Default swaption pricing and counterparty credit risk valuation

Final project of Credit Risk

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Contents

1 Introduction

In this project, we build up a Monte Carlo similation scheme, which is implemented by using a Python program, to price CDS, default swaption and calculate exposures and CVA for these products. The main work focuses on three parts, hazard rate model calibration, default swaption pricing and counterparty credit exposure valuation.

2 Theory Background

In this section, the frame of credit default swap(CDS) and default swaption pricing will be stated. It first starts with survival probability calculation and CIR hazard model calibration. Then Monte Carlo Simulation method to price CDS and default swaption will follow.

2.1 Default Probability and Hazard Rate

Compared with other financial derivatives,for examples, equity, foreign exchange, and interest rates, credit risk product involves another dimension, default probability. Hence, we need some model to measure hazard rate and survival probability under certain assumptions. Following the framework established by Jarrow and Turnbull (1995) and later in Lando (1998), the default time τ for the portfolio is the first arrival time of a Poisson process, the hazard rate $\lambda(t)$ is definded as:

$$\lim_{dt \to 0} = \frac{1}{dt} Pr(\tau \leq t + dt | \tau > t) = \lambda(t) \tag{1}$$

Hence, if we assume the hazard rate is deterministic, with n evenly divided interval of time 0 to T, the probability is approximated as:

$$Pr(\tau > T) = \exp\left(- \int_0^T \lambda(t) dt \right) \tag{2}$$

In this project, we assume the hazard rate model follow a Cox process, of which the randomness come from two sides, stochastic process of $\lambda(t)$ and Poisson jump process. The cumulated intensity or cumulated hazard rate or Hazard function is defined as:

$$\Lambda(t) := \int_0^t \lambda_u du \tag{3}$$

Then by Poisson process, the first jump time, integral by cumulative hazard rate, follow an exponential distribution which is independent of the current market information as:

$$\Lambda(\tau) := \xi = exponential(1) \tag{4}$$

To sum up, the survival probability would be:

$$P(\tau > t) = P(\xi > \int_0^t \lambda_u du) = E\left[P(\xi > \int_0^t \lambda_u du | F^\lambda)\right] = E\left[e^{-\int_0^t \lambda_u du}\right] \tag{5}$$

2.2 Hazard Rate Model

If we assume $\lambda(t)$ follow a CIR process,

$$d\lambda_t = k(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t, \quad 2k\theta \geq \sigma^2 \tag{6}$$

The survival probability has analytical closed form as,

$$Q(t,T) = A(T-t)e^{-B(T-t)\lambda_t} \tag{7}$$

where $\beta = \sqrt{k^2 + 2\sigma^2}$ and

$$A(s) = \left[\frac{2\beta e^{\frac{(\beta+k)s}{2}}}{(\beta+k)(e^{\beta s}-1)+2\beta}\right]^{\frac{2k\theta}{\sigma^2}}, \quad B(s) = \frac{2(e^{\beta s - 1})}{(\beta+k)(e^{\beta s}-1)+2\beta}$$

This formula is almost same as zero coupon bond pricing under CIR model as hazard rate and short rate follow same type stochastic process and exact same calculation formula.

2.3 Credit Default Swap

According to Dominic O'Kane [1],"A credit default swap is a bilateral over-the-counter contract whose purpose is to protect one party, the protection buyer, from the loss from par on a specified face value of bonds or loans following the default of their issuer. This protection lasts from the effective date2 which falls on the calendar date after the trade date, until a specified maturity date also known as the scheduled termination date. The specified bonds and loans which are protected are known as the deliverable obligations and the issuer is known as the reference entity."

The contract can be The contract can be divided into two parts: The protection leg: if a default event happens, part of loss will be paid by protection seller; premium leg: to get the insurance or protection payment for default event, protection buyer would pay regularly in certain frequency (usually 3 month). However, if default event happens, the payment will be terminated.

