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Seasonal integration and cointegration for monthly data

A documentation of the test procedure and the respective "R" package.

Contents

1. Introduction

Prerequisite for seasonal cointegration is that time series X and Y are integrated at the same order and there exists a stationary linear combination of both time series on a certain seasonal frequency. Therefore in order to test for cointegration one has to test for integration at every frequency in each time series. For this purpose the test from Hylleberg, Engle, Granger and Yoo (1990) can be used for quarterly data. Since we are dealing with a higher frequency we have to use the extended HEGY test for monthly data (Beaulieu & Miron, 1993). The test from Engle, Granger, Hylleberg and Lee (1993) can be used to test for seasonal cointegration in quarterly data. There exists no seasonal cointegration test for monthly data. Thus this work focuses on formalizing an extended test procedure of the EGHL test using monthly data. Furthermore, specific features of the respective **R** function are introduced.

2. Frequency

Due to the fact that this work deals with monthly data, there are more frequencies to consider than for quarterly data. A data set consisting of monthly observations implies possible patterns for 11 seasonal frequencies and one long run frequency. The long run frequency is also referred to as the zero frequency. Table 2 contains the cycles per year and the period in months for all seasonal frequencies. Every cycle is associated with one cycle in the unit circle. The zero frequency passes one cycle within an infinite number of months. Every seasonal frequency can be illustrated with an respective sine wave[1]. It can be shown that the number of peaks of the sine wave during a year is the same for two frequencies each. Here those frequencies are referred to as frequency pairs.

3. Test for integration

In order to test if a time series is integrated at a certain frequency one can use the HEGY test (Hylleberg et al., 1990). This test allows to check each frequency for unit root without taking a stand on whether other frequencies have unit root or not. A series X_t is integrated if it has an unit root in its autoregressive representation. An integrated time series can be expressed as

$$\varphi(L)X_t = \epsilon_t \tag{1}$$

[1] See page 6-7 in the presentation belonging to this documentation.

where ϵ_t is a white noise process and $\varphi(L)$ is a polynomial in the backshift operator. For monthly data, $\varphi(L)$ is associated with $(1 - L^{12})$. The highest exponent of this polynomial indicates 12 possible unit roots. Thus equation 1 can be used to generate a seasonal ARIMA $(0,0,0)(0,1,0)_{12}$ process, which is integrated at all frequencies . This polynomial can be factorized into a root decomposition of 7 terms

$$\begin{aligned}(1 - L^{12}) = &(1 - L)(1 + L)(1 + L^2)(1 + L + L^2) \\ &\times (1 - L + L^2)(1 + \sqrt{3}L + L^2)(1 - \sqrt{3}L + L^2).\end{aligned} \tag{2}$$

Selected terms of latter decomposition are used for constructing a test for seasonal cointegration. The term associated with the zero frequency is $(1 - L)$ and $(1 + L)$ for 6 cycles per year. The terms $(1 + L^2)$, $(1 + L + L^2)$, $(1 - L + L^2)$, $(1 + \sqrt{3}L + L^2)$ and $(1 - \sqrt{3}L + L^2)$ are associated with 3 and 9, 4 and 8, 2 and 10, 5 and 7, 1 and 11 cycles per year, respectively. Further factorization gives

$$\begin{aligned}(1 - L^{12}) = &(1 - L)(1 + L)(1 - iL)(1 + iL) \\ &\times [1 + \frac{1}{2}(1 + i3^{\frac{1}{2}}L)][1 + \frac{1}{2}(1 - i3^{\frac{1}{2}})L][1 - \frac{1}{2}(1 - i3^{\frac{1}{2}})L] \\ &\times [1 - \frac{1}{2}(1 + i3^{\frac{1}{2}})L][1 + \frac{1}{2}(3^{\frac{1}{2}} + i)L][1 + \frac{1}{2}(3^{\frac{1}{2}} - i)L] \\ &\times [1 - \frac{1}{2}(3^{\frac{1}{2}} - i)L][1 - \frac{1}{2}(3^{\frac{1}{2}} + i)L].\end{aligned} \tag{3}$$

