Asset Pricing: Assignment 2

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(a) Merton Proposition

In this section we replicate the derivation to the following optimization problem. Let μ and Σ be the expected return of risky assets and the respective covariance matrix. Then, we optimize

$$\min_w \frac{1}{2} w^t \Sigma w \tag{1}$$

with constraints

$$\begin{aligned} \mu^t w &= \mathbb{1} \\ \mathbb{1}^t w &= 1 \end{aligned}$$

We proceed my applying the method of Lagrange multipliers and we get the following equivalent optimization problem

$$\min_w \frac{1}{2} w^t \Sigma w + \lambda_1(\mu_p - \mu^t w) + \lambda_2(1 - \mathbb{1}^t w) + \tag{2}$$

We take the derivative with respect to W and set equal to zero. And we get the first order condition.

$$0 = \Sigma w - \lambda_1 \mu - \lambda_2 \mathbb{1} \tag{3}$$

Thus, it follows that

$$w = \lambda_1 \Sigma^{-1} \mu + \lambda_2 \Sigma^{-1} \mathbb{1}$$

We multiply by μ the above equation and we get and also by $\mathbb{1}$

$$\mu^t w = \lambda_1 \mu^t \Sigma^{-1} \mu + \lambda_2 \mu^t \Sigma^{-1} \mathbb{1}$$

$$\mathbb{1}^t w = \lambda_1 \mathbb{1}^t \Sigma^{-1} \mu + \lambda_2 \mathbb{1}^t \Sigma^{-1} - \mathbb{1}$$

Now, define the following scalars,

$$\begin{aligned} A &:= \mu^t \Sigma^{-1} \mu \\ B &:= \mu^t \Sigma^{-1} \mathbb{1} \\ C &:= \mathbb{1}^t \Sigma^{-1} \mathbb{1} \end{aligned}$$

Thus, we rewrite the linear problem as follows,

$$\begin{aligned} \mu_p &= \lambda_1 A + \lambda_2 B \\ 1 &= \lambda_1 B + \lambda_2 C \end{aligned}$$

If we assume μ is a non-zero vector, we note that A and C are positive real numbers. This, follows from the fact that Σ is positive definite and therefore Σ^{-1} is also a positive definite matrix. Since Σ^{-1} is positive definite it follows that the second derivative with respect to w is positive, thus, the solution to the equiation is a minima, as desired.

It is also the case that $D > 0$. By positive definiteness of Σ^{-1} we have that

$$\begin{aligned} &(B\mu - A\mathbb{1})\Sigma^{-1}(B\mu - A\mathbb{1}) \\ &= B^2(\mu\Sigma^{-1}\mu) - 2AB(\mathbb{1}\Sigma^{-1}\mu + A^2\mathbb{1}^t\Sigma^{-1}\mathbb{1} \\ &= A^2B - 2A^2B + B^2A \\ &= A(AC - B^2) = AD > 0 \end{aligned}$$

Thus, since $A > 0$ it follows that D must also be positive. Note that we not claim a specific sign of B.
We solve the simple linar problem above and we get and we get that

$$\begin{aligned} \lambda_1 &= \frac{C\mu_p - B}{D} \\ \lambda_2 &= \frac{A - B\mu_p}{D} \end{aligned}$$

(b) Tobin's Separation Theorem

Tobin's Separation Theorem states that *'The relative portfolio fraction is independent of the choice of the targeted portfolio returns μ_p'*.

We know the expression for the vector w. Thus, to get the fractions we just normalize the vector.

$$\frac{w}{\mathbb{1}^t w} = \frac{\lambda_1\Sigma^{-1}(\mu - R_f)\mathbb{1}}{\lambda_1\mathbb{1}^t\Sigma^{-1}(\mu - R_f)\mathbb{1}} = \frac{\Sigma^{-1}(\mu - R_f)\mathbb{1}}{\mathbb{1}^t\Sigma^{-1}(\mu - R_f)\mathbb{1}}$$

Since the expression for the fractions of the portfolio does not contain μ_p, then the theorem follows.

(c) Covariance of Mean Variance portfolio

Let w_p be the weights of a mean-variance portfolio and w_q the weights for a general portfolio. Let R be the returns of the stocks where $R \sim \mathcal{N}(\mu, \Sigma)$ and let $\mu \neq k\mathbb{1}$ for some $k \in \mathbb{R}$ Then it follows that