Firstly note that, the risky zero coupon bond paid at t_n with possibiliy of default is as:

$$Z(\hat{t, t_n}) = E_t \left[\exp\left(- \int_t^{t_n} r_s ds \right) 1_{\tau > t_n} \right] \tag{8}$$

Under the assumption of independent short rate r_t and λ_t, we have $Z(\hat{t, t_n}) = Z(t, t_n)Q(t, t_n)$ and the risk annuity:

$$S_0 \sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Z(t, t_n) Q(t, t_n) \tag{9}$$

However, there is still some probabilities that during coupon period, default event happens, hence the value of premiun accrued is

$$S_0 \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t, s)(-dQ(t, s)) \tag{10}$$

To sum up,

$$PremiumLegPV = S_0 * RA(t, T) \tag{11}$$

and

$$RA(t, T) = \sum_{n=1}^{N} \Delta(t_{n-1}, t_n) Z(t, t_n) Q(t, t_n) + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s) Z(t, s)(-dQ(t, s)) \tag{12}$$

which can be approximated by

$$RA = \frac{1}{2} \sum_{i=1}^{n} Z(t, T_i) \delta(T_{i-1}, T_i)(Q(t, T_{i-1} + Q(t, T_i))) \tag{13}$$

Similarly, the protection leg is defined as:

$$D(\hat{t, T}) = (1 - R) E_t \left[\exp(- \int_t^{\tau} r_s ds) 1_{\tau \leq T} \right] = (1 - R) \int_t^T Z(t, s)(-dQ(t, s)) \tag{14}$$

which can be approximated by

$$DL = (1 - R) \sum_{i=1}^{n} Z(t, T_i)(Q(t, T_{i-1}) - Q(t, T_i)) \tag{15}$$

At the end, the value of CDS is by

$$CDSPrice = DL - S_0 * RA \tag{16}$$

2.4 Default swaption

The default swaption is a kind of over-the-counter option with forward start CDS as underlying. It gives the holder the right, but not the obligation to enter into a credit default swap starting on some future date at a spread fixed today. According to Dominic O'Kane, we can calculate the present value of payer CDS option using following formulas:

$$V_{Pay}(0) = E^Q[\frac{1}{D(0,t_E)} 1_{\tau > t_E} \max[S(t_E,T) - K, 0] RA(t_E,T)] \tag{17}$$

where t_E is the expiry of option, RA is the risk annuity defined in the precious section, T is the maturity of the forward CDS, K is the strike, $S(t_E)$ is the par spread of the forward CDS starting at t_E, D is the discount factor. Similarly, the receiver CDS option is defined as

$$V_{Rec}(0) = E^Q[\frac{1}{D(0,t_E)} 1_{\tau > t_E} max[K - S(t_E,T), 0] RA(t_E,T)] \tag{18}$$

The market standard approach to price default swaption is to assume that the P measure dynamics of CDS spread is a simple lognormal process with volatility σ (Here the P measure is the measure with numeraire $A(t) = 1_{\tau > t_E} RA(t, t_E, T)$, t_E is the start date of CDS):

$$dS(t) = \sigma S(t) dW^P(t)$$

Under this risky annuity measure P, we can re-express the present value of payer and receiver :

$$V_Pay(0) = RA(0, t_E, T) E^P[\max[S(t_E,T) - K, 0]]$$
$$V_Pay(0) = RA(0, t_E, T) E^P[\max[K - S(t_E,T), 0]]$$

Then under this measure we can derive the Black formula:

$$V_{Pay}(0) = RA(0, t_E, T)(S(0, t_E, T)\Phi(d_1) - K\Phi(d_2))$$

where

$$d_1 = \frac{ln(S(0, t_E, T/K)) + \frac{1}{2}\sigma^2 t_E}{\sigma\sqrt{t_E}}$$
$$d_2 = \frac{ln(S(0, t_E, T/K)) - \frac{1}{2}\sigma^2 t_E}{\sigma\sqrt{t_E}}$$

And

$$V_Rec(0) = RA(0, t_E, T)(K\Phi(-d_2) - S(0, t_E, T)\Phi(-d_1)) \tag{19}$$

2.5 Counterparty credit exposure

According to Cesari (2009)[3], counterparty credit exposure is the amount an institution could potentially lose in the event of one of its counterparties defaulting. Exposures can be computed by simulating in different scenarios and at different times in the future, the value of the transactions with the given counterparty, and then by using some chosen metrics to characterize the value distributions which have been generated.

