

Input: Weighted directed graph G with labeled source s and target t.

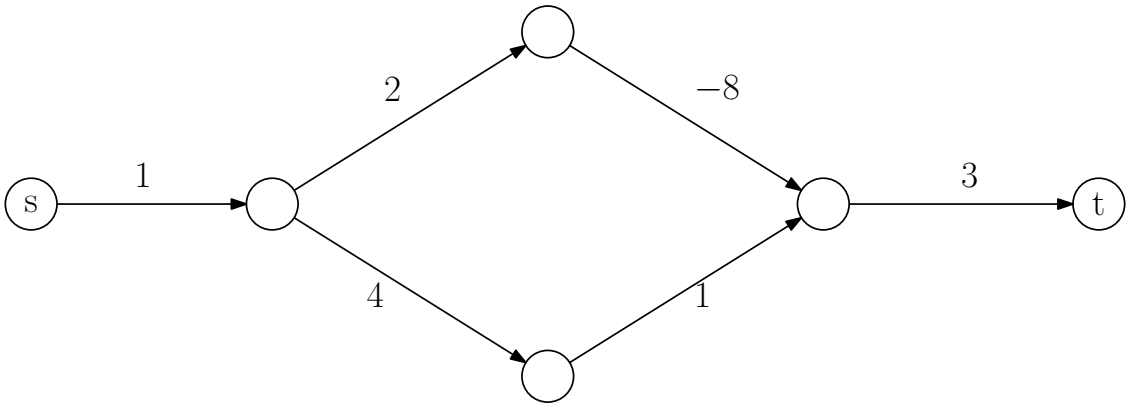
Output: Find shortest $s - t$ path, i.e. directed path P from s to t that minimized

$$w(P) = \sum_{w \to v \in P} w(u \to v)$$

Note that we can allow negative weights, but cannot be negative cycles.

A example of no SP is shown in fig. 13.

Figure 13: Example for No Shortest Path



9.9.1 Single Source Shortest Path (SSSP)

Almost every algorithm to find shortest $s - t$ path find shortest path to every vertex. We can represent all shortest paths from s using a spanning tree rooted at s.

Why a tree satisfied?

Suppose shortest $s - w$ path passes through v and shortest $s - t$ path passes through v. The subpath of a shortest path is a shortest path.

9.9.2 SSSP Algorithm

Most SSSP algorithms have same structure.

For every vertex v, store two values.

- $dist(v)$ = length of tentative shortest sv path, or ∞ if no $s - v$ path exists.
- $pred(v)$ = predecessor of v in tentative shortest $s - v$ path tree.

Predecessor pointers define tentative shortest path tree.

Initially, we know $dist(s) = 0$, $pred(s) = null$, and for every $v \neq s$, initially $dist(v) = \infty$, $pred(v) = null$ to indicate we don't know if there is $s - v$ path.

As we run algorithm $dist(v)$ values may decrease, but always upper bound tree distances.

During algorithm execution, an edge $u \to v$ is "tense" if $dist(v) + w(u \to v) < dist(v)$. If $u \to v$ tense, clearly shortest path from s to v is wrong, since $s - u$ path followed by $u \to v$ is shortest.

9.9.3 Generic SSSP algorithm

The general idea is:

Repeated find tense edge and relax it.

The relax operation is defined as algorithm 44.

Algorithm 44 Relax Operation in SSSP Algorithm

```
1: procedure RELAX(u → v)
2:     dist(v) = dist(u) + w(u → v)
3:     pred(v) = u
4: end procedure
```

The algorithm stops when no tense edges. This can prove the following:

1) $dist(v) \geq$ true distance from s to v at all times.

2) Algorithm halts, i.e. at some finite time, no more tense.

3) When algorithm halts, all $dist(v)$ values correct.

Proof: For 1), prove $\forall v \in V$, $dist(v)$ is either ∞ or the length of $s - v$ walk.

Use induction on number of relaxations.

- Consider $v \in V$, if v not updated in previous relax, then holds by induction.
- If v updated, then $dist(v) = dist(u) + w(u \to v)$, and $dist(u)$ is the length of walk by induction.

For 2).

In 1) can actually prove $dist(v)$ is ∞ or length of $s - v$ path (not just walk).

There finite number of paths, and distances only decrease, thus, the algorithm must stop at some finite time.

For 3).

Suppose otherwise, and let v be vertex in true shortest path tree, s.t. $dist(v)$ wrong, and v closest to s in tree (in terms of edges).

