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М.А.Браун

THEORY OF STRONG INTERACTIONS

Учебно-методическое пособие

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- Рецензент: зав. кафедрой ядерной физики, д.ф.м.н., профессор К.А. Гриднев

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М.А.Браун
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В учебно-методическом пособии излагаются основы современной теории сильных взаимодействий элементарных частиц и квантовой хромодинамики. Пособие предназначено для студентов 5-7-го курсов, аспирантов и соискателей, специализирующихся в области теоретической физики.

1 Theory of Strong Interactions

1.1 Introduction

The strong interaction has been known since the beginning of the 20th century, when the compound structure of atomic nucleus was established. This interaction is responsible for formation of the nucleus. It showed properties which radically distinguish it from the two previously known interactions, gravitational and electromagnetic. The first property is that it is a short-range interaction: it only acts between particles located at distances of the order 10^{-13} cm. So it is absent in the world of macroscopic objects. Is second property is that it is a strong interaction: the strong force acting between two protons at distance of the order 10^{-13} cm is roughly a thousand times stronger that the electromagnetic repulsion between them. Finally, unlike,say, the gravitational interaction, the strong interaction acts not between all known particles, but between a group of them, called hadrons. Hadrons include the well-known protons, neutrons, composite structures made of them, nuclei, and a good many of newly discovered particles. However photons, electrons, muons, gravitons and some other particles do not take part in the strong interaction.

According to our present knowledge all observed hadrons are in fact not purely elementary. They are built of certain 'prehadrons', called quarks. The quarks exist in 6 varieties, called flavours. They can be grouped in three generations (u,d) ('up' and 'down'), (c,s) ('charmed' and 'strange') and (t,b)('top' and 'bottom', ot 'beautiful'). All quarks are fermions with spin 1/2 and positive parity (by definition). Quarks u,c and t have their electromagnetic charge equal to $(2/3)e$, where e is the proton electromagnetic charge. Quarks d,s and b have their electromagnetic charge equal to $-(1/3)e$.

In its turn, the quark of each flavour is supposed to exist in three states, which differ by a certain new quantum number, colour. So the wave function of a quark has a structure $q^f_\alpha(x)$, where flavour $f = u, d, c, s, t, b$ and color $\alpha = 1, 2, 3$. Sometimes, for illustrative purposes, the three colour states are marked as 'red', 'yellow' and 'blue', which, of course, has nothing to do with the relevant colours in our everyday life.

It is the presence of colour which is supposed to generate the strong interaction. In a sense, colour plays the role of charge for the strong interaction. However it is a more complicated charge than the electromagnetic one. As we shall see roughly speaking it is a vestor charge as compared to the scalar electromagnetic charge. The strong interaction feels only color but ignores flavour. So quarks of any flavour strongly interact absolutely in the same manner. It is supposed that the strong interaction is invariant under unitary transformations of the three color states of each flavoured quark

$$q'^f_\alpha = U_{\alpha\beta} q'^f_\beta, \quad \det U = 1 \tag{1}$$

A peculiar property of quarks is that they cannot be observed as free particles. This property ('confinement') is supposed to originate from the fact that strong interaction between coloured objects infinitely rises with the distance between them. So to split, say, a quark-antiquark pair into its constituents one needs an infinite amount of energy. As a result only colourless objects, invariant under transformation (1), can be observed experimentally.

There are two evident ways to build a colourless object from coloured quarks. One is to take a colourless object composed from a quark-antiquark pair

$$M^{f,\bar f} = \bar q^{\bar f}_\alpha q^f_\alpha \tag{2}$$

The resulting hadron has an integer spin and is called a meson. Mesons can have different flavour contents $f, \bar{f}$ and also different spins, which may come both from the intrinsic spins of the pair (0 or 1) and from their relative rotation (orbital angular momentum). In this way one predicts existence of a multitude of mesons with generally rising spin (and mass).

The second way to build a colorless state is to form a fully antisymmetric state of three quarks

$$B^{f_1,f_2,f_3} = \epsilon_{\alpha,\beta\gamma} q_\alpha^{f_1} q_\beta^{f_2} q_\gamma^{f_3} \tag{3}$$

This hadron has a semi-integer spin and is called baryon. Again one can have a multitude of baryons depending on the three chosen flavours and resulting spin.

To discuss the hadrons observed in laboratory we have to say a few words about the quark masses. Since the free quarks cannot be seen, the notion of their mass is not completely clear. One may discern two sorts of mass, the current mass, which enters into the Lagrangian of the strong interaction , and constituent mass, which may be estimated from the mass of the composite hadron, provided the binding energy is not too large. Leaving aside the complicated theoretical deliberations, we first mention that for the so-called heavy quarks, c,b and t, the two definitions practically coincide and qive values 1.5, 4.5 and 180 GeV respectively. For the rest, light quarks, the current and constituent masses are found quite different. For u,d and s quarks the current mases are 4,7 and 150 MeV, respectively, while their constituen masses are 300, 300 and 450 MeV, respectively.

However, whatever the definition, there is a strong difference in masses between light and heavy quarks, and, as a consequence, between hadron built of exclusively light quarks and those which include at least one of the heavy quarks. Naturally most common hadrons are those built of light quarks, which either exist in nature as such or can be produced already at small accelerator energies.

The most known hadrons are of course the nucleons, the proton, p=uud, and neutron, n=udd, with spin 1/2 and masses $m_p \simeq m_n = 940$ MeV. They are building elements for all nuclei. The proton is stable and it is the only stable hadron. The free neutron decays by the weak interaction into p+e+$\bar{\nu}$. Substituting u or d by s-quarks one obtains a family of hyperons with spin 1/2: Λ^0=uds (m=1115 Mev),$\Sigma^{0,\pm}$ =uus,uds,dds (m=1190 MeV), $\Xi^{0,-}$=uss,dss (m=1315 MeV). All of them decay by the weak interaction. More baryons are obtained adding the three quark spins into 3/2, e.g. $\Delta^{-,0,+,++}$=ddd,udd,uud,uuu (m=1240 MeV), or $\Omega^- = sss$ (m=1670 MeV). The first decays by the strong interaction and so is visible only as a resonance in the scattering amplitudes, the second decays by the weak interaction.

The lightest meson is the triplet $\pi^{0,\pm} = \bar{d}u, \bar{u}d, 1/\sqrt{2}(\bar{u}u + \bar{d}d)$ with spin 0, negative parity and mass $m_\pi = 140$ MeV. This anomalously small mass allows to consider a theory with mass $m_\pi = 0$ as a first approximation. In this limit new, chiral, symmetry arises, which allows to make certain non-trivial predictions, which constitute the content of the so-called chiral dynamics. Changing u or d quarks or antiquarks into s or anti-s one obtains a family of K-mesons with mass $m = 450$ MeV. All these mesons are unstable, decaying by the weak interaction, with the exception of π^0 which decays by the electromagnetic interaction and so has a very short lifetime. Summing the spins of the quark-antiquark pair into 1, one obtains spin 1 mesons $\omega^0 = 1/\sqrt{2}(\bar{u}u - \bar{d}d)$, m=780 MeV, and $\rho^{0,\pm} = 1/\sqrt{2}(\bar{u}u + \bar{d}d), \bar{d}u, \bar{u}d$, m=760 MeV.

Special attention is to be drawn to spin 1 mesons with hidden non-trivial quantum numbers: $\phi^0 = \bar{s}s$, m=960 MeV, $J/\psi = \bar{c}c$, mass=3.1 GeV and Upsilon=$\bar{b}b$, m=9 GeV.

These mesons are clearly visible in the cross-section for e^+e^- annihilation. They served to discover c and b quarks.

The strong interaction, as mentioned, is only a consequence of the presence of colour. It does not feel the flavor. This means, that first of all, flavour is conserved in the strong interaction. As a result we have 6 conserved quantities in the strong interaction: the number of the differences $u-\bar{u}$, $d-\bar{d}$, $c-\bar{c}$, $s-\bar{s}$, $t-\bar{t}$ and $b-\bar{b}$. Second, quarks of all flavours strongly interact in the same way. This means that , if all quarks had the same mass, the strong interaction would be symmetric under the unitary transformation of all the flavours, which form the group $SU(6)_F$. One may expect such a full flavour symmetry at energies much higher than any quark mass. At present energies this symmetry is of course badly broken by quark masses. However there remains a flavour symmetry $SU(2)_F$ due to the approximately equal constituent masses of u and d-quarks, the isospin symmetry, well known in the low-energy nuclear physics.

The hadrons are not point-like particles but have a finite spatial extention. Because of that, in spite of being colorless, the interact strongly due to the fact that strong interaction between their constituents is not fully compensated. However the resulting strong interaction between hadrons turns out to be short-ranged.

The fact that the fundamental entities for the strong interaction, quarks, are not observed experimentally makes the study of the strong interaction very difficult. In fact we observe only its non-compensated trace in the interaction of the observed hadrons. Also the very strength of the interaction does not allow to indiscriminately use the perturbation theory, which remains up to the present the only effective tool to get prediction from the quantum field theory. Happily it has been discovered that the strength of the strong interaction depends on the distance between the interacting particles and goes down (albeit slowly) as this distance diminishes. So very strong at nuclear distances, the strong interaction becomes weaker at smaller distances, or transferred momenta much larger that the typical nuclear ones (of the order $0.2 \div 0.3$ GeV/c). This opens a possibility to use the perturbation theory for process dominated by these small distances or large transferred momenta ('hard processes')

Accordingly the theory of strong interactions can be split into three parts, which study different regions of transferred momenta. At small energies and thus transferred momenta the dominant processes are elastic scattering of protons and neutrons and formation and scattering of various nuclei. The complicated quark structure of hadrons is irrelevant. So the basic tool is just the ordinary (non-relativistic) quantum mechanics with certain potentials, which describe the interaction of nucleons. The upper energy, which allows such a treatment, is fixed by the possibility of pion production. In the next energy range up to $4 \div 5$ GeV, the quark structure is still not felt, but the spectrum of hadrons which are formed is becoming numerous. Most of them are seen as resonances in the scattering amplitudes. So the standard tool is to approximate the process by just creating a particle-resonance and its subsequent decay. Of course this leads to a very phenomenological description of the interaction, which uses the experimental information about the position and lifetime of various hadrons, which appear as possible resonances.

A more fundamental approach can be taken at very high energies. Here one can separate a whole class of processes or events in which hadron and their constituent quarks are brought to very small relative distances. These hard processes can be treated by the standard perturbation method applied to the quantum field theory which supposedly describes the strong interaction, the Quantum Chromodynamics (QCD). To such process belong: e^+e^- annihilation into hadrons, the so-called deep inelastic e-p scattering (DIS), scattering of hadrons at large angles etc. Unfortunately a wide class of events is dominated by large distances between participants ('soft processes') or involve, apart from small distances, also large ones. Then one cannot make predictions based only on the fundamental theory, but has instead, to bring in some amount of phenomenological material.

Our course will consist of two main parts. In the first we shall study some general properties of strong interaction (any interaction in fact), which constitute a basis for both purely phenomenological treatment and perturbative approach. In this part the sort of interacting particles, as well as underlying quantum field will be irrelevant. For simplicity we shall consider fictitious neutral scalar particles with properties resembling those of the physical nucleons and pions in that the mass of our nucleon is taken to be much large than that of our pion and that the interaction proceed by emission of a pion from nucleon: N$\rightarrow$N+π. The second part of our course will be wholly devoted to QCD and its application to the study of hard processes in the strong interaction.

1.2 General properties of scattering amplitudes

1.2.1 Cross-sections and amplitudes

Consider a process of collision of a pair of particles with momenta k_1 and k_2 in which in the final state $n \geq 2$ particles are produced with momenta k'_i, $i = 1, 2, ...n$. For simplicity we assume all particle to be scalar (spinless) but admit some of them belong to different species. The differential cross-section for this process is given by

$$d\sigma(k'_i) = \frac{1}{J_{12}}\Big|\mathcal{A}(k'_i|k_1,k_2)\Big|^2 d\tau'_n(k_1+k_2) \tag{4}$$

Here

$$J_{12} = 4\sqrt{(k_1k_2)^2 - m_1^2m_2^2} \tag{5}$$

is the invariant flux, m_1 and m_2 are the masses of colliding particles, $d\tau'_n$ is the invariant phase volume for the final particles restricted by conservation of energy-momentum:

$$d\tau'_n(k_1+k_2) = (2\pi)^4\delta^4(\sum_{i=1}^n k'_i - k_1 - k_2)\frac{1}{n_1!n_2!...n_s!}\prod_{i=1}^n \frac{d^3k'_i}{(2\pi)^3 2k_{i0}} \tag{6}$$

where the factorial factor refers to identical particles of the total number of s different species, $\sum n_i = n$, and the smooth function $\mathcal{A}$ is the scattering amplitude, which will be our main object throughout this lectures.

The amplitude is related to the scattering matrix S by the relation

$$S(k'_i|k_1,k_2) =< \Psi^{out}_{k'_1,...k'_n}|\Psi^{in}_{k_1,k_2} >= \delta(k'_i|k_1,k_2) + i(2\pi)^4\delta^4(\sum_{i=1}^n k'_i - k_1 - k_2)\mathcal{A}(k'_i|k_1,k_2) \tag{7}$$

Here

$$\delta(k'_i|k_1,k_2) = \delta_{2n}(2\pi)^6\sqrt{16k_{10}k_{20}k'_{10}k'_{20}}\frac{1}{2}\Big(\delta^3(k_1-k'_1)\delta^3(k_2-k'_2)+\delta^3(k_1-k'_2)\delta^3(k_2-k'_1)\Big) \tag{8}$$

is the relativistic invariant δ-function, that is the unit matrix in our basis.

An important particular case is when $n = 2$ and we have what is called a binary reaction. Then

$$d\tau_2(k_1 + k_2) = \frac{1}{16\pi^2}\frac{k'}{W}d\Omega' \tag{9}$$

where W, k' and Ω' are the energy, momentum and solid angle of the final particles in the center-of-mass (c.m.) system.

The cross-section (4) is called exclusive. It refers to the experimental setup in which momenta of all final particles are measured. In fact, with a multitude of particles produced at high energies, this cross-section is difficult to obtain and moreover, if found, it depends on so many variables that its theoretical analysis is hopeless. For this reason a more realistic setup is to fix momenta of a few final particles allowing for the rest to have any momenta. A typical example is the single inclusive cross-section which implies that in the experiment only the momentum of a single particle of a given sort is measured. Such an experiment is denoted as a process

$$1 + 2 \to 1' + X \tag{10}$$

where X means all the unobserved particles. The single inclusive cross-section is expressed via the amplitude by the formula

$$d\sigma(k_1', X) = \frac{1}{J_{12}}\frac{d^3k_1'}{(2\pi)^3 k_{10}'}\sum_n \int d\tau_n'(k_1 + k_2 - k_1')\Big|\mathcal{A}(k_1', X|k_1, k_2)\Big|^2 \tag{11}$$

The maximally inclusive cross-section is the total one, which experimentally is determined just by the decrease in the incoming flux:

$$d\sigma^{tot} = \frac{1}{J_{12}}\sum_n \int d\tau_n'(k_1 + k_2)\Big|\mathcal{A}(k_i'|k_1, k_2)\Big|^2 \tag{12}$$

1.2.2 Relativistic invariance

Let $U(\Lambda)$ be the unitary operator acting on the states which corresponds to the generalized Lorentz transformations Λ (that is including spatial rotations). Then Lorentz-invariance implies that

$$< \Psi^{out}_{k_1',...k_n'}|\Psi^{in}_{k_1,k_2} >=< \Psi^{out}_{k_1',...k_n'}|U^{-1}(\Lambda)U(\Lambda)\Psi^{in}_{k_1,k_2} >=< \Psi^{out}_{\Lambda k_1',...\Lambda k_n'}|\Psi^{in}_{\Lambda k_1,\Lambda k_2} > \tag{13}$$

As a result we find

$$\mathcal{A}(\Lambda k_i'|\Lambda k_1, \Lambda k_2) = \mathcal{A}(k_i'|k_1, k_2) \tag{14}$$

That is the amplitude does not change if all momenta are Lorentz-transformed. Of course, if particles have spins one has additionally to rotate the spins.

Apart from relativistic invariance the amplitude is symmetric under a few discrete transformations. For the strong interaction these are C (charge conjugation), P (parity) and T (time reversal). The first two are trivially derived similar to the Lorentz-invariance. Charge conjugation C consists in changing all particles (initial and final) into antiparticles and vice versa. Parity P consists in reflecting all three spatial axes, under which the momenta $\mathbf{k}$ change sign but spin projections σ do not

$$P: \quad (\mathbf{k}, \sigma) \to (-\mathbf{k}, \sigma) \tag{15}$$

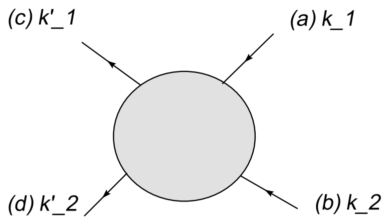

Figure 1: Binary amplitude

Somewhat non-trivial is the time reversal, which changes sign of both momenta and spin projections, interchanges the initial and final states and also introduces a certain phase factor for particles with spins:

$$T: \quad |\Psi^{in}_{\mathbf{k}_1,\sigma_1,\dots} > \to < \Psi^{out}_{-\mathbf{k}_1,-\sigma_1}| \prod_i (-1)^{s_i-\sigma_i} \tag{16}$$

Here s_i is the spin of the i-th particle. As a result of T-invariance we find (for spinless particles)

$$\mathcal{A}(\mathbf{k}'_i|\mathbf{k}_1,\mathbf{k}_2) = \mathcal{A}(-\mathbf{k}_1,-\mathbf{k}_2|-\mathbf{k}'_i) = \mathcal{A}(\mathbf{k}_1,\mathbf{k}_2|\mathbf{k}'_i) \tag{17}$$

The second equality follows from the P-invariance. So combined PT invariance implies that the amplitude does not change if we interchange initial and final states (for spinless particles). Note that this relation has practical sense only if $n = 2$, that is for binary reactions. Otherwise we relate the initial amplitude with a non-physical one which corresponds to the initial state containing more than two particles.

Relativistic invariance implies that the amplitude is an explicit functions of scalar products of 4-momenta of all participating particles (initial plus final). However one has to remember that the number of such products is generally larger than the number of independent variables. If the total number of participating particles is $N = 2+n$ then the total number of independent variables is $3N-10$. Indeed the total number of independent 3-momenta is $3N$,which are restricted by the energy-momentum conservation to $3N-4$. We have then 6 Lorentz transformations which allow to put one of the momentum equal to zero, direct another momentum along the z axis and the third in the xz plane, which removes 6 variables. This means that between the scalar products of 4-momenta there exist many algebraic relations.

The important case as before is the binary reaction with $n = 2$ and $N = 4$ (a four-legged amplitude). It depends on $3 \cdot 4 - 10 = 2$ independent variables, for which one stndardly chooses the energy and scattering angle in the c.m. system. However to make the relativistic invariance explicit it is convenient to introduce different variables ('the Mandelstam variables'). s, t and u, which are standardly defined as follows (see Fig. 1)

$$s = (k_1+k_2)^2 = (k'_1+k'_2)^2, \quad t = (k_1-k'_1)^2 = (k_2-k'_2)^2, \quad u = (k_1-k'_2)^2 = (k_2-k'_1)^2 \tag{18}$$

They are three, so one expects to find a relation between them. Indeed one finds

$$s+t+u = m_1^2 + m_2^2 + 2k_1k_2 + m_1^2 + {m'_1}^2 - 2k_1k'_1 + m_1^2 + {m'_2}^2 - 2k_1k'_2$$
$$= 3m_1^2 + m_2^2 + {m'_1}^2 + {m'_2}^2 - 2k_1(k'_1 + k'_2 - k_2) = m_1^2 + m_2^2 + {m'_1}^2 + {m'_2}^2$$

So the sum of s, t and u is equal to the sum of mass squares of participating 4 particles.

$$s+t+u = m_1^2 + m_2^2 + {m'_1}^2 + {m'_2}^2 \tag{19}$$

The Mandelstam variables have a transparent physical meaning. In the c.m. system $\mathbf{k}_1 + \mathbf{k}_2 = 0$ and $s = W^2$, that is the square of c.m. energy. Obviously for the process to take place s has to lie above the threshold

$$s > \max\{(m_1 + m_2)^2, (m_1' + m_2')^2\} \tag{20}$$

The meaning of t is clearer for elastic processes, when the masses of particles do not change: $m_1' = m_1$ and $m_2' = m_2$. Then in the c.m. system the momentum does not change its magnitude and only rotates: $|\mathbf{k}'| = |\mathbf{k}|$. Then $k_{10} = k_{10}'$ and $t = -(\mathbf{k}_1 - \mathbf{k}_2)^2$ that is the transferred momentum squared with a minus sign. In terms of the scattering angle

$$t = -2\mathbf{k}^2(1 - \cos\theta) \tag{21}$$

and so is negative and satisfies

$$-4\mathbf{k}^2 < t < 0 \tag{22}$$

Variable u has a less transparent meaning, since it includes a contribution from the non-equal zero components. Its limits can be derived from (19)

$$2(m_1^2 + m_2^2) - (\sqrt{(\mathbf{k}^2 + m_1^2)(\mathbf{k}^2 + m_2^2)} < u < (m_1 - m_2)^2 \tag{23}$$

Inequalities (20), (21) and (22) mark what is known as the physical region for the elastic scattering process

$$a(k_1) + b(k_2) \to a(k_1') + b(k_2') \tag{24}$$

1.2.3 Unitarity

In terms of the scattering matrix it is formulated as

$$SS^\dagger = S^\dagger S = 1 \tag{25}$$

Taking this operator relations between two-particle scattering states $< k_1', k_2'|$ and $|k_1, k_2 >$ we obtain for the corresponding amplitudes

$$i\mathcal{A}^*(k_1, k_2|k_1', k_2') - i\mathcal{A}(k_1', k_2'|k_1, k_2) = \sum_n \int d\tau_n \mathcal{A}^*(n|k_1', k_2')\mathcal{A}(n|k_1, k_2) \tag{26}$$

Using the PT invariance (for spinless particles) we find

$$\mathcal{A}^*(k_1, k_2|k_1', k_2') = \mathcal{A}^*(k_1', k_2'|k_1, k_2)$$

which allows to rewrite (26) as

$$2\mathrm{Im}\,\mathcal{A}(k_1', k_2'|k_1, k_2) = \sum_n \int d\tau_n \mathcal{A}^*(n|k_1', k_2')\mathcal{A}(n|k_1, k_2) \tag{27}$$

This general unitarity relation gives for the forward scattering

$$2\mathrm{Im}\,\mathcal{A}(k_1, k_2|k_1, k_2) = \sum_n \int d\tau_n \Big|\mathcal{A}(n|k_1, k_2)\Big|^2 = J_{12}\sigma^{tot} \tag{28}$$

which is the famous 'optical theorem'. Note that this relation allows to find the fully inclusive, total cross-section from the knowledge of only the $2 \to 2$ forward elastic amplitude and thus avoiding to sum all the exclusive amplitudes. Similar relations can be obtained also for single and multiple inclusive cross-sections.

A particularly simple case is the two-particle unitarity, which follows from the intermediate states with $n = 2$ in 27

$$\operatorname{Im} \mathcal{A}(k_1', k_2'|k_1, k_2) = \frac{k''}{32\pi^2 W} \int d\Omega'' \mathcal{A}^*(k_1'', k_2''|k_1', k_2')\mathcal{A}(k_1'', k_2''|k_1, k_2) \tag{29}$$

For light particles, like pions and nucleons, it is exactly fulfilled for energies below the threshold of new particle production.

One has to stress that in terms of Mandelstam variables s, t and u the unitarity relation is obtained only in the physical region, where all the momenta are real.

1.2.4 Crossing

From the perturbation theory it is known that a given Feynman diagram describes different processes depending on the sign of the zero components of external particle momenta. A change $k \to -k$ for an external line corresponds to the transition from the initial particle to the final antiparticle and vise versa. For particles with spins one has to additionally transform the relevant spinors. To be concrete consider a binary amplitude for the scattering of scalar particles (Fig. 1)

$$a(k_1) + b(k_2) \to c(k_1') + d(k_2') \tag{30}$$

The property of crossing means that the same Feynman diagrams and hence the amplitude will also describe two more processes

$$a(k_1) + \bar{c}(-k_1') \to \bar{b}(-k_2) + d(k_2') \tag{31}$$

and

$$a(k_1) + \bar{d}(-k_2') \to c(k_1') + \bar{b}(-k_2) \tag{32}$$

For the first reaction variable s plays the role of the square of the c.m. energy. For the second one this role is played by variable t, and for the third by variable u. Correspondingly the three reactions are denoted by channels: s-channel for (30), t-channel for (31) and u-channel for (32).

In fact the amplitudes in the three channel depend on their variables in different regions which do not overlap. The s-channel amplitude implies that all zero components of k_1, k_2, k_1' and k_2' are positive. For the t-channel k_1' and k_2 have to possess negative zero components. For the u-channel k_2' and k_2 have to possess negative zero components. The Mandelstam variables s, t, u have to belong to the physical regions of each channel, which do not overlap. In particular for the s channel conditions (20-22) are to be satisfied, which imply that s is positive and t is negative. However for the t channel one immediately concludes that t has to be positive and s and u predominantly negative (for elastic scattering their regions is given by (22).

Since the physical regions of the three channels do not overlap, the crossing by itself does not lead to some constructive consequence for the amplitude. In fact one meets with three different functions $\mathcal{A}(s, t, u)$ corresponding to the amplitude in different physical regions. However if $\mathcal{A}$ is given in a certain analytic form, e.g. by the corresponding Feynman diagrams, one find a possibility to analytically continue $\mathcal{A}(s,t,u)$ from one physical region to another. Then one discovers a relation between the amplitudes for generally different physical processes. For a particular case when some of the channels coincide one finds a symmetry: after continuation one has to find the same function.

Thus the property of crossing acquires its full meaning in relation with the analyticity properties of the amplitude.

The crossing is of course trivially generalized to multi-legged amplitudes. Each external particle may be considered both as the incoming particle or outgoing antiparticle and vice versa. Correspondingly one obtains amplitudes for different processes depending on the values of the generalized energies and transferred momenta, which now may involve several particles. Again something constructive may follow if one is able to analytically continue the amplitude from one physical region to another.

1.2.5 Analyticity

This property is the most complicated. It involves several points. First, as we shall see the amplitude indeed allows for an analytic continuation from one physical region to another. Second, it turns out that the amplitude is in general an analytic function of its variables in the whole complex plane, except for isolated singularity points of the pole or branch points type. Third, the position of these singularities and discontinuities around them are determined by the unitarity relation.

These properties will be derived in the following section. Here we only stress its importance. As one knows, the analytical functions is totally determined by its singularity poits, residues at its poles and discontinuities along the cuts. Once these can be found from the unitarity, a possibility arises to find the amplitude uniquely only from unitarity for different channels and analyticity, without recurring to any dynamical background, related to the Lagrangian approach in the framework of the quantum field theory. However this maximalistic idea fails due to complexity of analytic properties of multiparticle amplitudes and the growth of amplitudes for particles with spins at infinitely large momenta.

1.3 Analytic Properties in the Perturbation Theory

Derivation of analytic properties of scattering amplitudes from fundamental principles, illustrated in the preceding section is difficult even for simplest amplitudes and becomes impossible for more general ones. So most of the information necessary for practical applications comes from the study of the expressions obtained in perturbation approach, that is from Feynman diagrams. Again spins are inessential and we shall restrict ourselves to interaction of spinless particles.

The simplest Feynman diagrams do not include integration over momenta of intermediate particles. These are the so-called tree diagrams. Analytic properties of tree diagrams are easily established, since they are just products of propagators of intermediate particles. Each one of them is just a pole in an invariant variable which is a sum of a certain combination of external particle momenta at the point corresponding to the mass squared of the intermediate particle. So the amplitude is an analytic function of all possible invariant variables, having poles in sums of external momenta at intermediate particles mass squares.

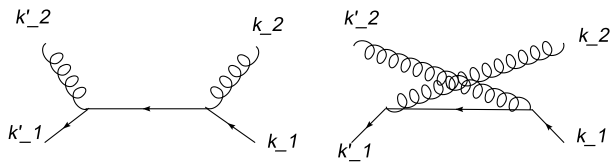

Figure 2: Tree diagrams for elastic π-N scattering

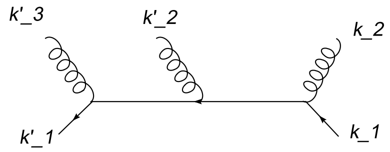

Figure 3: A tree diagram for pion production in π-N collisions

A simple example is the elastic πN amplitude studied in the preceding section. The two possible tree diagrams for it are shown in Fig.2. Each one is just a propagator of the intermediate nucleon. They give a sum of two poles:

$$\frac{g^2}{m^2 - s} + \frac{g^2}{m^2 - u} \tag{33}$$

For a more complicated amplitude for the pion production

$$N(k_1) + \pi(k_2) \to N(k'_1) + \pi(k'_2) + \pi(k'_3) \tag{34}$$

one of a set of possible tree diagrams is shown in Fig. 3. It is a product of two intermediate nucleon propagators and so a double pole

$$g^3 \frac{1}{m^2 - (k_1 + k_2)^2} m^2 - (k_1 + k_2 - k'_3)^2 \tag{35}$$

1.3.1 Loop diagrams. Landau's rules

More complicated , loop contributions to the amplitude involve integration over intermediate particle momenta. To study their analytic properties certain practical rules have been established . L.D.Landau invented rules which allow to determine positions of singularities of loop amplitudes in their external momentum variables.

The contribution of a general Feynman diagram to the amplitude can be written in the form of a multiple integral

$$\mathcal{A} = \int \prod_{i=1}^{L} d^4 k_i \prod_{j=1}^{l} \frac{1}{A_j} \tag{36}$$

where $A_j = m^2 - q_j^2$ is the propagator corresponding to the jth internal line. The total number of internal lines is l. Integration is performed over L loop 4-momenta k_i, $i = 1, ...L$. In fact the momenta q_i of intermediate particles are linear combinations of the loop momenta and external particle momenta.

To illustrate consider a simple one-loop diagram shown in Fig. 4. It is convenient

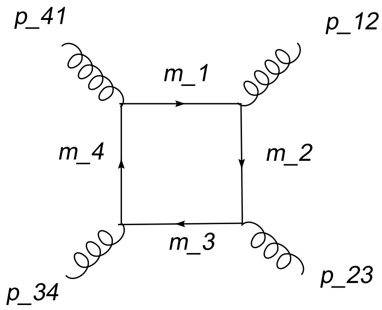

Figure 4: An one-loop diagram

to assume all external momenta as incoming have the maximal symmetry :$p_{12} + p_{23} + p_{34} + p_{41} = 0$. If the loop momentum is k then one may choose $q_1 = k$, $q_2 = k + p_{12}$, $q_3 = k + p_{12} + p_{23}$ and $q_4 = k + p_{12} + p_{23} + p_{34}$.

One can present the amplitude in the Feynman parametrization, using the identity

$$\prod_{j=1}^{l} \frac{1}{A_j} = (l-1)! \int_0^\infty \prod_{j=1}^{l} d\alpha_j \delta(1 - \sum_{j=1}^{l} \alpha_j) \frac{1}{(\sum_{j=1}^{l} \alpha_j A_j)^l} \tag{37}$$

For our purpose it is desirable to exclude the δ- function without spoiling the symmetry in all variables. To this end we insert the identity

$$1 = \int_0^\infty d\lambda e^{-\lambda}$$

into (37) and change variables $\alpha_j \to \lambda \alpha_j$ Integration over λ with the help of δ-function gives

$$\prod_{j=1}^{l} \frac{1}{A_j} = (l-1)! \int_0^\infty \frac{\prod_{j=1}^{l} \alpha_j e^{-1/\sum \alpha_j}}{\sum \alpha_j (\sum_{j=1}^{l} \alpha_j A_j)^l} \tag{38}$$

Now we have to analyze possible singularities of the amplitude presented in the form (38) and considered as a function of external momenta.

To start we consider a function $f(z)$ of the complex variable z presented as an integral over another complex variable ξ along a certain curve:

$$f(z) = \int_a^b d\xi g(z, \xi) \tag{39}$$

Let $g(z, \xi)$ be an analytic function of its variables, which has isolated singular points. One can distinguish between two possibilities. Considered as a function of z function $g(z, \xi)$ may have singularities at points $z = z_s$ which do not depend on ξ ('fixed singular points'). In this case obviously $f(z)$ given by (39) will also have singularities at the same fixed points. However this case has no relation with the structure of the integral (38) and so not very interesting. The second possibility is that the position of the singularity of $g(z, \xi)$ in variable z depends on variable ξ ('moving singular points'). In this case it is advantageous consider $g(z, \xi)$ as a function of the integration variable ξ, which has a singularity at $\xi = \xi_s(z)$ depending on z. As z varies the singular point $\xi_s(z)$ moves in the complex ξ-plane. If $\xi_s(z)$ touches the integration contour in (39) this generally does not cause a singularity of $f(z)$, since the contour can be displaced to avoid the singularity ξ_s.

A singularity arises only when such a displacement becomes impossible. This may occur in two cases.

1) The singular point $\xi_s(z)$ touches the initial or final integration points a or b. Equations $\xi_s(z)=a$ or $\xi_s(z)=b$ determine the position of this so-called end-point singularity in z.

2) Two different singularities $\xi_{s1}(z)$ and $\xi_{s2}(z)$ touch the integration contour from different side. This generates a 'pinch singularity' whose position is determined by the equation $\xi_{s1}(z)=\xi_{s2}(z)$ In a particularly important point function $g(x,\xi)$ has a form of $1/h(z,\xi)$ where $h(z,\xi)$ is an analytic function which may have isolated zeros. Then coincidence of two zeros at the same point implies two equations

$$h(z,\xi)=0, \quad \frac{\partial h(z,\xi)}{\partial \xi}=0 \tag{40}$$

which determine the pinch singularity.

These considerations can be easily generalized to multiple integrals. Let us consider a function $f(z)$ presented as a double integral

$$f(z)=\int_{a_2}^{b_2} d\xi_2 \int_{a_1}^{b_1} d\xi_1 \frac{1}{h(z,\xi_1,\xi_2)}=\int_{a_2}^{b_2} d\xi_2 g(z,\xi_2) \tag{41}$$

where

$$g(z,\xi_2)=\int_{a_1}^{b_1} d\xi_1 \frac{1}{h(z,\xi_1,\xi_2)} \tag{42}$$

Function $g(z,\xi_2)$ becomes singular either if a singularity in ξ_1 touches the end points a_1 or b_1 or if two singularities in ξ_1 coincide, that is if

$$h(z,\xi_1,\xi_2)=0, \quad \frac{\partial h(z,\xi_1,\xi_2)}{\partial \xi_1}=0 \tag{43}$$

In the last case, expressing from the second equation $\xi_1=\xi_1(z,\xi_2)$ we find the equation for the singularity of $g(z,\xi_2)$ as

$$h(z,\xi_1(\xi_2),\xi_2)=0 \tag{44}$$

Now a pinch singularity of $f(z)$ arises if two singularities of $g(z,\xi_2)$ coincide, that id when additionally

$$\frac{\partial h(z,\xi_1(\xi_2),\xi_2)}{\partial \xi_2}=0 \tag{45}$$

However

$$\frac{\partial h(z,\xi_1(\xi_2),\xi_2)}{\partial \xi_2}=\frac{\partial h(z,\xi_1(\xi_2),\xi_2)}{\partial \xi_2}\Big|_{fixed\ \xi_1}+\frac{\partial h(z,\xi_1(\xi_2),\xi_2)}{\partial \xi_1}\frac{\partial \xi_1(\xi_2))}{\partial \xi_2}=0 \tag{46}$$

However the last derivative is zero, so that the final conditions for the pinch singularity are

$$h(z,\xi_1,\xi_2)=0, \quad \frac{\partial h(z,\xi_1,\xi_2)}{\partial \xi_1)}=0, \quad \frac{\partial h(z,\xi_1,\xi_2)}{\partial \xi_2}=0. \tag{47}$$

Any of the two equations with derivatives can be substituted by a condition for the end-point singularity, $\xi_1=a_1$ or $\xi_1=b_1$ or $\xi_2=a_2$ or $\xi_2=b_2$.

In the general case when

$$f(z) = \int \prod_{i=1}^{n} d\xi_i \frac{1}{h(z, \xi_1, ...\xi_n)} \tag{48}$$

the conditions for a singularity in z are

$$h(z, \xi_1, ...\xi_n) = 0, \quad \frac{\partial h(z, \xi_1, ...\xi_n)}{\partial \xi_i} = 0, \quad (\xi_i = a_i \quad \text{or} \quad b_i) \tag{49}$$

Now we apply these conditions to the expression (38) for the amplitude. The integral involves two sets of variables: $\alpha_1, ...\alpha_l$ for each internal line and $k_1, ...k_L$ for each integration loop. The role of function h in (49) is played by function $\sum_{j=1}^{l} \alpha_j A_j$ Applying (49) to variables α_j we obtain conditions:

$$A_j = 0, \quad \text{or} \quad \alpha_j = 0, \quad \text{for each line} \tag{50}$$

Due to these confitions we do not need to additionally require $h = 0$ Applying (49) to variables k_i we obtain

$$\sum_{j=1}^{l} \alpha_j \frac{\partial A_j}{\partial k_i} = 0 \tag{51}$$

Since q_j is a linear function of k_i in a given loop with a coefficient which may be chosen to be $+1$ with a suitable choice of direction of q_j, this relation can also be written as

$$\sum \alpha_j q_j = 0, \quad \text{for each integration loop} \tag{52}$$

Relations (50)-(52) are the Landau rules to determine the position of the singularity of the amplitude.

