Sequential Decision-Making for Inventory Control using Bayesian Decision Theory

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Abstract

Inventory control is a core aspect of supply chain management, balancing customer demand fulfillment against the costs of holding and ordering inventory. This paper addresses inventory management through the lens of Bayesian decision theory, enabling probabilistic decision-making under uncertainty. By establishing a framework for sequential decision-making over an infinite time horizon, an optimal solution tailored to a specific asymmetric cost function is derived, offering insights for practical implementations in inventory management.

1 Introduction

Inventory management is a critical component of supply chain management that involves overseeing the ordering, storage, and use of a company's inventory, including raw materials, components, and finished goods. Effective inventory control ensures that a company has adequate stock to meet demand while minimizing costs associated with holding inventory. In practice, this often requires a balance between minimizing stockouts, which disrupt service levels, and limiting overstocking, which incurs unnecessary holding costs.

The coordination of inventory and transportation decisions has gained significant attention over recent decades, driven by the need for specific practices like vendor-managed inventory (VMI), third-party logistics (3PL), and time-definite delivery (TDD) (see, e.g., (Çetinkaya and Lee, 2000; Alumur, 2012; Gürler, 2014)). These programs aim to optimize the balance between inventory holding costs and transportation costs. Traditional inventory models often assume immediate delivery to meet demand, yet this can be inefficient due to the fixed costs of transportation, prompting companies to adopt shipment consolidation policies that merge smaller demands into larger, less frequent shipments (Çetinkaya and Bookbinder, 2003; Higginson and Bookbinder, 1994).

Three commonly implemented shipment consolidation policies—quantity-based, time-based, and hybrid—are particularly useful in improving cost efficiency by regulating shipment size or timing. Each policy type has distinct impacts on performance metrics, such as delay penalties, average inventory, and annual costs, guiding companies toward designing more efficient inventory-transportation systems (Çetinkaya, 2005; Wei, 2020; Çetinkaya and Lee, 2006).

While these approaches provide valuable frameworks, they do not explicitly account for the uncertainties inherent in demand fluctuations or the associated asymmetric costs of overstocking and understocking. In settings where inventory decisions must be made sequentially, uncertainty can be more effectively managed through Bayesian decision theory, which offers a probabilistic approach to decision-making under uncertainty. This framework allows for more refined, sequential adjustments to inventory based on demand forecasting, which dynamically incorporates new data over time.

In this study, Bayesian decision theory is utilized to address a sequential inventory control problem. Specifically, an optimal decision-making framework is derived based on a discounted asymmetric cost function, considering the impact of decisions made infinitely far into the future. Our model involves a decision-maker (henceforth referred to as the Robot) who manages stock levels over time in an interactive environment shaped by a random demand process. Through this setup, a mathematically rigorous and computationally feasible solution for optimal inventory decisions that account for uncertainty and asymmetric costs is derived.

2 Sequential Decision-Making Framework

To formalize the sequential decision-making problem, consider a stock management scenario in which the Robot makes decisions at each time step, interacting with an entity termed Nature, which represents environmental randomness (Lavalle, 2006). At each iteration t, the Robot decides on an action from its action space Ω_U, while Nature removes a certain number of units from stock, denoted $s_t \in \Omega_S$. The goal is to manage the stock optimally, balancing the costs of overstocking ($N > 0$) and understocking ($N < 0$), given that each action taken has a probabilistic effect on the Robot's objectives.

The Robot's decisions are based on historical data D, which contains past stock removals and other relevant demand features. Given an initial stock N_0, the stock level at time t evolves according to

$$\begin{aligned} N_t &\equiv N_0 + \sum_{t'=1}^{t} (U_{t'-L} - s_{t'}) \\ &= N_0 + \upsilon_t - \zeta_t \end{aligned}, \tag{1}$$

where $U_{t'-L}$ represents the decision made at an earlier time due to a potential lag L and $\zeta_t \equiv \sum_{t'=1}^{t} s_{t'}$ and $\upsilon_t \equiv \sum_{t'=1}^{t} U_{t'-L}$. To support probabilistic decision-making, the Robot uses a forecast based on D in the form

$$p(s_1, s_2, \ldots | D, I), \tag{2}$$

where I denotes any additional background information (Sivia and Skilling, 2006). This probabilistic forecast, combined with the Robot's asymmetric cost function, guides the sequence of decisions over time.