Let $s, \ldots, u, v$ be shortest path, true distance to v is$dist(u) + w(u \to v)$. By 1), $dist(v) > truedistance$, so $dist(v) > truedistance = dist(u) + w(u \to v)$, which means tense edge still exist. So the algorithm shouldn't halts, hence result in contradiction. □

Note that during the process, we look at a local statement, in the end result in a global statement.

So, how do we determine tense edges and what order to relax?

One way to do this is to mimic traverse. The algorithm is shown in algorithm 45.

Algorithm 45 Generic SSSP Algorithm

```
1: procedure InitSSSP(s)
2:     dist(s) = 0
3:     pred(s) = null
4:     for all s ≠ v do
5:         dist(v) = ∞
6:         pred(v) = null
7:     end for
8: end procedure
9: procedure GenericSSSP(s)
10:    InitSSSP (s)
11:    Put s in bag
12:    while bag not empty do
13:        Take u from bag
14:        for all u → v do
15:            if u → v is tense then
16:                Relax (u → v)
17:                Put v in bag
18:            end if
19:        end for
20:    end while
21: end procedure
```

Initially, inly tense edge are those leaving s. Each time we relax an edge $u \to v$, a new edge can become tense, but those edge must have v as the origin. Hence all tense edges would be seen before bag empty.

The question is: How do we implement the bag?

If bag is stack, could run in exponential time. And here come the Dijkstra's Algorithm.

9.9.4 Dijkstra's Algorithm

Use a minimum priority queue on $dist(u)$ values to implement the bag. If no negative edge weights, we can prove vertices removed from bag in increasing order of distance from s.

It follows each vertex removed (or inserted) from bag at most once. Each time an edge relaxed, algorithm performs decrease key on priority queue. Since each vertex inserted at most once, at most one decrease key per edge. So, total number insertions/deletions is $\mathcal{O}(n)$.

Fibonacci heaps has $\mathcal{O}(1)$ decrease key time, $\mathcal{O}(\log n)$ insert/delete time. Thus in total

$$\mathcal{O}(m + n \log n)$$

For regular binary heap, every operation takes $\mathcal{O}(\log n)$ time. Thus in total

$$\mathcal{O}((n + m) \log n) = \mathcal{O}(m \log n)$$

9.9.5 Bellman-Ford Algorithm

Use a standard FIFO queue to implement the bag.

The correctness follows since GenericSSSP (S) is correct.

Algorithm 46 Bellman-Ford Algorithm

```
1: procedure Bellman-Ford(s)
2:     InitSSSP (s)
3:     loopRepeat |V| times:
4:         for every edge u → v do
5:             if u → v is tense then
6:                 Relax (u → v)
7:             end if
8:         end for
9:     end loop
10:    for every edge u → v do
11:        if u → v is tense then
12:            return "Negative Cycle"
13:        end if
14:    end for
15: end procedure
```

In other word, the algorithm relax all tense edges and repeat.

The running time if $\mathcal{O}(mn)$.

Is the algorithm correct?

We can prove that, after i repeat phases, $dist(v)$ is length of shortest walk from s to v using at most i edges. This statement holds true regardless of negative edges.

If negative cycle exists, there is always tense edges, since it can always get shorter walk. Hence Bellman-Ford Algorithm can detect negative cycles.

9.9.6 Some Intuition on SSSP

Dijkstra's algorithm: sends out wave based on true distance.
Bellman-Ford algorithm: sends out wave based on number of edges.

9.10 Max Flows and Min Cuts

Example first:

Given a network of nodes, where each connection has some bandwidth.

1. How much data per unit time can be sent from s to t?
2. How much bandwidth must be removed, so that s cannot reach t?

The answer for both questions is the same! Max flow = Min cut.

9.10.1 Flows

Given a directed graph G, with source s, target t. An $(s,t) - flow$ is a function: $f : E \to \mathbb{R}_{\geq 0}$ which satisfies following conservation at all $v \neq s, t$

$$\sum_u f(u \to v) = \sum_w f(v \to w)$$

which means flow into v and flow out of v.

The "value" of the flows, denoted by $|f|$ is the net flow leaving s:

$$|f| = \sum_w f(s \to w) - \sum_u f(u \to s)$$

To simplify notation, define

$$\Delta(f(v)) = \sum_u f(u \to v) - \sum_w f(v \to w)$$

as the total net flow out of any vertex v.

By conservation:

$$\sum_v \Delta(f(v)) = \Delta(f(s)) + \Delta(f(t)) = 0$$

since any flow leaving vertex must enter some other vertex.