Application of these rules to concrete amplitude will be presented in the last subsection of this section. Here just a few preliminary comments. Conditions (50) imply that in the study of singularities each internal article either should be considered as physical, $q_j^2 = m_j^2$, or the corresponding line should be dropped, $\alpha_j = 0$, with the attached vertices joined together. In the latter case we get a new, reduced diagram which should be treated by the same method. The possibility to put any $\alpha_j = 0$ thus implies that a given diagram contains all the singularities corresponding to its reduced diagrams and also certain proper singularities, obtained when none of the α_j is put to zero. Some reduced diagrams corresponding to the original diagram in Fig. 4 are shown in Fig. 5. The second condition (52) means that in the study of singularities all momenta in the main and reduced diagrams should be taken coplanar.

It is important that the Landau rules are only the necessary conditions for the existence of singularity. In our derivation we mentioned that the pinch occurs only when the two singularities touch the integration contour from opposite sides. If they touch it from the same side there obviously no singularity occur. Thus some of the singularities determined by the Landau rules may be non-existent on the physical sheet of the amplitude. They can then be found on unphysical sheets reached by analytic continuation from the physical one. In particular physical singularities require that the values of α_j found from the Landau rules be non-negative in correspondence with the integration region in (38). Analytic continuation may demand distortion of this region and then the singularity may appear also with some negative values of α_j.

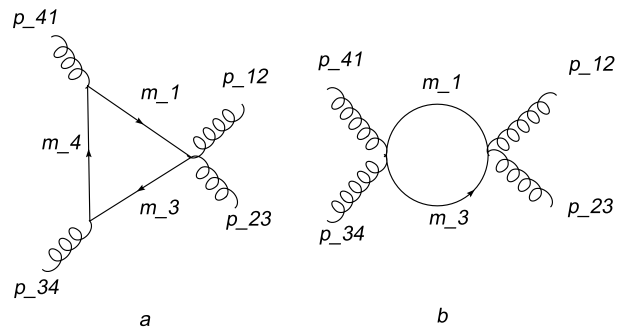

Figure 5: Reduced diagrams corresponding to Fig. 4

1.3.2 Loop diagrams. Cutkosky's rules

The Landau rules only tell us the position of the singularity. Normally the singularity is a branch point. The discontinuity of the amplitude across the cut which starts from this branch point is determined by the Cutkosky rules. They are based on the technique in which all masses corresponding to internal lines are taken different (also in the physical amplitude masses take on just a few values).

Suppose we want to study the discontinuity across the cut which related to the branch point corresponding to a given reduced diagram, with only certain number $n \leq l$ lines remaining from the original amplitudes which we enumerate $1, ..., n$. Present the amplitude in the form

$$\mathcal{A} = \int_{a_1(z)}^{b_1(z)} \frac{dq_1^2}{m_1^2 - q_1^2} \int_{a_2(z,q_1^2)}^{b_1(z,q_1^2)} \frac{dq_2^2}{m_2^2 - q_2^2} \cdots \int_{a_n(z,q_1^2,...q_{n-1}^2)}^{b_1(z,q_1^2,...q_{n-1}^2)} \frac{dq_n^2}{m_n^2 - q_1^2} \mathcal{A}_n(z, q_1^2 ... q_n^2) \tag{53}$$

where we separated the first n denominators A_i which correspond to the lines remained in the reduced diagrams and stored all the rest denominators together with the determinant for the transition from loop momenta to variables q_i^2, $i = 1, ...n$ into function $\mathcal{A}_n$. The external variable (or variables) are denoted as z.

We begin with the study of integration over q_1^2 presenting (53) as

$$\mathcal{A} = \int_{a_1(z)}^{b_1(z)} \frac{dq_1^2}{m_1 - q_1^2} \mathcal{A}_1(z, q_1^2) \tag{54}$$

Note the characteristic feature of the singularity in question: it should depend on all masses $m_1, ...m_n$ and only on them. Let us study the integral (54) from this point of view. It can have singularity in three cases:

1) The singularity of the denominator at $q_1^2 = m_1^2$ may coincide with the endpoints $a_1(z)$ or $b_1(z)$.

2) A singularity of $\mathcal{A}_1(z, q_1^2)$ at $q_1^2 = q_{1s}^2$ may coincide with the endpoints $a_1(z)$ or $b_1(z)$.

3) This singularity at $q_1^2 = q_{1s}^2$ may coincide with the singularity of the denominator: $q_{1s}^2 = m_1^2$.

It is trivial to see that a singularity depending on all n masses $m_1, ...m_n$ can only arise in the third case. In the first case it depends only on m_1 and in the second it does not depend on m_1. So our singularity is a pinch shown graphically in Fig. 6. Obviously we can deform the integration contour in the q_1^2 plane as shown in Fig. 7 separating the

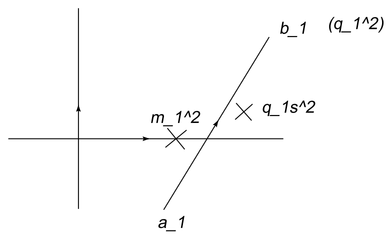

Figure 6: Pinching two singularities

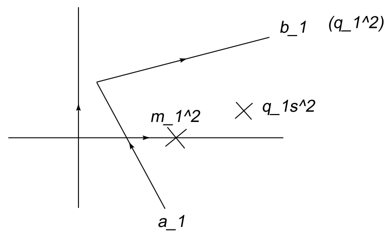

Figure 7: The new integration contour avoiding the pinch

contribution from the pole at $q_1^2 = m_1^2$ and the rest integral which has no singularity;

$$\mathcal{A} = \pm 2\pi i \mathcal{A}_1(z, m_1^2) + \int_{C_1} \frac{dq_1^2}{m_1^2 - q_1^2} \mathcal{A}_1(z, q_1^2) \tag{55}$$

We do not know the true location of singularities, which leads to an undetermined sign of the pole contribution.

Since the integral in (55) has no singularity, the singular part of the amplitude (and the branch cut discontinuity) is contained in the pole contribution proportional to $\mathcal{A}_1(z, m_1^2)$. So next we study this pole term presenting

$$\mathcal{A}_1(z, m_1^2) = \int_{a_2(z,m_1^2)}^{b_1(z,m_1^2)} \frac{dq_2^2}{m_2 - q_1^2} \mathcal{A}_2(z, m_1^2, q_2^2) \tag{56}$$

We repeat the same study as for the initial amplitude and find that from all three possibility for the singularity only the pinch between the pole at $q_2^2 = m_2^2$ and a singularity of $\mathcal{A}_2(z, m_1^2, q_2^2)$ at $q_{2s}^2(z, m_1^2)$ gives the desired singularity in z depending on all masses $m_1, ...m_n$. Again we present $\mathcal{A}_2(z, m_1^2)$ as a sum of the contribution from the pole term and an integral which has no singularity:

$$\mathcal{A}_2(z, m_1^2) = \pm 2\pi i \mathcal{A}_2(z, m_1^2, m_2^2) + \quad \text{regular} \quad \text{term} \tag{57}$$

Next we present $\mathcal{A}_2(z, m_1^2, m_2^2)$ as an integral over dq_3^2 from $\mathcal{A}_3(z, m_1^2, m_2^2, q_3^2)$ and do the same reasoning after which we repeat this procedure until we arrive at

$$\mathcal{A}_{n-1}(z, m_1^2, ...m_{n-1}^2) = \int_{a_n(z,m_1^2,...m_{n-1}^2)}^{b_n(z,m_1^2,...m_{n-1}^2)} \frac{dq_n^2}{m_n^2 - q_n^2} \mathcal{A}_n(z, m_1^2, ...m_{n-1}^2) \tag{58}$$

. Now from the three possibilities for the singularity the only suitable is the coincidence of the singularity at $q_n^2 = m_n^2$ with one of the endpoints

$$a_n(z, m_1^2, ...m_{n-1}^2) = m_n^2 \quad \text{or} \quad b_n(z, m_1^2, ...m_{n-1}^2) = m_n^2 \tag{59}$$

Other two possibilities, which involve a singularity of $\mathcal{A}_n$, depend on masses m_j with $j > n$ which enter $\mathcal{A}_n$ and so correspond to different reduced diagrams.

Let $b_n(z_s) = m_n^2$ (we suppress other arguments in b_n). Then at z close to the singularity point z_s we have

$$b_n(z) = m_n^2 + (z - z_s)b_n'(z_s) + .. \tag{60}$$

As z moves around the singular point $b_n(z)$ also moves around m_n^2 (if $b_n'(z_s) \neq 0$). So we find the discontinuity

$$\text{Disc}\, \mathcal{A}_{n-1}(z) = \pm 2\pi \mathcal{A}_n(z, m_1^2, ...m_n^2) \tag{61}$$

Again the sign is indeterminate, since the sign of $b_n'(z_s)$ is unknown.

Combining all our results we find the discontinuity of the amplitude

$$\text{Disc}\, \mathcal{A}(z) = (\pm 2\pi)^n \mathcal{A}_n(z, m_1^2, ...m_n^2) \tag{62}$$

This means that to find the discontinuity one has to substitute all particle propagators remaining in the reduced diagram according to

$$\frac{1}{m^2 - q^2} \to \pm 2\pi i \delta(m^2 - q^2) \tag{63}$$

This is the content of the Cutkosky rules.

It is instructive to see how one can determine the position of the singularity point in this approach. As mentioned the condition for a singularity is

$$b_n(z, m_1^2, ...m_{n-1}^2) = m_n^2 \tag{64}$$

or $a_n = m_n^2$). So one encounters the problem to find $b_n(z, m_1^2, ...m_{n-1}^2)$. To this one has to find an extremum of q_n^2 at fixed values of $q_1^2, ...q_{n-1}^2$. The actual variables are the 4 =momenta k_1. So one has an equation for determination of the extremum

$$\frac{\partial q_n^2}{\partial k_i} + \gamma_1 \frac{\partial q_1^2}{\partial k_i} + \cdots \gamma_{n-1} \frac{\partial q_{n-1}^2}{\partial k_i} = 0 \tag{65}$$

with γ_i $i = 1, ...n-1$ being the Lagrange multipliers and under complementary conditions $q_i^2 = m_i^2$, $i = 1, ...n-1, n$. Obviously these equations coincide with the Landau rules.

1.3.3 Unitary and non-unitary singularities

Let us apply the found rules for the study some simple examples. For the box diagram of Fig. 4 the simplest non-trivial reduced diagrams are shown in Fig. 5. Let us consider one of them shown in Fig. 8 with more natural notations for particle external momenta. The Landau equation for it are

$$\alpha_1 q_1 - \alpha_2 q_3 = 0, \quad q_1^2 = m_1^2, \quad q_3^2 = m_3^2 \tag{66}$$

(the negative sign in the first equations comes from the direction of q_3 in the integration loop, opposite to that of q_1. From the first equation we find $q_3 = q_1(\alpha_1/\alpha_3)$. Squaring this we find:

$$m_3^2 = \Big(\frac{\alpha_1}{\alpha_3}\Big)^2 m_1^2 \tag{67}$$

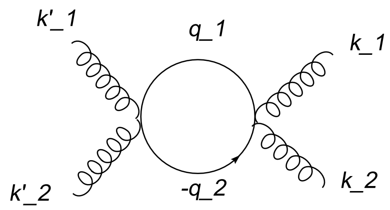

Figure 8: Simple reduced diagram for binary scattering

which gives

$$\frac{\alpha_3}{\alpha_1} = \frac{m_1}{m_3} \tag{68}$$

On the other hand, $k_1 + k_2 = q_1 + q_3 = q_1(1 + m_3/m_1)$. Taking its square modulus we find the position of singularity

$$s = (k_1 + k_2)^2 = m_1^2\Big(1 + \frac{m_3}{m_1})\Big)^2 = (m_1 + m_3)^2 \tag{69}$$

This is precisely the threshold of the reaction according to the unitarity relation with the intermediate state of two particles with masses m_1 and m_2.

Now we try to evaluate the discontinuity across the cut starting from this singularity using the Cutkosky rules. The amplitude corresponding to Fig. 6 is given by

$$\mathcal{A} = g^4 \int \frac{d^2k}{(2\pi)^4 i} \prod_{j=1}^{4} \frac{1}{m_j^2 - q_j^2} \tag{70}$$

where one can choose $q_1 = k$, $q_2 = k - k_1$, $q_3 = k - k_1 - k_2$, $q_4 = k - k_1'$. According to the Cutkosky rules the discontinuity across the cut related to the branch point (69 will be obtained if we substitute the propagators left in the corresponding reduced diagram as indicated in (63). This gives

$$\mathrm{Disc}\,\mathcal{A} = g^4 \int \frac{d^2k}{(2\pi)^4 i}(2\pi i)\delta(m_1^2 - q_1^2)(2\pi i)\delta(m_3^2 - q_3^2)\frac{1}{m_2^2 - q_2^2}\frac{1}{m_4^2 - q_4^2} \tag{71}$$

We transform it into

$$\mathrm{Disc}\,\mathcal{A} = i \int \frac{d^3q_1}{(2\pi)^3 2q_{10}}\frac{d^3q_3}{(2\pi)^3 2q_{30}}(2\pi)^4\delta(q_1 + q_3 - k_1 - k_2)\Big(g^2\frac{1}{m_2^2 - q_2^2}\Big)\Big(g^2\frac{1}{m_4^2 - q_4^2}\Big) \tag{72}$$

Obviously this is just the lowest order contribution from the two-particle state to the unitarity relation

$$\mathrm{Disc}\,\mathcal{A} = i \int d\tau_2(k_1 - k_2)\mathcal{A}^*(k_1' + k_3' \to q_1 + q_3)\mathcal{A}(k_1 + k_2 \to q_1 + q_3) \tag{73}$$

with

$$\mathcal{A}(k_1 + k_2 \to q_1 + q_3) = g^2\frac{1}{m_2^2 - q_2^2}, \quad \mathcal{A}(k_1' + k_2' \to q_1 + q_3) = g^2\frac{1}{m_4^2 - q_4^2} \tag{74}$$

Thus the singularity and discontinuity of the reduced diagram of Fig. 9 fully correspond to those which follow from the unitarity relation.

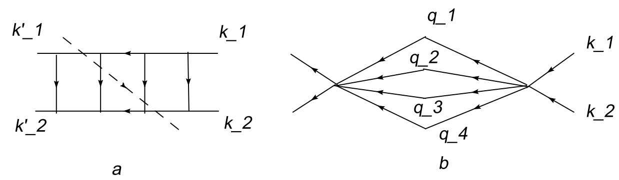

Figure 9: A ladder diagram and its 'fish' reduced diagram

This result can be generalized to more complicated amplitudes with the reduced amplitude of the Fig. 8 type ('fish amplitude'). Consider the ladder diagram and its reduced amplitude shown in Fig. 9.a and b respectively. The internal momenta which remain in the reduced amplitude are marked by a dashed line ('a cut') in the amplitude of Fig. 9.a. Their momenta for convenience are denoted $q_1, ...q_4$. Now the reduced diagram contains 3 loops and correspondingly we have 3 equations from the Landau rules:

$$\alpha_1 q_1 - \alpha_i q_i = 0, \quad q_1^2 = m_1^2, \quad q_i^2 = m_i^2, \quad i = 2,3,4 \tag{75}$$

Similarly to (68) we find $q_i = q_1\alpha_1/\alpha_i$, $i = 2,3,4$

$$\frac{\alpha_i}{\alpha_1} = \frac{m_1}{m_i}, \quad i = 2,3,4 \tag{76}$$

and, using $k_1 + k_2 = q_1 + \sum_2^4 q_i = q_1(1 + \sum_2^4 m_i/m_1)$ we find the position of singularity

$$s = (k_1 + k_2)^2 = m_1^2\Big(1 + \sum_2^4 \frac{m_i}{m_1})\Big)^2 = (m_1 + m_2 + m_3 + m_4)^2 \tag{77}$$

This is a threshold for the transition of two initial particles into 4 intermediate ones $k_1 + k_2 \to \sum_1^4 q_i$ corresponding to the contribution to the unitarity relation from 4 intermediate particles. Now according to the Cutkosky rules we find the discontinuity related to this threshold

$$\mathrm{Disc}\,\mathcal{A} = \int \prod_{j=1}^4 \frac{d^4 q_j}{(2\pi)^4 i}(2\pi i)\delta(m_j^2 - q_j^2)(2\pi)^4 i\delta(\sum_{j=1}^4 q_j - k_1 - k_2)$$

$$\mathcal{A}^*(k_1' + k_2' \to \sum_{j=1}^4 q_j)\mathcal{A}(k_1 + k_2 \to \sum_{j=1}^4 q_j) \tag{78}$$

where we denoted parts of the amplitude suppressed in the reduced amplitude $\mathcal{A}(k_1 + k_2 \to \sum_{j=1}^4 q_j)$ to the right of the cutting line and $\mathcal{A}^*(k_1' + k_2' \to \sum_{j=1}^4 q_j)$ to the left of the cutting line. Note that the left amplitude should be taken conjugate, which is related to the fact that it itself may have branch points in s (if the internal masses are adequate, which is usually the case) and a careful study of interplay of the following branch cuts revels that it has to be taken on the opposite side of the cut as compared to the right amplitude. In fact this is clear from the start because the discontinuity should be pure imaginary. Of course discontinuity (78) is nothing but the contribution to the imaginary part of the amplitude from the 4-particle intermediate state:

$$\mathrm{Disc}\,\mathcal{A} = i\int d\tau_4(k_1 + k_2)\mathcal{A}^*(k_1' + k_2' \to \sum_{j=1}^4 q_j)\mathcal{A}(k_1 + k_2 \to \sum_{j=1}^4 q_j) \tag{79}$$

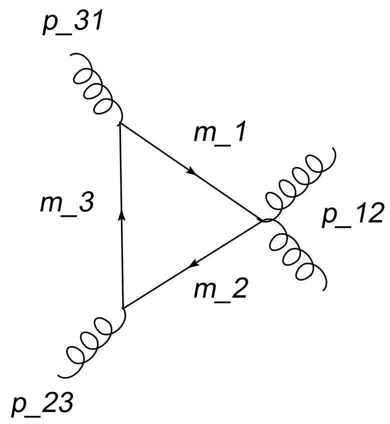

Figure 10: The triangle diagram

Thus the reduced amplitude Fig. 9b contains the singularity of the original amplitude dictated by the unitarity relation.

This a general result: for any amplitude the singularities following from the unitarity, their positions and discontinuities are described by reduced amplitudes of the 'fish' type.

In contrast, more complicated reduced diagrams correspond to singularities, which are not related to the unitarity relation. Fortunately, as we shall presently see, these non-unitary singularities as a rule do not appear in the physical amplitudes but only in their analytic continuation to the second and higher sheets.

To be concrete we study the simplest case of a triangle singularity of the original amplitude in Fig. 4 corresponding to the reduced amplitude shown in Fig. 10. We denoted $p_{12} = p_{14} + p42 = p_{13} + p_{32}$. Our energetic variable $s = (p_{14} + p42)^2 = p_{12}^2$. We take q_1 for the loop integration momentum. Other internal lines momenta are $q_3 = q_1 + p_{13}$, $q_2 = q_3 + p_{32}$.

The Landau equations are

$$\sum_{i=1}^{3} \alpha_i q_i, \quad \alpha_i > 0, \quad q_i^2 = m_i^2 \tag{80}$$

Multiplying this equation by q_j, $j = 1, 2, 3$ we get three scalar equations

$$\sum_{i=1}^{3} \alpha_i q_i q_j, \quad j = 1, 2, 3 \tag{81}$$

We define

$$q_i q_j = m_i m_j y_{ij}, \quad y_{ii} = 1, \quad i, j = 1, 2, 3 \tag{82}$$

Eq. (81) in terms of y_{ij} is rewritten as

$$\sum_{i=1}^{3} \beta_i y_{ij}, \quad j = 1, 2, 3, \beta_i = m_i \alpha_i > 0 \tag{83}$$

Note that

$$(q_i - q_j)^2 = p_{ij}^2 = m_i^2 + m_j^2 - 2q_i q_j = m_i^2 + m_j^2 - 2m_i m_j y_{ij}$$

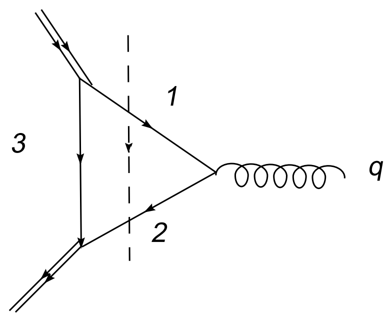

Figure 11: The deuteron form-factor

so that for $i \neq j$

$$y_{ij} = \frac{m_i^2 + m_j^2 - p_{ij}^2}{2m_i m_j}, \quad i \neq j, \quad i,j = 1,2,3 \tag{84}$$

The Landau condition evidently reduces to

$$\det ||y_{ik}|| = 0 \tag{85}$$

which explicitly gives an equation

$$1 + 2y_{12}y_{23}y_{31} - y_{12}^2 - y_{23}^2 - y_{31}^2 = 0 \tag{86}$$

However we have to take into account that β_i $i = 1,2,3$ solving the system (83) should all be positive. This immediately implies that from the three non-trivial y_{ij} $i \neq j$ at least two have to be negative. Otherwise the equation which does not contain the only negative y_{ij} cannot be satisfied by positive β.

Precisely this condition prohibits existence of the triangular singularity for most amplitudes. In fact, in our case this means that at least one of the y_{13} or y_{23} related to the vertexes attached to external particles have to be negative. Other negative y may be supplied by y_{12}, which is not fixed by particle masses. However condition that $y_{13} < 0$ or $y_{23} < 0$ cannot be satisfied for common amplitudes for the scattering of elementary particles with realistic masses. For, say, $y_{13} < 0$ one should have $p_{13}^2 > m_2^2 + m_3^2$, which never happens with normal elementary particles. Say for the vertex N$\to$ N+π we have $p_{13}^2 = m^2$ and $m_2^2 + m_3^2 = m^2 = \mu^2 > p_{13}^2$. The same situation is realized in all amplitudes with simple hadrons. In all these cases the triangular singularity exists only on the second or higher sheets of the complex plane and not on the physical sheet.

Still there exist cases when the triangular singularity appears in the physical amplitude. This happens when one or several external particles are weakly bound systems, for which one has the mass M with

$$M^2 > m_2^2 + m_3^2, \quad \text{although} \quad M < m_2 + m_3 \tag{87}$$

(the second condition is the requirement of stability of particle M). The typical example is the deuteron D with mass $M_D = 2m - \epsilon$, $\epsilon << m$ and so for the process D$\to$ N+N we have $M_D^2 \simeq 4m^2 - 4m\epsilon > 2m^2$ so that the corresponding y is well negative.

To illustrate non-unitary singularities consider the deuteron form-factor, the simplest diagram for which is shown in Fig. 11. The role of p_{12}^2 is in practice played by the

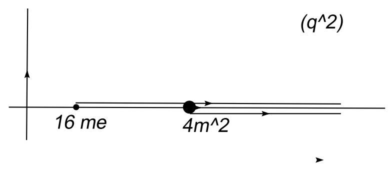

Figure 12: Normal and anomalous cuts in the q^2-plane for the deuteron form-factor

momentum squared transferred from the electron in the reaction

$$e(k_2) + D(k_1) \to e(k_2') + D(k_1') \tag{88}$$

Compared to Fig. 10, $p_{12}^2 = (k_2 - k_2')^2$, $p_{13} = k_1$, $p_{23} = -k_1'$. We find

$$y_{13} = y_{23} = \frac{2m^2 - M^2}{2m^2} \simeq -1 + 2\frac{\epsilon}{m} < 0 \tag{89}$$

The Landau equation (86) determines the singular point in y_{12}

$$1 + 2y_{12}\Big(1 - 4\frac{\epsilon}{m}\Big) - y_{12}^2 - 2\Big(1 - 4\frac{\epsilon}{m}\Big) = 0 \tag{90}$$

We transform it like follows

$$(1 - y_{12})(1 + y_{12}) - 2(1 - y_{12})\Big(1 - 4\frac{\epsilon}{m}\Big) = (1 - y_{12})\Big[1 + y_{12} - 2\Big(1 - 4\frac{\epsilon}{m}\Big)\Big] = 0 \tag{91}$$

We find two solutions

$$y_{12} = 1, \quad y_{12} = \Big(1 - 4\frac{\epsilon}{m}\Big) - 1 = 1 - 8\frac{\epsilon}{m} \tag{92}$$

The first solution leads to $p_{12}^2 = 0$. This solution appeared only due to the symmetry of the diagram. It lies on the second sheet of the complex p_{12}^2 plane, since its position does not depend on the value of M^2, which can be taken small when the singularity cannot exist in the physical amplitude. So the triangular singularity in the physical amplitude is determined by the second solution, which gives

$$p_{12}^2 = 2m^2 - 2m^2\Big(1 - 8\frac{\epsilon}{m}\Big) = 16m\epsilon \tag{93}$$

In the complex p_{12}^2 plane this singularity lies much lower than the unitary singularity at $p_{12}^2 = 4m^2$ (see Fig. 12).

It can be found that the triangular singularity coincides with the singularity found in the quantum mechanical treatment of the deuteron as a composite particle made of two nucleons. There it is provided by the structure of the deuteron wave function. So non-unitary singularities found in Feynman diagrams for particles with small binding energies are equivalent to the structure of the corresponding amplitudes in terms of the quantum mechanical wave functions.

1.4 Dispersion relations

The study of Feynman diagrams tells us that scattering amplitudes are analytic functions of any invariant variable with singularities determined by the Landau equations and

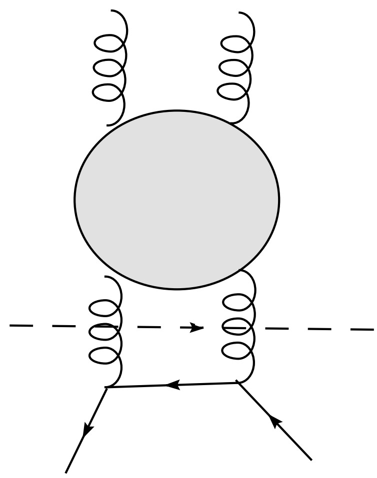

Figure 13: Contribution from the intermediate two-pion state to the unitarity in the t-channel

discontinuities determined by the Cutkosky rules. In absence of anomalous, non-unitary singularities, the discontinuities are given by the generalized unitarity relations, which include non-phyiscal regions).

Returning to our example of πN scattering amplitude we find that at fixed t and thus varying s and u the amplitude $\mathcal{A}(s,t)$ is an analytic function with poles at $s = m^2$ and $u = m^2$ and unitary cuts at $s \geq (m+\mu)^2$ and $u \geq (m+\mu)^2$. In the complex s-plane the phyisical amplitude is given by the values of $\mathcal{A}(s,t)$ on the real axis above the right-hand cut in the s-channel and below the left-hand cut in the u-channel. The discontinuities across the cuts $2i\mathcal{A}_s$ and $-2i\mathcal{A}_u$ are given by the unitarity relations in the s- and u-channels, respectively.

As an example, in the elastic region $(m+\mu)^2 < s < (m+2\mu)^2$ one finds

$$\mathcal{A}_s(s,t) = \frac{k}{32\pi^2 W} \int d\Omega_1 \mathcal{A}^*(s,t_1)\mathcal{A}(s,t_2) \tag{94}$$

where

$$t = -2k^2(1-\cos\theta), \quad t_1 = -2k^2(1-\cos\theta_1), \quad t_2 = -2k^2(1-\cos\theta_2),$$
$$d\Omega_1 = d\cos\theta_1 d\phi_1, \quad \cos\theta_2 = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos\phi_1$$

and k_2 id the c.m. momentum squared. We stress that this is a generalized unitarity equation, which is assumed to be valid at any fixed t including values outside the physical region $-4k^2 < t < 0$. In the unphysical region one finds $|\cos\theta| > 1$ so that $\cos\theta_2$ turns out to be complex. So in the integrand there appears $\mathcal{A}(s,t)$ at complex values of t. This means that to find $\mathcal{A}_s$ in the unphysical region one has to be able to analytically continue $\mathcal{A}s,t$ in the variable t at fixed s.

From the perturbation theory we conclude that this is indeed possible. At fixed s variables t and u are varying. The amplitude is found to be analytic function of these variables with singularities along the real axis at old points $u = m^2$, $u \geq (m+\mu^2)$ and new point $t > 4\mu^2$ dictated by the unitarity in the t-channel (see Fig. 13). Note that this last singularity lies in the deep unphysical region, since the physical region starts from $t > 4m^2 >> 4\mu^2$.

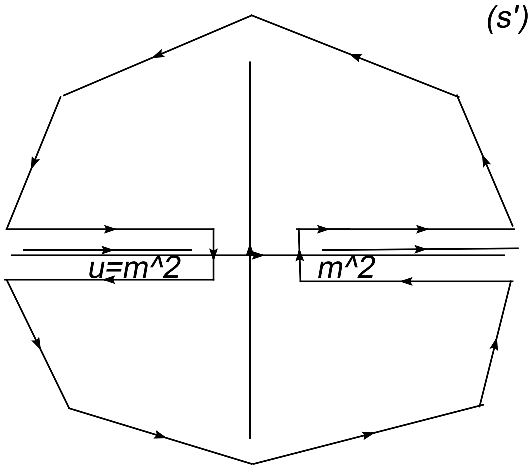

Figure 14: Integration contour in the s'-plane

Likewise the amplitude can be considered as an analytic function of varying variables s and t at fixed u. In this case we have meet with already mentioned singularities at $s = m^2$, $s \geq (m+\mu)^2$ and $t \geq 4\mu^2$.

It is well-known that the analytic function is completely determined by its singularities. Dispersion relation is a technical tool to restore the amplitude from given discontinuities across its cuts.

Let us again consider our $\mathcal{A}(s,t)$ for πN scattering. Let us fix t and study $\mathcal{A}(s,t)$ as a function of complex s and u. Let us first assume that $\mathcal{A}(s,t)$ vanishes at $|s| \to \infty$:

$$\mathcal{A}(s,t) \to 0, \quad \text{at} \quad |s| \to \infty \tag{95}$$

Consider the relation

$$\int_C \frac{ds' \mathcal{A}(s',t)}{s'-s} = 2\pi i \mathcal{A}s,t \tag{96}$$

where s is complex and the integral is taken along the closed contour C, shown in Fig. 14, which embraces the cuts of the amplitude $\mathcal{A}(s,t)$ and closes on the large circle. The relation (96) is valid since $\mathcal{A}(s,t)$ is an analytic function inside the integration contour, so that the only singularity of the integrand is a pole at $s' = s$. Due to (95) the integral along the large circle goes to zero as it radius grows and in the limit we find that only the contribution from the discontinuities along the cuts remain in (96). So we find the so called dispersion relation

$$\begin{aligned}\mathcal{A}(s,t) &= \frac{1}{\pi}\int_{m^2}^{\infty} \frac{ds' \mathcal{A}_s(s',t)}{s'-s} - \frac{1}{\pi}\int_{-\infty}^{s(u=m^2)} \frac{ds' \mathcal{A}_u(u(s'),t)}{s'-s} \\ &= \frac{1}{\pi}\int_{m^2}^{\infty} \frac{ds' \mathcal{A}_s(s',t)}{s'-s} + \frac{1}{\pi}\int_{m^2}^{\infty} \frac{du' \mathcal{A}_u(s',t)}{u'-u}\end{aligned} \tag{97}$$

Here $u(s) = 2(m^2+\mu^2) - s - t$ and $u' = u(s') = 2(m^2+\mu^2) - s' - t$ so that $s' - s = u - u'$. In fact at $s < (m+\mu)^2$ and $u < (m+\mu)^2$ the only contribution comes from pole terms:

$$\mathcal{A}(s,t) = \frac{g^2}{m^2-s} + \frac{g^2}{m^2-u} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{ds' \mathcal{A}_s(s',t)}{s'-s} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{du' \mathcal{A}_u(s',t)}{u'-u} \tag{98}$$

Note that the crossing-symmetry leads to the equality of the two absorptive parts: $\mathcal{A}_u(u,t) = \mathcal{A}_s(u,t)$

If the amplitude does not vanish at $|s| to \infty$ then one constructs the so called dispersion relation with subtractions. Let us assume, instead of (95) that there exists a integer N such than

$$\frac{\mathcal{A}(s,t)}{(s-s_0)^N} \to 0 \quad \text{at} \quad |s| \to \infty \tag{99}$$

where s_0 is an arbitrary finite point. Then the procedure employed above is repeated for the function $\mathcal{A}(s,t)/(s-s_0^N$. As compared to (96) we obtain additional terms coming from the N-fold pole at s_0:

$$\int_C \frac{ds'\mathcal{A}(s',t)}{(s'-s_0)^N(s'-s)} = 2\pi i \Big\{ \frac{\mathcal{A}(s,t)}{(s-s_0)^N} + \frac{1}{(N-1)!} \Big[\Big(\frac{d}{ds'}\Big)^{N-1} \frac{\mathcal{A}(s',t)}{s'-s} \Big]_{s'=s_0} \Big\} \tag{100}$$

With the circle included on C extending to infinity we get the dispersion relation with subtractions

$$\mathcal{A}s,t = \frac{(s-s_0)^N}{\pi} \Big(\int_{m^2}^{\infty} \frac{ds'\mathcal{A}_s(s',t)}{(s'-s-0)^N(s'-s)} - \int_{-\infty}^{s(u=m^2)} \frac{ds'\mathcal{A}_u(u(s'),t)}{(s'-s_0)^N(s'-s)} \Big) + P_{N-1}(s) \tag{101}$$

where $P_{N-1}(s)$ is a polynomial in s of order $N-1$. It represents N first terms in the expansion of $\mathcal{A}(s,t)$ in the Tailor series around the point $s = s_0$. If one defines u_0 by the equality $s - s_0 = u_0 - u$ then (101) can be rewritten in a more symmetric form

$$\mathcal{A}s,t = \frac{(s-s_0)^N}{\pi} \int_{m^2}^{\infty} \frac{ds'\mathcal{A}_s(s',t)}{(s'-s-0)^N(s'-s)}$$

$$+ \frac{(u-u_0)^N}{\pi} \int_{m^2}^{\infty} \frac{du'\mathcal{A}_u(u',t)}{(u'-u_0)^N(u'-u)} \Big) + P_{N-1}(s) \tag{102}$$

1.5 The Froissart theorem

1.5.1 Partial wave expansion

In this section, for simplicity, we shall consider an especially simple case when all masses of the scattering particles are equal. An adequate physical example is $\pi-\pi$ elastic scattering:

$$\pi(k_1) + \pi(k_2) \to \pi(k_1') + \pi(k_2') \tag{103}$$

In this case the relation between the c.m. energy and momentum is simplified to

$$k^2 = \frac{1}{4}(s - 4\mu^2) \tag{104}$$

The transferred momenta squared are

$$t = -2k^2(1-\cos\theta), \quad u = -2k^2(1+\cos\theta) \tag{105}$$

where θ is the scattering angle in the c.m. system. They vary in the interval

$$-4k^2 < t, u < 0 \tag{106}$$

with $t = 0$ for the forward scattering and $u = 0$ for the backward scattering. For convenience we denote the cosine of the scattering angle as

$$z = \cos\theta = 1 + \frac{t}{2k^2} = -1 - \frac{u}{2k^2} \tag{107}$$

Let us expand the scattering amplitude at fixed s in the Legendre series in z:

$$\mathcal{A}\Big(s, t(z)\Big) = \sum_{l=0}(2l+1)a_l(s)P_l(z) \tag{108}$$

Coefficients $a_l(s)$ are scattering partial waves. They can be found from the amplitude according to

$$a_l(s) = \frac{1}{2}\int_{-1}^{1}\mathcal{A}\Big(s, t(z)\Big)P_l(z) \tag{109}$$

Partial wave expansion radically simplifies the elastic unitarity relation:

$$\mathrm{Im}\,\mathcal{A}(s,t) = \frac{k}{32\pi^2\sqrt{s}}\int d\Omega_1 \mathcal{A}^*(s,t_1)\mathcal{A}(s,t_2) \tag{110}$$

valid in the interval $4\mu^2 < s < 16\mu^2$. Here

$$d\Omega_1 = dz_1 d\phi_1, \quad t_1 = -2k^2(1-z_1),$$

$$t_2 = -2k^2(1-z_2), \quad z_2 = zz_1 + \cos\phi_1\sqrt{(1-z^2)(1-z_1^2)}$$

If we put here the expansion (108) and use

$$\int d\Omega_1 P_{l-1}(z_1)P_{l_2}(z_2) = \frac{4\pi}{2l+1}\delta_{l_1 l_2}P_{l_1}(z) \tag{111}$$

we find that the unitarity relation (110) is diagonalized in the orbital momentum l

$$\mathrm{Im}\,a_l(s) = \frac{k}{8\pi\sqrt{s}}\Big|a_l(s)\Big|^2 \equiv \rho(s)\Big|a_l(s)\Big|^2 \tag{112}$$

As is well known this relation is solved by introduction of scattering phases. We find Im $a_l^{-1}(s) = -\rho(s)$ so that one can take

$$a_l^{-1}(s) = \rho(s)(\cot\delta_l(s) - i), \quad \text{or} \quad a_l(s) = \frac{1}{\rho(s)}e^{i\delta_l(s)}\sin\delta_l(s) \tag{113}$$

The phases $\delta_l(s)$ remain real while the scattering is elastic and acquire a negative imaginary part above the threshold of inelastic channels.

Above the inelastic threshold to the right-hand side of (110) inelastic contributions are added. It is important that at fixed orbital momentum they are positive, since both the initial and final two-particle states are essentially non-degenerate: at fixed l they only differ in their z projection l_z, which is conserved. So both the initial and final states are given by Ψ_{l,l_z} and the contribution from inelastic states has the structure

$$\sum_n |<\Psi_n|\Psi_{l,l_z}>|^2 > 0$$

As a result we have an inequality

$$\operatorname{Im} a_l(s) \geq \rho(s) \Big| a_l(s) \Big|^2 \tag{114}$$

which is valid at all values of energy $\sqrt{s}$.

