The Conditional Expected Market Return *

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Abstract

We derive lower bounds on the conditional expected excess market return and expected log market returns. The bounds are related to a volatility index, a skewness index, and a kurtosis index. The bounds can be calculated in real time at any date using the cross-section of option prices. The bounds require no-arbitrage assumptions, but do not depend on any distributional assumptions about market returns or past observations. The bounds are highly volatile, positively skewed, and exhibit fat tails. They imply that the term structure of equity returns is decreasing during turbulent times and increasing during normal times, and that the expected excess market return is on average 5%.

KEY WORDS: Equity Risk Premium, Risk Neutral Moments, Preferences

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1. Introduction

The expected excess market return, commonly known as the market risk premium, is the compensation that investors require for holding the market portfolio. Over the years, researchers have developed many approaches to estimate the expected excess market return. Early efforts focused on estimating the market risk premium through the lens of the capital asset pricing model (CAPM) with differing conclusions. For instance, Black, Jensen, and Scholes (1983) and Fama and MacBeth (1973) find evidence for a positive market risk premium in the pre-1969 period, however, Fama and French (1992) illustrate potential issues with their statistically insignificant and negative estimate of the market risk premium.

More recently, researchers have looked to other techniques and models for estimating the market risk premium. Avdis and Wachter (2017) use a maximum likelihood method to estimate the unconditional equity premium from observed asset prices. Schneider (2018) and Schneider and Trojani (2017) use no-arbitrage conditions to provide estimate of the expected market return. Such estimates typically rely on historical return observations, or make assumptions about the distribution of returns, investor preferences, or both. Ross (2015) uses the recovery theorem to show how physical moments of returns can be extracted from the risk neutral measure. Borovicka, Hansen, and Scheinkman (2016) provide conditions under which the recovery theorem of Ross holds. Similarly, Martin (2017) shows that the risk-neutral variance of the simple return discounted by the risk-free return is a lower bound for the conditional expected excess return as long as his negative correlation condition (NCC) holds. Additionally, he provides evidence that the bound is tight, which implies that the NCC holds and that the correlation it describes is unconditionally close to zero. Martin's work does not account for the effect of higher-order risk-neutral simple return moments (i.e., skewness, kurtosis, and other higher moments).[1]

In this paper, we construct new bounds on the conditional expected excess market return that are func-

[1]Gormsen and Jensen (2018) use the Martin (2017, Appendix) approach and assumption that investors have power utility to estimate ex ante market higher order physical moments from their risk-neutral counterparts estimated from options prices. Unlike Gormsen and Jensen (2018), we do not rely on the assumption that investors have power utility. Their focus is on documenting the behavior of estimated ex ante physical moments and their relation to proposed macroeconomic risk mechanisms that have been proposed in the literature, whereas our focus is to use information in options prices to estimate bounds on expected market returns and conditional losses directly.

tions of higher-order risk-neutral simple return moments. We provide evidence that our bound measures are tight, and therefore consider them as measures of the conditional expected excess market return. They can be computed in real time at any date by using a cross-section of option prices. They perform similarly to the Martin (2017) measure in forecasting regressions at short horizons (30, 60, and 90 days) with out-of-sample R-squared values of 0.2%, 0.9%, and 1.1%, respectively. The corresponding values based on the Martin (2017) bound are 0.3%, 0.9%, and 1.0%. At longer horizons, our bounds perform significantly better than the Martin (2017). For instance, our bound achieves out-of-sample R-squared values of 7.1% and 5.8% in forecasting regressions at horizons of 180 and 360 days, respectively, whereas the Martin (2017) bound has corresponding R-squared values of 4.3% and 1.3%, respectively.[2] These results indicate that accounting for higher moments is relatively important for constructing measures of expected returns, particularly at longer horizons.

We also find evidence that the term structure of expected returns is time-varying. It is upward sloping during normal times, but downward sloping during turbulent times. This observation is consistent with other recent evidence in Bansal, Miller, and Yaron (2017). Additionally, our term structure is increasing more with maturity during normal times and decreasing more with maturity during turbulent times compared to that implied by the Martin (2017) bound. To the extent that expected returns are relatively high at long horizons during normal times and low at long horizons during turbulent times relative to values implied by the Martin (2017), this could explain the relatively better forecasting power of our bound at longer horizons described above.

We begin by developing theoretical results that do not rely on the NCC and illustrate how expected excess market returns are related to the entire risk-neutral distribution of returns. Our bound is theoretically motivated and does not rely on any distributional assumption about excess returns. Under the assumption of no-arbitrage, we derive a simple closed-form expression for the conditional expected excess market return as a function of all moments of the risk-neutral distribution of simple returns. Risk neutral moments of the distribution of the market return are expressed in terms of option prices for a given maturity.

[2]These results are based on our bound measure that does not require preference parameter estimation.

We impose only mild assumptions investor preferences to construct our bound. In one specification, we use the data to estimate preference parameters to construct a measure of expected returns. In a second specification, we use mild assumptions about these preference parameters to construct an alternative measure of expected returns that requires no estimation. In this sense, our estimates are almost model-free. These results do depend on preferences being time separable, however, in an extension we show how these results are related to analogous results where investors have time inseparable preferences.

The key difference between the conditional expected excess return derived in this paper and that derived in Martin (2017) is that our measure incorporates the effects of high order risk-neutral moments. We show that these moments (particularly skewness and kurtosis) are empirically important for forecasting returns and lead to significant deviations from expectations formed using risk neutral variance alone. These results imply that the conditional expected excess return is even more highly time-varying than previously thought. The difference between the bound derived in this paper and that derived in Martin (2017) varies significantly over time, is highly volatile, and skewed. The difference is particularly large during crisis periods. For example, the difference was approximately 18% in November 2008. The bound derived in this paper also exhibits fatter tails than that derived in Martin (2017). On average, the conditional expected excess market return using our lower bound measure is 5.2%. This is 0.7% higher than that based on the measure derived in Martin (2017), which we found to be approximately 4.5% in our sample.[3] Avdis and Wachter (2017) estimate the unconditional expected excess return to be 5.1% using a maximum likelihood method. They note that their estimate is lower than previous estimates (using historical data) of 6.4%, which is economically significant. Importantly, we cannot reject the null hypothesis that our lower bound is tight at all maturities investigated. Our bound can be used to better approximate the conditional expected excess market return during normal and turbulent times.

We further derive an upper bound on the conditional expected excess market return, which represents the maximum risk premium investors would expect for holding the market portfolio. The upper bound is

[3] Martin and Wagner (2016) and Kadan and Tang (2015) estimate the expected returns on individual stocks using similar methods and options on individual stocks.

time-varying, highly volatile, positively skewed, and also exhibits fat tails. The upper bound is approximately 92% at the peak of the 2008 crisis and is, on average, 8.6%.

We also derive an identity that relates the expected excess log return to observable options prices. In particular, we derive an identity that relates the conditional expected excess log return to the risk-neutral distribution of returns. Fitting with the theme of previously described results, we find that the lower bound on the conditional expected excess log return is time-varying, highly volatile, positively skewed, and exhibits fat tails. At the peak of the 2008 crisis, the lower bound on the conditional expected excess log return was approximately 48%. We also test whether the conditional expected excess log return derived in this paper is tight, and we cannot reject this null hypothesis at all maturities. The lower bound on the conditional expected excess log return is on average 3%.

The rest of the paper is organized as follows. Section 2 provides theoretical motivation for our proposed lower bounds on expected excess returns. In Section 3, we quantify the lower bound on the expected excess market return and show that we cannot reject that this bound is tight. Section 4 discusses the sources of the difference between our bound and the bound on the expected excess return derived in Martin (2017), and presents empirical evidence for the difference. We also derive the lower bound on expected excess log returns and show that we cannot reject the null that this bound is also tight. Section 5 concludes.

2. Theoretical Framework

In this section, we theoretically show how to use risk-neutral conditional moments of the S&P 500 to derive the conditional expected market return and higher-order conditional physical moments of the S&P 500. We then use truncated risk-neutral conditional moments to provide an upper bound on the conditional expected market return. Expressions for the expected excess return and conditional moments are derived without making any time-series assumptions about the distribution of the market return or using past return observations. We can use the cross-section of options prices at any date to construct these measures. We use the terms risk premium and expected excess return interchangeably herein. All proofs appear in the

Appendix.

2.1. *Conditional Expected Excess Market Return*

Denote the gross return on an individual stock from time t to T by $R_{i,t\to T}$ and that on the risk-free asset $R_{f,t\to T}$. Let $M_{t\to T}$ be the stochastic discount factor (SDF) from time t to T. We define risk-neutral and physical moments of the market return as

$$\mathbb{M}^{*(n)}_{t\to T} = \mathbb{E}^{*}_{t}\left[(R_{M,t\to T} - R_{f,t\to T})^{n}\right] \text{ and } \mathbb{M}^{(n)}_{t\to T} = \mathbb{E}_{t}\left[(R_{M,t\to T} - R_{f,t\to T})^{n}\right] \text{ when } n \geq 1, \tag{1}$$

respectively. The case with $n = 1$ corresponds to the conditional expected excess market return. In this paper, we refer the third and fourth moments of the simple market return as the skewness and kurtosis of the market return, respectively. We use asterisks to denote quantities calculated under the risk-neutral measure. In Result 1, we derive the expected excess return on any asset under the risk-neutral measure.

Result 1 *The expected excess return on any asset i is*

$$\mathbb{E}_{t}\left(R_{i,t\to T}\right) - R_{f,t\to T} = \frac{1}{R_{f,t\to T}} Cov^{*}_{t}\left(R_{i,t\to T}, \frac{1}{M_{t\to T}}\right). \tag{2}$$

The expected excess return on any asset is determined by the discounted value of the covariance between its return and the inverse of the SDF under the risk-neutral measure. Identity (2) depends only on the assumption that there is no arbitrage, which implies the existence of an SDF. The expected excess return on an asset is positive if and only if the correlation between the asset return and the inverse of the SDF is positive under the risk-neutral measure.

We consider an economy where the representative agent has a generic utility function u and maximizes her expected utility, in a one-period economy with no consumption, according to

$$\max_{\substack{\omega_t \\ W_T = W_t\left(R_{f,t} + \omega_t^{\perp}\left(R_{t\to T} - R_{f,t\to T}\right)\right)}} \mathbb{E}_t\left(u\left[W_T\right]\right), \tag{3}$$

where $\omega_t = (\omega_{i,t})_{i=1,\ldots,n}$ is a vector of portfolio weights, and $R_{t\to T} = (R_{i,t\to T})_{i=1,\ldots,n}$ is a vector of returns on risky assets. We denote the investor's initial wealth by W_t and the investor's terminal wealth by W_T. We assume that the first and second derivatives of the utility function exist. In addition, $u'[.] > 0$ and $u''[.] < 0$. These assumptions lead to the following result:

Result 2 *The conditional expected excess return on an asset, consistent with the first-order conditions of (3), is*

$$\mathbb{E}_t\left(R_{i,t\to T}\right) - R_{f,t\to T} = Cov_t^*\left(R_{i,t\to T}, \frac{\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}}{\mathbb{E}_t^*\left(\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}\right)}\right). \tag{4}$$

We use identity (4) to derive the conditional expected excess market return and higher physical moments of the excess market return.

Result 3 *The risk premium on the n^{th} order moment of the excess return is determined by the covariance under the risk-neutral measure of the return moment and the inverse of the marginal utility*

$$\mathbb{M}_{t\to T}^{(n)} - \mathbb{M}_{t\to T}^{*(n)} = \mathbb{COV}_t^*\left(\left(R_{M,t\to T} - R_{f,t\to T}\right)^n, \frac{\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}}{\mathbb{E}_t^*\left(\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}\right)}\right). \tag{5}$$

Result 3 is general in a one-period economy with utility over final wealth. It applies to any utility function in such a setting. We do not assume that the initial wealth is unity. Three remarks are in order. Although we can estimate this covariance for a generic utility function using Taylor expansions, for the case in which investors have log utility, CRRA utility, or exponential utility, closed-form expressions of (5) are available.

Remark 1 *In the special case of log utility, $u[x] = \log x$, expression (5) becomes*

$$\mathbb{M}_{t\to T}^{(n)} - \mathbb{M}_{t\to T}^{*(n)} = \frac{1}{R_{f,t\to T}} \mathbb{M}_{t\to T}^{*(n+1)}. \tag{6}$$

In the case where $n = 1$, $\mathbb{M}_{t\to T}^{(n)}$ is the expected excess market return and $\mathbb{M}_{t\to T}^{*(n)} = 0$, and (6) reduces to the result from Martin (2017):

$$\mathbb{E}_t \left[R_{M,t\to T} - R_{f,t\to T}\right] = \frac{1}{R_{f,t\to T}} \mathbb{M}_{t\to T}^{*(2)} = \frac{1}{R_{f,t\to T}} Var_t^* \left(R_{M,t\to T}\right). \tag{7}$$

Remark 2 *In the special case of CRRA utility, $u[x] = \frac{x^{1-\alpha}}{1-\alpha}$ (where α is the relative risk aversion), expression (5) becomes*

$$\mathbb{M}_{t\to T}^{(n)} - \mathbb{M}_{t\to T}^{*(n)} = \frac{1}{R_{f,t\to T}\mathbb{E}_t^*\left(R_{M,t\to T}^{\alpha}\right)} \mathbb{COV}_t^* \left(\left(R_{M,t\to T} - R_{f,t\to T}\right)^n, R_{M,t\to T}^{\alpha}\right). \tag{8}$$

The closed-form expression for the risk-neutral covariance is available in Appendix B. Expression (8) is similar to Result 8 in the Internet Appendix of Martin (2017). Our Results 2 and 3 are general, and are not specific to the CRRA utility described in Martin (2017, Appendix).

Remark 3 *In the special case of exponential utility, $u[x] = 1 - e^{-\alpha x}$ (where α is a constant that represents the degree of risk aversion), expression (5) becomes*

$$\mathbb{M}_{t\to T}^{(n)} - \mathbb{M}_{t\to T}^{*(n)} = \frac{1}{R_{f,t\to T}\mathbb{E}_t^*\left(e^{\alpha R_{M,t\to T}}\right)} \mathbb{COV}_t^* \left(\left(R_{M,t\to T} - R_{f,t\to T}\right)^n, e^{\alpha R_{M,t\to T}}\right).$$

The closed-form expression for the risk-neutral covariance is available in Appendix B.

We show that when investors have log utility, CRRA utility, or exponential utility, closed-form expressions of all moments of the excess return can be derived. More generally, when the functional form of the utility function is available, the closed-form expressions of (5) can be derived. When the functional form of the utility is not available, we can use a Taylor expansion series of the inverse marginal utility to characterize the expected excess return in (5).

We assume that high-order derivatives of the marginal utility exist and define

$$f[x] = \frac{u'[W_t x_0]}{u'[W_t x]} \text{ with } x = R_{M,t\to T} \text{ and } x_0 = R_{f,t\to T}. \tag{9}$$

Taylor expansion series of $f[.]$ around $x = x_0$ yields

$$f[x] = 1 + \sum_{k=1}^{\infty} \theta_k (x - x_0)^k \text{ with } \theta_k = \frac{1}{k!}\left(\frac{\partial^k f[x]}{\partial^k x}\right)_{x=x_0}. \tag{10}$$

Expression (5), together with (10), lead to

$$\mathbb{M}^{(n)}_{t\to T} - \mathbb{M}^{*(n)}_{t\to T} = \frac{\sum_{k=1}^{\infty} \theta_k \left(\mathbb{M}^{*(n+k)}_{t\to T} - \mathbb{M}^{*(n)}_{t\to T}\mathbb{M}^{*(k)}_{t\to T}\right)}{1 + \sum_{k=1}^{\infty} \theta_k \mathbb{M}^{*(k)}_{t\to T}}. \tag{11}$$

The general expression (11) holds when all derivatives of the utility function exist, and all assumptions required for Result 3 holds (namely, that the economy is a one-period economy with utility over final wealth). We further show that when terms in (11) associated with $k > 3$ are neglected, the right-hand side of (11) represents a lower (upper) bound on $\mathbb{M}^{(n)}_{t\to T} - \mathbb{M}^{*(n)}_{t\to T}$ when n is odd (even). To do so, we first characterize a few preference parameters. We define the risk tolerance τ, the skew tolerance ρ, and the kurtosis tolerance κ as

$$\begin{aligned}
\tau &= -\frac{u'[W_t R_{f,t\to T}]}{W_t R_{f,t\to T} u''[W_t R_{f,t\to T}]}, && (12)\\
\rho &= \frac{1}{2!}\frac{W_t R_{f,t\to T} u'''[W_t R_{f,t\to T}]}{u''[W_t R_{f,t\to T}]}\frac{u'[W_t R_{f,t\to T}]}{W_t R_{f,t\to T} u''[W_t R_{f,t\to T}]}, \textit{and} && (13)\\
\kappa &= \frac{1}{3!}\frac{W_t R_{f,t\to T} u''''[W_t R_{f,t\to T}]}{u'''[W_t R_{f,t\to T}]}\frac{W_t R_{f,t\to T} u'''[W_t R_{f,t\to T}]}{u''[W_t R_{f,t\to T}]}\left(\frac{u'[W_t R_{f,t\to T}]}{W_t R_{f,t\to T} u''[W_t R_{f,t\to T}]}\right)^2. && (14)
\end{aligned}$$

The preference parameters τ, ρ, and κ are time-varying. They depend on the risk-free return and the initial wealth. The risk tolerance $\tau = \frac{1}{\mathcal{A}}$ is a measure of the inverse of the investor's absolute risk aversion $\mathcal{A}$. The skewness preference parameter can alternatively be written (up to a constant) as the ratio of prudence to risk aversion

$$\rho = \frac{1}{2}\frac{\mathcal{P}}{\mathcal{A}}, \tag{15}$$

where $\mathcal{P}$ measures the investor's absolute prudence. Also, the kurtosis preference parameter can be expressed as

$$\kappa = \frac{1}{2}\frac{\mathcal{P}}{\mathcal{A}}\frac{1}{3}\frac{\mathcal{T}}{\mathcal{A}}, \tag{16}$$

where $\mathcal{T}$ is defined as the absolute temperance (see e.g, Eeckhoudt and Schlesinger (2006), Deck and Schlesinger (2014), and Noussair, Trautmann, and VanDeKuilen (2014)). Gollier (2001) shows that the option to purchase assets increases the marginal value of wealth if prudence is larger than twice the risk aversion. This is equivalent to stating that $\rho \geq 1$. This implies that the absolute prudence is higher than 2 when the absolute risk aversion is higher than 1. When prudence is higher than twice the risk aversion and temperance is higher than three times prudence, the parameter $\kappa \geq 1$. When temperance is higher than three times prudence and risk aversion is higher than 1, then temperance is higher than 3. Noussair, Trautmann, Van De Kuilen (2014, Table 3, page 335 and Table B1, page 351) use different scenarios to provide empirical evidence that risk aversion, prudence, and temperance are approximately equal or higher than 3. The preference parameters τ, ρ, and κ are functions of the initial wealth W_t, and the risk-free rate.

In Section 3, we first assume that the preference parameters τ, ρ, and κ are constant. Second, we avoid estimating preference parameters by imposing restrictions on them. An alternative approach is to perform a Taylor expansion series of the preference parameters around $W_t = 1$. This gives rise to more than three parameters to be estimated and also requires an estimate of the aggregate wealth at each point in time. We avoid following this route.

In addition to our relatively standard assumptions about a one-period economy, we need two additional assumptions to derive bounds on the moments of the excess market return.

Assumption 1 *Risk-neutral moments satisfy the following inequalities*

$$\mathbb{M}^{*(k)}_{t\to T} \quad \leq \quad 0 \quad \textit{if } k \textit{ is odd.}$$

This assumption is necessary to derive our bounds. To motivate the assumption, first note that empirical estimates of the third, fifth, and seventh moments of the realized market returns are typically negative. Since odd moments (for instance, skewness) are associated with unfavorable market conditions that market participants would like to insure against, it is reasonable that they have negative risk premia. For instance, Kozhan, Neuberger, and Schneider (2013) find a negative skewness premium, which is equivalent to the average risk-neutral skewness being even more negative than the average realized skewness. We similarly expect negative risk premia on higher odd moments. Paired with the observation that realized odd moments of the market return tend to be negative, this intuition implies that risk-neutral odd moments will be even more negative.

Consistent with this intuition, we find empirical evidence that odd risk-neutral moments are negative. In our data set, almost every observation of daily risk-neutral skewness is negative at the 1-month horizon (98% at the 1-year horizon). 99% of daily estimates of $\mathbb{M}^{*(5)}_{t\to T}$ at the 1-month horizon are negative (97% at the 1-year horizon). Further, 99% of daily estimates of $\mathbb{M}^{*(7)}_{t\to T}$ at the 1-month horizon also are negative (94% at the 1-year horizon). This suggests that when k is odd, the quasi majority of daily estimates of $\mathbb{M}^{*(k)}_{t\to T}$ are negative.

Assumption 2 *Preference parameters*

$$\theta_k = \frac{a_k}{R^k_{f,t\to T}}, \tag{17}$$

with

$$a_1 = \frac{1}{\tau},\ a_2 = \frac{(1-\rho)}{\tau^2},\ a_3 = \frac{(1-2\rho+\kappa)}{\tau^3}. \tag{18}$$

satisfy the following inequalities

$$\theta_k \quad \leq \quad 0 \quad \textit{if k is even and} \quad \theta_k \quad \geq \quad 0 \quad \textit{if k is odd.}$$

To understand why Assumption 2 is reasonable, we assume that Assumption 1 holds. Investors who fully invest in the market portfolio receive compensation for exposure of their portfolio to risk neutral moments $\mathbb{M}^{*(k)}_{t\to T}$. Thus, the contribution of each risk neutral moment to the expected return is positive. For example, empirical evidence shows that the price of the market skewness is negative. Since the quantity of risk (market skewness) is also negative, the contribution of the market skewness measured as the price of risk times the quantity of risk is positive (see (A9)-(A10) in the Appendix). Additionally, Assumption 2 can be further motivated by Eeckhoudt and Schlesinger (2006).

Result 4 *Under Assumptions 1 and 2, the conditional expected value $\mathbb{M}^{(n)}_{t\to T}$ satisfies the following inequality:*

$$\mathbb{M}^{(n)}_{t\to T} - \mathbb{M}^{*(n)}_{t\to T} \quad \geq \quad B^{(n)}_{t\to T} \textit{ if n is odd and} \tag{19}$$

$$\mathbb{M}^{(n)}_{t\to T} - \mathbb{M}^{*(n)}_{t\to T} \quad \leq \quad B^{(n)}_{t\to T} \textit{ if n is even,} \tag{20}$$

with

$$B^{(n)}_{t\to T} = \frac{\sum_{k=1}^{3} \theta_k \left(\mathbb{M}^{*(n+k)}_{t\to T} - \mathbb{M}^{*(n)}_{t\to T}\mathbb{M}^{*(k)}_{t\to T} \right)}{1 + \sum_{k=1}^{3} \theta_k \mathbb{M}^{*(k)}_{t\to T}} \tag{21}$$

When n is odd, we denote the bound using $LB^{(n)}_{t\to T} = B^{(n)}_{t\to T}$ to emphasize that it represents a lower bound. When n is even, we denote the bound using $UB^{(n)}_{t\to T} = B^{(n)}_{t\to T}$ to emphasize that it represents an upper bound.

In Equation (21), the θ_k coefficients satisfy the restrictions $\theta_1 > 0$, $\theta_2 \leq 0$, and $\theta_3 \geq 0$. To identify the sign of the coefficients θ_k for $k = 1,2,3$, we set n=1 and show that (21) reduces to the expected excess market return:

$$\mathbb{E}_t\left(R_{M,t\to T} - R_{f,t\to T}\right) \geq LB^{(1)}_{t\to T} = \frac{\theta_1 \mathbb{M}^{*(2)}_{t\to T} + \theta_2 \mathbb{M}^{*(3)}_{t\to T} + \theta_3 \mathbb{M}^{*(4)}_{t\to T}}{1 + \theta_2 \mathbb{M}^{*(2)}_{t\to T} + \theta_3 \mathbb{M}^{*(3)}_{t\to T}}. \tag{22}$$

The right-hand side of (22) is a lower bound on the expected excess market return. The expected excess return on the market (22) is determined by the risk-neutral moments and preference parameters. Because investors demand compensation for assets that exhibit negative skewness ($\mathbb{M}^{*(3)}_{t\to T}$) and positive kurtosis ($\mathbb{M}^{*(4)}_{t\to T}$) (see e.g., Harvey and Siddique (2000), Dittmar (2002), Chabi-Yo (2012)), one should expect

$$\frac{\theta_2}{1 + \theta_2 \mathbb{M}^{*(2)}_{t\to T} + \theta_3 \mathbb{M}^{*(3)}_{t\to T}} \leq 0 \text{ and } \frac{\theta_3}{1 + \theta_2 \mathbb{M}^{*(2)}_{t\to T} + \theta_3 \mathbb{M}^{*(3)}_{t\to T}} \geq 0. \tag{23}$$

The denominator of (22) is the expected value of the inverse of the marginal utility (9), hence it should be a positive quantity. Consequently, from (23), one should expect $\theta_2 \leq 0$ and $\theta_3 \geq 0$.