Hence,

$$|f| = \Delta(f(s)) = -\Delta(f(t))$$

Also, given capacity function $c : E \to \mathbb{R}^{\geq 0}$ that assigns a non-negative capacity $c(e)$ to each edge e. A flow is "feasible" (with respect to c) if $f(e) \leq c(e), \forall e \in E$. A flow f saturates e if $f(e) = c(e)$ and avoid edge e if $f(e) = 0$. The max flow problem is to *find feasible f s.t. $|f|$ is maximized.*

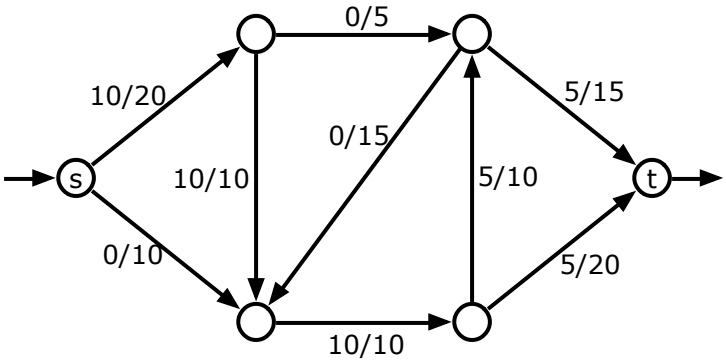


Figure 14: An $(s,t) - flow$ with value 10. Each edge is labeled with its flow/capacity.

9.10.2 Cuts

An $(s,t) - cut$ is a vertex bi-partition S, T, i.e. $S \cap T = \emptyset$ and $S \cup T = V$, s.t. $s \in S$ and $t \in T$.

The capacity of a cut is the sum of capacities of edges starting in S and ending in T

$$\|S,T\| = \sum_{v \in S} \sum_{w \in T} c(v \to w)$$

The min cut problem is to *find cut whos capacity is as small as possible.*

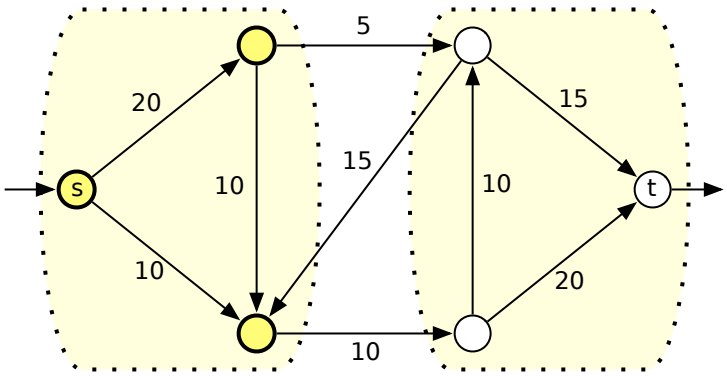


Figure 15: An $(s,t) - cut$ with capacity 15. Each edge is labeled with its capacity.

Intuitively, for any feasible $(s,t) - flow$ and any $(s,t) - cut$, $|f| \leq \|S,T\|$. The proof is

as follow:

$$
\begin{aligned}
|f| &= \sum_w f(s \to w) - \sum_u f(u \to s) && \text{by definition} \\
&= \sum_{v \in S} \left(\sum_w f(v \to w) - \sum_u f(u \to v) \right) && \text{by the conservation constraint} \\
&= \sum_{v \in S} \left(\sum_{w \in T} f(v \to w) - \sum_{u \in T} f(u \to v) \right) && \text{removing duplicate edges} \\
&\leq \sum_{v \in S} \sum_{w \in T} f(v \to w) && \text{since } f(u \to v) \geq 0 \\
&\leq \sum_{v \in S} \sum_{w \in T} c(v \to w) && \text{since } f(u \to v) \leq c(u \to v) \\
&= \|S, T\| && \text{by definition}
\end{aligned}
$$

Observation:

$|f| = \|S, T\|$ if and only if:

- f avoids every edge from T to S;
- f saturates every edge from S to T.

If $|f| = \|S, T\|$, then f is a max flow and S, T is a min cut. A max flow saturates all min cuts simultaneously.

9.10.3 The Maxflow MinCut Theorem

Theorem 9.10.1 (The Maxflow Mincut Theorem)**.** *In any flow network with source s and target t, the value of the maximum $(s,t) - flow$ is equal to the capacity of the minimum $(s,t) - cut$.*

Proof:

To make proof easier, assume capacity function is reduced, i.e. for $v, w \in V$, either $c(u \to v) = 0$ or $c(v \to u) = 0$ (only one of the edges exist). This assumption is easy to enforce, see fig. 16.

