Reduced-Form Credit Risk Models for Zero-Coupon Bonds Pricing

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Chapter 1

Introduction

In this report, I analyze reduced-form credit risk models without considering default correlations among firms, and implement credit risk valuation in defaultable bonds pricing.

Credit risk, also known as default risk, refers to the risk of debt obligors falling to make payments as required. There are two types of models generally used in credit risk modelling: structural and reduced-form models. For structural models, the risk of default and recovery rate are usually based on the relationship between firms' asset value and liabilities. Compared with structural models, in reduced-form models default intensity and recovery rate are exogenously given by stochastic processes without considering firms' value structures. Because of the difficulty of implementation and calibration of structural models, I prefer to reduced-form models in this project.

In section 2, I theoretically present reduced-form modelling with a concentration on default intensity. Beginning with deriving default intensities from Poisson processes, then survival probabilities are defined and further used for pricing defaultable bonds. Finally I describe a basic affine process modelling default intensities as a general case.

In section 3, I focus on the implementation of reduced-form models in defaultable zero-coupon bonds pricing. During pricing, I treat both interest rate and default intensity as stochastic processes from CIR models, as a simplified affine process, which is generally used in short-term rate modelling. Finally I numerically analyze the impacts of recovery rate and model parameters on the prices of defaultable zero-coupon bonds.

Chapter 2

Default Intensity Models

2.1 Information Framework

Based on a complete set of business models, I need to specify first the probability measure used in this report is risk-neutral measure. We could define the probability space as $\Pi(\Omega, \mathcal{F}, \mathcal{P})$, then under this probability measure, assume the investors are indifferent between investing in money market account or in any other asset [1]. Thus I use β_t to denote the money market account which is used as numeraire, and the definition is as follows:

$$\beta_t = exp(\int_0^t r_s ds) \tag{2.1}$$

where r_s is $\mathcal{F}_t$-adapted interest-rate. Risk-neutral measure is meaningful in three ways which will be used in credit risk valuation further:

Firstly, zero-coupon bonds discounted by numeraire are $(\mathcal{F}_t, P)$-martingales;

Secondly, we could assume there are J variates $X_t = (X_{1,t}, \ldots, X_{J,t})$, therefore X_t is a Markov process with respect to $\mathcal{F}_t$, which could be used to represent economic variables affecting default intensity;

Thirdly, I could also assume there exists a counting process N_t which could be used to describe the arrivals of default. If N_t jumps to 1 from 0, default happens.

2.2 Defaultable Zero-Coupon Bond Pricing

Generally zero-coupon bonds we refer to are default-free, which makes the pricing simpler. Here we begin to take the default into account of zero-coupon bonds pricing. Default intensity is the most important issue in pricing defaultable zero-coupon bonds. Thus reduced-form models are also known as default intensity models.

Generally in credit risk modelling, default is represented by a binary variable which takes value1 or 0 to indicate default happening or not. Thus we usually use the first jump of Poisson process to model default arrivals. Default time τ is defined as follows in our models:

$$\tau = inf\{t \in R^+ \mid N_t > 0\} \tag{2.2}$$

where $N_t = \sum_h 1\{\tau_h < t\}$ is a counting process. And $N_t - N_s$ follows a Poisson distribution with parameter $(t-s)\lambda$, for $s < t$, which is consistent with what we have discussed in information framework in the first section.

Default intensity is defined as the instantaneous default probability. And based on the definition of default arrival times, default intensity could be defined as the conditional probability of default arrival at time $\tau \in (t, t+h)$, given no default happens before time t:

$$\lim_{h \to 0} \frac{P[\tau \in (t, t+h) \mid \tau > t]}{h} = \frac{f(t)}{1 - F(t)} = \lambda_t \tag{2.3}$$

where $F(t) = P(\tau < t)$, probability of default happening before time t, and $f(t)$ is the density of $F(t)$. In general, survival probability represents no default happens during a specific time period, which is given by $P(\tau > t) = 1 - F(t) = P(N_t = 0) = E[exp(-\int_0^t \lambda_s ds)]$. In a more general case, the probability of survival from time t to time T is defined as follows:

$$s\,(t, T) = P[\tau > T \mid \tau > t] = E[exp(-\int_t^T \lambda_s ds) \mid \mathcal{F}_t] \tag{2.4}$$

Now we could use the derived survival probability to price defaultable zero-coupon bond. The market is assumed to be complete and no-arbitrage. Start from pricing default-free zero-coupon bond. Previously in formula (1), the money market account value is defined as β_t. Correspondingly the discounted default-free zero-coupon bond $P(t,T)/\beta_t$ is a $(\mathcal{F}_t, P)$-martingale. The price of default-free zero-coupon bond at time t with maturity time T, face value 1 is as follows:

$$P\,(t, T) = \beta_t E[\frac{P(T,T)}{\beta_T} \mid \mathcal{F}_t] = E[exp(-\int_t^T r_s ds) \mid \mathcal{F}_t] \tag{2.5}$$

where $P(T,T)$, the price of default-free zero-coupon bond at time T, equals the face value.