For equation 3 $(1 - L^{12}) = 0$ holds if at least one term of those twelve terms is zero. In this case, we know that a root exists for at least one term of the decomposition. Due to the form of the polynomial every term has a root on the unit circle. As you can see for the unit roots in table 1, each corresponding to a certain term of latter decomposition, two roots are real numbers and the others are either a pure imaginary number or complex numbers. Let S be the number of observations per year. One way to derive all unit roots for any S is

$$\begin{aligned}&Z^n = 1 = \exp(i2\pi j) = \cos(2\pi j) + i\sin(2\pi j) \\ \Leftrightarrow \quad &Z = 1^{\frac{1}{S}} = \exp(\frac{i2\pi j}{S}) = [\cos(2\pi j) + i\sin(2\pi j)]^{\frac{1}{S}} \\ &\quad = \cos(\frac{2\pi j}{S}) + \mathrm{i}\sin(\frac{2\pi j}{S}), \text{ with } j = 0, ..., S - 1.\end{aligned}$$

Inserting $j = 0, ..., S-1$ with $S = 12$ separately in the previous equations gives

$$\begin{aligned}
Z_1 &= exp(0) = 1,\\
Z_2 &= exp(\frac{i\pi}{6}) = cos(\frac{\pi}{6}) + isin(\frac{\pi}{6}) = \frac{1}{2}(\sqrt{3}+i),\\
Z_3 &= exp(\frac{i\pi}{3}) = cos(\frac{\pi}{3}) + isin(\frac{\pi}{3}) = \frac{1}{2}(1+\sqrt{3}i),\\
Z_4 &= exp(\frac{i\pi}{2}) = cos(\frac{\pi}{2}) + isin(\frac{\pi}{2}) = i,\\
Z_5 &= exp(\frac{i2\pi}{3}) = cos(\frac{2\pi}{3}) + isin(\frac{2\pi}{3}) = -\frac{1}{2}(1-\sqrt{3}i),\\
Z_6 &= exp(\frac{5\pi}{6}) = cos(\frac{5\pi}{6}) + isin(\frac{5\pi}{6}) = -\frac{1}{2}(\sqrt{3}-i),\\
Z_7 &= exp(\pi i) = cos(\pi) + isin(\pi) = -1,\\
Z_8 &= exp(\frac{i7\pi}{6}) = cos(-\frac{5\pi}{6}) + isin(\frac{5\pi}{6}) = -\frac{1}{2}(\sqrt{3}+i),\\
Z_9 &= exp(\frac{i4\pi}{3}) = cos(-\frac{2\pi}{3}) + isin(-\frac{2\pi}{3}) = -\frac{1}{2}(1+\sqrt{3}i),\\
Z_{10} &= exp(\frac{i3\pi}{2}) = cos(-\frac{\pi}{2}) + isin(-\frac{\pi}{2}) = -i,\\
Z_{11} &= exp(\frac{i5\pi}{3}) = cos(-\frac{\pi}{3}) + isin(-\frac{\pi}{3}) = \frac{1}{2}(1-\sqrt{3}i),\\
Z_{12} &= exp(\frac{i11\pi}{6}) = cos(-\frac{\pi}{6}) + isin(-\frac{\pi}{6}) = \frac{1}{2}(\sqrt{3}-i).
\end{aligned}$$

Those unit roots, the frequency in terms of π and cycles per year are illustrated in table 1. One can rewrite the polynomial $\varphi(L)$ due to Lagrange expansion which is often used in approximation theory as

$$\varphi(L) = \sum_{k=1}^{S} \lambda_k \Delta(L) \frac{1-\delta_k(L)}{\delta_k(L)} + \Delta(L)\varphi^*(L), \tag{4}$$

where

$$\delta_k(L) = 1 - \frac{1}{\theta_k} L, \quad \lambda_k = \frac{\varphi(\theta_k)}{\prod\limits_{j\neq k} \delta_j(\theta_k)}, \quad \Delta(L) = \prod_{k=1}^{S} \delta_k(L). \tag{5}$$

