Asset Pricing: Choice under Uncertainty

Basic Definitions

Markus Leippold

Spring 2019

University of Zurich
Department of Banking and Finance

Table of contents

Expected Utility and Risk Aversion

Can we measure utility?

- **Ordinal utility**: invariant to monotonically increasing transformations.
- **Cardinal utility**: invariant to positive affine (increasing linear), but not to nonlinear transformations.

Asset pricing uses mostly **von Neumann-Morgenstern utility theory**:

- States $s = 1, \ldots, S$ associated with outcome $x_s \in X$ having probability p_s (lotteries L).
  - Continuity axiom:
    Given $L_a \succeq L_b \succeq L_c$, then $\exists \alpha \in [0,1]$ such that $L^b \sim \alpha L^a + (1-\alpha) L^c$, i.e., we get an **ordinal utility function**.
  - Independence axiom: $L^a \succeq L^b \implies \alpha L^a + (1-\alpha) L^c \succeq \alpha L^b + (1-\alpha) L^c$ for all possible L^c, which implies that the preference functional is **linear in probabilities**.
- Then, we can define scalars u_s (utilities), such that $L^a \succeq L^b \implies \sum_{s=1}^{S} p_s^a u_s \geq \sum_{s=1}^{S} p_s^b u_s$, i.e., we have implicitly defined a **cardinal utility function**.

What does it mean to be risk averse?

- Assume a cardinal utility and that the argument of the utility function is wealth, i.e., we have a single consumption good.
- Hence, we have a static two-period model where all wealth is liquidated and consumed in the second period, after uncertainty is resolved.

Definition (Risk Aversion)

An agent is risk averse if she (weakly) dislikes all zero-mean risk at all levels of wealth. That is, for all initial wealth levels W_0 and risk X with $\mathbb{E}[X] = 0$, $\mathbb{E}[u(W_0 + X)] \leq u(W_0)$.

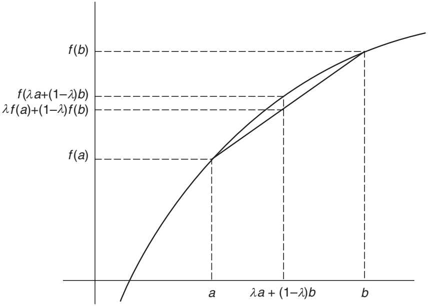
The **coefficient of absolute risk aversion** is defined by

$$A(W_0) = \frac{-u''(W_0)}{u'(W_0)}.$$

Risk Aversion (cont.)

Risk aversion is equivalent to the **concavity of the utility function**:

- Set $Z = W_0 + X$ and show $\mathbb{E}[u(Z)] \leq u(\mathbb{E}[Z])$ via Jensen's inequality.



Jensen Inequality

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, X an integrable real-valued random variable and φ a concave function. Then $\varphi(\mathbb{E}[X]) \geq \mathbb{E}[\varphi(X)]$.

Comparing risk aversions

If agent 1 is more risk-averse than agent 2, then utility function 1 is a concave transformation of the utility function 2 and has a higher coefficient of absolute risk aversion at all levels of initial wealth.

Proof. Define $\phi(X) = u_1(u_2^{-1}(X))$, so $u_1(z) = \phi(u_2(z))$, $u_1'(z) = \phi'(u_2(z))u_2'(z)$ and $\phi' > 0$, and from $u_1''(z)$ we get $\phi'' = \frac{u_1'}{u_2'^2}(A_2 - A_1)$. Hence, higher absolute risk aversion of agent 1 is equivalent to concavity of ϕ. Consider a risk X for which $\mathbb{E}[u_2(W_0 + X)] \leq u_2(W_0)$. Then, if agent 1 is more risk averse, we must also have $\mathbb{E}[u_1(W_0 + X)] \leq u_1(W_0)$. Since $\phi' > 0$, $u_1(W_0) = \phi(u_2(W_0)) \geq \phi(\mathbb{E}[u_2(W_0 + X)])$. Then, if 1 is more risk averse than 2, we need to have $\mathbb{E}[\phi(u_2(W_0 + X))] \leq \phi(\mathbb{E}[u_2(W_0 + X)])$. By Jensen's inequality, this last expression is equivalent to the concavity of ϕ. ∎

To directly compare risk aversions, we can study the amount an agent is prepared to pay to avoid a zero-mean risk.