3 Formulation of the Cost Function and Optimal Policy

The Robot's objective is to formulate a sequence of decision rules, called a policy, $\xi = \{U_0(D), U_1(D), \ldots\}$, where each $U_j(D) = u_j$ minimizes the expected cost

$$\xi^* = \arg\min_{\xi} \mathbb{E}[C|D, I], \tag{3}$$

where the cost C is assumed to be discounted viz

$$C = \sum_{t=1}^{\infty} \gamma_{\text{disc}}^{t} \left(h_t 1_{N_t > 0} + c_t (1_{N_t > 0} - 1) \right) N_t, \tag{4}$$

where $\gamma_{\text{disc}} \in [0, 1]$ is a discount factor, and h and c represent storage (holding) and understocking costs, respectively. The optimal decisions satisfy the first-order condition

$$\frac{d}{dU_m} \mathbb{E}[C|D, I] \Big|_{U_m = U_m^*} \stackrel{!}{=} 0 \quad \forall m, \tag{5}$$

which leads to the optimal decision criterion (see Appendix A for derivation).

Theorem 1 (Optimal Policy Rule for Inventory Control)**.** *Given the sequential decision-making framework and the asymmetric cost function in equation* (4)*, the optimal policy for the Robot at each time step t is defined by*

$$p(N_t^* > 0|D, I) = \frac{c_t}{c_t + h_t}, \tag{6}$$

where

$$N_t^* = N_0 + \upsilon_t^* - \zeta_t. \tag{7}$$

4 Optimal Policy under Conditional Independence

Equation (6) can be written

$$\begin{aligned} p(N_t^* > 0|D, I) &= \sum_{s_1,\ldots s_t} 1_{N_t \geq 0} p(s_1, \ldots s_t|D, I) \sum_{s_{t+1},\ldots} p(s_{t+1}, \ldots |s_1, \ldots s_t, D, I) \\ &= \sum_{\zeta_t=0}^{N_0+\Upsilon_t} p(\zeta_t|D, I) \sum_{s_{t+1},\ldots} p(s_{t+1}, \ldots |\zeta_t, D, I) \end{aligned} \tag{8}$$

Assuming conditional independence $p(s_{t+1}, \ldots |\zeta_t, D, I) = p(s_{t+1}, \ldots |D, I)$

$$p(N_t^* > 0|D, I) = \sum_{\zeta_t=0}^{N_0+\upsilon_t} p(\zeta_t|D, I). \tag{9}$$

Combining eqation (9) with theorem 1 implies that the optimal policy is related to quantiles of ζ_t viz

$$\upsilon_t^* = Q_{\frac{c_t}{c_t+h_t}} \zeta_t - N_0, \tag{10}$$

where $Q_q(X)$ denotes the q-quantile of the random variable X.

4.1 Policy Efficiency Ratio

In order to gauge the effect of using the optimal policy (equation (10)), the associated expected cost is compared to the expected cost associated to a baseline policy via the policy efficiency ratio (PER)

$$\text{PER} \equiv \frac{\mathbb{E}[C^*|D, I]}{\mathbb{E}[C'|D, I]}, \tag{11}$$

where C^* is the cost function with the optimal policy and C' is the baseline policy defined viz

Definition 1 (baseline policy)**.** *The baseline policy consists of ordering the expected value* $\mathbb{E}[s_t|D,I]$ *if* $N_t < \mathbb{E}[s_t|D,I]$.