Given a portfolio of positions traded with a counterparty, in order to compute the counterparty credit exposure at time t, first we need to model the distribution of the portfolio value computed at time $t \geq 0$ and seen from today, which is denoted as V_t:

$$V_t = N_t \sum_{T_i > t}^{T_n} E\left[\frac{X_{T_i}}{N_{T_i}}\middle|\mathcal{F}_t\right] \tag{20}$$

where T_i, $i = 1, \cdots, n$, are cashflow payments dates. T_n is the maturity of the trade. X_s is the (generally stochastic) payment amount at time s, and $X_s = 0$ if s is not a member of T_i. N_t indicates the numeraire, and E is the expectation in the numeraire measure. By using (1), regular Monte Carlo framework can be implemented to derive the future value distribution of the trade.

Typical risk metrics used include: (1) the mean, (2) the 97.5 percent or 99 percent quantile, called Potential Future Exposure (PFE), and (3) the mean of the positive part of the distribution, referred to as the Expected Positive Exposure (EPE).

The PFE is defined as

$$PFE(\alpha, t) = q_{\alpha,t} = \inf x : P(V_t \leq x) \geq \alpha \tag{21}$$

where α is the given confidence level, q is the value of quantile. The graph of PFE(α, t) as a function of t is known as the exposure profile of the trade.

The EPE is defined as

$$EPE(t) = E\left[\max(V_t, 0)\right] \tag{22}$$

where the expectation can be taken under the real or pricing measure depending on the usage of EPE.

2.6 CVA

According to Cesari (2009)[3], Credit Valuation Adjustment (CVA) is the price of hedging the counterparty credit risk. Assuming that future cash flow values are independent from defaults, CVA can be defined as the value of a credit default swap with notional being the EPE profile of the underlying transaction. Suppose for simplicity that the EPE profile is a piecewise constant function over a time interval $[T_{i-1}, T_i)$ then

$$CVA = (1 - R) \sum_i EPE(i) D(0, T_i) PD(T_{i-1}, T_i) \tag{23}$$

where $PD(T_{i-1}, T_i)$ is the marginal default probability in the time interval $[T_{i-1}, T_i)$ and $D(0, T_i)$ the discount zero bond maturing at time T_i with notional $1.R$ is the recovery rate. CVA corresponds to a portfolio of forward starting CDSs with piecewise constant notional. The CVA depends on the level of exposure as well on the credit spread of the counterparty.

3 Implementation and results

3.1 Global assumptions and inputs

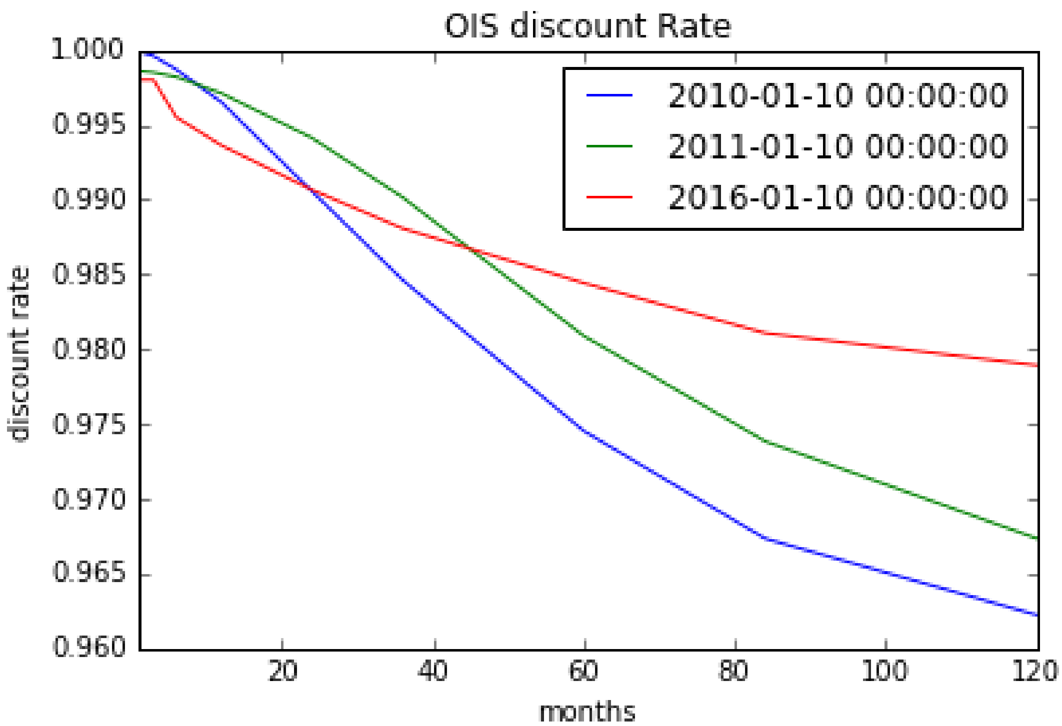
CDS start date: 2011-1-10
CDS end date: 2016-1-10
Swaption start date: 2010-1-10
Recovery rate: 0.4
Market spread (BP): 39.3, 40.40, 71.47, 130, 202.25, 265.43, 346.89, 378.25 [1]
Tenors: 6 months , 1 year, 2 year, 3 year, 4 year, 5 year, 7 year, 10 year
initial guess for CIR parameters (λ_0, κ, θ, σ): 0.03, 1, 0.1, 0.03
simNumber: 30