This reformulation enables us to test if the roots of the polynomial lie on the unit circle against the alternative that they lie outside the unit circle. The zero frequency unit root plus the $S-1$ seasonal unit roots are expressed with θ_k. The remaining roots outside the unit circle are presented with $\varphi^*(L)$. Inserting equation 4 into equation 1 gives the HEGY regression for monthly data[2]

$$y_{13t} = \sum_{k=1}^{12} \pi_k y_{k,t-1} + \text{augmentations} + \epsilon_t. \tag{6}$$

The regressors are filtered time series and can explicitly be expressed as

$$\begin{aligned}
y_{1t} &= (1 + L + L^2 + L^3 + L^4 + L^5 + L^6 + L^7 + L^8 + L^9 + L^{10} + L^{11})X_t,\\
y_{2t} &= -(1 - L + L^2 - L^3 + L^4 - L^5 + L^6 - L^7 + L^8 - L^9 + L^{10} - L^{11})X_t,\\
y_{3t} &= -(L - L^3 + L^5 - L^7 + L^9 - L^{11})X_t,\\
y_{4t} &= -(1 - L^2 + L^4 - L^6 + L^8 - L^{10})X_t,\\
y_{5t} &= -\tfrac{1}{2}(1 + L - 2L^2 + L^3 + L^4 - 2L^5 + L^6 + L^7 - 2L^8 + L^9 + L^{10} - 2L^{11})X_t,\\
y_{6t} &= \tfrac{\sqrt{3}}{2}(1 - L + L^3 - L^4 + L^6 - L^7 + L^9 - L^{10})X_t,\\
y_{7t} &= \tfrac{1}{2}(1 - L - 2L^2 - L^3 + L^4 + 2L^5 + L^6 - L^7 - 2L^8 - L^9 + L^{10} + 2L^{11})X_t,\\
y_{8t} &= -\tfrac{\sqrt{3}}{2}(1 + L - L^3 - L^4 + L^6 + L^7 - L^9 - L^{10})X_t,\\
y_{9t} &= -\tfrac{1}{2}(\sqrt{3} - L + L^3 - \sqrt{3}L^4 + 2L^5 - \sqrt{3}L^6 + L^7 - L^9 + \sqrt{3}L^{10} - 2L^{11})X_t,\\
y_{10t} &= \tfrac{1}{2}(1 - \sqrt{3}L + 2L^2 - \sqrt{3}L^3 + L^4 - L^6 + \sqrt{3}L^7 - 2L^8 + \sqrt{3}L^9 - L^{10})X_t,\\
y_{11t} &= \tfrac{1}{2}(\sqrt{3} + L - L^3 - \sqrt{3}L^4 - 2L^5 - \sqrt{3}L^6 - L^7 + L^9 - \sqrt{3}L^{10} + 2L^{11})X_t,\\
y_{12t} &= -\tfrac{1}{2}(1 + \sqrt{3}L + 2L^2 + \sqrt{3}L^3 + L^4 - L^6 - \sqrt{3}L^7 - 2L^8 - \sqrt{3}L^9 - L^{10})X_t,\\
y_{13t} &= \Delta_{12}X_t = (1 - L^{12})X_t.
\end{aligned}$$

Each y_{kt} for $k = 1, ..., 12$ is a filtered time series for all frequencies except the one of interest. A filtered time series y_{kt} will only contain a pattern of the frequency[3] associated with k. The OLS method can be used in order to estimate the HEGY regressions. To test if the time series of interest has a seasonal unit root at a specific frequency, one has to test if $\varphi(\theta_k) = 0$ and therefore that $\pi_k = 0$. Thus a left sided t-test can be used for the zero and bimonthly unit root

$$H_0 : \pi_k = 0 \quad \text{vs.} \quad H_1 : \pi_k < 0 \quad \text{for} \quad k = 1, 2. \tag{7}$$