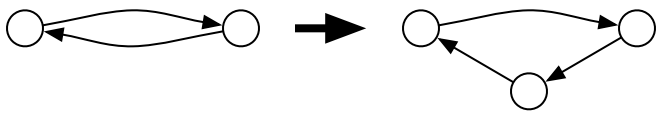


Figure 16: Enforcing the one-direction assumption

Let f be any feasible flow. Define a new capacity function $c_f : V \times V \to \mathbb{R}$, called the residual capacity, as follows:

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

because edge is reduced, an pair can only in one situation.

Since $f \geq 0$ and $f \leq c$, the residual capacities are always non-negative. Note that, we can have $c_f(u \to v) > 0$, for $u \to v \notin E$. Thus, define residual graph as $G_f = (V, E_f)$, where E_f is the set of edges with positive residual capacity. Notice that the residual capacities are not necessarily reduced, it can have both $c_f(u \to v) > 0$ and $c_f(v \to u) > 0$, as shown in fig. 17

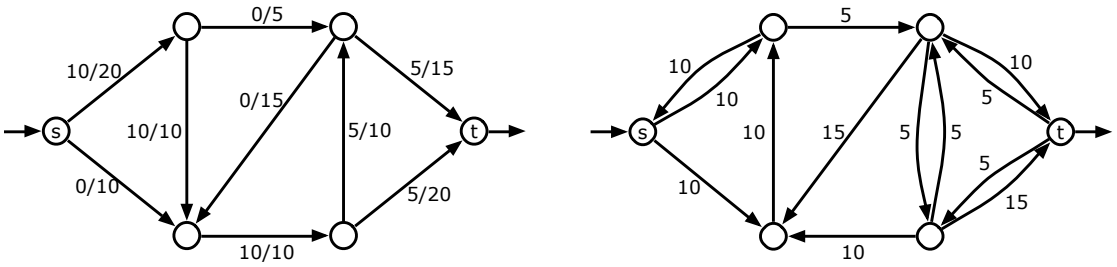


Figure 17: A flow f in a weighted graph G and the corresponding residual graph G_f.

Case 1 There is no path from s to t.
Let S be the vertices reachable from s in G_f, and let $T = V \setminus S$.
Then (S, T) is clearly a cut.
For any $u \in S$ and $v \in T$, we have

$$c_f(u \to v) = (c(u \to v) - f(u \to v)) + f(v \to u) = 0 \ ,$$

- If $u \to v \in E$, then $c(u \to v) = 0$, i.e. saturated.
- if $v \to u \in E$, then $f(v \to u) = 0$, i,e, avoided.

So f avoids every edge from T to S and saturates all edges from S to T.
Hence, $|f| = \|S, T\| \Rightarrow f$ is max flow, S, T is min cut.

Case 2 There is a path $s = v_0 \to v_1 \to \cdots \to v_r = t$ in G_f.
Refer to such path as an augmenting path. Let $F = \min_i c_f(v_i \to v_{i+1})$ denote the maximum amount can pass through this augmenting path. Define a new

flow function $f' : E \to \mathbb{R}$ as follows:

$$f'(u \to v) = \begin{cases} f(u \to v) + F & \text{if } u \to v \text{ is in the augmenting path} \\ f(u \to v) - F & \text{if } v \to u \text{ is in the augmenting path} \\ f(u \to v) & \text{otherwise} \end{cases} \tag{16}$$

To prove that the flow f' is feasible with respect to original capacities c, we need to verify that $f' \geq 0$ and $f' \leq c$ for all edges.

Consider and edge $u \to v \in G$.

- If $u \to v$ is in augmenting path, then $f'(u \to v) > f(u \to v) \geq 0$ and

$$\begin{aligned} f'(u \to v) &= f(u \to v) + F && \text{by definition of } f' \\ &\leq f(u \to v) + c_f(u \to v) && \text{by definition of } F \\ &= f(u \to v) + c(u \to v) - f(u \to v) && \text{by definition of } c_f \\ &= c(u \to v) \end{aligned}$$

- If $v \to u$ is in the augmenting path, then $f'(u \to v) < f(u \to v) \leq c(u \to v)$, which implies that

$$\begin{aligned} f'(u \to v) &= f(u \to v) - F && \text{by definition of } f' \\ &\geq f(u \to v) - c_f(u \to v) && \text{by definition of } F \\ &= f(u \to v) - f(u \to v) && \text{by definition of } c_f \\ &= 0 \end{aligned}$$

- If $u \to v$ nor $v \to u$ is in augmenting path, nothing change.

