Resampling

Asset Management: Advanced Investments

Markus Leippold

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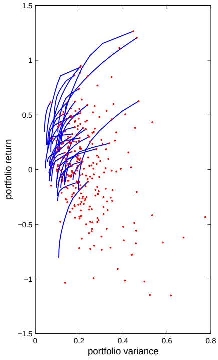
University of Zurich
Department of Banking and Finance

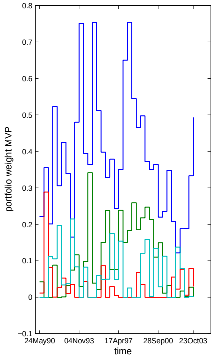
Outline

Lack of acceptance of MPT among practitioners

- A huge MV critique: Estimation error.
- MV portfolios often seem counterintuitive, inexplicable and overly sensitive to the input parameters.
- MV portfolios are sensitive to small changes in input data.
  - Chopra et al. (1993): slight changes to estimates of expected returns or risk produce vastly different MV optimized portfolios.
  - Best and Grauer (1991) analyze sensitivity of optimal portfolios to changes in expected return estimates.
  - Instability of the optimal portfolio weights (Jobson and Korkie (1981), Britten-Jones (1999)).
- Jobson and Korkie (1981): equal-weighted portfolio have greater Sharpe ratio than optimal MV portfolio computed using estimated inputs.
- Broadie (1993): estimated efficient frontier overestimates expected returns of portfolios for varying levels of estimation errors.

Markowitz, naively applied:

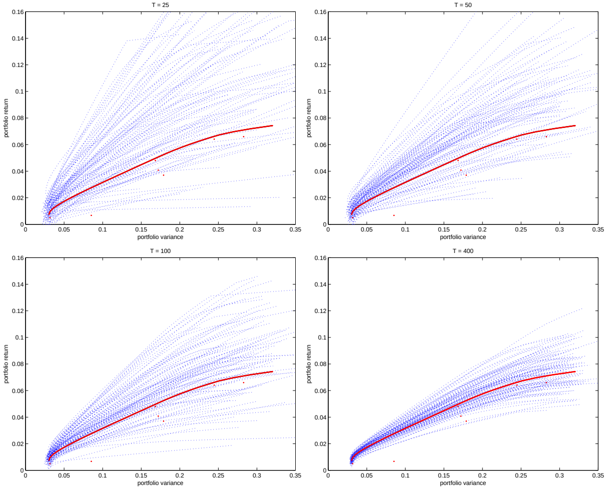




- In theory, portfolio weights should not move.
- Efficient frontiers move a lot.
- There is a high rebalancing activity.
- Theory is clearly inconsistent with reality.
- Example International diversification with USA, JAP, KOR, HKN, SIG, KLP, JAK, TAI, MEX, CHI, BRA.

Estimation Error in MV

red = estiamted original EF
blue = simulated EF using estimated parameters



the longer the time series for estimation of expected return and or risk the less estimation risk hence the less variability of the newly estimated efficient frontiers

The Cause:

- Because of the ill-effects of estimation errors on optimal portfolios, portfolio optimization has been called 'error maximization' (See Michaud (1989)).
- Most of the estimation risk is due to errors in estimates of expected returns, and not in the estimates of risk (e.g., Chopra and Ziemba (2011)).
- Many portfolio managers concur, saying that their confidence in risk estimates is much greater than their confidence in expected return estimates.
- To cope with the effect of estimation errors in the estimates of expected returns, attempts have been made to create better and more stable mean-variance optimal portfolios by using expected return estimators that have a better behavior when used in the context of the MV framework.

The Role of Inverse of Σ

- We already know the solution of an unconstrained quadratic optimization problem, $\boldsymbol{w} = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu}$, where λ is a risk aversion parameter.
- From Stevens (1998), we know that we can write

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_{11}(1-R_1^2)} & -\frac{\beta_{12}}{\sigma_{11}(1-R_1^2)} & \cdots & -\frac{\beta_{1n}}{\sigma_{11}(1-R_1^2)} \\ -\frac{\beta_{21}}{\sigma_{22}(1-R_2^2)} & \frac{1}{\sigma_{22}(1-R_2^2)} & \cdots & -\frac{\beta_{2n}}{\sigma_{22}(1-R_2^2)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\beta_{n1}}{\sigma_{nn}(1-R_n^2)} & -\frac{\beta_{n2}}{\sigma_{nn}(1-R_n^2)} & \cdots & \frac{1}{\sigma_{nn}(1-R_n^2)} \end{bmatrix},$$

where the β_{ij} arise from the regression

$$r_i = \alpha_i + \sum_{j \neq i} \beta_{ij} r_j + \epsilon_i,$$

with r-squared values given by R_i^2.