Similarly, he price of defaultable zero-coupon bond at time$t$ with maturity T, face value 1, recovery rate R is follows:

$$Q\,(t, T) = \beta_t E[\frac{Q(T,T)}{\beta_T} \mid \mathcal{F}_t] = \beta_t E[\frac{1\{\tau > T\}}{\beta_T} \mid \mathcal{F}_t] + \beta_t E[\frac{R_\tau}{\beta_\tau} \mid \mathcal{F}_t] \tag{2.6}$$

$$Q\,(t, T) = E[exp(-\int_t^T (r_s + \lambda_s) ds) \mid \mathcal{F}_t] + E[\int_t^T R_s \lambda_s exp(-\int_t^s (r_u + \lambda_u) du) ds \mid \mathcal{F}_t] \tag{2.7}$$

assuming $\tau > t$ and the expectations are finite. In the above formula, the first term is the payment at time T discounted back to time t, under the probability of no default happening from time t to T; the second term is the recovery payment from τ to t under the probability of default happening at time τ. Notice that the probability of default at time τ is that default intensity at time τ times the probability of no default happens before time τ.

2.3 Default Intensity Modelling

Default intensity plays a very important role in defaultable zero-coupon bond pricing based on the pricing formula (2.7) we have derive in Section 2.1. The choices of default intensity directly impact the results of bonds pricing. Default intensity is typically treated in several ways: constant, linear or quadratic polynomials of the time, or stochastic processes. In this project I prefer to assuming default intensity follows a stochastic processes. We need to consider what kind of stochastic processes could be qualified to model default intensity. Recalling survival probability $s(t,T)$ in formula (2.4) and price of default-free zero-coupon bond $P(t,T)$ in formula (2.5), both of them are expressed as the expectations of discounted payments. Meanwhile we could find the only difference between these terms is the discounted rate: λ_t is used in $s(t, T)$, r_t is used in $P(t, T)$. Therefore we could consider modelling default intensity by well-known short-rate models, which have been maturely developed and widely implemented in market over years. Moreover, the interest rate and default intensity share some important characteristics which are introduced in detail by Schonbucher (2003).

In this project, I introduce a basic affine process as a general framework. Consider J variables at time t affecting default intensity, $X_t = (X_{1,t}, \ldots, X_{J,t})$, which is a martingale as discussed in Information Framework. and $X_{j,t}$ follows a basic affine process as follows:

$$dX_{j,t} = \alpha_j(\theta_j - X_{j,t})dt + \sigma_j\sqrt{X_{j,t}}dW_{j,t} + dJ_{j,t} \tag{2.8}$$

for $j = 1, \ldots, J$, α_j is the mean-reversion rate, θ_j is long-term mean, $W_{j,t}$ is a $\mathcal{F}_t$-adapted Brownian motion, $J_{j,t}$ is a jump process with jump size following exponential distribution, jump times following Poisson distribution (jump sizes, jump times for different states are both independent, meanwhile jump time is independent of jump size). Therefore two more parameters are added into this stochastic process, μ_j, the mean of the exponential distribution; and γ_j, the intensity of the Poisson distribution. Thus we consider use affine functions of $X_{j,t}$ to model default intensity: $\lambda_t = \phi(X_t) = a_0 + a_1X_{1,t} + \ldots + a_JX_{J,t}$. Meanwhile we could also make interest rate r_t as an affine function of stochastic variables X_t.

There are many advantages of modelling default intensity by affine processes, which are introduced by Duffie, Pan and Singleton (2000). Briefly taking advantages in these several ways:

Firstly, we could add correlations between default intensity and interest rate easily by correlated Brownian motion terms.

