Distance to default package

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This package provides fast functions to work with the Merton's distance to default model. We will only briefly cover the model here. See e.g., [5] for a more complete coverage. Denote the observed market values by S_t and unobserved asset values by V_t. We assume that V_t follows a geometric Brownian motion

$$dV_t = \mu V_t \,\mathrm{d}t + \sigma V_t \,\mathrm{d}W_t$$

We observe the asset values over increments of dt in time. Let V_k denote the value at $t_0 + k \cdot dt$. Thus,

$$V_{k+1} = V_k \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma W_t\right)$$

We further let r denote the risk free rate, D_t denote debt due at time $t + T$. Then

$$\begin{aligned} C(V_t, D_t, T, \sigma, r) &= V_t N(d_1) - D_t \exp\left(-rT\right) N(d_1 - \sigma\sqrt{T}) \\ d_1 &= \frac{\log(V_t) - \log D_t + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \qquad (1) \\ S_t &= C(V_t, D_t, T, \sigma, r) \qquad (2) \end{aligned}$$

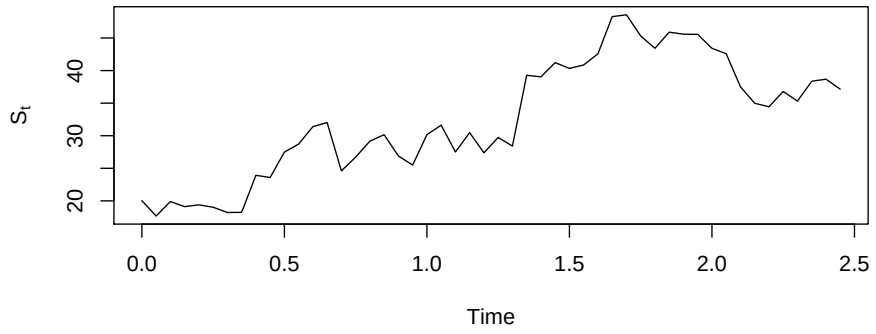
where C is a European call option price, T is the time to maturity, D_t is the debt to due at time $T + t$, and r is the risk free rate. Common choices tend to be $T = 1$ year and D_t as the short term debt plus half of the long term debt. d_1 in equation (1) is the so-called distance-to-default which is the name of the package. It is a very good predictor of default risk despite it's simplicity [see e.g., 1].

Equation (2) can be computed with the `BS_call` function. Further, the `get_underlying` function can be used to invert call option price in equation (2)

```
library(DtD)
(S <- BS_call(100, 90, 1, .1, .3))

## [1] 22.51

get_underlying(S, 90, 1, .1, .3)

## [1] 100
```

To illustrate the above then we can simulate the underlying and transform the data into the stock price as follows

```
# assign parameters
vol <- .1
mu  <- .05
dt  <- .05
V_0 <- 100
t.  <- (1:50 - 1) * dt
D   <- c(rep(80, 27), rep( 70, length(t.) - 27))
r   <- c(rep( 0, 13), rep(.02, length(t.) - 13))

# simulate underlying
set.seed(seed <- 83992673)
V <- V_0 * exp(
  (mu - vol^2/2) * t. + cumsum(c(
    0, rnorm(length(t.) - 1, sd = vol * sqrt(dt)))))

# compute stock price
S <- BS_call(V, D, T. = 1, r, vol)
plot(t., S, type = "l", xlab = "Time", ylab = expression(S[t]))
```



Despite that the model assume a constant risk free rate than we let it vary in this example. We end by plotting the stock price. Further, we can confirm that we the same underlying after transforming back

```
all.equal(V, get_underlying(S, D, 1, r, vol))

## [1] TRUE
```

We could also have used the simulation function in the package

```r
set.seed(seed) # use same seed
sims <- BS_sim(
  vol = vol, mu = mu, dt = dt, V_0 = V_0, D = D, r = r, T. = 1)

isTRUE(all.equal(sims$V, V))

## [1] TRUE

isTRUE(all.equal(sims$S, S))

## [1] TRUE
```

1 Drift and volatility estimation

There are a few ways to estimate the volatility, σ, and drift, μ. This package only includes the iterative procedure and maximum likelihood method covered in [4, 6, 3, 1]. We denote the former as the "iterative" method and the latter as the MLE. We have not implemented the method where one solves two simultaneous equation as it will be based on two measurements and may be quite variable [as mentioned in 2].