Now we are going to use analytic properties of the amplitude as a function of t. We can write a dispersion relation (without subtractions) at fixed s:

$$\mathcal{A}(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{dt' \mathcal{A}_t(s,t')}{t'-t} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{du' \mathcal{A}_u(s,u')}{u'-u} \tag{115}$$

where of course $u = 4\mu^2 - s - t$ and $u' = 4\mu^2 - s - t'$. Both t and u are just linear function of z. Therefore one can rewrite this dispersion relation in terms of z:

$$\mathcal{A}(s,t) = \frac{1}{\pi}\int_{z_0(s)}^{\infty} \frac{dz' \mathcal{A}_t(s,t(z'))}{z'-z} - \frac{1}{\pi}\int_{-\infty}^{-z_0(s)} \frac{dz' \mathcal{A}_u(s,u(z'))}{z'-z} \tag{116}$$

where

$$z_0(s) = 1 = \frac{2\mu^2}{k^2} \tag{117}$$

Putting this representation into (109) we get

$$a_l(s) = \frac{1}{\pi}\int_{z_0(s)}^{\infty} dz \mathcal{A}_t(s,t(z)) Q_l(z) - \frac{1}{\pi}\int_{-\infty}^{-z_0(s)} dz \mathcal{A}_u(s,u(z)) Q_l(z) \tag{118}$$

where $Q_l(z)$ is the Legendre function of the second kind, defined by

$$Q_l(z) = \frac{1}{2}\int_{-1}^{1} \frac{dz' P_l(z')}{z-z'} \tag{119}$$

At fixed integer l function $Q_l(z)$ is an analytic function with a cut at $-1 < z < 1$. It has the property

$$Q_l(-z) = (-1)^{l+1} Q_l(z) \tag{120}$$

and the following asymptotic properties. At large $|z|$ at fixed l with Re $l > 0$

$$P_l(z) \sim z^l, \quad Q_l(z) \sim z^{-l-1}. \tag{121}$$

At large $|l|$ with Re $l > 0$ at $|z| < 1$, $z = \cos\phi$

$$P_l(z) \sim \frac{1}{\sqrt{l}} \cos\Big((l+1/2)\phi - \pi/4\Big), \quad Q_l(z+i0)\frac{1}{\sqrt{l}} \exp\Big((l+1/2)\phi - \pi/4\Big) \tag{122}$$

at $z > 1$, $z = \cosh\alpha$

$$P_l(z) \sim \frac{1}{\sqrt{l}} e^{(l+1)\alpha}, \quad Q_l(z) \sim \frac{1}{\sqrt{l}} e^{-(l+1)\alpha} \tag{123}$$

To avoid negative values of z we can use (120) and rewrite (118) as

$$a_l(s) = \frac{1}{\pi}\int_{z_0(s)}^{\infty} dz Q_l(z) \Big(\mathcal{A}_t(s,t(z)) Q_l(z) + (-1)^l \mathcal{A}_u(s,u(-z)) \Big) \tag{124}$$

It is convenient to introduce the amplitudes with a fixed signature $\pm$:

$$a_l^{(\pm)}(s) = \frac{1}{\pi}\int_{z_0(s)}^{\infty} dz Q_l(z)\Big(\mathcal{A}_t(s,t(z))Q_l(z) \pm \mathcal{A}_u(s,u(-z))\Big) \tag{125}$$

The positive signature amplitude $a_l^{(+)}(s)$ coincides with the physical amplitude $a_l(s)$ at even values of l, the negative signature amplitude $a_l^{(-)}(s)$ at odd values of l.

From (124) we can extract, first of all, the threshold behaviour of the partial wave amplitude. We return to variable t in (125)

$$a_l^{(\pm)}(s) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{dt}{2k^2} Q_l\Big(1+\frac{t}{2k^2}\Big)\Big(\mathcal{A}_t(s,t)Q_l(z) \pm \mathcal{A}_u(s,t)\Big) \tag{126}$$

where we use $u(-z) = t(z)$. As $k^2 \to 0$ we find from (126)

$$a_l^{(\pm)}(s) \sim k^{2l} \tag{127}$$

For our purpose the most important conclusion is about the behaviour of the partial wave amplitudes at large $l > 0$. From (126) we find

$$a_l^{(\pm)}(s) \sim \frac{1}{\sqrt{l}} e^{-(l+1)\alpha_0} f(s) \tag{128}$$

where

$$\cosh \alpha_0 = 1 + \frac{2\mu^2}{k^2} \tag{129}$$

and $f(s) \sim \mathcal{A}_t(s, 4\mu^2), \mathcal{A}_u(s, 4\mu^2)$

1.5.2 High-energy behaviour

At $s >> \mu^2$ we find $k^2 = s/4$ and obviously $\alpha_0 << 1$. So

$$\cosh \alpha_0 = 1 + \frac{1}{2}\alpha_0^2 = 1 + \frac{8\mu^2}{s}$$

from which we find

$$\alpha_0 = \frac{4\mu}{\sqrt{s}} \tag{130}$$

so that at both l and s large we have the asymptotic

$$a_l^{(\pm)}(s) \sim \frac{1}{\sqrt{l}} e^{-(l+1)4\mu/\sqrt{s}} f(s) \tag{131}$$

This asymptotic plus the unitarity relation form the basis for derivation of the restriction on the high-energy behavour of the scattering amplitude, known as the Froissart theorem. One additional fundamental ingredient is the assumption that function $f(s)$ in (128) and (131) may grow with s not faster than a certain power of s. This assumption is equivalent to the assumption that the amplitude does not have an essential singularity at infinity, which would spoil the possibility to pass to the Eucledean formulation.

To start the derivation we first use the unitarity property (114). For a reduced amplitude $\tilde{a}_l(s) = \rho(s)a_l(s)$ we find an inequality

$$\mathrm{Im}\,\tilde{a}_l(s) \geq |\tilde{a}_l(s)|^2 \tag{132}$$

Introducing real and imaginary parts $\tilde{a}_l(s) = x + iy$ we rewrite it as

$$y \geq x^2 + y^2 \tag{133}$$

Solution of the equation $y = x^2 + y^2$ gives $y = 1/2 \pm \sqrt{1/4 - x^2}$ from which we conclude

$$|x| < \frac{1}{2}, \quad 0 \leq y \leq 1, \quad x^2 + y^2 \leq 1$$

which transform into the inequality

$$|a_l(s)| \leq \frac{1}{\rho(s)} \to 16\pi \quad \text{at} \quad s >> \mu^2 \tag{134}$$

Now we pass to our partial wave expansion (108) at large s. Using the properties of the Legendre polynomials we have

$$\Big|\mathcal{A}(s,t)\Big| \leq \sum_{l=0}(2l+1)|a_l(s)| \tag{135}$$

Here we split the summation in two parts $0 \leq l \leq \Lambda(s) - 1$ and $\Lambda(s) \leq l < \infty$, where we assume that $\Lambda(s) >> 1$. In the first sum we use the unitarity limitation (134): to find

$$S_1 \equiv \sum_{l=0}^{\Lambda-1}(2l+1)|a_l(s)| \leq 16\pi \sum_{l=0}^{\Lambda-1}(2l+1) \simeq 16\pi C_1 \Lambda^2(s) \tag{136}$$

where C_1 is an inessential constant and we used $\Lambda >> 1$.

In the second sum we use the asymptotical behaviour (131):

$$S_2 \equiv \sum_{l=\Lambda}^{\infty}(2l+1)|a_l(s)| \sim \sum_{l=\Lambda}^{\infty}(2l+1)\frac{1}{\sqrt{l}}e^{-(l+1)4\mu/\sqrt{s}+\ln f(s)} \tag{137}$$

Here we put $l = l' + \lambda$ and we choose

$$\Lambda(s) = \frac{\sqrt{s}}{4\mu}\ln f(s) \tag{138}$$

to cancel the second term in the exponent in (137). Note that at large s

$$\Lambda(s) \sim c\sqrt{s}\ln s \tag{139}$$

where c is some constant. We get

$$S_2 = 2\sum_{l'=0}^{\infty}\sqrt{l' + \Lambda}e^{-l'4\mu/\sqrt{s}} \leq 2\sum_{l'=0}^{\infty}(\sqrt{l'} + \sqrt{\Lambda})e^{-l'4\mu/\sqrt{s}} \tag{140}$$

where we once more used $\Lambda >> 1$. The two sums which appear may be estimated by substituting the sums by integrations:

$$S_2 \leq 2\int_0^{\infty} dl'(\sqrt{l'}+\sqrt{\Lambda})e^{-l'4\mu/\sqrt{s}} \sim 2\Big(\sqrt{\frac{\sqrt{s}}{4\mu}}+\sqrt{\Lambda}\Big)\frac{\sqrt{s}}{4\mu} < 16\pi C_1\Lambda^2(s) \tag{141}$$

So the second sum is less than the first and we obtain the restriction of Froissart

$$\Big|\mathcal{A}(s,t)\Big| \leq Cs\ln^2 s \tag{142}$$

Recalling the optical theorem and the asymptotic $J_{12}(s) \to 2s$ at $s >> \mu^2$ we find limitation on the behaviour of the total cross-section at high energies

$$\sigma^{tot}(s) \leq C\ln^2 s \tag{143}$$

One can prove that at t strictly smaller than zero a more stringent limitation is valid

$$\Big|\mathcal{A}(s,t)\Big| \leq C's\ln s, \quad t \neq 0, \quad t < 0 \tag{144}$$

1.6 Complex angular momenta

1.6.1 Definition and properties

For reasons which will be clearer later, we shall study the partial wave expansion in the t-channel:

$$\mathcal{A}(s,t) = \sum_{l=0}^{\infty}(2l+1)a_l(t)P_l(z_t) \tag{145}$$

where z_t is the cosine of the scattering angle in the t-channel:

$$s = -2k_t^2(1-z_t), \quad u = -2k_t^2(1+z_t), \quad k_t^2 = \frac{1}{4}(t-4\mu^2) \tag{146}$$

At fixed t the s- dependence is concentrated in Legendre polynomials and is explicit and simple. The basic idea is to generalize this formula to be valid at large values of s to describe the scattering at high energy $\sqrt{s}$ and fixed t. In this form however it does not suit to this aim: the Legendre expansion (145) converges in the ellipse in the z_t plane with focuses at points ± 1 and limited by the nearest singularity in z_t which corresponds to $s = 4\mu^2$ or $u = 4\mu^2$. It certainly does not converge at large values of $|s|$ and thus large $|z_t|$.

The technical instrument for the extension of representaton (145) to large values of s is transition to complex angular momenta. Partial waves with complex angular momenta can be introduced starting from one of the two formulas for their values at integer l, which we had in the previous section, (109) and (148)

$$a_l(t) = \frac{1}{2}\int_{-1}^{1} dz_t\mathcal{A}\Big(s(z_t),t\Big)P_l(z_t) \tag{147}$$

$$a_l^{(\pm)}(t) = \frac{1}{\pi}\int_{1+2\mu^2/k_t^2}^{\infty} dz_tQ_l(z_t)\Big(\mathcal{A}_s(s(z),t) \pm \mathcal{A}_u(u(-z_t),t)\Big) \tag{148}$$

Both formulas admit analytic continuation from integer values of l to any complex values of l, since $P_l(z_t)$ and $Q_l(z_t)$ are both analytic functions of l at fixed $z_t \neq \pm 1$. Function $Q_l(z_t)$ has simple poles at negative integer $l = -1, -2, ...$, So in principle both representations may serve to define partial waves with complex angular momenta. At first sight the simpler formula (147) is preferable, since the integration in it goes over a finite interval and no problem with convergence arise. However in reality it is the second formula (148) which leads to constructive results. In fact one must have in mind that continuation from integer values to complex ones is not unique. Obviously one can add to any chosen continuation an arbitrary analytic function proportional to $\sin(\pi l)$ and so vanishing at integer l. So the problem is to choose a particular continuation suitable for our purpose. As we shall see, this continuation is realized by Eq. (148).

Eq. (148) defines an analytic function of l provided the integral converges. If at large values of z_t the function in the integrand (the signatured absorptive partes)

$$\mathcal{A}_s^{\pm}(s,t) = \mathcal{A}_s(s,t) \pm \mathcal{A}_u(u,t)\Big) \sim s^N \tag{149}$$

then the total dependence of the integrand on z^t at large z_t has the form $\sim z_t^{-l-1+N}$ and obviously the integral converges while

$$\mathrm{Re}\, l > N \tag{150}$$

So Eq. (148) defines an analytic function of l in the right half-plane restricted by (150). At very large l in this half-plane the continued partial amplitudes behave as derived in the previous section

$$a^{(\pm)}(t) \sim \frac{1}{\sqrt{l}} e^{-(l+1)\alpha_0} f(t), \quad \cosh\alpha_0 = 1 + \frac{2\mu^2}{k_t^2} \tag{151}$$

and thus decrease in the right half-plane at large Re l.

It is important to have in mind that generally $N = N(t)$ and so the analyticity region or equivalently position of singularities in the l plane depend on t. We already have some information about the possible behaviour of the amplitudes (and so their absorptive parts) as function of s at physical values of $t \leq 0$. According to the Froissart theorem, at these values of t $\mathcal{A}_s^{(\pm)} < Cs\ln^2 s$. This means that in any case at $t \leq 0$ the partial waves $a_l^{(\pm)}$ are defined by (148) and analytic to the right of the line Re l=1.

For the signatured amplitudes the Pomeranchuk theorem implies that at high energies the positive signature absorptive part, defined by (149) grows faster that the negative signature one:

$$\frac{\mathcal{A}_s^{(-)}(s,t)}{\mathcal{A}_s^{(+)}(s,t)} \to 0, \quad s \to \infty \tag{152}$$

and as a consequence, at $t \leq 0$ the negative signature partial waves $a_l^{(-)}(t)$ are generally analytic in a wider region of l than $a_l^{(+)}(t)$.

The advantage of the continuation by means of (148) follows from the fact that it is unique in the sense, that it is the only continuation which decreases in the right-half plane of l. This follows from the Karlsson theorem:

Karlsson theorem

If function $f(l)$ is analytic at Re $l \geq N$, has asymptotic $\mathcal{O}(e^{\kappa}|l|)$ with $\kappa < \pi$ and $f(l) = 0$ at $l = N, N+1, N+2, ...$ then it is equal to zero: $f(l) = 0$

Because of this property it is not reasonable to analytically continue the partial waves using (147).

Uniqueness of the continuation with the mentioned properties allows to demonstrate many useful properties of the partial waves with complex l defined by (148).

In particular we can prove the unitarity relation in the t-channel for complex values of l. To do this we first define a new function, also analytic in l, using the complex conjugate absorptive parts in (148):

$$\bar{a}_l^{(\pm)}(t) = \frac{1}{\pi}\int_{1+2\mu^2/k_t^2}^{\infty} dz_t Q_l(z_t)\mathcal{A}_s^{\pm *}(s(z),t) \tag{153}$$

It is analytic in l in the same region as $a_l^{(\pm)}(t)$ and coincides with it when $\mathcal{A}_s^{(\pm)}(s(z),t)$ is real. It is easy to see that $\mathcal{A}_s^{(\pm)}(s(z),t)$ is real in the interval

$$-4\mu^2 < t < 4\mu^2$$

Indeed $\mathcal{A}_s^{(\pm)}(s(z),t)$ is the discontinuity of the amplitude across the cut $s > 4\mu^2$ and bears all the singularity of the amplitude as a function of t and u, that is at $t > 4\mu^2$ and $u > \mu^2$, where it becomes complex. If we choose $t < 4\mu^2$ possible singularities come only from the region $u > 4\mu^2$, that is

$$4\mu^2 - s - t > 4\mu^2 \quad \text{or} \quad s + t < 0$$

However if $t > -4\mu^2$ we have $s + t > 0$ and we are outside the singularities, so that $\mathcal{A}_s^{(\pm)}(s(z),t)$ remains real. So we find

$$\bar{a}_l^{(\pm)}(t) = a_l^{(\pm)}(t), \quad \text{at} \quad -4\mu^2 < t < 4\mu^2 \tag{154}$$

and arbitrary complex l. We also note that at real l and z_t function $Q_l(z_t)$ is real. From this we conclude that at real z_t $Q_{l^*}(z_t) = Q_l^*(z_t)$, and consequently

$$[\bar{a}_l^{(\pm)}]^*(t) = a_{l^*}^{(\pm)}(t), \quad \text{at} \quad \text{real} \quad t \tag{155}$$

Now consider the elastic unitarity in the region $4\mu^2 < t < 16\mu^2$ at integer values of l of the appropriate signature:

$$a_l^{(\pm)}(t) - \bar{a}_l^{(\pm)}(t) = 2i\rho(t)a_l^{(\pm)}(t)\bar{a}_l^{(\pm)}(t), \quad \text{at} \quad 4\mu^2 < t < 16\mu^2 \tag{156}$$

where $\rho(t) = k_t/2\pi\sqrt{t}$ and we used (155) at integer l. Considered as functions of complex l both the left- and right-hand sides of (156) are analytic in the right half-plane and decrease as $\mathrm{Re}\,l \to \infty$. Therefore they are equal at any complex l. Thus the unitarity relation (156) is valid at any complex l in the analyticity region.

1.6.2 Regge poles

Now let us try to understand what kind of singularities may exist in partial waves in the complex l plane, which limit the analyticity region. As in the previous studies we can distingush between the fixed singularity points $l = l_s$ which do not depend on t and moving singularities $l = l_s(t)$ whose position depend on t.

Elastic unitarity in the form (156) prohibits existence of fixed singularities. In fact assume that the partial amplitude $a_l(t)$ (of any signature, which we suppress in the following) has a fixed pole at $l = l_0$

$$a_l(t) \sim \frac{b(t)}{l - l_0}$$

where l_0 is t-independent. Taking $|t| < 4\mu^2$ we also find

$$\bar{a}_l(t) = a_l(t) \sim \frac{b(t)}{l - l_0}, \quad |t| < 4\mu^2$$

Now we continue this relation in t to the interval $t > 4\mu^2$. Since at $t = 4\mu^2$ the amplitude has a branch point we find that at $t > 4\mu^2$

$$\bar{a}_l(t) = \sim \frac{b^*(t)}{l - l_0}, \quad t > 4\mu^2$$

with a complex conjugate b but the same l_0. If we put (1.6.2) and (1.6.2) into (156) we encounter a contradiction: the left-had part has a simple pole at $l = l_0$ but the right-hand part a double pole. This conclusion can be easily generalized to any type of singularities.

Thus if we assume (156) the only possible singularities may only be moving $l = l_s(t)$. Moreover if we assume (156) true at any values of $t > 4\mu^2$ these singularity may only be poles at $l = \alpha(t)$ which are called Regge poles. Indeed singularities in l depending on t can also be considered as singularities in t depending on l. From the definition of $a_l(t)$, Eq. (148) one can conclude that in the analyticity region of l and as a function of t on the physical sheet of complex t plane it cannot have any l-dependent singularities (the only singularities come either from those of $\mathcal{A}_s^{(\pm)}(t, s(z_t)$ or from $Q_l(z_t)$ whose position does not depend on l). However it can have such singularities on the second sheet of the complex t plane related with the unitarity cut at $t > 4\mu^2$. Expressing from unitarity the amplitude above the unitary cut in terms of the same amplitude below the cut

$$a_l^{(\pm)}(t) = \frac{\bar{a}_l^{(\pm)}(t)}{1 - 2i\rho(t)\bar{a}_l^{(\pm)}(t)} \tag{157}$$

we see that the right-hand side is analytic below the cut and gives the analytic continuation of the amplitude to the second sheet of the complex t plane. We observe that the amplitude is analytic on this second sheet except for possible poles at points where

$$1 - 2i\rho(t)\bar{a}_l^{(\pm)}(t) = 0 \tag{158}$$

These equation defines the so-called Regge trajectory $l = \alpha^{(\pm)}(t)$, which determines the position of a moving pole of the amplitude $a_l^{(\pm)}(t)$. These poles stay on the unphysical sheet of the complex t plane and so are not visible in the analyticity region of l plane. However with the change of complex l they may appear on the physical sheet and then become actual singularities of the amplitude.

In particular, with a change of interaction, the poles may appear on the physical sheet of t from the start. Then at integer l they have to be situated below the threshold $t = 4\mu^2$ and correspond to bound states in the t channel. With the change of l one obtains a family of bound states of spin l when the trajectory passes integer values of l of the appropriate signature. If the poles in t remain on the second sheet at $t > 4\mu^2$ the bound states transform into resonances.

Thus a Regge trajectory $\alpha^{(\pm)}(t)$ describes a family of particles with rising spins J=0,2,4... for positive signature and J=1,3,5,.. for negative signature with different (and normally rising) masses $m_J = \sqrt{t_J}$ determined by the equation $J = \alpha^{(\pm)}(t_J)$. If $t_J < \mu^2$ these particles are stable and if $t_J > 4\mu^2$ they are unstable, with a finite width and are rather resonances in the scattering cross-section. So a Regge trajectory unites into a family a group of particles superficially not related to one another but in fact belonging to rotational excitations of the basic particle. From the unitarity equation continued to complex value of l it follows that the contribution of the Regge pole to the amplitude $a_l^{(\pm)}$ preserves the same structure which is valid for normal bound states or resonances at integer values of l, that is it has a factorizable residue with factors which can be interpreted as a coupling of the intermediate particle with the initial and final ones. Therefore the Regge trajectory may be considered as the position of a pole corresponding to propagation of a certain quasi-particle with variable spin and mass, which is called Reggeon.

Apart from the signature, Reggeons carry all the usual internal quantum numbers corresponding to the property of the t-channel in which they appear. In our simple case they obviously have the vacuum quantum numbers. However if the initial (and final) particle have non-trivial quantum numbers, electric charge, parity, isospin, strangeness, charm etc, the intermediate Reggeon will have the same quantum numbers.

To conclude, we have to mention that the conclusion that the only possible singularities of the complex angular momentum amplitudes may only be poles heavily rests on the assumption that the unitarity relation is reduced to its elastic contribution. Inclusion of inelastic channel allows to admit singularities of the branch-point type. Physically they correspond to the appearance of more complicated structures than a single Reggeon in the intermediate states, two, three and more Reggeons.A detailed study of these Regge cut singularities is beyond the scope of the present lectures. In the following we shall mention situations when inclusion of these singularities may be important.

1.6.3 Gribov-Froissart representation and high-energy asymptotic

To convert expansion (145) into an expression suitable for the study of large values of s and consequently of z_t let us transform it into an integral in the complex l plane following Sommerfeld-Watson. We define the signatured amplitude by a similar expansion in terms of signatured partial waves:

$$\mathcal{A}^{(\pm)}(s,t) = \sum_{l=0}^{\infty}(2l+1)a_l^{(\pm)}(t)P_l(z_t) \tag{159}$$

We transform this sum into an integral

$$\mathcal{A}^{(\pm)}(s,t) = \frac{i}{2}\int_C \frac{dl(2l+1)}{\sin\pi l}P_l(-z_t)a_l^{(\pm)}(t) \tag{160}$$

where contour C is shown in Fig. 15 a. At $|z_t| < 1$ the integrand decreases at large Re l, since function $P_l(-z_t) \sim (1\sqrt{l})\cos(\phi)$ with $|\phi| < \pi$ so that the ratio $P_l(-z_t)/\sin\pi l$ is limited and $a_l^{(\pm)}(t)$ decreases. This allows to deform the contour to pass parallel to the imaginary axis as shown in Fig. 15 b, so that (160) transforms into

$$\mathcal{A}^{(\pm)}(s,t) = \frac{i}{2}\int_{\eta-i\infty}^{\eta+i\infty} \frac{dl(2l+1)}{\sin\pi l}P_l(-z_t)a_l^{(\pm)}(t) \tag{161}$$

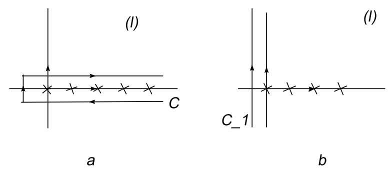

Figure 15: Original and deformed contour of integration in the complex l-plane

This new integral proves to converge at any values of $|z_t|$ including large values. In fact, since Rel is limited both $P_l(-z_t)$ and $a_l^{(\pm)}(t)$ are limited in the integrand and $\sin\pi l$ exponentially rises. So the integrand decreases at large Iml and the integral converges. Representation (161) is known as Gribov-Froissart representation for the scattering amplitude.

Importance of this representation consists in its simple dependence on the energetic variable s at $s \to \infty$. It is concentrated in the Legendre function $P_l(-z_t)$. We recall that $z_t = 1 + s/2k_t^2 = 1 + 2s/(t - 4\mu^2)$. In the physical region of the s channel at large energies $t < 0$ and we find $-z_t \sim 2s/(4\mu^2 - t)$, so that $P_l(-z_t) \sim (-z_t)^l \sim s^l$. So at large s the the behaviour of the amplitude in the Gribov-Froissart form is determined by the rightmost singularities of the partial waves $a_l^{(\pm)}$, which prevent moving the contour to the left. If this leading singularity occurs at $l = l_s$ then at large s the amplitude behaves as s^{l_s} and grows as $s^{\mathrm{Re}\, l_s}$. So we find that the behaviour of the amplitude at large energies is related to the analytical properties of the partial waves in the crossed channel considered as function of the complex angular momentum and is determined by its singularities with the maximal value of their real part.

If this leading singularity is a Regge pole

$$A^{(\pm)} \sim \frac{\beta(t)}{l - \alpha(t)} \tag{162}$$

then its contribution to the amplitude (161) is obviously

$$\mathcal{A}_R^{(\pm)}(s,t) = -\pi(2\alpha(t)+1)\beta(t)\frac{P_\alpha(-z_t)}{\sin\pi\alpha(t)} = C(t)s^{\alpha(t)} \tag{163}$$

Up to now we considered the signatured amplitude. Passing to the normal amplitude we find:

$$\mathcal{A}(s,t) = \sum_{l=0}^{\infty}(2l+1)P_l(z_t)a_l(t) = \frac{1}{2}\sum_{l=0}^{\infty}(2l+1)\Big(P_l(z_t) + P_l(-z_t)\Big)a_l^{(+)}(t)$$

$$+\frac{1}{2}\sum_{l=0}^{\infty}(2l+1)\Big(P_l(z_t) - P_l(-z_t)\Big)a_l^{(-)}(t) = \frac{1}{2}\sum_{\xi=\pm}\sum_{l=0}^{\infty}(2l+1)\Big(P_l(z_t) + \xi P_l(-z_t)\Big)a_l^{(\xi)}(t) \tag{164}$$

Applying the Sommerfeld-Watson transformation and deforming the contour as before we get

$$\mathcal{A}(s,t) = \frac{i}{4}\sum_{\xi=\pm}\int_{\eta-i\infty}^{\eta+i\infty} dl(2l+1)a_l^{(\xi)}(t)\frac{P_l(-z_t) + \xi P_l(z_t)}{\sin\pi l} \tag{165}$$

The contribution from a Regge pole of a given signature $\xi = \pm$ will be

$$\mathcal{A}_R(s,t) = -\pi(4\alpha(t)+1)\beta(t)\frac{P_\alpha(-z_t) + \xi P_l(z_t)}{\sin\pi\alpha(t)} \tag{166}$$

As we have seen, in the physical region of the s channel $z_t = -2s/(4m^2 - t)$ and is negative with a small negative imaginary part (above the unitary cut in s). At complex l function $P_l(z_t)$ has a cut at $-\infty < z_t < -1$ and in this region

$$P_l(-z_t \pm i0) = e^{\pm i\pi l} P_l(z_t) - \frac{2}{\pi} Q_l(z_t) \sin \pi l \tag{167}$$

Function $Q_l(z_t)$ decreases at large z_t so in the physical region of the s channel at large s and negative z_t we get

$$P_l(z_t) \simeq e^{-i\pi l} P_l(-z_t) \tag{168}$$

Thus the contribution from a Regge pole becomes

$$\mathcal{A}_R(s,t) = -\pi(4\alpha(t)+1)\beta(t) P_{\alpha(t)}(-z_t) \frac{1+\xi e^{-i\pi\alpha(t)}}{\sin \pi\alpha(t)} \tag{169}$$

The 'signature factor'

$$\zeta(\alpha(t)) = \frac{1+\xi e^{-i\pi\alpha(t)}}{\sin \pi\alpha(t)} \tag{170}$$

guarantees exclusion of the poles of the wrong signature in the amplitude. Indeed at $\alpha^{(\xi)}(t) = n$, n=0,1,2,.. we have $1+\xi e^{-i\pi\alpha(t)} = 1+\xi(-1)^n$. So if $\xi = +$ the pole occurs only at even values of n and if $\xi = -$ the pole occurs only at odd values of n. At large s from (169) we find the asymptotical behaviour

$$\mathcal{A}_R(s,t) \sim -\pi(4\alpha(t)+1)\beta(t)(2\mu^2 - t/2)^{-\alpha(t)} \zeta(\alpha(t)) s^{\alpha(t)} \tag{171}$$

The Froissart limitation implies that at $t \leq 0$ Re $\alpha(t) \leq 1$.

1.6.4 Application to physical processes. The Pomeron

According to the previous subsection we derive a relation between the existence of particles and resonances at $t > 0$ in the t channel and the asymptotic of the amplitude at high energies in the s-channel. Knowledge of the spectrum of particles observed in the t channel at $t > 0$ allows to determine the corresponding Regge trajectories on which the rotational particle levels lie. The experimental data show that at comparatively small values of t all Regge trajectories are linear functions of t and with a good approximation can be presented as:

$$\alpha(t) = \alpha(0) + \alpha' t \tag{172}$$

The numbers $\alpha(0)$ and α', are called intercept and slope, respectively. For all the channels except for the vacuum channel, the value of α' is practically the same and very close to 1 (GeV/c)$^{-2}$. Intercepts are all smaller than unity and different for different channels. According to the asymptotic (171) the leading contribution to the total cross-section determined by $\mathcal{A}(s, t = 0)$ comes from the trajectory with the highest intercept. Some representative trajectories derived from the observed spectrum of mesons are shown in Fig. 16 in the so-called Chew-Frautschi plot which shows Re$\alpha(t)$ as a function of t. It is remarkable that the trajectories for the two different signatures coincide: particles appear on them at when the trajectory passes through an integer value, irrespective of whether it is even or odd. The trajectory with the highest intercept is the $\omega - \rho$ trajectory on which lie the well-known vector mesons ω and ρ Its intercept is close to 0.5. Next follows

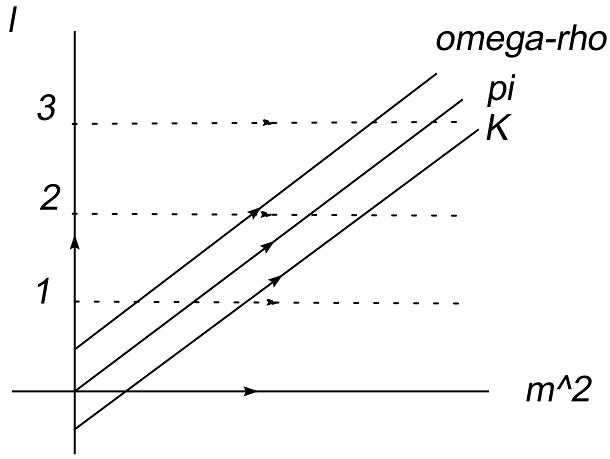

Figure 16: The Chew-Frautschi plot

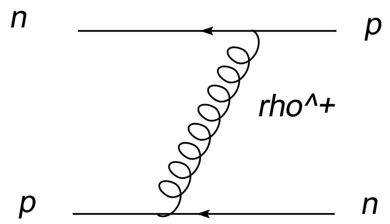

Figure 17: The pn exchange amplitude

the π trajectory with intercept close to zero. Further down lies the K trajectory with a negative intercept.

As we observe intercepts of all trajectories derived from the observed particles lie below unity. So the amplitudes which are described by the exchange of the corresponding Reggeons in the t channels grow with s more slowly than s and the relevant cross-sections fall with s. Note that practically all the observed particles and consequently the corresponding Reggeons have non-trivial internal quantum numbers (charge, isospin, strangeness etc). The exchange of such Reggeons implies the exchange of quantum numbers between the colliding particles. An example of such exchange reaction is the binary collision of the proton and neutron in which the proton becomes the neutron and vice versa. Such a reaction can indeed go via the exchange of an $\omega - \rho$ Reggeon, as illustrated in Fig. 17. According to (171) the corresponding amplitude at large s behaves as $\sqrt{s}$ and the cross-section decreases as $1/\sqrt{s}$. Experimental study of this and similar exchange reactions has confirmed that their cross-sections indeed decrease at high energies in full correspondence with the predictions following from the picture, in which they go via the exchange of an $\omega - \rho$ Reggeon. In general description of all exchange reaction is well described by exchanges of Reggeons derived from the observed particle spectrum in the relevant t-channels.

A striking exception is the total cross-section for which the t channel has to bear vacuum quantum numbers. Experimentally there has not been found any sign of particles with vacuum quantum numbers which may be related to a Regge trajectory with an intercept close to unity to account for the experimental behaviour of the total cross-section, which certainly does not decrease but rather grows with energy. This behaviour implies existence of a singularity in the positive signature vacuum t channel amplitude $a_l^{(+)}(t)$ which has been called Pomeranchuk singularity, or simply Pomeron. The simplest assumption is that the Pomeron is just a Regge pole with intercept $\alpha(0) = 1$. In this case the forward scattering amplitude will grow linearly with s at large s and the cross-sections will tend to a constant value. Experimental data, as mentioned, rather indicate that cross-sections grow with s. Since such a growth cannot be powerlike according to the Froissart theorem, but at most be proprtional to $\ln^2 s$, this implies that in fact the Pomeron is not a simple pole but a more complicated singularity. At present the exact nature of the pomeron is not fully understood. So we shall discuss some consequences following from the assumption that it is just a simple pole with unity intercept, that is with the trajectory

$$\alpha_P(t) = 1 + \alpha'_P t \tag{173}$$

No physical particles have been observed lying on this trajectory. This can be explained by the fact that α'_P is small as compared to non-vacuum trajectories. From some (not very reliable) experimental data to be discussed in the following one can conclude that

$$\alpha'_P \simeq 0.2\ (\mathrm{GeV/c})^{-2} \tag{174}$$

Turning to the expression (170) for the signature factor we find at small t for the Pomeron

$$\zeta_P(t) = \frac{1 - e^{-i\pi\alpha'_P t}}{-\sin\pi\alpha'_P t} = -i \tag{175}$$

So at small t the contribution from the Pomeron to the amplitude is

$$\mathcal{A}_P(s,t) = i\frac{\pi}{4}3\beta_P(0)\Big(\frac{s}{2\mu^2}\Big)^{1+\alpha'_P t} = icse^{\alpha'_P t\ln(s/s_0)} \tag{176}$$

where we left only that t dependence which is increase by the factor growing with s. As we observe the amplitude is pure imaginary and at $t = 0$ grows with s linearly, which correspond to a constant total cross-section ($\sigma^{tot} = c$). It is instructive to calculate the differential cross-section for elastic scattering. According to the general formulas

$$\frac{d\sigma^{el}}{d\Omega} = \frac{1}{J_{12}}\frac{k}{16\pi^2\sqrt{s}}\Big|\mathcal{A}_P(s,t)\Big|^2 = \frac{1}{64\pi^2 s}\Big|\mathcal{A}_P(s,t)\Big|^2 \tag{177}$$

where we have used that at $s \to \infty$ we have $k/\sqrt{s} \to 1/2$, $J_{12} \to 2s$. It is convenient to present

$$\frac{d\sigma^{el}}{d\Omega} = \frac{d\sigma^{el}}{2\pi dz} \simeq \frac{s}{4\pi}\frac{d\sigma^e l}{dt} \tag{178}$$

from which we find

$$\frac{d\sigma^e l}{dt} = \frac{1}{16\pi s^2}\Big|\mathcal{A}_P(s,t)\Big|^2 = \frac{c^2}{16\pi}e^{2\alpha'_P t\ln(s/s_0)} \tag{179}$$

We see that the width $b(s) = 1/\Big(2\alpha'_P\ln(s/s_0) + const\Big)$ of the forward cone in t is decreasing with s as $1/\ln s$. With the growth of energy particles are emitted in the cone which becomes narrower. The experimental observations give contradictory results. The value

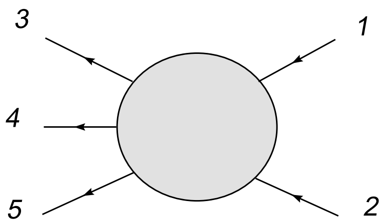

Figure 18: Production amplitude

(174) has been extracted from some data. But other data are compatible even with $\alpha' = 0$ when no change in the cone width occurs with the growth of s.

The total elastic cross-section is going down at high energies:

$$\sigma^{el} \simeq \int_{-\infty}^{0} dt \frac{d\sigma^{el}}{dt} = \frac{c^2}{16\pi} \int_{-\infty}^{0} dt e^{t/b} = \frac{c^2}{16\pi} b(s) \tag{180}$$

Thus the contribution of the elastic channel decreases. as compared to the inelastic ones.

1.7 Multiregge asymptotic and inclusive cross-sections

1.7.1 Production amplitude and its high-energy asymptotic

Let us study a general production amplitude, illustrated in Fig. 18. According to our general rule it depends on 5 variables. For these variables we may, for instance, choose 3 energetic variables $s_{12} = (k_1 + k_2)^2$, $s_{34} = (k_3 + k_4)^2$, $s_{45} + (k_4 + k_5)^2$ and two variables of the transferred momentum type $t_{13} = (k_1 - k_3)^2$, $t_{25} = (k_2 - k_5)^2$ (the concrete choice of the final momenta k_3, k_4 and k_5 will be done later). However in the following sometimes it will be more convenient to choose other sets of 5 variables.