The baseline policy represents an (R,Q) policy (Bartmann and Beckmann, 1992; Axsater, 2006) with the reorder point $N_t = 0$ and batch quantity $\mathbb{E}[s_t|D,I]$. This policy is a simple approach to keeping the stock as close as possible to the reorder point while favoring a positive stock. Assuming i) $p(\zeta_t|D,I)$ follows a Poisson distribution, ii) $k_t = k, c_t = c$ are constant and conditional independence, the PER can be approximated (see appendix B for derivation) viz

$$\frac{\mathbb{E}[C^*|D,I]}{\mathbb{E}[C'|D,I]} \approx \frac{4}{2+\frac{h}{c}+\frac{c}{h}}. \tag{12}$$

========= to this point ===========

Equation (12) provides a rule of thumb to gauge the expected cost reduction by using the optimal policy (equation (10)) over a conventional approach. In typical industrial settings $c \gg h$ and so the expected reduction in cost will be modest, however, possibly still significant depending on the scale and elements included in the cost model. Figure 1 show the PER (equation (12)) plotted as a function of h, c.

4.1.1 Example: Numerical Policy Efficiency Ratio

To provide an example of implementation (see source code on Github) and test equation (12) on an anecdotal level, consider the case where $N_0 = 37$, $L = 6$ and $U_j \in \mathbb{Z}$, $0 \leq U_j \leq 0$ for any time step. Data are mocked from a Poisson distribution with rate parameter $\lambda = 3$. 300 random samples are drawn to represent a mock time series, with the last 52 reserved as test data and the remaining as training data. Based on the training data, the λ is estimated using MLE and based on a Poisson distribution with this estimated rate parameter, a forecast of dimensions (number_of_samples, forecast_horizon), with number_of_samples=5 000 and forecast_horizon=52, is drawn. The optimal and baseline policies applied to yield decision over the forecast horizon. Using these decisions, the actual costs are computed using the test data. This is done 5 000 times to yield an estimate of the PER for a particular h, c. This process is repeated for $c \in \mathbb{Z}$, $1 \leq c \leq 100$ and $h \in \mathbb{Z}$, $1 \leq c \leq 20$, yielding the PERs shown in figure 2. Figure 2 clearly show the same pattern as figure 1, however, with slight differences (as expected for an approximation); i) the maximum slightly exceeds 1 and ii) the minimum significantly deviates from 0. The former is likely due to random fluctuations from approximating the expected cost from 5 000 samples. Increasing the sample size gradually brings the maximum closer to 1.