For the remaining complex root pairs a joint F-statistic is used with

$$H_0 : \pi_k = \pi_{k+1} = 0 \quad \text{vs.} \quad H_1 : \text{not } H_0 \quad \text{for} \quad k = 3, 5, 7, 9, 11.$$

[2]See Beaulieu and Miron (1993) p. 322 ff. for technical details.
[3]See table 1 for frequencies associated with index k

The design of the HEGY regression enables us to use two separate tests which accounts for the flexibility of this approach. To take a proper test decision the test statistics are compared to simulated finite sample distributions under the respective null hypothesis. Those null distributions were simulated with Monte Carlo runs and are explained in more detail in chapter 5.2.1. In both cases the null hypothesis stands for the existence of unit root. In other words, the time series is not stationary and thus integrated. Furthermore, it is possible to regress on additional lags of y_{13}. Those are captured by the augmentations term in equation 6. Without augmentations this regression often fails to produce white noise residuals. Hence, there are often some significant lags in the ACF of the residuals left over. By using augmentations these can be eliminated[4].

Table 1: Seasonal unit roots for each frequency

Seasonal unit root	Frequency in terms of pi	Frequency in terms of cycles/year	Index
-1	π	6	$k = 2$
$\pm i$	$\pm\frac{\pi}{2}$	3, 9	$k = 3, 4$
$-\frac{1}{2}(1 \pm \sqrt{3}i)$	$\pm\frac{2\pi}{3}$	4, 8	$k = 5, 6$
$\frac{1}{2}(1 \pm \sqrt{3}i)$	$\pm\frac{\pi}{3}$	2, 10	$k = 7, 8$
$-\frac{1}{2}(\sqrt{3} \mp i)$	$\pm\frac{5\pi}{6}$	5, 7	$k = 9, 10$
$\frac{1}{2}(\sqrt{3} \pm i)$	$\pm\frac{\pi}{6}$	1, 11	$k = 11, 12$

Each root corresponds to a term of the polynomial decomposition in equation 3. The unit root solving the zero frequency term $(1 - L)$ for 0 is 1. The index asoociated with the zero frequency is $k = 1$.

4. Test for cointegration

Let X and Y be the time series of interest containing twelve observations per year. To test for seasonal cointegration using monthly data, one should first use a seasonal integration test to find out if both time series are integrated at a certain frequency. The HEGY test in chapter 3 can be used for this purpose. Normally, both time series have to be integrated at the same order for a certain frequency. If this is the case, the EGHL test can be used (Engle et al., 1993). As you can see in equation 2 the polynomial can be factorized in 7 terms. The first two terms are associated with the zero and $\frac{6}{12}$ frequency. Every term of the five remained terms is associated with a pair of frequencies. To construct a polynomial, which filters for all frequencies except

[4] Note that the test results are highly sensible to the use of augmentations. Further explanations to the use of augmentations can be found in chapter 5.

the frequency of interest, we can multiply the respective terms and will get 7 filter representations and one representation of the original polynomial