$$
\begin{aligned}
Cov(w_p^t R, w_q^t R) &= E\left[w_p^t (R-\mu)(R-\mu)^t w_q\right] \\
&= w_p^t \Sigma w_q \\
&= \left(\lambda_1 \Sigma^{-1}\mu + \lambda_2 \Sigma^{-}1\mathbb{1}\right)^t \Sigma w_q \\
&= \lambda_1 \mu^t w_q + \lambda_2 \mathbb{1}^t w_q \\
&= \lambda_1 \mu_q + \lambda_2 \\
&= \frac{C\mu_p - B}{D}\mu_q + \frac{A - B\mu_p}{D} \\
&= \frac{C}{D}\left(\mu_p\mu_q - \frac{B}{C}\mu_q - \frac{B}{C}\mu_p + \frac{B^2}{C^2} - \frac{B^2}{C^2}\right) + \frac{A}{D} \\
&= \frac{C}{D}\left(\mu_q - \frac{B}{C}\mu_q\right)\left(\mu_p - \frac{B}{C}\mu_p\right) + \frac{A}{D} - \frac{B^2}{C^2} \\
&= \frac{C}{D}\left(\mu_q - \frac{B}{C}\mu_q\right)\left(\mu_p - \frac{B}{C}\mu_p\right) + \frac{AC - B^2}{CD} \\
&= \frac{C}{D}\left(\mu_q - \frac{B}{C}\mu_q\right)\left(\mu_p - \frac{B}{C}\mu_p\right) + \frac{1}{C}
\end{aligned}
$$

If $\mu = k\mathbb{1}$ for some $k \in \mathbb{1}$ then we have that

$$
\begin{aligned}
A &= k^2 C \\
B &= kC \\
D &= k^2C^2 - (kC)^2 = 0
\end{aligned}
$$

Since $D = 0$ in this case we have that that the covariance blows up to positive or negative infinity.

(d) Minimum Variance Portfolio

To find the portfolio with minimum variance we take the derivative of the variance with respect to μ_p and set equal to zero

$$
\begin{aligned}
Var(R_p) &= \frac{C}{D}\left(\mu_p - \frac{B}{C}\right)^2 + \frac{1}{C} \\
\frac{\partial}{\partial \mu_p}\left(Var(R_p)\right) &= \frac{2C}{D}\left(\mu_p - \frac{B}{C}\right) \\
\frac{\partial^2}{\partial \mu_p^2}\left(Var(R_p)\right) &= \frac{2C}{D} > 0
\end{aligned}
$$

Since the second derivative is positive it follows that the minimal variance portfolio has $\mu_p = B/C$. We plug back in μ_p into the expresion for $Var(R_p)$ and $W*$ and we get

$$Var(R_p) = \frac{1}{C}$$

$$w^* = \frac{\Sigma^- 1\mathbb{1}}{C}$$

(e) Hedge Portfolio

Let $z^* = \Sigma^{-1}\frac{C\mu - B\mathbb{1}}{D}$ then we show that w_p from the Section 1 can be rewritten as

$$w_p = w_g + (\mu_p - \mu_g)Z^*$$

$$\begin{aligned}
&w_g + (\mu_p - \mu_g)Z^* \\
&= \frac{\Sigma^{-1}\mathbb{1}}{C} + \left(\mu_p - \frac{B}{C}\right)\Sigma^{-1}\left(\frac{C\mu - B\mathbb{1}}{D}\right) \\
&= \frac{\Sigma^{-1}\mathbb{1}}{C} + \left(\frac{C\mu_p - B}{D}\right)\Sigma^{-1}\mu - \left(\frac{B\mu_p - B^2/C}{D}\right)\Sigma^{-1}\mathbb{1} \\
&= \left(\frac{C\mu_p - B}{D}\right)\Sigma^{-1}\mu + \left(\frac{1}{C} - \frac{B\mu_p - B^2/C}{D}\right)\Sigma^{-1}\mathbb{1} \\
&= \left(\frac{C\mu_p - B}{D}\right)\Sigma^{-1}\mu + \left(\frac{D + B^2 - B/c\mu_p}{CD}\right)\Sigma^{-1}\mathbb{1} \\
&= \left(\frac{C\mu_p - B}{D}\right)\Sigma^{-1}\mu + \left(\frac{A - B\mu_p}{D}\right)\Sigma^{-1}\mathbb{1} \\
&= \lambda_1\Sigma^{-1}\mu + \lambda_2\Sigma^{-1}\mathbb{1}
\end{aligned}$$

(f, g, h) Zero Beta

Answer to the question is in the slides.

(i) Mean Variance with Liabilities

Now we solve the following minimization problem

$$\min_{\boldsymbol{w}} \mathrm{Var}\left(R_p - \frac{1}{f}R_l\right)$$

subject to $w^t\mathbb{1} = 1$ and

$$E\left(R_p - \frac{1}{f}R_l\right) = \overline{r}$$

As seen in the slides we rewrite the problem as

$$\min_{\boldsymbol{w}}\left(\frac{1}{2}\boldsymbol{w}^\top\Sigma\boldsymbol{w} - \frac{1}{f}\boldsymbol{\zeta}^\top\boldsymbol{w}\right)$$

subject to

$$\begin{aligned}\boldsymbol{\mu}^\top \boldsymbol{w} &= \overline{r} + \frac{1}{f}\mathbb{E}\left(R_l\right) =: \hat{r} \\ \boldsymbol{w}^\top \mathbb{1} &= 1\end{aligned}$$

We proceed as in the slides and we use Laplace multipliers to get the following FOC

$$\begin{aligned}\Sigma w - \frac{1}{f}\zeta - \lambda\mu - \delta 1 = 0 \\ \mu^\top w = \hat{r} \\ 1^\top w = 1\end{aligned}$$