3.2 Discounting curve

Instead of calibrating and simulating short rate, we download OIS data from Quandl directly to calculate discount factors. At each day in the time horizon in need, we have an OIS yield curve with tenors 1 month, 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years and 30 years. Based on these rates, linear interpolation is used to get daily discounting curves.

[1]DELL (BB+) 2016-12-16 CDS spread. Data source: Bloomberg

In the following figure, OIS discounting curves for swaption start date, forward CDS start date and end date are demonstrated[2].

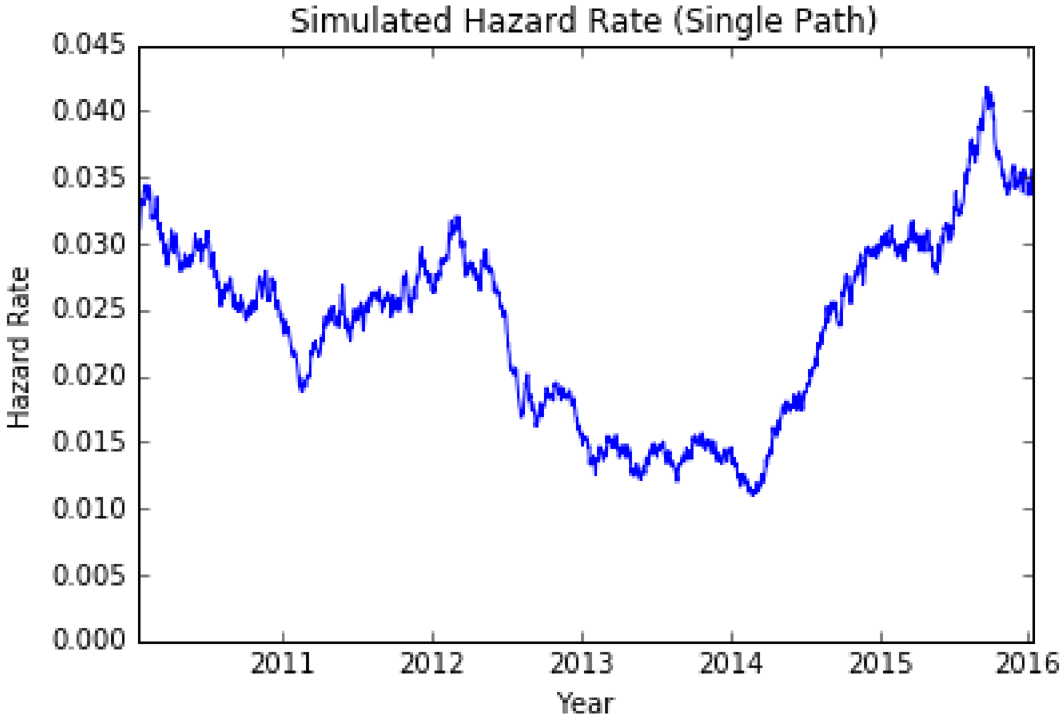


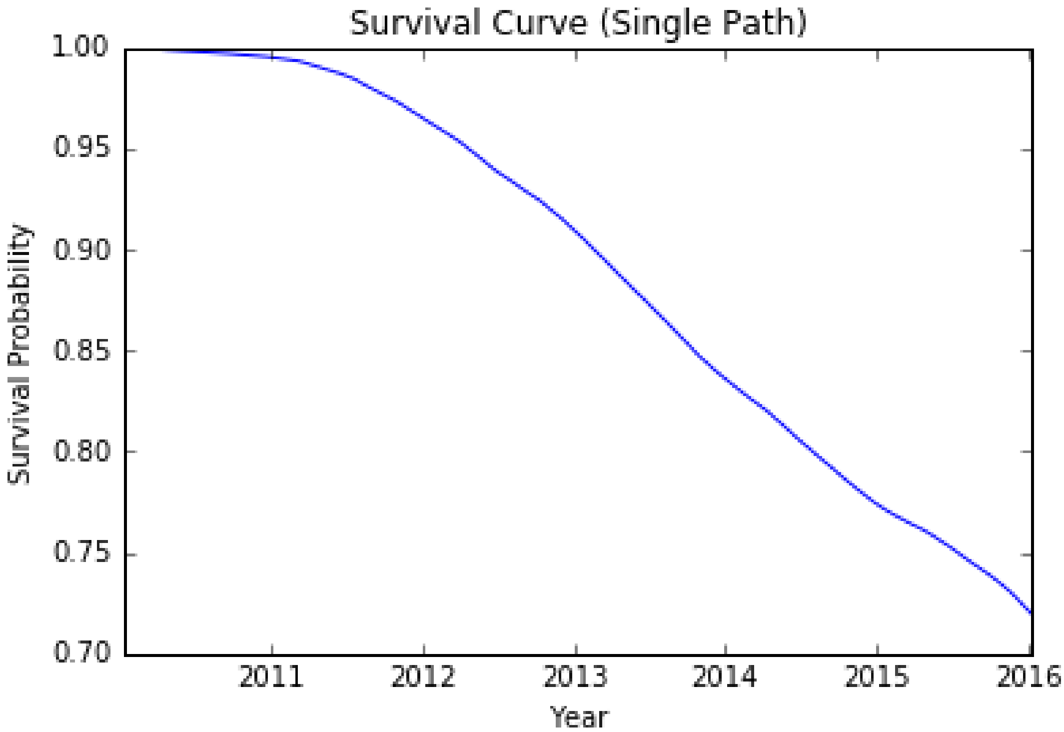
3.3 survival curve calibration and simulation