Secondly, since X_t is a Markov process which is mentioned in Chapter 1, the discounted expectation is also a Markov process based on the basic affine function as follows:

$$E[exp(-\int_s^t \phi(X_u)du \times G(X_t) \mid \mathcal{F}_s] = H(X_s) \tag{2.9}$$

Moreover, if $\phi(x)$ and $g(x)$ are both affine functions of X_T, where X_t is a basic affine process, we can further derive $H(X_s)$ to a close-form as follows:

$$H(X_s) = exp(\upsilon(s,t) + \omega(s,t))X_s \tag{2.10}$$

for $0 \leq s \leq t$. And $\upsilon(s, t)$, $\omega(s, t)$ are also in close-forms with parameters calibrated by market data.

For now we could price the building blocks $P(t, T), s(t, T), Q(t, T)$ derived in Section2.2 by the specific default intensity introduced here.

Chapter 3

Implementation of Credit Risk Valuation in Defaultable Zero-Coupon Bond Pricing

3.1 Simulation of CIR model

From Chapter2, we have derived default intensity in the forms of basic affine processes. But during model implementation, we need consider the complexity of parameter calibration and the efficiency of simulation. I simplify the basic affine model to CIR model by deleting the jump terms $dJ_{j,t}$, since this term brings two more parameters into our model and make simulation more time-consuming. Moreover I only consider interest rate and default intensity without considering other variables affecting them, correspondingly the variate $X_t = (X_{1,t}, \ldots, X_{J,t})$ previously discussed becomes $X_t = (\lambda_t, r_t)$. In general simplified formulas from Chapter 2 are generalized as follows, which could be easily implemented in further simulation:

$$dr_t = \alpha_r(\theta_r - r_t)dt + \sigma_r\sqrt{r_t}dW_{r,t} \tag{3.1}$$

$$d\lambda_t = \alpha_\lambda(\theta_\lambda - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dW_{\lambda,t} \tag{3.2}$$

$$dW_{r,t}dW_{\lambda,t} = \rho dt \tag{3.3}$$

where parameters (α, θ, σ) are governed by market data, and default intensity is correlated with interest rate by ρ in formula (3.3).

In practice, there are usually two ways used in simulating diffusion processes: Euler method and transition density method. For Euler method, the most attractive advantage in simulating CIR processes for our model is that correlations caused by ρ could be easily added, and applicable for most diffusion processes. In contrast of Euler method, transition density method is more accurate in simulation but only applicable to solve SDE with specific restrictions, like CIR models. In this section I implement simulations of default density by transition density method. I focus to test the reactions of default intensity to different parameters (α, θ, σ) in CIR processes.

3.1.1 Impact of mean-reversion rate

For comparisons,I have generated 9 realizations of default intensities with $\alpha = 0.2+0.1j$, for $j = 1, \ldots, 9$, $\theta = 1$, $\sigma = 0.1$, initial value $\lambda_0 = 10$, time steps $N = 5000$. The result is shown in the following Figure 3.1: Simulations of default intensity with different mean-reversion rates α, and we could see that all 9 paths behave as expected: starting at the initial value of 10, and then tending to their long-term mean value $\theta = 1$ in different quickness with respect to different α. Moreover larger the α is, more rapidly the default intensities decrease to the long-term mean.

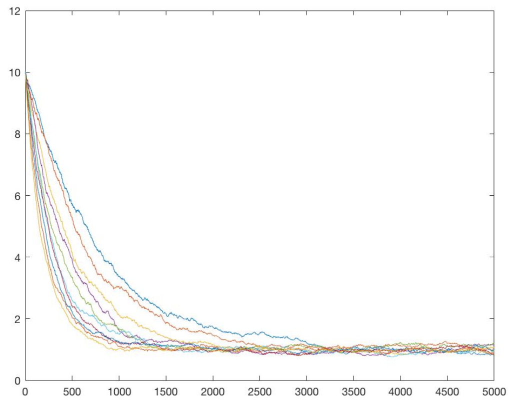


Figure 3.1: Simulations of default intensity with different α

3.1.2 Impact of long-term mean

For comparisons,I have generated 9 paths of default intensities with $\theta = 1+1j$, for $j = 1, \ldots, 9$, $\alpha = 0.2$, $\sigma = 0.1$, initial value $\lambda_0 = 10$, time steps $N = 5000$. From the following Figure 3.2: Simulations of default intensity with different long-term means θ, we could see that all simulations: starting at the initial value of 10, and then tending to different long-term mean values.