Definition (Risk Premium)

The risk premium $\pi(W_0, u, X)$ is the greatest amount an agent equipped with W_0 and utility function u is willing to pay to avoid a zero-mean risk X, i.e., π solves

$$\mathbb{E}(u(W_0 + X)) = u(W_0 - \pi).$$

Definition (Certainty Equivalent)

For risks with nonzero mean μ, the certainty equivalent is given by

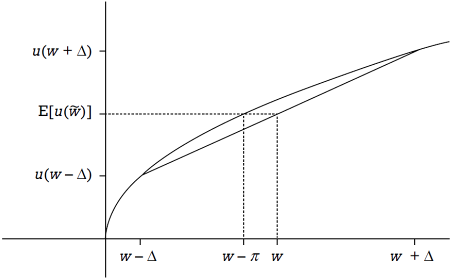
$$\mathbb{E}(u(W_0 + \mu + X)) = u(W_0 + CE).$$

Clearly,

$$CE(W_0, \mu + X) = \mu - \pi(W_0 + \mu, X).$$

Universität Zürich[UZH]

Graphical Illustration:



- Assume the random variable $\tilde{w}$ takes value $w \pm \Delta$ with equal probability.
- Expected utility is the height of the line segment connecting the points $(w \pm \Delta, u(w \pm \Delta))$ at its midpoint w.
- The CE of $\tilde{w}$ is $w - \pi$ shown on the horizontal axis.
- The amount a u-person would pay to obtain the mean instead of the gamble is π, the distance between the two vertical dotted lines.

Arrow-Pratt Approximation

Consider a zero mean risk $Y = kX$ with some scalar k. Write the risk premium as $g(k) = \pi(W_0, kX)$. Then

$$\mathbb{E}[u(W_0 + kX)] = u(W_0 - g(k)).$$

Take the first derivative with respect to k:

$$\mathbb{E}[Xu'(W_0 + kX)] = -g'(k)u'(W_0 - g(k)).$$

At $k = 0$, $\mathbb{E}[Xu'(W_0 + kX)] = \mathbb{E}[Xu'(W_0)] = \mathbb{E}[X]u'(W_0)$. Hence, $g'(0) = 0$. Taking the second derivative

$$\mathbb{E}[X^2 u''(W_0 + kX)] = g'(k)^2 u''(W_0 - g(k)) - g''(k)u'(W_0 - g(k)),$$

implies

$$g''(0) = -\frac{u''(W_0)}{u'(W_0)}\mathbb{E}[X^2] = A(W_0)\mathbb{E}[X^2].$$

Therefore,

$$\pi = g(k) \approx \frac{1}{2} A(W_0) k^2 \mathbb{E}[X^2] = \frac{1}{2} A(W_0) \mathbb{E}[Y^2].$$

Remarks:

- The above result states that the**risk premium is proportional to absolute risk aversion**.
- We can also say that the risk premium is proportional to the variance of the gamble, which therefore is called **second-order risk aversion**.
- There are different theories of preferences that imply **first-order risk aversion**, with a risk premium proportional to the standard deviation of the gamble.

✎ First-order risk aversion entails much larger risk premia for small gambles.

Example

Let w be your wealth and consider flipping a coin where you win 10% of Ww if the coin comes up heads and lose 10% of W if it comes up tails. This is a large gamble, so the above approximation may not be very good. The standard deviation of the random variable Y defined as $Y = \pm 0.1$ with equal probabilities is 0.1. Hence, our approximation dictates that an investor would pay approximately

$$\frac{1}{2} A(W) \times 1\%$$

of her wealth to avoid the gamble. If she would pay exactly 2% of her wealth to avoid this 10% gamble, then we would say that $A(W) = 4$.