In this study, we assume the hazard rate process follows CIR model. To calibrate parameters for this model, we use the logic of minimizing the sum of the square differences between market spreads and model spreads for each tenor, i.e., $\min \sum_i (S_i^{market} - S_i^{model})^2$. To calculate the model spreads, discounting curve derived from OIS and analytic solution of CIR survival probabilities (see section 2.1) are used. The calibrated parameters of CIR hazard rate model then are used as inputs of MonteCarlo simulation (daily time step is used). The survival probability from t to $t+s$ $(s>0)$on the survival curve can be derived from $Q(t,t+s) = \exp\left(-\int_t^{t+s} \lambda_u du\right)$.

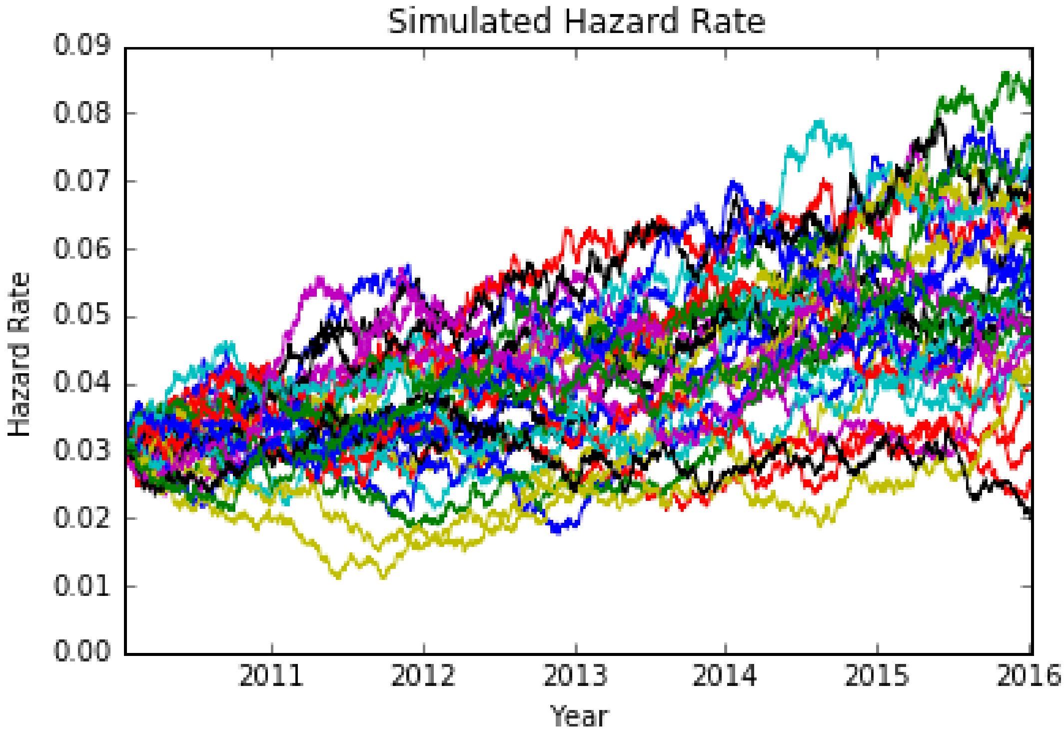
The calibrated parameters for hazard rate using CIR are κ =0.070, θ =0.096, σ =0.042. In the following figures, one path of the simulated hazard rate and survival curve are demonstrated.