Under the expected utility framework, we expect $\frac{\partial^k\left(u'[x]\right)}{\partial^k x} \leq 0$ if k is odd, and $\frac{\partial^k\left(u'[x]\right)}{\partial^k x} \geq 0$ if k is even (see e.g., Eeckhoudt and Schlesinger (2006)). Thus, under mild conditions one can argue that $\frac{\partial^k(f[x])}{\partial^k x} \geq 0$ if k is odd, and $\frac{\partial^k(f[x])}{\partial^k x} \leq 0$ if k is even. Provided that $\frac{\partial^k(f[x])}{\partial^k x} \geq 0$ if k is odd and $\frac{\partial^k(f[x])}{\partial^k x} \leq 0$ if k is even, the remaining terms in the expansion (11) are positive. Inequality constraints $a_2 \leq 0$ and $a_3 \geq 0$ are consistent with and $\frac{\partial^2(f[x])}{\partial^2 x} \leq 0$ and $\frac{\partial^3(f[x])}{\partial^3 x} \geq 0$. The bound $LB^{(1)}_{t\to T}$ will be tight if the contribution of moments associated preference parameters with $k > 3$ are small relative to those for which $k \leq 3$. We test this in the empirical section and find evidence that the bound is tight. We provide an additional testable restriction below.

Result 5 *Under Assumptions 1 and 2, if the preference parameters satisfy the restrictions*

$$\frac{1}{\tau} \geq 1,\ \frac{(1-\rho)}{\tau^2} \leq -1,\ \frac{(1-2\rho+\kappa)}{\tau^3} \geq 1, \tag{24}$$

it follows that the expected excess return on the market (4) is bounded:

$$\mathbb{E}_t\left[R_{M,t\to T} - R_{f,t\to T}\right] \geq LBR^{(1)}_{t\to T} \tag{25}$$

with

$$LBR^{(1)}_{t\to T} = \frac{\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T} - \frac{1}{R^2_{f,t\to T}}\mathbb{M}^{*(3)}_{t\to T} + \frac{1}{R^3_{f,t\to T}}\mathbb{M}^{*(4)}_{t\to T}}{1 - \frac{1}{R^2_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T} + \frac{1}{R^3_{f,t\to T}}\mathbb{M}^{*(3)}_{t\to T}}. \tag{26}$$

We refer to $LBR^{(1)}_{t\to T}$ as the restricted lower bound. If preference restrictions in (24) are satisfied, the expected excess return on the market is bounded by (26) and this lower bound on the expected excess market return can be computed without estimating any preference parameters. As with the unrestricted bound in (22), we also investigate whether this restricted version is tight in the empirical section.

The theoretical bounds in Results 4 and 5 place a lower bound on the expected excess market return. In contrast, the Hansen and Jagannathan (1991, HJ) framework places an upper bound on the expected excess market return. Similarly to the HJ, we apply the Cauchy-Schwartz inequality to identity (2) and show

$$\mathbb{E}\left(R_{M,t\to T} - R_{f,t\to T}\right) \leq \frac{1}{R_{f,t\to T}}\left(\mathbb{M}^{*(2)}_{t\to T}\right)^{\frac{1}{2}}\left(\mathbb{VAR}^*_t\left(1/M_{t\to T}\right)\right)^{\frac{1}{2}}.$$

Together, this inequality with our bound on the expected excess market return yields the following inequalities

$$B^{(1)}_{t\to T} \leq \mathbb{E}\left(R_{M,t\to T} - R_{f,t\to T}\right) \leq \frac{1}{R_{f,t\to T}}\left(\mathbb{M}^{*(2)}_{t\to T}\right)^{\frac{1}{2}}\left(\mathbb{VAR}^*_t\left(1/M_{t\to T}\right)\right)^{\frac{1}{2}}$$

where $B^{(1)}_{t\to T}$ is defined in Result 4. The left-hand inequality is the bound on the conditional expected excess market return. It relates the expected excess market return to a directly observable quantity (namely, options prices), which can be used to infer relevant information from the risk-neutral distribution of market returns; but it requires preference parameters as input or restrictions on preference parameters. However, the right-hand inequality, similar to the Hansen-Jagannathan bound, relates two quantities: an observable quantity which is the discounted risk neutral market volatility, and an unobservable quantity which is the risk neutral volatility of $1/M_{t\to T}$.

To provide further insight on the tightness of the bound, we use the information in the extreme left tail of the distribution of the S&P 500 index to derive an upper bound on the conditional expected excess market return that can also be computed using observable quantities (unlike the HJ bound). This upper bound, together with the lower bound, further helps assess whether the lower bound is informative and tight. In addition, it sheds light on how information related to the extreme tails of the S&P 500 return distribution helps approximate the expected excess return. To proceed with our analysis, we begin by decomposing the conditional expected excess return on the market as

$$\mathbb{E}_t\left(R_{M,t\to T}-R_{f,t\to T}\right)=\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}\geq k_0}\right)+\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}\leq k_0}\right), \quad (27)$$

where k_0 is given. The second term in the right-hand side of (27) is always negative, provided that $k_0 \leq R_{f,t\to T}$. To understand why this term is negative, note that

$$\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}\leq k_0}\right)=\underbrace{\mathbb{E}_t\left(\left(R_{M,t\to T}-k_0\right)1_{R_{M,t\to T}\leq k_0}\right)}_{\leq 0}+\underbrace{\left(k_0-R_{f,t\to T}\right)\mathbb{E}_t\left(1_{R_{M,t\to T}\leq k_0}\right)}_{\leq 0}. \quad (28)$$

Thus, we exploit the negative sign of (28) to derive an upper bound on the expected excess market return. To proceed, we define

$$\mathbb{M}^{*(n)}_{t\to T}\left[k_0\right]=\mathbb{E}^*_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^n 1_{\{R_{M,t\to T}\leq k_0\}}\right) \text{ when } n>0,$$

where $\mathbb{M}^{*(1)}_{t\to T}[k_0]$ is the truncated mean.

The risk-neutral quantities $\mathbb{M}^{*(2)}_{t\to T}[k_0]$, $\mathbb{M}^{*(3)}_{t\to T}[k_0]$, and $\mathbb{M}^{*(4)}_{t\to T}[k_0]$ represent the truncated second, third, and fourth moments, respectively. One should expect $\mathbb{M}^{*(1)}_{t\to T}[k_0]\leq 0$, $\mathbb{M}^{*(2)}_{t\to T}[k_0]\geq 0$, $\mathbb{M}^{*(3)}_{t\to T}[k_0]\leq 0$, and $\mathbb{M}^{*(4)}_{t\to T}[k_0]\geq 0$. We then use identity (27) to show the following result.

Result 6 *Under Assumptions 1 and 2, the conditional expected excess return on the market has an upper bound,*

$$\mathbb{E}_t\left(R_{M,t\to T}-R_{f,t\to T}\right)\leq\frac{-\mathbb{M}^{*((1))}_{t\to T}\left[k_0\right]+\sum_{k=1}^{\infty}\theta_k\left(\mathbb{M}^{*(k+1)}_{t\to T}-\mathbb{M}^{*(k+1)}_{t\to T}\left[k_0\right]\right)}{1+\sum_{k=1}^{\infty}\theta_k\mathbb{M}^{*(k)}_{t\to T}}. \quad (29)$$

Furthermore,

$$\mathbb{E}_t\left(R_{M,t\to T} - R_{f,t\to T}\right) \leq \widetilde{UB}^{(1)}_{t\to T}, \tag{30}$$

where

$$\widetilde{UB}^{(1)}_{t\to T} = \frac{-\mathbb{M}^{*(1)}_{t\to T}[k_0] + \sum_{k=1}^{3}\theta_k\left(\mathbb{M}^{*(k+1)}_{t\to T} - \mathbb{M}^{*(k+1)}_{t\to T}[k_0]\right)}{1+\sum_{k=1}^{3}\theta_k \mathbb{M}^{*(k)}_{t\to T}}. \tag{31}$$

If the preference parameters τ*,* ρ*,* κ *satisfy the restrictions*

$$\frac{1}{\tau} = 1,\ \frac{(1-\rho)}{\tau^2} = -1,\ \frac{(1-2\rho+\kappa)}{\tau^3} = 1, \tag{32}$$

it follows that

$$\widetilde{UBR}^{(1)}_{t\to T} = \frac{-\mathbb{M}^{*(1)}_{t\to T}[k_0] + \sum_{k=1}^{3}\frac{(-1)^{k+1}}{R^k_{f,t\to T}}\left(\mathbb{M}^{*(k+1)}_{t\to T} - \mathbb{M}^{*(k+1)}_{t\to T}[k_0]\right)}{1+\sum_{k=1}^{3}\frac{(-1)^{k+1}}{R^k_{f,t\to T}} \mathbb{M}^{*(k)}_{t\to T}}. \tag{33}$$

Result 6 shows that the upper bound on the conditional expected excess market return is determined by the risk-neutral moments of the simple market return and the truncated risk-neutral moments. Closed-form expressions of the truncated moments are available in Appendix B. Together, the lower and upper bounds provide investors with useful real-time bounds of expected excess market returns. More importantly, if the bounds are tight they provide an approximation of the true conditional expected excess market return.

3. Empirical results

In this section, we provide empirical support for the theoretical results derived in Section 2. We estimate our bounds in two ways: (i) by using estimates of preference parameters as input, and (ii) without using preference parameters, but imposing restrictions on them. When bounds are estimated by using preference parameters as input, we assume that the preference parameters τ, ρ, and κ are not time-varying.[4] We begin

[4]One could use Taylor expansion series to expand preference parameters around the initial wealth. But doing so gives rise to more than 3 parameters to be estimated. As a result, for parsimony, we decide to not assume they are time-varying.

by estimating preference parameters and the a_k coefficients implied by Result 4 and then test the tightness of the bound measures implied in (22), (26), (31), and (33) by running forecasting regressions.

3.1. Data

We use S&P 500 index options obtained from OptionMetrics to compute all risk-neutral moments and truncated risk-neutral moments according to equations (B3) and (B4) in Appendix B. We use discretized versions of these equations (available in the Internet Appendix) to compute our measures of risk-neutral moments. The data covers the period from January 1996 to August 2015. We apply the following filters to mitigate the effect of recording errors and option illiquidity. First, we remove options for which the bid or ask price violates arbitrage bounds (call prices must be less than the index price, and put prices must be less than the strike multiplied by the risk-free bond price); second, we remove options for which the ask price is lower than the bid, the bid price is zero, or the bid-ask spread is less than the minimum tick size ($0.05 for options trading lower than $3, and $0.10 otherwise); and finally, we remove options with no open interest. Risk-free rates are computed using the OptionMetrics zero coupon yield curve database.

3.2. Risk-neutral Moment and Truncated Risk-Neutral Moment calculations

We compute risk-neutral moments at five maturities (30, 60, 90, 180, and 360 days). Since we are interested in constant-maturity moments and options are not typically available with the desired maturities on a given day, we use an interpolation procedure to estimate moments at desired maturities from available maturities. We first use discretized versions of the spanning formulae in Appendix B to compute implied moments for listed maturities. We then linearly interpolate between these traded maturities to obtain moments at our maturities of interest. Figure B.1 in the Internet Appendix plots the variance, skewness, and kurtosis of the simple market return under the risk-neutral measure for all five maturities. As expected, the risk-neutral moments are time-varying and more volatile during crisis periods than in normal times. Also, Figure B.3 plots the truncated risk-neutral first, second, third, and fourth moments for all five maturities.

To estimate truncated moments, we set $k_0 = 0.80$. A low threshold k_0 tightens the upper bound. Due to the availability of the option data, we use $k_0 = 0.80$ since there are few options available when $k_0 < 0.80$. As expected, the first and third truncated moments are negative. The second and fourth truncated moments are positive. All truncated moments are time-varying and highly volatile.

3.3. Estimating τ*,* ρ*, and* κ *using three moment restrictions*

We now use the expressions in (19)–(20) to estimate the preference parameters τ, ρ, and κ. To identify all three parameters, we use three different moment restrictions ($n = 1$, $n = 2$, and $n = 3$). We estimate the parameters using both individual maturities and then using all maturities, which we find increases the precision of the estimates. The inequalities in (19)–(20) imply that we can use the following regressions to estimate the preference parameters

$$R_{M,t\to T} - R_{f,t\to T} = \alpha_1 + LB^{(1)}_{t\to T} + \varepsilon^{(1)}_{t\to T}, \tag{34}$$

$$(R_{M,t\to T} - R_{f,t\to T})^2 - \mathbb{M}^{*(2)}_{t\to T} = \alpha_2 + UB^{(2)}_{t\to T} + \varepsilon^{(2)}_{t\to T}, \text{ and} \tag{35}$$

$$(R_{M,t\to T} - R_{f,t\to T})^3 - \mathbb{M}^{*(3)}_{t\to T} = \alpha_3 + LB^{(3)}_{t\to T} + \varepsilon^{(3)}_{t\to T}, \tag{36}$$

where the bounds are defined in (19)–(20) and assumed to be tight. α_i are constants assumed to be the same across all maturities, but differ across each bound expression as indicated above. Results are similar whether we include the constant term or not. Results are also similar whether we use a single constant term across all maturities or maturity-specific constants. The bounds and $\mathbb{M}^{*(n)}_{t\to T}$ are quantities computed at time t while the excess return $(R_{M,t\to T} - R_{f,t\to T})^n$ are computed at from time t to T. We use a two-stage nonlinear least squares regression to estimate the preference parameters. Namely, in the first stage, we use uniform weights on each moment restriction equation to estimate the preference parameters. We then weight the errors in the second stage by the corresponding residual standard deviations from the first stage. The purpose of this weighting is to give less weight to data with relatively high volatility in the second stage estimation. Since $LB^{(1)}_{t\to T}$ and $LB^{(3)}_{t\to T}$ are lower bounds, one should expect $\alpha_1 \geq 0$ and $\alpha_3 \geq 0$. In contrast, $UB^{(2)}_{t\to T}$ is an upper bound and one should expect $\alpha_2 \leq 0$.

Six sets of estimation results are presented in Tables 1 and 2 using progressively longer maturities in each estimation in the first five columns. Table 1 presents the estimates of the preference parameters τ, ρ, and κ. Table 2 presents the corresponding estimates of lower bound coefficients a_1, a_2, and a_3. Noting that the estimates using individual maturities are noisy and imprecise, we present results using all maturities for estimation in the last columns of each table. This helps to increase the precision of the estimates. We use these estimation results to compute lower bounds used in the remainder of the paper. T-statistics are computed using standard errors from a block bootstrap where the block length is three times the maturity used in each estimation (three times 360 calendar days in the case of the estimation that uses all maturities).

As expected, all preference parameters τ, ρ, and κ are positive. The risk tolerance (τ), skewness preference (ρ), and kurtosis preference (κ) in the preferred specification (estimation using all maturities) are 0.97, 2.32, and 3.50, respectively. In Table 2, the coefficient a_1 is positive, while a_2 is negative. The coefficient a_3 is negative but not statistically significant. The last column in Table 2 presents preference parameter estimates when we use all available data. Using all maturities in this estimation is important for establishing the statistical significance of the estimates. The estimate of the risk aversion coefficient a_1 is 1.03 and is statistically significant at the 5% level with a t-statistic of 1.98. The estimate of the skewness preference coefficient a_2 is -1.39 and is statistically significant at the 5% level with a t-statistic of -2.05. We also test whether the coefficients a_1, a_2, and a_3 are statistically indistinguishable from 1, -1, and 1, respectively. We cannot reject the null hypotheses that the coefficients are, individually, statistically indistinguishable from 1, -1, and 1. Further, we cannot reject the joint hypothesis that $(a_1, a_2, a_3) = (1, -1, 1)$. These tests are relevant for justifying the assumptions underlying Result 5, which allows us to compute bounds without preference parameter estimates.

3.4. *Expected Excess Returns and Forecasting Regressions*

Next, we compute the lower and upper bounds on the expected excess market return in two ways. We first use the estimates of preference parameters in the last column of Table 1 to compute the lower bound $LB^{(1)}_{t\to T}$ and upper bound $\widetilde{UB}^{(1)}_{t\to T}$. Closed-form expressions for these bounds are in (19)–(21) and(31), respectively. Second, we compute lower and upper bounds by restricting the preference parameters. These bounds are labeled $LBR^{(1)}_{t\to T}$ and $\widetilde{UBR}^{(1)}_{t\to T}$ respectively. Expressions for these bounds are displayed in (26) and (33). All results are reported in percentages. For brevity, we present results for annualized 30-day expected excess returns. Results for maturities higher than 30 days are reported in the Internet Appendix.

3.4.1. *Lower and Upper Bounds Computed Using Estimated Preferences*

The top graph in Figure 1 plots the lower bound, the upper bound, and also the Martin (2017) lower bound. Both the lower and upper bounds on the market expected excess return are time-varying and more volatile during crisis periods than in normal times. The second graph in Figure 1 shows the difference between the upper and the lower bounds. The bottom graph of Figure 1 plots the difference between our lower bound $LB^{(1)}_{t\to T}$ and the lower bound derived in Martin (2017). All bounds in Figure 1 are computed for the 30-day maturity (plots for other maturities are provided in the Online Appendix). As shown in this graph, our conditional lower bound is always tighter than the Martin (2017) bound (i.e. it is always lower than the Martin (2017) bound)). The difference between the two bounds is time-varying and is more pronounced in periods of crisis or disasters than in normal times. This is intuitive since our lower bound exploits a richer set of information from the risk-neutral return distribution. Particularly, it is dependent on risk-neutral skewness and kurtosis, both of which increase in magnitude during turbulent times leading to higher bounds.

Table 3 presents the mean, standard deviation, and quantiles of the distribution of the upper and lower bounds for various maturities. Panel A of Table 3 presents the summary statistics for the upper bound. The upper bound is volatile, positively skewed, and exhibits fat tails. At the 30-day maturity, the mean of the upper bound for the full sample is approximately 8.57%. The upper bound varies from a minimum of 1.52% to a maximum of 92.93%. The maximum value of the upper bound is realized in October 2008. The standard deviation, skewness, and kurtosis of the upper bound are 8.63, 3.74, and 23.14, respectively. Results are similar for other investment horizons. Panel B presents the summary statistics for the lower bound. Similar to the upper bound, the lower bound is also volatile, positively skewed, and exhibits fat tails. At the 30-day maturity, the mean of the lower bound for the full sample is approximately 5.22%. The lower bound varies from a minimum of 0.73% to a maximum of 64.22%. The standard deviation, skewness, and kurtosis of the upper bound are 5.20, 4.33, and 31.76, respectively.

In Panel C, we report the summary statistics of difference between the lower bound $LB^{(1)}_{t\to T}$ and the lower bound derived in Martin (2017). For comparison, we also report the lower bound derived in Martin (2017) in Table 4. As shown in Panel C of Table 3, at the 30-day maturity, the difference is highly volatile and varies from a minimum of 0.07% to a maximum of 14.45%. Further, the difference in the lower bounds is positively skewed (with a skewness of approximately 5.40) and exhibits fat tails (with a kurtosis of approximately 48.42). The difference between our lower bound and the Martin (2017) lower bound is more skewed and has fatter tails than the lower bound on the expected excess return. The summary statistics reveal that, on average, our conditional lower bound is slightly higher than the Martin (2017) bound by approximately 1%. However, the time-varying conditional lower bound $LB^{(1)}_{t\to T}$ is significantly higher than the lower bound derived in Martin (2017). Results are similar when 60, 90, 180, and 360 days to maturity are used. This further shows that, in addition to the risk-neutral variance, higher risk-neutral moments of simple returns contain a rich set of information that enables us to tighten the lower bound and, hence, obtain a more accurate measure of the conditional expected excess return on the market.

Next, we test the ability of the lower bound $LB^{(1)}_{t\to T}$ and upper bound $\widetilde{UB}^{(1)}_{t\to T}$ measures to forecast the market return at various horizons by estimating standard forecasting regressions of the following form:

$$R_{M,t\to T} - R_{f,t\to T} = a_T + b_T X_{t\to T} + e_{t,T}, \tag{37}$$

where $X_{t\to T}$ represents the forecasting variable of interest corresponding the maturity of interest.[5] The coefficients a_T and b_T are maturity-specific estimates. We follow Goyal and Welch (2008) and Campbell and Thompson (2008) and compute the out-of-sample performance of the forecasts. The out-of-sample R-squared statistic is computed using

$$R^2_{OOS} = 1 - \frac{\sum_{t=1}^{T}(r_t - \hat{r}_t)^2}{\sum_{t=1}^{T}(r_t - \bar{r}_t)^2}, \tag{38}$$

where $\hat{r}_t$ is the forecasted return and $\bar{r}_t$ is the average historical return up to the forecast date. We use compounded S&P 500 daily returns (less the compounded risk-free rate) from CRSP beginning in 1926 to compute the expected excess market returns at each maturity. Market excess returns are computed for each maturity and then averaged over the entire historical period up to the forecasting date. Results are reported in Table 5.

Panel A of Table 5 displays results when the upper bound $\widetilde{UB}^{(1)}_{t\to T}$ is used. The null hypothesis $(\hat{a}_T, \hat{b}_T) = (0,1)$ cannot be rejected at the 10% significance level for any maturity. The out-of-sample R-squared statistic, labeled $R^2_{pseudoOOS}$ (7.9%), is highest at the 180-day maturity and is approximately equal to 2.7% at the 360-day maturity.[6] In contrast, the out-of-sample R-squared is small at the shorter maturities (from 30 days to 90 days). The out-of-sample R-squared values are similar to in-sample R-squared values regardless of maturity.

Panel B of Table 5 reports the coefficient estimates when the lower bound $LB^{(1)}_{t\to T}$ is used. The null hypothesis that $(\hat{a}_T, \hat{b}_T) = (0,1)$ cannot be rejected at the 10% significance level for any maturity. The out-of-sample R-squared value (6.7%) is the highest at the 180-day maturity and is approximately equal to 4.6% at the 360-day maturity. The out-of-sample R-squared value is approximately 0 at the 30-day maturity and 1% at the 90-day maturity.

[5]The estimated conditional moments we recover from the data are imperfectly correlated with the true regressors we would ideally like to use. We do recognize that the regressors are estimates of conditional moments from partial data, and this can bias the inference in favor of not rejecting the (0; 1) null.

[6]Note that we designate R-squared values that use full-sample preference parameter estimates to compute the bound at each date as pseudo out-of-sample R-squared values.

In Panel C, we report the coefficient estimates when the lower bound of Martin (2017) is used. Consistent with Martin (2017), we cannot reject the null hypothesis that $(\hat{a}_T, \hat{b}_T) = (0,1)$. The out-of-sample R-squared value is the highest (4.3%) at the 180-day maturity and is approximately 1.3% at the 360-day maturity. Although similar at shorter maturities, the out-of-sample R-squared values obtained using the lower bound $LB^{(1)}_{t\to T}$ and the upper bound $\widetilde{UB}^{(1)}_{t\to T}$ are higher than those using the Martin (2017) bound.

3.4.2. Lower and Upper Bounds Computed without Using Preference Parameters

In this section, we compute the lower and upper bounds without estimating the preference parameters. We use the closed-form expressions of the lower bound (26) and upper bound (33), respectively. The lower bound is labeled $LBR^{(1)}_{t\to T}$, while the upper bound is labeled $\widetilde{UBR}^{(1)}_{t\to T}$. These bounds are almost preference-free in the sense that they do not rely on specific preference parameters, although they do rely on the restrictions imposed by (24). They are functions of the risk-free return, risk-neutral moments of simple returns, and truncated risk-neutral moments of simple returns. The top graph in Figure 6 displays the bounds $LBR^{(1)}_{t\to T}$ and $\widetilde{UBR}^{(1)}_{t\to T}$, respectively. We also show the Martin (2017) bound for comparison.

The conditional expected excess return bounds are similar to the conditional bounds reported in Figure 1. The lower and upper bounds are time-varying, highly volatile and show peaks during periods of turbulence. The graph in the middle shows the difference between the upper and lower bounds. The bottom graph in Figure 6 displays the difference between our bound and the Martin (2017) bound. The difference between the two bounds is more pronounced during turbulent times than during normal times. We obtain similar results for other maturities (see the Internet Appendix).

Table 6 reports the summary statistics. Panel A of this table reports the mean, standard deviation, skewness, kurtosis, and quintiles of the upper bound $\widetilde{UBR}^{(1)}_{t\to T}$. At the 30-day maturity, the mean of the upper bound is approximately 8.52%, while the standard deviation over the whole sample is 8.76. The upper bound is highly skewed (with a skewness of 3.9) and exhibits fatter tails (with a kurtosis of 24.40). These results are comparable to results in Panel A of Table 3. The table also presents similar results for higher maturities.

Panel B of Table 6 presents summary statistics for the lower bound $LBR^{(1)}_{t\to T}$. At the 30-day maturity, the mean of this bound over the whole sample is 5.15%. This bound is highly volatile (with a standard deviation of 5.36%), highly skewed (with a skewness of 4.64), and also exhibits fat tails (with a kurtosis of 35.95). The bound varies from a minimum of 0.70% to a maximum of 70.30%. The lower bound results are also similar to the lower bound results displayed in Panel B of Table 3. The table also presents similar results for higher maturities.

In Panel C of Table 6, we present the summary statistics for the difference between the lower bound $LBR^{(1)}_{t\to T}$ and the lower bound reported in Martin (2017). The mean of the difference in bounds is approximately 0.64%. The standard deviation is 1.11%. The difference in the bounds is highly skewed (with a skewness of approximately 6.75) and exhibits fat tails (with a kurtosis of 69.89). It varies from a minimum of 0.04% to a maximum of 18.41%. The table also presents similar results for higher maturities.