We shall consider it at high energies when

$$s_{12} = (k_1 + k_2) \to \infty \tag{181}$$

To simplify we assume all particle massless, since at high energies non-zero masses will generally be unimportant and assuming

$$k_i^2 = 0, \quad i = 1, 2, ...5 \tag{182}$$

will simplify the kinematics. In the c.m. system $\mathbf{k}_1 + \mathbf{k}_2 = 0$ we direct the initial particles along the z-axis $k_{1\perp} = k_{2\perp} = 0$. Then obviously $k_{10} = k_{20} = k_{1z} > 0$.

It is convenient to introduce light-cone variables. For any 4-momentum k we define

$$k_{pm} = \frac{1}{\sqrt{2}}(k_0 \pm k_z) \tag{183}$$

They are positive for physical particles. Then $k_{1+} = sqrt2k_{1z}$, $k_{1-} = 0$ and $k_{2+} = 0$, $k_{2-} = \sqrt{2}k_{1z} = k_{1+}$ variable s_{12} in terms of these light-cone momenta is written as

$$s_{12} = 2k_1k_2 = 2k_{1+}k_{2-} = 2k_{1+}^2 \tag{184}$$

Now let us consider final momenta k_3, k_4 and k_5. Their sum is

$$k_3 + k_4 + k_5 = k_1 + k_2 \tag{185}$$

Since k_{1+} and k_{2-} tend to infinity, at least one of the three momenta k_3, k_4 and k_5 has to possess a large '+'- component and at least another one has to posses a large '-'- component.

All the experimental data indicate that at large energies the produced particles have their transverse momenta much smaller and growing with energy much weaker than the longitudinal ones. In the first approximation one can safely assume that their $k_\perp^2$ stays finite at large energies (in fact it slowly grows with energy as its logarithm). From condition $k_i^2 = 0$ we find $2k_{i+}2k_{i-} = -k_{i\perp}^2$ which means that if $k_{i+} \to \infty$ then $k_{i-} \to 0$ and vice versa if $k_{i-} \to \infty$ then $k_{i+} \to 0$. So one of the three final momenta, let it be k_3, has a maximal '+' component, which tends to ∞ and a minimal '-'-component, which tends to zero, and another of these momenta let it be k_5, has a maximal '-' component, which tends to ∞ and a minimal '+'-component, which tends to zero:

$$k_{5+}, k_{4+} < k_{3+} \to \infty, \quad k_{5-}, k_{4-} > k_{3-} \to 0, \quad k_{3-}, k_{4-} < k_{5-} \to \infty, \quad k_{3+}, k_{4+} > k_{5+} \to 0 \tag{186}$$

Let us study the relative energetic variable $s_{34} = (k_3 + k_4)^2$. In terms of light-cone variables

$$s_{34} = 2k_3k_4 = 2(k_{3+}k_{4-} + k_{3-}k_{4+} + (k_3k_4)_\perp) \tag{187}$$

Using the mass-shell condition we rewrite it as follows

$$s_{34} = -k_{4\perp}^2 \frac{k_{3+}}{k_{4+}} - k_{3\perp}^2 \frac{k_{4+}}{k_{3+}} + (k_3k_4)_\perp \tag{188}$$

We may distinguish between two cases:

1) If the ratio k_{3+}/k_{4+} is finite, which implies $k_{3+} \sim k_{4+}$, then s_{34} is finite.

2) If $k_{3+}/k_{4+} >> 1$, which means that $k_{4+} << k_{3+}$, then

$$s_{34} \simeq -k_{4\perp}^2 \frac{k_{3+}}{k_{4+}} = 2k_{3+}k_{4-} \to \infty \tag{189}$$

Similarly we consider the energetic variable s_{45} which we present as

$$s_{45} = -k_{4\perp}^2 \frac{k_{5-}}{k_{4-}} - k_{5\perp}^2 \frac{k_{4-}}{k_{5-}} + (k_3k_4)_\perp \tag{190}$$

Again two cases are possible

3) If the ratio k_{4-}/k_{5-} is finite, which implies $k_{4-} \sim k_{5-}$, then s_{45} is finite.

4) If $k_{5-}/k_{4-} >> 1$, which means that $k_{4-} << k_{5-}$, then

$$s_{45} \simeq k_{4\perp}^2 \frac{k_{5-}}{k_{4-}} = 2k_{5-}k_{4+} \to \infty \tag{191}$$

Combining these possibilities we find that cases 1) and 3) are incompatible. So we have only three possibilities ; 1), 3) and 2)+4) together. In other words:

1) s_{34} is finite, s_{45}, s_{35} are of the same order as s_{12} This region is called fragmentation region of the projectile. In fact the initial large momentum k_{1+} is split in two parts of the same order $k_{3+} + k_{4+}$ as if the initial projectile particle split in two particles with momenta k_3 and k_4

2) s_{45} is finite, s_{34}, s_{35} are of the same order as s_{12} This region is called fragmentation region of the target. Now the initial large momentum k_{2-} is split in two parts of the same

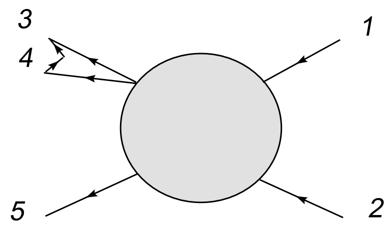

Figure 19: Particles 3 and 4 form a quasi-particle with mass $(k_3 + k_4)^2$

order $k_{4-} + k_{5-}$ as if the initial target particle split in two particles with momenta k_4 and k_5

3) $s_{34}, s_{45} \to \infty$. Then from (189) and (191) we find

$$s_{34}s_{35} = -k_{4\perp}^2 s_{12} \tag{192}$$

This region is called central.

Let us study the asymptotic of the amplitude in these three regions. For simplicity we shall assume that all singularities of the partial waves amplitudes to be introduced are Regge poles.

We begin with the projectile fragmentation region. Adequate variables in this case are $s_{12}, s_{34}, t_{25}, t_{13}$ and t_{14}. Since the total mass of the final particles 3 and 4 $\sqrt{s_{34}}$ is finite, we can consider the whole diagram as a binary process in which the initial particles 1 and 2 go into the final particle 5 and a compound particle formed by particles 3 and 4 with mass $\sqrt{s_{34}}$:

$$1 + 2 \to (34) + 5 \tag{193}$$

(see Fig. 19). In the t-channel in its center of mass the initial and final momenta now are different, k_t and k'_t due to difference in masses. From the $2\bar{5}$ end we have the standard relation $t_{25} = (k_2 - k_5)^2 = 4k_t^2$, so that $k_t = \sqrt{t_{25}}/2$ From the $1(\bar{3}\bar{4})$ end we find $t_{25} = (k_1 - k_3 - k_4)^2 = (k'_t + \sqrt{s_{34} + k_t'^2})^2$ or $\sqrt{s_{34} + k_t'^2} = \sqrt{t_{25}} - k'_t$ or $s_{34} = t_{25} - 2k'_t\sqrt{t_{25}}$, so that

$$k'_t = \frac{1}{2}\Big(\sqrt{t_{25}} - \frac{s_{34}}{\sqrt{t_{25}}}\Big) \tag{194}$$

(with $s_{34} = 0$ this passes into k_t). Now we consider s_{12} as a function of the scattering angle and momenta k_t and k'_t

$$s_{12} = (k_1 + k_2)^2 = (k_t + k'_t)^2 - (k_t^2 + k_t'^2 + 2k_t k'_t z_t) == -2k_t k'_t(1 - z_t) \tag{195}$$

where z_t is the cosine of the scattering angle in the t-channel. As before we expand the amplitude in the Legendre polynomials of z_t at fixed s_{34}, t_{25}, t_{13} and t_{14}:

$$\mathcal{A}(s_{12}(z_t), t_{25}, s_{34}, t_{13}, t_{14}) = \sum_{l=0}(2l+1)P_l(z_t)a_l(s_{34}, t_{25}, t_{13}, t_{14}) \tag{196}$$

Next we pass to complex angular momenta and transform this expansion into the Gribov-Froisart representation. At $s_{12} \to \infty$, as before, $-z_t \to \infty$ and the asymptotic will be determined by the rightmost singularities in the complex l-plane. If we assume that this rightmost singularity is a Regge pole at $l = \alpha(t_{25}$ then we find an asymptotic similar to (169)

$$\mathcal{A}_R(s_{12}, t_{25}s_{34}, t_{13}, t_{14}) = -\pi(4\alpha(t_{25})+1)\gamma_{R25}(t_{25})\gamma_{R134}(t_{25}, s_{34}, t_{13}, t_{14})P_{\alpha(t_{25})}(-z_t)\zeta(\alpha(t_{25})) \tag{197}$$

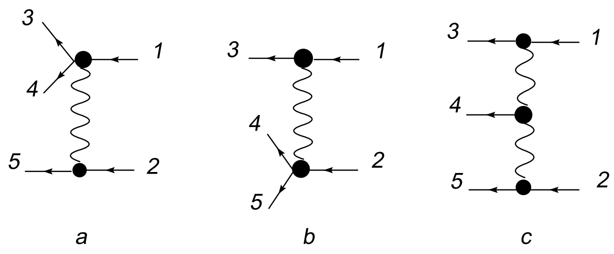

Figure 20: Single and double Regge limits

It is important that the residue at the pole factorizes into a coupling constant λ_{R25} of the Reggeon with particles 2 and 5, on the one side, and a coupling constant λ_{R134} with particles 1,3 and 4 on the other side, in accordance with the unitarity relation in the t channel. Each coupling naturally depends only on variables related to the coupled particles. The asymptotic (197), called a single-Regge limit, is illustrated graphically in Fig. 20 *a*. Note that the theory cannot tell much about the structure of the coupling vertexes: it only predicts the asymptotic dependence on the large s_{12}

A similar asymptotic can be derived in the second fragmentation region, that of the projectile. One then obtains an asymptotic of the same form as (197) with substitutions

$$t_{25}, s_{34}, t_{13}, t_{14} \to t_{13}, s_{45}, t_{25}, t_{24} \tag{198}$$

It is illustrated in Fig. 20 *b*.

We finally study the central region. Adequate variables are now $s_{34}, s_{45}, t_{13}, t_{25}$ and $k_{4\perp}^2$. In this case one considers two different t channels, $1\bar{3}$ with its energetic variable $t_{13} = (k_1 - k_3)^2$ and $2\bar{5}$ channel with its energetic variable $t_{25} = (k_2 - k_5)^2$. In the $1\bar{3}$ channel the initial particles are 1 and 3 with zero mass and final are 4 and (25) with masses zero and t_{25} respectively. So in this channel the c.m. momenta are $k_{13} = \sqrt{t_{13}}/2$ and $k'_{13} = (1/2)(\sqrt{t_{13}} - t_{25}/\sqrt{t_{13}})$. Variable s_{34} will be expressed by these momenta and the scattering angle as before

$$s_{34} = -2k_{13}k'_{13}(1 - z_{13}) \tag{199}$$

In the channel $2\bar{5}$ we shall find similar formulas with the interchange t_{13} and t_{25} and $s_{34} \to s_{54}$. In particular

$$s_{54} = -2k_{25}k'_{25}(1 - z_{25}) \tag{200}$$

with $k_{25} = \sqrt{t_{25}}/2$ and $k'_{25} = (1/2)(\sqrt{t_{25}} - t_{13}/\sqrt{t_{25}})$. Now we consider the amplitude as a function of z_{13} and z_{25} at fixed t_{13}, t_{25} and $k_{4\perp}^2$ and expand it in the double series of the Legendre polynomials of z_{13} and z_{25}:

$$\begin{aligned} &\mathcal{A}(s_{34}(z_{13}), s_{54}(z_{25})t_{13}, t_{25}, k_{4\perp}^2) \\ &= \sum_{l_1=0} \sum_{l_2=0} (2l_1+1)(2l_2+1)P_{l_1}(z_{13})P_{l_2}(z_{25})a_{l_1,l_2}(t_{13}, t_{25}, k_{4\perp}^2) \end{aligned} \tag{201}$$

Next we pass to complex angular momenta l_1 and l_2 and as before transform this expansion into double Gribov-Froissart representation. The asymptotic at large values s_{34} and s_{45} will be determined by the rightmost singularities in the complex l_1- and l_2 planes. If these

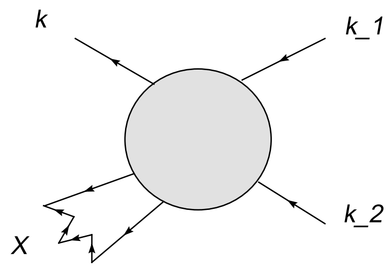

Figure 21: Amplitude for the inclusive cross-section

singularities are Regge poles in the two t channels at $l_1 = \alpha_1(t_{13})$ and $l_2 = \alpha_2(t_{25})$ we get an asymptotic (double Regge limit)

$$\mathcal{A}_{R_1R_2}(s_{34}, s_{45}, t_{13}, k_{4\perp}^2) = \pi^2(4\alpha_1(t_{13})+1)(4\alpha_2(t_{25})+1)$$

$$\gamma_{R_113}(t_{13})\gamma_{R_225}(t_{25})\gamma_{RR4}(t_{13}, t_{25}, k_{4\perp}^2)P_{\alpha_1(t_{13})}(-z_{13})P_{\alpha_2(t_{25})}(-z_{25})\zeta(\alpha_1(t_{13})\zeta(\alpha_2(t_{25})) \quad (202)$$

It is illustrated in Fig. 20 c.

1.7.2 Inclusive cross-sections. Generalities

In this subsection we shall study the single inclusive cross section for particle production at high energies. It corresponds to the reaction

$$a(k_1) + b(k_2) \to c(k) + X \quad (203)$$

where particle c with momentum k is registered in the experiment, all other produced particle being unregistered and denoted as X. Graphically this process is shown in Fig. 21 where unregistered particles are combined in the wide arrow. Analytically the inclusive cross-section is given by the formula (11), which we rewrite in the notation corresponding to (203) as

$$I(k) \equiv \frac{(2\pi)^3 k_0 d\sigma(k, X)}{d^3k} = \frac{1}{J_{12}} \sum_n \int d\tau_n'(k_1 + k_2 - k)\Big|\mathcal{A}(k, X|k_1, k_2)\Big|^2 \quad (204)$$

We first comment on the kinematics of the inclusive reactions. Already from Fig. 21 one observes that the reaction is described by the same structure as for any binary reaction with the only difference that the total mass corresponding to the unobserved produced particles, the 'missing mass' M is not fixed and is an extra variable

$$M^2 = (k_1 + k_2 - k)^2 \quad (205)$$

So the process is characterized by three independent variables. For these one may choose the old Mandelstam variables $s = (k_1 + k_2)^2$, $t_1 = (k_1 - k)^2$ and M^2. However this choice does not take into account the already mentioned fact that produced particles carry their transverse momenta much smaller than the longitudinal ones. For this reason usually the inclusive cross-section is characterized by different sets of variables.

For simplicity we shall again take masses of the three observed particles a,b and c all equal to zero: $k_1^2 = k_2^2 = k^2 = 0$. Then in the c.m. system, using the light-cone variables, we find as before

$$k_{1+} = k_{2-}, \quad k_{1\perp} = k_{2\perp} = 0, \quad s = 2k_{1+}^2 \quad (206)$$

The observed final particle may be characterized by its transverse momentum $k_\perp$ and Feynman scaling variables

$$x_+ = \frac{k_+}{k_{1+}}, \quad x_- = \frac{k_-}{k_{2-}} \tag{207}$$

Both $x_\pm$ are positive. At high energies their product is small:

$$x_+x_- = \frac{k_+k_-}{k_{1+}k_{2-}} = -\frac{k_\perp^2}{s} \to 0 \tag{208}$$

The missing mass is related to $x_\pm$ by the relation

$$M^2 = 2(k_1 + k_2 - k)_+(k_1 + k_2 - k)_- + (k_1 + k_2 - k)_\perp^2$$

$$= 2(k_1 - k)_+(k_2 - k)_- + k_\perp^2 = s(1 - x+)(1 - x_-) + k_\perp^2. \tag{209}$$

From this we conclude that both $x_\pm \leq 1$. Taking into account (208) we also see that if one of the two scaling variables, say x_+ is fixed at some positive value different from zero, then at large s the other $x_- \to 0$ and $M^2 \simeq s(1 - x_+)$. Since $k_z = (k_+ - k_-)/\sqrt{2} = k_{1+}(x_+ - x_-)/\sqrt{2}$ in this case the produced particle is moving fast along the direction of the incident particle a(k_1), that is in the forward direction. Momentum k_+ is of the order of the initial particle momentum k_{1+}, so that the observed particle is born in the fragmentation region of the particle a(k_1).

If the fixed variable is x_- then $x_+ \to 0$ and the same formula shows that $k_z < 0$ and the observed particle is moving in the backward direction. The observed particle is born in the fragmentation region of the target particle b(k_2)

However there exists a kinematical situation when both x_+ and x_- are small. Then on he one hand $k_+, << k_{1+}$ and on the other $k_- << k_{2-}$. The particle is born in the central region. As we shall see, at high energies the majority of particles are born precisely in this region. Scaling variables are badly suited to describe this region, since they reduce it to a point $x_\pm \sim 0$. A better description is give by the rapidity variable, defined as

$$y = \frac{1}{2}\ln\frac{k_+}{k_-} \tag{210}$$

One can transform this expression as

$$y = \frac{1}{2}\ln\frac{2k_+^2}{2k_+k_-} = \frac{1}{2}\ln\frac{x_+^2 s}{-k_\perp^2} = \ln\frac{x^+\sqrt{s}}{|k_\perp|} \tag{211}$$

or as

$$y = \frac{1}{2}\ln\frac{2k_+k_-}{2k_-^2 k} = \frac{1}{2}\ln\frac{-k_\perp^2}{x_-^2 s} = \ln\frac{|k_\perp|}{x_-\sqrt{s}} \tag{212}$$

From these formulas it follows that the rapidity varies in the interval

$$-\ln\frac{\sqrt{s}}{|k_\perp|} \leq y \leq \ln\frac{\sqrt{s}}{|k_\perp|} \tag{213}$$

In the fragmentation regions it is large, of the order $\ln\frac{\sqrt{s}}{k_\perp|}$. It is finite when both $x\pm$ are small, of the order $1/\sqrt{s}$ which corresponds to the central region. Note that rapidity is a natural generalization of the velocity in the non-relativistic mechanics. Under the Lorentz transformation in the direction of the z axis all rapidities change by the same constant, just as velocities under the Galilean transformation.

Thus in the central region suitable variables for the inclusive cross-sections are s, y and $k_\perp$. The phase volume takes an especially simple form in these variables:

$$\frac{d^3k}{(2\pi)^3 2k_0} = \frac{dyd^2k_\perp}{8\pi^3} \tag{214}$$

Let us turn now to the expression (204) for the inclusive cross-section. The right-hand side of this expression looks very similar to the right-hand side of the unitarity relation for the crossed process in the forward direction

$$a(k_1) + b(k_2) + \bar{c}(-k) \to a(k_1) + b(k_2) + \bar{c}(-k) \tag{215}$$

Indeed repeating the derivation of unitarity for the initial and final states containing the three particles we get, similarly to (26)

$$\begin{aligned} i\mathcal{A}^*(k_1, k_2, -k|k_1, k_2, -k) - i\mathcal{A}(k_1, k_2 - k|k_1, k_2, -k) &= 2\mathrm{Im}\,\mathcal{A}(k_1, k_2, -k|k_1, k_2, -k) \\ &= \sum_n \int d\tau_n(k_1 + k_2 - k)\Big|\mathcal{A}(n|k_1, k_2, -k)\Big|^2 \end{aligned} \tag{216}$$

Of course in this relation it is assumed that $k_0 < 0$, which corresponds to the initial particle $\bar{c}(-k)$. Note that from the point of view of the analytic properties the twice imaginary part (216) gives just the discontinuity divided by i of the amplitude considered as a function of $M^2 = (k_1 + k_2 - k)^2$ on its right-hand unitarity cut. So we can also rewrite (216) as

$$\mathrm{Disc}_{M^2>0}\mathcal{A}(k_1, k_2, -k|k_1, k_2, -k) = i\sum_n \int d\tau_n(k_1 + k_2 - k)\Big|\mathcal{A}(n|k_1, k_2, -k)\Big|^2 \tag{217}$$

If we analytically continue this relation in k_0 from negative values to positive ones, then in the right-hand side the amplitudes $\mathcal{A}(n|k_1, k_2, -k)$ describing transition of the initial three-particle state into intermediate states $|n>$ go over into crossed amplitudes $\mathcal{A}(k, n|k_1, k_2)$ which describe transitions of the initial two-particle state into intermediate states $|k, n>$ containing particle $c(k)$ and other particles labeled as n. So the right-hand side of (217) goes over into the right-hand side of Eq. (204) multiplied by iJ_{12}. Thus we get

$$\begin{aligned} I(k) &= \frac{1}{iJ_{12}}\mathrm{Disc}_{M^2>0}\mathcal{A}(k_1, k_2, -k|k_1, k_2, -k) \\ &= \frac{2}{J_{12}}\mathrm{Im}\,\mathcal{A}(k_1, k_2, -k|k_1, k_2, -k), \quad M^2, k_0 > 0 \end{aligned} \tag{218}$$

This relation is known as a generalized optical theorem by A.Mueller. It allows to find the single inclusive cross-section from the imaginary part of the three-to-three forward amplitude, in full similarity with the standard optical theorem. Note that (218) implies a possibility of analytic continuation of the standard unitarity relation in k_0 and thus in variables M^2, $t_1 = (k_1 - k)^2$ and $t_2 = (k_2 - k)^2$ related to k_0. The question about the branches in all these variables is not immediately clear, but is actually solved by the condition that the right-hand side should be pure imaginary with a positive imaginary part. Graphically the generalized optical theorem can be illustrated as shown in Fig. 22, where it is shown how the square modulus of the production amplitudes $\mathcal{A}(k, n|k_1, k_2)$ are summed into the discontinuity of the three-to-three by Cutkosky rules.

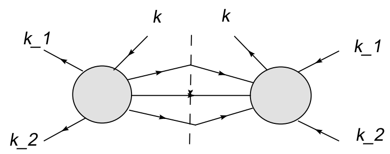

Figure 22: Optical theorem for inclusive cross-sections

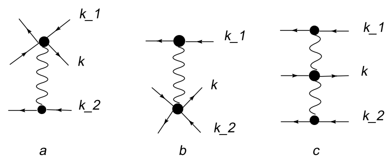

Figure 23: Regge description in the fragmentation and central regions

1.7.3 Inclusive cross-sections. High-energy asymptotic

According to the generalized optical theorem to find the asymptotic of the inclusive cross-section we have to find the asymptotic of the imaginary part of the amplitude $\mathcal{A}(k_1, k_2, -k|k_1, k_2, -k)$ at $M^2 > 0$ above the cut and divide it $J_{12}/2 \simeq s$. Applying our previous results for the production amplitude, trivially generalized to our three-to-three amplitude we distinguish between three kinematical regions.

If x_+ is finite (smaller than unity) and correspondingly $x_- \to 0$ then k_+ is of the same order as k_{1+} and we are in the fragmentation region of the projectile. In the b$\bar{b}$ channel we can consider the process as the transition of the initial b$\bar{b}$ pair into the quasi-particle a$\bar{c}$ with the total momentum $k_1 - k$ and its antiparticle with a finite mass $t_1 = (k_1 - k)^2$. The natural variables in this kinematics are the "energy" squared of these colliding particles in the s channel M^2 and characteristics of the observed particle c x_+ and $k_\perp^2$. We recall that the collision energy s is related to M^2 by $M^2 = s(1 - x_+)$ in this kinematics. At high M^2 the three-to-three amplitude will be described by the single-Regge limit (Fig. 23 a). Repeating the same derivation as for the production amplitude we expand the amplitude in the b$\bar{b}$ channel in partial waves. considering the particles (a$\bar{c}$) as a single particle with mass squared $t_1 = (k_1 - k)^2 = k_\perp^2/x_+$. To avoid a singularity in the b$\bar{b}$ channel at zero energy t we preserve a non-zero mass m_b for particle b. Then the two c.m. momenta in the b$\bar{b}$ -channel are $k_t = (1/2)\sqrt{t - m_b^2}$ and $k'_t = (1/2)\sqrt{t - t_1}$. We find

$$M^2 = -2k_t k' t(1 - z_t) \tag{219}$$

which at $t = 0$ gives $M^2 = -(1/2)m_b\sqrt{t_1}(1 - z_t)$ or

$$z_t = 1 + 2M^2/m_b\sqrt{t_1} \tag{220}$$

If the rightmost singularity in the relevant l-plane is the vacuum Regge pole (pomeron) then similar to (197) we get the asymptotic

$$\mathrm{Im}\,\mathcal{A}_P = 3\pi\gamma_{Pbb}(0)\gamma_{Pacac}(x_+, k_\perp^2)\Big(\frac{2M^2}{m_b\sqrt{t_1}}\Big)^{\alpha_P(0)} \tag{221}$$

where we put $\alpha_P(0) = 1$ in all places except when it appears in the exponent. The inclusive cross-section in this region results as

$$I(k) = f_{ac}(x_+, k_\perp^2)s^{\alpha_P(0)-1} \tag{222}$$

where f_{ac} is a function depending on x_+ and $k_\perp^2$ and on the type of particles a and c. If $\alpha_P(0)$ is exactly unity then we get essentially that the imaginary part grows linearly with s and the inclusive cross-section is a constant. As one may expect, the Regge approach only allows to find the asymptotic dependence on the large variable, other dependencies remaining unspecified.

In absolutely a similar manner we may study the case when x_- is fixed and smaller than unity and $s \to \infty$. Then k_- is of the same order as k_{2-} and we are in the fragmentation region of the target. The same derivation as above shows that if at large s and $M^2 = s(1-x_-)$

$$I(k) = f_{bc}(x_-, k_\perp^2) s^{\alpha_P(0)-1} \tag{223}$$

and does not depend on energy if $\alpha_P(0) = 1$. This situation is illustrated in Fig. 24 *b*

Now we pass to the case when both x_+ and x_- are small (central region). In this case it is convenient to use variables $t_1 = (k_1 - k)^2$, $t_2 = (k_2 - k)^2$ and $k_\perp^2$. Similarly to the case of production amplitude we find in this region

$$t_1 = -2kk_1 = \frac{k_\perp^2}{x_+}, \quad t_2 = \frac{k_\perp^2}{x_-} \tag{224}$$

so that both t_1 and t_2 are large and, using (208)

$$t_1 t_2 = -k_\perp^2 s \tag{225}$$

Next we expand the amplitude in partial waves in the two crossed channels $a\bar{a}$ and $b\bar{b}$ as was done for the production amplitude. At zero energy in these channels we get for the cosine of scattering angles

$$z_{a\bar{a}} = 1 + 2\frac{t_1}{m_a m_c}, \quad z_{b\bar{b}} = 1 + 2\frac{t_2}{m_b m_c} \tag{226}$$

At large t_1 and t_2 the asymptotic will be determined by the rightmost singularities in the two relevant complex angular momentum plane. It both these singularities correspond to Regge poles (pomerons) then the asymptotic will have the form of the double Regge limit (cf (202) (see Fig. 23 *c*)

$$\mathcal{A}_{PP} = 9\pi^2 \gamma_{Paa}(0)\gamma_{Pbb}(0)\gamma_{PPc}(k_\perp^2) P_{\alpha_P(0)}(-z_{a\bar{a}}) P_{\alpha_P(0)}(-z_{b\bar{b}}) \tag{227}$$

where we again have put $\alpha(0) = 1$ everywhere except when it is enhanced by large factors. Using (226) we find at large t_1 and t_2

$$(-z_{a\bar{a}})(-z_{b\bar{b}}) \sim \frac{4t_1 t_2}{m_a m_b m_c^2} = -s\frac{k_\perp^2}{m_a m_b m_c^2} \tag{228}$$

So the asymptotic of (227) is

$$\mathcal{A}_{PP} = f_{central}(k_\perp^2) s^{\alpha_P(0)} \tag{229}$$

and the inclusive cross-section in the central region is

$$I(k) = f_{central}(k_\perp^2) s^{\alpha_P(0)-1} \tag{230}$$

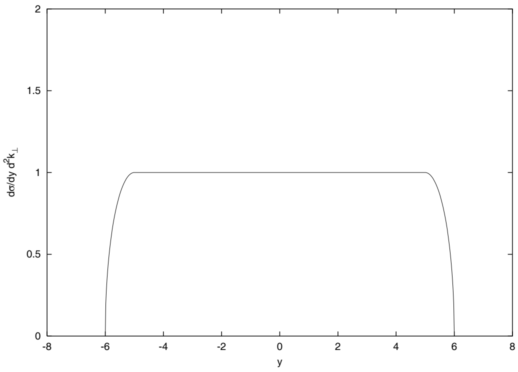

Figure 24: Inclusive cross-sections as a function of rapidity

This distribution does not depend on rapidity of the produced particles. So at fixed $k_\perp^2$ it is flat while the rapidity is finite and so has a form shown in Fig. 24 with a central plateau and decrease in the two fragmentation regions. The plateau height grows with energy as $s^{\alpha_P(0)-1}$ and if $\alpha_P(0) = 1$ stays fixed.

Integrated over $k_\perp$ and rapidity $I(k)$ gives the average multiplicity μ times the total cross-section. Since at large s the fragmentation regions are relatively unimportant, we find

$$\mu(s)\sigma^{tot}(s) = \int_{y_1}^{y_2} dy \frac{d^2 k_\perp}{8\pi^3} I(k) = const\, s^{\alpha_P(0)-1} \ln \frac{s}{< |k_\perp|^2 >} \tag{231}$$

where we used (213) for the limits of integration over rapidity y_1 and y_2 and substituted $|k_\perp|^2$ by its average value $< |k_\perp|^2 >$. Under the same approximation obviously

$$\sigma^{tot}(s) = const\, s^{\alpha_P(0)-1} \tag{232}$$

so that we get

$$\mu(s) = const\, \ln \frac{s}{s_0} \tag{233}$$

where s_0 is some fixed quantity with the dimension of energy squared.

2 Quantum Chromodynamics

2.1 Lagrangian and quantization

We recall that according to the modern thinking all hadrons are composed of quarks, which are fermions with spin 1/2 exiting in 6 species called flavours. They are grouped in three generations (u,d), (c,s) and (t,b) with the first quarks in each pair having the electric charge $2/3e$ and the second one $-1/3e$. The quarks u,d,s are light, with masses (current) 4,7 and 150 MeV. The rest are heavy, the masses of c-,b- and t-quarks being 1.5, 4.5 and 180 GeV. The hadrons are built of quarks of different flavours as composites of qqq, with half integer spin (baryons) or of $q\bar{q}$, with integer spin (mesons).

Already these first experimental conclusions gave reason to suspect that quarks possess some additional internal quantum number. Indeed the well- known resonance state Δ^{++} with spin 3/2 has to be composed from three u-quarks in a symmetric state, contrary to the Fermi-statistics. To remedy this contradiction it was assumed that quarks, apart from flavour, posses a new quantum number, colour, which allows to form a fully antisymmetric state of quarks of the same flavour. It is clear that the number of states of different colour has to be equal or larger than 3. All experimental consequences of this conjecture are consistent with the assumption that this number is exactly three ("red, yellow and blue"). The dynamic theory based on this assumption is naturally called Quantum Chromodynamics.

Thus the quark of any flavour exists in 3 different colour states: $q_{f,\alpha}$, $f = u, d, s, c, t, b$, $\alpha = 1, 2, 3$. It is assumed that colour itself does not show in strong interactions. In other word, the theory is invariant under unitary transformation of quark wave function with different colours. The group of such transformation with determinant equal to unity is $SU(3)_c$. It does not influence flavour at all and is supposed to be an exact symmetry of the world.

The basic $SU(3)_c$ spinor q_α transforms under infinitesimal transformations of the group as

$$\delta q = it^a\xi^a q \equiv i(tf)q, \quad a = 1, 2, ...8 \tag{234}$$

Here xi^a are 8 small (real) parameters of the transformation and t^a are 8 generators of the $SU(3)_c$ in the spinor representation. They obey the standard commutation rules

$$[t^a, t^b] = if^{abc}t^c \quad a, b, c = 1, 2, ...8 \tag{235}$$

where f^{abc} are fully antisymmetric structure constant of the $SU(3)$ and normalized according to

$$\mathrm{Tr}\{t^at^b\} = \frac{1}{2}\delta_{ab}, \quad a = 1, 2, ...8 \tag{236}$$

We also define a scalar product in the color space as $AB = A^aB^a$. Invariance of the theory under the transformation of the $SU(3)_c$ group means conservation of 8 total colours of the world Q_α which obey the same commutation relations

$$[Q^a, Q^b] = if^{abc}Q^c \quad a, b, c = 1, 2, ...8 \tag{237}$$

and commute with the total Hamiltonian H

$$[Q_\alpha, H] = 0 \tag{238}$$

All hadrons are supposed to be singlets under $SU(3)_c$ having $Q_\alpha = 0$. They can be formed from quarks in two different ways

$$\bar{q}_\alpha q_\alpha, \quad \text{or} \quad \epsilon_{\alpha\beta\gamma} q_\alpha q_\beta q_\gamma$$

where q are quarks of any (possibly different) flavours and $\epsilon_{\alpha\beta\gamma}$ is a numerical completely antisymmetric tensor. The first structure gives baryons, the second gives mesons.

The Quantum Chromodynamics (QCD) emerges when one imposes a condition that colour is conserved not only globally but also locally. This implies that invariance under the $SU(3)_c$ group is to fulfill also for transformations with parameters $f^a(x)$ depending on the space-time point. As a result one obtains a non-Abelian gauge theory with 8 gauge bosons called gluons. Let us follow the standard path for constructing this sort of theory. The transformation of the basic quark spinor preserves its form under local transformations:

$$\delta q(x) = i(t\xi(x))q(x), \quad a = 1, 2, ...8 \tag{239}$$

However the derivative $\partial q(x)$ acquires an additional term:

$$\delta \partial q(x) = i(t\xi(x))\partial q(x) + i(t\partial\xi(x)) \cdot q(x) \tag{240}$$

To compensate one introduces 8 vector fields G^a_μ and defines a generalized derivative

$$D = \partial - ig(tG) \tag{241}$$

One chooses the form of the transformation of G in accordance with its global property as a colour vector and also to obtain simple transformation properties of $Dq(x)$

$$\delta G^a(x) = f^{abc}\xi^b(x)G^c(x) + \frac{1}{g}\partial\xi^a(x) \equiv D^{ab}\xi^b(x) \tag{242}$$

Here, similar to (241)

$$D^{ab} = \delta^{ab}\partial + f^{abc}G^c = \delta^{ab}\partial - ig(TG) \tag{243}$$

is the generalized derivative in the self-adjoint representation with the generators

$$T^a_{bc} = -if^{abc} \tag{244}$$

It is trivial to demonstrate that $Dq(x)$ transforms in the same way as $q(x)$, that is does not feel the x-dependence of $\xi(x)$

$$\delta Dq(x) = i(t\xi(x))Dq(x), \quad a = 1, 2, ...8 \tag{245}$$

Indeed

$$\delta Dq = i(t\xi)\partial q + i(t\partial\xi) \cdot q - ig(TG)i(t\xi)q - igt^a f^{abc}\xi^b G^c q - -i(t\partial\xi)q$$

The two terms with $\partial\xi$ obviously cancel and the rest give

$$g(tG)(t\xi)q + igf^{abc}t^a\xi^b G^c q + i(t\xi)\partial q \tag{246}$$

We change the order of t's in the first term

$$(tG)(t\xi) = (t\xi)(tG) + [(tG), (t\xi)] = (t\xi)(tG) + if^{cba}G^c\xi^b t^a$$

The last term here cancels the second term in (246) so we find

$$[i(t\xi)\partial + (t\xi)(tG) = i(t\xi)Dq$$

which gives (245). As a result we find that the product $\bar{q}Dq$ is invariant under the local transformations of the colour group.

We have only to find an appropriate Lagrangian density for the gluons themselves. The appropriate method is to construct quantities which transform in a simple manner under the local colour group "gauge transformations"), in the same way as under global transformation with constant parameters ξ^a. The result has also to involve derivatives in x of no more than second order. An immediate candidate is

$$X^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu \tag{247}$$

which is an antisymmetric tensor in the coordinate space and vector in the colour space. Its transformation under the local colour transformation contain additional terms with derivatives of $\xi(x)$:

$$\delta X^a_{\mu\nu} = f^{abc}\partial_\mu xi^b \cdot G^c\nu - f^{abc}\partial_\nu xi^b \cdot G^c\mu + \frac{1}{g}f^{abc}\partial_\mu\partial_\nu\xi^a - \frac{1}{g}f^{abc}\partial_\nu\partial_\mu\xi^a \tag{248}$$

The last two terms with the second derivative cancel, but terms with the first derivative remain. Let us compare this with the transformation of another global colour vector

$$Y^a_{\mu\nu} = f^{abc}G^b_\mu G^c_\nu \tag{249}$$

which is also an antisymmetric tensor in the coordinate space. Again leaving only terms with derivatives of $\xi(x)$ we find

$$\delta Y^a_{\mu\nu} = \frac{1}{g}f^{abc}\partial_\mu\xi^b G^c_\nu + \frac{1}{g}f^{abc}G_\mu\partial_\nu\xi^c \tag{250}$$

Comparing with (248) we see that in the difference

$$G^a_{\mu\nu} = X^a_{\mu\nu} - gY^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - gf^{abc}G^b_\mu G^c_\nu \tag{251}$$

all terms with derivatives of ξ cancel, so that $G^a_{\mu\nu}$ transforms under local transformations in exactly the same way as under global transformations, that is as a colour vector. As a result the product $G^a_{\mu\nu}G^{\mu\nu}_a$ is invariant under gauge transformations (and under Lorentz transformations) and can serve to construct the QCD Lagrangian density.