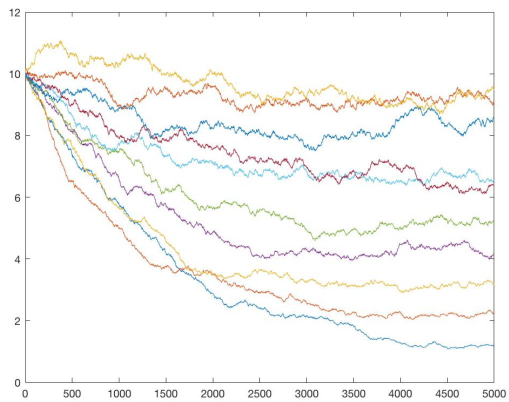


Figure 3.2: Simulations of default intensity with different θ

3.1.3 Impact of volatility

I set volatility $\sigma = 0.01 + 0.05j$, for $j = 1, \ldots, 9$, meanwhile fix other parameters as in the previous cases. From the following Figure 3.3: Simulations of default intensity with different volatility θ, we could see that the amplitudes of volatility differ much between different paths.

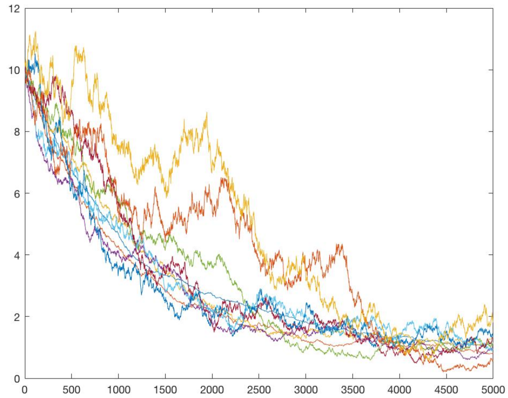


Figure 3.3: Simulations of default intensity with different σ

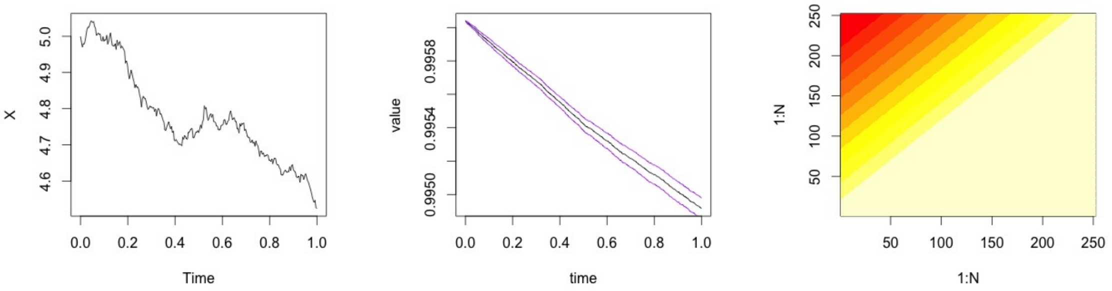
3.2 Simulations of Survival Probability and Defaultable Zero-Coupon Bond Price

In this section, survival probability and price of defaultable zero-coupon bond are simulated with the mean of 1000 realizations of default intensity and interest rate generated from CIR processes. Parameters estimates are from Duffie (1999) and Brigo and Afonsi (2005) [2].

3.2.1 Survival probability with different parameters estimates

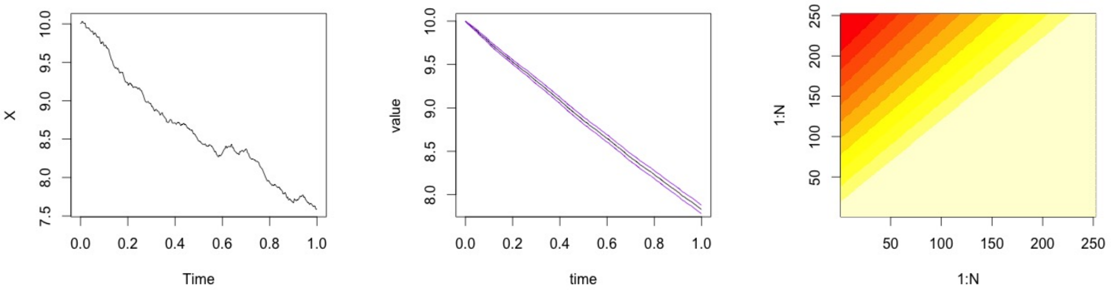
Firstly I implement model simulation with the parameters estimates from Duffie (1999):$\alpha = 0.238, \theta = 2.35, \sigma = 0.074$. Meanwhile, I set maturity time $T = 1$, time steps $N = 252$, number of simulations $m = 1000$. The results are shown as follows in Figure 3.4: Simulations with Duffie (1999) parameters estimates. In (a), one realization of default intensity is given. In (b), one simulation of survival probability and its confidence interval are given, which is generalized from the mean of default intensity's exponential function.We could see that in (b) the curve of $s(t,T)$ has a similar pattern with that of λ_t in (a), but more smoothly decreasing from initial value to long-term mean. Meanwhile survival probability is relative stable since the confidence interval is very narrow and close to the mean estimation which could be seen from (b). The matrix of logarithm of survival probability is shown in (c), which is also known as level curves. In order to see the transitional states between different time periods, logarithm of survival probability is used since exponential functions decrease rapidly [2]. We could see the convexity of $s(t,T)$.