To assess the tightness of the almost preference-free lower bound, we run predictive regressions by using the specification (37), where $X_{t\to T}$ represents either the lower bound $LBR^{(1)}_{t\to T}$ or the upper bound $\widetilde{UBR}^{(1)}_{t\to T}$. Panel A of Table 7 shows the results for the lower bound. The null hypothesis that $(\hat{a}_T, \hat{b}_T) = (0,1)$ cannot be rejected at all maturities with p-values above 0.10. The R-squared values are relatively low at short maturities (30 days to 90 days) and high at the 180-day and 360-day maturities. At the 180-day maturity, the out-of-sample R-squared value of 8.1% is comparable to the in-sample R-squared value of 7.7%. At the 360-day maturity, the out-of-sample R-squared value is 2.8% while the in-sample R-squared value is 1.9%. Panel B of Table 7 presents the results for the upper bound. The null hypothesis that $(\hat{a}_T, \hat{b}_T) = (0,1)$ cannot be rejected at the 10% significance level for any maturity. The out-of-sample R-squared value is approximately 7.1% at the 180-day maturity and 5.8% at the 360-day maturity. In Panel C, we present the forecasting regression results when the Martin (2017) bound is used. The null hypothesis that $(\hat{a}_T, \hat{b}_T) = (0,1)$ cannot be rejected at any maturity. The out-of-sample R-squared value is approximately 4.3% at the 180-day maturity and 1.3% at the 360-day maturity. As with the preference-based bounds, our preference-free bounds help explain more variation in returns than the bound computed using only risk-neutral variance.

3.4.3. Implications for the Conditional Term Structure of Equity Risk Premia

We briefly investigate the implications of our bounds for the term structure of equity risk premia. The left (right) panel in Figure 2 shows time series plots of the lower bound at different maturities when preference parameters are (not) used. One observation from these figures is that the term structure of the lower bound on conditional equity risk premia is upward sloping during normal times and downward slopping during crisis or turbulent periods. This observation is consistent with other recent evidence in Bansal, Miller, and Yaron (2017).[7]

To further illustrate this result, we plot the term structure implied by the lower bound at specific dates and averaged across NBER non-recession and recession periods in Figure 3.[8] This makes it more clear that the term structure is, on average, increasing during normal times and decreasing during turbulent times according to our measure.[9] We also provide the average difference between the term structure implied by our lower bound measure and that implied by the Martin (2017) measure in Figure 4. From this figure, it is clear that our bound implies expected returns that are higher than the Martin (2017) at all maturities. Additionally, our term structure is increasing relatively more with maturity during normal times, and decreasing more with maturity during turbulent times. To the extent that expected returns are relatively high at long horizons during normal times and low at long horizons during turbulent times relative to values implied by the Martin (2017), this could explain the relatively better forecasting power of our bound at longer horizons. In this way, our bounds imply that expected returns mean revert more quickly than what

[7]This result is not necessarily inconsistent with the finding in Gormsen (2017) that the term structure is upward sloping in good times and downward sloping in bad times. We each use different notions of the "term structure". In our case, we measure expected yields to maturity on a single asset (the SP 500), whereas Gormsen (2017) measures holding period returns to assets with different maturities.

[8]Results are similar when using high- and low-volatility periods identified in Sichert (2018) instead of the NBER recession/non-recession periods. We provide plots for these dates similar to those in Figure 3 in the Internet Appendix.

[9]A recent article by Koijen and vanBinsbergen (2017) reviews several papers that have investigated the unconditional term structure of equity risk premia.

is implied by the Martin (2017) bound.

In order to formally test whether the term structure patterns we observe are statistically significant, we consider regressions of the form

$$LB^{(1)}_{t\to t+T} - LB_{t\to t+30} = a \cdot I\{Rec_t\} + b \cdot [1 - I\{Rec_t\}] + \varepsilon^{(1)}_{t\to t+T}, \tag{39}$$

where $I\{Rec_t\}$ is an indicator for whether the economy is in a recession at date t, and $T \in \{60, 90\,180, 360\}$. Results from these regressions can be found below in Table 8, Panel A. The estimated a coefficients are negative at all horizons, consistent with the monotonically decreasing term structure during recessions documented in Figure 3. Additionally, the estimates become statistically significant at standard levels at and above the 90-day horizon, confirming that the decreasing term structure pattern is statistically significant in the sense that the implied (annualized) expected returns are lower at longer horizons than at the 30-day horizon during recessions. The estimated b coefficients are positive and statistically significant at all horizons, consistent with the monotonically increasing term structure during non-recessions documented in Figure 3. The fact that these estimates are statistically significant at standard levels across all horizons confirms that the increasing term structure pattern is statistically significant in the sense that the implied (annualized) expected returns are higher at longer horizons than at the 30-day horizon during non-recession periods.

We also run a block bootstrap to estimate the conditional average differences, $LB^{(1)}_{t\to t+T} - LB_{t\to t+30}$. This provides us with estimated parameter distributions for a and b in regression (39) from which we can test three related null hypotheses. Namely, we test (i) $H_0 : a >= 0$, $H_1 : a < 0$; (ii) $H_0 : b <= 0$, $H_1 : b > 0$; and (iii) $H_0 : a >= 0\, or\, b <= 0$, $H_1 : a < 0\, and\, b > 0$. We compute p-values for these tests by summing the number of estimated bootstrapped parameters that satisfy the null hypotheses. We take 10,000 draws from each series using a block length of three times T and compute the average value of $LB^{(1)}_{t\to t+T} - LB_{t\to t+30}$ conditional on being in a bootstrapped recession period, or a bootstrapped non-recession period. Results to these tests are provided in Table 8, Panel B.

A rejection of the null hypothesis that $a >= 0$ at a specific horizon, T, provides evidence that the expected (annualized) returns at horizon T are lower than those at the 30-day horizon in a statistically significant manner. We cannot reject the null at standard significance levels for $T = 60$, however, we can reject the null for $T = 90$, 180, & 360. These results are consistent with the implications of t-statistics from the regressions, and confirm the decreasing term structure pattern at maturities at and above 90 days during recessions.

A rejection of the null hypothesis that $b <= 0$ at a specific horizon, T, provides evidence that the expected (annualized) returns at horizon T are higher than those at the 30-day horizon in a statistically significant manner. We can reject this null at all horizons and at standard significance levels. These results are consistent with the implications of t-statistics from the regressions, and confirm the increasing term structure pattern during non-recession periods.

A rejection of the null hypothesis that $a >= 0 or b <= 0$ at a specific horizon, T, provides evidence that the term structure is jointly decreasing during recessions and increasing during non-recession periods. We cannot reject the null at standard significance levels for $T = 60$, however, we can reject the null for $T = 90$, 180, & 360. These results are consistent with the implications of t-statistics from the regressions, and are consistent with results from the previous two tests.

Figure 5 presents time series plots of the upper bound on the equity risk premia. The left (right) panel in Figure 5 shows time series plots of the upper bound at different maturities when preference parameters are (not) used. This figure also shows that the term structure implied by our upper bound on conditional equity risk premia is also upward sloping during normal times and downward sloping during crisis or turbulent periods.

These results are consistent with a world in which expected returns follow a mean-reverting process, and are high during turbulent times and relatively low during normal times. In this sense, our conditional term structure results are consistent with standard equilibrium asset pricing models where expected returns are mean reverting such as those based on habit preferences (Campbell and Cochrane (1999)) and long-run risk (Bansal and Yaron (2004)). Additionally, our finding that the unconditional term structure is only slightly upward sloping can be used to help discipline the calibration of such models, however, this finding is less strong than our conditional results due to the relatively short time period for which we have data and the fact that a deep recession was realized during this period.

4. Further Discussion

4.1. The Negative Correlation Condition (NCC)

Martin (2017) uses the NCC assumption that $\mathbb{COV}_t\left(M_{t\to T}R_{M,t\to T}, R_{M,t\to T}\right) \leq 0$ to derive the expected excess return on the market. Martin (2017) provides a number of examples of standard theoretical models in which the NCC holds. He also cannot reject the null that his bound is tight (this provides evidence that $\mathbb{COV}\left(M_{t\to T}R_{M,t\to T}, R_{M,t\to T}\right) \simeq 0$ unconditionally). Our results indicate that the conditional covariance $\mathbb{COV}_t\left(M_{t\to T}R_{M,t\to T}, R_{M,t\to T}\right)$ is time-varying and is pronounced during crisis periods or turbulent periods. This plays a key role in explaining the difference between the lower bound on the conditional expected excess market return and the conditional lower bound derived in Martin (2017). To understand our argument, note that

$$\mathbb{COV}_t\left(M_{t\to T}R_{M,t\to T}, R_{M,t\to T}\right) = \frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T} - \left(\mathbb{E}_t\left(R_{M,t\to T}\right) - R_{f,t\to T}\right). \tag{40}$$

Using the lower bound on the expected excess return provided in (19), we provide an upper bound on the covariance term

$$\mathbb{COV}_t\left(M_{t\to T}R_{M,t\to T}, R_{M,t\to T}\right) \leq \frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T} - LB^{(1)}_{t\to T}. \tag{41}$$

Similarly, we use the upper bound on the expected excess market return (6) to derive a lower bound on the covariance term

$$\mathbb{COV}_t\left(M_{t\to T}R_{M,t\to T}, R_{M,t\to T}\right) \geq \frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T} - \widetilde{UB}^{(1)}_{t\to T}. \tag{42}$$

Together, (41) and (42) produce bounds on the covariance term

$$\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}-\widetilde{UB}^{(1)}_{t\to T}\leq \mathbb{COV}_t\left(M_{t\to T}R_{M,t\to T},R_{M,t\to T}\right)\leq \frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}-LB^{(1)}_{t\to T}. \tag{43}$$

These bounds are plotted for the 30-day maturity in Figure 8. Covariance bounds for other maturities are provided in the Online Appendix. These plots confirm that the covariance term is time-varying, and is particularly large in magnitude during turbulent periods. Therefore, accounting for higher moments is likely important in achieving tight bounds on the market expected return, particularly during these times. Summary statistics for these bounds are provided in Table 9. These summary statistics indicate that the average risk premium is between 0.06% and 0.30% higher than indicated by the risk-neutral variance alone at the 30-day maturity (0.6% and 3.6% annualized). Similarly, the summary statistics indicate that the expected excess market return is between 1.3% and 3.3% higher at the 360-day maturity. The bounds are negatively skewed indicating that the expected excess market return is more positively skewed than the risk-neutral variance implies.

4.2. *Conditional Expected Excess Log Market Returns*

A number of asset pricing models have focussed on the expected excess log market returns rather than expected excess simple market return. With this in mind, we derive some additional results using log returns. We use a Taylor expansion series of $\log\left(R_{M,t\to T}\right)$ around $R_{f,t\to T}$ to show the following:

Result 7 *The conditional expected excess log market return admits a lower bound,*

$$\mathbb{E}_t\left(\log\left(R_{M,t\to T}\right)-\log\left(R_{f,t\to T}\right)\right)\geq \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{nR^n_{f,t\to T}}\left(B^{(n)}_{t\to T}+\mathbb{M}^{*(n)}_{t\to T}\right). \tag{44}$$

Up to the fourth order approximation of $\log\left(R_{M,t\to T}\right)$ *around* $\log\left(R_{f,t\to T}\right)$*,*

$$\mathbb{E}_t\left(\log\left(R_{M,t\to T}\right)-\log\left(R_{f,t\to T}\right)\right)\geq LLB_{t\to T}=\sum_{n=1}^{4}\frac{(-1)^{n+1}}{nR^n_{f,t\to T}}\left(B^{(n)}_{t\to T}+\mathbb{M}^{*(n)}_{t\to T}\right). \tag{45}$$

Under the preference restrictions (24),

$$\mathbb{E}_t\left(\log\left(R_{M,t\to T}\right)-\log\left(R_{f,t\to T}\right)\right)\geq LLBR_{t\to T}=\sum_{n=1}^{4}\frac{(-1)^{n+1}}{nR_{f,t\to T}^{n}}\left(BR_{t\to T}^{(n)}+\mathbb{M}_{t\to T}^{*(n)}\right). \tag{46}$$

Where $BR_{t\to T}^{(n)}$ represents the restricted bounds.

Figure 7 presents the lower bound on the conditional expected excess log return. The bounds are annualized and reported in percentages. The graph in the left panel of Figure 7 displays the estimated bounds when parameter values in the last column of Table 1 are used. The graph in the right panel of Figure 7 displays the estimated bounds without using any preference parameters. In both cases, the lower bounds on the expected excess return are positive, time-varying, highly volatile, skewed, and exhibit fat tails. Table 10 presents the summary statistics. Panel A of Table 10 reports summary statistics of the lower bound when (45) is used. At the 30-day maturity, the unconditional average of the lower bound is approximately 3.21%. The bound varies significantly, having a standard deviation of 3.55%. The lower bound varies from a minimum of 0.40% to a maximum of 47.65% . The bound exhibits positive skewness (with a skewness of 4.84) and fat tails (with a kurtosis of 38.84). Results are similar when other maturities are used. Panel B of Table 10 reports summary statistics of the lower bound when the bound in identity (46) is used. At the 30-day maturity, the unconditional average of the conditional lower bound is approximately 3.14%. The bound varies significantly, having a standard deviation of 3.73%. It varies from a minimum of 0.38% to a maximum of 52.17% . The bound exhibits positive skewness (with a skewness of 5.32) and fat tails (with a kurtosis of 42.41). We obtain similar results for longer maturities.

Since (45) and (46) represent bounds on the expected excess log return, one may ask whether these bounds are tight. To address this issue, we investigate the tightness of the bounds by running the forecasting regressions:

$$\log(R_{M,t\to T})-\log(R_{f,t\to T})=a_T+b_T LLB_t+e_{t,T} \tag{47}$$

and

$$\log(R_{M,t\to T}) - \log(R_{f,t\to T}) = a_T + b_T LLBR_t + e_{t,T}, \tag{48}$$

respectively. Panel A of Table 11 presents estimated coefficients when the specification (47) is used. The null hypothesis that $(a_T, b_T) = (0,1)$ is only rejected for the 180-day maturity. The adjusted R-squared value is 8% at the 180-day maturity and approximately 5.4% at the 360-day maturity. The out-of-sample R-squared value is approximately 5.5% (5.3%) at the 180-day (360-day) maturity. Panel B of Table 11 presents estimated coefficients when the specification (48) is used. The null hypothesis that $(a_T, b_T) = (0,1)$ cannot be rejected for all maturities except for the 180-day maturity. The adjusted R-squared value is 7.7% at the 180-day maturity and approximately 5.7% at the 360-day maturity. The out-of-sample R-squared value is approximately 6.1% (6.4%) at the 180-day (360-day) maturity. The results in Panels A and B of Table 11 suggest that the bounds on the expected excess log returns (45)–(46) are tight.

4.3. Forecasting Squared Excess Market Returns and Return Variance

There is a large literature related to volatility forecasting. We do not attempt to summarize it here, but we note that Result 4 can be used to compute an upper bound on expected squared excess market return (or any power of excess market returns for that matter), which motivates the following forecasting regression:

$$(R_{M,t\to T} - R_{f,t\to T})^2 = a_T + b_T \left(\mathbb{M}^{*(2)}_{t\to T} + UB^{(2)}_{t\to T} \right) + e_{t,T}, \tag{49}$$

where a_T and b_T are maturity-specific estimates. Table 12 summarizes the regression results at all horizons using the estimated preference parameters from Table 1 to compute θ_k (Panel A). Results are also provided when imposing restrictions in (32) (Panel B). At each horizon, the measure from Result 4 is able to explain a relatively large fraction of the variation in squared excess market returns with in-sample R-squared values ranging from 7.5-12.6% across both specifications. Although we reject the expectations hypothesis at shorter horizons (30, 60, and 90 days), we cannot reject it at longer horizons (180 and 360 days). These results suggest that the upper bound on the expected square excess market return is tight for longer horizons.

We can also use Result 4 to construct an upper bound for the market variance over any horizon

$$VAR_t\left(R_{M,t\to T}\right) = VAR_t\left(R_{M,t\to T} - R_{f,t\to T}\right) = \mathbb{M}^{(2)}_{t\to T} - \left(\mathbb{M}^{(1)}_{t\to T}\right)^2.$$

Since $UB^{(2)}_{t\to T}$ is an upper bound and $LB^{(1)}_{t\to T}$ is a lower bound, then a variance upper bound can be constructed as $\mathbb{M}^{*(2)}_{t\to T} + UB^{(2)}_{t\to T} - \left[LB^{(1)}_{t\to T}\right]^2$. We construct a measure of realized market variance by summing squared daily returns over any horizon, which we denote by $RVAR_{t\to T}$ and then run the following forecasting regressions:

$$RVAR_{t\to T} = a_T + b_T\left(\mathbb{M}^{*(2)}_{t\to T} + UB^{(2)}_{t\to T} - \left[LB^{(1)}_{t\to T}\right]^2\right) + e_{t,T}, \tag{50}$$

where a_T and b_T are forecasting period-specific estimates. Table 13 summarizes the regression results at all horizons using the estimated preference parameters from Table 1 to compute θ_k (Panel A). Results are also provided when imposing restrictions in (32) (Panel B). Under both specifications, the explained (in-sample) variation decreases with horizon, which is related to the persistence of volatility at shorter horizons. We can only reject the expectations hypothesis at the 5% significance level in the case of the 30-day horizon under the restricted (Panel B) specification. We reject the expectations hypothesis at the 10% significance level in the case of the 30-day horizon under both specifications. These results indicate that the variance upper bound is tight for all but the shortest maturities investigated.

4.4. A Link with Recursive Preference Framework

Although our main results depend on the single-period economy assumption, we can derive similar results in a recursive utility framework. For example, if we assume that the market return is equivalent to the return on the aggregate consumption claim in the recursive utility framework from Epstein and Zin

(1989), then the SDF has the form

$$m_{t\to T} = \upsilon^{'}\left[c_{t\to T}\right]\vartheta^{'}\left[R_{M,t\to T}\right], \tag{51}$$

where $c_{t\to T}$ is the consumption growth. In Epstein and Zin (1989), $\upsilon^{'}$ and $\vartheta^{'}$ are given by

$$\upsilon^{'}\left[c_{t\to T}\right] = \beta^{*\theta^{*}}\left(c_{t\to T}\right)^{-\frac{\theta^{*}}{\psi^{*}}} \text{ and } \vartheta^{'}\left[R_{M,t\to T}\right] = R_{M,t\to T}^{\theta^{*}-1} \text{ with } \theta^{*} = \frac{1-\gamma^{*}}{1-\rho^{*}}$$

respectively. Here $\rho^{*} = \frac{1}{\psi^{*}} \geq 0$ where ψ^{*} is the Elasticity of Inter-temporal Substitution (EIS), γ^{*} is the relative risk aversion, and β^{*} is the subjective discount factor.

Result 8 *Under the Epstein and Zin (1989) preferences,*

$$\mathbb{E}_t\left[R_{M,t\to T}\right] - R_{f,t\to T} = \mathbb{COV}_t^{*}\left[\frac{\frac{1}{\vartheta^{'}\left[R_{M,t\to T}\right]}}{\mathbb{E}_t^{*}\left[\frac{1}{\vartheta^{'}\left[R_{M,t\to T}\right]}\right]}, R_{M,t\to T}\right] - \mathbb{COV}_t^{*}\left(m_{t\to T}^{P}, R_{M,t\to T}\right). \tag{52}$$

where $m_{t\to T}^{P}$ and $m_{t\to T}^{T}$ are defined by

$$m_{t\to T}^{P} = \frac{\upsilon^{'}\left[c_{t\to T}\right]}{\mathbb{E}_t\left[\upsilon^{'}\left[c_{t\to T}\right]\right]} \text{ and } m_{t\to T}^{T} = \frac{1}{R_{f,t\to T}}\mathbb{E}_t^{*}\left[\left(\vartheta^{'}\left[R_{M,t\to T}\right]\right)^{-1}\right]\vartheta^{'}\left[R_{M,t\to T}\right].$$

and the SDF can be decomposed as

$$m_{t\to T} = m_{t\to T}^{P} \times m_{t\to T}^{T}.$$

Provided that the covariance $\mathbb{COV}_t^{*}\left(m_{t\to T}^{P}, R_{M,t\to T}\right)$ under the risk neutral measure is negative, it follows that

$$\mathbb{E}_t\left[R_{M,t\to T}\right] - R_{f,t\to T} \geq \mathbb{COV}_t^{*}\left[\frac{\frac{1}{\vartheta^{'}\left[R_{M,t\to T}\right]}}{\mathbb{E}_t^{*}\left[\frac{1}{\vartheta^{'}\left[R_{M,t\to T}\right]}\right]}, R_{M,t\to T}\right]. \tag{53}$$

The right hand side of this inequality is a bound on the expected excess market return and is similar to our Results 2 and 3. Since consumption does not co-move contemporaneously with the stock market return under the physical measure (e.g., see Hall (1978)), $\mathbb{COV}_t\left(m_{t\to T}^{P}, R_{M,t\to T}\right) = 0$, one should expect

$\mathbb{COV}_t^*\left(m_{t\to T}^P, R_{M,t\to T}\right)$ to be close to zero. Thus, the expected excess market return in recursive utility frameworks is bounded by the right hand side of (53).

5. Conclusion

We have presented theoretical and empirical results on the conditional expected excess market return and conditional expected excess log market returns that exploited information from the whole risk-neutral distribution of the market return to build our theory.

We derive theoretical lower and upper bounds on the conditional expected excess market return without making any assumptions about the distribution of the market return. Our bounds do require the assumption of no arbitrage and, in the case of the restricted bounds, mild assumptions about preference parameters. The bounds can be computed in real time by using a cross-section of option prices. Empirical results indicate that the bounds are highly volatile, are positively skewed, and exhibit fat tails. Our estimate of the bounds are tight and, hence, can be used as a measure of the conditional expected excess market return. The estimated lower bound on the conditional expected excess return is approximately 5.2% on average. We also show that the upper bound on the conditional expected excess return is approximately 8.5% on average.

We theoretically and empirically show that the lower bound derived in this paper is significantly different, over time and across maturities, than the bound derived in Martin (2017). The difference between the lower bound and Martin's bound is more pronounced during turbulent periods than normal periods. Our bound has similar forecasting power as the Martin (2017) bound at 1-3 month horizons as measured by the R-squared value, however, our bound performs significantly better at horizons of 6 and 12 months.

We also find that our bound implies a counter-cyclical term structure for the expected excess market return. During normal times the term structure is increasing, and during turbulent times it is decreasing. The term structure slopes are also larger than those implied by the Martin (2017) bound, which could be a source of our bound's better forecasting power at longer horizons.

We further derive a lower bound on the expected excess log market return and empirically show that it varies over time, is positively skewed, and exhibits fat tails. The bound on the expected excess log return is always lower than the bound on the expected excess simple return. The estimates of the conditional expected excess log return lower bound are tight; that is, it can be used to better approximate the conditional expected excess log return. On average, the conditional expected excess log return is approximately 3%. Our bounds are useful for computing the conditional expected excess market return in real time and also testing a large number of asset pricing models that have focussed in estimating the expected excess market return.