- Hence, we can write the optimal portfolio value as

$$w_i = \frac{1}{\lambda} \frac{\mu_i - \sum_{j \neq i} \beta_{ij} \mu_j}{\sigma_{ii}(1 - R_i^2)} = \frac{1}{\lambda} \frac{\alpha_i}{\sigma_{\epsilon_i}^2}.$$

- The numerator corresponds to the excess return after regression hedging, i.e., the excess return after the reward for implicit exposure to other assets has been taken out.
- The denominator corresponds to the non-hedgeable risk, or the variance of the error term in the regression equation.
- Since the regression tries to minimize the variance of the error term, the mean-variance rule tries to put maximum weight into those assets that are similar to all other assets but have very small return advantage.

Search for Portfolio Weight Stability:

1. Black and Litterman (1992) model - Bayesian shrinkage of the expected returns.
2. Ledoit and Wolf (2004) shrunk the covariance matrix.
3. Michaud (1989) suggested resampling the frontier - in effect a shrinkage estimator too.
4. Jagannathan and Ma (2003) – "Why imposing the wrong constraints helps" – proves that the inclusion of weight constraints and shrinking the sample risk matrix are equivalent.

Resampling Methods in the Real World

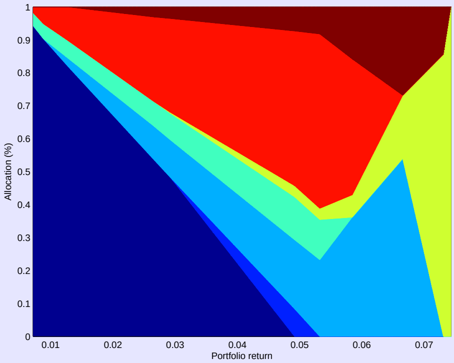
- $(\boldsymbol{\mu}, \Sigma)$ used in MV analysis estimated from single time series sample do not equal true population moments.
- Even if data is iid, there is sampling error in estimates.
- Optimization exacerbates effect of sampling error as it takes advantage of unusually high means and low variances.
- Jorion (1992) addresses "Portfolio Optimization in Practice" and proposes the first resampling method.
- Michaud (1989) has built a business around resampling. Implemented in Northfield optimizers.
- Ibbotson Associates also uses a resampling technique.
- The procedure described below has U.S. Patent #6,003,018 by Michaud et al., December 19, 1999.

Resampling Methods – How it works

- Given estimates $\boldsymbol{\mu}_0^*$ and Σ_0^* based on T observations of excess returns.
- Draw repeatedly (m times) from (joint normal) distribution given by point estimates.
- Generate: $(\boldsymbol{\mu}_1^*, \Sigma_1^*),\ (\boldsymbol{\mu}_2^*, \Sigma_2^*), \ldots, (\boldsymbol{\mu}_m^*, \Sigma_m^*)$.
- For each $i = 1, \ldots, m$, calculate portfolio weights $\boldsymbol{w}_{ij}$ based on $(\boldsymbol{\mu}_i^*, \Sigma_i^*)$ for set of target μ_j^p, $j = 1, \ldots, P$.
- Evaluate $\sigma_{ij}^p = \sqrt{\boldsymbol{w}_{ij}^\top \Sigma_0^* \boldsymbol{w}_{ij}}$ and $\mu_{ij}^p = \boldsymbol{w}_{ij}^\top \boldsymbol{\mu}_0^*$.
- Resulting portfolio mean-standard-deviation pairs lie below the efficient frontier.
- Statistically equal weights $\boldsymbol{w}_{ij}$ and $\boldsymbol{w}_{0j}$.
- Scatter plots show considerable variation unless length of time series T is very long (under iid returns).