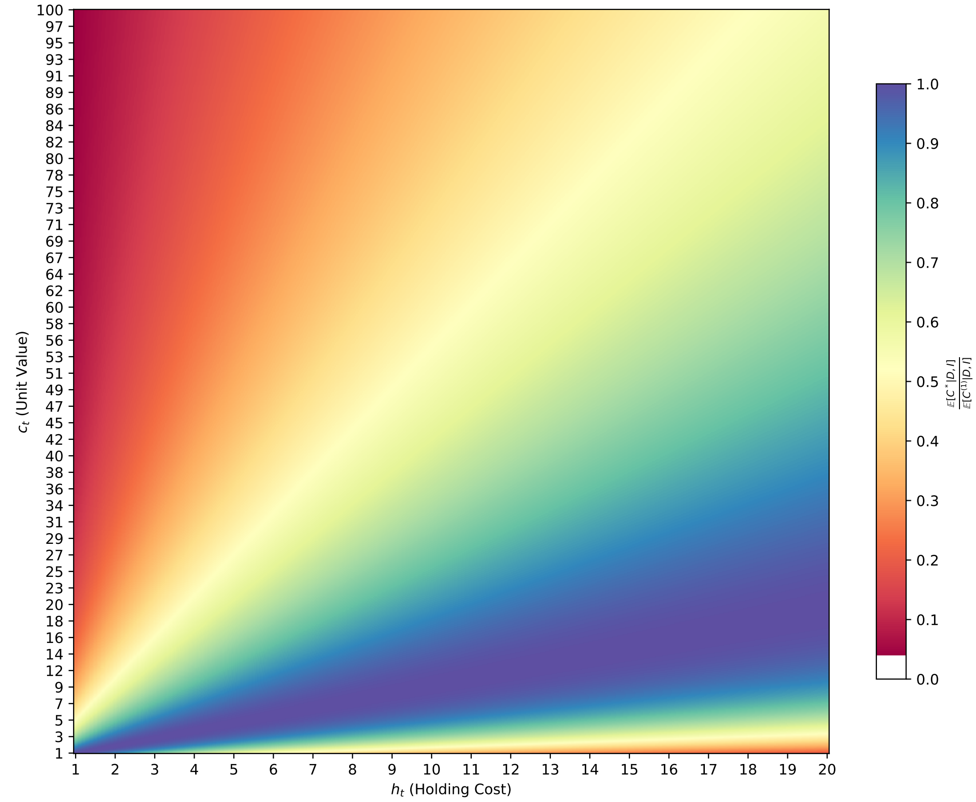

Figure 1: Heatmap of the analytical policy efficiency ratio (equation (12)).

========= TO THIS POINT ========== The latter? Is it an artefact, or due to numerical limitations? In the upper limit inaccuracy seems to be ~ 0.04. In the lower limit we have ~ 0.2... It can be inaccuracies of the approximation. I have defined the scale of the approximation based on deduction. Perhaps this is not perfectly accurate?

5 Conclusion

The derived decision rule offers a structured approach to managing inventory in a stochastic environment using Bayesian decision theory. The probabilistic framework accounts for demand uncertainty, guiding optimal actions based on expected costs. This work contributes to the field of inventory management by presenting an analytically derived solution, supporting practical applications in real-world inventory systems.

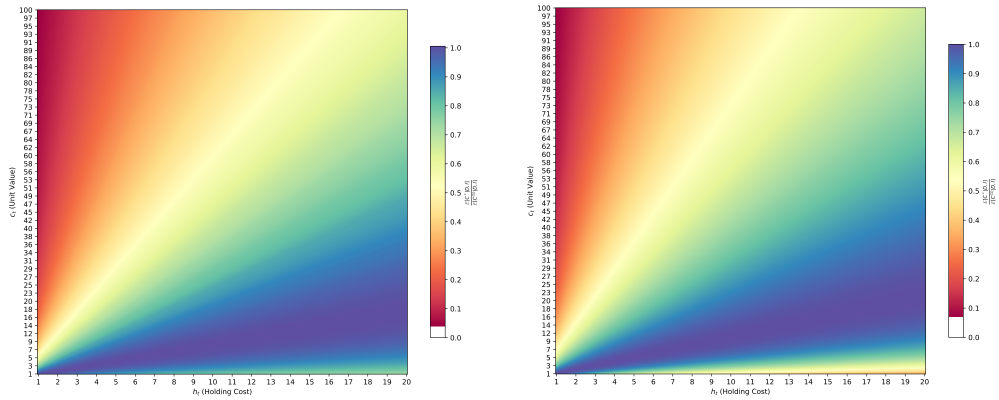

Figure 2: Heatmap of the numerical policy efficiency ratio calculated via simulation.

1. A few words on the intuitive interpretation of the cost function? The number of units overstocked multiplied with the cost of overstocking per unit. The number of units understocked multiplied with the value of each unit. The latter represents the lost value.

2. Mention the benefit of the analytical solution; the computation speed relative to a numerical optimization is highly beneficial at scale.

3. Compress the recursive relation to m-notation.

4. The baseline policy is an (R,Q) policy with $R = 0$ and $Q = \mathbb{E}[s_t|D, I]$.

5. Is our result in any of the inventory control books?

6. will the relationship between baseline and optimal policy depend on forecasting method? I would say yes. How do we handle this?

7. if there is around the same cost for over/under stocking, there is a $30 - 40\%$ reduction in costs with the optimal policy compared to the baseline. In the limit of $c \gg h$, the reduction in cost approach 0% and the gains are minor. This is the relevant limit in most cases, where the value of the unit significantly outweigh the holding cost. This is relevant, however, it is underlined, that the baseline policy is always equal or worse (statistically, meaning the expected cost is always lower. Expected cost is over a distribution. draws from that distribution can fall either way) than the optimal policy.