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A. Appendix

Proof of Result 1. The expected return on any asset i can be expressed as

$$\begin{aligned}
\mathbb{E}_t\left(R_{i,t\to T}\right) &= \mathbb{E}_t\left(M_{t\to T}\frac{R_{i,t\to T}}{M_{t\to T}}\right) \\
&= \left(\mathbb{E}_t\left(M_{t\to T}\right)\right)\mathbb{E}_t\left(\frac{M_{t\to T}}{\mathbb{E}_t\left(M_{t\to T}\right)}\frac{R_{i,t\to T}}{M_{t\to T}}\right) \\
&= \left(\mathbb{E}_t\left(M_{t\to T}\right)\right)\mathbb{E}_t^*\left(\frac{R_{i,t\to T}}{M_{t\to T}}\right) \\
&= \frac{1}{R_{f,t\to T}}Cov_t^*\left(R_{i,t\to T},\frac{1}{M_{t\to T}}\right)+R_{f,t\to T}\ \text{(since } \mathbb{E}_t\left(M_{t\to T}\right)=1/R_{f,t\to T}\text{)}
\end{aligned}$$

■

Proof of Result 2. The first-order condition (FOC) takes the form

$$\mathbb{E}_t\left(u^{'}\left[W_T\right]\left(R_{i,t\to T}-R_{f,t\to T}\right)\right)=0, \tag{A1}$$

which implies that the SDF has the form

$$M_{t\to T}=\frac{u^{'}\left[W_T\right]}{R_{f,t\to T}\mathbb{E}_t\left(u^{'}\left[W_T\right]\right)}. \tag{A2}$$

As a result, the SDF (A2) has the form

$$M_{t\to T}=\zeta_t u^{'}\left[W_t R_{M,t\to T}\right]\ \text{with}\ \varsigma_t=\frac{1}{R_{f,t\to T}\mathbb{E}_t\left(u^{'}\left[W_t R_{M,t\to T}\right]\right)}. \tag{A3}$$

Observe that the time-varying parameter ζ_t is important in quantifying the behavior of the SDF.

$$M_{t\to T}\left(u^{'}\left[W_t R_{M,t\to T}\right]\right)^{-1}=\zeta_t,$$

and its expected value (since it is a constant) equals

$$(\mathbb{E}_t [M_{t\to T}])\, \mathbb{E}_t \left[\frac{M_{t\to T}}{(\mathbb{E}_t [M_{t\to T}])} \left(u^{'} [W_t R_{M,t\to T}] \right)^{-1} \right] = \zeta_t,$$

which, after using the Radon Nikodym transformation measure, can be simplified to

$$\frac{1}{R_{f,t\to T}} \mathbb{E}_t^* \left[\left(u^{'} [W_t R_{M,t\to T}] \right)^{-1} \right] = \zeta_t.$$

Thus, the SDF simplifies to

$$M_{t\to T} = \frac{1}{R_{f,t\to T}} \mathbb{E}_t^* \left[\left(u^{'} [W_t R_{M,t\to T}] \right)^{-1} \right] u^{'} [W_t R_{M,t\to T}]. \tag{A4}$$

We then replace the expression of the SDF into the expected excess return formula in Result 1 and show that

$$\mathbb{E}_t (R_{i,t\to T}) - R_{f,t\to T} = Cov_t^* \left(R_{i,t\to T}, \frac{\frac{u^{'}[W_t R_{f,t\to T}]}{u^{'}[W_t R_{M,t\to T}]}}{\mathbb{E}_t^* \left(\frac{u^{'}[W_t R_{f,t\to T}]}{u^{'}[W_t R_{M,t\to T}]} \right)} \right). \tag{A5}$$

■

Proof of the Result 3. More generally, we define the return on n^{th} moments as

$$R_{i,t\to T} = \frac{(R_{M,t\to T} - R_{f,t\to T})^n}{\mathbb{E}_t [M_{t\to T} (R_{M,t\to T} - R_{f,t\to T})^n]} \text{ for } n > 0. \tag{A6}$$

This return can, alternatively, be written as

$$R_{i,t\to T} = R_{f,t\to T} \frac{(R_{M,t\to T} - R_{f,t\to T})^n}{\mathbb{E}_t^* [(R_{M,t\to T} - R_{f,t\to T})^n]}. \tag{A7}$$

Together, (A7) with (4) yields (after simplifications)

$$\mathbb{M}^{(n)}_{t\to T} - \mathbb{M}^{*(n)}_{t\to T} = Cov^*_t\left(\left(R_{M,t\to T} - R_{f,t\to T}\right)^n, \frac{\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}}{\mathbb{E}^*_t\left(\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}\right)} \right). \tag{A8}$$

■

Making the Case for Assumption 2. Result 2 applied to the market return shows that the expected excess return on the market is

$$\begin{aligned}
\mathbb{E}_t\left(R_{M,t\to T}\right) - R_{f,t\to T} &= Cov^*_t\left(R_{M,t\to T}, \frac{\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}}{\mathbb{E}^*_t\left(\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}\right)} \right). \\
&= \frac{1}{\mathbb{E}^*_t\left(\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}\right)} Cov^*_t\left(R_{M,t\to T}, \frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]} \right)
\end{aligned}$$

Using the approximation (A11), the expected excess market return is

$$\mathbb{E}_t\left(R_{M,t\to T}\right) - R_{f,t\to T} = \lambda^*_t \theta_1 \mathbb{M}^{*(2)}_{t\to T} + \lambda^*_t \theta_2 \mathbb{M}^{*(3)}_{t\to T} + \lambda^*_t \theta_3 \mathbb{M}^{*(4)}_{t\to T} \tag{A9}$$

with

$$\lambda^*_t = \frac{1}{\mathbb{E}^*_t\left(\frac{u'\left[W_t R_{f,t\to T}\right]}{u'\left[W_t R_{M,t\to T}\right]}\right)} > 0 \tag{A10}$$

Assume that Assumption 1 holds. Then investors receive compensation for exposure to risk neutral moments $\mathbb{M}^{*(2)}_{t\to T}$, $\mathbb{M}^{*(3)}_{t\to T}$, and $\mathbb{M}^{*(4)}_{t\to T}$ if $\theta_1 > 0$, $\theta_2 < 0$, and $\theta_3 > 0$. ■

Proof of Result 4. Recall that the identity (5) is

$$\mathbb{E}_t\left(\left(R_{M,t\to T} - R_{f,t\to T}\right)^n\right) - \mathbb{E}^*_t\left[\left(R_{M,t\to T} - R_{f,t\to T}\right)^n\right] = \frac{\sum_{k=1}^{\infty} \theta_k Cov^*_t\left(\left(R_{M,t\to T} - R_{f,t\to T}\right)^n, \left(R_{M,t\to T} - R_{f,t\to T}\right)^k\right)}{1 + \sum_{k=1}^{\infty} \theta_k \mathbb{E}^*_t\left(\left(R_{M,t\to T} - R_{f,t\to T}\right)^k\right)}.$$

Now, let's determine θ_1, θ_2, and θ_3. Observe that the Taylor expansion series of $\frac{u'[W_t x_0]}{u'[W_t x]}$ around $x_0 = R_{f,t\to T}$ gives

$$\begin{aligned}\frac{u'[W_t x_0]}{u'[W_t x]} &\simeq 1 - \frac{1}{1!}\frac{W_t u''[W_t x_0]}{u'[W_t x_0]}(x - x_0)\\ &+\frac{1}{2!}W_t^2\frac{\left(2\left(u''[W_t x_0]\right)^2 - u'''[W_t x_0]\,u'[W_t x_0]\right)}{\left(u'[W_t x_0]\right)^2}(x-x_0)^2\\ &+\frac{1}{3!}W_t^3\frac{\left\{\begin{array}{c}\left(3u''[W_t x_0]\,u'''[W_t x_0] - u''''[W_t x_0]\,u'[W_t x_0]\right)\left(u'[W_t x_0]\right)\\ -3\left(2\left(u''[W_t x_0]\right)^2 - u'''[W_t x_0]\,u'[W_t x_0]\right)u''[W_t x_0]\end{array}\right\}}{\left(u'[W_t x_0]\right)^3}(x-x_0)^3+\dots.\end{aligned}$$

This expression can be simplified as

$$\frac{u'[W_t R_{f,t\to T}]}{u'[W_t x]} = 1 + \frac{1}{\tau R_{f,t\to T}}(x - R_{f,t\to T}) + \frac{(1-\rho)}{\tau^2 R_{f,t\to T}^2}(x - R_{f,t\to T})^2 \tag{A11}$$

$$+\frac{(1-2\rho+\kappa)}{\tau^3 R_{f,t\to T}^3}(x - R_{f,t\to T})^3 + \dots. \tag{A12}$$

where

$$\tau = -\frac{u'[W_t R_{f,t\to T}]}{W_t R_{f,t\to T}\,u''[W_t R_{f,t\to T}]}, \tag{A13}$$

$$\rho = \frac{1}{2!}\frac{W_t R_{f,t\to T}\,u'''[W_t R_{f,t\to T}]}{u''[W_t R_{f,t\to T}]}\frac{u'[W_t R_{f,t\to T}]}{W_t R_{f,t\to T}\,u''[W_t R_{f,t\to T}]}, \textit{and} \tag{A14}$$

$$\kappa = \frac{1}{3!}\frac{W_t R_{f,t\to T}\,u''''[W_t R_{f,t\to T}]}{u'''[W_t R_{f,t\to T}]}\frac{W_t R_{f,t\to T}\,u'''[W_t R_{f,t\to T}]}{u''[W_t R_{f,t\to T}]}\left(\frac{u'[W_t R_{f,t\to T}]}{W_t R_{f,t\to T}\,u''[W_t R_{f,t\to T}]}\right)^2. \tag{A15}$$

Thus, the Taylor expansion series of $f[x]$ around $x = x_0$ yields

$$f[x] = 1 + \theta_1(x - x_0) + \theta_2(x-x_0)^2 + \theta_3(x-x_0)^3 + \dots,$$

where

$$\theta_1 = \frac{1}{\tau R_{f,t\to T}},\ \theta_2 = \frac{(1-\rho)}{\tau^2 R_{f,t\to T}^2},\ \theta_3 = \frac{(1-2\rho+\kappa)}{\tau^3 R_{f,t\to T}^3}$$

Observe that

$$\mathbb{E}_t\left((R_{M,t\to T} - R_{f,t\to T})^n\right) - \mathbb{E}_t^*\left[(R_{M,t\to T} - R_{f,t\to T})^n\right] = \frac{\sum_{k=1}^{\infty}\theta_k\left(\mathbb{M}_{t\to T}^{*(n+k)} - \mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)}\right)}{1+\sum_{k=1}^{\infty}\theta_k\mathbb{M}_{t\to T}^{*(k)}}.$$

Under Assumptions 1 and 2,

$$\theta_k\mathbb{M}_{t\to T}^{*(k)} \leq 0 \text{ for any } k.$$

and

$$\begin{aligned}
\mathbb{COV}_t^*\left((R_{M,t\to T} - R_{f,t\to T})^n, (R_{M,t\to T} - R_{f,t\to T})^k\right) &\geq 0 \quad \text{if } n \text{ and } k \text{ are odd or if they are both even, and} \\
\mathbb{COV}_t^*\left((R_{M,t\to T} - R_{f,t\to T})^n, (R_{M,t\to T} - R_{f,t\to T})^k\right) &\leq 0 \quad \text{if } n \text{ is odd and } k \text{ is even or if } n \text{ is even and } k \text{ is odd.}
\end{aligned}$$

Next,

- Assume that n is odd. If k is odd, $\theta_k \geq 0$, and

  $$\mathbb{M}_{t\to T}^{*(n+k)} - \mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)} = \mathbb{COV}_t^*\left((R_{M,t\to T} - R_{f,t\to T})^n, (R_{M,t\to T} - R_{f,t\to T})^k\right) \geq 0.$$

  This implies

  $$\theta_k\left(\mathbb{M}_{t\to T}^{*(n+k)} - \mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)}\right) \geq 0 \text{ for any } k.$$

- Assume that n is odd. If k is even, $\theta_k \leq 0$, and

  $$\mathbb{M}_{t\to T}^{*(n+k)} - \mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)} = \mathbb{COV}_t^*\left((R_{M,t\to T} - R_{f,t\to T})^n, (R_{M,t\to T} - R_{f,t\to T})^k\right) \leq 0.$$

  This implies

  $$\theta_k\left(\mathbb{M}_{t\to T}^{*(n+k)} - \mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)}\right) \geq 0 \text{ for any } k.$$

Therefore, if n is odd, all terms $\theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right)$ are positive for $k = 1, \ldots$.

Hence,

$$\begin{aligned} \sum_{k=1}^{\infty} \theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right) &\geq \sum_{k=1}^{3} \theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right) \text{ and} \\ 1 + \sum_{k=1}^{\infty} \theta_k \mathbb{M}^{*(k)}_{t \to T} &\leq 1 + \sum_{k=1}^{3} \theta_k \mathbb{M}^{*(k)}_{t \to T}. \end{aligned}$$

It follows, from the above two inequalities, that

$$\mathbb{M}^{(n)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} = \frac{\sum_{k=1}^{\infty} \theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right)}{1 + \sum_{k=1}^{\infty} \theta_k \mathbb{M}^{*(k)}_{t \to T}} \geq \frac{\sum_{k=1}^{3} \theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right)}{1 + \sum_{k=1}^{3} \theta_k \mathbb{M}^{*(k)}_{t \to T}}.$$

Now,

- Assume that n is even. If k is odd, $\theta_k \geq 0$, and

$$\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} = \mathbb{COV}^{*}_{t} \left(\left(R_{M,t \to T} - R_{f,t \to T} \right)^{n}, \left(R_{M,t \to T} - R_{f,t \to T} \right)^{k} \right) \leq 0.$$

Thus,

$$\theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right) \leq 0 \text{ for any } k.$$

- Assume that n is even. If k is even, $\theta_k \leq 0$, and

$$\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} = \mathbb{COV}^{*}_{t} \left(\left(R_{M,t \to T} - R_{f,t \to T} \right)^{n}, \left(R_{M,t \to T} - R_{f,t \to T} \right)^{k} \right) \geq 0.$$

Thus,

$$\theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right) \leq 0 \text{ for any } k.$$

Therefore, if n is odd, all terms $\theta_k \left(\mathbb{M}^{*(n+k)}_{t \to T} - \mathbb{M}^{*(n)}_{t \to T} \mathbb{M}^{*(k)}_{t \to T} \right)$ are positive for $k = 1, \ldots$.

Hence,

$$\begin{aligned}\sum_{k=1}^{\infty}\theta_k\left(\mathbb{M}_{t\to T}^{*(n+k)}-\mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)}\right) &\leq \sum_{k=1}^{3}\theta_k\left(\mathbb{M}_{t\to T}^{*(n+k)}-\mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)}\right)\leq 0\\ 1+\sum_{k=1}^{\infty}\theta_k\mathbb{M}_{t\to T}^{*(k)} &\leq 1+\sum_{k=1}^{3}\theta_k\mathbb{M}_{t\to T}^{*(k)}.\end{aligned}$$

It follows that

$$\mathbb{M}_{t\to T}^{(n)}-\mathbb{M}_{t\to T}^{*(n)}=\frac{\sum_{k=1}^{\infty}\theta_k\left(\mathbb{M}_{t\to T}^{*(n+k)}-\mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)}\right)}{1+\sum_{k=1}^{\infty}\theta_k\mathbb{M}_{t\to T}^{*(k)}}\leq\frac{\sum_{k=1}^{3}\theta_k\left(\mathbb{M}_{t\to T}^{*(n+k)}-\mathbb{M}_{t\to T}^{*(n)}\mathbb{M}_{t\to T}^{*(k)}\right)}{1+\sum_{k=1}^{3}\theta_k\mathbb{M}_{t\to T}^{*(k)}}.$$

This ends the proof. ■

Proof of Result 5. From Result 4, it follows that

$$\mathbb{E}_t\left(R_{M,t\to T}-R_{f,t\to T}\right)\geq\frac{\sum_{k=1}^{3}\theta_k\mathbb{M}_{t\to T}^{*(k+1)}}{1+\sum_{k=1}^{3}\theta_k\mathbb{M}_{t\to T}^{*(k)}}.$$

Using the preference restrictions in (24), it follows that

$$\theta_1\geq\frac{1}{R_{f,t\to T}},\ \theta_2\leq-\frac{1}{R_{f,t\to T}^2},\text{ and }\theta_3\geq\frac{1}{R_{f,t\to T}^3}.$$

Since

$$\mathbb{M}_{t\to T}^{*(2)}\geq 0,\ \mathbb{M}_{t\to T}^{*(3)}\leq 0,\text{ and }\mathbb{M}_{t\to T}^{*(4)}\geq 0,$$

it follows that

$$\theta_1\mathbb{M}_{t\to T}^{*(2)}\geq\frac{1}{R_{f,t\to T}}\mathbb{M}_{t\to T}^{*(2)},\ \theta_2\mathbb{M}_{t\to T}^{*(3)}\geq-\frac{1}{R_{f,t\to T}^2}\mathbb{M}_{t\to T}^{*(3)},\text{ and }\theta_3\mathbb{M}_{t\to T}^{*(4)}\geq\frac{1}{R_{f,t\to T}^3}\mathbb{M}_{t\to T}^{*(4)}$$

and

$$\theta_1 \mathbb{M}^{*(1)}_{t\to T} = 0,\ \theta_2 \mathbb{M}^{*(2)}_{t\to T} \leq -\frac{1}{R^2_{f,t\to T}} \mathbb{M}^{*(2)}_{t\to T},\ \text{and}\ \theta_3 \leq \frac{1}{R^3_{f,t\to T}} \mathbb{M}^{*(3)}_{t\to T}.$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{3} \theta_k \mathbb{M}^{*(k+1)}_{t\to T} &\geq \frac{1}{R_{f,t\to T}} \mathbb{M}^{*(2)}_{t\to T} - \frac{1}{R^2_{f,t\to T}} \mathbb{M}^{*(3)}_{t\to T} + \frac{1}{R^3_{f,t\to T}} \mathbb{M}^{*(4)}_{t\to T} \text{ and} && \text{(A16)} \\
\sum_{k=1}^{3} \theta_k \mathbb{M}^{*(k)}_{t\to T} &\leq -\frac{1}{R^2_{f,t\to T}} \mathbb{M}^{*(2)}_{t\to T} + \frac{1}{R^3_{f,t\to T}} \mathbb{M}^{*(3)}_{t\to T}. && \text{(A17)}
\end{aligned}$$

Together, (A16) and (A17) imply

$$\mathbb{E}_t\left(R_{M,t\to T} - R_{f,t\to T}\right) \geq \frac{\frac{1}{R_{f,t\to T}} \mathbb{M}^{*(2)}_{t\to T} - \frac{1}{R^2_{f,t\to T}} \mathbb{M}^{*(3)}_{t\to T} + \frac{1}{R^3_{f,t\to T}} \mathbb{M}^{*(4)}_{t\to T}}{1 - \frac{1}{R^2_{f,t\to T}} \mathbb{M}^{*(2)}_{t\to T} + \frac{1}{R^3_{f,t\to T}} \mathbb{M}^{*(3)}_{t\to T}}.$$

This ends the proof. ■

Proof of Result 6. We use the following identity to prove our results:

$$\begin{aligned}
\left(R_{M,t\to T} - R_{f,t\to T}\right) &= \left(R_{M,t\to T} - R_{f,t\to T}\right) 1_{R_{M,t\to T} > k_0} + \left(R_{M,t\to T} - R_{f,t\to T}\right) 1_{R_{M,t\to T} < k_0} \\
&= \left(R_{M,t\to T} - R_{f,t\to T}\right) 1_{R_{M,t\to T} > k_0} + \left(R_{M,t\to T} - k_0\right) 1_{R_{M,t\to T} < k_0} + \left(k_0 - R_{f,t\to T}\right) 1_{R_{M,t\to T} < k_0}.
\end{aligned}$$

Thus, the expected value of the expression above is

$$\mathbb{E}_t\left(R_{M,t\to T} - R_{f,t\to T}\right) = \mathbb{E}_t\left(\left(R_{M,t\to T} - R_{f,t\to T}\right) 1_{R_{M,t\to T} > k_0}\right) + \mathbb{E}_t\left(\left(R_{M,t\to T} - k_0\right) 1_{R_{M,t\to T} < k_0}\right) + \mathbb{E}_t\left(\left(k_0 - R_{f,t\to T}\right) 1_{R_{M,t\to T} < k_0}\right).$$

Since $k_0 \leq R_{f,t\to T}$, it follows that

$$\mathbb{E}_t\left(\left(k_0 - R_{f,t\to T}\right) 1_{R_{M,t\to T} < k_0}\right) \leq 0, \text{ and} \mathbb{E}_t\left(\left(R_{M,t\to T} - k_0\right) 1_{R_{M,t\to T} < k_0}\right) \leq 0. \qquad \text{(A18)}$$

Hence,

$$\mathbb{E}_t\left(R_{M,t\to T}-R_{f,t\to T}\right)\leq\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right).$$

Now, we compute $\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right)$. We have

$$\begin{aligned}\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right) &= \left(\mathbb{E}_t\left(M_{t\to T}\right)\right)\mathbb{E}_t\left(\frac{M_{t\to T}}{\mathbb{E}_t\left(M_{t\to T}\right)}\frac{1}{M_{t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right)\\ &= \left(\mathbb{E}_t\left(M_{t\to T}\right)\right)\mathbb{E}_t^*\left(\frac{1}{M_{t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right)\\ &= \frac{1}{R_{f,t\to T}}\mathbb{E}_t^*\left(\frac{1}{M_{t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right). \qquad \text{(A19)}\end{aligned}$$

Observe that

$$\frac{1}{M_{t\to T}}=R_{f\to T}\frac{\frac{u'\left[W_tR_{f,t\to T}\right]}{u'\left[W_tR_{M,t\to T}\right]}}{\mathbb{E}_t^*\left(\frac{u'\left[W_tR_{f,t\to T}\right]}{u'\left[W_tR_{M,t\to T}\right]}\right)}. \qquad \text{(A20)}$$

Thus, we replace (A20) in (A19) and obtain

$$\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right)=\mathbb{E}_t^*\left(\frac{\frac{u'\left[W_tR_{f,t\to T}\right]}{u'\left[W_tR_{M,t\to T}\right]}}{\mathbb{E}_t^*\left(\frac{u'\left[W_tR_{f,t\to T}\right]}{u'\left[W_tR_{M,t\to T}\right]}\right)}\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right). \qquad \text{(A21)}$$

Recall the identity

$$\frac{\frac{u'\left[W_tR_{f,t\to T}\right]}{u'\left[W_tR_{M,t\to T}\right]}}{\mathbb{E}_t^*\left(\frac{u'\left[W_tR_{f,t\to T}\right]}{u'\left[W_tR_{M,t\to T}\right]}\right)}=\frac{f\left[x\right]}{\mathbb{E}_t^*\left(f\left[x\right]\right)},$$

where the function $f[x]$ is defined in (10). Therefore, (A21) can be expressed as

$$\mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right)=\mathbb{E}_t^*\left(\frac{\left(1+\sum_{k=1}^{\infty}\theta_k\left(R_{M,t\to T}-R_{f,t\to T}\right)^k\right)}{\mathbb{E}_t^*\left(1+\sum_{k=1}^{\infty}\theta_k\left(R_{M,t\to T}-R_{f,t\to T}\right)^k\right)}\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right),$$

which simplifies to

$$
\begin{aligned}
& \mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right) \\
= \quad & \frac{\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right)+\sum\limits_{k=1}^{\infty}\theta_k\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^{k+1}1_{R_{M,t\to T}\leq k_0}\right)}{1+\sum\limits_{k=1}^{\infty}\theta_k\mathbb{M}_{t\to T}^{*(k)}}.
\end{aligned}
$$

The above expression further simplifies to

$$
\begin{aligned}
& \mathbb{E}_t\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right) \\
= \quad & \frac{\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)1_{R_{M,t\to T}>k_0}\right)+\sum\limits_{k=1}^{\infty}\theta_k\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^{k+1}\left(1-1_{R_{M,t\to T}\leq k_0}\right)\right)}{1+\sum\limits_{k=1}^{\infty}\theta_k\mathbb{M}_{t\to T}^{*(k)}} \\
= \quad & \frac{-\mathbb{M}_{t\to T}^{*(1)}\left[k_0\right]+\sum\limits_{k=1}^{\infty}\theta_k\left(\mathbb{M}_{t\to T}^{*(k+1)}-\mathbb{M}_{t\to T}^{*(k+1)}\left[k_0\right]\right)}{1+\sum\limits_{k=1}^{\infty}a_k\mathbb{M}_{t\to T}^{*(k)}}.
\end{aligned}
$$

Finally, the expected excess return has an upper bound,

$$
\mathbb{E}_t\left(R_{M,t\to T}-R_{f,t\to T}\right)\leq\frac{-\mathbb{M}_{t\to T}^{*(1)}\left[k_0\right]+\sum\limits_{k=1}^{\infty}\theta_k\left(\mathbb{M}_{t\to T}^{*(k+1)}-\mathbb{M}_{t\to T}^{*(k+1)}\left[k_0\right]\right)}{1+\sum\limits_{k=1}^{\infty}\theta_k\mathbb{M}_{t\to T}^{*(k)}}.
$$

Up to the third-order expansion of the inverse of the marginal utility,

$$
\mathbb{E}_t\left(R_{M,t\to T}-R_{f,t\to T}\right)\leq\frac{-\mathbb{M}_{t\to T}^{*(1)}\left[k_0\right]+\sum\limits_{k=1}^{3}\theta_k\left(\mathbb{M}_{t\to T}^{*(k+1)}-\mathbb{M}_{t\to T}^{*(k+1)}\left[k_0\right]\right)}{1+\sum\limits_{k=1}^{3}\theta_k\mathbb{M}_{t\to T}^{*(k)}}. \tag{A22}
$$

Under the restrictions in (24), inequality (A22) simplifies to

$$
\mathbb{E}_t\left(R_{M,t\to T}-R_{f,t\to T}\right)\leq\frac{-\mathbb{M}_{t\to T}^{*(1)}\left[k_0\right]+\sum\limits_{k=1}^{3}\frac{(-1)^{k+1}}{R_{f,t\to T}^{k}}\left(\mathbb{M}_{t\to T}^{*(k+1)}-\mathbb{M}_{t\to T}^{*(k+1)}\left[k_0\right]\right)}{1+\sum\limits_{k=1}^{3}\frac{(-1)^{k+1}}{R_{f,t\to T}^{k}}\mathbb{M}_{t\to T}^{*(k)}}.
$$

This ends the proof. ■

Proof of Result 7. The Taylor expansion of $\log\left(R_{M,t\to T}\right)$ around $R_{M,t\to T}=R_{f,t\to T}$ produces

$$\log\left(R_{M,t\to T}\right)=\log\left(R_{f,t\to T}\right)+\sum_{k=1}^{\infty}\frac{1}{k}\frac{(-1)^{k+1}}{R_{f,t\to T}^{k}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^{k}.$$

Now, take the expected value of the above quantity and obtain

$$\mathbb{E}_t\left(\log\left(R_{M,t\to T}\right)-\log\left(R_{f,t\to T}\right)\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{kR_{f,t\to T}^{k}}\left(\mathbb{M}_{t\to T}^{(k)}-\mathbb{M}_{t\to T}^{*(k)}\right)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{kR_{f,t\to T}^{k}}\mathbb{M}_{t\to T}^{*(k)}.$$