Example with 8 asset:

Given the estimated returns and the covariance matrix, the simple MV allocation may look as follows:



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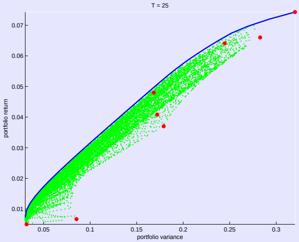
Resampled efficient frontier:

We can now use our resampling method to construct a whole set of new efficient frontiers:



Comments on Resampling

- In the standard MV allocation, only a few assets play a large role in the portfolio.
- Also, weights jump as risk tolerance increases.
- With resampling, we construct a whole set of efficient frontiers, which are close to the original efficient frontier.
- Important aspect: How much randomness is coming from means and how much from covariances?
- One may repeat resampling calculating portfolio weights holding the means or the covariances fixed at the sample moments ($\boldsymbol{\mu}_0^*$ or Σ_0^*, respectively).

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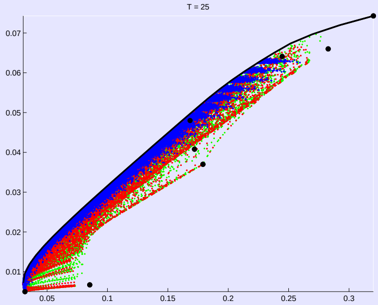
Estimation errors in mean/covariance only:

We can resample by assuming that we know the true mean (blue frontiers) or assuming that we know the true covariance matrix (red frontiers).

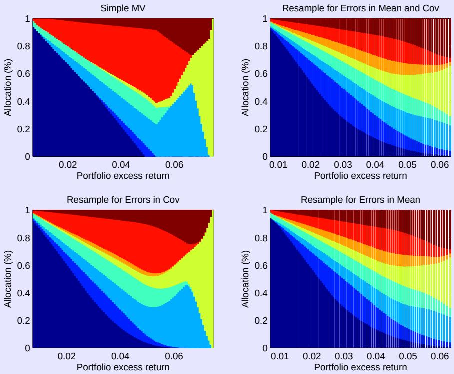


Determining resampled weights:

- Basic question: how should one choose a portfolio as resampling generates scatter plot of portfolios but what is optimal?
- Ad hoc but widely used approach consists of averaging resampled weights for given portfolio rank.
- We take for the lth portfolio on the efficient frontier just the mean from the resampling exercise:

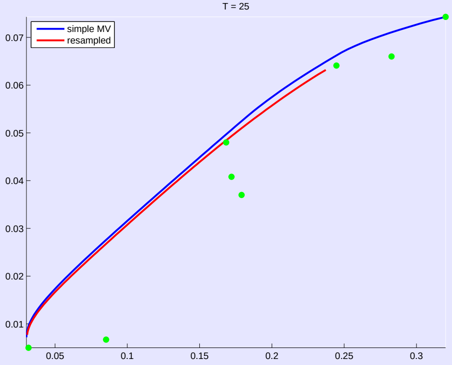
$$\widetilde{\boldsymbol{w}_l} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{w}_{il}.$$

Resampled portfolio weights:



Resampled efficient frontier:

Resampled portfolio weights change in a smooth way as risk tolerance changes. Large differences may arise to the MV weights. However, the efficient frontiers are very close.



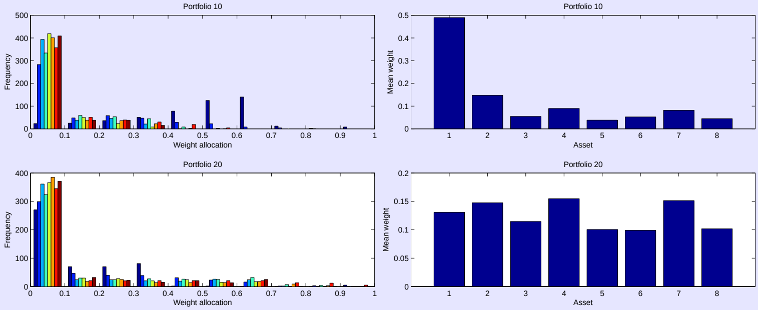
Observation:

- Resampling is most effective when correcting for errors in means.
- Portfolios based on resampling are close in μ-σ space to standard efficient frontier portfolios, but they are far apart in "weight-space".

Some obvious concerns:

- Portfolio weight averages may reflect a few extreme observations.
- Also there may be strange comparative statics.
- If a given asset becomes more volatile, it will sometimes have large positive returns and sometimes large negative returns, so the portfolio unconstrained weights will move about a lot.
- If there are short-selling constraints, the average weight on the asset may go up!