Universität Zürich[UZH]

Multiplicative Risk:

- In principle, we could define a multiplicative risk as $W = W_0(1+kX) = W_0(1+Y)$.
- Defining $\hat{\pi}$ the share of wealth one pays to avoid this risk, i.e., $\hat{\pi} = \pi(W_0, W_0 Y)/W_0$, we get

$$\hat{\pi} \approx \frac{1}{2} W_0 A(W_0) k^2 \mathbb{E}[X^2] = \frac{1}{2} R(W_0) \mathbb{E}[Y^2],$$

  where $R(W_0)$ is the coefficient of relative risk aversion.

General Remark: Most of the time, we use 'risk premium' in a different way, meaning the extra expected return an investor earns from holding a risky asset.

Linear Risk Tolerance (LRT)

HARA (or LRT) utility functions build a highly tractable class, which is defined through linear risk tolerance:

$$T(W) = \frac{1}{A(W)} = \frac{W}{1-\gamma} + \frac{b}{a}.$$

A utility function $u(W)$ has this property, and thus is a HARA utility function, if and only if it has the form

$$u(W) = \frac{1-\gamma}{\gamma}\left(\frac{aW}{1-\gamma} + b\right)^{\gamma},$$

with restrictions on wealth and the parameters such that $a > 0, b + \frac{aW}{1-\gamma} > 0$.

Universität Zürich[UZH]

Special Cases:

- When $\gamma \to 1$, we have a **linear utility function**.
- When $\gamma \to 0$, we get a **logarithmic utility function** $u(W) = \ln(aW + b)$.
- **Utility is quadratic** (an implausible though mathematically tractable case, with increasing absolute risk aversion) if $\gamma = 2$.
- **Exponential utility function**, which has constant absolute risk aversion, occurs if $b = 1$ and $\gamma \to -\infty$.
- **Power utility function**, if $\gamma < 1$ and $a = 1 - \gamma$ and isoelastic utility function, with constant relative risk aversion, occurs if, further, $b = 0$.

Linear Risk Tolerance (LRT) (cont.)

Most often, utility functions are categorized according to whether their risk aversion is decreasing, constant, increasing:

- **Absolute risk aversion** is decreasing (equivalently $T(W) > 0$), if and only if γ is finite and less than 1; this is considered the empirically plausible case, since it implies that an investor will put more funds into risky assets the more funds are available to invest. **Constant absolute risk aversion** occurs as γ goes to positive or negative infinity, and the particularly implausible case of increasing absolute risk aversion occurs if γ is greater than one and finite.
- **Relative risk aversion** $R(W) = WA(W)$ is increasing, if $b > 0$ (for $\gamma \neq 1$), constant if $b = 0$, and decreasing if $b < 0$ (for $-\infty < \gamma < 1$).

Important Subclasses:

1. **Constant Absolute Risk Aversion (CARA):**
   - Every CARA utility function is a monotone affine transformation of the negative exponential utility function

     $$u(W) = -\exp(-\gamma W),$$

     where γ is the (constant) **absolut risk aversion**.
   - For the risk premium, we note that $u(W_0 - \pi) = -e^{-\gamma W_0} e^{\gamma \pi}$. Hence, $u(W_0 - \pi) = \mathbb{E}[u(W_0 + X)]$ implies

     $$\pi = \frac{1}{\gamma} \log \mathbb{E}[e^{-\gamma X}].$$

   ☞ CARA investor will pay the same to avoid a fair gamble, irrespective of her initial wealth. Unreasonable.

   ✎ If $X \sim N(0,\sigma^2)$, we get $\pi = \frac{1}{2}\gamma\sigma^2$, i.e., the Arrow-Pratt approximation becomes exact.

2. **Constant Relative Risk Aversion (CRRA):**
   - Any CRRA utility function is a DARA utility function.
   - Any monotone CRRA utility function is a montone affine transformation of

$$u(W) = \begin{cases} \frac{1}{1-\gamma} W^{1-\gamma}, & \gamma \neq 1, \\ \log(W), & \gamma = 1. \end{cases}$$

✎ Here, γ is now the coefficient of **relative risk aversion** for the CRRA utility function.

☞ The fraction of wealth an individual would be willing to pay to avoid a gamble that is proportional to initial wealth is independent of wealth, i.e., we can show from

$$u((1-\pi)W_0) = \mathbb{E}[u((1+X)W_0)]$$

that π will be independent of W.