Together with the quark part, the Lagrangian density for QCD is constructed as

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \sum_f \bar{q}_f(i\hat{D} + m_f)q_f \tag{252}$$

where the sum goes over over flavours. From the start it is clear that flavours do not change in the strong interaction. Gluons are massless particles. Quarks emit and absorb gluons very much in the same way as charged particles emit and absorb photons in the Quantum Electrodynamics. However in contrast to the latter, gluons also interact between themselves, as seen from the structure of the square of $G^a_{\mu\nu}$: there appear triple and quartic interactions of gluons, the first containing a derivative.

Expression (252) is the Lagrangian of the classical QCD. To quantize the easiest way is to use the generating functional of the Green functions in terms of the functional integral

$$Z(j_i) = \int DGDq_fD\bar{q}_fe^{iS(G,q_f,\bar{q}_f)+iS_E(G,q_f,\bar{q}_f,j_i)} \tag{253}$$

Here the classical action S is

$$S(G,q_f,\bar{q}_f) = \int d^4xL \tag{254}$$

and the external action, depending on external sources j_i is

$$S_E(G,q_f,\bar{q}_f) = \int d^4xL_E \tag{255}$$

Both actions are invariant under local transformation of the $SU(3)_c$ group. Let us denote the fields generically as $u = \{G,q_f,\bar{q}_f\}$ and as u_ξ the fields gauge transformed with parameters $\xi^a(x)$ (not necessarily small). Then we have

$$S(u_\xi) = S(u) \quad S_E(u_\xi,j_i) = S_E(u,j_i) \tag{256}$$

The latter condition reflects the requirement that the physical significance have only gauge invariant Green functions. So L_E should depend only on colour invariant combinations of the field.

The invariance of the action under transformations with 8 arbitrary functions $\xi^a(x)$ means that the integrand in (253) in fact does not depend on 8 functions in the functional space and, as a consequence, the functional integral involves an infinite volume of integration over these functions as a factor and so is badly defined. To define it one has to restrict integrations prohibiting those which involve gauge transformations. Following Faddeev-Popov (FP)let us do it by requiring the fields to obey 8 conditions fixing the gauge:

$$F^a(u(x)) = C^a(x), \quad a = 1,...8 \tag{257}$$

with the requirement that the equation for $xi^a(x)$

$$F^a(u_\xi(x)) = C^a(x) \tag{258}$$

has $\xi_(x) = 0$ as the only solution. The functional integral over $\xi^a(x)$, as a functional of the fields u

$$\int D\xi\prod_x\delta(F^a(u_\xi(x) - C^a(x) = \Big(\det\frac{\delta F^a(x)}{\delta\xi^b(x}\Big|_{\xi=0}\Big)^{-1} \equiv \Delta^{-1}(u) \tag{259}$$

defines the so-called FP functional determinant. Obviously it is gauge invariant

$$\Delta(u_\xi) = \Delta(u) \tag{260}$$

Using (259) we introduce this determinant into the functional integral (253):

$$Z(j) = \int Due^{iS(u)+iS_E(u,j)}\Delta(u)\int D\xi\prod_x\delta\Big(F^a(u_\xi(x)) - C^a(x)\Big) \tag{261}$$

Now we pass to integration over the fields u_ξ. Using gauge invariance of both the action and the FP determinant we find

$$Z(j) = \int Due^{iS(u)+iS_E(u,j)}\Delta(u)\prod_x \delta\Big(F^a(u(x)) - C^a(x)\Big) \cdot \int D\xi \tag{262}$$

The gauge volume $\int D\xi$ is separated as a constant factor and can be dropped. Obviously the remaining functional integral is independent of the choice of function $C^a(x)$. So we may integrate over $C^a(x)$ with an arbitrary weight factor $\exp iS_1(C) = \exp i \int d^4xL(C^a(x))$ to obtain

$$Z(j) = \int Due^{iS(u)+iS_1(F(u))+iS_E(u,j)}\Delta(u) \tag{263}$$

Finally we use the well known result that integration over anticommuting fields $\chi^a(x)$ and $\bar\chi^a(x)$:

$$\int D\chi D\bar\chi e^{i\int d^4xd^4y\bar\chi^a(x)M^{ab}(x,y)\chi^b(y)} = const \cdot \det M^{ab}(x,y) \tag{264}$$

This allows to substitute the FP determinant by integration over auxiliary anticommuting fields $\chi^a(x)$ and $\bar\chi^a(x)$ called ghosts:

$$Z(j) = \int DuD\chi D\bar\chi e^{iS(u)+iS_1(F(u))+iS_2(u,\chi,c\bar{h}i)+iS_E(u,j)} \tag{265}$$

where

$$S_2 = \int L_2, \quad L_2 = \bar\chi^a(x)\frac{\delta F^a(x)}{\delta\xi^b(x)}\chi^b(x) \tag{266}$$

This expression gives the final generating functional for the QCD.

The concrete choice of function F, the choice of gauge, may be made differently. The standard relativistic invariant choice ('Lorentz gauge') is

$$F^a = \partial G^a \tag{267}$$

As a result $\delta F^a = (1/g)\partial D^{ab}\xi^b$, so that $M^{ab} = (1/g)\partial D^{ab}$ Standardly one rescales $\chi \to \sqrt{g}\chi$ $\bar\chi \to \sqrt{g}\bar\chi$ to find

$$L_2 = \bar\chi\partial D\chi = \bar\chi\partial^2\chi - gf^{abc}\bar\chi^a\partial(G^c\chi^b) \tag{268}$$

From this one concludes that 8 ghosts are massless as the gluons and interact with the latter emitting and absorbing gluons with a derivative interaction. As to S_1 the standard choice leading to a renormalizable theory is

$$S_1 = \int d^4xL_1, \quad L_1 = -\frac{1}{2\alpha}F^aF^a = -\frac{1}{2\alpha}\partial G^a\partial G^a \tag{269}$$

where α is a numerical parameter.

2.2 Feynman rules. General properties and problems

The Feynman rules for perturbation theory are readily read from different parts of the QCD Lagrangian. The relevant propagators and verteces are illustrated in Fig. 25.

The propagators for quarks and ghosts are simple and not different from any quantum field theory.

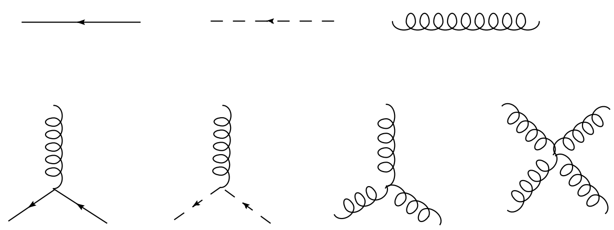

Figure 25: Elements of the diagrammatic technique

The propagator of the quark of flavour f and momentum k is

$$D_q(k) = \frac{-i}{m_f - \hat{k}} \tag{270}$$

The propagator of the ghost of colour a and momentum k is

$$D_{gh}^{ab} = \frac{i\delta_{ab}}{k^2} \tag{271}$$

To find the propagator of the gluon D_g we have to study the part of the Lagrangian quadratic in the gluon field

$$\frac{1}{2}G_\mu^a(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)G_\nu^a + \frac{1}{2\alpha}G_\mu^a\partial^\mu\partial^\nu G_\nu^a = \frac{1}{2}G_\mu^a\Big(\partial^2 g^{\mu\nu} - (1-1\alpha)\partial^\mu\partial^\nu\Big)G_\nu^a \tag{272}$$

By definition of the propagator this is just

$$-\frac{1}{2}G_\mu^a\Big(iD_g^{-1}\Big)^{\mu\nu}G_\nu^a$$

Passing to the momentum space we find that the inverse propagator of the gluon with momentum k is

$$\Big(iD_g^{-1}\Big)^{\mu\nu}(k) = k^2 g^{\mu\nu} - \Big(1 - \frac{1}{\alpha}\Big)k^\mu k^\nu \tag{273}$$

To invert the matrix of the inverse propagator we introduce projectors onto the spaces parallel and orthogonal to vector k:

$$P_\perp^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \quad P_\|^{\mu\nu} = \frac{k^\mu k^\nu}{k^2} \tag{274}$$

They are orthogonal and squared give the projectors themselves. We have found

$$iD_g^{-1}(k) = k^2 P_\perp + \frac{1}{\alpha}k^2 P_\| \tag{275}$$

which gives for the matrix of the propagator

$$iD_g(k) = \frac{1}{k^2}P_\perp + \frac{\alpha}{k^2}P_\| \tag{276}$$

Explicitly with color indexes

$$D_{\mu\nu}^{ab}(k) = -i\delta_{ab}\Big\{\frac{g_{\mu\nu}}{k^2} - (1-\alpha)\frac{k^\mu k^\nu}{k^4}\Big\} \tag{277}$$

The simplest choice is $\alpha = 1$ which corresponds to frequently used Feynman gauge:

$$D^{ab}_{\mu\nu}(k) = -i\delta_{ab}\frac{g_{\mu\nu}}{k^2} \tag{278}$$

From the theoretical point of view the transverse gauge $\alpha = 0$ has some advantages

$$D^{ab}_{\mu\nu}(k) = -i\delta_{ab}P^{\perp}_{\mu\nu}(k) \tag{279}$$

The interaction vertexes are read from the corresponding non-linear terms in the Lagrangian. The term $\bar{q}_f i\hat{D}q_f$ generate the vertex V_{qg} for the interaction of the quarks with the gluon of color a and polarization μ

$$V^a_\mu = g\gamma_\mu t^a \tag{280}$$

The term $-gf^{abc}\bar{\chi}\partial^\mu(\chi^b G^c_\mu)$ gives rise to the vertex for the interaction of the gluon of color b with the incoming ghost of colour c and outgoing ghost of colour a and momentum p

$$V^{abc}_\mu = igf^{abc}p_\mu \tag{281}$$

Note that this vertex is not symmetric in the incoming and outgoing ghosts. The triple gluon interaction comes from the interference of linear and quadratic terms in the product GG. Assuming all the three gluons outgoing with momenta, colour and polarizations (q, b, μ),(p, a, λ) and $(r, c\nu)$ we find a symmetric triple gluon vertex

$$V^{abc}_{\nu\lambda\mu} = igf^{abc}[(p-q)_\nu g_{\nu\lambda} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda}] \tag{282}$$

There finally is the quartic gluon interaction. It does not depend on the gluon momenta. If colours and polarizations are (a, λ), (b, μ), (c, ν) and $(d\sigma)$ then the quartic verex is

$$V^{abcd}\lambda\mu\nu\sigma = -ig^2[f^{abe}f^{cde}(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu})$$

$$+f^{ace}f^{bde}(g_{\lambda\mu}g_{\nu\sigma} - g_{\lambda\sigma}g_{\mu\nu}) + f^{ade}f^{cbe}(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\mu}g_{\sigma\nu})] \tag{283}$$

Inspection of these rules allows to make the first conclusions about the sort of quantum field theory which is the QCD. By construction (gauge fixing) the theory is now not invariant under local gauge transformations but preserves the invariance under global transformation of the $SU(3)_c$ group. However there remains a strong consequence of the initial invariance under local gauge transformations: all couplings are fixed by it. All quarks of any flavour and the triple gluon interaction have the same coupling constant g and the quartic gluon coupling has the strength g^2. Unlike photons, gluons interact with each other. Also new auxiliary scalar particles obeying the Fermi statistics, ghosts, appear in the theory and interact with gluons.

As a result of the presence of self-interacting zero-mass particles the theory possesses strong infrared divergences, which reveal themselves when one tries to go to the mass-shell for gluons and ghosts. It is important that the off-mass-shell Green function are free from infrared divergence. However when one tries to set external masses of gluons and ghosts to zero, the corresponding quantities diverge. In fact consider the integral which appears in the gluon self-mass (see Fig. 26)

$$\int \frac{d^4q}{q^2(q-k)^2} = \int_0^1 dx \int \frac{d^4q}{[x(q-k)^2 + (1-x)q^2]^2} = \int_0^1 dx \int \frac{d^4q}{[(q^2 + x(1-x)k^2]^2}$$

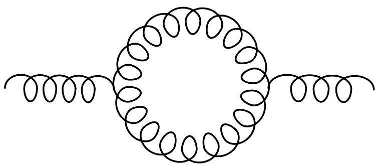

Figure 26: A contribution to the gluon self-mass

This integral is logarithmically divergent in the untraviolet at any value of k^2 but this divergence is eliminated by the mass renormalization. However at $k^2 = 0$ there additionally appears an infrared divergence, so that even renormalized self-mass does not exist on the mass shell. The behaviour of full Green function for gluons and ghosts around their mass shell in absence of interaction $k^2 = 0$ is in fact unknown. It is generally postulated that it is such that the massless gluon and ghosts are excluded from the physical spectrum and so cannot be observed in nature. This property is assumed to be also true for any objects with a non-trivial colour and is called 'confinement of colour'. Coloured particles can only appear as intermediate virtual particles and so are confined inside the observed hadrons. The property of confinement has not been proved so far but is supported by the experimental facts.

From the point of view of ultrviolet divergencies, the QCD belongs to the class of renormalizable quantum field theories. It possesses a finite number of skeleton divergent diagrams, of the same type as in the Quantum Electrodynamics. These are the self-masses of gluons and ghosts with the order of divergence $\omega = 2$, the quark self-mass with $\omega = 1$, vertex parts for the triple and quartic interaction of gluons and gluons with ghosts with $\omega = 1$ and for the interaction of quarks with gluons with $\omega = 0$. They all are eliminated by renormalization. However this procedure in the QCD has certain problems.

Technical problems arise, first,from the impossibility to use the gluon and ghost mass shells as the subtraction points in the renormalization procedure, since the self-masses and vertex parts do not exist there. This obstacle can be easily overcome by noting that in fact any value of k^2 can be used for renormalization as well. E.g. one may fix the renormalized Green function by the condition that

$$k^2 G_R(k^2) = 1, \quad \text{at} \quad k^2 = -\mu^2 \tag{284}$$

with μ^2 is a suitably chosen renormalization point, and similarly for vertex parts. The second problem is related to the necessity to preserve the consequences of gauge invariance at the intermediate stage when the theory is regularized at large momenta. The best know way (which also solves the first problem) is to use the so-called dimensional regularization, when the momentum integrals are considered to be in the dimension $d = 4 - \epsilon$. The ultraviolet divergence then are revealed as pole terms at $\epsilon = 0$. The renormalized quantities then can be defined as the result of dropping all pole terms.

However there is a more profound problem related to the preservation of gauge invariance in renormalization. Roughly speaking we have to guarantee that all the renormalized coupling constants be related among themselves in the same manner as in the unrenormalized theory (in particular, be equal for triple verteces). This is not trivial, since we encounter several different vertex parts and it is not at all clear that one can eliminate ultraviolet divergence in all of them renormalizing only one common coupling constant. As we shall see this is indeed possible, since the divergent terms in different vertex parts are related to each other by gauge invariance. This point will be studied in the following sections.

2.3 BRST invariance

We recall the generating functional for the QCD:

$$Z(j) = \int DuD\chi D\bar{\chi} e^{iS^{tot}} \tag{285}$$

$$S^{tot} = \int d^4x, \quad S^{tot} = S + S_1 + S_2 + S_E = \int d^4x(L + L_1 + L_2 + L_E) \tag{286}$$

where

$$L = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \sum_f \bar{q}_f (i\hat{D} + m_f) q \tag{287}$$

$$L_1 = -\frac{1}{2\alpha} F^a F^a, \quad L_2 = g\bar{\chi}^a \frac{\delta F^a}{\delta \xi^b}_{\xi=0} \chi^b \tag{288}$$

Action S^{tot} is no more gauge invariant due to the fixation of gauge. However it turns out that it possesses an invariance under gauge transformations of a very special sort, the BRST transformations, which also include the ghosts. This invariance is more conveniently expressed if we somewhat transform this action to linearize it in F^a. We present

$$e^{iS_1} = \exp i \int d^4x \Big(-\frac{1}{2\alpha} F^a F^a \Big) = \int Dh e^{i\tilde{S}_1} \tag{289}$$

with

$$\tilde{S}_1 = \int d^4x \tilde{L}_1, \quad \tilde{L}_1 = \frac{1}{2}\alpha h_a^2 + h_a F_a \tag{290}$$

Then the generating functional is

$$Z(j) = \int DuD\chi D\bar{\chi} Dh e^{i\tilde{S}^{tot}} \tag{291}$$

where $\tilde{S}^{tot}$ differs from S^{tot} by the substitution of L_1 by $\tilde{L}_1$.

For the physical fields G and q the BRST transformation is just the usual gauge transformation with a gauge function

$$\xi = g\chi\epsilon \tag{292}$$

where ϵ is an anticommuting number, which anticommutes with itself and all anticommuting fields, q_f, $\bar{q}_f$, χ and $\bar{\chi}$. Naturally the product $\chi\epsilon$ commutes with all fields as any gauge function should do. So under the BRST transformation:

$$\delta G = D\chi\epsilon, \quad \delta q = ig(t\chi)\epsilon q = -ig(t\chi)q\epsilon \tag{293}$$

Additionally one postulates transformation for the ghost fields and h:

$$\delta h = 0, \quad \delta\bar{\chi} = h\epsilon, \quad \delta\chi^a = \frac{1}{2} g f^{abc} \chi^b \chi^s \epsilon \tag{294}$$

Before we find that action $\tilde{S}^{tot}$ is invariant under these transformations we can demonstrate that they are nilpotent. Let for any field u, by definition

$$\delta u = su\epsilon \tag{295}$$

where s plays the role of the generator of the BRST transformations. Then we it can be demonstrated that

$$s^2 = 0 \tag{296}$$

by studying

$$\delta(su) = s^2 u\epsilon \tag{297}$$

Now we note that

$$\delta(\bar{\chi}F) = \delta\bar{\chi} \cdot F + \bar{\chi}\delta F \tag{298}$$

However

$$\delta F^a = \frac{\delta F^a}{\delta \xi^b} g\chi\epsilon$$

so that

$$\delta(\bar{\chi}F) = h\epsilon F + g\chi^a \frac{\delta F^a}{\delta \xi^b}\chi\epsilon$$

and

$$s(\bar{\chi}F) = hF + g\chi^a \frac{\delta F^a}{\delta \xi^b}\chi \tag{299}$$

We also have

$$\delta(\bar{\chi}h) = h\epsilon h = h^2\epsilon, \quad s(\bar{\chi}h) = h^2 \tag{300}$$

This shows that

$$\tilde{L}_1 + L_2 = s\Big(\bar{\chi}F + \frac{1}{2}\alpha\bar{\chi}h\Big) \tag{301}$$

so that

$$\tilde{L}^{tot} = L + L_E + s\Big(\bar{\chi}F + \frac{1}{2}\alpha\bar{\chi}h\Big) \tag{302}$$

Since $L + L_E$ are gauge invariant we have $s(L + L_E) = 0$. Nilpotency of s allows to prove that

$$s\tilde{L}^{tot} = s^2\Big(\bar{\chi}F + \frac{1}{2}\alpha\bar{\chi}h\Big) = 0 \tag{303}$$

Thus we have demonstrated that the theory is invariant under the BRST transformations.

2.4 Renormalization

2.4.1 Formal procedure and problems

Both the fields and the parameters which appear in the generating functional are not the observable ones but "bare". The physical, renormalized fields are related to them by renormalization. In absence of ultraviolet divergencies one can manipulate with bare quantities and aftewards determine the physical quantities considering concrete reactions and comparing the resulting predictions with experimental data. Divergencies do not allow this procedure, since the bare fields and parameters turn out to be divergent. Renormalization consists in expressing the theory directly in terms of renormalized quantities and avoid using the bare ones altogether.

The first step in renormalization procedure is to pass to renormalized fields, which are related to the bare ones multiplicatively. For any field u one defines the renormalized field by

$$u = Z_u^{1/2} u_R, \quad u = G, q, \bar{q}, \chi \tag{304}$$

(by tradition the numerical factor is called $Z^{1/2}$). In the future we shall have to study the Green function of our basic fields u (non gauge invariant) for which purpose one chooses

$$L_E = \sum_u u j_u, \quad u = G, q, \bar{q}, \chi, \bar{\chi} \tag{305}$$

Renormalization of the fields is to be accompanied by the inverse renormalization of the external currents

$$j_{Ru} = Z_u^{1/2} j_u \ \text{ so that } \ L_E = \sum_u u_R j_R \tag{306}$$

It has also be accompanied by the renormalization of the gauge parameter α to preserve the choice of gauge

$$\alpha = Z_G \alpha_R \tag{307}$$

In terms of renormalized field the QCD Lagrangian takes the form

$$L^{tot} = L_0 + L_I + L_E \tag{308}$$

where L_0 is a part quadratic in fields

$$L_0 = -\frac{1}{4} Z_G (\partial \mu G_{R\nu} - \partial_\nu G_{R\mu})^2 - -\frac{1}{2\alpha} (\partial G)^2 + \sum_f Z_{q_f} \bar{q}_{fR} (i\hat{\partial} + m_f) q_{fR} + Z_\chi \bar{\chi}_R \partial^2 \chi_R \tag{309}$$

L_I is the interaction part

$$L_I = -\frac{1}{2} Z_G^{3/2} g f^{abc} (\partial_\mu G_{R\mu} - \partial_\nu G_{R\mu}) G_R^\mu G_R^\nu - \frac{1}{4} Z_G^2 g^2 [f^{abc} G_{R\mu} G_{R\nu}]^2$$
$$+ \sum_f Z_{q_f} Z_G^{1/2} g \bar{q}_{fR} (t\hat{G}_R) q_{fR} - Z_\chi Z_G^{1/2} f^{abc} \bar{\chi}_R^a \partial (G_R^b \chi_R^c) \tag{310}$$

and L_E is the external action given by (306).

Next step is renormalization of the quark masses. One puts for each flavour

$$Z_q m = m_R + \delta m \tag{311}$$

The (infinite) quantity δm is the mass renormalization constant.

Finally one has to renormalize the coupling. In contrast to the QED this can be made in different ways, since the coupling g appears in different places. Typically one takes the coefficient in a given interaction and defines it as the renormalized coupling g_R times an infinite vertex renormalization constant Z_1 (traditional notation). E.g. using the triple gluon interaction one can define

$$g Z_G^{3/2} = g_R Z_{1G} \tag{312}$$

Alternatively one can use the quark-gluon interaction defining

$$g Z_q Z_G^{1/2} = g_R Z_{1q} \tag{313}$$

or the ghost-gluon interaction

$$gZ_\chi Z_G^{1/2} = g_R Z_{1\chi} \tag{314}$$

Obviously with the fixed and unique renormalized coupling constant g_R different Z_1 are not independent but related to each other a

$$Z_{1G}Z_G^{-1} = Z_{1q}Z_q^{-1} = Z_{1\chi}Z_\chi^{-1} \tag{315}$$

With any definition one the splits $Z_u = 1 + (Z_u - 1)$ and $Z_1 = 1 + (Z_1 - 1)$ and assumes that quantities δm, $Z_u - 1$ and $Z_1 - 1$ are of the order in g_R higher than the first. Then in perturbation theory in higher orders one finds counterterms proportional to these quantities which may be chosen to cancel ultraviolet divergencies found in this order.

One immediately sees a problem mentioned before. The theory has divergencies coming from each type of interaction, that is, in each vertex part in the perturbation theory. To cancel all these divergencies one has only one independent renormalization constant Z_1 defined by a particular vertex part. However different vertex parts will have different divergent contributions which depend on the way one regularizes the theory and defines their finite parts. It is not clear how one can eliminate all these divergencies introducing a single counterterm and moreover guarantee relations (315) It is clear that if one introduces indiscriminately separate counterterms for each vertex choosing them from only the condition to eliminate the diveregence then the relation (315) will be generally violated. As a result the renormalized coupling constants for different vertexes will be different and gauge invariance will be blatantly broken.

We are going to demonstrate that there exists such a choice of the renormalization procedure and subtraction points which, on the one hand, guarantees fulfillment of relations (315) and, on the other, eliminates all ultraviolet divergencies. We shall choose the vertex renormalization constants to satisfy (315) from the start and show that we can remove all the ultraviolet divergencies in all vertex parts. This is possible only because there exist certain relations between the divergent terms in different vertex parts, so that in fact there is only one independent divergence which can be eliminated by a single renormalization constant Z_1. This is a consequence of a general relation between different Green function in the theory, which generalize the well-known Ward identities in the QED, and bear the name of Slavnov-Tailor identities.

2.4.2 Slavnov-Tailor identities

To derive the Slavnov-Tailor identities we make use of the BRST invariance. Consider the generating functional with the external action linear in fields:

$$Z(j) = \int DuD\chi d\bar{\chi}e^{iS+iS_1+iS_2+iS_E} \tag{316}$$

where

$$L_E = jG + \bar{\eta}q + \bar{q}\eta + \bar{\chi}\zeta + \bar{\zeta}\chi \tag{317}$$

and $j, \bar{\eta}, \eta, \bar{\zeta}, \zeta$ is a set of external sources which generate Green function for the fields $G, q, \bar{q}, \chi, \bar{\chi}$ respectively. Note that this external action and with it the whole generating functional is not gauge and BRST invariant. Now we BRST transform the fields over which the integration is done in $Z(j)$. Its Jacobian is unity, since all transformation except that of χ are shifts of the variables. The matrix of the transformation of the field χ is obviously

$$\frac{d\chi'^a}{d\chi^b} = \delta_{ab} - gf^{acb}\chi^c\epsilon \tag{318}$$

which corresponds to the unitary rotation in the colour space with angle $\chi^c\epsilon$ whose determinant is unity. After the transformation actions S, S_1 and S_2 do not change and the only part which changes is S_E. So we find an identity

$$\int DuD\chi D\bar{\chi}e^{iS+iS_1+iS_2+iS_E}\Big(e^{i\delta S_E} - 1\Big) = 0 \tag{319}$$

For brevity we denote the functional averaging with weight $\exp(iS + iS_1 + iS_2 + iS_E)$ by the symbol $< ... >$:

$$< \Phi(u, \chi\bar{\chi}) >\equiv \int DuD\chi D\bar{\chi}\Phi(u, \chi\bar{\chi})e^{iS+iS_1+iS_2+iS_E} \tag{320}$$

We also omit the sign of integration over the coordinate space except in case when the coordinates are explicitly indicated:

$$\delta S_E = \int d^4x\delta L_E = jD\chi\epsilon + ig\bar{\eta}(t\chi)\epsilon q - ig\bar{q}(t\chi)\epsilon\eta - \frac{1}{\alpha}\partial G\epsilon\zeta + \frac{1}{2}g\bar{\zeta}^a f^{abc}\chi^b\chi^c\epsilon \tag{321}$$

From (319) we find

$$< jD\chi - ig\bar{\eta}(t\chi)q + ig\bar{q}(t\chi)\eta + \frac{1}{\alpha}\partial G\zeta + \frac{1}{2}g\bar{\zeta}^a f^{abc}\chi^b\chi^c >= 0 \tag{322}$$

Identity (322) is true for any values of external sources. We differentiate over $\zeta(x)$ and out $\zeta = z\bar{e}ta = 0$ afterwards to obtain

$$< -i\bar{\chi}(x)\Big(jD\chi - ig\bar{\eta}(t\chi)q + ig\bar{q}(t\chi)\eta\Big) + \frac{1}{\alpha}\partial G(x) >= 0 \tag{323}$$

This is the generating functional (unrenormalized) for the Slavnov-Tailor identities.

To renormalize it we have to take into account some important immediate consequences. Differentiating over $j(y)$ and putting then all sources zero we find

$$< iG(y)\frac{1}{\alpha}\partial G(x) - i\bar{\chi}(x)D\chi(y) >= 0 \tag{324}$$

Differentiating in y:

$$< \frac{1}{\alpha}\partial G(y)\partial G(x) >=< \bar{\chi}(x)\partial D\chi(y) >= -i\delta^4(x-y) \tag{325}$$

The latter equality follows from the fact that all dependence on $\bar{\chi}$ and χ in the Lagrangian has the form $\bar{\chi}(x)\partial D\chi(x)$ so that taking as variables $\bar{\chi}$ and $\chi' = \partial D\chi$ we shall have an integral

$$\int D\chi' D\bar{\chi}\chi'(y)\bar{\chi}(x)e^{i\int d^4x\bar{\chi}\chi'} = i\delta^4(x-y)$$

In the momentum space we find for the full gluon Green function $\Delta^{tot}_{\mu\nu}$

$$-i\frac{1}{\alpha}k^\mu k^\nu\Delta^{tot}_{\mu\nu} = -i$$

So that presenting the Green function as a sum of its transverse and longitudinal parts

$$\Delta^{tot}(k) = \Delta_{\perp}^{tot}(k^2)P_{\perp}(k) + \Delta_{L}^{tot})k^2)P_{||}(k) \tag{326}$$

we find

$$\frac{1}{\alpha}k^2\Delta_L^{tot} = 1, \quad \text{or} \quad \Delta_L^{tot} = \frac{\alpha}{k^2} = \Delta_L \tag{327}$$

where Δ is the Green function in absence of interaction (the gluon propagator) In other words, the longitudinal part of the gluon Green function is not changed by the interaction.

From this, as in QED, it follows that the gluon polarization operator $Pi(k)$ defined by the matrix Dyson equation

$$\Delta^{tot}(k) = \Delta(k) + \Delta(k)\Pi\Delta^{tot}(k) \tag{328}$$

is purely transverse:

$$\Pi^{\mu\nu}(k) = \Pi(k^2)P_{\perp}^{\mu\nu} \tag{329}$$

Also since $P_{\perp}^{\mu\nu}(k)$ has a pole at $k^2 = 0$ the polarization operator has to have a zero at $k^2 = 0$

$$\Pi(k^2)_{k^2=0} = 0 \tag{330}$$

As a result, in absence of infrared divergencies the gluon mass remains equal to zero after introduction of interaction since from the Dyson equation

$$\Delta^{tot}(k^2) = \frac{1}{k^2 - \Pi(k^2)} \tag{331}$$

and has a zero at $k^2 = 0$.

Taking into account that the longitudinal part of the Gluon field is in a sense trivial, we split the external gluon current into the transverse and longitudinal parts.

$$j = j_{\perp} + J_L, \quad j_{\perp}^{\mu} = P_{\perp}^{\mu\nu}j_{\nu}, \quad j_L^{\mu} = P_L^{\mu\nu}j_{\nu} \tag{332}$$

Of course in the coordinate space both projectors become non-local. E.g.

$$P_L^{\mu\nu} = \partial^{\mu}\partial^{\nu}\partial^{-2} \tag{333}$$

where ∂^{-2} is an integral operator with the kernel $\Delta_0(x-y)$ which is the propagator of the massless scalar field. So we find

$$jD\chi = j_{\perp}D\chi + j_L D\chi \tag{334}$$

From the second term we find a contribution inside 323

$$< i\bar{\chi}(x)\Big(\partial j \cdot \partial^{-2}\partial D\chi\Big) > \tag{335}$$

However we have seen that $< \bar{\chi}(x)\partial D\chi(y) >= -i\delta^4(x-y)$, so that (reflong1) becomes

$$\int d^4yd^4z\partial j(y) \cdot \partial^{-2}(y-z)\delta^4(x-z) = \int d^4yj(y)\partial\Delta_0(y-x)$$

This is a numerical function depending only on the longitudinal current, which may be safely put to zero.

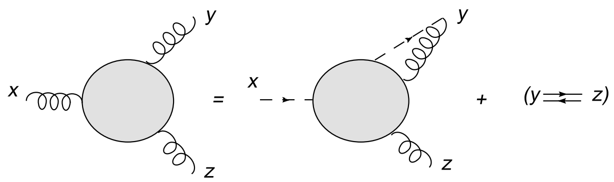

Figure 27: Graphical content of the Slavnov-Tailor identity (339)

In the part of (334) with the transverse current we may drop the derivative in D:

$$j_\perp D\chi = -j^a_\perp g f^{abc} G^c \chi^b \tag{336}$$

so that (323) takes the form

$$< \frac{1}{\alpha}\partial G(x) + ig\bar\chi(x)\Big(j^a_\perp f^{abc} G^c \chi^b + i\bar\eta(t\chi)q - i\bar q(t\chi)\eta\Big) >= 0 \tag{337}$$

Now we are ready to renormalize this identity. We recall that the products of the field and the corresponding source conserve their form in the renormalized quantities. So $jG = j_R G_R$, $\bar\eta q = \bar\eta_R q_R$ and $\bar q\eta = \bar q_R \eta_R$. The first term in (337) is rewritten as $Z_g^{-1/2} G_R(x)/\alpha_R$. The pair $\bar\chi\chi$ in the rest part gives a factor Z_χ in terms of renormalized fields. Finally we define the renormalized charge from the ghost-gluon interaction (314) which gives a factor for this part $Z_\chi^{-1} z_g^{-1} Z_{1\chi}$. As a result, multiplying by $Z_g^{1/2}$ we find the renormalized generating functional as

$$< \frac{1}{\alpha_R}\partial G_R(x) + ig_R Z_{1\chi}\bar\chi_R(x)\Big(j^a_{R\perp} f^{abc} G^c_R \chi^b_R + i\bar\eta_R(t\chi_R)q_R - i\bar q_R(t\chi_R)\eta_R\Big) >= 0 \tag{338}$$

In the following we shall use almost exclusively the renormalized fields and omit subindex R.

Differentiating various times (338) in external sources and then putting them to zero one obtains an (infinite) set of Slavnov-Tailor identities for the QCD. We shall only derive the simplest of them, which are necessary to prove the renormalizability of the theory.

Differentiating (338) over $j_\perp$ twice, at points y and z and putting then the sources equal to zero we find

$$< \frac{1}{\alpha}\partial G^d(x)\cdot G^a_\perp(y) G^b_\perp(z) >$$
$$= igZ_{1\chi} < \bar\chi^d(x)\Big(f^{adc}G^c(y)\chi^d(y)G^b_\perp(z) + f^{bdc}G^c(z)\chi^d(z)G^a_\perp(y)\Big) >= 0 \tag{339}$$

The first term in this identity is the standard triple gluon Green function with two transverse and one longitudinal gluons. The two rest terms correspond to a Green function in which the longitudinal gluon is substituted by an incoming ghost, one of the transverse gluons is retained and the other substituted by a pair gluon-ghost at the same point. This point looks like the ghost-gluon interaction vertex but without the derivative in the ghost line. The identity (338) is illustrated in Fig.27. Its contents in the lowest order is shown in Fig.28 and in the next, third order, in Fig.29.

It is remarkable, that in contrast to the QED, this Slavnov-Tailor identity relates not the physical Green functions among themselves but rather a physical (triple gluon) Green function with some artificial structures which do not appear in the perturbation theory at all.

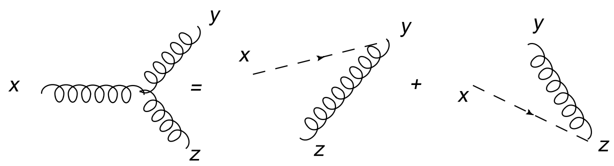

Figure 28: Leading order of the Slavnov-Tailor identity (339)

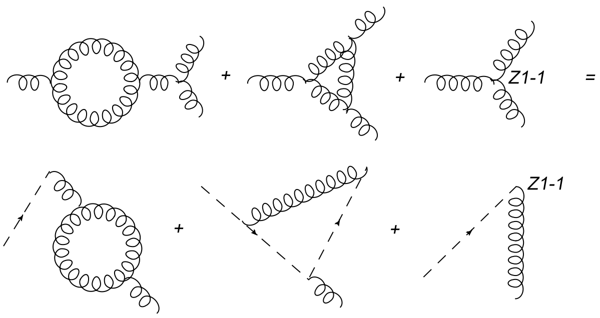

Figure 29: Next-to-leading order of the Slavnov-Tailor identity (339)

One can obtain a similar identity for the Green function with quarks. Differentiating (338) over $\eta(y)$ from the right and over $\bar{\eta}(z)$ from the left and putting then the sources equal to zero we get

$$< \frac{1}{\alpha}\partial G^d(x)\bar{q}(y)q(z) >$$

$$= igZ_{1\chi} < \bar{\chi}^d(x)\bar{q}(y)\Big(t\chi(y)\Big)q(z) - \bar{\chi}^d(x)\bar{q}(y)\Big(t\chi(z)\Big)q(z) >= 0 \tag{340}$$

The first term is the standard Green function with two quark legs and one longitudinal gluon leg. In the rest two terms the longitudinal gluon is again substituted by the ghost, one of the quark is retained and the other is substituted by a pair ghost-quark at the same point. Graphically (340) is illustrated in Fig.30. Its contents in the lowest order is shown in Fig.31 and in the third order in Fig. 32.

Now we shall use these Slavnov-Tailor identities to demonstrate renormalizability of the QCD.

2.4.3 Renormalizability of the QCD

Our main goal is to show that having at our disposal only one renormalization constant for all vertex parts we can eliminate all the vertex divergencies. We shall choose for this single renormalization constant $Z_{1\chi}$ corresponding to the gluon-ghost vertex part. It will determine the common renormalized coupling constant according to (314). Then the rest

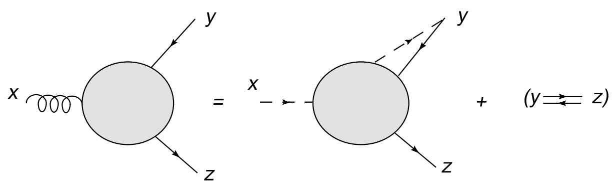

Figure 30: Graphical content of the Slavnov-Tailor identity (340)

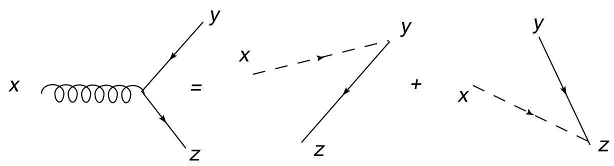

Figure 31: Leading order of the Slavnov-Tailor identity (340)

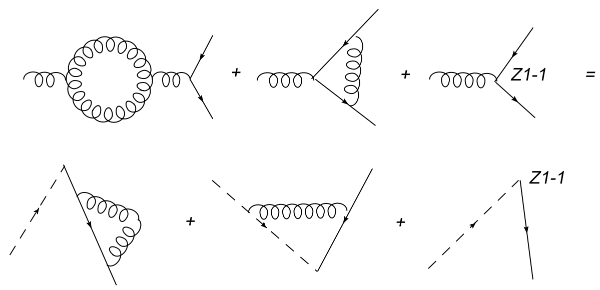

Figure 32: Next-to-leading order of the Slavnov-Tailor identity (340)

of vertex renormalization constant will be automatically found from relation (315). We shall demonstrate that then divergencies in all vertex parts will be eliminated. We shall limit ourselves with the lowest non-trivial order when the divergent contributions come from only one loop.