Distribution of resampled weights:



Using Resampling to Compare Portfolios

- Resampling provides a distribution of portfolio weights, allowing us to test whether two portfolios are statistically different.
- Difference or distance may be measured in terms of variation in the portfolio weights $\boldsymbol{w} \in \mathbb{R}^{n\times 1}$. Resampled weight vectors allow to calculate a covariance matrix Ω.
- Given a particular portfolio (e.g. current allocation), $\boldsymbol{w}_p$, calculate the **Mahalanobis distance**:

$$(\boldsymbol{w}_p - \boldsymbol{w}_i)^\top \Omega^{-1} (\boldsymbol{w}_p - \boldsymbol{w}_i)$$

  which is χ_2 with k-degrees of freedom.
- Intuition of test statistic: small weight differences for highly correlated assets might be of greater significance than large weight differences for assets with low correlation.

Constant Density Ellipsoids

- In low dimensional case, one can calculate density ellipsoids to assess the distance between portfolios.
- Assume two assets with

$$(w_1, w_2)^\top = \frac{1}{\lambda}\Sigma^{-1}\boldsymbol{\mu},$$

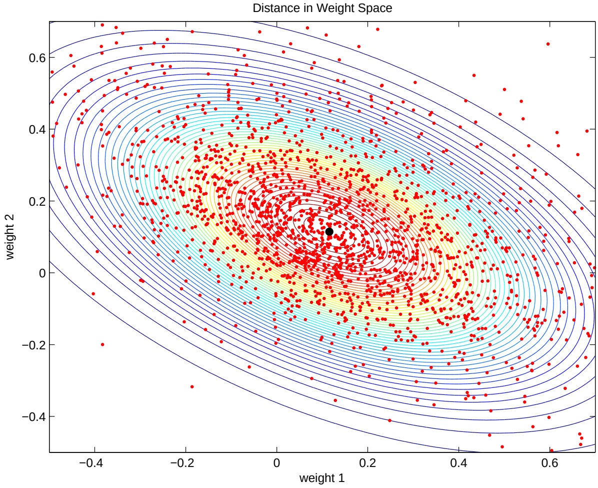
where λ is the risk aversion.

- These are based on the joint asymptotic distribution of the weights:

$$p(w_1 - w_1^*, w_2 - w_2^*) = (2\pi)^{-1}|\Omega|^{-\frac{1}{2}} e^{\left(-\frac{1}{2}(\boldsymbol{w}-\boldsymbol{w}^*)^\top \Omega^{-1}(\boldsymbol{w}-\boldsymbol{w}^*)\right)},$$

where Ω is the covariance matrix of the weight distribution.

Distance of Portfolio Weights



Distance to return distribution

- Problem: Introducing long-only constraints invalidates the normality assumption!
- Distance between two portfolios may also be measured by looking at the distance in return distribution.
- A statistic for this is:

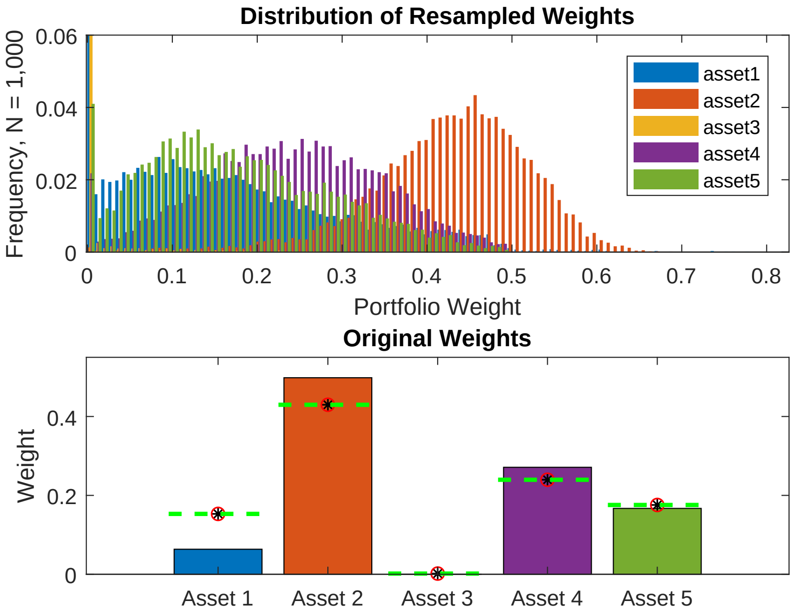
$$(\boldsymbol{w}_p - \boldsymbol{w}_i)^\top \Sigma_0^* (\boldsymbol{w}_p - \boldsymbol{w}_i) \geq TE_\alpha^2 \tag{1}$$

- To test use resampling to generate the distribution of the test statistic.
- This is equivalent to the squared tracking error (volatility of return differences between portfolios $\boldsymbol{w}_i$ and $\boldsymbol{w}_p$).