Using the results in (19)–(20) it follows that

$$\mathbb{E}_t\left(\log\left(R_{M,t\to T}\right)-\log\left(R_{f,t\to T}\right)\right)\geq\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{kR_{f,t\to T}^{k}}\left(B_{t\to T}^{(k)}+\mathbb{M}_{t\to T}^{*(k)}\right).$$

Up to the fourth-order approximation of $\log\left(R_{M,t\to T}\right)$ around $R_{M,t\to T}=R_{f,t\to T}$, we have

$$\mathbb{E}_t\left(\log\left(R_{M,t\to T}\right)-\log\left(R_{f,t\to T}\right)\right)\geq\sum_{k=1}^{4}\frac{(-1)^{k+1}}{kR_{f,t\to T}^{k}}\left(B_{t\to T}^{(k)}+\mathbb{M}_{t\to T}^{*(k)}\right).$$

This ends the proof. ■

Proof of Result 8. It follows from (51) that

$$m_{t\to T}\left(\vartheta'\left[R_{M,t\to T}\right]\right)^{-1}=\upsilon'\left[c_{t\to T}\right]. \tag{A23}$$

The expected value of (A23) is

$$\begin{aligned}\mathbb{E}_t\left(\upsilon'\left[c_{t\to T}\right]\right) &= \mathbb{E}_t\left[m_{t\to T}\left(\vartheta'\left[R_{M,t\to T}\right]\right)^{-1}\right],\\ &= \frac{1}{R_{f,t\to T}}\mathbb{E}_t^{*}\left[\left(\vartheta'\left[R_{M,t\to T}\right]\right)^{-1}\right].\end{aligned}$$

The expected market return is

$$\mathbb{E}_t\left[R_{M,t\to T}\right] = \mathbb{E}_t\left[m^T_{t\to T}\frac{R_{M,t\to T}}{m^T_{t\to T}}\right]. \tag{A24}$$

Observe that

$$m^T_{t\to T} = m_{t\to T} - \left(m^P_{t\to T} - \mathbb{E}_t\left[m^P_{t\to T}\right]\right)m^T_{t\to T}, \tag{A25}$$

and replace identity (A25) in (A24) and show

$$\begin{aligned}
\mathbb{E}_t\left[R_{M,t\to T}\right] &= \mathbb{E}_t\left[\left(m_{t\to T} - \left(m^P_{t\to T} - \mathbb{E}_t\left[m^P_{t\to T}\right]\right)m^T_{t\to T}\right)\frac{R_{M,t\to T}}{m^T_{t\to T}}\right] \\
&= \mathbb{E}_t\left[m_{t\to T}\frac{R_{M,t\to T}}{m^T_{t\to T}}\right] - \mathbb{E}_t\left[\left(m^P_{t\to T} - \mathbb{E}_t\left[m^P_{t\to T}\right]\right)R_{M,t\to T}\right] \\
&= \left(\mathbb{E}_t\left(m_{t\to T}\right)\right)\mathbb{E}_t\left[\frac{m_{t\to T}}{\mathbb{E}_t\left(m_{t\to T}\right)}\frac{R_{M,t\to T}}{m^T_{t\to T}}\right] - \mathbb{E}_t\left[\left(m^P_{t\to T} - \mathbb{E}_t\left[m^P_{t\to T}\right]\right)R_{M,t\to T}\right] \\
&= \left(\mathbb{E}_t\left(m_{t\to T}\right)\right)\mathbb{E}^*_t\left[\frac{R_{M,t\to T}}{m^T_{t\to T}}\right] - \mathbb{COV}_t\left(m^P_{t\to T}, R_{M,t\to T}\right) \\
&= \frac{1}{R_{f,t\to T}}\left(\mathbb{COV}^*_t\left[\frac{1}{m^T_{t\to T}}, R_{M,t\to T}\right] + \mathbb{E}^*_t\left[\frac{1}{m^T_{t\to T}}\right]R_{f,t\to T}\right) - \mathbb{COV}_t\left(m^P_{t\to T}, R_{M,t\to T}\right).
\end{aligned}$$

Notice that the inverse of $m^T_{t\to T}$ is

$$\frac{1}{m^T_{t\to T}} = \frac{\frac{R_{f,t\to T}}{\vartheta'\left[R_{M,t\to T}\right]}}{\mathbb{E}^*_t\left[\frac{1}{\vartheta'\left[R_{M,t\to T}\right]}\right]} \tag{A26}$$

Thus, the expected value of $\frac{1}{m^T_{t\to T}}$ under the risk neutral measure is

$$\mathbb{E}^*_t\left[\frac{1}{m^T_{t\to T}}\right] = R_{f,t\to T},$$

and, the expected excess return simplifies to

$$\begin{aligned}
\mathbb{E}_t\left[R_{M,t\to T}\right] &= \frac{1}{R_{f,t\to T}}\left(\mathbb{COV}^*_t\left[\frac{1}{m^T_{t\to T}}, R_{M,t\to T}\right] + R^2_{f,t\to T}\right) - \mathbb{COV}^*_t\left(m^P_{t\to T}, R_{M,t\to T}\right) \\
&= \frac{1}{R_{f,t\to T}}\mathbb{COV}^*_t\left[\frac{1}{m^T_{t\to T}}, R_{M,t\to T}\right] + R_{f,t\to T} - \mathbb{COV}^*_t\left(m^P_{t\to T}, R_{M,t\to T}\right)
\end{aligned}$$

The expected excess return is

$$\mathbb{E}_t\left[R_{M,t\to T}\right]-R_{f,t\to T}=\frac{1}{R_{f,t\to T}}\mathbb{COV}_t^*\left[\frac{1}{m_{t\to T}^T},R_{M,t\to T}\right]-\mathbb{COV}_t^*\left(m_{t\to T}^P,R_{M,t\to T}\right) \tag{A27}$$

We replace $\frac{1}{m_{t\to T}^T}$ by its expression (A26) and use (A27) to show that the expected excess return on the market is given as in (52). This ends the proof. ■

B. Appendix

Proof of the closed-form expressions for risk-neutral covariances in Remarks 2 and 3.

The proofs use the spanning formula of Carr and Madan (2001) and Bakshi, Kapadia, and Madan (2003):

$$h[x]=h[x_0]+(x-x_0)\,h_x[x_0]+\int_{x_0}^{\infty}h_{xx}[K]\,(x-K)^+\,dK+\int_0^{x_0}h_{xx}[K]\,(K-x)^+\,dK, \tag{B1}$$

where

$$h_x[x_0]=\left(\frac{\partial h[x]}{\partial x}\right)_{x=x_0}\text{ and }h_{xx}[K]=\left(\frac{\partial^2 h[x]}{\partial^2 x}\right)_{x=K},$$

and, applying the expectation operator to (B1), yields

$$\mathbb{E}_t^*\left(h[x]\right)=h[x_0]+h_x[x_0]\left(\mathbb{E}_t^*\left(x-x_0\right)\right)+R_{f,t\to T}\left(\int_{x_0}^{\infty}h_{xx}[K]\,C_t[K]\,dK+\int_0^{x_0}h_{xx}[K]\,P_t[K]\,dK\right). \tag{B2}$$

Regarding Remark 2, observe that

$$\mathbb{COV}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^n,R_{M,t\to T}^{\alpha}\right)=\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^n R_{M,t\to T}^{\alpha}\right)-\mathbb{M}_{t\to T}^{*(n)}\mathbb{E}_t^*\left(R_{M,t\to T}^{\alpha}\right).$$

Denote

$$h[S_T]=\left(\frac{S_T}{S_t}-R_{f,t\to T}\right)^n\left(\frac{S_T}{S_t}\right)^{\alpha}.$$

Expression $\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^n R_{M,t\to T}^{\alpha}\right)$ can be computed by using by using (B2) if we also denote

$$h\left[S_T\right]=\left(\frac{S_T}{S_t}\right)^{\alpha}.$$

Expression $\mathbb{E}_t^*\left(R_{M,t\to T}^{\alpha}\right)$ can be computed by using by using (B2). Note that Martin (2017) derives similar results in his Online Appendix for the case where $n=1$ and investors have power utility. In a similar way, regarding Remark 3,

$$\mathbb{COV}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^n, e^{\alpha R_{M,t\to T}}\right)=\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^n e^{\alpha R_{M,t\to T}}\right)-\mathbb{M}_{t\to T}^{*(n)}\mathbb{E}_t^*\left(e^{\alpha R_{M,t\to T}}\right).$$

Denote

$$h\left[S_T\right]=\left(\frac{S_T}{S_t}-R_{f,t\to T}\right)^n e^{\alpha\left(\frac{S_T}{S_t}\right)}.$$

Expression $\mathbb{E}_t^*\left(\left(R_{M,t\to T}-R_{f,t\to T}\right)^n e^{\alpha R_{M,t\to T}}\right)$ can be computed by using by using (B2) if we also denote

$$h\left[S_T\right]=e^{\alpha\left(\frac{S_T}{S_t}\right)}.$$

Expression $\mathbb{E}_t^*\left(e^{\alpha\left(\frac{S_T}{S_t}\right)}\right)$ can be computed by using by using (B2). ■

Formulas of the risk-neutral moments of the market return. Given that

$$R_{M,t\to T}=\frac{S_T}{S_t},$$

using the spanning formula of Carr and Madan (2001),

$$\left(R_{M,t\to T}-R_{f,t\to T}\right)^n=\frac{n\left(n-1\right)}{S_t^2}\left\{\begin{array}{c}\int_{R_{f,t\to T}S_t}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(S_T-K\right)^+dK\\+\int_0^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(K-S_T\right)^+dK\end{array}\right\}.$$

Hence,

$$\mathbb{E}_t^*\left((R_{M,t\to T}-R_{f,t\to T})^n\right)=\frac{n(n-1)}{S_t^2}\left\{\begin{array}{c}\int_{R_{f,t\to T}S_t}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\mathbb{E}_t^*(S_T-K)^+dK\\ +\int_0^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\mathbb{E}_t^*(K-S_T)^+dK\end{array}\right\}.$$

Finally

$$\mathbb{E}_t^*\left((R_{M,t\to T}-R_{f,t\to T})^n\right)=\frac{n(n-1)R_{f,t\to T}}{S_t^2}\left\{\begin{array}{c}\int_{R_{f,t\to T}S_t}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}C_t[K]\,dK\\ +\int_0^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}P_t[K]\,dK\end{array}\right\}. \quad \text{(B3)}$$

■

Formulas of the risk-neutral moments of the market in down market. Again, using the spanning formula of Carr and Madan (2001), we have

$$\begin{aligned}(R_{M,t\to T}-R_{f,t\to T})^n &= (k_0-R_{f,t\to T})^n+n(k_0-R_{f,t\to T})^{n-1}\left(\frac{S_T}{S_t}-k_0\right)\\ &\quad+\frac{n(n-1)}{S_t^2}\left\{\begin{array}{c}\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(S_T-K)^+dK\\ +\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(K-S_T)^+dK\end{array}\right\},\end{aligned}$$

and, hence,

$$\begin{aligned}(R_{M,t\to T}-R_{f,t\to T})^n 1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} &= (k_0-R_{f,t\to T})^n 1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}+n(k_0-R_{f,t\to T})^{n-1}\left(\frac{S_T}{S_t}-k_0\right)1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}\\ &\quad+\frac{n(n-1)}{S_t^2}\left\{\begin{array}{c}\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(S_T-K)^+1_{\{S_T\le k_0S_t\}}dK\\ +\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(K-S_T)^+1_{\{S_T\le k_0S_t\}}dK\end{array}\right\}\\ &= (k_0-R_{f,t\to T})^n 1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}+n(k_0-R_{f,t\to T})^{n-1}\left(\frac{S_T}{S_t}-k_0\right)1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}\\ &\quad+\frac{n(n-1)}{S_t^2}\left\{\begin{array}{c}\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(S_T-K)\,1_{S_T\ge K}1_{\{S_T\le k_0S_t\}}dK\\ +\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(K-S_T)\,1_{S_T\le K}1_{\{S_T\le k_0S_t\}}dK\end{array}\right\}.\end{aligned}$$

Hence,

$$\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} = \left(k_0-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}+n\left(k_0-R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t}-k_0\right)1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} +\frac{n\left(n-1\right)}{S_t^2}\left\{\begin{array}{c}\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(S_T-K\right)\left(1-1_{S_T\le K}\right)1_{\left\{S_T\le k_0S_t\right\}}dK\\ +\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(K-S_T\right)1_{S_T\le\min(k_0S_t,K)}dK\end{array}\right\},$$

which simplifies to

$$\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} = \left(k_0-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}+n\left(k_0-R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t}-k_0\right)1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} +\frac{n\left(n-1\right)}{S_t^2}\left\{\begin{array}{c}\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(S_T-K\right)1_{\left\{S_T\le k_0S_t\right\}}dK\\ -\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(S_T-K\right)1_{S_T\le\min(k_0S_t,K)}dK\\ +\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(K-S_T\right)1_{S_T\le\min(k_0S_t,K)}dK\end{array}\right\}$$

and

$$\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} = \left(k_0-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}+n\left(k_0-R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t}-k_0\right)1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} +\frac{n\left(n-1\right)}{S_t^2}\left\{\begin{array}{c}-\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(k_0S_t-S_T\right)^{+}dK\\ +\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(k_0S_t-S_T\right)^{+}dK\\ \left(1_{\left\{S_T\le k_0S_t\right\}}\right)\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(k_0S_t-K\right)dK\\ -\left(1_{S_T\le k_0S_t}\right)\int_{S_tk_0}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(k_0S_t-K\right)dK\\ +\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(K-S_T\right)^{+}dK\end{array}\right\}.$$

The above expression simplifies to

$$\begin{aligned}\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}} &= \left(k_0-R_{f,t\to T}\right)^{n}1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}\\ &\quad+n\left(k_0-R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t}-k_0\right)1_{\left\{\frac{S_T}{S_t}\le k_0\right\}}\\ &\quad+\frac{n\left(n-1\right)}{S_t^2}\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(K-S_T\right)^{+}dK.\end{aligned}$$

Now, we take the expectation under the risk-neutral measure:

$$\begin{aligned}
\mathbb{E}_t^* \left[\left(R_{M,t\to T} - R_{f,t\to T} \right)^n 1_{\left\{ \frac{S_T}{S_t} \leq k_0 \right\}} \right] &= \left(k_0 - R_{f,t\to T} \right)^n \mathbb{E}_t^* \left[1_{\left\{ \frac{S_T}{S_t} \leq k_0 \right\}} \right] \\
&\quad + n \left(k_0 - R_{f,t\to T} \right)^{n-1} \mathbb{E}_t^* \left[\left(\frac{S_T}{S_t} - k_0 \right) 1_{\left\{ \frac{S_T}{S_t} \leq k_0 \right\}} \right] \\
&\quad + \frac{n(n-1)}{S_t^2} \int_0^{S_t k_0} \left(\frac{K}{S_t} - R_{f,t\to T} \right)^{n-2} \mathbb{E}_t^* \left[\left(K - S_T \right)^+ \right] dK,
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\mathbb{E}_t^* \left[\left(R_{M,t\to T} - R_{f,t\to T} \right)^n 1_{\left\{ \frac{S_T}{S_t} \leq k_0 \right\}} \right] &= \left(k_0 - R_{f,t\to T} \right)^n \mathrm{Prob}_t^* \left[\frac{S_T}{S_t} \leq k_0 \right] \\
&\quad + n \left(k_0 - R_{f,t\to T} \right)^{n-1} \frac{1}{S_t} \mathbb{E}_t^* \left[\left(S_T - k_0 S_t \right) 1_{S_T \leq k_0 S_t} \right] \\
&\quad + \frac{n(n-1)}{S_t^2} \int_0^{S_t k_0} \left(\frac{K}{S_t} - R_{f,t\to T} \right)^{n-2} \mathbb{E}_t^* \left[\left(K - S_T \right)^+ \right] dK,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_t^* \left[\left(R_{M,t\to T} - R_{f,t\to T} \right)^n 1_{\left\{ \frac{S_T}{S_t} \leq k_0 \right\}} \right] &= \left(k_0 - R_{f,t\to T} \right)^n \mathrm{Prob}_t^* \left[\frac{S_T}{S_t} \leq k_0 \right] \\
&\quad + n \left(k_0 - R_{f,t\to T} \right)^{n-1} \frac{1}{S_t} \mathbb{E}_t^* \left[\left(S_T - k_0 S_t \right) 1_{S_T \leq k_0 S_t} \right] \\
&\quad + \frac{n(n-1)}{S_t^2} \int_0^{S_t k_0} \left(\frac{K}{S_t} - R_{f,t\to T} \right)^{n-2} \mathbb{E}_t^* \left[\left(K - S_T \right)^+ \right] dK,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_t^* \left[\left(R_{M,t\to T} - R_{f,t\to T} \right)^n 1_{\left\{ \frac{S_T}{S_t} \leq k_0 \right\}} \right] &= \left(k_0 - R_{f,t\to T} \right)^n \mathrm{Prob}_t^* \left[\frac{S_T}{S_t} \leq k_0 \right] \\
&\quad - n \left(k_0 - R_{f,t\to T} \right)^{n-1} \frac{R_{f,t\to T}}{S_t} P_t \left[k_0 S_t \right] \\
&\quad + \frac{n(n-1) R_{f,t\to T}}{S_t^2} \int_0^{S_t k_0} \left(\frac{K}{S_t} - R_{f,t\to T} \right)^{n-2} P_t [K] \, dK.
\end{aligned}$$

Finally,

$$\mathbb{E}_t^*\left[\left(R_{M,t\to T}-R_{f,t\to T}\right)^n 1_{\left\{\frac{S_T}{S_t}\leq k_0\right\}}\right] = \left(k_0-R_{f,t\to T}\right)^n \text{Prob}_t^*\left[\frac{S_T}{S_t}\leq k_0\right]-n\left(k_0-R_{f,t\to T}\right)^{n-1}\frac{R_{f,t\to T}}{S_t}P_t\left[k_0S_t\right] + \frac{n\left(n-1\right)R_{f,t\to T}}{S_t^2}\int_0^{S_tk_0}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}P_t\left[K\right]dK, \tag{B4}$$

where $\text{Prob}_t^*\left[\frac{S_T}{S_t}\leq k_0\right]$ is the risk-neutral probability, $P_t\left[k_0S_t\right]$ is the price at time t of a put with strike k_0S_t, and $P_t\left[K\right]$ is the price at time t of a put with strike K. ■

Maturity (days)	30	60	90	180	360	All
$100\alpha_1$	-0.017	0.507	-0.171	-1.083	3.873	0.419
$t(\alpha_1)$	-0.041	0.637	-0.152	-0.376	0.865	0.821
$1000\alpha_2$	-0.511	-1.522	-1.021	2.899	5.888	-1.003
$t(\alpha_2)$	-2.535	-2.923	-1.224	0.635	1.247	-3.320
$10000\alpha_3$	0.117	1.879	-0.120	-17.950	23.521	0.915
$t(\alpha_3)$	0.222	0.947	-0.035	-0.821	0.504	1.585
τ	0.691	3.206	0.751	0.596	2.870	0.974
$t(\tau)$	0.067	0.172	0.069	0.081	0.324	0.244
ρ	3.305	36.518	3.101	1.872	10.253	2.321
$t(\rho)$	0.002	0.012	0.001	0.001	0.014	0.013
κ	3.466	59.205	3.344	2.369	55.881	3.503
$t(\kappa)$	0.000	0.002	0.000	0.000	0.003	0.001

Table 1
Nonlinear least squares regression estimates of preference parameters (τ, ρ, and κ) using three moment restrictions from Result 4 (corresponding to $n = 1$, $n = 2$, and $n = 3$). Results in the first five columns use daily data from single maturities listed in the column header. The last column uses daily data from all maturities. T-statistics are based on standard errors from 1,000 block bootstrap simulations where the block lengths are three times the maturity length used in each estimation. P-values are computed assuming that each t-statistic follows a standard normal distribution under the null.

Maturity (days)	30	60	90	180	360	All
a_1	1.447	0.312	1.332	1.677	0.348	1.026
$t(a_1)$	0.874	0.235	1.308	1.770	0.493	1.978
$p(a_1 = 1)$	0.787	0.604	0.745	0.475	0.357	0.959
a_2	-4.825	-3.456	-3.725	-2.453	-1.124	-1.391
$t(a_2)$	-1.069	-1.356	-1.262	-1.415	-1.692	-2.048
$p(a_2 = -1)$	0.397	0.335	0.356	0.402	0.852	0.565
a_3	-6.495	-0.389	-4.385	-1.771	1.539	-0.150
$t(a_3)$	-1.067	-0.092	-1.024	-0.601	0.848	-0.144
$p(a_3 = 1)$	0.218	0.743	0.209	0.347	0.766	0.268
$p(a_1 = 1, a_2 = -1, a_3 = 1)$	0.220	0.755	0.404	0.427	0.649	0.654

Table 2
Nonlinear least squares regression estimates of the coefficients a_1, a_2, and a_3 from Result 4 using preference parameter estimates from Table 1. T-statistics are based on standard errors from 1,000 block bootstrap simulations where the block lengths are three times the maturity length used in each estimation. P-values for individual tests are computed assuming that each t-statistic follows a standard normal distribution under the null. P-values for the joint test are computed based on a Chi-squared statistic using the parameter covariance matrix estimated using the bootstrap results.

Panel A: $\widetilde{UB}^{(1)}_{t\to T}$ summary statistics													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	8.565	8.633	3.742	23.140	1.519	1.836	2.549	3.650	6.220	9.577	16.977	52.124	92.928
60	8.437	7.001	2.903	15.667	1.601	1.802	2.870	3.992	6.501	10.213	15.741	36.516	73.569
90	8.554	6.262	2.572	13.178	1.844	2.072	3.113	4.246	7.060	10.672	15.250	33.587	59.657
180	8.148	4.154	1.419	6.096	1.958	2.659	3.755	4.917	7.406	10.265	13.294	23.278	32.511
360	7.253	2.805	1.069	4.787	2.073	3.088	4.053	5.304	6.778	8.722	11.087	16.060	21.657
Panel B: $LB^{(1)}_{t\to T}$ summary statistics													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	5.219	5.207	4.333	31.757	0.727	1.195	1.655	2.325	3.847	6.039	9.801	27.874	64.218
60	5.377	4.600	3.562	22.477	1.102	1.365	1.895	2.618	4.176	6.405	9.597	25.402	56.310
90	5.483	4.374	3.434	21.024	0.437	1.486	2.070	2.768	4.411	6.609	9.461	24.875	50.257
180	5.567	3.285	1.919	8.412	1.191	1.819	2.429	3.164	4.836	6.845	9.343	17.962	25.649
360	5.411	2.731	1.646	7.406	0.537	1.730	2.649	3.564	4.808	6.664	8.873	15.268	22.085
Panel C: $LB^{(1)}_{t\to T} - \frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}$ summary statistics													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	0.714	0.946	5.405	48.418	0.066	0.110	0.181	0.267	0.438	0.772	1.422	4.704	14.454
60	0.868	0.939	3.967	25.574	0.105	0.152	0.250	0.366	0.587	0.990	1.651	4.914	10.221
90	0.993	1.031	4.091	27.073	-1.108	0.186	0.300	0.440	0.702	1.157	1.901	5.573	11.005
180	1.172	0.833	1.601	6.446	-2.619	0.235	0.443	0.624	0.903	1.459	2.277	4.090	5.489
360	1.328	0.847	1.484	6.127	-3.129	0.021	0.580	0.765	1.060	1.608	2.544	4.203	5.805

Table 3: Summary statistics for unrestricted bounds (annualized and in %).

	$\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}$ **lower bound summary statistics**												
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	4.506	4.305	4.198	30.283	0.661	1.068	1.468	2.051	3.416	5.274	8.330	23.028	54.943
60	4.509	3.700	3.476	21.886	0.997	1.202	1.650	2.236	3.604	5.407	7.983	20.327	46.089
90	4.490	3.394	3.232	19.253	0.351	1.285	1.748	2.300	3.712	5.414	7.622	19.416	39.252
180	4.395	2.545	2.019	9.388	0.764	1.489	1.966	2.487	3.855	5.491	7.233	14.165	20.596
360	4.084	2.036	1.780	8.673	0.225	1.391	2.048	2.640	3.712	5.042	6.493	11.946	17.399

Table 4: Summary statistics for lower bound using $\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}$ (annualized and in %).