8. PER is highly dependent on the policy. If the reference level is moved away from $N_t = 0$, for example, the plot completely changes and the blue area shifts to the top left corner.

1. S. Axsäter. Inventory control

2. Gasthaus paper

3. inspiration for introduction

4. probabilistic trucks

5. Amazon paper (inspiration)

A Minimization of Expected Cost

To determine the policy, ξ^*, that minimize the expected cost $\mathbb{E}[C|D, I]$, the derivative of the cost function with respect to ξ is needed

$$\frac{dC}{dU_m} = \sum_{t=1}^{\infty} \gamma_{\text{disc}}^t \left(h_t 1_{N_t>0} + c_t(1_{N_t>0} - 1)\right) \frac{dN_t}{dU_m} \tag{13}$$

where

$$\begin{aligned} \frac{dN_q}{dU_m} &= \sum_{t'=1}^{t} \frac{dU_{t'-L}}{dU_m} \\ &= \sum_{t'=1}^{t} \delta_{t'-L,m} \end{aligned} \tag{14}$$

Using equation (14) in equation (13)

$$\frac{dC}{dU_m} = \sum_{t=1}^{\infty} \gamma_{\text{disc}}^t \Big(h_t 1_{N_t \geq 0} + c_t(1_{N_t \geq 0} - 1) \Big) \sum_{t'=1}^{t} \delta_{t'-L,m}. \tag{15}$$

For some generic function g_t

$$\begin{aligned} \sum_{t=1}^{\infty} g_t \sum_{t'=1}^{t} \delta_{t'-L,m} &= g_1 \delta_{1-L,m} + g_2(\delta_{1-L,m} + \delta_{2-L,m}) + \dots \\ &= \sum_{t=L+m}^{\infty} g_t \end{aligned} \tag{16}$$

meaning

$$\frac{dC}{dU_m} = \sum_{t=L+m}^{\infty} \gamma_{\text{disc}}^t (h_t 1_{N_t \geq 0} + c_t(1_{N_t \geq 0} - 1)). \tag{17}$$

Combining equations (5) and (17)

$$\sum_{s_1, s_2, \dots} \sum_{t=L+m}^{\infty} \gamma_{\text{disc}}^t (h_t 1_{N_t \geq 0} + c_t(1_{N_t \geq 0} - 1)) p(s_1, s_2, \dots | D, I) \stackrel{!}{=} 0 \quad \forall m \tag{18}$$

The sums can be evaluated viz

$$\begin{aligned} \sum_{s_1, s_2, \dots} 1_{N_t \geq 0} p(s_1, s_2, \dots | D, I) &= p(N_t \geq 0 | D, I), \\ \sum_{s_1, s_2, \dots} p(s_1, s_2, \dots | D, I) &= 1. \end{aligned} \tag{19}$$

Let

$$\psi_t \equiv (h_t + c_t)p(N_t \geq 0|D, I) - c_t, \tag{20}$$

then

$$\begin{aligned} \frac{d}{dU_m}\mathbb{E}[C|D,I] &= \sum_{t=L+m}^{\infty} \gamma_{\text{disc}}^t \psi_t \\ &\stackrel{!}{=} 0 \quad \forall m \end{aligned} \tag{21}$$

A recursion relation can be derived viz

$$\begin{aligned} \frac{d}{dU_0}\mathbb{E}[C|D,I] &= \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \psi_t \\ &= \gamma_{\text{disc}}^L \psi_L + \sum_{t=L+1}^{\infty} \gamma_{\text{disc}}^t \psi_t \\ &= \gamma_{\text{disc}}^L \psi_L + \frac{d}{dU_1}\mathbb{E}[C|D,I] \\ &= \gamma_{\text{disc}}^L \psi_L + \gamma_{\text{disc}}^{L+1} \psi_{L+1} + \frac{d}{dU_2}\mathbb{E}[C|D,I] \\ &= \dots \\ &\stackrel{!}{=} 0 \end{aligned} \tag{22}$$

meaning

$$\frac{d}{dU_m}\mathbb{E}[C|D,I] = \gamma_{\text{disc}}^{L+m} \psi_{L+m} + \frac{d}{dU_{m+1}}\mathbb{E}[C|D,I]. \tag{23}$$