The estimated parameters from Brigo and Afonsi (2005) are: $\alpha = 0.25, \theta = 0.05, \sigma = 0.1$ [2]. Compared with Duffie (1999), volatility of default intensity is lower thus the confidence interval of survival probability



(a) λ_t (b) $s(0,T)$ (c) Log Matrix of $s(0,T)$

Figure 3.4: Simulation with Duffie (1999) parameters estimates



(a) λ_t (b) $s(0,T)$ (c) Log Matrix of $s(0,T)$

Figure 3.5: Simulation with Brigo and Afonsi (2005) parameters estimates

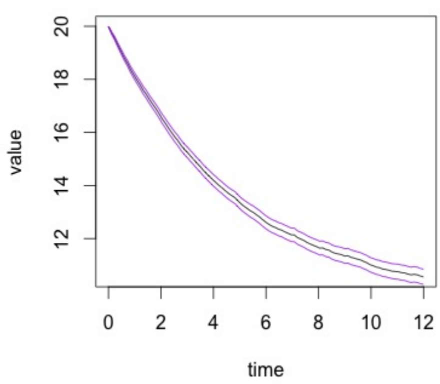
is more narrow and closer to the mean estimations.

3.2.2 Simulation of Defaultable Zero-Coupon Bond Pricing

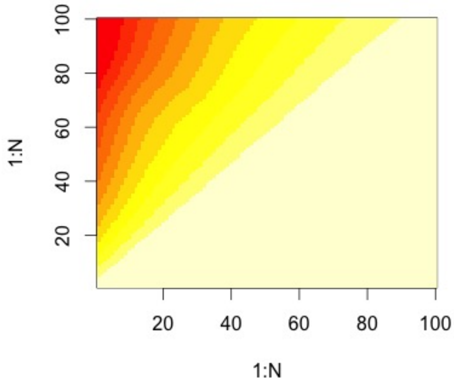
In this section, pricing defaultable zero-coupon bonds is based on the results of: simulation of default intensity with parameters estimates from Duffie (1999); simulation of interest rate with parameters estimates from Brigo and Afonsi (2005). Here I set time maturity $T = 12$, time steps $N = 100$, number of simulations $m = 1000$, time interval $\Delta t = \frac{T}{N}$. Implementing the dedaultable zero-coupon bond pricing formula (2.4) into simulation, we could get the following equation:

$$Q\,(0,T) = \frac{1}{m}\sum_{j=1}^{m}[exp[-\sum_{i=1}^{N}(r_{t_{i,j}} + \lambda_{t_{i,j}})\Delta t_{i,j}] + \sum_{i=s}^{N}[R_{t_{s,j}}\lambda_{t_{s,j}}exp(-\sum_{i=1}^{s}(r_{t_{i,j}} + \lambda_{t_{i,j}})\Delta t_{i,j})\Delta t_{s,j}]] \tag{3.4}$$

The results of simulation is shown as follows in Figure 3.6: Simulation of Defaultable Zero-Coupon Bond Pricing, $Q(0,T)$ is changing with the time periods T. And we could see that the estimation should be a good estimator of the accurate price when the confidence interval is very close to the estimated mean, the level curve decrease rapidly. In (b), the level curves are kind of different from previous one. it seems that passing from a state to anther takes longer time in final states. Since here interval time Δt is larger than previous ones, intermediate states are longer. Meanwhile if intermediate states are longer, CIR process



(a) $Q(0,T)$



(b) Log Matrix of $Q(0,T)$

Figure 3.6: Simulation of Defaultable Zero-Coupon Bond Pricing

reaches their long-term mean faster [2].