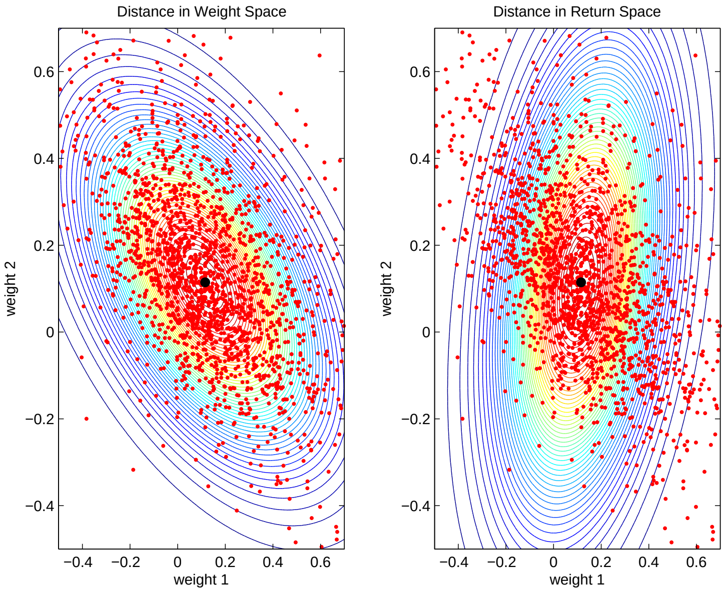
Distance to return distribution - Procedure:

1. Define reference portfolio, $\boldsymbol{w_p}$.
2. Calculate test statistic in (1) for all resampled portfolio weights.
3. See if the test statistic based on an optimal set of weights assuming no sampling error is greater than the α-quantile of the test statistic distribution obtained in 2.
4. This tells us if the optimized weights are statistically significantly different from the current portfolio.

Distance of Portfolio Weights



Distance of Portfolio Weights



Outline

- Regression approach of Britten-Jones (1999) provides exact finite-sample procedure for testing hypotheses about the weights of mean-variance efficient portfolios.
- Optimal portfolio weights can be calculated using simple regression software.
- Regression approach provides answers to:
  - Are the weights in my portfolio the result of one or two abnormal events?
  - How well are my portfolio weights estimated? Are some positions critical to the portfolio performance?
  - Are my holdings sub-optimal? Is the evidence strong enough to warrant a trade?
  - How different are two portfolios? Is one more efficient than the other? Or are they both efficient but just offer different risk-return trade-offs?
  - What are the costs in terms of portfolio efficiency of my investment constraints?

Regression Approach - Setup

- Let $\boldsymbol{R}_t$ be the vector of n risky excess returns.
- We make $\mathcal{R} \in \mathbb{R}^{T \times n}$ observations.
- $\boldsymbol{R}_t^\top \boldsymbol{w}$ is the excess return on the portfolio with weights $\boldsymbol{w}$.
- Assume that there exists a risk-less interest rate.
- Weights do not need to sum up to one.
- Perform an OLS regression of $\mathbf{1}$ on excess returns $\mathcal{R}$. Why?
  - We are projecting the unit vector on to the subspace generated by the columns of the return matrix $\mathcal{R}$.
  - Since the unit vector has no variance among its elements, we are finding a linear combination of asset returns that are constant over time, except for the residual error.
  - Also, by forcing the regression through the origin, we are maximizing the Sharpe ratio.

Regression Approach - Main Result

- OLS regression of $\mathbf{1}$ on excess returns $\boldsymbol{R}$ without an intercept, $\mathbf{1} = \mathcal{R}\boldsymbol{w} + \boldsymbol{u}$, results in an estimated coefficient vector

$$\widehat{\boldsymbol{w}} = \left(\mathcal{R}^\top \mathcal{R}\right)^{-1} \mathcal{R}^\top \mathbf{1},$$

  that is a set of risky-asset only portfolio weights for a sample efficient portfolio.