The proof is simplest in the transverse gauge $\alpha = 0$. In this gauge in the gluon-ghost vertex the derivative acting on the outgoing ghost propagator can be transferred on the incoming ghost propagator, since the derivative acting on the gluon propagator is zero. Therefore in diagrams with ghosts for each external pair χ and $\bar{\chi}$ one can separate two external momenta, which effectively decreases the order of divergence by two units. As a result, all diagrams with external ghosts become convergent, except the ghost self-mass diagram. The ghost self mass $\Sigma_\chi(k^2)$ acquires the form

$$\Sigma(k^2) = k^2 F_\chi(k^2) \tag{341}$$

where $F(k^2)$ diverges only logarithmically. This divergence is compensated by the renormalization of the gluon wave function, which introduces interaction in the form $(Z_{chi} - 1)\bar{\chi}\partial^2\chi$ and thus gives an additional contribution to the self-mass

$$\delta\Sigma_\chi(k^2) = k^2(Z_\chi - 1) \tag{342}$$

Requirement of the finiteness of the sum $\Sigma_\chi + \delta\Sigma_\chi$ determines the (divergent part of) Z_χ. Its finite part is in principle arbitrary and is fixed by the concrete definition of the renormalized mass (e.g. by the choice of the renormalization point μ^2 in the definition (284). Note that the renormalized ghost mass formally remains equal to zero.

Other self-masses are renormalized in the same manner as in the QED. As we have seen, the gluon self-mass is purely transverse:

$$\Sigma_G^{\mu\nu}(k) = \Big(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\Big)\Pi(k^2) \tag{343}$$

where $\Pi(0) = 0$, so that we can substitute $\Pi(k^2) \to \Pi(k^2) - \Pi(0) = k^2 F_G(k^2)$ This liquidates the formal quadratic divergence in Π leaving only a logarithmic one. This is eliminated by the gluon wave function renormalization which adds a contribution to the self mass

$$\delta\Sigma_G^{\mu\nu}(k) = \Big(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\Big)(Z_G - 1)k^2 \tag{344}$$

One then chooses the divergent part of Z_G to cancel the logarithmic divergence in $F_G(k^2)$. Its finite part remains arbitrary and is fixed by the concrete renormalization conditions.

The quark self mass, as is well-known, has two different logarithmically divergent contributions

$$\Sigma_q(k) = mA(k^2) + \hat{k}F_q(k^2) \tag{345}$$

The first divergency is eliminated by the mass renormalization which introduces an additional contribution

$$\delta_1\Sigma_q(k) = \delta m \tag{346}$$

The second divergency is eliminated by the quark wave function renormalization which gives additionally

$$\delta_2\Sigma_q(k) = (Z_q - 1)\hat{k} \tag{347}$$

Divergent parts of δm and $Z_q - 1$ are determined by the condition that the total self-mass be finite.

So after the renormalization of the quark masses and all the fields all self-masses become finite.

Now we pass to the vertex parts. We start from the gluon-ghost vertex part for which we have our chosen renormalization constant $z_{1\chi}$. The formal order of divergence for this vertex part is unity. But in our gauge $\alpha = 0$ it decreases by two units and becomes negative, which implies that the diagram is in fact convergent. As a result the renormalization constant $Z_{1\chi}$ proves to be finite (and determined by the concrete renormalization conditions, e.g. by the chosen renormalization point).

Consider now the triple gluon vertex part V_{3G}. We turn to the Slavnov-Tailor identity (339). On the left-hand side we have this vertex part coupled to the three external propagators. The propagator for the longitudinal gluon $\alpha k^m u k^\nu/k^4$ will be multiplied by factor $k_\mu/alpha$ and so give k^ν/k^2 applied to the contribution without the longitudinal external line. As we see the result does not depend on α. Inspecting the diagrams in the third order shown in Fig. 29 we find that the right part of the identity is finite. It involves either the (renormalized) mass terms, or the finite contribution from factor $(Z_{1\chi} - 1)$ or diagrams, which have a structure similar to the gluon ghost vertex part but without derivatives, which is explicitly finite. Therefore the left part of the identity is also finite. The left-hand part, apart from finite renormalized self-masses, contains the triple gluon vertex part together with its renormalizing counterterm equal to the bare 3G vertex multiplied by $(Z_{1,G} - 1)g$, all multiplied by the momentum k of the longitudinal gluon: $k_\nu V_{3G}^{\nu\lambda\rho}$ From symmetry requirements and dimensional properties ($\dim V_{3G} = 1$) the (possibly) divergent part of the vertex V_{3G} may include two terms with different color structures

$$\mathrm{div}V^{abc}_{3G,\nu\lambda\rho} = c_f f^{abc}(g_{\nu\lambda}(k_1-k_2)_\rho + g_{\lambda\rho}(k_2-k_3)_\nu + g_{\nu\rho}(k_3-k_1)_\lambda + c_d d^{abc}(g_{\nu\lambda}k_{3\rho} + g_{\lambda\rho}k_{1\nu} + g_{\nu\rho}k_{2\lambda}) \tag{348}$$

where $k_1 + k_2 + k_3 = 0$ and $c_{f,d}$ are two (possibly) logarithmically divergent constants. If we multiply (348) by k^ν we still are left with many different structures which all have to be finite. From this it follows that both constants c_f and c_d are in fact finite.

Let us finally study the quark-gluon vertex part V_q. We consider the second Slavnov-Tailor identity (340). Again on the right-hand side all contributions turn out to be finite. A new diagram with a pair quark-ghost at the same point is finite: its formal order of divergence is zero, but we can separate the momentum of the external ghost to lower it to minus unity. So the left-hand part has also to be finite. This means that $l_\mu V^a_{q\mu}(k, k+l)$ is finite, where V_q is the quark-gluon vertex part together with its renormalizing counterterm $(Z_{1q} - 1)g\gamma_\mu t^a$ and k and $k + l$ are the quark momenta. The divergent part of the quark-gluon vertex part has a structure

$$\mathrm{div} V^a_{q\mu} = c\gamma_\mu t^a \tag{349}$$

where c is a constant, possibly logarithmically divergent. Multiplying by l_μ we find that c is in fact finite.

Thus we have demonstrated that all triple vertex parts are made finite by renormalization. The quartic vertex parts are all finite as they are, except the quartic gluon vertex V_{4G} To prove that the latter vertex part is also finite one has to derive a new Slavnov-Tailor identity identity, containing four gluons on the left-hand part (performing one new differentiation in the current $j_\perp$). Then similarly to the triple vertex case, on the right-hand side there appear only finite contributions, from which it will follow that also V_{4G} is finite.

2.5 Renormalization group. The running coupling constant

2.5.1 Kallan-Symanzik equation

The multiplicative transformation of fields which enters the renormalization procedure has an intrinsic arbitrariness involving a finite numerical factor. Changing this factor has a meaning of changing the renormalization conditions, e.g. the renormalization point μ^2. Choosing a new value of μ^2 implies a redefinition of the physical (renormalized) coupling constant g, which therefore is in fact a function of μ. The set of finite multiplicative transformation of the fields form the renormalization group under which the theory remains invariant. This invariance allows to find the relation between the coupling constant and μ. In special cases when physical observables involve only one variable with dimension (e.g. the momentum) the dimensional analysis then allows to find the dependence on this single variable.

To simplify the discussion we first study the simple model of a scalar massless self-interacting field with a Lagrangian

$$L = \frac{1}{2}\phi_0 \partial^2 \phi_0 - \frac{1}{4!} g_0 \phi_0^4 \tag{350}$$

Here ϕ_0 is the unrenormalized field and g_0 is the unrenormalized coupling constant. The renormalized quantities are defined in the standard manner

$$\phi = Z^{1/2}\phi_0, \quad Z_1 g = Z^2 g_0 \tag{351}$$

where Z and z_1 are the (divergent) renormalization constant for the field and the quartic vertex. We introduce unrenormalized Green functions

$$G_0^{(n)}(x_1,...x_n) =< T\{\phi_0(x_1)...\phi_0(x_n)\} > \tag{352}$$

and renormalized ones by the same formula with renormalized fields. They are related as

$$G^{(n)} = Z^{-n/2}G_0^{(n)} \tag{353}$$

Here and in the following we often suppress the arguments of the functions when they are obvious. It is convenient to define single-particle irreducible parts of the Green functions ('amputated Green functions') separating from $G^{(n)}$ n external functions $G^{(2)}$ putting in the momentum space

$$G_0^{(n)}(k_1,k_n) = V_0^{(n)}(k_1,...k_n)\prod_{i=1}^{n} G_0^{(2)}(k_i) \tag{354}$$

and similarly for the renormalized Green functions. Then the relation between renormalized and unrenormalized quantities becomes inverse:

$$V^{(n)} = Z^{n/2}V_0^{(n)} \tag{355}$$

The physical coupling constant is typically defined by the requirement

$$V^{(4)}(k_i) = g \quad \text{at} \quad k_ik_j = -\mu^2\Big(\delta_{ij} - \frac{1}{4}\Big) \quad i,j = 1,2,3,4 \tag{356}$$

(one takes into account that for massless particles $\sum_{i,j} k_ik_j = 0$) with a single renormalization point μ. If one introduces the vertex part $\Gamma^{(4)}$ then the relations

$$V^{(4)} = g\Gamma^{(4)}, V_0^{(4)} = g_0\Gamma_0^{(4)} \tag{357}$$

lead to the standard result

$$\Gamma_0^{(4)}(k_i) = Z_1^{-1} \quad \text{at} \quad k_ik_j = -\mu^2\Big(\delta_{ij} - \frac{1}{4}\Big) \tag{358}$$

Let us consider the relation between the renormalized and unrenormalized quantities in more detail, taking into account that the latter are only determined in the theory regularized in the unltraviolet. It is essential that such regularization requires introduction of a regularizing parameter with dimension. A simple (but not unique) possibility is just to introduce an ultraviolet cutoff Λ for integration momenta. Unrenormalized quantities will then become functions of Λ which do not exist in the limit $\Lambda \to \infty$. In contrast renormalized quantities will admit this limit after which they become independent of Λ.

In particular the finite renormalized coupling constant is independent of Λ. But it will obviously depend on the normalization point μ. To specify this dependence we shall first take two different values of the renormalization point μ and $\mu^{(0)}$. The renormalization condition (351) together with the definition (358 will express each of the corresponding coupling constants g and $g(0)$ as

$$g = g(g_0,\mu,\Lambda), \quad g^{(0)} = (g_0,\mu^{(0)},\Lambda) \tag{359}$$

Note that both are finite. From the second relation we can determine g_0 as a function of $g^{(0)}$ and $\mu^{(0)}$:

$$g_0 = g_0(g^{(0)}, \mu^{(0)}, \Lambda) \tag{360}$$

Of course it is divergent. Putting this into the first relation (359) we find

$$g = g\Big(g_0(g^{(0)}, \mu^{(0)}, \Lambda), \mu, \Lambda\Big) = g(g^{(0)}, \mu, \mu^{(0)}, \Lambda) = g(g^{(0)}, \mu, \mu^{(0)}) \tag{361}$$

The last equality follows from the finiteness of g, which admits taking the limit $\Lambda \to \infty$. However g is dimensionless, so that it has to depend only on the ratio of the two μ's with the dimension of mass:

$$g = g\Big(g^{(0)}, \frac{\mu}{\mu^{(0)}}\Big) \tag{362}$$

This is the equation which explicitly shows the dependence of the renormalized coupling function on the renormalization point. Considered as a function of μ, $g(\mu)$ is called the running coupling constant. In the following we shall take the logarithmic derivative in μ and put $\mu = \mu^{(0)}$ afterwards. Then we find

$$\mu\frac{\partial g}{\partial \mu} = \frac{\mu}{\mu^{(0)}}\frac{\partial g(g^{(0)}, z)}{\partial z}\Big|_{z=\mu/\mu^0} \tag{363}$$

Putting $z = 1$ we obtain

$$\mu\frac{\partial g}{\partial \mu}\Big|_{\mu=\mu^{(0)}} = \frac{\partial g(g, z)}{\partial z}\Big|_{z=1} \equiv \beta(g) \tag{364}$$

which is a function of only one variable g.

Now we differentiate over μ relation (355) keeping $g^{(0)}$ and $\mu^{(0)}$ or equivalently g_0 and Λ fixed. The unrenormalized Green function then does not depend on μ at all. The renormalized one depends on μ both explicitly and through the μ-dependence of g. So we get

$$\Big(\mu\frac{\partial}{\partial \mu} + \beta(g)\frac{\partial}{\partial g}\Big)V^{(n)} = V_0^n \mu\frac{\partial}{\partial \mu}Z^{n/2} \tag{365}$$

We have

$$\mu\frac{\partial}{\partial \mu}Z^{n/2} = \mu\frac{\partial}{\partial \mu}e^{(n/2)\ln Z} = \frac{n}{2}\mu\frac{\partial}{\partial \mu}\ln Z \cdot e^{(n/2)\ln Z} \equiv n\gamma Z^{n/2} \tag{366}$$

where we define the so-called anomalous dimension

$$\gamma = \mu\frac{\partial}{\partial \mu}\frac{1}{2}\ln Z \tag{367}$$

Transferring everything to the left-hand side we obtain the Kallan-Symanzik equation

$$\Big(\mu\frac{\partial}{\partial \mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma\Big)V^{(n)} = 0 \tag{368}$$

One concludes from this equation that the anomalous dimension γ is a finite quantity, since the first two terms are obviously finite. Since by construction it is dimensionless, it cannot depend on μ explicitly but is only a function of the coupling constant g: $\gamma = \gamma(g)$

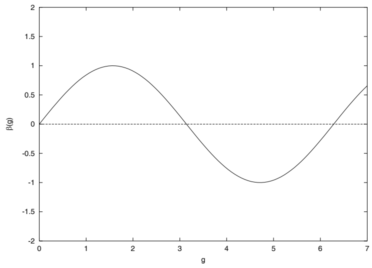

Figure 33: Schematic behaviour of $\beta(g)$ with a positive derivative at $g = 0$

2.5.2 The running coupling

Formally the running coupling is determined by the equation (364)

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g) \tag{369}$$

If the β-function is known the running coupling is trivially found. Let $\mu = \mu_0 e^t$ where t is a dimensionless variable going from $-\infty$ to $+\infty$. Then the equation takes the form

$$\frac{\partial g(t)}{\partial t} = \beta(g) \tag{370}$$

with a solution

$$t = \int_{g(0)}^{g(t)} \frac{dg'}{\beta g'} \tag{371}$$

which implicitly determines function $g(t)$.

Let us study some general properties of this implicit solution. Since $(-\infty < t < +\infty)$ the integral on the right-hand side of (371) has to diverge at least twice. This can occur either due to going to $\pm\infty$ of the upper limit $g(t)$ or due to vanishing of the denominator $\beta(g)$ at some points, called fixed points g^*: $\beta(g^*) = 0$. Note that one such fixed point is known: it is $g = 0$, since in any theory perturbative $\beta(g)$ starts from the second or third order in g.

It turns out that the behaviour of $g(t)$ crucially depends on the sign of the derivative $d\beta(g)/dg$ at $g = 0$. Consider the case when this derivative is positive, the behaviour of $\beta(g)$ qualitatively shown in Fig. 33. One easily finds that in region I at $t \to -\infty$ $g(t) \to 0$ and at $t \to +\infty$ $g(t) \to g_1$. In region II at $t \to -\infty$ $g(t) \to g_2$ and at $t \to +\infty$ $g(t) \to g_1$.

The case with $d\beta(g = 0)/dg < 0$ is illustrated in Fig. 34. Here the behaviour of the running coupling is opposite. In region I at $t \to -\infty$ $g(t) \to g_1$ and at $t \to +\infty$ $g(t) \to 0$. In region II at $t \to -\infty$ $g(t) \to g_1$ and at $t \to +\infty$ $g(t) \to g_2$.

Special interest is naturally drawn to the situations when the running coupling becomes small $g(t) \to 0$, since in this case one may hope to apply the perturbation theory. This happens at very small values of μ if $d\beta(g = 0)/dg > 0$ and very large μ if $d\beta(g = 0)/dg < 0$. The first case unfortunately has little relevance, since in realistic theories, with small values of μ, the influence of other mass parameters becomes dominant and completely overshadows these predictions. In contrast the case when $d\beta(g = 0)/dg < 0$ and μ is large has direct and most important consequences, since then the influence of other mass parameters becomes negligible. In this case we shall show that physical processes at

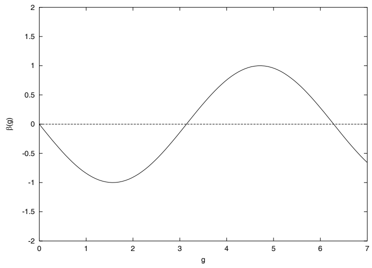

Figure 34: Schematic behaviour of $\beta(g)$ with a negative derivative at $g = 0$

large momenta become governed by the running coupling at values of μ of the order of these momenta and since then $g(t) \to 0$ interaction decreases at large momenta. This phenomenon is known as 'asymptotic freedom'.

It turned out that in the QCD precisely this situation is realized. So in the QCD phenomena occurring at large momenta admit application of the perturbation theory. We shall see that in the following by directly calculating the β-function at small values of g.

2.6 Solution of the Callan-Symanzik equation. High momentum asymptotic

Solution of the Callan-Symanzik equation can be easily found. Putting again $\mu = \mu_0 e^t$ we are to solve the equation

$$\Big(\frac{\partial}{pdt} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\Big)V(n) = 0 \tag{372}$$

We introduce a function $\bar{g}(t)$ by the equation

$$\frac{d\bar{g}(t)}{dt} = \beta(\bar{g}(t)) \tag{373}$$

and seek the solution in the form

$$V^{(n)}(g,t) = f(\bar{g}(t), t), \quad \text{with} \;\; \bar{g}(t) = g \tag{374}$$

Then we find

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \beta(\bar{g}(t))\frac{\partial f}{\partial g} = \frac{\partial V^{(n)}}{\partial t} + \beta(g)\frac{\partial V^{(n)}}{\partial g} = n\gamma(\bar{g}(t))f \tag{375}$$

with an obvious solution

$$f(t) = f(0)e^{n\int_0^t dt'\gamma(\bar{g}(t'))} \tag{376}$$

or in terms of μ and g:

$$V^{(n)}(\mu, g) = v^{(n)}(\mu_0, g_0)e^{n\int_0^t dt'\gamma(\bar{g}(t'))} \tag{377}$$

where g and g_0 are related by

$$t = \int_{g_0}^{g} \frac{dg'}{\beta(g')} \tag{378}$$

Now we pass to the study of the dependence of amputated Green functions $V^{(n)}(p_i,\mu,g)$ on the assumption that all momenta p_i are simultaneously rescaled as

$$p_i \to p_i e^t. \tag{379}$$

Our basic observation is that due to dimensional properties

$$V^{(n)}(p_i e^t,\mu,g) = e^{\Gamma t} V^{(n)}(p_i,\mu e^{-t},g) \tag{380}$$

where Γ is the canonical dimension of the Green function. Once again we introduce function $\bar{g}(t)$ with a different boundary condition $\bar{g}(-t) = g$ Then we find

$$V^{(n)}(p_i,\mu e^{-t},\bar{g}(-t)) = V^{(n)}(p_i,-t)$$

$$= V^{(n)}(p_i,0) e^{n\int_0^{-n} dt'\gamma(\bar{g}(t'))} = V^{(n)}(p_i,0) e^{-n\int_0^{n} dt'\gamma(\bar{g}(-t'))} \tag{381}$$

Here $V^{(n)}(p_i,0) = V^{(0)}(p_i,\mu,g_0)$ where

$$-t = \int_{g_0}^{\bar{g}(-t)} \frac{dg'}{\beta(g)} \tag{382}$$

or

$$t = \int_{\bar{g}(-t)}^{g_0} \frac{dg'}{\beta(g)} \equiv \int_g^{g(t)} \frac{dg'}{\beta(g)} \tag{383}$$

Here at fixed lower limit g the upper limit g_0 becomes a function of t, which we denote as $g(t)$. From this we find

$$t'' = t - t' = \int_g^{\bar{g}(-t')} \frac{dg'}{\beta(g)} \equiv \int_g^{t''} \frac{dg'}{\beta(g)} \tag{384}$$

where we naturally denote $\bar{g}(-t') = g(t'')$ This allows to rewrite

$$\int_0^t dt'\gamma(\bar{g}(-t')) = \int_0^t dt''\gamma(g(t'')) \tag{385}$$

So our final result is

$$V^{(n)}(p_i e^t,\mu,g) = e^{\Gamma t} V^{(n)}(p_i,\mu,g(t)) e^{-n\int_0^t dt''\gamma(g(t''))} \tag{386}$$

where $g(t)$ is the running coupling constant is defined by (383)

2.6.1 The Callan-Symanzik equation and the running coupling in the QCD

In the QCD obviously we have the Green functions which may involve different numbers of quark and gluon legs $V^{(n_q,n_G)}$. So the relation (355) will contain different powers of the renormalization constants Z_q and Z_G Also we encounter a new dimensionless gauge parameter α, which renormalizes according to $\alpha_0 = Z_G\alpha$, so that the renormalized α also depends on μ. Accordingly, taking the derivative $\mu d/d\mu$ we shall find a new contribution

$$\mu\frac{\partial\alpha}{\partial\mu}\frac{\partial}{\partial\alpha}$$

However
$$\mu\frac{\partial\alpha}{\partial\mu} = 2\cdot\frac{1}{2}\mu\frac{\partial\ln Z_G}{\partial\mu}\frac{\partial\alpha}{\partial\ln Z_G} = 2\gamma_G\alpha\frac{\partial\ln\alpha}{\partial\ln Z_G} = -2\gamma_G\alpha \equiv \delta$$
So we find the Callan-Synanzik equation in the QCD

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \delta\frac{\partial}{\partial\alpha} - n_q\gamma_q - n_G\gamma_G\Big)V^{(n_q,n_G)}(p_i,\mu,g,\alpha) = 0 \tag{387}$$

Here β, δ, γ_q and γ_G are all functions of g and α. Note that in the transverse gauge $\alpha = 0$ and so also $\delta = 0$. This means that the Callan-Symanzik equation is compatible with the purely transverse gauge.

In the perturbation theory at small values of g one finds in the QCD

$$\beta(g) = -bg^3 + \mathcal{O}(g^5) \tag{388}$$

where for the total number of flavours N_f

$$b = \frac{1}{48\pi^2}(33 - 2N_f) > 0 \quad \text{unless} \quad N_f > 16 \tag{389}$$

Thus with the realistic $N_f = 6$ b is positive and the QCD belongs to a theory with asymptotic freedom, where the coupling constant decreases with energy. In fact we solve the equation for the running coupling as

$$t = -\int_g^{g(t)}\frac{dg}{bg^3} = \frac{1}{2b}\Big(\frac{1}{g^2(t)} - \frac{1}{g^2}\Big) \tag{390}$$

which gives
$$g^2(t) = \frac{g^2}{1 + g^2 2bt} \tag{391}$$
Relating t with the change of momentum scale s

$$2t = \ln\frac{Q^2}{Q_0^2} \tag{392}$$

we finally find the relation between the running coupling constant at momentum scales Q_0 and Q as
$$g^2(Q^2) = \frac{g^2(Q_0^2)}{1 + g^2(Q_0)^2 b\ln\frac{Q^2}{q_0^2}} \tag{393}$$

This relation can also be rewritten as

$$g^2(Q^2) = \frac{1}{g^{-2}(Q_0) + b\ln\frac{Q^2}{q_0^2}} = \frac{1}{b\ln\frac{Q^2}{\Lambda^2_{QCD}}} \tag{394}$$

where we defined
$$g^{-2}(Q_0) - b\ln Q_0^2 = -\ln\Lambda^2_{QCD} \tag{395}$$
In this way we see that at large values of Q^2 the dimensionless QCD coupling constant can be fully determined by the mass parameter Λ_{QCD} ('dimensional transmutation').

Finally let us see how the Green function behave when their momentum variables p_i are all equally increased from scale Q_0 to scale Q. The anomalous dimensions start from terms of the order g^2 at small g:

$$\gamma_u = -\frac{1}{2}\zeta_u g^2 + \mathcal{O}(g^4) \tag{396}$$

Correspondingly the factor corresponding to the leg with the anomalous dimension γ_u is

$$\begin{aligned}\exp\Big\{-\int_0^t dt' \gamma_u\Big(g(t')\Big)\Big\} &= \exp\Big\{\frac{1}{2}\zeta_u \int_0^t dt' \frac{1}{2bt'}\Big\} \\ &= \exp\Big(\frac{zeta_u}{4b}\ln t\Big) = \Big(\frac{1}{2}\ln\frac{Q^2}{Q_0^2}\Big)^{\frac{\zeta_u}{4b}}\end{aligned} \tag{397}$$

This factor will appear in the Green function for each leg $u = q, G$.

2.7 Calculation of the β-function in the QCD

By definition

$$\beta(g) = \frac{\partial g}{\partial \mu}, \quad \gamma(g) = \frac{1}{2}\frac{\partial \ln Z}{\partial \mu}$$

at fixed bare coupling constant g_0 and ultraviolet cutoff Λ. In terms of bare coupling and renormalization constants

$$g = g_0 Z_G^{3/2} Z_{1G}^{-1} = g_0 Z_G^{1/2} Z_q Z_{1q}^{-1} = g_0 Z_G^{1/2} Z_\chi Z_{1\chi}^{-1} \tag{398}$$

This definition generally depends on the renormalization prescription and chosen gauge. However in preturbation approach one can demonstrate that the two first terms, corresponding to one- and two-loop approximation do not depend on gauge or on the renormalization recipe. Indeed let g and $\tilde{g}$ be two renormalized coupling constant introduced in different gauges or renormalization prescriptions. One can obviously express one through the other in the first two orders as

$$\tilde{g} = g(1 + ag^2) \quad \text{or} \quad g = \tilde{g}(1 - a\tilde{g}^2) \tag{399}$$

The two β-functions are obviously related by

$$\tilde{\beta}(\tilde{g}) = \mu\frac{\partial \tilde{g}}{\partial \mu} = \mu\frac{\partial g}{\partial \mu}\frac{\partial \tilde{g}}{\partial g} = \beta(g)\frac{\partial \tilde{g}}{\partial g} \tag{400}$$

In the first two orders we have

$$\beta(g) = -bg^3 + cg^5 \tag{401}$$

Then we find

$$\begin{aligned}\tilde{\beta}(\tilde{g}) = (-bg^3 + cg^5)(1 + 3ag^2) &= \Big(-b\tilde{g}^3(1 - 3ag^2) + c\tilde{g}^5\Big)(1 + 3a\tilde{g}^2) \\ &= -b\tilde{g}^3 + c\tilde{g}^5\end{aligned} \tag{402}$$

so that the two first term do not change. We are going to calculate only the first term corresponding to one-loop approximation. Then we can use any gauge or renormalization prescription.

In the perturbation theory in the single loop approximation we have for any Z

$$Z = 1 + g^2\zeta \ln\frac{\Lambda}{\mu} + finite\ terms \simeq 1 + g_0^2\zeta \ln\frac{\Lambda}{\mu} + finite\ terms \tag{403}$$

where ζ and $finite\ terms$ are some constants independent of μ, since they are dimensionless. From this we immediately find that $\partial/\partial\ln\mu = -\partial/\partial\ln\Lambda$ and so

$$\gamma_u = -\frac{1}{2}\frac{\partial \ln Z_u}{\partial \ln\Lambda} = -\frac{1}{2}g_0^2\zeta_u \simeq -\frac{1}{2}g^2\zeta_u \tag{404}$$

The importance of this result is that it is independent of the renormalization prescription and only based on the knowledge of the divergent part of Z.

In the same manner we observe that

$$g = g_0 + bg_0^3 \ln\frac{\Lambda}{\mu} \tag{405}$$

so that

$$\beta = -\frac{\partial g}{\partial\ln\Lambda} = -bg^3 \tag{406}$$

Relations (398) give

$$b = \frac{3}{2}\zeta_G - \zeta_{1G} = \frac{1}{2}\zeta_G + \zeta_q - \zeta_{1q} = \frac{1}{2}\zeta_G\zeta - \chi - \zeta_{1\chi} \tag{407}$$

So to find ζ's and b we have to study the divergent terms in the self-masses for ζ_u and vertex parts for ζ_{1u}, $u = G, q$ and χ.

We can use any of the definitions of g from (398 to find b. To have some contact with analogous calculations in the QED we shall use its definitions through the vertex part quark-gluon. Then we have to know ζ_q, ζ_G and ζ_{1g}. To simplify we shall use the Feynman gauge.

2.7.1 Quark self-mass

The divergent part of the quark self mass $\Sigma_q(p)$ has a structure

$$\mathrm{div}\Sigma_q(p) = c_1(\Lambda)m + c_2(\Lambda)\hat{p} \tag{408}$$

where both c_1 and c_2 are logarithmically divergent.. Taking as a mass renormlization recipe the requirement $\Sigma_q(\hat{p} = m) = 0$ we obtain as a result

$$\mathrm{div}\Big(\Sigma_q(p) - \Sigma_q(\hat{p} = m)\Big) = c_2(\Lambda)(\hat{p} - m) \tag{409}$$

This has to be made finite by the addition of the term coming from the quark wave function renormalization $(Z_q - 1)(\hat{p} - m)$ wherefrom we find

$$Z_q - 1 = -c_2(\Lambda) \tag{410}$$

and to find the divergent part of Z_q we have to find the divergent term proportional to $\hat{p}$ in the self-mass $\Sigma_q(p)$. It does not depend on the quark mass, so we can put $m = 0$ from the start. Then in the second order in g

$$\Sigma_q(p) = -g^2\int\frac{d^4l}{(2\pi)4i}t^a\gamma_\alpha\frac{\hat{l}}{l^2}\gamma^\alpha t^a\frac{1}{(l-p)^2} \tag{411}$$

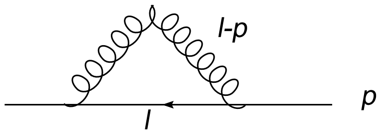

Figure 35: The quark self-mass in the lowest order

(see Fig. 35).

The invariant sum $t^a t^a$ is proportional to the unit 3×3 color matrix: $t^a t^a = c \cdot 1$. Taking the trace with the standard normalization

$$\mathrm{Tr}\{t^a t^b\} = \frac{1}{2}\delta_{ab} \tag{412}$$

we obtain

$$t^a t^a = \frac{4}{3} \tag{413}$$

Now the $\gamma-$matrices:

$$\gamma_\alpha \hat{l} \gamma^\alpha = (2l_\alpha - \hat{l}\gamma_\alpha)\gamma^\alpha = 2\hat{l} - 4\hat{l} = -2\hat{l} \tag{414}$$

So we find

$$\Sigma_q(p) = \frac{8}{3}g^2 \int \frac{d^4 l}{(2\pi)4i} \frac{\hat{l}}{l^2(l-p)^2} \tag{415}$$

The divergent contribution comes from the region $l >> p$ so that we can expand

$$\frac{1}{(l-p)^2} = \frac{1}{l^2}\Big(1 + 2\frac{pl}{l^2} + ...\Big) \tag{416}$$

to find

$$\mathrm{div}\Sigma_q(p) = \frac{8}{3}g^2 \int \frac{d^4 l}{(2\pi)4i} \frac{\hat{l}}{l^4}\Big(1 + q\frac{pl}{l^2}\Big) \tag{417}$$

The first term is evidently zero. In the second term we find an integral

$$\int \frac{d^4 l}{(2\pi)^4 i} \frac{l_\alpha l_\beta}{l^6} = \frac{1}{4}g_{\alpha\beta} \int \frac{d^4 l}{(2\pi)^4 i} \frac{1}{l^4} \tag{418}$$

The last logarithmically divergent integral is standardly calculated by the rotation to the Eucledean momenta with a cutoff Λ at large Eucledean $|l^2|$ to give

$$I(\Lambda) = \int \frac{d^4 l}{(2\pi)^4 i} \frac{1}{l^4} = \frac{2\pi^2}{(2\pi)^4} \int^\Lambda \frac{dl}{l} = \frac{1}{8\pi^2} \ln \Lambda \tag{419}$$

So we find

$$\mathrm{div}\Sigma_q(p) = \frac{4}{3}g^2 \hat{p} I(\Lambda) \tag{420}$$

and consequently the divergent part of Z_q is

$$Z_q - 1 = -\frac{4}{3}g^2 I(\lambda) = -g^2 \frac{8}{3}\frac{1}{16\pi^2} \ln\frac{\Lambda}{\mu} \tag{421}$$

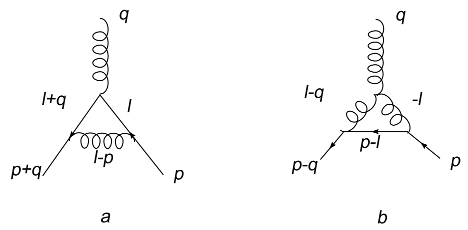

Figure 36: The quark-gluon vertex part in the second order

2.7.2 Vertex part quark-gluon

The divergent part of the quark-gluon vertex has a structure $c(\lambda)t^a\gamma_\alpha$. So to find it we can put the quark mass and all the external momenta equal to zero. In the second order the quark-gluon vertex part has two contributions shown in Fig. 36 a, b.

The first (Fig. 36 a) is given by

$$\Gamma^a_{1,\alpha} = g^2 \int \frac{d^4l}{(2\pi)^4 i} t^b \gamma_\mu \frac{\hat{l}}{l^2} t^a \gamma_\alpha \frac{\hat{l}}{l^2} t^b \gamma^\mu \frac{1}{l^2} \tag{422}$$

(of course this is only its divergent contribution, which we do not specify explicitly in the following). The colour factor is

$$t^b t^a t^b = t^b t^b t^a + i f^{abc} t^b t^c$$

$$t^b t^b t^a = \frac{4}{3} t^a$$

$$i f^{abc} t^b t^c = i f^{acb} t^c t^b = -i f^{abc} t^c t^b = \frac{1}{2} i^2 f^{abc} f^{bcd} t^d = -\frac{3}{2} t^a$$

where we have used that

$$f^{abc} f^{dbc} = 3\delta_{ab} \tag{423}$$

Thus we have found the colour factor to be

$$t^b t^a t^b = \Big(\frac{4}{3} - \frac{3}{2}\Big) t^a \tag{424}$$

Now the γ matrices.

$$\gamma_\mu \hat{l} \gamma_\alpha \hat{l} \gamma^\mu = \gamma_\mu (2l_\alpha - \gamma_\alpha \hat{l}) \hat{l} \gamma^\mu = 2l_\alpha(-2\hat{l}) - l^2(-2\gamma_\alpha) = 2(\gamma_\alpha l^2 - 2l_\alpha \hat{l})$$

We get for the vertex

$$\Gamma^a_{1,\alpha} = 2g^2 \Big(\frac{4}{3} - \frac{3}{2}\Big) t^a \int \frac{d^4l}{(2\pi)^4 i} \frac{1}{l^6} (\gamma_\alpha l^2 - 2l_\alpha \hat{l}) = g^2 \Big(\frac{4}{3} - \frac{3}{2}\Big) t^a \gamma_\alpha I(\Lambda) \tag{425}$$

So we find the contribution from the diagram Fig. 36 a

$$Z^{(1)}_{1q} - 1 = -g^2 \Big(\frac{4}{3} - \frac{3}{2}\Big) I(\Lambda) \tag{426}$$

Now the second contribution from the diagram shown in Fig. 36 b. Putting all external momenta equal to zero we have

$$\Gamma^a_{2\lambda} = -ig^2 \int \frac{d^4l}{(2\pi)^4 i} f^{abc} \Big[l_\nu g_{\lambda\mu} - 2l_\lambda g_{\mu\nu} + l_\mu g_{\nu\lambda}\Big] t^c \gamma_\mu \frac{-\hat{l}}{l^2} t^b \gamma^\mu \frac{1}{l^4} \tag{427}$$

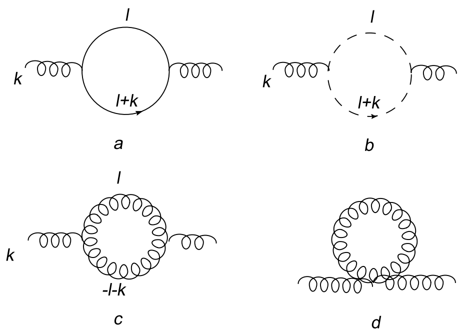

Figure 37: The gluon self-mass in the second order

The colour factor is

$$if^{abc}t^ct^b = \frac{3}{2}t^a$$

The spin factor is

$$\Big[l_\nu g_{\lambda\mu} - 2l_\lambda g_{\mu\nu} + l_\mu g_{\nu\lambda}\Big]\gamma^\nu \hat{l}\gamma^\mu = l^2\gamma_\lambda - 2l_\lambda\gamma_\nu\hat{l}\gamma^\nu + l^2\gamma_\lambda = 2l^2\gamma_\lambda + 4l_\lambda\hat{l}$$

So we find

$$\Gamma^a_{2\lambda} = g^2\frac{3}{2}\cdot 2\cdot t^a\int\frac{d^4l}{(2\pi)^4 i}\frac{1}{l^6}(l^2\gamma_\lambda + 2l_\lambda\hat{l}) = g^2\frac{9}{2}t^a\gamma_\lambda I(\Lambda) \tag{428}$$

and the second part of the renormlization constant is

$$Z^{(2)}_{1q} - 1 = -g^2\frac{9}{2}I(\Lambda) \tag{429}$$

In the sum we find

$$Z^{(1)}_{1q} - 1 = g^2\Big(3 - \frac{4}{3}\Big)I(\Lambda) \tag{430}$$

2.7.3 Gluon polarization operator

To find Z_G we have to study the gluon polarization operator $\Pi^{ab}_{\mu\nu}(k)$ which, in the second order, is represented by 4 Feynman diagrams shown in Fig. 37 $a-d$. We shall use that it is diagonal in colour and pure transverse

$$\Pi^{ab}_{\mu\nu} = \delta_{ab}P^\perp_{\mu\nu}\Pi(k^2) \tag{431}$$

and we also know that $\Pi(k^2=0)=0$, so that we can substitute

$$\Pi(k^2) \to \Pi(k^2) - \Pi(0) \tag{432}$$

Multiplying (431) by $g^{\mu\nu}\delta_{ab}$ and summing over the indeces we get

$$\Pi(k^2) = \frac{1}{24}\Big(\Pi^{aa\mu}_{\mu}(k) - \Pi^{aa\mu}_{\mu}(0)\Big) \tag{433}$$

It is clear from the start that the diagram of Fig. 37 d will give no contribution since it does not depend on k.