Finally, we observe that only the first edge in augmenting path leaves s, so $|f'| = |f| + F > |f|$. Hence, f is not a max flow.

We proved: *For feasible f, let S be reachable set from s in G_f, f is max flow, if and only if $t \notin S$. If $t \notin S$, $|f| = \|S, V \setminus S\|$ and so $\|S, V \setminus S\|$ is min cut.* $\square$

9.10.4 Ford-Fulkerson Algorithm

Maxflow Mincut theorem proof gives an algorithm to compute max flow and hence min cut.

1. Initially set $G_f = G$;

2. Find any s, t path P in G_f;

3. Augment f along P, update G_f and repeat until no s, t path.

The question is: how long does this algorithm takes?

Theorem 9.10.2 (Integrality Theorem)**.** *If all capacities in a flow network are integers, then there is a maximum flow such that the flow through every edge is an integer.*

Analysis: Let f^* be max flow. Each time Ford-Fulkerson's augments, flow increases by at least 1. Thus, augmenting takes $\mathcal{O}(|E|)$ time. Total time: $\mathcal{O}(|E||f^*|)$ time.

However, Ford-Fulkerson might alternate between pushing 1 unit of flow along the augmenting path back and forth, as shown in fig. 18.

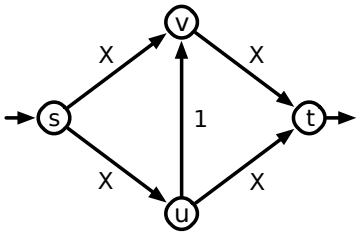


Figure 18: A bad example for the Ford-Fulkerson algorithm

To avoid this situation, we can choose (Edmands-Karp algorithm)

- path with largest bottleneck;
- path with shortest number of edges.

For more information, see Chapter 23.6 in Jeff's notes.

In conclusion, Ford-Fulkerson's algorithm takes $\mathcal{O}(X)$ time where the maximum flow value $|f^*| = 2X$, but the input size is $\mathcal{O}(\log X)$, thus, Ford-Fulkerson is actually exponential in the input size.

9.10.5 Application of Max Flow

- Edge disjoint path (edp).
  Given directed grpah G, want max number of edge disjoint $s - t$ paths (note that it may share vertices). Let K be the max number.
  Solution: Assign capacity 1 to all edges, compute max flow f^*, then $|f^*| = k$.

- Vertex capacities.
  Also given vertex capacities, $c(v)$ for each $v \in V$, want max flow with additional requirement that total flow entering $v \leq c(v)$ (and therefore flow out of $v \leq c(v)$).
  Solution: Use standard flow, but replace each v with v_{in} and v_{out}, with edge $v_{in} \to v_{out}$ of capacity $c(v)$. All incoming edges to v now to v_{in}, all outgoing edges now from v_{out}.

- Maximum Matching in Bipartite graphs.
  Given an undirected graph $G = (V, E)$ is Bipartite. If V can be partitioned into two sets A, B, so that all edges are from A to B (A, B are independent set). A matching is a subset $M \subseteq E$ s.t every vertex in $G' = (V, M)$ has degree ≤ 1 (i.e. edges in M are non-adjacent). A maximum matching is matching s.t $|M|$ is maximized.
  Can find max matching in bipartite G as follow:

  1. orient all edges from u to W.
  2. adding source s, with directed edges from all of A.
  3. adding sink t, with directed edges from all of B.
  4. assign capacity 1 to all edges.
  5. compute max flow, saturated A to B edges are max matching.

10 NP-Hardness

10.1 A Game You Can't Win

Given a box with n binary switches and a light bulb. Inside the box is a boolean circuit, s.t

- AND, OR, NOT gates, as shown in fig. 19;
- n input wires, one per switch;
- Gates connected to single output, i.e the light bulb.