$$\begin{aligned}
\Theta_1 &= (1+L)(1+L^2)(1+L+L^2)(1-L+L^2)(1+\sqrt{3}L+L^2)(1-\sqrt{3}L+L^2),\\
\Theta_2 &= -(1-L)(1+L^2)(1+L+L^2)(1-L+L^2)(1+\sqrt{3}L+L^2)(1-\sqrt{3}L+L^2),\\
\Theta_3 &= -(1-L)(1+L)(1+L+L^2)(1-L+L^2)(1+\sqrt{3}L+L^2)(1-\sqrt{3}L+L^2),\\
\Theta_4 &= -(1-L)(1+L)(1+L^2)(1-L+L^2)(1+\sqrt{3}L+L^2)(1-\sqrt{3}L+L^2),\\
\Theta_5 &= -(1-L)(1+L)(1+L^2)(1+L+L^2)(1+\sqrt{3}L+L^2)(1-\sqrt{3}L+L^2),\\
\Theta_6 &= -(1-L)(1+L)(1+L^2)(1+L+L^2)(1-L+L^2)(1-\sqrt{3}L+L^2),\\
\Theta_7 &= -(1-L)(1+L)(1+L^2)(1+L+L^2)(1-L+L^2)(1+\sqrt{3}L+L^2)(1-\sqrt{3}L+L^2),\\
\Theta_8 &= (1-L)(1+L)(1+L^2)(1+L+L^2)(1-L+L^2)(1+\sqrt{3}L+L^2)(1-\sqrt{3}L+L^2)\\
&= (1-L^{12}).
\end{aligned}$$

The filters Θ_k for $k = 1, ..., 7$ are associated with the frequencies $\{0; \frac{6}{12}; (\frac{3}{12}, \frac{9}{12}); (\frac{4}{12}, \frac{8}{12}); (\frac{2}{12}, \frac{10}{12}); (\frac{5}{12}, \frac{7}{12}); (\frac{1}{12}, \frac{11}{12})\}$. Formalizing these simplifies the use of a proper test procedure. To get Θ_1 one simply omits $(1-L)$ from equation 2, i.e. the term that would filter out the zero frequency. Therefore, by applying Θ_1 only the zero frequency remains in the time series. Assuming both time series have an unit root at the zero frequency, one would test for cointegration at this frequency by regressing

$$\Theta_1 Y_t = \alpha_1 + \beta_1 \Theta_1 X_t + u_t. \tag{8}$$

Both time series are cointegrated if $\hat{u}_t$ is $I(0)$. Θ_1 filters X and Y for all frequencies except the zero frequency. Thus only the zero frequency will be left over in the residuals from equation 8. In order to examine the presence of stationary residuals, an auxiliary regression in line with the ADF test () is done

$$(1-L)\hat{u}_t = \lambda_1 \hat{u}_{t-1} + \text{augmentations} + e_t. \tag{9}$$

Remember, that we require by definition that there exists a stationary linear combination of both time series. Thus we need to examine if these residuals are stationary. As done in equation 9, this can be done by filtering the remaining frequency with the respective term of the polynomial factorization and using an OLS regression[5]. To test for cointegration at the $\frac{6}{12}$ frequency one regresses

$$\Theta_2 Y_t = \alpha_2 + \beta_2 \Theta_2 X_t + v_t, \tag{10}$$

[5]Note that the same explanation can be used for the $\frac{6}{12}$ and the remaining 5 pair frequencies.

with the auxiliary regression

$$(1+L)\hat{v}_t = -\lambda_2 \hat{v}_{t-1} + \text{augmentations} + c_t. \tag{11}$$

As you can see, for the $\frac{6}{12}$ frequency the basic part of equation 11 is regressing $(1+L)\hat{v}_t$ on $-\hat{v}_{t-1}$. This auxiliary regression differs from equation 9 in terms of the filter and the minus sign. As mentioned by Engle et al. (1993) on page 283 it is crucial to use a minus sign in order for the distribution to be the same as for $(1-L)\hat{u}_t$ on $\hat{u}_t$. Otherwise, a mirror image of this distribution should be used. To get a proper test decision, the t-statistic of λ_1 and λ_2 is compared to a respective simulated finite sample distribution under the null hypothesis of

$$H_0 : \lambda_i = 0 \quad \text{vs.} \quad H_1 : \lambda_i < 0 \quad \text{for} \quad i = 1, 2.$$