- The portfolio $\widehat{\boldsymbol{w}}$ can be interpreted as coming closest to a portfolio with zero risk (the vector of ones has no volatility) and unit return, which would present an arbitrage opportunity.

- The normalized coefficients $\widehat{\boldsymbol{w}}/(\mathbf{1}^\top \widehat{\boldsymbol{w}})$ is thus the familiar MV tangency portfolio

$$\boldsymbol{w}_T = \frac{\widehat{\Sigma}^{-1}\widehat{\boldsymbol{\mu}}}{\mathbf{1}^\top \widehat{\Sigma}^{-1}\widehat{\boldsymbol{\mu}}},$$

  where the sample mean and covariance matrix

$$\widehat{\boldsymbol{\mu}} = \mathcal{R}^\top \mathbf{1}/T, \quad \widehat{\Sigma} = \left(\mathcal{R} - \mathbf{1}\widehat{\boldsymbol{\mu}}^\top\right)^\top \left(\mathcal{R} - \mathbf{1}\widehat{\boldsymbol{\mu}}^\top\right)/T,$$

  are used as parameters.

- The derivation of the above result is as follows. We start from

$$\widehat{\boldsymbol{w}} = \left(\mathcal{R}^\top \mathcal{R}\frac{1}{T}\right)^{-1} \frac{1}{T}\mathcal{R}^\top \mathbf{1} = \left(\mathcal{R}^\top \mathcal{R}\frac{1}{T}\right)^{-1} \widehat{\boldsymbol{\mu}}.$$

- Then, since we can write the covariance matrix as

$$
\begin{aligned}
\widehat{\Sigma} &= \left(\mathcal{R} - \mathbf{1}\widehat{\boldsymbol{\mu}}^\top\right)^\top \left(\mathcal{R} - \mathbf{1}\widehat{\boldsymbol{\mu}}^\top\right) / T \\
&= \frac{\mathcal{R}^\top\mathcal{R} - 2T\widehat{\boldsymbol{\mu}}\widehat{\boldsymbol{\mu}}^\top + \widehat{\boldsymbol{\mu}}\mathbf{1}^\top\mathbf{1}\widehat{\boldsymbol{\mu}}^\top}{T} \\
&= \frac{\mathcal{R}^\top\mathcal{R}}{T} - \widehat{\boldsymbol{\mu}}\widehat{\boldsymbol{\mu}}^\top,
\end{aligned}
$$

we have $\frac{\mathcal{R}^\top\mathcal{R}}{T} = \widehat{\Sigma} + \widehat{\boldsymbol{\mu}}\widehat{\boldsymbol{\mu}}^\top$.

- Therefore,

$$
\begin{aligned}
\widehat{\boldsymbol{w}} = \left(\mathcal{R}^\top \mathcal{R}\frac{1}{T}\right)^{-1} \widehat{\boldsymbol{\mu}} &= \left(\widehat{\Sigma} + \widehat{\boldsymbol{\mu}}\widehat{\boldsymbol{\mu}}^\top\right)^{-1} \widehat{\boldsymbol{\mu}} \\
&= \left(\widehat{\Sigma}^{-1} - \frac{\widehat{\Sigma}^{-1}\widehat{\boldsymbol{\mu}}\widehat{\boldsymbol{\mu}}^\top\widehat{\Sigma}^{-1}}{1+\widehat{\boldsymbol{\mu}}^\top\widehat{\Sigma}^{-1}\widehat{\boldsymbol{\mu}}}\right)\widehat{\boldsymbol{\mu}} \\
&= \frac{\widehat{\Sigma}^{-1}\widehat{\boldsymbol{\mu}}}{1+\widehat{\boldsymbol{\mu}}\widehat{\Sigma}^{-1}\widehat{\boldsymbol{\mu}}},
\end{aligned}
$$

where in the second line, we made use of Sherman-Morrison formula.