Since all derivatives are required to be simultaneously zero,

$$\gamma_{\text{disc}}^j \psi_j \stackrel{!}{=} 0 \quad \forall j \Rightarrow \psi_j = 0 \tag{24}$$

meaning

$$p(N_t^* \geq 0|D, I) = \frac{c_t}{c_t + h_t}, \tag{25}$$

where

$$N_t^* \equiv N_0 + \upsilon_t^* - \zeta_t \tag{26}$$

denote the units on stock given optimal decisions.

B Policy Efficiency Ratio

In this appendix, the policy efficiency ratio (PER, equation (11)) is calculated under the assumption of i) $p(\zeta_t|D,I)$ follows a Poisson distribution, ii) $k_t = k, c_t = c$ are constant and conditional independence between ζ_t. The first step consist of re-writing the expected cost

$$\begin{aligned}\mathbb{E}[C|D,I] &= \sum_{t=1}^{\infty}\sum_{\zeta_t=0}^{\infty}\gamma_{\text{disc}}^{t}\left(h_t 1_{N_t>0} + c_t(1_{N_t>0}-1)\right)N_t p(\zeta_t|D,I)\\ &= \sum_{t=1}^{\infty}\sum_{\zeta_t=0}^{\infty}\gamma_{\text{disc}}^{t}\Big((h_t+c_t)(N_0+\upsilon_t)1_{N_t>0}\\ &\qquad - (h_t+c_t)1_{N_t>0}\zeta_t - c_t(N_0+\upsilon_t) + c_t\zeta_t\Big)p(\zeta_t|D,I).\end{aligned} \tag{27}$$

The sums over ζ_t can be expressed as follows

$$\begin{aligned}\sum_{\zeta_t=0}^{\infty}1_{N_t>0}p(\zeta_t|D,I) &= \sum_{\zeta_t=0}^{N_0+\upsilon_t}p(\zeta_t|D,I)\\ &= \frac{\Gamma(N_0+\upsilon_t+1,\lambda_t)}{\Gamma(N_0+\upsilon_t+1)},\\ \sum_{\zeta_t=0}^{\infty}1_{N_t>0}\zeta_t p(\zeta_t|D,I) &= \sum_{\zeta_t=0}^{N_0+\upsilon_t}\zeta_t p(\zeta_t|D,I)\\ &= \lambda_t\frac{\Gamma(N_0+\upsilon_t,\lambda_t)}{\Gamma(N_0+\upsilon_t)},\\ \sum_{\zeta_t=0}^{\infty}\zeta_t p(\zeta_t|D,I) &= \lambda_t,\\ \sum_{\zeta_t=0}^{\infty}p(\zeta_t|D,I) &= 1.\end{aligned} \tag{28}$$

Using an approximation from Bartmann and Beckmann (1992)

$$\frac{\Gamma(x,y)}{\Gamma(x)} \simeq \frac{1}{1+e^{-\frac{m}{\sqrt{y}}(x-y)}} \tag{29}$$

with $m \simeq 1.8$. Applying this yields

$$\begin{aligned}\sum_{\zeta_t=0}^{\infty} 1_{N_t>0}\, p(\zeta_t|D,I) &\simeq \frac{1}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+\upsilon_t+1-\lambda_t)}},\\ \sum_{\zeta_t=0}^{\infty} 1_{N_t>0}\,\zeta_t\, p(\zeta_t|D,I) &\simeq \lambda_t \frac{1}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+\upsilon_t-\lambda_t)}}.\end{aligned} \tag{30}$$