Panel A: $\widetilde{UB}^{(1)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}_T$	0.002	0.001	0.002	-0.012	0.032
$t(\hat{a}_T)$	0.430	0.094	0.175	-0.485	0.510
$\hat{b}_T$	0.523	0.631	0.635	1.419	0.765
$t(\hat{b}_T)$	0.806	0.917	0.880	2.979	1.147
$\chi^2_2(a_T = 0, b_T = 1)$	0.591	0.624	0.443	0.897	0.271
$p(\chi^2_2)$	0.744	0.732	0.801	0.639	0.873
R^2_{IS}	0.006	0.012	0.015	0.076	0.018
$R^2_{pseudoOOS}$	0.002	0.009	0.011	0.079	0.027

Panel B: $LB^{(1)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}$	0.003	0.004	0.006	-0.025	0.005
$t(\hat{a})$	0.866	0.468	0.441	-0.944	0.072
$\hat{b}$	0.660	0.750	0.713	2.051	1.268
$t(\hat{b})$	0.710	0.801	0.688	3.256	1.613
$\chi^2_2(\alpha_0 = 0, \alpha_1 = 1)$	0.941	0.246	0.216	2.954	0.614
$p(\chi^2)$	0.625	0.884	0.897	0.228	0.736
R^2_{IS}	0.003	0.007	0.009	0.077	0.038
$R^2_{pseudoOOS}$	0.003	0.010	0.014	0.067	0.046

Panel C: $\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}$	0.003	0.004	0.006	-0.017	0.015
$t(\hat{a})$	0.734	0.480	0.485	-0.671	0.229
$\hat{b}$	0.869	0.873	0.811	2.234	1.418
$t(\hat{b})$	0.768	0.758	0.621	2.552	1.241
$\chi^2_2(\alpha_0 = 0, \alpha_1 = 1)$	1.052	0.435	0.446	2.177	1.062
$p(\chi^2)$	0.591	0.804	0.800	0.337	0.588
R^2_{IS}	0.004	0.007	0.007	0.055	0.027
R^2_{OOS}	0.003	0.009	0.010	0.043	0.013

Table 5

Univariate forecasting regressions using proposed forecasting variables. Panel A presents results for the upper bound $\widetilde{UB}^{(1)}_{t\to T}$. Panel B presents results when the lower bound $LB^{(1)}_{t\to T}$ is used to predict the excess market return. Panel C presents results using $\mathbb{M}^{*(2)}_{t\to T}$. We use the estimated preference parameters from Table 2 to estimate both the lower and upper bounds. Standard errors in both panels are obtained using Newey and West (1987) standard errors where the window is the same as the investment horizon.

Panel A: $\widetilde{UBR}^{(1)}_{t\to T}$ **summary statistics**													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	8.519	8.759	3.852	24.403	1.479	1.781	2.501	3.583	6.148	9.486	16.943	52.217	96.060
60	8.425	7.165	3.057	17.300	1.557	1.764	2.824	3.936	6.456	10.162	15.762	37.399	80.690
90	8.576	6.460	2.747	14.822	1.811	2.037	3.072	4.199	7.035	10.683	15.344	34.325	66.682
180	8.237	4.334	1.531	6.696	1.920	2.624	3.737	4.902	7.466	10.359	13.550	24.385	34.763
360	7.410	3.031	1.254	5.777	2.053	3.064	4.051	5.317	6.869	8.980	11.624	16.888	25.339
Panel B: $LBR^{(1)}_{t\to T}$ **summary statistics**													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	5.150	5.362	4.644	35.952	0.699	1.157	1.602	2.253	3.750	5.900	9.678	28.593	70.295
60	5.354	4.859	3.957	27.258	1.065	1.319	1.840	2.548	4.089	6.290	9.563	26.819	65.625
90	5.516	4.712	3.824	25.291	0.430	1.443	2.017	2.705	4.364	6.579	9.559	26.686	59.082
180	5.748	3.711	2.354	11.669	1.141	1.781	2.396	3.146	4.884	7.025	9.768	20.132	32.599
360	5.763	3.291	2.202	11.216	0.670	1.726	2.650	3.635	5.005	7.083	9.650	18.208	29.160
Panel C: $LBR^{(1)}_{t\to T} - \frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}$ **summary statistics**													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	0.644	1.112	6.747	69.893	0.039	0.070	0.127	0.192	0.340	0.643	1.302	5.438	18.407
60	0.844	1.209	5.538	48.329	0.068	0.108	0.190	0.295	0.500	0.903	1.607	6.318	19.535
90	1.027	1.378	5.344	43.307	-0.615	0.142	0.250	0.380	0.645	1.131	1.972	7.328	19.830
180	1.352	1.225	2.989	16.786	-0.652	0.250	0.418	0.612	0.949	1.626	2.644	6.100	12.493
360	1.679	1.340	2.698	14.367	-0.255	0.252	0.592	0.862	1.253	2.080	3.268	6.811	12.238

Table 6: Summary statistics for restricted bounds (annualized and in %).

Panel A: $\widetilde{UBR}^{(1)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}_T$	0.002	0.001	0.003	-0.011	0.033
$t\left(\hat{a}_T\right)$	0.456	0.130	0.230	-0.436	0.558
$\hat{b}_T$	0.515	0.610	0.600	1.367	0.722
$t\left(\hat{b}_T\right)$	0.805	0.905	0.850	3.013	1.162
$\chi^2_2\left(a_T=0, b_T=1\right)$	0.613	0.645	0.489	0.755	0.312
$p\left(\chi^2_2\right)$	0.736	0.724	0.783	0.686	0.856
R^2_{IS}	0.006	0.011	0.014	0.077	0.019
R^2_{OOS}	0.002	0.008	0.009	0.081	0.028
Panel B: $LBR^{(1)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}$	0.003	0.004	0.007	-0.019	0.007
$t\left(\hat{a}\right)$	0.944	0.567	0.554	-0.792	0.117
$\hat{b}$	0.628	0.686	0.623	1.803	1.146
$t\left(\hat{b}\right)$	0.698	0.772	0.641	3.230	1.865
$\chi^2_2\left(\alpha_0=0, \alpha_1=1\right)$	1.059	0.337	0.316	2.147	0.404
$p\left(\chi^2\right)$	0.589	0.845	0.854	0.342	0.817
R^2_{IS}	0.003	0.007	0.008	0.076	0.045
R^2_{OOS}	0.002	0.009	0.011	0.071	0.058
Panel C: $\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}$	0.003	0.004	0.006	-0.017	0.015
$t\left(\hat{a}\right)$	0.734	0.480	0.485	-0.671	0.229
$\hat{b}$	0.869	0.873	0.811	2.234	1.418
$t\left(\hat{b}\right)$	0.768	0.758	0.621	2.552	1.241
$\chi^2_2\left(\alpha_0=0, \alpha_1=1\right)$	1.052	0.435	0.446	2.177	1.062
$p\left(\chi^2\right)$	0.591	0.804	0.800	0.337	0.588
R^2_{IS}	0.004	0.007	0.007	0.055	0.027
R^2_{OOS}	0.003	0.009	0.010	0.043	0.013

Table 7
Univariate forecasting regressions using proposed forecasting variables. Panel A presents results for the upper bound $\widetilde{UBR}^{(1)}_{t\to T}$. Panel B presents results when the lower bound $LBR^{(1)}_{t\to T}$ is used to predict the excess market return. Panel C presents results using $\mathbb{M}^{*(2)}_{t\to T}$. Standard errors in both panels are obtained using Newey and West (1987) standard errors where the window is the same as the investment horizon.

Panel A: Regression results				
T	**60**	**90**	**180**	**360**
$\hat{a}$	-0.006	-0.010	-0.032	-0.049
$t(\hat{a})$	-1.345	-2.158	-2.761	-3.177
$\hat{b}$	0.002	0.004	0.008	0.008
$t(\hat{b})$	4.728	4.035	3.367	2.450
R^2	0.053	0.083	0.150	0.197
Panel B: Hypothesis testing results				
$p(a >= 0)$	0.174	0.075	0.009	0.003
$p(b <= 0)$	0.000	0.000	0.000	0.006
$p(a >= 0\, or\, b <= 0)$	0.174	0.075	0.009	0.008

Table 8

Term structure regressions and tests. Panel A: Term structure regressions results according to the specification in equation (39), namely $LB^{(1)}_{t\to t+T} - LB_{t\to t+30} = a \cdot I\{Rec_t\} + b \cdot [1 - I\{Rec_t\}] + \varepsilon^{(1)}_{t\to t+T}$. Standard errors are calculated according to the Newey-West (1987) procedure where the window is three times the maturity, T, in each regression. P-values for the associated null hypotheses are computed using 10,000 block bootstrapped simulations for each horizon where block lengths are equal to three times T.

Panel A: NCC covariance upper bound ($\left(\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T} - LB^{(1)}_{t\to T}\right) * 100$) summary statistics													
Horizon (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	-0.059	0.078	-5.405	48.418	-1.188	-0.387	-0.117	-0.063	-0.036	-0.022	-0.015	-0.009	-0.005
60	-0.143	0.154	-3.967	25.574	-1.680	-0.808	-0.271	-0.163	-0.096	-0.060	-0.041	-0.025	-0.017
90	-0.245	0.254	-4.091	27.073	-2.713	-1.374	-0.469	-0.285	-0.173	-0.109	-0.074	-0.046	0.273
180	-0.578	0.411	-1.601	6.446	-2.707	-2.017	-1.123	-0.720	-0.445	-0.308	-0.219	-0.116	1.291
360	-1.309	0.835	-1.484	6.127	-5.725	-4.146	-2.509	-1.586	-1.045	-0.755	-0.572	-0.021	3.086
Panel B: NCC covariance lower bound ($\left(\frac{1}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T} - \widetilde{UB}^{(1)}_{t\to T}\right) * 100$) summary statistics													
Horizon (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	-0.297	0.342	-3.698	22.850	-3.662	-1.950	-0.652	-0.334	-0.197	-0.106	-0.066	-0.033	-0.017
60	-0.629	0.545	-2.445	11.277	-4.517	-2.744	-1.263	-0.768	-0.463	-0.285	-0.185	-0.084	-0.036
90	-0.986	0.714	-1.962	8.455	-5.398	-3.718	-1.842	-1.282	-0.808	-0.483	-0.319	-0.179	-0.142
180	-1.905	0.929	-1.304	6.287	-7.384	-4.754	-3.104	-2.403	-1.746	-1.219	-0.870	-0.539	-0.453
360	-3.280	1.085	-0.713	3.608	-8.952	-6.336	-4.807	-3.861	-3.159	-2.545	-1.964	-1.472	-0.824
Panel C: NCC covariance bound difference (multiplied by 100) summary statistics													
Horizon (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	0.230	0.265	3.424	19.731	0.002	0.009	0.047	0.078	0.152	0.266	0.508	1.402	2.499
60	0.480	0.401	2.073	8.442	0.001	0.048	0.134	0.219	0.357	0.600	0.978	2.005	2.867
90	0.735	0.490	1.379	5.130	0.080	0.119	0.237	0.365	0.609	0.989	1.369	2.471	2.808
180	1.324	0.601	1.513	8.967	0.289	0.372	0.641	0.888	1.269	1.660	2.021	3.256	5.351
360	1.988	0.600	0.671	3.380	0.076	0.941	1.274	1.588	1.895	2.327	2.847	3.540	4.791

Table 9: Summary statistics for NCC covariance bounds from equation (43) (multiplied by 100).

Panel A: $LLB^{(1)}_{t\to T}$ summary statistics													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	3.210	3.552	4.838	38.444	0.404	0.676	0.963	1.360	2.238	3.617	6.063	18.688	47.653
60	3.443	3.302	4.041	27.581	0.616	0.786	1.137	1.587	2.551	3.978	6.191	17.975	42.876
90	3.638	3.309	3.986	26.460	0.295	0.881	1.273	1.757	2.807	4.290	6.407	18.379	39.502
180	3.913	2.596	2.191	10.079	-0.296	1.146	1.581	2.158	3.225	4.740	6.852	13.867	21.317
360	4.016	2.352	1.925	8.639	0.020	1.091	1.782	2.510	3.387	4.888	7.021	12.208	19.134
Panel B: $LLBR^{(1)}_{t\to T}$ **summary statistics**													
Maturity (days)	Mean	Std. Dev.	Skew.	Kurt.	Min.	1%	10%	25%	50%	75%	90%	99%	Max
30	3.136	3.732	5.320	45.411	0.375	0.635	0.907	1.283	2.130	3.475	5.919	19.281	52.174
60	3.420	3.602	4.607	34.923	0.577	0.738	1.074	1.513	2.461	3.872	6.142	19.369	51.971
90	3.678	3.703	4.505	32.540	0.282	0.837	1.217	1.693	2.743	4.256	6.493	20.736	48.029
180	4.110	3.043	2.676	14.083	0.757	1.105	1.544	2.155	3.270	4.936	7.314	15.906	27.973
360	4.398	2.924	2.359	11.730	0.274	1.128	1.790	2.552	3.585	5.355	7.909	15.360	25.392

Table 10: Summary statistics for the log-return lower bounds (annualized and in %).

Panel A: $LLB^{(1)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}_T$	0.003	0.003	0.005	-0.028	-0.014
$t\left(\hat{a}_T\right)$	1.004	0.430	0.392	-1.084	-0.220
$\hat{b}_T$	0.549	0.873	0.816	2.708	1.724
$t\left(\hat{b}_T\right)$	0.414	0.691	0.603	3.782	2.243
$\chi^2_2\left(a_T = 0, b_T = 1\right)$	1.222	0.272	0.214	6.826	2.924
$p\left(\chi^2_2\right)$	0.543	0.873	0.898	0.033	0.232
R^2_{IS}	0.001	0.005	0.007	0.080	0.054
$R^2_{pseudoOOS}$	0.000	0.006	0.009	0.055	0.053

Panel B: $LLBR^{(1)}_{t\to T}$					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}$	0.004	0.004	0.006	-0.022	-0.007
$t\left(\hat{a}\right)$	1.118	0.561	0.539	-0.905	-0.126
$\hat{b}$	0.483	0.754	0.657	2.277	1.429
$t\left(\hat{b}\right)$	0.386	0.654	0.541	3.642	2.421
$\chi^2_2\left(\alpha_0 = 0, \alpha_1 = 1\right)$	1.393	0.361	0.312	4.677	1.886
$p\left(\chi^2\right)$	0.498	0.835	0.855	0.096	0.389
R^2_{IS}	0.001	0.004	0.005	0.077	0.057
$R^2_{pseudoOOS}$	-0.001	0.005	0.007	0.061	0.064

Table 11
Univariate forecasting regressions using the log-return bounds. Panel A uses the unrestricted bound from (45) and Panel B uses the restricted bound from (46). Standard errors in both panels are obtained using Newey and West (1987) standard errors where the window is the same as the investment horizon.

Panel A: Squared return forecasting regressions (unrestricted)					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}_T$	0.000	0.001	0.001	-0.001	0.009
$t\left(\hat{a}_T\right)$	0.556	0.767	0.366	-0.328	0.697
$\hat{b}_T$	0.651	0.583	0.692	1.059	1.010
$t\left(\hat{b}_T\right)$	3.689	2.966	3.972	4.199	2.484
$\chi^2_2\left(a_T = 0, b_T = 1\right)$	13.016	10.232	6.980	0.112	2.221
$p\left(\chi^2_2\right)$	0.001	0.006	0.030	0.946	0.329
R^2_{IS}	0.115	0.075	0.097	0.118	0.099
$R^2_{pseudoOOS}$	0.089	0.091	0.158	0.176	0.238
Panel B: Squared return forecasting regressions (restricted)					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}_T$	0.000	0.001	0.001	0.000	0.009
$t\left(\hat{a}_T\right)$	0.560	0.784	0.360	-0.054	0.829
$\hat{b}_T$	0.648	0.576	0.691	0.980	0.996
$t\left(\hat{b}_T\right)$	3.699	2.940	3.939	3.988	3.194
$\chi^2_2\left(a_T = 0, b_T = 1\right)$	13.417	10.642	7.135	0.056	2.272
$p\left(\chi^2_2\right)$	0.001	0.005	0.028	0.972	0.321
R^2_{IS}	0.115	0.075	0.098	0.112	0.126
R^2_{OOS}	0.088	0.088	0.158	0.171	0.261

Table 12
Univariate forecasting regressions for squared excess market returns based on the regression specification in Equation (49). Panel A uses the estimated preference parameters from Table 1 to compute θ_k. Panel B imposes restrictions in (32) on θ_k. Standard errors in both panels are obtained using Newey and West (1987) standard errors where the window is the same as the investment horizon.

Panel A: Variance forecasting regressions (unrestricted)					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}_T$	-0.001	-0.002	-0.002	-0.001	0.010
$t\left(\hat{a}_T\right)$	-1.705	-1.515	-1.205	-0.294	1.230
$\hat{b}_T$	1.370	1.350	1.338	1.340	1.094
$t\left(\hat{b}_T\right)$	4.827	4.719	4.645	3.821	4.492
$\chi_2^2\left(a_T=0, b_T=1\right)$	5.855	2.752	1.492	1.797	3.967
$p\left(\chi_2^2\right)$	0.054	0.253	0.474	0.407	0.138
R_{IS}^2	0.460	0.335	0.262	0.172	0.109
$R_{pseudoOOS}^2$	0.428	0.322	0.255	0.153	0.258

Panel B: Variance forecasting regressions (restricted)					
Maturity (days)	**30**	**60**	**90**	**180**	**360**
$\hat{a}_T$	-0.001	-0.002	-0.002	-0.001	0.012
$t\left(\hat{a}_T\right)$	-1.692	-1.545	-1.196	-0.367	1.527
$\hat{b}_T$	1.363	1.360	1.348	1.362	1.043
$t\left(\hat{b}_T\right)$	4.806	4.683	4.494	3.882	4.491
$\chi_2^2\left(a_T=0, b_T=1\right)$	6.042	2.947	1.458	1.886	4.282
$p\left(\chi_2^2\right)$	0.049	0.229	0.482	0.389	0.118
R_{IS}^2	0.457	0.340	0.260	0.182	0.113
R_{OOS}^2	0.427	0.325	0.253	0.161	0.253

Table 13
Univariate market return variance forecasting regressions based on the regression specification in Equation (50). Panel A uses the estimated preference parameters from Table 1 to compute θ_k. Panel B imposes restrictions in (32) on θ_k. Standard errors in both panels are obtained using Newey and West (1987) standard errors where the window is the same as the investment horizon.

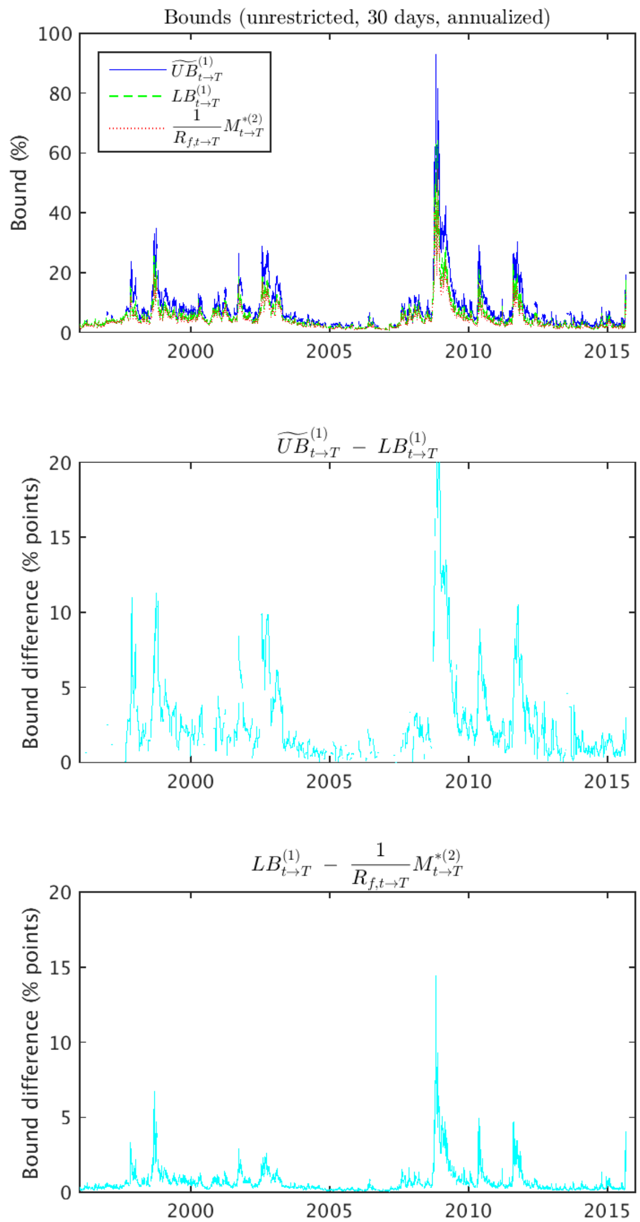


Fig. 1. **Unrestricted Bounds on the Expected Excess Market Return:** The bound measures are computed using the 30-day maturity options and estimated parameter values from Table 1. The bounds are annualized and reported in percentages. The top graph presents the upper bound (31), the lower bound (22) and the lower bound derived in Martin (2017). The graph in the middle shows the difference between the upper bound and the lower bound. The bottom graph presents the difference between the lower bound and the Martin (2017) bound.

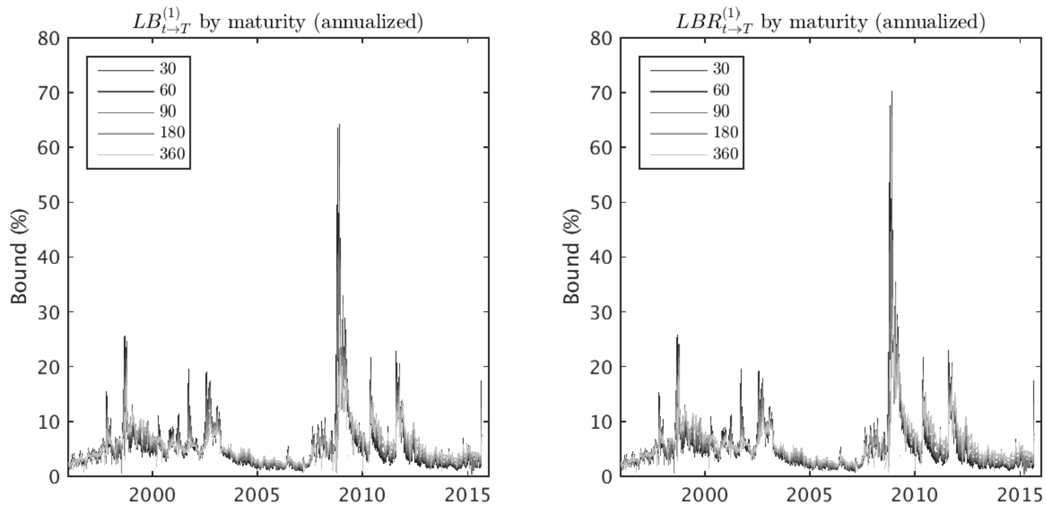


Fig. 2. **Unrestricted and Restricted Lower Bounds on the Expected Excess Market Return:** The bound measures are computed for all five maturities considered herein. The unrestricted bounds are estimated using parameter values from Table 1. The bounds are annualized and reported in percentages. The left graph presents the unrestricted bounds (22), and the right graph presents the restricted bounds (26).

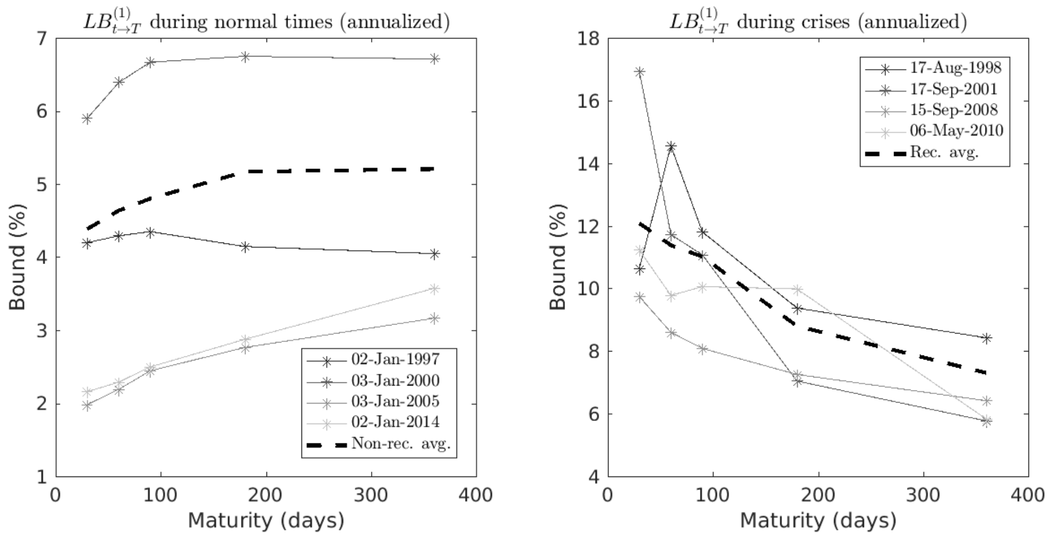


Fig. 3. **Unrestricted Lower Bound Term Structures:** This figure illustrates the term structure behavior implied by the unrestricted lower bound (22) during normal (left) and turbulent (right) times. In the figure on the left, we plot the implied term structure at four ”normal” dates (these were chosen simply as the first date for which we have data during years that were not particularly turbulent). The bold dotted line represents the average lower bound (by maturity) over all non-NBER recession dates. In the figure on the right, we plot the implied term structure at four ”turbulent” dates. These include the Russian debt crisis, the September 11 terrorist attacks, the Lehman Brothers bankruptcy, and the Flash Crash, respectively. The bold dotted line represents the average lower bound (by maturity) over all NBER recession dates. The unrestricted bounds are estimated using parameter values from Table 1. The bounds are annualized and reported in percentages.