Thus, we get that

$$w = \frac{1}{f}\Sigma^{-1}\zeta - \lambda\Sigma^{-1}\mu - \delta\Sigma^{-1}\mathbb{1} = 0$$

Using the fact that $\mathbb{1}^1 w = 1$ we get

$$\delta = \frac{1 - \frac{1}{f}\mathbf{1}^\top\Sigma^{-1}\zeta - \lambda\mathbf{1}\Sigma^{-1}\mu}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}} = \frac{1 - \frac{1}{f}\mathbf{1}^\top\Sigma^{-1}\zeta}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}} - \lambda\frac{\mathbf{1}\Sigma^{-1}\mu}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}$$

$$w = \frac{1}{f}\Sigma^{-1}\zeta + \delta\Sigma^{-1}1 + \lambda\Sigma^{-1}\mu$$

Then we plug w into $\mu^T w = \hat{r}$ and get

$$\lambda = \frac{\hat{r} - \frac{1}{f}\mu^\top\Sigma^{-1}\zeta - \delta\mu^\top\Sigma^{-1}\mathbb{1}}{\mu^\top\Sigma^{-1}\mu}$$

$$\lambda = \left(\frac{\hat{r} - \frac{1}{f}\mu^\top\Sigma^{-1}\zeta - \frac{1-\frac{1}{f}\mathbf{1}^\top\Sigma^{-1}\zeta}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}\mu^\top\Sigma^{-1}\mathbb{1}}{\mu^\top\Sigma^{-1}\mu}\right) \Big/ \left(1 - \left(\frac{\mu^\top\Sigma^{-1}\mathbb{1}}{\mathbb{1}^\top\Sigma^{-1}\mathbb{1}}\right)^2\right)$$

Let

$$E = 1 - \left(\frac{\mu^\top\Sigma^{-1}\mathbb{1}}{\mathbb{1}^\top\Sigma^{-1}\mathbb{1}}\right)^2$$

$$F = \frac{1}{f}\mathbf{1}^\top\Sigma^{-1}\zeta$$

Thus,

$$\lambda = \frac{\hat{r}}{AE} - \frac{F + \frac{1-F}{C}}{AE} = \frac{\hat{r}}{AE} - \frac{CF + 1 - F}{AEC}$$

$$\delta = \frac{1-F}{C} - \lambda\frac{B}{C} = \frac{1-F}{C} - \left(\frac{\hat{r}}{AE} - \frac{CF+1-F}{AEC}\right)\frac{B}{C}$$

$$w = F - \left(\frac{\hat{r}}{AE} - \frac{CF+1-F}{AEC}\right)\Sigma^{-1}\mu - \left(\frac{1-F}{C} - \left(\frac{\hat{r}}{AE} - \frac{CF+1-F}{AEC}\right)\frac{B}{C}\right)\Sigma^{-1}\mathbb{1}$$

$$= F - \left(\frac{\hat{r}}{AE} - \frac{CF+1-F}{AEC}\right)\Sigma^{-1}\mu - \left(\frac{1-F}{C} - \left(\frac{B\hat{r}}{ACE} - \frac{B(CF+1-F)}{AEC^2}\right)\right)\Sigma^{-1}\mathbb{1}$$

$$= F + \left(\frac{CF+1-F}{AEC}\right)\Sigma^{-1}\mu - \left(\frac{1-F}{C} + \frac{1-F}{C}\right)\Sigma^{-1} + \hat{r}\left(\frac{B\Sigma^{-1}\mathbb{1}}{ACE} - \frac{\Sigma^{-1}\mu}{AE}\right)$$

Now let

$$G = F + \left(\frac{CF+1-F}{AEC}\right)\Sigma^{-1}\mu - \left(\frac{1-F}{C} + \frac{1-F}{C}\right)\Sigma^{-1}$$

$$H = \left(\frac{B\Sigma^{-1}\mathbb{1}}{ACE} - \frac{\Sigma^{-1}\mu}{AE}\right)$$

$$w = G + \hat{r}H$$

Thus we have Var as a function of $\hat{r}$, therefore we take the derivative with respect to $\hat{r}$ and set to zero to find the optimal $\hat{r}$,

$$\frac{1}{2}\boldsymbol{w}^\top\Sigma\boldsymbol{w} - \frac{1}{f}\boldsymbol{\zeta}^\top\boldsymbol{w} = \frac{1}{2}\left(G+\hat{r}H\right)^\top\Sigma\left(G+\hat{r}H\right) - \frac{1}{f}\boldsymbol{\zeta}^\top\left(G+\hat{r}H\right)$$

$$0 = \hat{r}H^t\Sigma H + HG - \frac{1}{f}\zeta H$$

$$r^* = -\frac{HG - \frac{1}{f}\zeta H}{H^t\Sigma H}$$

Now we may put r^* in the expression of Var to get its values minimal variance portfolio. Let

$$w_{r^*} = G + Hr^*$$

$$Var_{r^*} = \left(\frac{1}{2}\boldsymbol{w_{r^*}}^\top\Sigma\boldsymbol{w_{r^*}} - \frac{1}{f}\boldsymbol{\zeta}^\top\boldsymbol{w_{r^*}}\right)$$