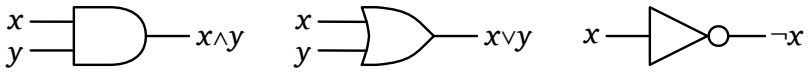


Figure 19: An AND, an OR, and a NOT gate

An example is shown in fig. 20

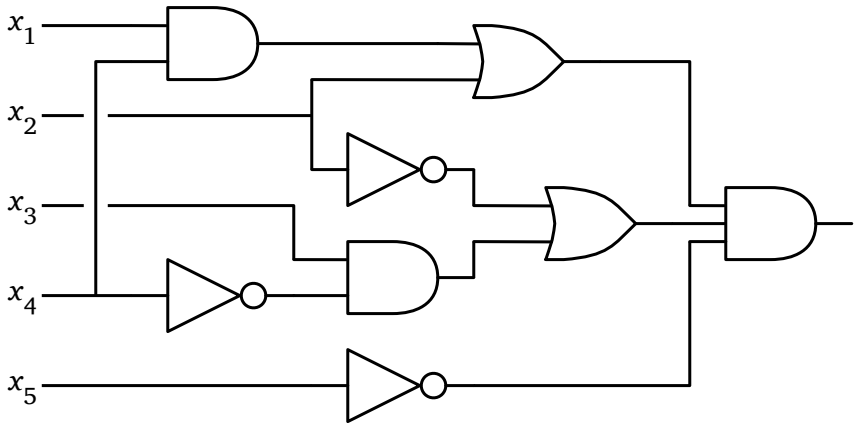


Figure 20: A boolean circuit

We want to know if there is a way to set switches s.t light bulb turns on. There are 2^n possible binary inputs. There is an $\mathcal{O}(n)$ size circuit which accepts a single strings and rejects all others. Hence, if you cannot see inside, in worst case you must try all 2^n strings.

Suppose now you can see inside box. Can you do better than guessing all strings? The answer is nobody knows for sure but many believe not.

The problem of determining if boolean circuit has satisfying assignment is all CIRCUITSAT.

10.2 P versus NP

Minimum for an algorithm to be efficient is that it runs in polynomial time, i.e $\mathcal{O}(n^c)$ for constant c, n is input size. In this course, we focused on problems with polynomial solutions where c is small.

- Some problems known to require exponential time, i.e $\mathcal{O}(c^n)$, $c > 1$.
- Some problems cannot even be solved (for example, Halting Problem).
- Some problems we don't know how hard.

NP-hard is a class most believe cannot be solved in polynomial time, but no one knows for sure.

A decision problem is a problem whose output is a single boolean value, i.e. YES/NO. For example: LIS was an optimization problem. → "Is there any increasing sequence of length $\geq k$?" is a decision problem.

P Set of decision solvable in polynomial time, i.e "efficiently solvable".

NP Decision problems s.t if answer in YES, then there is a proof that can be checked in polynomial time. → can verify YES if proof given to us.

Co-NP Decision problems s.t if answer in NO, then there is a proof that can be checked in polynomial time. → can verify NO if proof given to us.

CIRCUITSAT is in NP, if satisfying string, then given string, it is easy to verify.

Why do most believe $P \neq NP$? One intuition is that: solving problems is harder than verifying solutions.

P Problem in CLRS that you can solve quickly.

NP Problem in CLRS that you can verify solution quickly.

10.3 NP-Hard/NP-Complete

A problem Π is NP-hard if a polynomial time algorithm for Π implies polynomial time algorithm for all problems in NP.

Π is NP-hard $\leftrightarrow$ If Π solution in polynomial time, then P = NP.

Hence, most believe no NP-hard problem solvable in polynomial time.

Π is NP-complete if Π is NP-hard and $\Pi \in$ NP.

Thousands of problems have been shown to be NPC. How does one prove something NP-hard or NPC?

Theorem 10.3.1 (Cook-Levin Theorem)**.** CircuitSAT *is NPC.*

All problems that are shown to be NPC, reduce from CircuitSAT.

10.4 Reductions & SAT

To prove any problem other than the CircuitSAT is NP-hard, use reduction. Reducing problem A to problem B means solving A using solution for B. To prove B is NP-hard, reduce from known NP-hard problem A to problem B. Require reduction to be polynomial time in the A instance size.

Intuition: we are sure A is hard, Hence B must be hard, since otherwise it implies A can be solved quickly.

Often write: $B \geq_p A$, $\geq_p$ stands for B at least as hard as A ignoring polynomial factor.

A may still require more time to solve bu not more than polynomial factor.

Given a boolean formula, is there a satisfying assignment, i.e a binary assignment variables evaluating to be true. For example:

$$(a \vee b \vee c \vee \overline{d}) \Longleftrightarrow ((b \wedge \overline{c}) \vee \overline{(\overline{a} \Rightarrow d)} \vee (c \neq a \wedge b))$$

To prove SAT is NP-hard, we reduce from CircuitSAT. So given a boolean circuit (i.e. a DAG), we must convert to an equivalent boolean formula.