The expected cost can thus be written

$$\begin{aligned}\mathbb{E}[C|D,I] \simeq \sum_{t=1}^{\infty} \gamma_{\text{disc}}^t \Bigg((h_t+c_t) \Bigg[&\frac{N_0+\upsilon_t}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+\upsilon_t+1-\lambda_t)}} \\ &- \frac{\lambda_t}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+\upsilon_t-\lambda_t)}} \Bigg] - c_t(N_0+\upsilon_t-\lambda_t) \Bigg).\end{aligned} \tag{31}$$

Due to the lead time, no decisions can be implemented until $t \geq L$, leading to

$$\begin{aligned}\mathbb{E}[C|D,I] \simeq E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \Bigg((h_t+c_t) \Bigg[&\frac{N_0+\upsilon_t}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+\upsilon_t+1-\lambda_t)}} \\ &- \frac{\lambda_t}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+\upsilon_t-\lambda_t)}} \Bigg] - c_t(N_0+\upsilon_t-\lambda_t) \Bigg),\end{aligned} \tag{32}$$

where E_L is defined as

$$E_L \equiv \sum_{t=1}^{L-1} \gamma_{\text{disc}}^t \left(\frac{N_0(h_t+c_t)}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+1-\lambda_t)}} - \frac{\lambda_t(h_t+c_t)}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0-\lambda_t)}} - \frac{c_t}{h_t}(N_0-\lambda_t) \right). \tag{33}$$

B.1 Optimal Expected Cost

For the optimal cost, use that

$$\frac{c_t}{c_t+h_t} 1_{t\geq L} \simeq \frac{1}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(N_0+\upsilon_t+1-\lambda_t)}} \tag{34}$$

yielding the optimal expected cost

$$\begin{aligned}\mathbb{E}[C^*|D,I] &\simeq E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \left(c_t(N_0+\upsilon_t) - \lambda_t c_t \frac{c_t+h_t}{c_t+e^{\frac{m}{\sqrt{\lambda_t}}} h_t} - c_t(N_0+\upsilon_t-\lambda_t) \right) \\ &= E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \lambda_t h_t \frac{e^{\frac{m}{\sqrt{\lambda_t}}}-1}{e^{\frac{m}{\sqrt{\lambda_t}}}\frac{h_t}{c_t}+1}.\end{aligned} \tag{35}$$

B.2 Baseline Expected Cost

Finally, for the baseline cost, with $\upsilon_t = 1_{t\geq L}(\lambda_t - N_0 + R)$ with R the constant reorder point, the baseline expected cost is

$$\mathbb{E}[C'|D,I] \simeq E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \left(\frac{(h_t + c_t)(\lambda_t + R)}{1 + e^{-\frac{m}{\sqrt{\lambda_t}}(R+1)}} - \frac{(h_t + c_t)\lambda_t}{1 + e^{-\frac{m}{\sqrt{\lambda_t}}R}} - c_t R \right) \tag{36}$$

with

$$\begin{aligned} \lim_{R\to 0} \mathbb{E}[C'|D,I] &\simeq E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \frac{h_t \lambda_t}{2} \left(1 + \frac{c_t}{h_t}\right) \frac{e^{\frac{m}{\sqrt{\lambda_t}}} - 1}{e^{\frac{m}{\sqrt{\lambda_t}}} + 1}, \\ \lim_{R\to\infty} \mathbb{E}[C'|D,I] &\simeq E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t h_t R. \end{aligned} \tag{37}$$

B.3 Policy Efficiency Ratio

Given the expected cost of the optimal and baseline policies, the PER can be written

$$\frac{\mathbb{E}[C^*|D,I]}{\mathbb{E}[C'|D,I]} \simeq \frac{E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \lambda_t h_t \frac{e^{\frac{m}{\sqrt{\lambda_t}}} - 1}{e^{\frac{m}{\sqrt{\lambda_t}}} \frac{h_t}{c_t} + 1}}{E_L + \sum_{t=L}^{\infty} \gamma_{\text{disc}}^t h_t \left(\frac{(1+\frac{c_t}{h_t})(\lambda_t + R)}{1+e^{-\frac{m}{\sqrt{\lambda_t}}(R+1)}} - \frac{(1+\frac{c_t}{h_t})\lambda_t}{1+e^{-\frac{m}{\sqrt{\lambda_t}}R}} - \frac{c_t}{h_t} R \right)} \tag{38}$$