[2]The first point on the X-axis is 1 month, so the corresponding discount rate is not exact 1

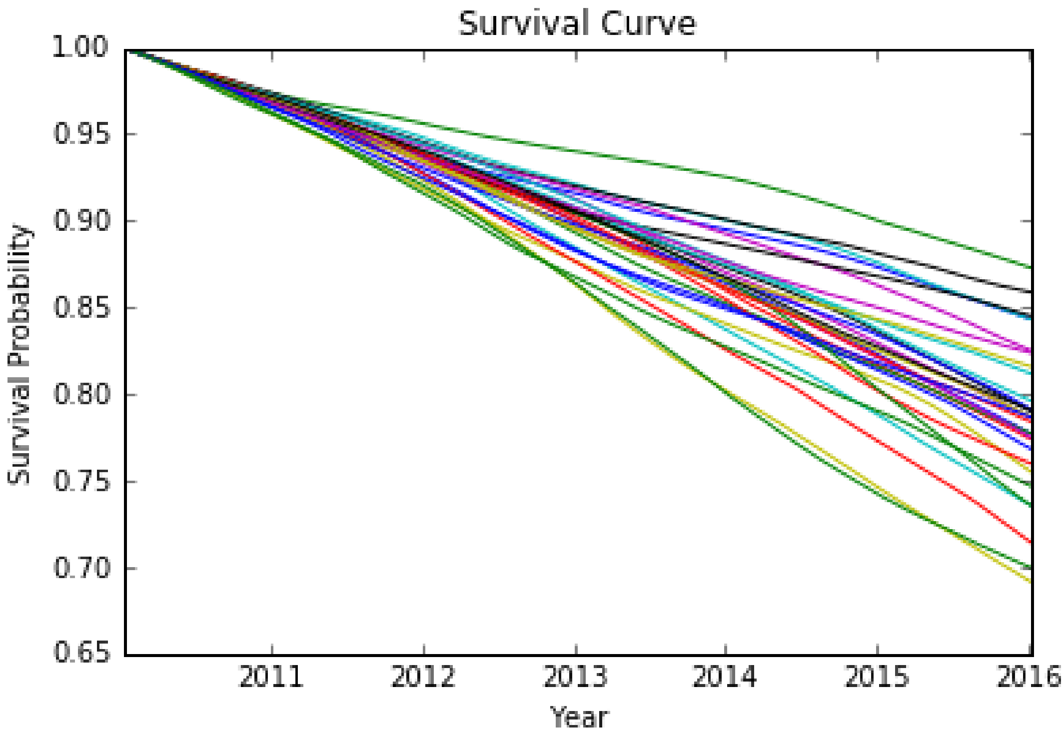




The following figures show the simulated paths for hazard rate and survival probability.