One solution: circuit has single output gate, so start at root gate, recursively write formula, for each subtree and combine formula with root gate operation. This method takes exponential time since DAG, not a tree.

Here is the polynomial time reduction:

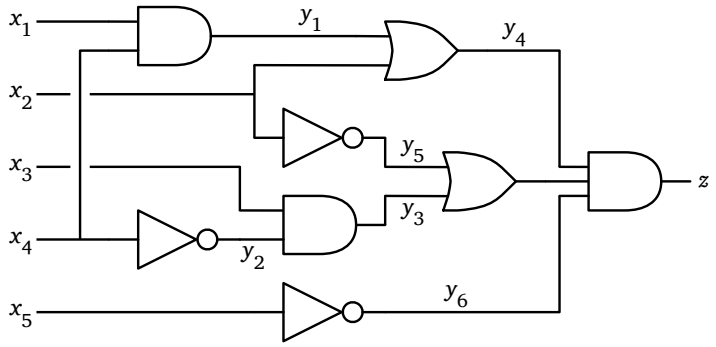
1. For each gate output wire create a variable: $y_1 \ldots y_n$ as gates outputs, $x_1 \ldots x_m$ as circuit input, z as circuit output.

2. For each gate wire clause saying inputs match output.

3. Take And of all these clauses and z.

4. This says all gates work properly and output is 1.

For example, fig. 21 shows the formula transformed from fig. 20.



$$(y_1 = x_1 \wedge x_4) \wedge (y_2 = \overline{x_4}) \wedge (y_3 = x_3 \wedge y_2) \wedge (y_4 = y_1 \wedge x_2) \wedge$$
$$(y_5 = \overline{x_2}) \wedge (y_6 = \overline{x_5}) \wedge (y_7 = y3 \vee y_5) \wedge (z = y_4 \wedge y_7 \wedge y_6) \wedge z$$

Figure 21: A boolean circuit with gate variables added, and an equivalent boolean formula

The original circuit is satisfiable if and only if the boolean formula is.

$\Longrightarrow$ Given satisfying assignment to circuit, can find satisfying assignment to formula by computing output of every gate.

$\Longleftarrow$ Given satisfying assignment to formula, just ignore y_is and z to get circuit input.

By this, we can produce formula from circuit in linear time. Thus, polynomial time reduction from CircuitSAT to SAT.

10.5 3SAT

Boolean formula is in conjunctive normal form (CNF) if it is the conjunction (And) of clause, each of which is a disjunction (Or) of literals, each of which is a variable of its negation.

A 3CNF formula is a CNF formula with exactly 3 literals per clause.

Theorem 10.5.1. *3SAT is NP-complete.*

Proof: To prove 3SAT $\in$ NP, given satisfying assignment, plug in values and evaluate.

To prove 3SAT is NP-hard: (reduce from CircuitSAT)

Given CircuitSAT instance, convert to 3SAT instance: Assume each gate has fan in at most 2.

1. Write circuit as formula with one clause per gate;

2. Change each gate clause to CNF:

$$a = b \wedge c \iff (a \vee \overline{b} \vee \overline{c}) \wedge (\overline{a} \vee b) \wedge (\overline{a} \vee c)$$
$$a = b \vee c \iff (\overline{a} \vee b \vee c) \wedge (a \vee \overline{b}) \wedge (a \vee \overline{c})$$
$$a = \overline{b} \iff (a \vee b) \wedge (\overline{a} \vee \overline{b})$$

3. Convert to 3CNF (exactly 3 literals per clause)

$$a \iff (a \vee x \vee y) \wedge (a \vee \overline{x} \vee y) \wedge (a \vee x \vee \overline{y}) \wedge (a \vee \overline{x} \vee \overline{y})$$
$$a \vee b \iff (a \vee b \vee x) \wedge (a \vee b \vee \overline{x})$$

Polynomial time reduction, hence 3SAT is NP-hard. □

10.6 Independent Set

Let G be an undirected graph. An independent set in G is a set of vertices with no edge between them.
Independent Set Problem: Given G and integer k, is there an independent set of size $\geq k$.
Max Independent Set Problem: Given G, find the size of largest independent set.

Proof: We prove the Independent Set Problem is NP-hard by reducing from 3SAT: Input is a 3SAT instance with x clauses. For each literal of each clause, create a vertex. Add an edge between two vertices if either:

- literals are in the same clause;
- literal correspond to variable and its negation.