The parameters can be estimated with the `BS_fit` function. The iterative method is used in the following call

```r
# simulate data
set.seed(52722174)
sims <- BS_sim(
  vol = .2, mu = .05, dt = 1/252, V_0 = 100, r = .01, T. = 1,
  # simulate firm that grows partly by lending
  D = 70 * (1 + .01 * (0:(252 * 4)) / 252))

# the sims data.frame has a time column. We need to pass this
head(sims$time, 6)

## [1] 0.000000 0.003968 0.007937 0.011905 0.015873 0.019841

# estimate parameters
it_est <- BS_fit(
  S = sims$S, D = sims$D, T. = sims$T, r = sims$r, time = sims$time,
  method = "iterative")
it_est

## $ests
##       mu      vol
## -0.06848  0.19701
##
## $n_iter
## [1] 19
```

```
##
## $success
## [1] TRUE
```

The volatility is quite close the actual value while the drift is a bit off. This may be due to the fact that the likelihood is flat in the drift. The maximum likelihood estimator is obtained by maximizing the observed log likelihood

$$\begin{aligned} L(\mu, \sigma, \vec{S}) \propto & -n \log\left(\sigma^2 dt\right) - \sum_{k=1}^{n} \frac{\left(\log \frac{C^{-1}(S_k, \sigma)}{C^{-1}(S_{k-1}, \sigma)} - \left(\mu - \sigma^2/2\right) dt\right)^2}{\sigma^2 dt} \\ & - 2 \sum_{k=1}^{n} \left(\log C^{-1}\left(S_k, \sigma\right) + \log \left|C'\left(C^{-1}\left(S_k, \sigma\right), \sigma\right)\right|\right) \end{aligned} \tag{3}$$

where C^{-1} is the inverse of the call option price in equation (2) and implicitly depend on D_t, T, and r. Notice that we need to use dt in (3) and the time to maturity, T, in C and C^{-1}. The last term in equation (3) follows from the change of variable

$$\begin{aligned} X &= h^{-1}(Y) \\ f_Y(y) &= f_X\left(h^{-1}(y)\right) \left|(h^{-1})'(y)\right| \\ &= f_X\left(h^{-1}(y)\right) \left|\frac{1}{h'\left(h^{-1}(y)\right)}\right| \end{aligned} \tag{4}$$

where f denotes a density and the subscript denotes which random variable the density is for. We can estimate the parameters with the MLE method as follows

```
mle_est <- BS_fit(
  S = sims$S, D = sims$D, T. = sims$T, r = sims$r, time = sims$time,
  method = "mle")
mle_est

## $ests
##       mu      vol
## -0.07094  0.19762
##
## $n_iter
## [1] 47
##
## $success
## [1] TRUE
```

The result are usually very similar although they need not to as far as I gather

```
it_est$est - mle_est$est

##         mu        vol
##  0.0024646 -0.0006042
```

The iterative method is faster though

```
library(microbenchmark)
with(sims,
  microbenchmark(
    iter = BS_fit(
      S = S, D = D, T. = T, r = r, time = time, method = "iterative"),
    mle = BS_fit(
      S = S, D = D, T. = T, r = r, time = time, method = "mle"),
    times = 5))

## Unit: milliseconds
##  expr   min    lq  mean median    uq   max neval cld
##  iter 270.2 271.2 275.0  272.0 279.7 281.9     5  a
##   mle 599.7 612.5 620.8  624.6 631.3 635.7     5   b
```

We can also estimate the parameters when there unequal time gaps in the data set

```
# drop random rows
sims <- sims[sort(sample.int(nrow(sims), 100L)), ]

# the gap lengths are not equal anymore
range(diff(sims$time))

## [1] 0.003968 0.174603

# estimate parameters
BS_fit(
  S = sims$S, D = sims$D, T. = sims$T, r = sims$r, time = sims$time,
  method = "iterative")

## $ests
##       mu      vol
## -0.05982  0.17937
##
## $n_iter
## [1] 20
##
## $success
## [1] TRUE
```

References

[1] Sreedhar T. Bharath and Tyler Shumway. Forecasting default with the merton distance to default model. *The Review of Financial Studies*, 21(3):1339–1369, 2008.

[2] Peter Crosbie. Modeling default risk. Technical report, Moody, December 2003.

[3] Jin-Chuan Duan, Geneviève Gauthier, and Jean-Guy Simonato. On the equivalence of the kmv and maximum likelihood methods for structural credit risk models. 2004.

[4] JinChuan Duan. Maximum likelihood estimation using price data of the derivative contract. *Mathematical Finance*, 4(2):155–167, 1994.

[5] David Lando. *Credit risk modeling: theory and applications*. Princeton University Press, 2009.

[6] Maria Vassalou and Yuhang Xing. Default risk in equity returns. *The Journal of Finance*, 59(2):831–868, 2004.