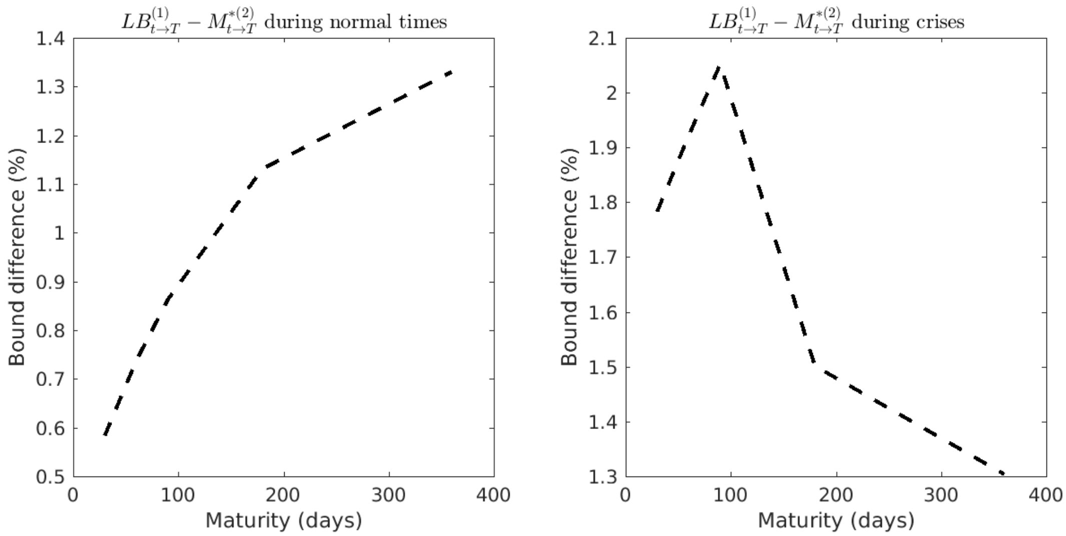


Fig. 4. **Term Structure Differences:** This figure plots the difference between the average conditional lower bound term structures in Figure 3 and that implied by the Martin (2017) during normal (left) and turbulent (right) times. The bold dotted line represents the average difference between our lower bound and the Martin (2017) (by maturity) over all non-NBER recession dates. In the figure on the right, we plot the average difference over all NBER recession dates. We use unrestricted lower bounds from equation (22) estimated using parameter values from Table 1. The bounds are annualized and reported in percentages.

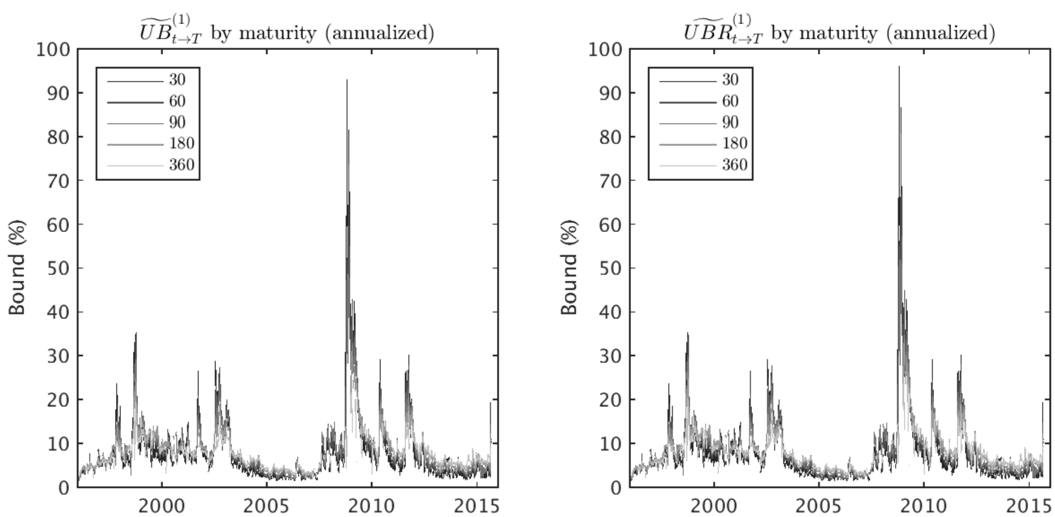


Fig. 5. **Unrestricted and Restricted Upper Bounds on the Expected Excess Market Return:** The bound measures are computed for all five maturities considered herein. The unrestricted bounds are estimated using parameter values from Table 1. The bounds are annualized and reported in percentages. The left graph presents the unrestricted bounds (31), and the right graph presents the restricted bounds (33).

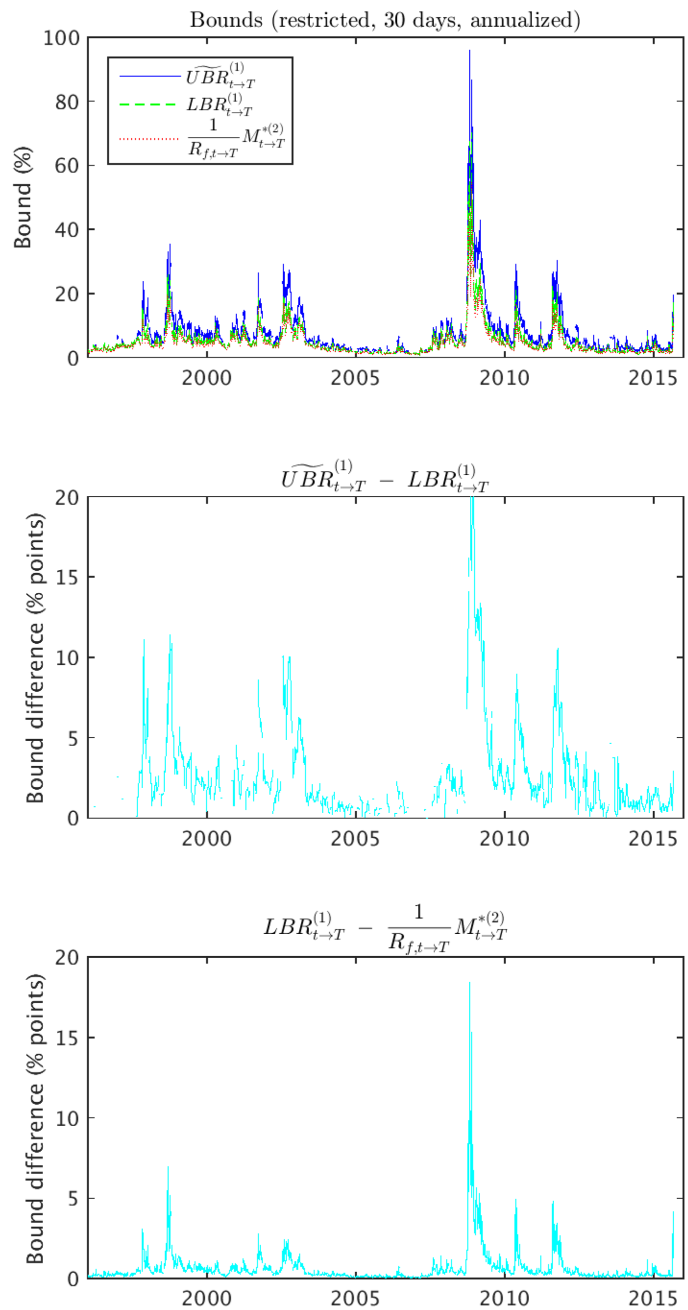


Fig. 6. **Restricted Bounds on the Expected Excess Market Return:** The bound measures are computed using the 30-day maturity options and estimated parameter values from Table 1. The bounds are annualized and reported in percentages. The top graph presents the upper bound (33), the lower bound (26) and the lower bound derived in Martin (2017). The graph in the middle shows the difference between the upper bound and the lower bound. The bottom graph presents the difference between the lower bound and the Martin (2017) bound.

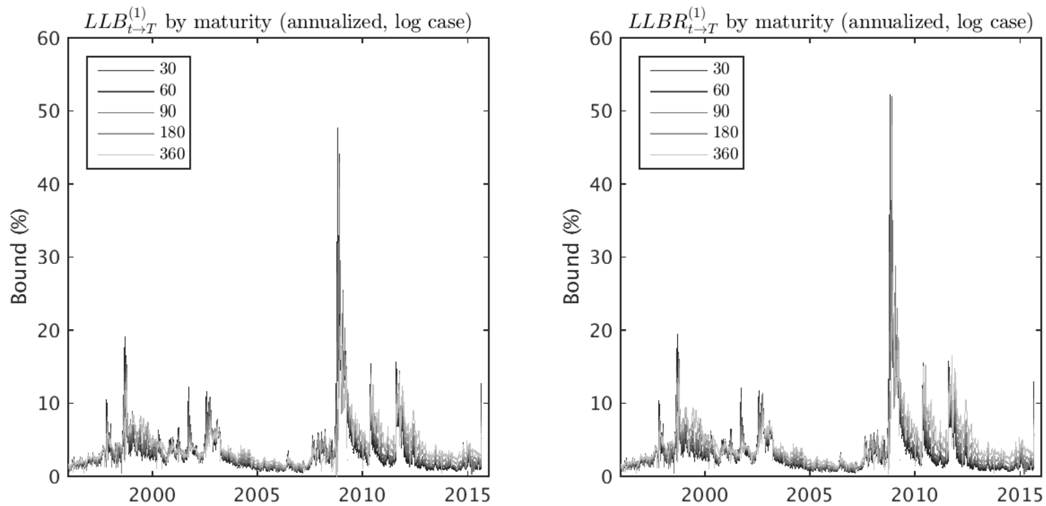


Fig. 7. **Unrestricted and Restricted Lower Bounds on the Expected Excess Log Market Return:** The bound measures are computed for all five maturities considered herein. The unrestricted bounds are estimated using parameter values from Table 1. The bounds are annualized and reported in percentages. The left graph presents the unrestricted bounds (45), and the right graph presents the restricted bounds (46).

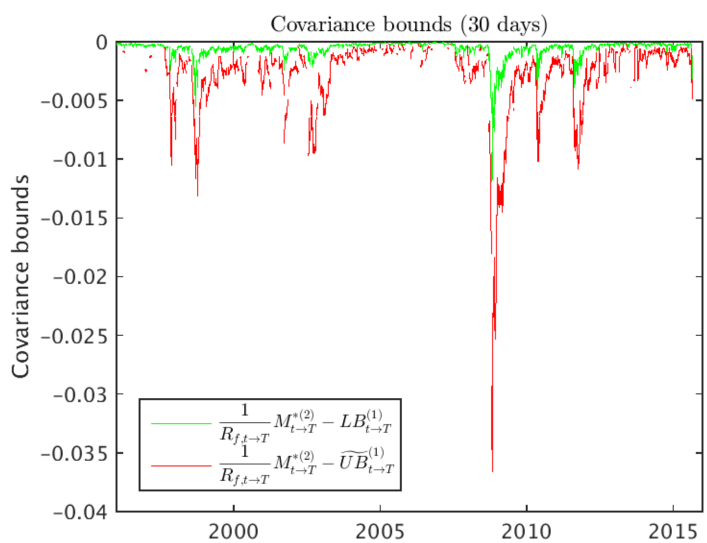


Fig. 8. Covariance bounds from inequality (43) at the 30-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.

Conditional Expected Market Return

Internet Appendix: Not for Publication

Abstract

This Internet Appendix presents additional results. Section A presents the approach used to approximate integrals that are used to compute various moments. Section B presents figures that display (i) risk-neutral moments and truncated risk-neutral moments, and (ii) the bounds on expected excess returns.

A. Approximation of Integrals Used to Compute the Risk-Neutral Moments

A.1. Risk-neutral Moments of Simple Returns

We discretize expressions for the risk-neutral moments of simple return (B3) as follows. Let $\{K_i\}$ represent all available out-of-the money strikes for a particular date and maturity where $i \in \{1,...,N\}$. Define the step size as follows:

$$\Delta I(K_i) \equiv \begin{cases} \frac{K_{i+1}-K_{i-1}}{2}, & for\, 0 \leq i \leq N\,(with\, K_{-1} \equiv 2K_0 - K_1\, and\, K_{N+1} \equiv 2K_N - K_{N-1} \\ 0, & else \end{cases}.$$

Then the risk-neutral moments can be approximated as

$$E_t^*\left[(R_{M,t\to T} - R_{f,t\to T})^n\right] \approx \frac{n(n-1)R_{f,t\to T}}{S_t^2} \left(\begin{array}{c} \sum\limits_{K_i \leq R_{f,t\to T}S_t} \left(\frac{K_i}{S_t} - R_{f,t\to T}\right)^{n-2} P_t[K_i]\,\Delta I(K_i) \\ + \sum\limits_{K_i > R_{f,t\to T}S_t} \left(\frac{K_i}{S_t} - R_{f,t\to T}\right)^{n-2} C_t[K_i]\,\Delta I(K_i) \end{array} \right).$$

A.2. Risk-neutral Moments in Down Market

In this subsection, we show how we approximate the integrals in Equation (B4). To approximate (B4), we first approximate $\text{Prob}_t^*\left[\frac{S_T}{S_t} \leq k_0\right]$. To compute $\text{Prob}_t^*\left[\frac{S_T}{S_t} \leq k_0\right]$, we first observe that, for $K = S_t k_0$

$$\left(k_0 - \frac{S_{t+1}}{S_t}\right)^+ = \left(\frac{S_t k_0 - S_{t+1}}{S_t}\right) 1_{S_t k_0 > S_{t+1}} = \left(\frac{K - S_{t+1}}{S_t}\right) 1_{K > S_{t+1}}. \tag{A1}$$

Thus

$$\frac{\partial (K - S_{t+1})^+}{\partial K} = 1_{K>S_{t+1}}, \text{ and} \frac{1}{S_t}\frac{\partial (K - S_{t+1})^+}{\partial K} = \frac{1}{S_t} E^*\left[1_{K>S_{t+1}}\right].$$

Therefore,

$$\frac{1}{S_t}\frac{\partial (K - S_{t+1})^+}{\partial K} = \frac{1}{S_t} E_t^*\left[1_{K>S_{t+1}}\right] = \frac{1}{S_t}\text{Prob}^*\left[S_{t+1} < K\right] \tag{A2}$$

and

$$\frac{\partial P_t[K]}{\partial K}=\frac{1}{R_{f,t}}\text{Prob}^*[S_{t+1}<K],$$

where $P_t[K]$ is the price of a put option with maturity K. We can use the center difference as a proxy for $\frac{\partial P_t[K]}{\partial K}$

$$\frac{\partial P_t[K]}{\partial K}=\left(\frac{P_t[K+\Delta]-P_t[K-\Delta]}{2\Delta}\right).$$

Hence,

$$\text{Prob}^*[S_{t+1}<K]=R_{f,t}\left(\frac{P_t[K+\Delta]-P_t[K-\Delta]}{2\Delta}\right).$$

Finally,

$$\text{Prob}^*[S_{t+1}<k_0S_t]=R_{f,t}\left(\frac{P_t[k_0S_t+\Delta]-P_t[k_0S_t-\Delta]}{2\Delta}\right).$$

We discretize the expressions as follows. Let K_- and K_+ be the first out-of-the-money put strike below and above k_0S_t, respectively. Define the step size $\Delta I(K_i)$ as above. Let $\{K_i\}$ be the set of strikes available that are below k_0S_t on a given date and for a given maturity. We approximate the risk-neutral probability as

$$Prob_t^*[S_{t+1}<k_0S_t]\approx R_{f,t}\left(\frac{P_t[K_+]-P_t[K_-]}{K_+-K_-}\right).$$

We approximate the price of a put with strike k_0S_t as the linearly interpolated price of the nearest two straddling puts:

$$P_t[k_0S_t]\approx\left(\frac{k_0S_t-K_-}{K_+-K_-}\right)P_t[K_-]+\left(\frac{K_+-k_0S_t}{K_+-K_-}\right)P_t[K_+].$$

Finally, we approximate the above integral using the discretization:

$$\int_0^{k_oS_t}\left(\frac{K}{S_t}-R_{f,t}\right)^{n-2}P_t[K]\,dK\approx\sum_{K_i\leq k_0S_t}\left(\frac{K}{S_t}-R_{f,t}\right)^{n-2}P_t[K_i]\,\Delta I(K_i).$$

B. Additional Figures

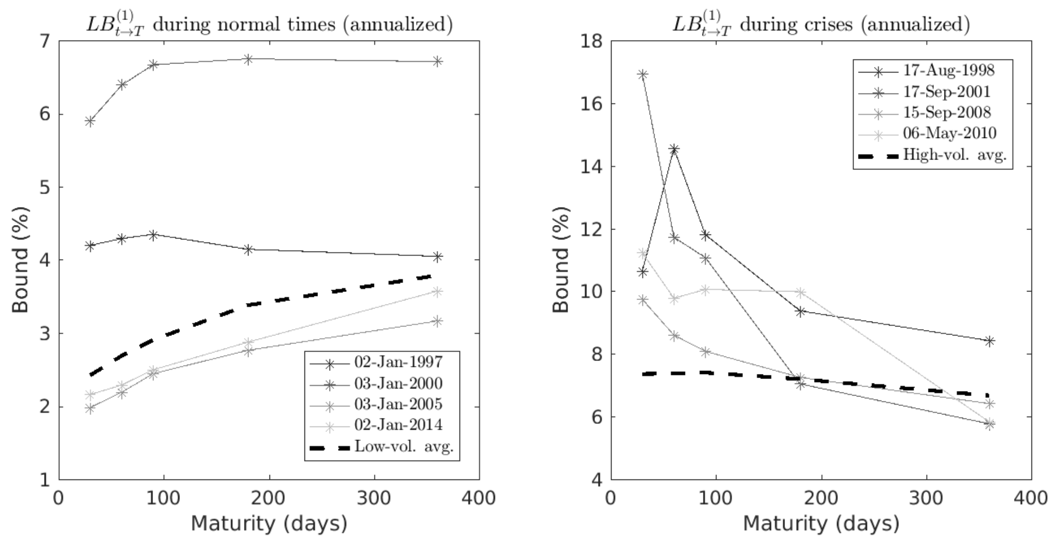


Fig. B.2. **Unrestricted Lower Bound Term Structures:** This figure illustrates the term structure behavior implied by the unrestricted lower bound (22) during normal (left) and turbulent (right) times. In the figure on the left, we plot the implied term structure at four "normal" dates (these were chosen simply as the first date for which we have data during years that were not particularly turbulent). The bold dotted line represents the average lower bound (by maturity) over the three low-volatility periods identified in Sichert (2018). In the figure on the right, we plot the implied term structure at four "turbulent" dates. These include the Russian debt crisis, the September 11 terrorist attacks, the Lehman Brothers bankruptcy, and the Flash Crash, respectively. The bold dotted line represents the average lower bound (by maturity) over the two high-volatility periods identified in Sichert (2018). The unrestricted bounds are estimated using parameter values from Table 1. The bounds are annualized and reported in percentages.

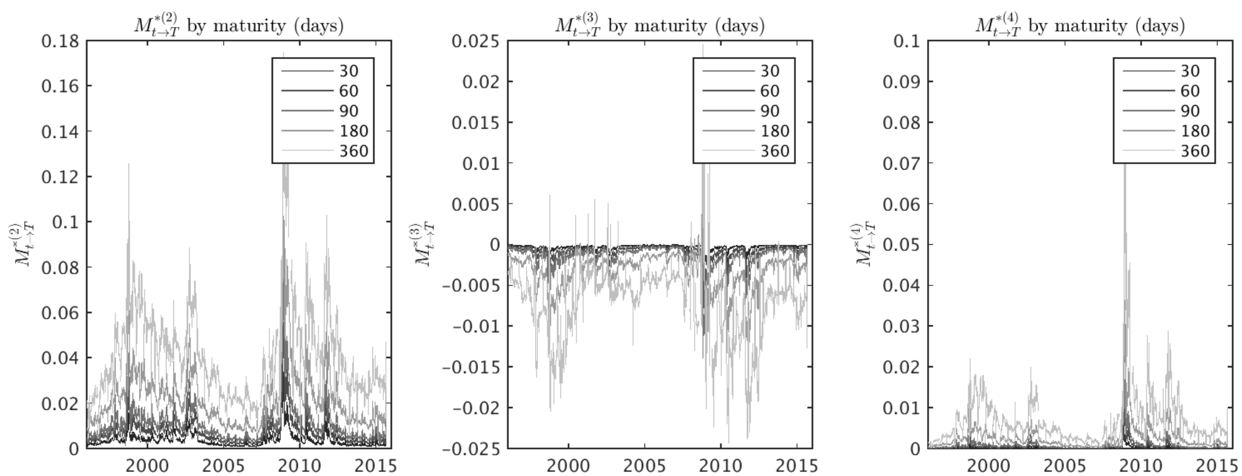


Fig. B.1: Risk-neutral simple return moment measures at maturities of 30, 60, 90, 180, and 360 days.

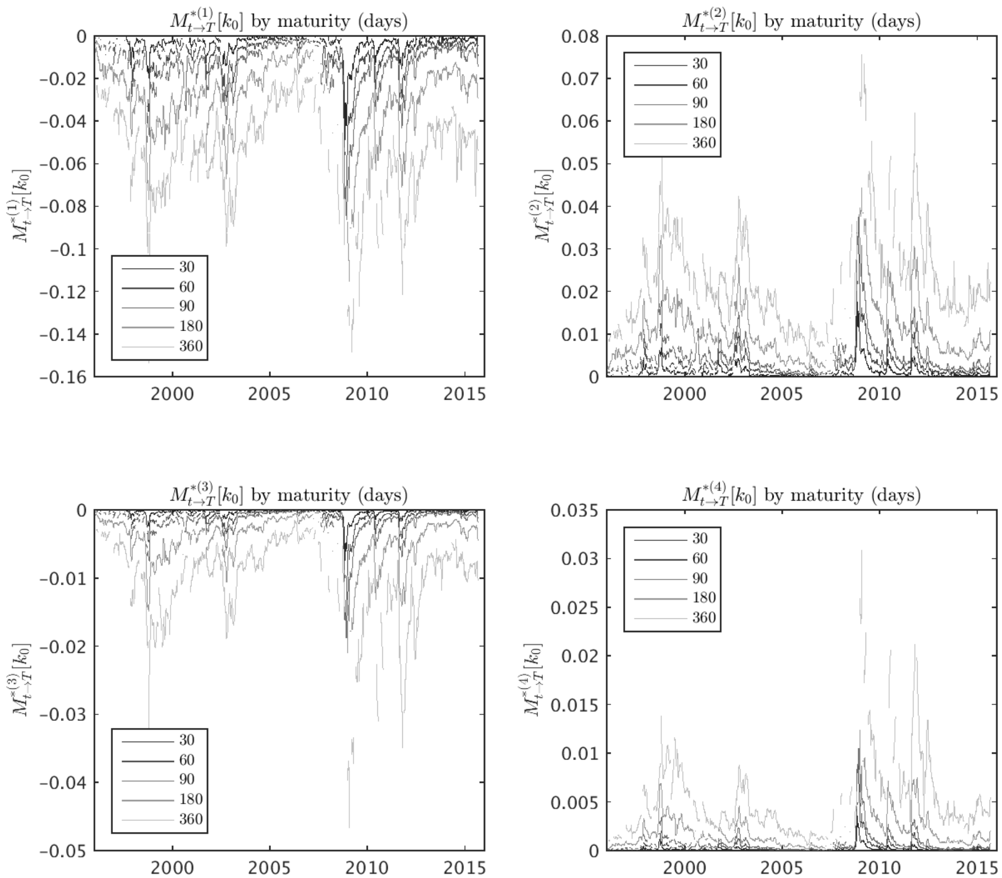
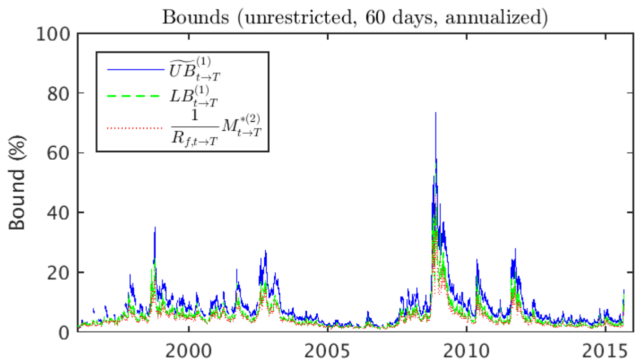
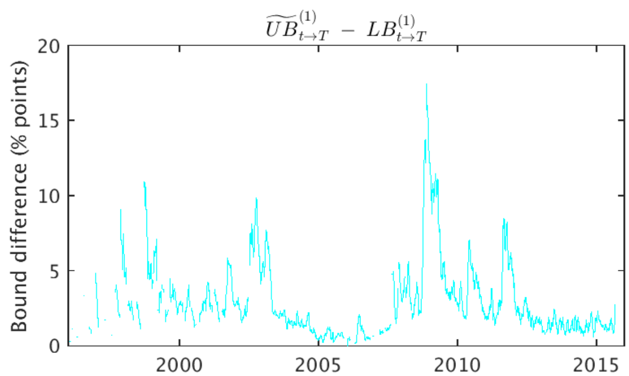


Fig. B.3: Truncated moment measures using $k_0 = 0.8$ at maturities of 30, 60, 90, 180, and 360 days. Raw data is averaged over 20 calendar days to reduce noise.