For example the following formula

$$(a \vee b \vee c) \wedge (b \vee \overline{c} \vee \overline{d}) \wedge (\overline{a} \vee c \vee d) \wedge (a \vee \overline{b} \vee \overline{d})$$

is transformed into the following graph:

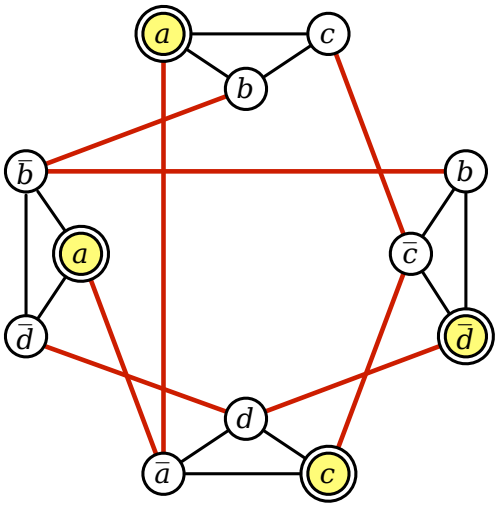


Figure 22: A graph derived from a 3CNF, and an independent set of size 4

The formula is satisfiable if and only if independent set of size x:

$\Longrightarrow$ To get satisfying assignment, set each literal from independent set to be true. Since contradictory literals connected by edge, this assignment consistent. The formula is evaluated to be true since each clause satisfied.

$\Longleftarrow$ If there is a satisfying assignment, choose one literal for each clause that is true. These literal are independent set in graph since from different clauses and consistent.

This proves IndepSet(G,K) is NP-hard.

Since given a subset, can easily check if it is independent set and size $\geq k$, IndepSet(G,K) $\in$ NP.

Hence, IndepSet in NP-complete. $\square$

10.7 Clique

A clique is a subset of vertices where every pair connected. MaxClique asks for size of largest clique.

Clique is NPC.

Define complement graph $\overline{G}$ as a vertex set, where edge is in $\overline{G}$ if and only if the edge is not in G. A set if vertices is independent set of G if and only if the same set of vertices is clique in $\overline{G}$.

10.8 Vertex Cover

A vertex cover is a set of vertices such that every edge in G is adjacent to vertex in set. Minimum vertex cover asks for size of smallest vertex cover. VertexCover asks if vertex cover of size $\leq k$.

Claim 10.8.1. *A set $I \subseteq V$ is independent set if and only if $V \setminus I$ is the vertex cover.*

$\implies$ Consider edge uv, cannot have both u and $v \in I$, so at least one endpoint in $V \setminus I$.

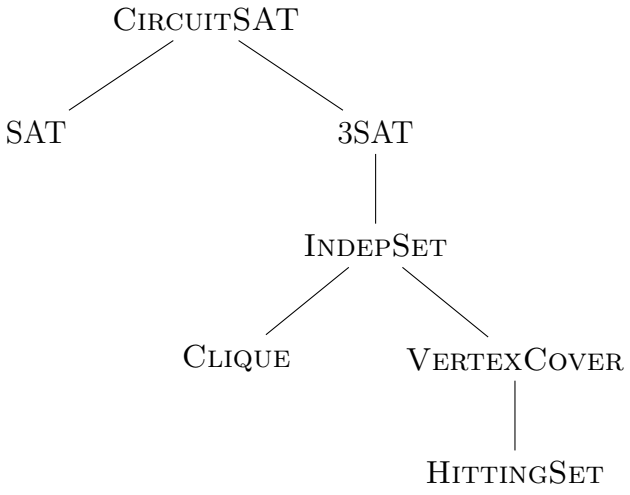
$\impliedby$ If i is not independent set, then for some edge uv, $u, v \in I$, hence uv not covered in $V \setminus I$, (not vertex cover).

Hence, IndepSet of size $\geq k$ if and only if vertex cover of size $\leq n - k$. I_{max} independent set if and only $V \setminus I$ is minimum vertex cover.

10.9 Some Intuition

The reduction tree of the problems discussed in the note and homework is shown in fig. 23.

Figure 23: Reduction Tree



Here are some other NPC problems:

- 3 Coloring
- Hamiltonian Path/Cycle
- If weighted, TS

And here are some Polynomial time problem:

- Edge Cover
- 2SAT
- 2 Coloring
- Euler tour in connected even degree vertex graph.

And here is the end of the lecture note.