Finally, the following regression is used for the five remaining frequency pairs

$$\Theta_k Y_t = \alpha + \beta_1 \Theta_k X_t + \beta_2 \Theta_k X_{t-1} + w_t \quad \text{for} \quad k = 3, 4, 5, 6, 7, \tag{12}$$

with the auxiliary regression

$$-\frac{\Theta_8}{\Theta_k} \hat{w}_t = -\gamma_1 \hat{w}_{t-2} - \gamma_2 \hat{w}_{t-1} + \text{augmentations} + z_t \quad \text{for} \quad k = 3, 4, 5, 6, 7. \tag{13}$$

Note that $-\frac{\Theta_8}{\Theta_k}$ for $k = 3, 4, 5, 6, 7$ is equal to the set of filters

$$\{(1+L^2), (1+L^2+L^3), (1-L+L^2), (1+\sqrt{3}L+L^2), (1-\sqrt{3}L+L^2)\},$$

and is associated with the frequencies $(\frac{3}{12}, \frac{9}{12}), (\frac{4}{12}, \frac{8}{12}), (\frac{2}{12}, \frac{10}{12}), (\frac{5}{12}, \frac{7}{12}), (\frac{1}{12}, \frac{11}{12})$, respectively. Also, one has to take into account that by testing for a seasonal pair of frequencies a regression on two independent variables in equation 12 as well as in the auxiliary regression 13 is being done. To test for cointegration at one of the pair frequencies a F-test can be used with the hypothesis

$$H_0 = \gamma_1 = \gamma_2 = 0 \quad \text{vs.} \quad H_1 : \text{not } H_0 \quad \text{for} \quad k = 3, 4, 5, 6, 7.$$

Again, critical values are taken out of simulated finite sample distributions under the null hypothesis. As for the zero and $\frac{6}{12}$ frequency the alternative hypothesis means that the respective residuals are stationary. Therefore, there exists a stationary linear combination of X and Y. Thus, we can not reject the null hypothesis of no cointegration between X and Y.

5. Augmentations

5.1. Procedure

As you can see, the HEGY test from chapter 3 as well the extended EGHL test from chapter 4 are both using augmentations in their regressions. This can be intuitively explained by taking the auxiliary regression 9 as an example. In many cases that regression fails to generate white noise residuals. Thus, we will often find significant lags in the ACF function of the residuals e_t. This problem can usually be avoided by including augmentations in the regression. Augmentations are lagged values of the dependent variable[6]. There exist many approaches how one can select appropriate augmentations. The seasonal cointegration function uses the following approach. First, one estimates the appropriate regression where augmentations might be useful. In each case, the augmentations consist of the first 24 lagged values of the respective dependent variable. Second, one determines the most insignificant augmentation based on p-values and removes it. Finally, this procedure is repeated as long as only significant augmentations are left or no significant augmentations are found.

5.2. Option

The user decides to either use the default option or the manual option. This selection is implemented because the test decision is sensible to the length of the time series in interest. Thus, the user will get the most precise results if he makes use of the "manual" option with a high number of Monte Carlo runs. As mentioned in the previous chapters, the HEGY test decisions as well the EGHL test decisions are based on test statistics of the simulated distributions under the null hypothesis. The default option uses an interpolation between time series of different lengths for calculating the p-values. In contrast, the manual option is based on a simulation with time series of the exact length as the time series in interest.

5.2.1. Default option

The default option makes use of simulated and already saved distributions under the null hypothesis. The simulation was conduct with 25.000 Monte Carlo runs. Each run contains four time series with $60, 120, 240$ and 1200 simulated observations of monthly data, respectively. A seasonal ARIMA (0,0,0)(0,1,0) process is used for simulating the

[6]Note that this argumentation can be applied for every equation with an augmentations term within this work.

time series

$$z_t = z_{t-12} + \epsilon_t, \text{ with } \epsilon_t \overset{iid}{\sim} N(0,1).$$

In contrast to the testing procedure, a different approach is applied to select proper augmentations. In each Monte Carlo run, every selected augmentation of the 24 possible augmentations is documented. Afterward, we calculate the percentage of every selected augmentation. The results from table 3 show that there is a certain pattern for every lag order of augmentation. The augmentations which are selected relatively often where used to simulate the null distribution to be used if the default option is chosen by the user. The calculation of the p-value is done as follows. First, the time series is assigned to the interval it belongs to. Then, the p-values for the boundaries of this interval are calculated. Finally, a linear interpolation between those two p-values is being done.