We start with the quark contribution shown in Fig. 37 a.

$$\Pi^{(q)} = -\frac{1}{24}g^2 \int \frac{d^4l}{(2\pi)^4 i} \mathrm{Tr}\Big\{t^a \gamma_\mu \frac{\hat{l}}{l^2} t^a \gamma^\mu \frac{\hat{l}+\hat{k}}{(l+k)^2}\Big\} - \Big(k=0\Big) \tag{434}$$

As before $t^a t^a = 4/3$ so that

$$\mathrm{Tr}\{t^a t^a\} = 4$$

Now $\gamma_\mu \hat{l} \gamma^\mu = -2\hat{l}$, so that we find a trace

$$-2\mathrm{Tr}\{\hat{l}(\hat{l}+\hat{k})\} = -8(l^2 + kl)$$

Thus we find

$$\Pi^{(q)} = -\frac{1}{24} \cdot 4 \cdot (-8) g^2 \int \frac{d^4l}{(2\pi)^4} \Big(\frac{l^2+kl}{l^2(l+k)^2} - \frac{1}{l^2}\Big)$$

In the integrand we find a factor

$$\frac{l^2+kl}{(l+k)^2} - 1 = -\frac{k^2+kl}{(l+k)^2} \simeq -\frac{k^2+kl}{l^2}\Big(1 - \frac{2kl}{l^2}\Big) = -\frac{k^2}{l^2} + \frac{2(kl)^2}{l^4}$$

Integration will substitute $(kl)^2 \to k^2 l^2/4$ so that in the sum we get $-k^2/(2l^2)$ This gives

$$\Pi^{(q)} = -\frac{1}{24} \cdot 4 \cdot (-8) \cdot (-\frac{1}{2}) g^2 k^2 I(\Lambda) = -g^2 k^2 \frac{2}{3} I(\Lambda) \tag{435}$$

Our first contribution to the gluon renormalization constant is therefore

$$Z_G^{(q)} - 1 = -g^2 \frac{2}{3} I(\Lambda) \tag{436}$$

The second piece comes from the ghost contribution shown in Fig. 37 b:

$$\Pi^{(\chi)} = -\frac{1}{24}g^2 \int \frac{d^4l}{(2\pi)^4 i} i f^{bac} i f^{cab} \frac{l(k+l)}{l^2(l+k)^2} - \Big(k=0\Big) \tag{437}$$

The colour factor is $if^{bac}if^{cab} = 24$. We find

$$\Pi^{(\chi)} = -\frac{1}{24} \cdot 24 g^2 \int \frac{d^4l}{(2\pi)^4 i} \Big(\frac{l(k+l)}{l^2(l+k)^2} - \frac{1}{l^2}\Big) = -\frac{3}{4}\Pi^{(q)} = k^2 g^2 \frac{1}{2} I(\Lambda) \tag{438}$$

Thus the second part of Z_G is

$$Z_G^{(\chi)} - 1 = g^2 \frac{1}{2} I(\Lambda) \tag{439}$$

Finally we study the gluon contribution shown in Fig. 37 c

$$\Pi^{(G)} = \frac{1}{24} \cdot \frac{1}{2} g^2 \int \frac{d^4l}{(2\pi)^4 i} i f^{abc}(-i f^{abc})$$

$$\Big[(k-l)_\nu g_{\mu\lambda} + (2l+k)_\lambda g_{\mu\nu} + (-l-2k)_\mu g_{\nu\lambda}\Big]^2 \frac{1}{l^2(l+k)^2} - \Big(k=0\Big) \tag{440}$$

Note the symmetry factor 1/2. The colour factor again gives $if^{abc}(-if^{abc}) = 24$. The square bracket squared gives

$$4(k-l)^2 + 4(2l+k)^2 + 4(l+2k)^2 + 2(k-l)(2l+k) - 2(k-l)(l+2k) - 2(2l+k)(l+2k)$$

$$= 3(k-l)^2 + 3(2l+k)^2 + 3(l+2k)^+(k-l+2l+k-l-2k) = 18(k^2+l^2+kl)$$

We obtain

$$\Pi^{(G)} = \frac{1}{24}\cdot 24\cdot 18\cdot \frac{1}{2}g^2\int \frac{d^4l}{(2\pi)^4 i}\frac{1}{l^2}\frac{-kl}{(k+l)^2} \tag{441}$$

Now the integration gives

$$\frac{kl}{(k+l)^2} = \frac{kl}{l^2}\Big(1-\frac{2kl}{l^2}\Big) \to -\frac{2(kl)^2}{l^4} \to -\frac{1}{2}\frac{k^2}{l^2} \tag{442}$$

Taking together all the factors we get

$$\Pi^{(G)} = k^2 g^2 \frac{9}{2} I(\Lambda) \tag{443}$$

which gives the final piece of $Z_G^{(G)}$

$$Z^{(G)} - 1 = \frac{9}{2} I(\Lambda) \tag{444}$$

2.7.4 β-function

Collecting our results and taking into account the expression for $I(\Lambda)$ we find

$$\zeta_q = -\frac{4}{3}\frac{1}{8\pi^2},\quad \zeta_{1q} = \Big(3-\frac{4}{3}\Big)\frac{1}{8\pi^2},\quad \zeta_G = (-\frac{2}{3}N_f + \frac{1}{2} + \frac{9}{2}\Big)\frac{1}{8\pi^2} \tag{445}$$

From (407) we find

$$b = \Big(\frac{11}{2} - \frac{1}{3}N_f\Big)\frac{1}{8\pi^2} \tag{446}$$

which gives (389).

2.8 e^+e^- annihilation into hadrons

As a first application of the perturbative QCD we shall consider the process of annihilation of a pair of leptons onto hadrons. To be concrete we take an electron-positron pair as initial leptons and the process in question is

$$e^+ = e^- \to \quad hadrons \tag{447}$$

(see Fig. 38).It is implied that the observer measures the total cross-section to produce any number of hadrons in the annihilation. It is to be noted, that apart from final hadronic states the pair may also go into leptonic states. The typical process in this respect is production of a $\mu^+\mu^-$ pair:

$$e^+ = e^- \to \ \mu^+ + \mu^- \tag{448}$$

whose cross-section is well-known from the standard perturbative approach in the QED. This process will serve as a basis for comparison. We shall assume that the c.m. energy $\sqrt{s} = q^2$ is large to be able to apply the asymptotic freedom in the QCD. The total cross-section of the reaction (447) can be written as a product of the leptonic and hadronic parts times the photon propagator. In the Feynman gauge we have

$$\sigma^{tot}(s) = \frac{1}{J}e^2 L_{\alpha\beta}H^{\alpha\beta}\frac{1}{q^4} \tag{449}$$

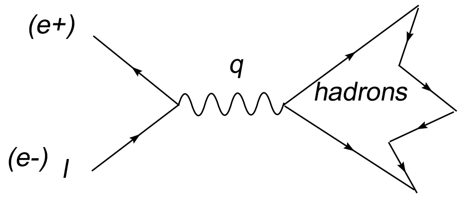

Figure 38: The process $e^+ = e^- \to \quad hadrons$

Here the leptonic part is

$$L_{\alpha\beta} = \frac{1}{4} \sum_{pol} [\bar{v}(-\bar{l})\gamma_\alpha u(l)]_{\xi\eta} \cdot [\bar{u} l \gamma_\beta v(-\bar{l})]_{\eta\xi} \tag{450}$$

where l and $\bar{l}$ are the momenta of the electron and positron, the sum is taken over polarizations and ξ and η are spinor indexes. The hadronic part is

$$H_{\alpha\beta} = \sum_n \int d\tau_n(q) \langle 0 | j^{em}_\alpha(0) | X_n \rangle \langle X_n | j^{em}_\beta(0) | 0 \rangle \tag{451}$$

where j^{em} is the hadronic electromagnetic current, $|0>$ and $|X_n>$ are the vacuum and intermediate hadronic state respectively. As usual $d\tau_n(q)$ is the phase volume element restricted by the energy-momentum conservation:

$$d\tau_n(q) = (2\pi)^4 \delta(p_n - q) d\tau_n \tag{452}$$

where $d\tau_n$ is the unrestricted phase volume element for n intermediate particles.

The leptonic part of (449) is known since

$$\sum_{pol} u(l)\bar{u}(l) = m + \hat{l}, \quad \sum_{pol} v(-\bar{l}\bar{v}(-\bar{l}) = \hat{\bar{l}} - m \tag{453}$$

so that

$$L_{\alpha\beta} = \frac{1}{4} \mathrm{Tr}\{\hat{\bar{l}}\gamma_\alpha \hat{l} \gamma_\beta\} = \bar{l}_\alpha l_\beta + l_\alpha \bar{l}_\beta - g_{\alpha\beta} l\bar{l} \tag{454}$$

In the hadronic part we present

$$(2\pi)^4 (p_n - q) = \int d^4x e^{ix(q-p_n)} \tag{455}$$

to lift the restriction on the energy-momentum of intermediate states. Using

$$e^{-ixp_n} \langle 0 | j^{em}(0) | X_n \rangle = \langle 0 | j^{em}(x) | X_n \rangle$$

and then summing over all states X_n we rewrite the hadronic tensor as

$$H_{\alpha\beta}(q) = \int d^4x e^{iqx} \langle 0 | j^{em}(x) j^{em}_\beta(0) | 0 \rangle \tag{456}$$

It is instructive to compare $H_{\alpha\beta}$ with the hadronic part of the photon polarization operator $\Pi^{(h)}_{\alpha\beta}$ which is

$$\Pi^{(h)}_{\alpha\beta}(q) = i \int d^4x e^{iqx} \langle 0 | \mathrm{T}\{j^{em}(x) j^{em}_\beta(0)\} | 0 \rangle \tag{457}$$

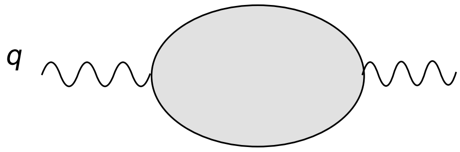

Figure 39: The hadronic part of the photon polarization operator.

and is graphically illustrated in Fig. 39. It is clear that in fact the hadronic tensor H is just the discontinuity of the polarization operator across its cut at $q^2 > 0$, which puts all intermediate hadrons on their mass-shell according to Cutkosky rules:

$$H_{\alpha\beta} = \frac{1}{i}\text{Disc }\Pi^{(h)}_{\alpha\beta} \quad \text{at} \quad q^2 > 0 \tag{458}$$

Both Π and H are pure transverse:

$$H_{\alpha\beta} = \Big(- g_{\alpha\beta} + \frac{q_\alpha q_\beta}{q^2}\Big) q^2 h(q^2), \quad \Pi^{(h)}_{\alpha\beta} = \Big(- g_{\alpha\beta} + \frac{q_\alpha q_\beta}{q^2}\Big) q^2 \pi(q^2) \tag{459}$$

and obviously

$$h(q^2) = \frac{1}{i}\text{Disc }\pi(q^2) \quad \text{at} \quad q^2 > 0 \tag{460}$$

The cross-section can be expressed directly via the dimensionless function $h(q^2)$.

$$L_{\alpha\beta}H^{\alpha\beta} = q^2 h(q^2)\Big(- g_{\alpha\beta} + \frac{q_\alpha q_\beta}{q^2}\Big)(\bar{l}_\alpha l_\beta + l_\alpha \bar{l}_\beta - g_{\alpha\beta} l\bar{l})$$

We define $p = (l - \bar{l})/2$. Then $(qp) = 0$, $l = q/2 + p$, $\bar{l} = q/2 - p$ and

$$\bar{l}_\alpha l_\beta + l_\alpha \bar{l}_\beta - g_{\alpha\beta} l\bar{l}) = -2p_\alpha p_\beta + \frac{1}{2}q_\alpha q_\beta - g_{\alpha\beta}(q^2/4 - p^2)$$

Then we find

$$L_{\alpha\beta}H^{\alpha\beta} = q^2 h(q^2)\Big(\frac{3}{4}q^2 - p^2\Big)$$

The c.m. energy $s = q^2 = 2l\bar{l}$, the incoming flux $J = 4l\bar{l}$ and also $p^2 = (l - \bar{l})^2 = -s/4$. So we find

$$\sigma^{tot}(s) = \frac{e^2}{4s} h(q^2) \tag{461}$$

Naive considerations tell that the dimensionless function $h(q^2)$ cannot depend on q^2 and so is a number. Then the cross-section has the order e^2/s and rapidly falls with s. However we know that renormalization introduces a new parameter with a dimension of mass, which transforms into the QCD-parameter Λ_{QCD}. Therefore we may expect a non-trivial dependence of h on q^2.

To study this dependence we set up the Callan-Symanzik equation for $h(q^2)$. This requires knowledge of its renormalization, which is made within the QED. The one-particle irreducible transverse photon Green function is defined as

$$V^{(2)} = G_\gamma^{-1} \tag{462}$$

so that we find

$$V^{(2)} = Z_\gamma V_0^{(2)} \tag{463}$$

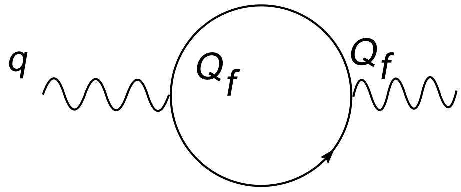

Figure 40: The lowest order contribution to the hadronic tensor

where Z_γ is the photon renormalization constant. Separating the transverse projector $P_\perp$ we have

$$V^{(2)}(q^2) = q^2 - \Pi(q^2), \quad V_0^{(2)}(q^2) = q^2 - \Pi_0(q^2) \tag{464}$$

where Π is the photon polarization operator. Its renormalization obviously goes as $q^2 - \Pi = Z_\gamma(q^2 - \Pi_0)$ or

$$\Pi(q^2) = Z_\gamma \Pi_0(q^2) - q^2(Z_\gamma - 1) \tag{465}$$

and is not purely multiplicative. However H is defined as a discontinuity in k^2 and the extra terms proportional to $(Z_\gamma - 1)$ are analytic in k^2 so that for function $h(k^2)$ we find

$$h(q^2) = Z_\gamma h_0(q^2) \tag{466}$$

The photon renormalization constant is different from unity in the second order in the electromagnetic coupling constant $e^2/(4\pi) = 1/137$. Neglecting this small electromagnetic correction we find

$$h(q^2) = h_0(q^2) \tag{467}$$

which implies that the photon anomalous dimension is zero (in fact it has order e^2). Therefore we find the Kallan-Symanzik equation for h:

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\Big)h(q^2, \mu, g) = 0 \tag{468}$$

and solving it, for $q^2 = q_0^2 e^{2t}$, $t >> 1$

$$h\Big(q_0^2 e^{2t}, \mu, g\Big) = h\Big(q_0^2, \mu, g(t)\Big) \tag{469}$$

where $g(t)$ is the running QCD coupling. In the limit $t \to \infty$ $g(t) \to 0$ so that at large q^2

$$h(q^2, \mu, g) = h(q_0^2, \mu, 0) + \mathcal{O}(1/\ln(q^2/q_0^2) \tag{470}$$

At $g = 0$ function h does not involve strong interactions and corresponds to the process in which the lepton pair annihilates into a pair of a quark and antiquark, of any flavour, which do not interact between themselves (Fig. 40). It coincides with the analogous function for the purely electromagnetic process $e_+e^- \to \mu^+\mu^-$ except for the difference in electromagnetic charges and the presence of colours and flavours. If the electromagnetic charge of the quark of flavour f is Q_f (in units e) then we find in the high-energy limit

$$R = \frac{\sigma^{tot}(e^+e^- \to hadrons)}{\sigma^{tot}(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2 \,\Big(+ \mathcal{O}(1/\ln(q^2/q_0^2)\Big) \tag{471}$$

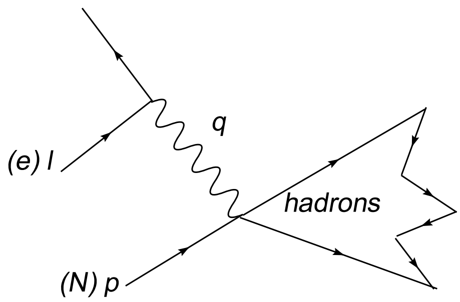

Figure 41: The deep inelastic lepton-hadron scattering

The number on the right-hand side in fact depends on the number of flavours which can be created at a given energy $\sqrt{s}$. In the limit $s \to \infty$ it includes all 6 flavours, so that

$$R = 3\Big(3\frac{4}{9} + 3\frac{1}{9}\Big) = \frac{15}{3}$$

However up to energies of the order of several tens of GeV production of b and t quarks is suppressed and we rather have

$$R = 3\Big(2\frac{4}{9} + 2\frac{1}{9}\Big) = \frac{10}{3}$$

This agrees quite well with the experimental data. Note that should we forget about the colour this number would be reduced to 10/9 in gross contradiction with the data. In this way the experimental data on lepton-annihilation into hadron clearly demonstrate that the colour exists and implies exactly 3 colour states for quarks.

2.9 Deep inelastic lepton-hadron scattering: kinematics

We pass to the process which represents a classical application of the perturbative QCD and has served to establish its validity as the theory of strong interaction, the deep inelastic lepton-hadron scattering (Fig. 41):

$$e(l) + N(p) \to e(l') + X(p') \tag{472}$$

where N represents the target nucleon and instead of the electron also μ and ν can be used as the projectile. Similarly to the preceding chapter the inclusive cross-section to observe the outgoing lepton can be factorized in to leptonic and hadronic parts;

$$d\sigma = \frac{1}{J}\frac{d^3l'}{(2\pi)^3 2l'_0} L_{\alpha\beta} H^{\alpha\beta}\frac{e^2}{q^4} \tag{473}$$

Neglecting the lepton mass we have $J = 4lp$. The c.m. energy squared for the reaction is $s = (l+p)^2 = m^2 + 2lp$ where m is the nucleon mass. We shall be interested in the region of high energies when $s >> m^2$, so that $s \simeq 2lp$.

For unpolarized initial particles the leptonic part is

$$L_{\alpha\beta} = \frac{1}{2}\sum_{pol}[\bar{u}(l')\gamma_\alpha u(l)]_{\xi\eta} \cdot [\bar{u}l\gamma_\beta u(l')]_{\eta\xi} \tag{474}$$

As for the lepton pair annihilation into hadrons, it is calculated explicitly. Neglecting the lepton mass

$$L_{\alpha\beta} = \frac{1}{2}\mathrm{Tr}\ \{\hat{l}'\gamma_\alpha\hat{l}\gamma_\beta\} = 2(l'_\alpha l_\beta + l_\alpha l'\beta - g_{\alpha\beta}ll') \tag{475}$$

The hadronic part is

$$H_{\alpha\beta} = \frac{1}{2}\sum_{pol}\sum_n \int d\tau_n(p+q-p_n)\langle N(p)|j^{em}_\alpha(0)|X_n\rangle\langle X_n|j^{em}_\beta(0)|N(p)\rangle \tag{476}$$

Up to an unimportant coefficient, it has an obvious meaning of the total cross-section for the virtual-photon-nucleon collision. It depends on two variables: the c.m. energy of the collision $(p+q)^2 = m^2 + 2pq + q^2$ and the photon virtuality q^2

It is easy to find that $q^2 = (l-l')^2 < 0$. Indeed in the antilab system $q^2 = 2m_e^2 - 2m_e\sqrt{m_e^2 + \mathbf{l}'^2} < 0$. Usually one introduces a positive $Q^2 = -q^2$. The missing mass squared $M^2 = (p+q)^2 = m^2 + 2pq + q^2 > m^2$ So one finds a restriction $2pq - Q^2 > 0$ from which it follows that the so-called scaling variable

$$0 \le x = \frac{Q^2}{2pq} \le 1 \tag{477}$$

The value $x = 1$ obviously follows when $M^2 = m^2$ and corresponds to purely elastic collision. In the general case we have

$$M^2 = 2pq(1-x) \equiv 2\nu(1-x) \tag{478}$$

where $2\nu = 2pq$ is approximately the c.m. energy squared for the photon-nucleon collision. The missing mass squared is a finite part of it and grows with ν. Deep inelastic scattering corresponds to exactly this situation when $\nu \to \infty$ and x is a fixed finite positive number smaller than unity. The standard variables which characterize the hadronic part of the deep inelastic scattering are x and Q^2.

As in the preceding chapter we lift the restriction on the energy-momentum of the intermediate hadronic states representing

$$(2\pi)\delta^4(q+p-p_n) = \int d^4z e^{iz(p+q-p_n)}$$

and using

$$e^{iz(p-p_n)}\langle N(p)|j^{em}_\alpha(0)|X_n\rangle = \langle N(p)|j^{em}_\alpha(z)|X_n\rangle$$

Summing over all intermediate states we find

$$H_{\alpha\beta} = \frac{1}{2}\sum_{pol}\int d^4z e^{iqz}\langle N(p)|j^{em}_\alpha(z)j^{em}\beta(0)|N(p)\rangle \tag{479}$$

This should be compared with the forward scattering amplitude for the virtual-photon-nucleon scattering (see Fig. 42)

$$\mathcal{A}_{\alpha\beta} = \frac{i}{2}\sum_{pol}\int d^4z e^{iqz}\langle N(p)|\mathrm{T}\ \{j^{em}_\alpha(z)j^{em}\beta(0)\}|N(p)\rangle \tag{480}$$

Obviously

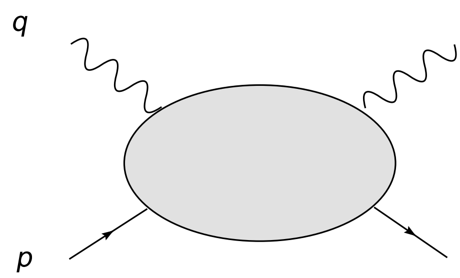

Figure 42: The forward $\gamma^* - N$ scattering amplitude

$$H_{\alpha\beta}\frac{1}{i}\mathrm{Disc}_{(q+p)^2>m^2}\ \mathcal{A}_{\alpha\beta} \tag{481}$$

Note that both H and $\mathcal{A}$ are dimensionless:

$$\dim j^{em}(x) = 3,\ \dim\langle N(p)||N(p)\rangle = -2,\quad \dim d^4x = -4 \tag{482}$$

The tensor structure of H and T is determined by current conservation:

$$q^\alpha H_{\alpha\beta} = H_{\alpha\beta}q^\beta = 0 \tag{483}$$

and similarly for $\mathcal{A}$ From two vectors p and q one can construct two tensors which are orthogonal to q:

$$g^{(q)}_{\alpha\beta} = g_{\alpha\beta} - \frac{q_\alpha q\beta}{q^2} \tag{484}$$

and

$$\Big(p_\alpha - q_\alpha\frac{pq}{q^2}\Big)\Big(p_\beta - q_\beta\frac{pq}{q^2}\Big) = p^{(q)}_\alpha p^{(q)}_\beta \tag{485}$$

Therefore H can be presented via two scalar dimensionless structure functions:

$$\frac{1}{2\pi}H_{\alpha\beta} = -g^{(q)}_{\alpha\beta}F_1(x,Q^2) + \frac{1}{\nu}p^{(q)}_\alpha p^{(q)}_\beta F_2(x,Q^2) \tag{486}$$

A similar representation is valid for $\mathcal{A}$ with two scalar functions $Z_{1,2}$.

The inclusive cross-section (473) can be directly expressed via these two structure functions. Omitting rather tedious derivation we find

$$d\sigma = \frac{1}{J}\frac{d^3l'}{(2\pi)^3 2l'_0}I \tag{487}$$

where

$$I = \frac{\pi e^4}{8mE^2E'\sin^4\frac{\theta}{2}}\Big(2F_1\sin^2\frac{\theta}{2} + \frac{m^2}{\nu}F_2\cos^2\frac{\theta}{2}\Big) \tag{488}$$

The two structure functions $F_{1,2}$ may be related to the cross-sections $\sigma_\perp$ and σ_L for the scattering off the nucleon of the virtual photons polarized in the plane transverse to the $p-q$ plane and in that plane respectively. For the transverse photon polarization vector one can take the transverse unit vectors $e_\perp$ orthogonal to both q and p and normalized as $e_\perp^2 = -1$. Then one immediately finds

$$\sigma_\perp = c\ F_1 \tag{489}$$

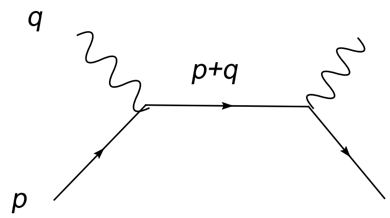

Figure 43: The simplest diagram for the hadronic tensor H

where the constant c depends on the choice of flux (which is unphysical for virtual photons). The longitudinal photon polarization vector is evidently proportional to $p^{(q)}$:

$$e_L = \frac{p^{(q)}}{\sqrt{-p^{(q)2}}}$$

Then one has

$$\sigma_L = c\Big(F_1 - \frac{p^{(q)2}}{2\nu}F_2\Big) = c\Big[F_1 - \Big(\frac{m^2}{\nu} - \frac{\nu}{q^2}\Big)F_2\Big] \simeq c\Big[F_1 - \frac{1}{2x}F_2\Big] \tag{490}$$

From naive considerations we again may expect that dimensionless functions $F_{1,2}(x, Q^2)$ do not depend on variable Q^2 with dimension mass squared but only on x. This situation is known as scaling. Again renormalization introduces parameter Λ_{QCD} with dimension, so one expects correction to the scaling picture following from the QCD dynamics. Experimental observation indeed show violation of scaling with the variation of Q^2. They also testify that the longitudinal cross-section σ_L is very small as compared to the transverse one, which implies that with a high precision one has

$$F_2(x, Q^2) = 2xF_1(x, Q^2) \tag{491}$$

This relation can be understood if one assumes that the particle interacting with the photon has spin 1/2. Indeed consider the simplest diagram for H in the lowest order of perturbation theory, shown in Fig. 43. It gives

$$H_{\alpha\beta} = \frac{1}{2}\sum_{pol} \bar{u}(p)\gamma_\alpha(m + \hat{p} + \hat{q})\gamma_\beta u(p)(2\pi)\delta(m^2 - (p+q)^2)$$

$$= 2\pi\frac{1}{2\nu}\delta(x-1)2\Big((p+q)_\alpha p_\beta + p\alpha(p+q)_\beta - g_{\alpha\beta}p(p+q)\Big) \tag{492}$$

From this we conclude

$$F_1 = \delta(x-1), \quad F_2 = 2\delta(x-1) = 2x\delta(x-1) \tag{493}$$

so that we indeed find $F_2 = 2xF_1$. Obviously if the charged particle has spin 0 we shall not find a term proportional to $g_{\alpha\beta}$ so that in that case $F_1 = 0$.

This fact gave a basis for the parton model of Feynman, in which it is assumed that the hadron is made of components, the charged sort of them having spin 1/2. Then relation (491) will be valid for the resulting structure functions.

To finish let us study the region of z which give the decisive contribution in the integral representations (479) or (480). Let us choose the system in which $p_\perp = q_- = 0$. Then $q^2 = q_\perp^2$ and $\nu = p_- q_+$. With a finite p_- we have $q_\perp \to \infty$ and $q_+ \to \infty$. In the exponent we have

$$qz = q_+ z_- + (qz)_\perp$$

So the important region is

$$z_- \to 0 \quad \text{and} \quad z_\perp \to 0 \tag{494}$$

As follows $z^2 \to 0$ although z does not go to zero, since z_+ is finite. Therefore to find the asymptotic of H or $\mathcal{A}$ in our kinematic region we have to study the matrix element in the integrand on the light cone $z^2 \to 0$. To this end we shall use the so-called operator expansion on the light cone, which will be studied in the next chapter.

2.10 Composite operators and their renormalization

To simplify we study composite operators in the thoery of a single scalar real field $\phi(x)$ (with a renormalizable interaction $\lambda\phi^4$). A composite operator is any local operator constructed from a finite number of fields $\phi(x)$ and their derivatives. Locality allows to always make a shift and relate the composite operator to the point $x = 0$. Examples of composite operators are ϕ^n, $\phi^{n-1}\partial_{\alpha_1}...\partial_{\alpha_s}\phi^{n_2}$, $\phi^{n-1}\partial_{\alpha_1}...\partial_{\alpha_s}(\partial^2)^m\phi^{n_2}$ and so on. Fundamental fields ϕ are assumed to be renormalized. Any composite operator O_i may be fully characterized by a set of Green functions which contain this operator and a number n of fundamental operators Φ:

$$G_n(x_1, ...x_n) =< \mathrm{T}\{O_i \phi(x_1)...\phi(x_n)\} > \tag{495}$$

Amputating the external pair Green functions for the fundamental fields we obtain the corresponding single-particle irreducible Green functions V_n

Each operator O_i may be characterized by the number l_i of operators ϕ which it contains. It is also a Lorentz tensor of a certain rank, corresponding to the number of Lorentz indexes $\alpha_1, ...\alpha_s$. Such a tensor is generally reducible in the tensor algebra: one can construct from it tensors of smaller ranks contracting it with invariant tensors $g_{\alpha\beta}$ and $\epsilon_{alpha\beta\gamma\delta}$. In fact we shall mostly deal with symmetric tensors so practically one can contract them with $g_{\alpha\beta}$. Subtracting results of such contraction from the original tensor one can make it irreducible, that is giving zero under such contractions. The rank of this irreducible tensor is called Lorentz spin s. Each composite operator can be presented as a sum of operators with a given spin s. E.g.

$$O = \phi\partial_\alpha\partial_\beta = \phi(\partial_\alpha\partial_\beta - (1/4)g_{\alpha\beta}\partial^2\phi + (1/4)g_{\alpha\beta}\phi\partial^2\phi = O_2 + (1/4)g_{\alpha\beta})O_0$$

where

$$O_2 = \phi(\partial_\alpha\partial_\beta - (1/4)g_{\alpha\beta}\partial^2\phi$$

is an irreducible operator of spin 2 and

$$O_0 = \phi\partial^2\phi$$

is an irreducible operator of spin 0. In the following we study exclusively operators with a given spin s. One more characteristic of a composite operator is its dimension d in units of mass. It is just a sum of dimensions of its components ϕ and ∂ each of them

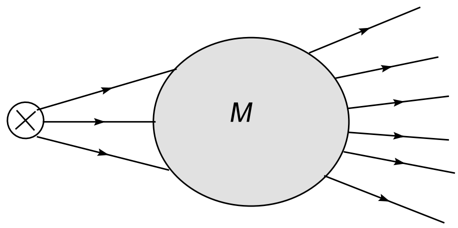

Figure 44: The schematic structure of the Green function V_n

contributing one unit. In the following an important role will be played by the twist t of the operator which is the difference between its dimension and spin

$$t = d - s$$

The twist does not change upon introduction of uncontracted derivatives. For all non-trivial operators we have

$$d \geq 2, \quad s \geq 0, \quad t \geq 2 \tag{496}$$

Let us study the irreducible Green functions $V(i)_n$ in the perturbation theory. They are represented by standard Feynman diagrams with a single vertex of a special kind. which corresponds to the operator O_i. To this vertex exactly l lines should be attached and it also has to contain the due number of derivatives and their contraction. In the momentum representation we can consider the momenta p_i of n external lines as independent. The momentum corresponding to the lines attached to the vertex O_i will then be just the sum of these external momenta. In the lowest approximation we obviously find that $n = l$ so that

$$V_n^{(0)} = \delta_{nl} T_{\alpha_1,\ldots\alpha_s}(p_i) P_q(p_i p_j) \tag{497}$$

where $T_{\alpha_1,\ldots\alpha_s}(p_i)$ is a certain irreducible tensor of rank s constructed from the external momenta and $P_q(p_i p_j)$ is a homogeneous polynomial of rank q constructed from scalar products of external momenta (corresponding to contracted derivatives). Obviously one has

$$t = l + 2q \tag{498}$$

(and does not depend on spin s). Considered as a set of all Green function for arbitrary n (497) represents our operator in the lowest order of the perturbation theory;

$$O_i \equiv V_n^{(0)} \tag{499}$$

The contribution of higher orders can be represented as a sum of all diagrams in which l lines attached to the vertex pass to n external lines with all possible interaction in between. In other words the vertex is attached to the Green function M_{ln} corresponding to the transition of l initial (off-shell) fundamental particles to n number of such particles (Fig. 44).

$$V_n(p_i) = V_n^{(0)}(p_i) + \int \prod_{i=1}^{l} d^4k_i (2\pi)^4 V_n^{(0)}(k_i) M_{ln}(k_1..k_l|p_1,..p_n) \tag{500}$$

or symbolically

$$O_i = O_i^{(0)} + O_i^{(0)} \otimes M \tag{501}$$

The Green function M is finite after renormalization. However integrations over the internal lines attached to the vertex of momenta k_i may introduce new divergence So the composite operator O_i defined by (501) is generally divergent in the ultraviolet and requires renormalization. This latter is much simpler that the standard renormalization procedure since the composite vertex does not participate in the interaction. It cannot be internal and (in our example) is just a single vertex in all Feynman diagrams.

To study renormalization we first have to find out the character of divergent terms. As usual it will heavily rest on the dimensional considerations. The initial Green function G_n in the coordinate space has evidently dimension $\dim G_n(x_1, ..x_n) = d + n$. in the momentum space its dimension is $d - 3n$ so that the on-particle irreducible Green function will have dimension $\dim V_n(p_i, ..p_n) = d - n$ Possible divergent terms are to be constructed only from the external momenta and ultraviolet cutoff Λ, since at large integration momenta all masses can be neglected. V_n has to be an irreducible tensor of rank s and so can be expanded in all possible tensors of rank s constructed from the external momenta $T^j_{\alpha_1,\ldots\alpha_s}(p_i)$. So the divergent part is

$$\mathrm{div} V_n = \sum_j T^j_{\alpha_1,\ldots\alpha_s}(p_i) C_j(\Lambda, p_i) \tag{502}$$

Since $\dim T_j = s$ we find that $\dim C_j = d - n - s = t - n$. This implies that C_j have a structure

$$C_j = \sum_{q=0} c_{jq} P^{(j)}_q(p_i p_j) \Lambda^{t-n-2q}$$

with summation over values of q until $t - n - 2q \geq 0$ One observes that the leading divergence correspond to $q = 0$ and has the form Λ^{t-n} It is fully determined by the twist of the operator and the number of external line. For fixed t one finds a finite number of divergent diagrams with $n \leq t$ and for $n = t$ the divergence is logarithmic.

It is to be stressed that for an operator O_i with given l, d, s and t and type of tensor T^i and polynomial P^i the divergent expression contains contributions with different l, d and t, tensors T^j and polynomials P^j. To make this more instructive we can rewrite the divergent part (502) as follows

$$\mathrm{div} V_n = \sum_{j, l_j, q_j} c_{j q_j} T^j_{\alpha_1,\ldots\alpha_s}(p_i) P^{(j)}_{q_j}(p_i p_j) \delta_{n l_j} \Lambda^{t - l_j - 2q_j} \tag{503}$$

Comparing with (497) we see that in the sum we find the zero-order vertex part of different composite operators O_j, which have the same spin but different values of number of lines l_j and twist $t_j = l_j + 2q_j$. So we find

$$\mathrm{div} V_n = \sum_j c_j \Lambda^{t - t_j} [V^{(0)}_j]_n \tag{504}$$

or, since this is true for any n simply

$$\mathrm{div} O_i = \sum_j c_{ij} \Lambda^{t_i - t_j} O^{(0)}_j \tag{505}$$

Summation obviously goes over all operators O_j with $t_j \leq t$. So in the divergent part of the composite operator there appear all bare operators with the same or smaller twist. This phenomenon is called operator mixing. The smaller the twist of the mixed operator

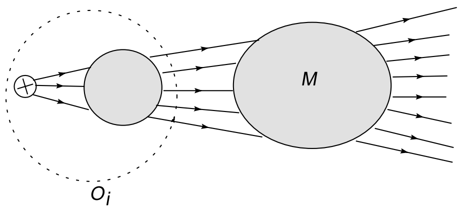

Figure 45: The schematic structure of V_n with the separated diagrams for the complete composite operator O_i

the stronger is the divergence. For the mixed operator of the same twist the divergence is logarithmic.