Assume that W is the mean of a gamble $\tilde{W}$ and that the utility function has a convergent Taylor series around W. Taking expectation of that Taylor series:

$$\mathbb{E}[u(\tilde{W})] = u(W) + \frac{1}{2}u''(W)\mathbb{E}[(\tilde{W} - W)^2] + \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(W)\mathbb{E}[(\tilde{W} - W)^n].$$

- Since $u''(W) < 0$, a risk-averse investor dislikes variance.
- Since for a DARA utility function $u'''(W) > 0$, a DARA investor prefers positive to negative skewness.
- Combination of DARA and decreasing absolute prudence ($-u'''/u''$, see Kimball (1990)) implies $u^{(4)} < 0$, hence an investor dislikes kurtosis.

Some Critique of Expected Utility Theory

Consider a set of lotteries, each of which involves drawing one ball from an urn containing 100 balls, labeled 0–99. The monetary prizes, for four different lotteries L^a,L^b, M^a, and M^b, that will be awarded for drawing each ball are (Allais, 1953):

	0	1–10	11–99
L^a	50	50	50
L^b	0	250	50
M^a	50	50	0
M^b	0	250	0

- Many people, confronted with this choice, prefer L^a to L^b.
- But $L^a \succeq L^b$ while $M^b \succeq M^a$ **violates the independence axiom**!
- Debate: abandone independence axiom or/and educate people?

Utility theory cannot **explain observed aversion to small gambles** without implying **ridiculous aversion to large gambles**. Follows from the fact that differentiable utility has **second-order risk aversion**.

Example:

- Suppose CARA utility with risk aversion α,

$$u(X) = -\exp(-\alpha X).$$

Suppose the agent rejects a gamble that gives 100 with probability 0.6 and -100 with probability 0.4, i.e.,

$$-p\exp(-100\alpha) - (1-p)\exp(100\alpha) < -1,$$

implying $\alpha > 1/100\ln(3/2)$.

- If indeed $\alpha > 1/100 \ln(3/2)$, then expected utility from gamble that gives -200 with probability 0.5 and any gain with probability 0.5 is at most

$$\frac{1}{2}\exp(-\infty) - \frac{1}{2}\exp(200\alpha) < -\frac{1}{2}\exp\left(2\ln(3/2)\right) = -9/8 < -1.$$

- The decision maker must reject this gamble which has unlimited upside! Absurd!
- Is the problem CARA utility? No: for any concave function, if decision-maker rejects small gambles over a fairly small range of wealth levels, imposes enough curvature on utility function to force absurd risk-aversion over large gambles.

Rabin Critique (cont.)

In the table below Rabin (2000) visualizes his critique:

"If averse to 50-50 lose \$100/gain g bets for all wealth levels, will turn down 50-50 lose L/gain G bets; G's entered in table."

L/g	\$101	\$105	\$110	\$125
\$400	400	420	550	1,250
\$1,000	1,010	1,570	∞	∞
\$4,000	5,750	∞	∞	∞
\$10,000	∞	∞	∞	∞

Note that an entry of ∞ implies that the agent will turn down the bet for any finite upside, no matter how large.

- “First-order” risk aversion by contrast with the “second-order” risk aversion implied by twice differentiable utility (Segal and Spivak, 1990)
- Kahneman and Tversky (1979) prospect theory: a preference function that is concave in the domain of gains and convex (risk-seeking) in the domain of losses, and subjective probabilities that are larger than objective probabilities when those probabilities are small.

$$u(x) = x^{\beta} \text{ for } x \geq 0,$$
$$u(x) = -\lambda |x|^{\beta} \text{ for } x \leq 0,$$

where $x = W_0 - W_{REF}$, the difference between wealth and the reference level of wealth; $0 < \beta < 1$, to get concavity for gains and convexity for losses; and $\lambda > 1$ to deliver a kink at the reference point.

- Instead of going into more details, I refer to a nice blog entry by John Cochrane.

Conservatism in updating beliefs

- Humans find logic hard, mathematics harder, and probability even more challenging.
- Example: Monty Hall problem.
- In updating probability distributions using evidence, a standard method uses conditional probability, namely the rule of Bayes. An experiment on belief revision has suggested that humans change their beliefs faster when using Bayesian methods than when using informal judgment.