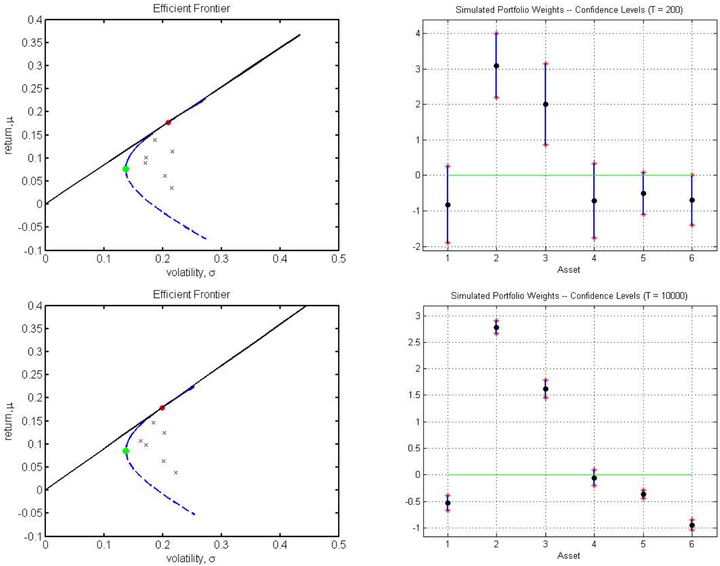
T = 3
n = number of monthly observations in the specific period

Table I

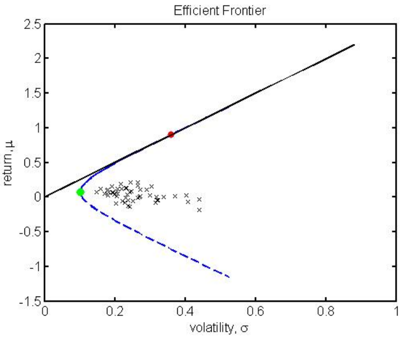
Estimates of a Global Tangency Portfolio

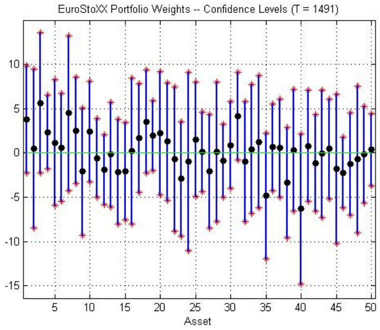
This table contains estimates of the weights of a global tangency portfolio, from the viewpoint of a U.S. investor. The monthly country returns are the Morgan Stanley Capital International (MSCI) country stock index returns converted into U.S. dollar returns using the MSCI exchange rate series. Excess returns are calculated by subtracting the one-month T-bill return obtained from Ibbotson Associates. Weights are in percentage form (i.e., they sum to 100). The *t*-statistics test the hypothesis that the country weight is zero. Standard errors (SE) are calculated under the null hypothesis of a zero weight, as the estimated weight divided by the *t*-statistic.

	1977–1996		1977–1986		1987–1996	
	Weight (*t*-statistic)	SE	Weight (*t*-statistic)	SE	Weight (*t*-statistic)	SE
Australia	12.8 (0.54)	23.6	6.8 (0.20)	33.6	21.6 (0.66)	32.6
Austria	3.0 (0.12)	25.5	−9.7 (−0.22)	44.1	22.5 (0.74)	30.3
Belgium	29.0 (0.83)	35.1	7.1 (0.15)	46.8	66.0 (1.21)	54.5
Canada	−45.2 (−1.16)	38.9	−32.7 (−0.64)	51.0	−68.9 (−1.10)	62.7
Denmark	14.2 (0.47)	30.2	−29.6 (−0.65)	45.3	68.8 (1.78)	38.7
France	1.2 (0.04)	28.7	−0.7 (−0.02)	37.3	−22.8 (−0.48)	47.3
Germany	−18.2 (−0.51)	35.4	9.4 (0.19)	49.4	−58.6 (−1.13)	52.1
Italy	5.9 (0.29)	20.2	22.2 (0.79)	27.9	−15.3 (−0.52)	29.5
Japan	5.6 (0.24)	23.4	57.7 (1.43)	40.4	−24.5 (−0.87)	28.1
U.K.	32.5 (1.01)	32.1	42.5 (0.99)	42.9	3.5 (0.07)	49.8
U.S.	59.3 (1.26)	47.0	27.0 (0.41)	65.2	107.9 (1.53)	70.6


Efficient Frontier
return, μ
volatility, σ
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
0.1
0.2
0.3
0.4
0.5
Simulated Portfolio Weights -- Confidence Levels (T = 200)
Asset
4
3
2
1
0
-1
-2
1
2
3
4
5
6
Efficient Frontier
return, μ
volatility, σ
Simulated Portfolio Weights -- Confidence Levels (T = 10000)
Asset
3
2.5
2
1.5
1
0.5
0
-0.5
-1

Application Eurostoxx





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