Assuming constant h, c, that E_L can be neglected in relative magnitude to the sums over infinity and using that

$$1 \leq e^{\frac{m}{\sqrt{\lambda_t}}} \lesssim 1.5 \Rightarrow e^{\frac{m}{\sqrt{\lambda_t}}} \approx k \tag{39}$$

the limit of $R \to 0$ yields

$$\begin{aligned} \lim_{R\to 0} \frac{\mathbb{E}[C^*|D,I]}{\mathbb{E}[C'|D,I]} &\approx 2 \frac{\frac{k-1}{k\frac{h}{c}+1}}{\left(1 + \frac{1}{\frac{h}{c}}\right)\frac{k-1}{1+k}} \frac{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \lambda_t}{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \lambda_t} \\ &= \frac{2(1+k)}{\left(1 + \frac{1}{\frac{h}{c}}\right)\left(k\frac{h}{c} + 1\right)} \\ &\simeq \frac{4}{2 + \frac{h}{c} + \frac{c}{h}} \end{aligned} \tag{40}$$

where for the last equality it was used that $k \simeq 1$. In the limit of $R \to \infty$

$$
\begin{aligned}
\lim_{R\to\infty} \frac{\mathbb{E}[C^*|D,I]}{\mathbb{E}[C'|D,I]} &\simeq \frac{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \lambda_t h_t \frac{e^{\frac{m}{\sqrt{\lambda_t}}}-1}{e^{\frac{m}{\sqrt{\lambda_t}}}\frac{h_t}{c_t}+1}}{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t h_t R} \\
&= \frac{1}{\frac{h}{c}+1} \frac{m}{R} \frac{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \sqrt{\lambda_t}}{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t}
\end{aligned} \tag{41}
$$

By definition

$$
\frac{m}{R} \frac{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t \sqrt{\lambda_t}}{\sum_{t=L}^{\infty} \gamma_{\text{disc}}^t} = 1 \tag{42}
$$

since $0 \leq \text{PER} \leq 1$. Therefore

$$
\lim_{R\to\infty} \frac{\mathbb{E}[C^*|D,I]}{\mathbb{E}[C'|D,I]} \simeq \frac{1}{\frac{h}{c}+1}. \tag{43}
$$

C Identities Related to the Poisson Distribution

The Poisson distribution with rate parameter λ describes the probability of a discrete random variable ζ taking integer values. Here, some key identities relevant for the PER (equation (11)) are explored. The cumulative probability of observing $\zeta \leq k$ given a Poisson rate λ is

$$
p(\zeta \leq k|\lambda) = e^{-\lambda} \sum_{j=0}^{k} \frac{\lambda^j}{j!}. \tag{44}
$$

This expression can also be written in terms of the incomplete gamma function, which is defined as

$$
\Gamma(s,x) = (s-1)! e^{-x} \sum_{j=0}^{s-1} \frac{x^j}{j!}, \tag{45}
$$

where $\Gamma(s)$ denotes the complete gamma function

$$
\Gamma(s) = (s-1)!. \tag{46}
$$

Using the incomplete gamma function, the cumulative probability $p(x \leq s|\lambda)$ can be expressed as

$$
p(x \leq s|\lambda) = \frac{\Gamma(s+1,x)}{\Gamma(s+1)}. \tag{47}
$$

Similarly, an expression for the conditional expectation of ζ, given that $\zeta \leq k$, can be derived viz

$$\begin{aligned} \mathbb{E}[\zeta|\zeta \leq k, \lambda] &= e^{-\lambda} \sum_{j=0}^{k} j \frac{\lambda^j}{j!} \\ &= e^{-\lambda} \sum_{j=0}^{k} \frac{\lambda^j}{(j-1)!} \\ &= \lambda e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} \\ &= \lambda \frac{\Gamma(s,x)}{\Gamma(s)}. \end{aligned} \tag{48}$$

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