Irrational deviations

- Behavioral finance has produced several generalized expected utility theories to account for instances where people's choices deviate from those predicted by expected utility theory.

- Particular theories include prospect theory, rank-dependent expected utility and cumulative prospect theory, etc..

Uncertain probabilities

- In practice there will be many situations where the probabilities are unknown, and one is operating under uncertainty.
- In economics, Knightian uncertainty or ambiguity may occur.
- This is particularly a problem when the expectation is dominated by rare extreme events, as in a long-tailed distribution.

Comparing Risks

Assuming concavity of the utility function, Rothschild and Stiglitz (1970) shows the equivalence of the following statements for risks with the same mean:

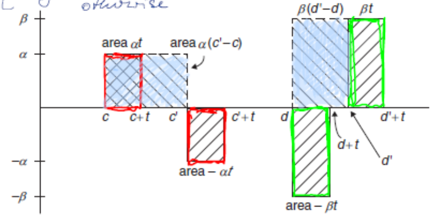
1. *All increasing and concave utility functions dislike the riskier distribution relative to the safer distribution.*

   Consider X and Y, with $\mathbb{E}[X] = \mathbb{E}[Y]$. Then X is weakly less risky than Y if for an increasing utility function, $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$.

2. *The riskier distribution has more weight in the tails than the safer distribution.*
   X is less risky than Y, if the density function $g(y)$ can be obtained from $f(x)$ by applying a **mean-preserving spread** (MPS) $s(x)$:

Comparing Risks with the Same Mean (cont.)

$$s(x) = \begin{cases} \alpha & \text{for } c < x < c+t \\ -\alpha & \text{for } c' < x < c'+t \\ \beta & \text{for } d < x < d+t \\ -\beta & \text{for } d' < x < d'+t \\ 0 & \text{otherwise} \end{cases} \quad \text{s.t.} \quad \begin{array}{l} \alpha, \beta, t > 0 \\ c+t < c' < c'+t < d < d+t < d' \\ \alpha(c'-c) = \beta(d'-d) \end{array}$$



To calculate the expected utility difference, we note:

$$\mathbb{E}[u(X)] - \mathbb{E}[u(Y)] = -\int u(z)s(z)dz$$

$$= -\alpha \int_c^{c+t} \left[u(z) - u(z+c'-c) - \frac{\beta}{\alpha}(u(z+d-c) - u(z+d'-c)) \right] dz.$$

Noting $\beta/\alpha = (c'-c)/(d'-d)$ and $u(z+h) - u(z) = u'(z*)h$ for some $z* \in [z, z+h]$, we get the desired result that $\mathbb{E}[u(X)] - \mathbb{E}[u(Y)] > 0$ from concavity.

3. *The riskier distribution can be obtained from the safer distribution by adding noise to it.*

   'Added noise' means that X is less risky than Y, if Y has the same distribution as $X + \epsilon$, where $\mathbb{E}[\epsilon|X] = 0$ for all values of X. We say that ϵ is a **fair game** with respect to X.

Remarks:

- The fair game condition is stronger than zero covariance, $\text{Cov}(\epsilon, X) = 0$. It is weaker than independence, $\text{Cov}(f(\epsilon), g(X)) = 0$ for all functions f and g. It is equivalent to $\text{Cov}(\epsilon, g(X)) = 0$ for all functions g.
- None of the three conditions are equivalent to Y having greater variance than X. While it is true that if Y is riskier than X then Y has greater variance. The reverse is not necessarily true, only for a restricted class of utility functions and distributions.

- The above arguments extend straightforwardly to the case where a riskier distribution, in the Rothschild-Stiglitz sense, is shifted downward and therefore has a lower mean.
- Based on the second-order stochastic dominance (SOSD) risk can be compared for distributions with different means.

Definition (First-order stochastic dominance)

X first-order stochastically dominates Y if $Y \sim X + \zeta$, $\zeta \leq 0$.
Equivalently, if $F(\cdot)$ is the cdf of X and $G(\cdot)$ is the cdf of Y, then X first-order stochastically dominates Y if $F(z) \leq G(z)$ for every z.
First-order stochastic dominance implies that every increasing utility function will prefer the distribution X.