And with the fixed maturity time $T = 12$, recovery rate $R = 0.4$ and $\rho = 0.1$, the estimation of defaultable zero-coupon bond with face value 1 from the mean of 1000 realizations is 0.049, compared the price of default-free zero-coupon bond with same face value is 0.567. Finally, I want to talk about how the correlation between $\lambda_t$and r_t, ρ affect the price in the simulations. Actually when I change ρ from 1 to -1, the price almost don't change. Maybe because λ_t and r_t always works in the same way since they are together as a discounted rate in our formula. For recovery rate, in practice, we usually take it as a constant from 10% to 40%, the prices of dafaultable zero-coupon bonds increase from 0.037 to 0.049, and we could see the reaction is pretty sensitive.

Bibliography

[1] Elizalde, Abel. "Credit risk models I: Default correlation in intensity models." Documentos de Trabajo (CEMFI) 5 (2006): 1.

[2] Pereira, Nelson Vieira. "Modeling Credit Risk: simulation of a reduced-form model." (2013).

Appendix : Related Programming Code

```
%Matlab code :
k=0.559;
r0=0.238;
sigma=0.074;
d=4*r0*k/sigma^2;
m=1000;
N=5000;
r=ones(N,m);
r(1,:)=10;
t=1/252;

for j=1:m
   if d >1
    for i =1 : N-1
        c= sigma^2*(1-exp(-k*(t*(i+1)-t*(i))))/(4*k);
        lamda=r(i,j)*(exp(-k*(t*(i+1)-t*(i))))/c;
        z=normrnd(0,1);
        x=chi2rnd(d-1,1,1);
        r(i+1,j)=c*((z+sqrt(lamda))^2+x);
    end
   else
       for i =1 : N-1
        c= sigma^2*(1-exp(-k*(t*(i+1)-t*(i))))/(4*k);
        lamda=r(i,j)*(exp(-k*(t*(i+1)-t*(i))))/c;
        n=poissrnd(lemda/2);
        x=chi2rnd(d+2*n,1,1);
        r(i+1,j)=c*x;
       end
   end
end
r=r(2:N,:);
sp=mean((r'));
plot(sp);


%Matlab code : Euler Discretization
k=0.25;
r0=0.05;
sigma=0.1;

k2=0.238;
x0=2.35;
sigma2=0.074;
```

```
m=1000;
N=252;
T=12;
dt=T/N;

r=ones(N,m);
x=ones(N,m);
rng(2);
z1=normrnd(0,1,[N,m]);
z2=normrnd(0,1,[N,m]);
x(1,:)=2.35;
r(1,:)=0.05;
rho = -1;
R=0.1;

for j=1:m
   for i=1:N-1
       r(i+1,j)=r(i,j)+k*(r0-
r(i,j))*dt+sigma*sqrt(r(i,j)*dt)*z1(i,j);
       x(i+1,j)=x(i,j)+k2*(x0-
x(i,j))*dt+sigma2*sqrt(x(i,j)*dt)*(rho*z1(i,j)+sqrt(1-
rho^2)*z2(i,j));
   end
end
r=r(2:N,:);
x=x(2:N,:);

DB1 = mean((exp(-dt*cumsum(r+x))));
DB2 = R*mean(exp(-log(x).*cumsum(x)*dt));
DB=mean(DB1+DB2);
NDB = mean(exp(-dt*sum(r)));
%y=[r,x];
%plot(y);
%plot(x);



% R code
X0=10
N=100
t0=0
T=12
M=1000
theta=c(0.559, 0.238, 0.074)
X0a=10
Na=100
t0a=0
Ta=12
```

```
Ma=1000
thetaa=c(0.514878, 0.082, 0.67)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=theta,
model="CIR")
Y <- sde.sim(X0=X0a, N=Na, M=Ma, t0=t0a, T=Ta, theta=thetaa,
model="CIR")
W=X+Y
dt=(T-t0)/N
W.mean = rowMeans(W)
W.sd = apply(W,1,sd)
plot(as.vector(time(W)),W.mean,type="l",xlab="time",ylab="value"
)
lines(as.vector(time(W)),W.mean + (1.96*W.sd)/sqrt(M),
col="purple")
lines(as.vector(time(W)),W.mean -
(1.96*W.sd)/sqrt(M),col="purple")
default <- function (i,j,W,dt) { if (j <= i) {
  return(1) }
  if (j == i+1) {
    return (mean(exp(-dt*W[j,])))
  }
  return (mean(exp(-dt*colSums(W[(i+1):j,])))) }
l = matrix(1,N,N)
for (i in 1:N) { for (j in 1:N) {
  l[i,j] = default(i, j, W, dt)
}
}
image (1:N,1:N,log(l))
```