We have to stress that the divergence we have found is the so-called proper divergence of the diagram, which correspond to the case when all the momenta joined at the vertex are equally large. Apart from this divergence, as in the usual treatment, there may be partial divergencies, when only some of these momenta are large and the rest remain finite. In this case the dimensional argument is not valid. However it is not difficult to find the general rule for separating these partial divergence. In fact if the divergent contribution comes from the integration only over some of the lines attached to the vertex then this divergent part will depend on internal momenta which are not large and on which the rest of the diagram will depend non-trivially. This corresponds to presenting the diagram in the form shown in Fig. 45 in which all the momenta to the right of the separated full O_i are finite. This brings us to the symbolic relation

$$O_i^{(0)} \otimes M = \sum_j c_{ij} \Lambda^{t_i - t_j} O_j^{(0)} + \sum_j c_{ij} \Lambda^{t_i - t_j} O_j^{(0)} \otimes M + \text{f.p.} O_i^{(0)} \tag{506}$$

where f.p. denote the finite part of the contribution. Using this and (501) we obtain

$$\text{f.p.} O_i = \sum_j \left(\delta_{ij} - c_{ij} \Lambda^{t_i - t_j}\right) O_j^{(0)} + \sum_j \left(\delta_{ij} - c_{ij} \Lambda^{t_i - t_j}\right) O_j^{(0)} \otimes M = \sum_j Z_{ij} O_j \tag{507}$$

where

$$Z_{ij} = \delta_{ij} - c_{ij} \Lambda^{t_i - t_j} \tag{508}$$

Thus taking for the renormalized operator O_{iR} the finite part of O_i we find their relation in the form of matrix multiplication

$$O_{iR} = \sum_j Z_{ij} O_j, \quad t_j \leq t_i \tag{509}$$

This rule generalizes the standard renormalization rule for the fundamental fields.

2.11 Operator expansion

In some applications one meets with products of operators in different space-time points, like $A(0)B(-x)$. It may be both a simple product or a T-product. We shall study the case of T-product not indicating this explicitly. Operators A and B are assumed to be renormalized.

One can write the standard expansion around $x=0$

$$B(-x)=\sum_{s=0}\frac{1}{s!}(-1)^s x^{\alpha_1}...x^{\alpha_s}\partial_{\alpha_1}...\partial_{\alpha_s}B(0) \tag{510}$$

and subsequently an expansion of $A(0)B(-x)$

$$A(0)B(-x)=\sum_{s=0}\frac{1}{s!}(-1)^s x^{\alpha_1}...x^{\alpha_s}A(0)\partial_{\alpha_1}...\partial_{\alpha_s}B(0) \tag{511}$$

in terms of composite operators $A(0)\partial_{\alpha_1}...\partial_{\alpha_s}B(0)$. These operators are however not irreducible under Lorentz indeces. We can present then as a sum of irreducible operators with spins s, $s-2$,...:

$$A(0)\partial_{\alpha_1}...\partial_{\alpha_s}B(0)=A(0)T^{(s)}_{\alpha_1...\alpha_s}(\partial)B(0)+g_{\alpha_1\alpha_2}T^{(s-2)}(\partial)B(0)+... \tag{512}$$

where $T^{(s)}$ is some irreducible Lorentz tensor of spin s. Then expansion (511) will be rewritten as

$$A(0)B(-x)=\sum_{s,q}A(0)T^{(s)}_{\alpha_1...\alpha_s}(\partial)B(0)x^{\alpha_1}...x^{\alpha_s}P_q(x^2) \tag{513}$$

where P_q is a polynomial of rank q

In fact the composite operators $A(0)T^{(s)}_{\alpha_1...\alpha_s}(\partial)B(0)$ do not exist although A and B are taken renormalized. Using the perturbative theory Wilson found a modification of this expansion valid after the composite operators in the sum are renormalized. One can easily recognize this modification in the simple example of $A=\phi^{n_1}$ and $B=\Phi^{n_2}$. Then already in the lowest order of perturbation theory for $A(0)B(-x)$ from contractions between operators $\phi(0)\phi(-x)$ one will find a sum of different operators multiplied by powers of $\Delta(-x)=(1/x^2)$. In higher orders of perturbation theory one finds Green functions in which these powers of $1/x^2$ are multiplied by functions finite at $x=0$ and admitting expansion in powers of x. So the net change is twofold. First, polynomials $P_q(x^2)$ are changed by functions of x^2 which may be singular at $x^2\to 0$. Second, apart from original operators $A(0)T^{(s)}_{\alpha_1...\alpha_s}(\partial)B(0)$ also other local operators appear. Renormalization will introduce still more local operators into the sum.

This brings us to the operator expansion as proposed by Wilson:

$$A(0)B(-x)=\sum_{i,s}O^{\alpha_1...\alpha_s}_{is}x^{\alpha_1}...x^{\alpha_s}C_{is}(x^2) \tag{514}$$

where summation goes over all values of Lorentz spin s and all local operators O_{is} with a given spin s. Wilson coefficients $C(x^2)$ are some functions of x^2 which may be singular at $x^2\to 0$. It is assumed that all operators in (514) are renormalized, so that Wilson coefficients are finite.

The benefit of the Wilson expansion consists in the possibility to extract the leading contribution to the product $A(0)B(-x)$ in the limit $x\to 0$ or $x^2\to 0$.

To start note that all operators entering the Wilson expansion may be assumed to have definite dimensions. Then we obtain a relation: $d_A+d_B=d_{is}+\dim C_{is}-s$ where d_A, d_B and d_{is} are dimensions of operators A, B and O_{is} respectively. So we obtain

$$\dim C_{is}=d_A+d_B-d_{is}+s=d_A+d_B-t_{is} \tag{515}$$

where t_{is} is the twist of operator O_{is}. The canonical dimension $\dim C_{is}$ determines its crude behaviour at $x^2 \to 0$

$$C_{is}(x^2) \sim \Big(\frac{1}{x^2}\Big)^{\dim C_{is}/2} \tag{516}$$

The greater the dimension the stronger is the singularity at $x^2 \to zero$. From (515) we conclude that this dominant singularity comes from the term in the Wilson expansion with operators O_{is} with a minimal twist (=2). As in the previous situations, in absence of any massive parameters this canonical dimension would completely determine the x-dependence of $C(x^2)$. However renormalization introduces a massive parameter μ which can be substituted by the QCD parameter Λ_{QCD}. This opens the way to corrections to the canonical behaviour in the form of logarithmic corrections containing $\log x^2\Lambda^2_{QCD}$. They can be determined by the Callan-Symanzik equation for the Wilson coefficients.

To construct it first we set up the Callan-Symanzik equation for the Green functions with composite operators. Let $V_n^{(A)}$ be the (one-particle irreducible) Green function with n fundamental operators and composite operator A. The renormalization procedure relates it with the bare Green function as

$$V_n^{(A)} = Z^{n/2}\sum_{A'} Z^{(A)}_{AA'}\big(V_n^{(A')}\big)_0 \tag{517}$$

where on the right-hand side appear the unrenormalized Green function with all operators A' which mix with A in the process of renormalization together with the matrix of renormalization constants $Z^{(A)}_{AA'}$ Applying the operator $\mu(\partial/\partial\mu)$ to this relation we shall obtain the Callan-Symanzik equation in the matrix form

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) - \gamma^{(A)}(g)\Big)V_n^{(A)} = 0 \tag{518}$$

Here V_n^A is considered as column of all Green functions $V(A)$ for different A. The first three terms in (518) are just unit matrices proportional to $\delta_{AA'}$. The non-trivial anomalous dimension $\gamma^{(A)}$ is defined as a matrix

$$\gamma^{(A)} = \mu\frac{\partial}{\partial\mu}Z^{(A)}\cdot {Z^{(A)}}^{-1} \tag{519}$$

Thus the Callan-Symanzik equation for composite operators has the same form as for fundamental operators with the only difference that due to mixing the anomalous dimension for the composite operator has a matrix structure.

The same is true when the Green functions include several composite operators at different space time points. In particular for the product $A(0)B(-x)$ we get the equation

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) - \gamma^{(A)}(g) - \gamma^{(B)})(g)\Big)V_n^{(AB)} = 0 \tag{520}$$

where anomalous dimension $\gamma^{(B)}$ is defined similarly to (519) through the matrix of renormalization constants of operators B.

Now we take the Wilson expansion for $A(0)B(-x)$. For brevity we include the product of $x^{\alpha_1,\dots\alpha_s}$ into the expansion coefficients and rewrite it just as

$$A(0)B(-x) = \sum_i O_iC_i(x) \tag{521}$$

where index i now also includes spin s. Correspondingly the Green function $V_n^{(AB)}$ will be presented as

$$V_n^{(AB)}(x) = \sum_i V_n^{(i)} C_i(x) \tag{522}$$

where $V_n^{(i)}$ is the Green function with a composite operator O_i. Differentiating this relation in μ we have to take into account that the Wilson coefficients depend on both g and μ. So putting relation (522) into the Callan-Symanzik equation (520) we get

$$\sum_i \Big\{ C_i(x)\Big(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)-\gamma^{(A)}(g)-\gamma^{(B)})(g)\Big)V_n^{(i)}+V_n^{(i)}\Big(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\Big)C_i(x)\Big\} = 0 \tag{523}$$

Now we recall the Callan-Symanzik equation for $V_n^{(i)}$

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) - \gamma^{(i)}(g)\Big)V_n^{(i)} = 0 \tag{524}$$

and putting this into (523) finally get the Callan-Zymanzik equation for the Wilson coefficients:

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma^{(i)}(g) - \gamma^{(A)}(g) - \gamma^{(B)}(g)\Big)C_i(x) = 0 \tag{525}$$

In this equation it is understood that $C_i(x)$ is in fact a matrix in three pairs of indeces: (AA'), (BB') and (ii'). Each of the three anomalous dimensions is a matrix in one pair and diagonal in two others.

From the Callan-Symanzik equation follows the behaviour of the coefficients $C_i(x)$ when $x \to 0$:

$$C_i(e^{-t}x, g) = e^{\Gamma t}\mathrm{P}\ \exp\Big\{ - \int_0^t dt'\Big(\gamma^{(A)}(g') + \gamma^{(B)}(g') - \gamma^{(i)}(g')\Big)\Big\}C_i\Big(x, g(t)\Big) \tag{526}$$

where the canonical dimension is $\Gamma = d_A + d_B - \dim O_i$.

2.12 Deep inelastic lepton-hadron scattering: evolution in Q^2

2.12.1 Moments of the structure functions

Returning to our formulas for the hadronic tensors $H_{\alpha\beta}$ or $\mathcal{A}_{\alpha\beta}$ in terms of the average in the nucleon state of the product of two electromagnetic currens, we have to study, say, operator

$$\mathrm{T}\ \{j_\alpha^{em}(z)j_\beta^{em}(0)\} \tag{527}$$

at $z^2 \to 0$. We expand this product in local operators at $z = 0$ in the form (514). Our first problem is how to associate the external indeces $\alpha\beta$ with the indeces $\alpha_1...\alpha_s$ which appear in the expansion. Note that the product $z^{\alpha_1}...z^{\alpha^s}C_{is}(z^2)$ in the expansion can be substituted by $\partial^{\alpha_1}....\partial^{\alpha_s}\tilde{C}_{is}(z^2)$. Indeed the difference contains terms proportional to $g_{\alpha_i\alpha_k}$ which give zero contracted with irreducible spin-s operators $O_{is}^{\alpha_1,...\alpha_s}$. But then the integral (480) each derivative ∂_{α_i} is equivalent to q_{α_i}. Associating the structure functions with terms proportional to either $g_{\alpha\beta}$ or $p_\alpha p_\beta$ we find that none of the derivatives ∂_{α_i} may carry indexes α or β. These indeces may appear only in the tensors g. This gives two possible structures in the expansion:

$$\mathrm{T}\ \{j_\alpha^{em}(z)j_\beta^{em}(0)\} = \sum_{is} O^{\alpha_1,...\alpha_s}\Big(- g_{\alpha\beta}\partial_{\alpha_1}...\partial_{\alpha_s}\tilde{C}^{(1)}(z^2) + g_{\alpha\alpha_1}g_{\beta\alpha_2}\partial_{\alpha_2}...\partial_{\alpha_s}\tilde{C}^{(2)}(z^2)$$

$$= \sum_{is} O^{\alpha_1,...\alpha_s}\Big(- g_{\alpha\beta} z_{\alpha_1}...z_{\alpha_s} C^{(1)}(z^2) + g_{\alpha\alpha_1} g_{\beta\alpha_2} z_{\alpha_2}...z_{\alpha_s} C^{(2)}(z^2) \qquad (528)$$

We put this representation into the matrix element and take into account that

$$\frac{1}{2}\sum_{pol}\langle N(p)|O_{is}^{\alpha_1...\alpha_s}|N(p)\rangle = o_{is}\Big(p_{\alpha_1}...p_{\alpha_s} - (trace\ terms)\Big) \qquad (529)$$

where $o_i s$ are some numbers; *trace terms* contain less numbers of p_s and include several metric tensors g necessary to make the whole construction irreducible. Inspecting (528) we see that the first term in the average will be proportional to $g_{\alpha\beta}$ and the second to $p_\alpha p_\beta$. So if

$$\frac{1}{2\pi}\mathcal{A}_{\alpha\beta} = -g_{\alpha\beta}^{(q)} Z_1(x, Q^2) + \frac{1}{\nu} p_\alpha^{(q)} p_\beta^{(q)} Z_2(x, Q^2) \qquad (530)$$

we find

$$Z_1 = \frac{i}{2\pi}\int d^4 z e^{iqz} \sum_{is} o_{is} C_{is}^{(1)}(z^2) z^{\alpha_1}...z^{\alpha_s}\Big(p_{\alpha_1}...p_{\alpha_s} - (trace\ terms)\Big) \qquad (531)$$

$$Z_2 = \frac{i}{2\pi}\nu\int d^4 z e^{iqz} \sum_{is} o_{is} C_{is}^{(2)}(z^2) z^{\alpha_3}...z^{\alpha_s}\Big(p_{\alpha_3}...p_{\alpha_s} - (trace\ terms)\Big) \qquad (532)$$

We change $z^{\alpha_1}...z^{\alpha_s}$ to

$$-i\frac{\partial}{\partial q_{\alpha_1}}... - i\frac{\partial}{\partial q_{\alpha_s}}$$

and these in turn to $(-2q_{\alpha_1})...(-2q_{\alpha_s})(\partial/\partial q^2)^s + (trace\ terms)$ The trace terms will give nothing since they are contracted with the irreducible product of p_s and we get

$$Z_1 = \frac{i}{2\pi}\sum_{is} o_{is}\Big(\nu^s - (trace\ terms)\Big)(-2i)^s\Big(\frac{\partial}{\partial q^2}\Big)^s\int d^4 z e^{iqz} C^{(1)_{is}(z^2} \qquad (533)$$

$$Z_2 = \frac{i}{2\pi}\nu\sum_{is} o_{is}\Big(\nu^{s-2} - (trace\ terms)\Big)(-2i)^{s-2}\Big(\frac{\partial}{\partial q^2}\Big)^{s-2}\int d^4 z e^{iqz} C^{(2)_{is}(z^2} \qquad (534)$$

At this moment it is instructive to study the canonical dimensions of the Fourier transforms of $C(1,2)_{is}$. Obviously in the coordinate space

$$\dim C^{(1)}(z^2) = 2\dim j^{em}(z) - t_{is} = 6 - t_{is}, \quad \dim C^{(2)}(z^2) = 2\dim j^{em}(x) - t_{is} - 2 = 4 - t_{is} \qquad (535)$$

where t_{is} is the twist of operator O_{is}. The Fourier transforms will have dimensions raised by 4

$$\dim C^{(1)}(q^2) = 2 - t_{is}, \quad \dim C^{(2)}(q^2) = -t_{is} \qquad (536)$$

The leading behaviour at large q^2 will come from the operators with the minimal twist equal to 2. Then $C^{(1)}(q^2) \sim const$ and $C^{(2)}(q^2) \sim 1/q^2$. Note also that the trace terms in (533)and (534) are subleading at large ν In fact in the simplest case of spin 2

$$p_{\alpha_1} p_{\alpha_2} - (trace\ terms) = p_{\alpha_1} p_{\alpha_2} - \frac{1}{4} g_{\alpha_1\alpha_2} m^2$$

so that multiplied by $q^{\alpha_1} q^{\alpha_2}$ we get

$$\nu^2 - \frac{1}{4} q^2 m^2 = \nu^2\Big(1 + x\frac{m^2}{4\nu}\Big)$$

The second term gives a small correction of the order m^2/Q^2.

Taking this into account we drop the trace terms in (533)and (534) and present each 2ν as Q^2/x. Then we define new coefficients

$$B^{(1)}_{is}(q^2) = io_{is}\Big(iq^2\frac{\partial}{\partial q^2}\Big)^s \int d^4z e^{iqz} C^{(1)}(z^2) \tag{537}$$

$$B^{(2)}_{is}(q^2) = io_{is}\Big(iq^2\frac{\partial}{\partial q^2}\Big)^{s-2}\Big(-\frac{1}{2}q^2\Big) \int d^4z e^{iqz} C^{(2)}(z^2) \tag{538}$$

to finally obtain

$$Z_1(x,Q^2) = \frac{1}{2\pi}\sum_{is} B^{(1)}_{is}(Q^2)\Big(\frac{1}{x}\Big)^s, \quad Z_2(x,Q^2) = \frac{1}{2\pi}\sum_{is} B^{(2)}_{is}(Q^2)\Big(\frac{1}{x}\Big)^{s-1} \tag{539}$$

Now we have to relate these expansion to the properties of the structure functions $F_{1,2}(x,Q^2)$ which are just the discontinuities of $Z_{1,2}(x,Q^2)$ considered as functions of ν at positive values $\nu > (1/2)Q^2$. We can consider this discontinuity as that across the cut in the variable x at $0 < x < 1$ and fixed Q^2. Obviously the forward scattering amplitude $\mathcal{A}_{\alpha\beta}$ considered as function of its energy $(p+q)^2$ at fixed $t = 0$ will also have a crossed cut at $(p-q)^2 > 0$, that is at $-1 < x < 0$. By crossing symmetry

$$T_{\alpha\beta}(p,q) = T_{\beta\alpha}(p-q) \tag{540}$$

which implies

$$Z_{1,2}(x,Q^2) = Z_{1,2}(-x,Q^2) \tag{541}$$

In the complex x plane we find a cut on the real axis at $-1 < x < 1$, so that we may write a dispersion relation (suppressing the fixed variable Q^2)

$$Z_{1,2}(x) = \frac{1}{2\pi i}\int_{-1}^{+1}\frac{dx'}{x-x'}\mathrm{Disc}_\nu Z_{1,2}(x') \tag{542}$$

From (541) we find

$$Z_{1,2}(-x) = \frac{1}{2\pi i}\int_{-1}^{+1}\frac{dx'}{-x-x'}\mathrm{Disc}_\nu Z_{1,2}(x') = \frac{1}{2\pi i}\int_{-1}^{+1}\frac{dx'}{-x+x'}\mathrm{Disc}_\nu Z_{1,2}(-x')$$

which implies that the structure functions can be defined at $x < 0$ as odd functions in x.

$$\mathrm{Disc}_\nu Z_{1,2}(x,Q^2) = iF_{1,2}(x,Q^2) = -iF_{1,2}(-x,Q^2) \tag{543}$$

With this definition we have

$$Z_{1,2}(x,Q^2) = \frac{1}{2\pi}\int_{-1}^{+1}\frac{dx'}{x-x'}F_{1,2}(x') \tag{544}$$

Expansion in values of spin s in (539) is in fact expansion of $Z_{1,2}$ around infinity in the x-plane. Taking $x >> x'$ in (544) we find

$$Z_{1,2}(x,Q^2) = \frac{1}{2\pi x}\sum_{s=0}\Big(\frac{1}{x}\Big)^s\int_{-1}^{+1} dx' x'^s F_{1,2}(x',Q^2) \tag{545}$$

Comparing this with expansions (539) we finally obtain

$$\int_{-1}^{1} dx x^{s-1} F_1(x,Q^2) = \sum_i B_{is}^{(1)}(Q^2), \quad \int_{-1}^{1} dx x^{s-2} F_2(x,Q^2) = \sum_i B_{is}^{(2)}(Q^2) \tag{546}$$

On the left-hand side there are so-called moments of the structure functions F_1 and F_2 considered as functions of x. They are directly observable (if not equal to zero by oddness of F's). These moments are expressed by a sum of (all) terms B_{is} for a given spin s related to a set of composite operators with a given s and minimal twist equal to 2. The number of such operators is finite (and small).

2.12.2 Evolution in Q^2 of the structure function moments

The behaviour of the coefficients $B_{is}^{(1,2)}(Q^2)$ at large Q^2 is governed by the Callan-Symanzik equation

$$B_s^{(1,2)}(Q_0^2 e^{2t}, g) = \mathrm{P}e^{-\int_0^t dt' (2\gamma^{em}(g(t') - \gamma_s(g(t'))} B_s^{(1,2)}(Q_0^2, g(t)) \tag{547}$$

Here the column $B_s^{(1,2)} = \{B_{is}^{(1,2)}\}$ is a set of all coefficients with a given spin s which correspond to operators O_{is} mixed by renormalization, γ^{em} is the anomalous dimension of the electromagnetic current and γ_s is the anomalous dimension matrix for the set of operators O_{is}. The anomalous dimension of the electromagnetic current is obviously zero. In fact $j^{em} = \partial^2 A$ where a is the electromagnetic field. Clearly there are no extra divergence related to the insertion of j^{em}. Presenting

$$\gamma_s = \zeta_s g^2 \tag{548}$$

we find the asymptotic of the set $B_s^{(1,2)}$ at large Q^2 as

$$B_s(Q^2, g) = \Big(\ln \frac{Q}{Q_0}\Big)^{\zeta/2b} B_s(Q_0^2, 0) \tag{549}$$

where matrix exponentiation and multiplication is implied and we recall that in QCD

$$b = \frac{1}{16\pi^2}\Big(11 - \frac{2}{3}N_f\Big)$$

To be more specific we have to find the operators of twist 2 in the QCD which may participate in the Wilson expansion of the product of two electromagnetic currents. Our guide will be gauge invariance of these operators implied by such invariance of the original product $j^{em}(z)j^{em}(0)$. One can construct three types of gauge invariant QCD operators of twist 2 and spin s: quark non-singlet, quark singlet and gluon:

$$O_s^{NS} = \Big[\bar{q}\lambda^F \gamma_{\alpha_1} iD_{\alpha_2}...iD_{\alpha_s} q\Big] \tag{550}$$

$$O_s^{S} = \Big[\bar{q}\gamma_{\alpha_1} iD_{\alpha_2}...iD_{\alpha_s} q\Big] \tag{551}$$

$$O_s^{G} = \frac{1}{2}\Big[G_{\alpha\alpha_1} iD_{\alpha_2}...iD_{\alpha_{s-1}} G^{\alpha}_{\alpha_s}\Big] \tag{552}$$

Here $D = \partial - igtG$ for quarks and $D = \partial - igTG$ for gluons. Matrix λ^F is some traceless matrix in quark flavours. Symbol [...] means symmetrization in indeces $\alpha_1 ... \alpha_s$ and addition of trace-terms necessary for making the operator irreducible in spin.

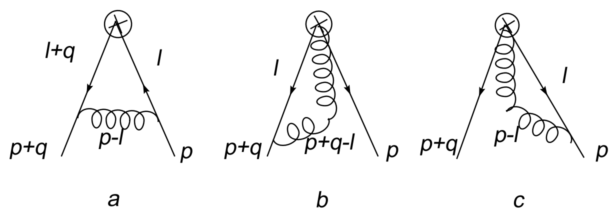

Figure 46: Single loop diagrams for the non-singlet renormalization constant Z^{NS}

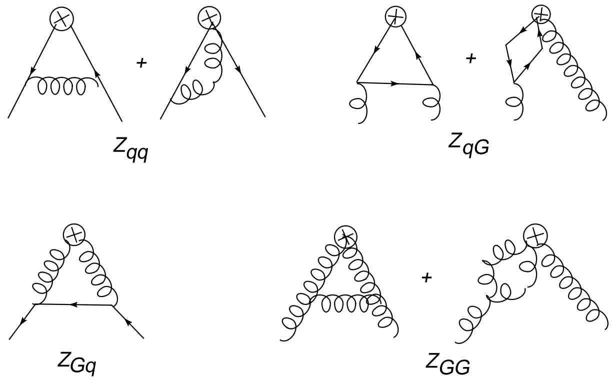

Figure 47: Single loop diagrams for the matrix of singlet renormalization constants

The non-singlet operator O^{NS} does not mix with others and is renormalized multiplicatively. In the single-loop approximation the contribution to the renormalization constant Z^{NS} comes from 3 diagrams shown in Fig 46. Calculations (see next subsection) give

$$\zeta_s^{NS} = -\frac{1}{8\pi^2}\frac{4}{3}\Big(1 - \frac{2}{s(s+1)} + 4\sum_{k=2}^{s}\frac{1}{k}\Big) \tag{553}$$

Note that $\zeta_1^{NS} = 0$ which corresponds to conservation of the flavour current. The asymptotic of the coefficients B_s^{NS} is

$$B_s^{NS}(Q^2) = B_s^{NS}(Q_0^2)\Big(\ln\frac{Q}{Q_0}\Big)^{p_s} \tag{554}$$

where

$$p_s = \frac{|\zeta^{NS}|}{2b} = \frac{4}{33 - 2N_f}\Big(1 - \frac{2}{s(s+1)} + 4\sum_{k=2}^{s}\frac{1}{k}\Big) \tag{555}$$

For $N_f = 4$ one finds $p_2^{NS} = 0.427$, $p_3^{NS} = 0.667$ and slowly grows with s.

Singlet and gluon operators mix under renormalization. In the single loop approximation one finds contributions to 4 components of the renormalization constant matrix illustrated in Fig. 47. Therefore to find the asymptotic of the coefficients B^S and B^G one has to diagonalize the corresponding renormalization constant matrix.

2.12.3 Calculation of anomalous dimensions for the non-singlet operator

The procedure to find the renormalization constants for composite operators in the one-loop approximation starts from the definition

$$V_n(g,\mu) = ZZ_\phi^{n/2}V_n^{(0)}(g_0,\Lambda) \tag{556}$$

where V_n and $V^{(0)}$ is the renormalized and unrenormalized Green functions with n external lines. Z is generally a matrix in the space of mixing operators. The inverse relation is

$$V_n^{(0)} g_0, \Lambda) = Z^{-1} Z_\phi^{-n/2} V_n(g, \mu) \tag{557}$$

Note that V_n is finite and $V_n^{(0)}$ is not.

In the single loop approximations, as we have seen, the maximal divergence of the Green function V_n is Λ^{t-n} where t is the twist. In our case $t = 2$ and the Green functions we are going to study have the minimal number of external lines $n = 2$. So the divergence is at most logarithmic. Accordingly we present

$$Z_\phi = 1 + \zeta_\phi g_0^2 \ln \frac{\Lambda}{\mu}, \quad Z = 1 + \zeta g_0^2 \ln \frac{\Lambda}{\mu} \tag{558}$$

Then the anomalous dimensions are

$$\gamma_\phi = \frac{\partial \ln Z_\phi}{\partial \ln \mu} = -g_0^2 \zeta_\phi \simeq -g^2 \zeta_\phi \tag{559}$$

and

$$\gamma = \frac{\partial Z}{\partial \ln \mu} Z^{-1} = -g_0^2 \zeta \simeq -g^2 \zeta \tag{560}$$

So to find the anomalous dimension it is sufficient to search for terms proportional to $ln\Lambda$ in the unrenormalized Green functions. We have, using (558)

$$V_n^{(0)} g_0, \Lambda) = V_n(g, \mu)\Big[1 - g_0^2 \ln \frac{\Lambda}{\mu}\Big(-\frac{n}{2}\zeta_\phi - \zeta\Big) \simeq V_n(g,\mu) - g^2 \ln \frac{\Lambda}{\mu}\Big(-\frac{n}{2}\zeta_\phi - \zeta\Big) V_{n0} \tag{561}$$

where V_{n0} is the Green function in the lowest (zero) order of perturbation theory. So the divergent part of $V_N(0)$ is

$$\mathrm{div} V_n^{(0)} = g^2 \ln \Lambda \tilde{\zeta} V_{n0} \tag{562}$$

where

$$\tilde{\zeta} = \zeta + \frac{n}{2}\zeta_\phi \tag{563}$$

and is generally a matrix to be applied to the column V_{n0} of all mixing operators. Formulas (562) and (563) give a practical recipe to find the anomalous dimension matrix coefficient ζ. One has to find the divergent part of the Green functions with composite operators from which coefficient $\tilde{\zeta}$ is to be extracted. Then, knowing the anomalous dimensions for the fundamental fields one calculates ζ from (563).

We shall limit ourselves with the calculation of the anomalous dimension for the non-singlet quark operator in the QCD. This operator does not mix with others and so ζ is just a number and not a matrix. The relevant diagrams for the Green function V_2 are shown in Fig. 46. Our aim is to find their divergent parts.

We start with the diagram of Fig. 46 a. It is given by the expression

$$V_2^{(a)} = g^2 \int \frac{d^4 l}{(2\pi)^4 i} t^b \gamma_\mu \frac{\hat{l} + \hat{q}}{(l+q)^2} \Big[\gamma_{\alpha_1} l_{\alpha_2} ... l_{\alpha_s}\Big] \frac{\hat{l}}{l^2} t^b \gamma^\mu \frac{1}{(l-p)^2} \tag{564}$$

where, as described, [...] means symmetrization in $\alpha_1, ...\alpha_s$ and subtraction of trace terms. In the following these operations will not always be explicitly shown. We are looking for a divergent part of the form

$$\mathrm{div} V_2^{(a)} = -g^2 \ln \Lambda \tilde{\zeta}^{(a)} \Big[\gamma_{\alpha_1} p_{\alpha_2} ... p_{\alpha_s}\Big] \tag{565}$$

Obviously it does not depend on q so that we can put $q = 0$. To simplify we also put $\hat{p} = 0$.

As we have already known $t^b t^b = 4/3$ and

$$\gamma_\mu \hat{l} \gamma_{\alpha_1} \hat{l} \gamma^\mu = 2(\gamma_{\alpha_1} l^2 - 2l_{\alpha_1} \hat{l})$$

So we get

$$V_2^{(a)} = \frac{8}{3} g^2 \int \frac{d^4 l}{(2\pi)^4 i} \frac{\gamma_{\alpha_1} l^2 - 2l_{\alpha_1} \hat{l}}{l^4 (l-p)^2} l_{\alpha_2} ... l_{\alpha_s} \tag{566}$$

Now we use the Feynman parametrization

$$\frac{1}{ab} = \int_0 dxdy \frac{\delta(1-x-y)}{(xa+yb)^2}, \quad \frac{1}{a^2 b} = 2\int_0 xdxdy \frac{\delta(1-x-y)}{(xa+yb)^3} \tag{567}$$

to write

$$V_2^{(a)} = \frac{16}{3} g^2 \int_0 xdxdy\delta(1-x-y) \int \frac{d^4 l}{(2\pi)^4 i} (\gamma_{\alpha_1} l^2 - 2l_{\alpha_1} \hat{l}) l_{\alpha_2} ... l_{\alpha_s} D^{-3} \tag{568}$$

where

$$D = xl^2 + y(l-p)^2 = l^2 - 2ylp = (l - yp)^2$$

In terms of $k = l - yp$ we have $l = k + yp$ and

$$V_2^{(a)} = \frac{16}{3} g^2 \int_0^1 (1-y) dy \int \frac{d^4 k}{(2\pi)^4 i k^6} \Big(\gamma_{\alpha_1}(k^2 + 2ykp) - 2(k_{\alpha_1} + yp_{\alpha_1})(\hat{k} + y\hat{p})\Big)$$

$$(k_{\alpha_2} + yp_{\alpha_2}) ... (k_{\alpha_s} + yp_{\alpha_s}) \tag{569}$$

Here we can drop the term with $\hat{p} = 0$. Let us classify the contributions by the terms which come from the last product.

If we take the term which contains the product of all yp then integration over k will effectively change

$$\gamma_{\alpha_1} k^2 - 2k_{\alpha_1} \hat{k} \to \frac{1}{2} \gamma_{\alpha_1} k^2$$

so that in the integrand we obtain a factor

$$\frac{1}{2} \gamma_{\alpha_1} p_{\alpha_2} ... p_{\alpha_s} k^2 (1-y) y^{s-1}$$

If we retain one of k_{α_j}, $j \geq 2$ then we have to retain terms odd in k in the first factor

$$\gamma_{\alpha_1} 2y(kp) - 2yp_{\alpha_1} \hat{k}$$

to be multiplied by

$$k_{\alpha_j} y^{s-2} \prod_{i=2, i \neq j} p_{\alpha_i}$$

Integration over k will substitute

$$(kp) k_{\alpha_j} \to \frac{1}{4} k^2 p_{\alpha_j} \quad \hat{k} k_{\alpha_j} \to \frac{1}{4} k^2 \gamma_{\alpha_j}$$

so that in the first bracket we find

$$\frac{1}{2}yk^2(\gamma_{\alpha_1}p_{\alpha_j}-\gamma_{\alpha_j}p_{\alpha_1})$$

which gives zero after symmetrization.

If we take more terms proportional to k from the last product we obviously will not get the structure we want, since there will not be sufficient number of p's. So in the end we obtain

$$\mathrm{div}V_2^{(a)}=\frac{16}{3}g^2\frac{1}{2}\gamma_{\alpha_1}p_{\alpha_2}...p_{\alpha_s}I(\Lambda)\int_0^1(1-y)y^{s-1}dy \tag{570}$$

where

$$I(\Lambda)=\int\frac{d^4k}{(2\pi)^4k^4}=\frac{1}{8\pi^2}\ln\Lambda$$

Doing the integral in y we find finally

$$\tilde{\zeta}^{(a)}=-\frac{8}{3}\frac{1}{s(s+1)}\frac{1}{8\pi^2} \tag{571}$$

Now contributions from diagrams Fig. 46 b and c. They are obviously equal. We consider Fig. 46 b. Its contribution is obtained by substituting in O^{NS} one of the operators D_α by $-igtG_\alpha$. The particular number of α does not matter due to symmetrization, but it is important at which place stands the original D_α: the derivatives to the right act on q and give its momentum, whereas the left derivatives act on the product Gq and give its total momentum, that is the momentum of $\bar{q}$. Accordingly we find

$$V_2^{(b)}=-g^2\sum_{j=2}^{s}\int\frac{d^4l}{(2\pi)^4i}t^b\Big[\gamma_{\alpha_1}\frac{\hat{l}}{l^2}\gamma_{\alpha_2}l_{\alpha_3}...l_{\alpha_j}p_{\alpha_{j+1}}...p_{\alpha_s}\Big]t^b\frac{1}{(l-p)^2} \tag{572}$$

Note that at $j=2$ there appears no factor l_α at all in the integrand. As before we put $q=\hat{p}=0$. Again $t^bt^b=4/3$. We also have

$$\gamma_{\alpha_1}\hat{l}\gamma_{\alpha_2}=\gamma_{\alpha_1}(2l_{\alpha_2}-\gamma_{\alpha_2}\hat{l})=2\gamma_{\alpha_1}l_{\alpha_2}-g_{\alpha_1\alpha_2}\hat{l}-\frac{1}{2}[\gamma_{\alpha_1},\gamma_{\alpha_2}]$$

We drop the term with $g_{\alpha_1\alpha_2}$ and the commutator will go after symmetrization. So we get

$$V_2^{(b)}=-\frac{8}{3}g^2\Big[\gamma_{\alpha_1}\sum_{j=2}^{s}p_{\alpha_{j+1}}...p_{\alpha_s}\int\frac{d^4l}{(2\pi)^4i}l_{\alpha_2}l_{\alpha_3}...l_{\alpha_j}\Big]\frac{1}{l^2(l-p)^2} \tag{573}$$

Using the Feynman parametrization we find

$$\int\frac{d^4l}{(2\pi)^4i}l_{\alpha_2}l_{\alpha_3}...l_{\alpha_j}\Big]\frac{1}{l^2(l-p)^2}=\int_0^1dy\int\frac{d^4k}{(2\pi)^4ik^4}(k_{\alpha_2}+yp_{\alpha_2})....(k_{\alpha_j}+yp_{\alpha_j})$$

$$\to p_{\alpha_2}...p_{\alpha_j}I(\Lambda)\int_0^1dyy^{j-1}=\frac{1}{j}p_{\alpha_2}...p_{\alpha_j}\frac{1}{8\pi^2}\ln\Lambda$$

Thus we get

$$\mathrm{div}V_2^{(b)}=-\frac{8}{3}g^2\Big[\gamma_{\alpha_1}p_{\alpha_2}...p_{\alpha_s}\Big]\frac{1}{8\pi^2}\ln\Lambda\sum_{j=2}^{s}\frac{1}{j} \tag{574}$$

so that

$$\tilde{\zeta}^{(b)} = \frac{8}{3}\frac{1}{8\pi^2}\sum_{j=2}^{s}\frac{1}{j} \tag{575}$$

The same contribution comes from Fig. 46 c. The total contribution from the three diagrams of Fig. 46 gives

$$\tilde{\zeta}_s^{NS} = -\frac{4}{3}\frac{1}{8\pi^2}\Big(\frac{2}{s(s+1)} - 4\sum_{k=2}^{s}\frac{1}{k}\Big) \tag{576}$$

In the Feynman gauge the quark anomalous dimension has coefficient ζ^q which can be extracted from our derivation of the β-function:

$$\zeta_s^q = \frac{4}{3}\frac{1}{8\pi^2}\frac{1}{2} \tag{577}$$

From this we find the final coefficient ζ_s^{NS}

$$\zeta_s^{NS} = \tilde{\zeta}_s^{NS} + 2\zeta_s^q = -\frac{4}{3}\frac{1}{8\pi^2}\Big(1 - \frac{2}{s(s+1)} + 4\sum_{k=2}^{s}\frac{1}{k}\Big) \tag{578}$$