Simulated Hazard Rate
Hazard Rate
Year
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
2011
2012
2013
2014
2015
2016

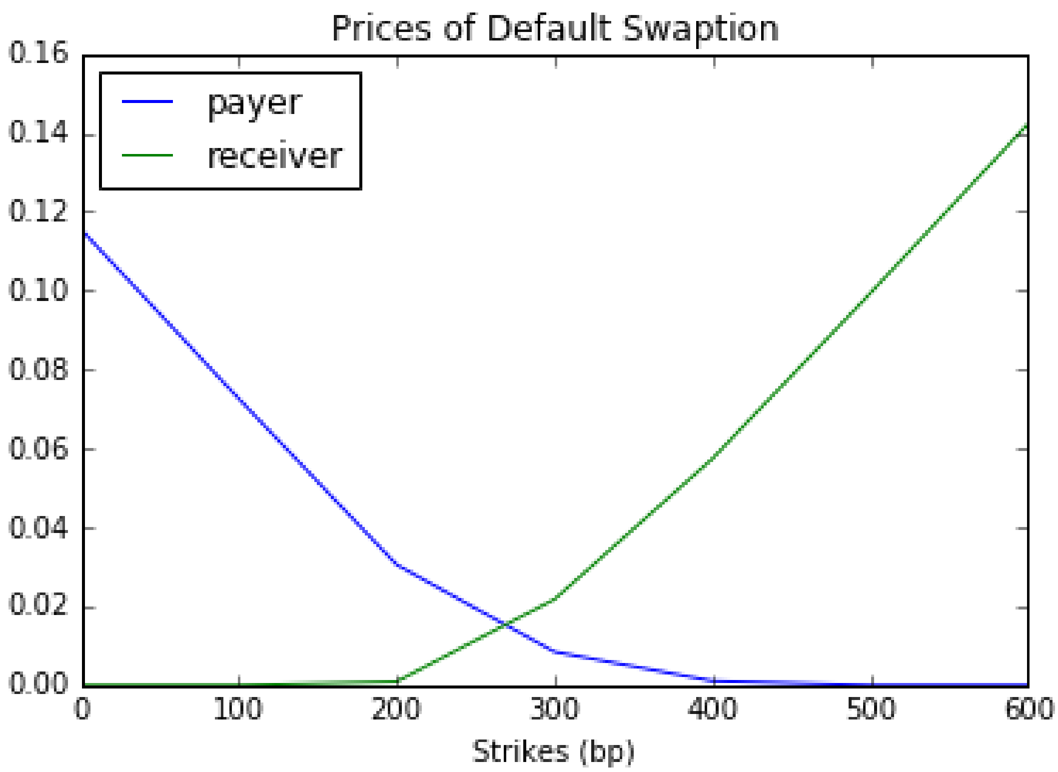


Survival Curve
Survival Probability
Year
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
2011
2012
2013
2014
2015
2016

3.4 Pricing CDS and Default swaption

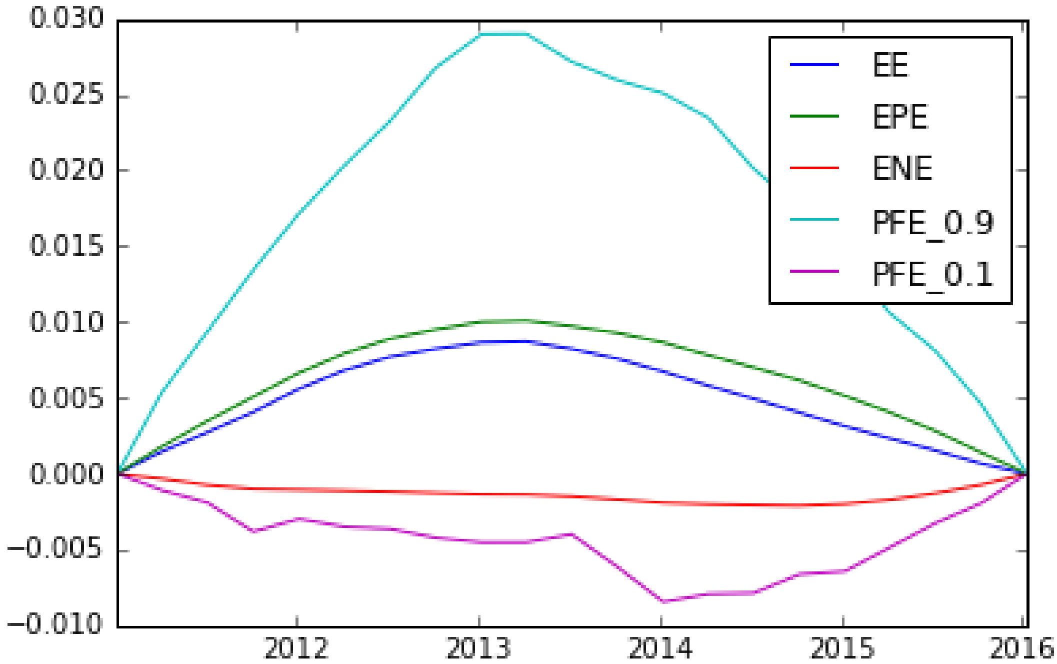
we use Monte Carlo Simulation to price CDS and default swaption. For each underlying survival curve, we use equation in section 2.2 to determine the breakeven spread of CDS. And then apply the method introduced in section 2.3 to derive the price of default swaption. Finally, simply calculate the average of these values to get the price of default swaption.