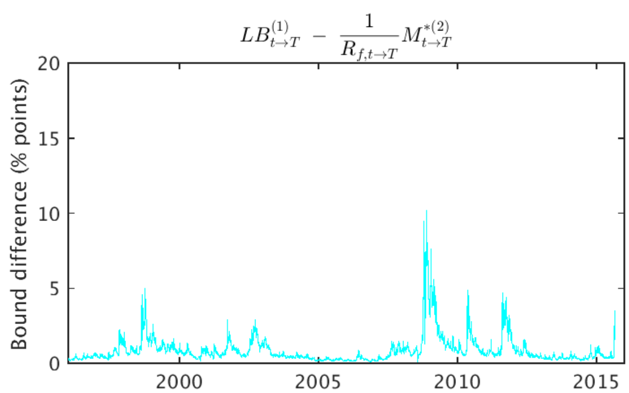


Fig. B.4. Unrestricted bound measures using the 60-day maturity maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (31) and (22), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

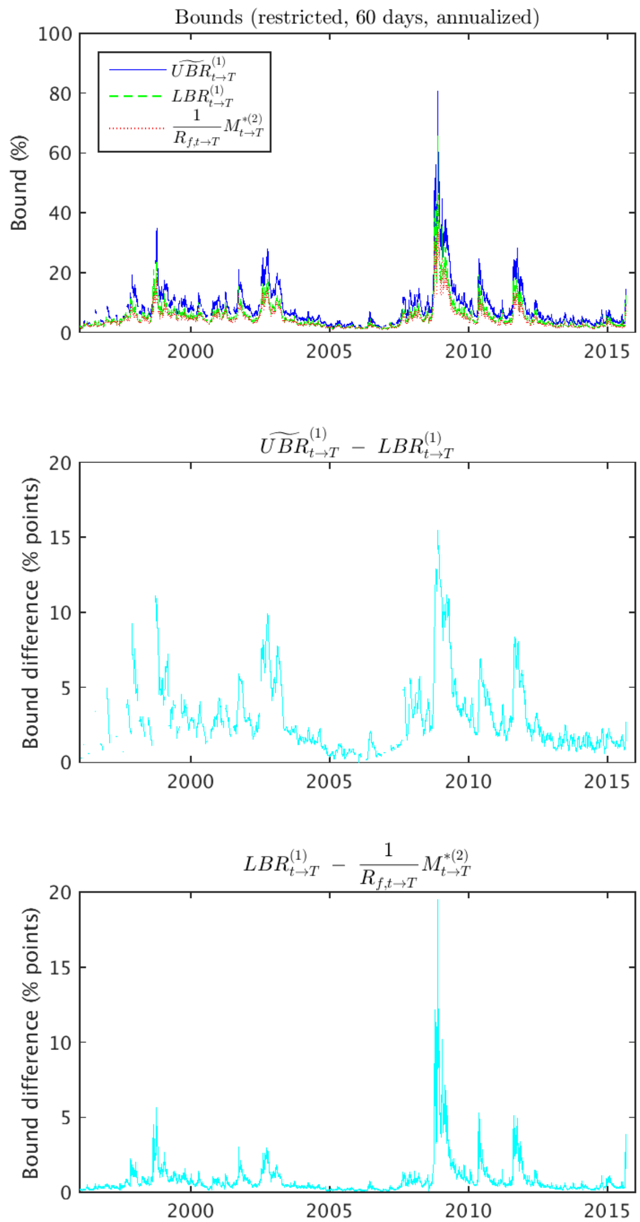


Fig. B.5. Restricted bound measures using the 60-day maturity maturity, annualized. Top: Upper and lower bound measures implied by 33 and 26, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

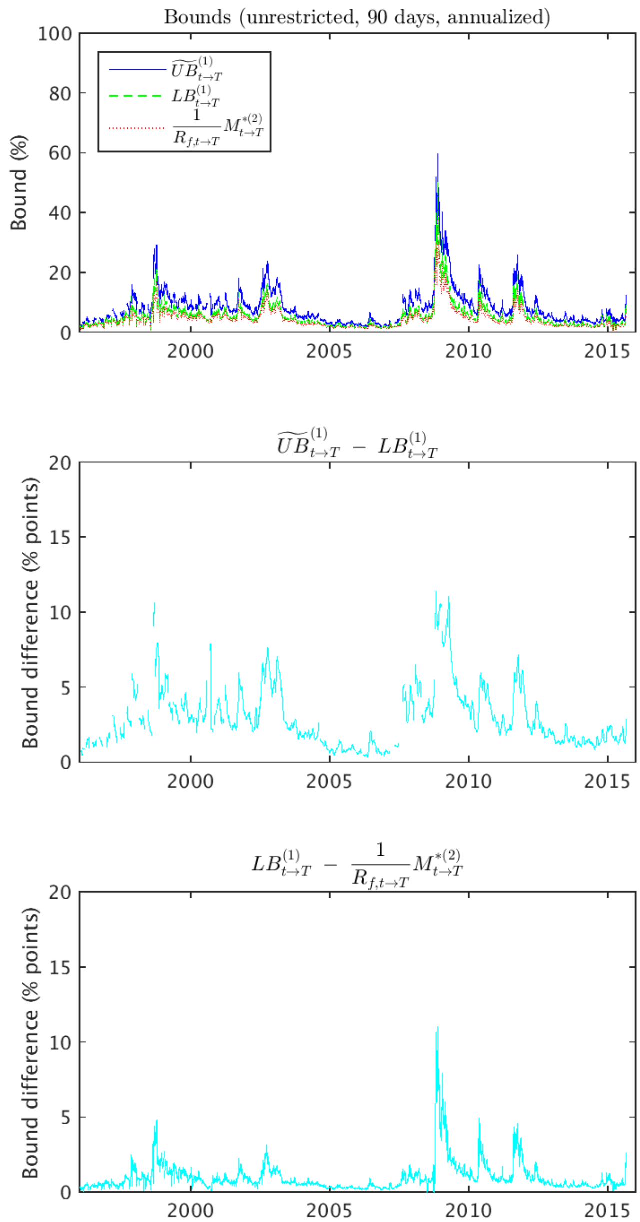


Fig. B.6. Unrestricted bound measures using the 90-day maturity maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (31) and (22), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

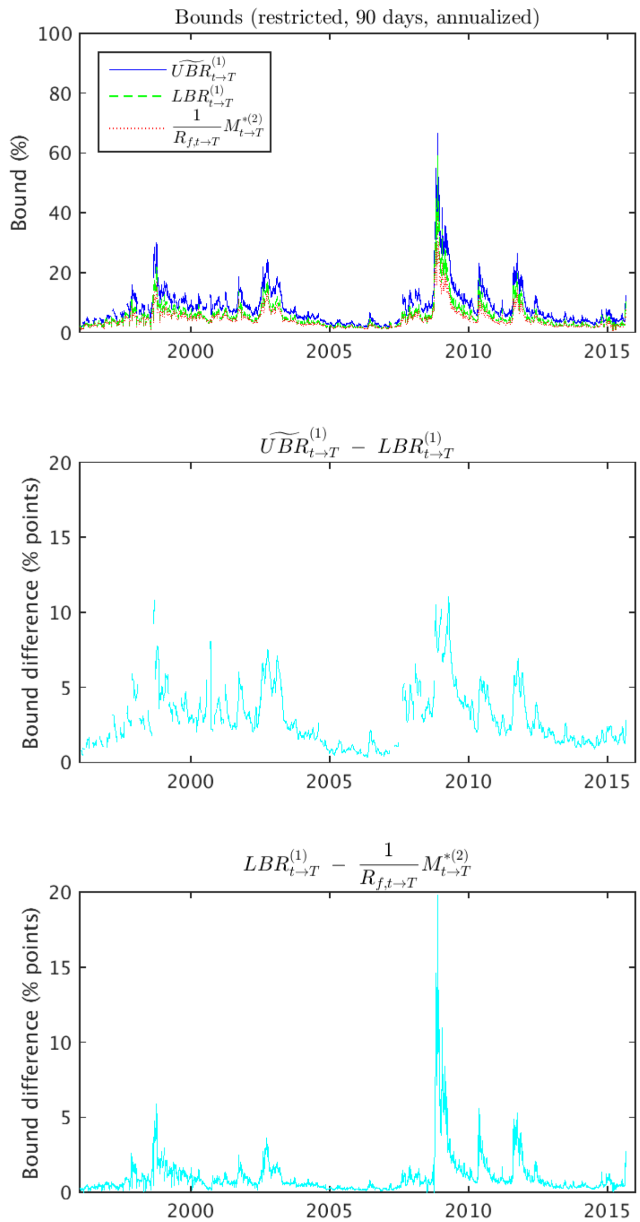


Fig. B.7. Restricted bound measures using the 90-day maturity maturity, annualized. Top: Upper and lower bound measures implied by 33 and 26, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

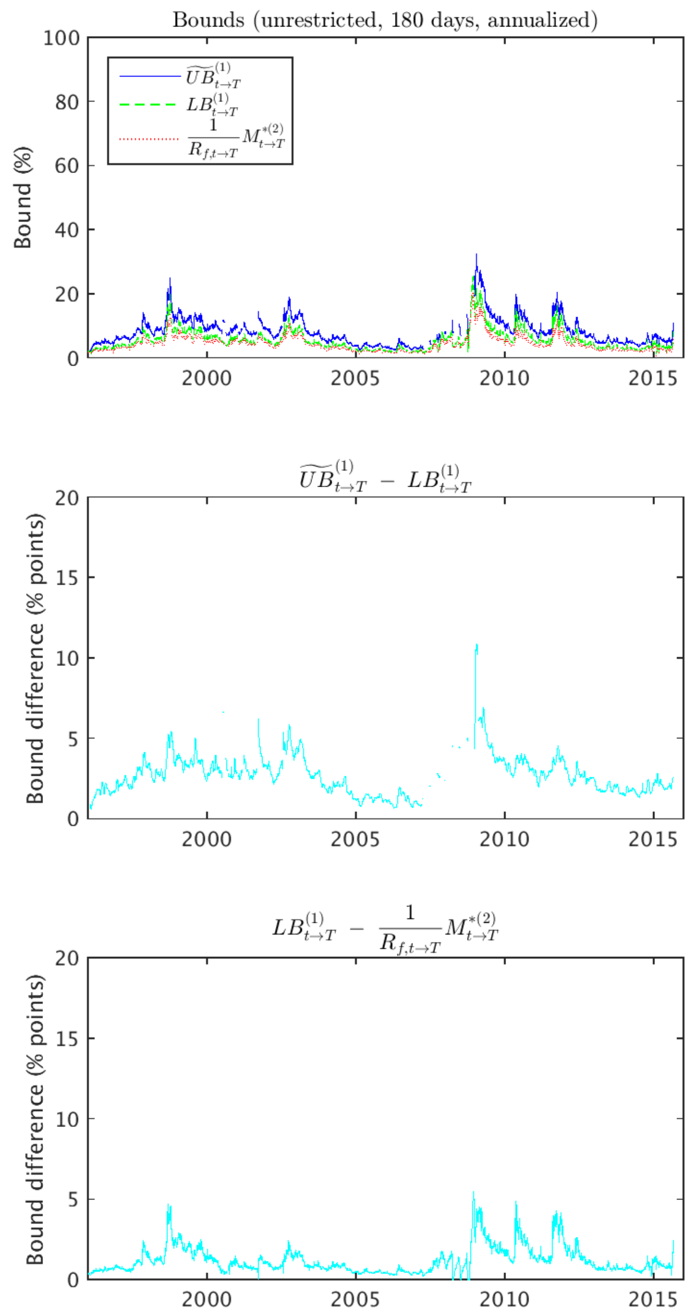


Fig. B.8. Unrestricted bound measures using the 180-day maturity maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (31) and (22), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

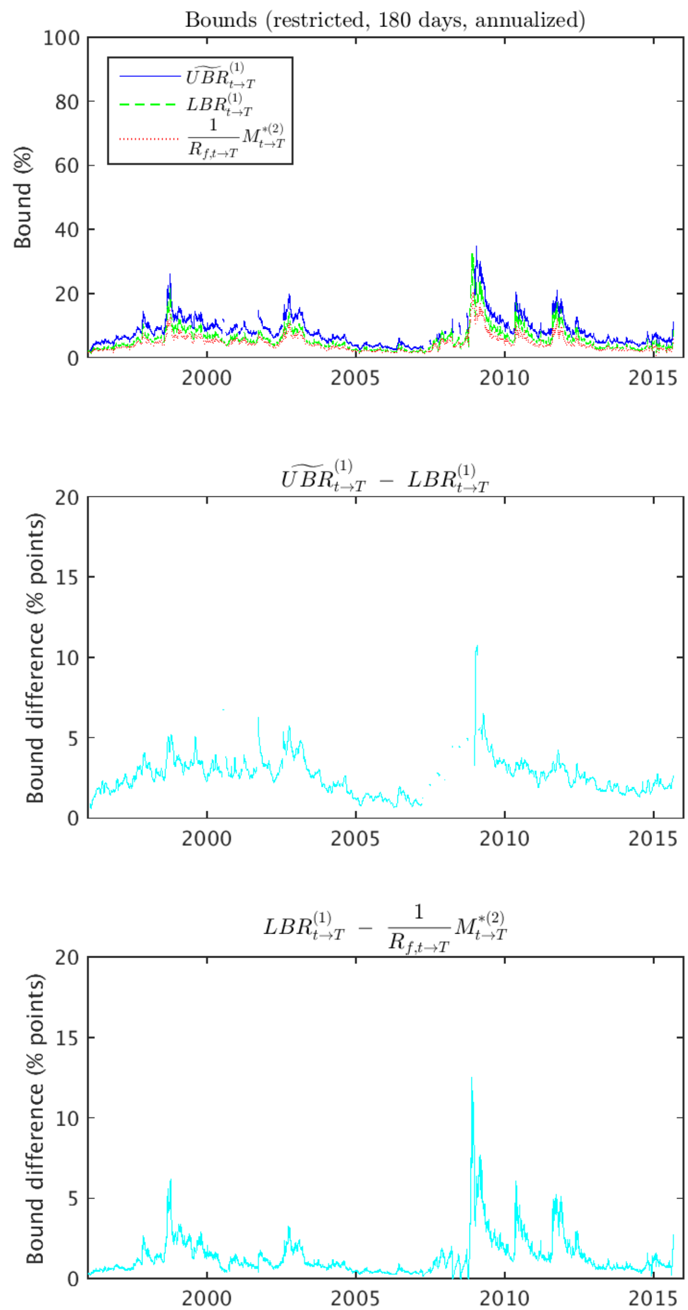


Fig. B.9. Restricted bound measures using the 180-day maturity maturity, annualized. Top: Upper and lower bound measures implied by 33 and 26, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

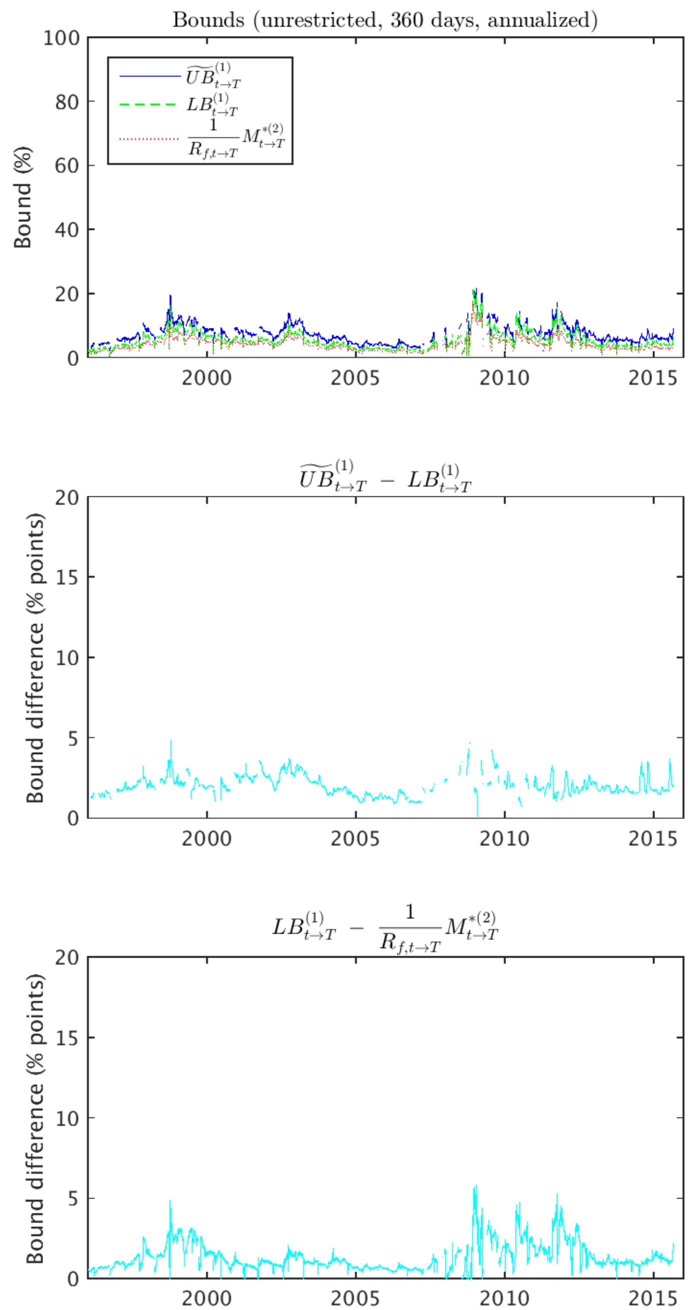


Fig. B.10. Unrestricted bound measures using the 360-day maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (31) and (22), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

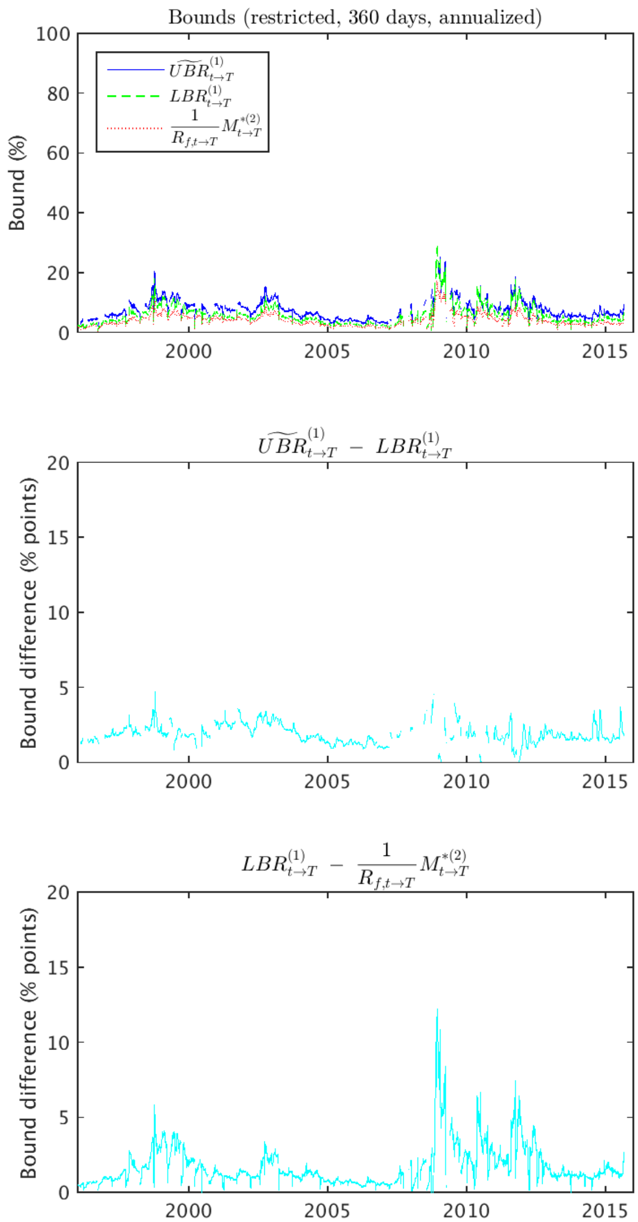


Fig. B.11. Restricted bound measures using the 360-day maturity, annualized. Top: Upper and lower bound measures implied by 33 and 26, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.

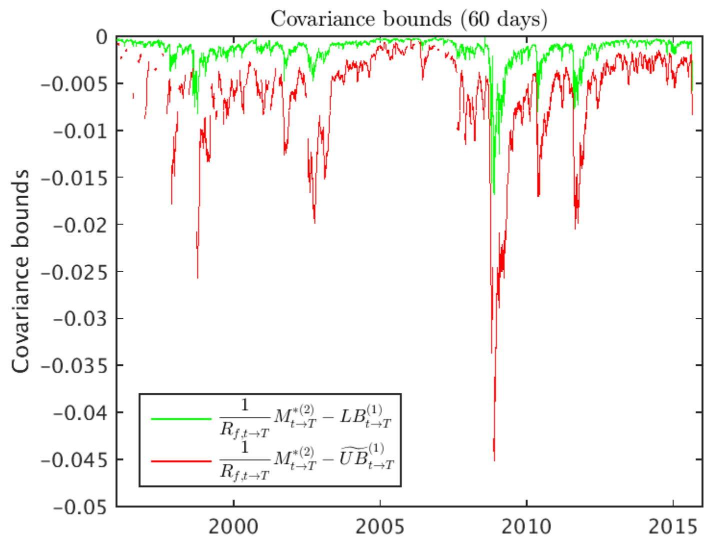


Fig. B.12. Covariance bounds from inequality (43) at the 60-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.

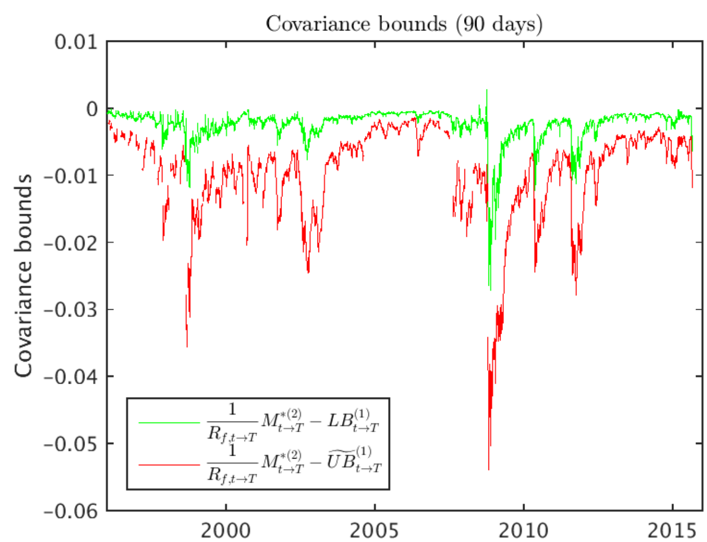


Fig. B.13. Covariance bounds from inequality (43) at the 90-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.

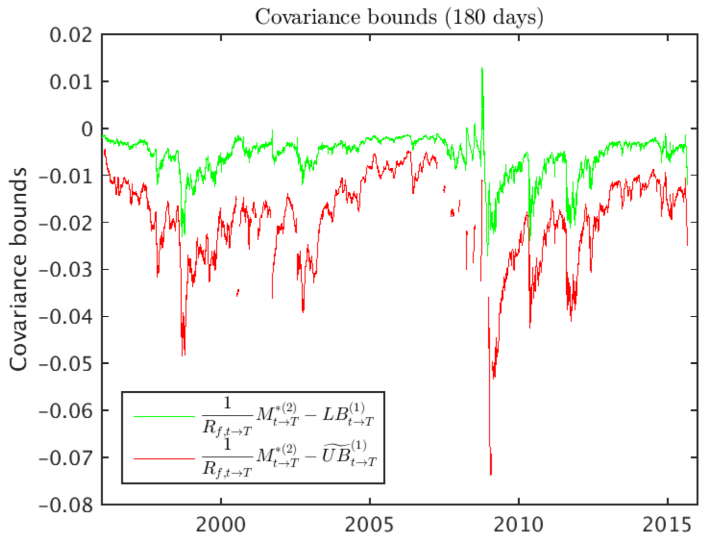


Fig. B.14. Covariance bounds from inequality (43) at the 180-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.

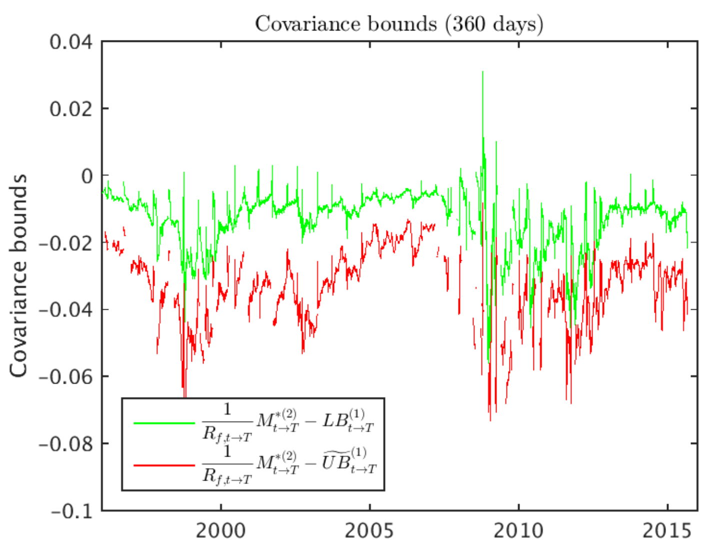


Fig. B.15. Covariance bounds from inequality (43) at the 180-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.