5.2.2. Manual option

The manual option does not use already simulated zero distributions. The user can decide how many Monte Carlo runs he wants to run for simulating the zero distribution. The simulation will then use the same length for the simulated time series as the time series of interest. Thus the p-values tend to be more exact and there will not be an inaccuracy due to the interpolation of p-values. An amount of approximately 1200 Monte Carlo runs will lead to precise results. In our experience, increasing the number of Monte Carlo runs beyond 1200 does barely change the null distribution. Thus the user has to make a trade-off between time and reliability. The augmentations selected with the procedure explained in chapter 5.1 are used.

6. Limitations

A central limitation of the EGHL test for monthly data is the asymmetry of the main regression it is based on. This means that, it makes a difference for the test result if you regress X on Y or Y on X. A possible explanation might be as follows. By taking regression 12 as an example, the direction of regressing X and Y will lead to different residuals w_t. Therefore the F-statistic of the auxiliary regression 13 will also be different. This effect becomes more important if the distinction in residuals differs enough to let the procedure not use the same augmentations. A difference in

F-statistics can result in different test decisions[7].

[7]You can see a specific example of this problem applied on a data set in the presentation on slides 27-30 belonging to this work.

A. Appendix

Table 2: Frequency

Frequency	Cycles per year	Period in months
$\frac{6}{12}$	6	2
$\{\frac{3}{12}, \frac{9}{12}\}$	3	4
$\{\frac{4}{12}, \frac{8}{12}\}$	4	3
$\{\frac{2}{12}, \frac{10}{12}\}$	2	6
$\{\frac{5}{12}, \frac{7}{12}\}$	5	2.5
$\{\frac{1}{12}, \frac{11}{12}\}$	1	12

Cycles per year is the number of times a cycle in the sine wave repeats within 12 months. Period is the length in months between two consecutive peaks.

Table 3: Percentage of selected augmentations for each frequency

	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag6	Lag 7	Lag 8	Lag 9	Lag 10	Lag 11	Lag 12	Lag 13
$\theta = 0$	0.050	0.051	0.051	0.048	0.050	0.048	0.056	0.053	0.050	0.047	0.049	0.050	0.0510
$\theta = \frac{6}{12}$	**0.150**	0.079	0.074	0.075	0.073	0.074	0.073	0.074	0.074	0.081	0.078	**0.125**	**0.118**
$\theta = \frac{3}{12}, \frac{9}{12}$	0.017	**0.059**	0.017	**0.064**	0.017	**0.069**	0.016	**0.075**	0.017	**0.076**	0.019	**0.087**	0.018
$\theta = \frac{4}{12}, \frac{8}{12}$	0.021	0.020	**0.141**	0.018	0.016	**0.094**	0.018	0.021	**0.080**	0.022	0.019	**0.089**	0.017
$\theta = \frac{2}{12}, \frac{10}{12}$	0.020	0.022	**0.150**	0.021	0.019	**0.102**	0.021	0.019	**0.076**	0.020	0.021	**0.091**	0.018
$\theta = \frac{5}{12}, \frac{7}{12}$	0.031	0.018	0.014	0.019	0.032	**0.161**	0.032	0.016	0.013	0.017	0.030	**0.101**	0.030
$\theta = \frac{1}{12}, \frac{11}{12}$	0.034	0.019	0.015	0.017	0.031	**0.171**	0.032	0.021	0.014	0.017	0.030	**0.1**	0.033

Higher values printed in bold are showing a pattern for every frequency. This table only shows lagged augmentations till lag 13 instead of 24.

B. References

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