The average par spread for the CDS setting in section 3.1 is calculated 257 bp. This is close to the market spread with the same tenor, which is 265 bp. Payer and receiver default Swaption prices at different strikes are shown in the following figure.



3.5 CDS exposure and CVA

CDS exposure metrics including EE, EPE, ENE, PFE(0.9), PFE(0.1) are shown in the following graph. The CVA is calculated as 0.0366



4 Sensitivity test

In this section, we implement the sensitivity testing with respect to the underlying credit spread curve. The base curve is defined in section 3.1. The scenarios we tested are listed in the following table.

Name	description
Base	-
Scen 1	parallel shock: + 10% at all tenors
Scen 2	parallel shock: − 10% at all tenors
Scen 3	key point shock: +10% only at tenor 6 month
Scen 4	key point shock: +10% only at tenor 3 year
Scen 5	key point shock: +10% only at tenor 10 year

The following table shows the sensitivity results. The first part lists the calibrated parameters under each scenario. The second part shows the payer (P) and receiver (R) default swaption prices at each strike (bp). The third part gives the average par spread of CDS and the standard deviation for multiple paths. The CVA value can be found in the fourth part.

flavor-strike	Base	scen 1	scen 2	scen 3	scen 4	scen 5
κ	0.070	0.065	0.077	0.049	0.065	0.089
θ	0.096	0.128	0.067	0.121	0.100	0.100
σ	0.042	0.057	0.025	0.041	0.040	0.092
spread-ave	265	273	230	240	263	281
spread-sd	70	83	61	60	62	137
P-0	0.116	0.130	0.095	0.111	0.118	0.129
P-100	0.073	0.088	0.052	0.068	0.076	0.089
P-200	0.031	0.047	0.011	0.029	0.034	0.052
P-300	0.008	0.017	0.000	0.005	0.006	0.027
P-400	0.001	0.005	0.000	0.001	0.000	0.013
P-500	0.000	0.001	0.000	0.000	0.000	0.003
P-600	0.000	0.000	0.000	0.000	0.000	0.001
R-0	0.000	0.000	0.000	0.000	0.000	0.000
R-100	0.000	0.000	0.000	0.000	0.000	0.001
R-200	0.001	0.001	0.003	0.004	0.001	0.007
R-300	0.022	0.013	0.036	0.023	0.016	0.024
R-400	0.057	0.044	0.079	0.062	0.053	0.053
R-500	0.099	0.082	0.123	0.104	0.095	0.085
R-600	0.142	0.124	0.167	0.147	0.138	0.125
CVA	0.037	0.056	0.026	0.039	0.044	0.065

From the results above, we can conclude that (1) decreasing spreads in credit curve parallelly has slightly more impact on default swaption prices than increasing. (2) The spreads with large tenors have more impact on the default swaption prices and CVA than small ones.

5 Conclusion and discussion

In this project, we use Monte Carlo simulation to price CDS swaption and calculate corresponding couterparty credit risk measures. Alternative approach is to use analytic formula by assuming some reasonable distribution for spread. To use this method, additional inputs of swaption volatility data are required. This will be done in future as comparision with the Monte Carlo simulation results.

Reference

[**1**] Dominic O'Kane, 2008. Modeling single-name and multi-names credit derivatives. JohnWiley & Sons.

[**2**] David Lando, 2004. Credit Risk Modeling: Theory and Applications. Princeton University Press.

[**3**] Giovanni Cesari, John Aquilina, Niels Charpillon, Zlatko Filipovic, Gordon Lee, Ion Manda, 2009. Modelling, Pricing, and Hedging Counterparty Credit Exposure: A Technical Guide. Springer.