Risks with Different Mean (cont.)

Definition (Second-order stochastic dominance)

X second-order stochastically dominates Y if Y has the distribution of $X + \zeta + \epsilon$, where $\zeta \leq 0$ and $\mathbb{E}[\epsilon | X + \zeta] = 0$. SOSD implies that every increasing, concave utility function will prefer the distribution X.

- SOSD offers an uncontroversial comparison of risks, but is only a partial order of gambles.
- To create a complete order, delivering a ranking of any two gambles, we need to specify utility functions and wealth levels.
- A complete order can be used to create a riskiness index, i.e., a mapping of a gamble to a real number that depends only on the attributes of the gamble itself (Aumann and Serrano, 2008; Foster and Hart, 2009).
- While riskiness indices lack the generality of SOSD, they can nonetheless be useful for descriptive and regulatory purposes.

The Rothschild-Stiglitz analysis can be used to prove the optimality of perfect diversification in a simple portfolio choice problem with identical risky assets:

- Consider n lotteries with *iid* payoffs $X_1, \ldots, X_n$.
- Choose for the lotteries weights $w_1, \ldots, w_n$ with $\sum_{i=1}^n w_i = 1$.
- Obvious choice: equally weighted portfolio with weights $w_i = 1/n$ for all i, having payoff is then $Z = 1/n \sum_{i=1}^n X_i$.

To prove optimality, we note that the payoff on any other strategy is

$$\sum_{i=1}^n w_i X_i = Z + \sum_{i=1}^n (w_i - 1/n) X_i = Z + \epsilon,$$

with $\mathbb{E}[\epsilon|Z] = \sum_{i=1}^n (w_i - 1/n) \mathbb{E}[X_i|Z] = k \sum_{i=1}^n (w_i - 1/n) = 0$. Hence, any other strategy has the same payoff as the 1/n portfolio, plus added noise. A concave utility function will therefore prefer 1/n.

Universität Zürich[UZH]

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'école américaine. *Econometrica*, 21:503–546.

Aumann, R. J. and Serrano, R. (2008). An economic index of riskiness. *Journal of Political Economy*, 116(5):810–836.

Campbell, J. Y. (2017). *Financial decisions and markets: a course in asset pricing*. Princeton University Press.

Foster, D. P. and Hart, S. (2009). An operational measure of riskiness. *Journal of Political Economy*, 117(5):785–814.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47:263–291.

Kimball, M. S. (1990). Precautionary saving in the small and in the large. *Econometrica: Journal of the Econometric Society*, pages 53–73.

Rabin, M. (2000). Risk aversion and expected-utility theory: A calibration theorem. *Econometrica*, 68(5):1281–1292.

Rothschild, M. and Stiglitz, J. E. (1970). Increasing risk i: A definition. *Journal of Economic Theory*, 2:225–243.

Segal, U. and Spivak, A. (1990). First order versus second order risk aversion. *Econometrica*, 51:111–125.

Stoll, H. R. (1978). The supply of dealer services in securities markets. *The Journal of Finance*, 33(4):1133–1151.

- Solve Problems 1.1 and 1.2 of Campbell (2017)
- As a version of the bid-ask spread model of Stoll (1978), consider an individual with CARA utility and assume $X,Y \sim N(\mu,\Sigma)$, $\mu = [\mu_X,\mu_Y]^\top$, $\Sigma = [\sigma_X^2,\rho\sigma_X\sigma_Y;\rho\sigma_X\sigma_Y,\sigma_Y^2]$.
  - Compute the maximum amount the individual would pay to obtain Y when starting with X, i.e., compute BID satisfying
$$\mathbb{E}[u(X)] = \mathbb{E}[u(X+Y-BID)].$$
  - Compute the minimum amount the individual would require to accept the payoff $-Y$ when starting with X, i.e., compute ASK satisfying
$$\mathbb{E}[u(X)] = \mathbb{E}[u(X-Y+ASK)].$$
  - Use the law of iterated expectations to show that if $\mathbb{E}[\epsilon|Y] = 0$, then $\text{Cov}(Y,\epsilon) = 0$.