Asset Management: Advanced Investments
Exercise Set 1 – Solutions

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Contents

1 Exercises Slide 35, Set 1: Markowitz approach

a) The Lagrangian problem of this optimization problem is

$$L = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \lambda\left(\mu_p - \boldsymbol{w}^\top \boldsymbol{\mu}\right) + \delta\left(1 - \mathbf{1}^\top \boldsymbol{w}\right)$$

with FOCs $\boldsymbol{w} = \Sigma^{-1}\left(\lambda\boldsymbol{\mu} + \delta\mathbf{1}\right)$, $0 = \mu_p - \mathbf{w}^\top\boldsymbol{\mu}$, $0 = 1 - \mathbf{1}^\top\mathbf{w}$. Stacking the last FOCs for the Lagrangian multipliers into a (2×1)-vector, we get

$$\begin{aligned}
\begin{pmatrix} \mu_p \\ 1 \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\mu}^\top \mathbf{w} \\ \mathbf{1}^\top \mathbf{w} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}^\top \Sigma^{-1}\left(\lambda\boldsymbol{\mu} + \delta\mathbf{1}\right) \\ \mathbf{1}^\top \Sigma^{-1}\left(\lambda\boldsymbol{\mu} + \delta\mathbf{1}\right) \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\mu}^\top \Sigma^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}^\top \Sigma^{-1}\mathbf{1} \\ \mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu} & \mathbf{1}^\top \Sigma^{-1}\mathbf{1} \end{pmatrix} \begin{pmatrix} \lambda \\ \delta \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda \\ \delta \end{pmatrix},
\end{aligned}$$

with the definitions

$$a = \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu} \quad b = \boldsymbol{\mu}^\top \Sigma^{-1} \mathbf{1}$$
$$c = \mathbf{1}^\top \Sigma^{-1} \mathbf{1} \quad d = ac - b^2.$$

If $d \neq 0$, this leads to

$$\begin{pmatrix} \lambda \\ \delta \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} \begin{pmatrix} \mu_p \\ 1 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} c\mu_p - b \\ a - \mu_p b \end{pmatrix}.$$

Finally we set

$$\boldsymbol{k}_1 = \frac{c\boldsymbol{\mu} - b\mathbf{1}}{d}, \qquad \boldsymbol{k}_2 = \frac{a\mathbf{1} - b\boldsymbol{\mu}}{d}$$

into the first FOC to get the result

$$\boldsymbol{w}_p = \Sigma^{-1} \left(\mu_p \boldsymbol{k}_1 + \boldsymbol{k}_2\right).$$

The second derivative of the Lagrangian is

$$\frac{\partial^2 L}{\partial \boldsymbol{w}^\top \partial \boldsymbol{w}} = \Sigma \geq 0,$$

showing that the function is convex, which means that we have indeed found a minimum.

We will show in c) that $d = 0 \Leftrightarrow \boldsymbol{\mu} = k\mathbf{1}$ for some $k \in \mathbb{R}$. But then we obtain for any portfolio $\boldsymbol{w}$,

$$\mu_p = \sum_{i=1}^n w_i \mu_i = k \sum_{i=1}^n w_i = k,$$

meaning that the efficient frontier only consists of one point. The corresponding minimum variance portfolio can be found with the Lagrangian $L = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \delta\left(1 - \mathbf{1}^\top \boldsymbol{w}\right)$ which leads to $\boldsymbol{w} = \Sigma^{-1}\delta\mathbf{1}$ and $1 = \mathbf{1}^\top \Sigma^{-1} \delta \mathbf{1}$, where the latter is equivalent to $\delta = \frac{1}{c}$. So in this case we have $\boldsymbol{w}_p = \frac{1}{c}\Sigma^{-1}\mathbf{1}$.

To check for the signs of the scalars a, b, c, and d, we proceed as follows. Since Σ is positive definite so is its inverse Σ^{-1}. This follows because in general a $N \times N$ matrix A is positive definite if and only if all its eigenvalues are positive, $\lambda_i > 0$ for all $i \in \{1, \cdots, N\}$. For this, we consider the eigendecomposition of A given by

$$A = Q\Lambda Q^{-1},$$

where Q consists of the eigenvectors and $\mathbf{\Lambda}$ is a diagonal matrix where the eigenvalues λ_i are on the diagonal. Its inverse is given by

$$A^{-1} = Q\Lambda^{-1}Q^{-1}_{.}$$

Because Λ^{-1} is a diagonal matrix, its inverse is

$$\left[\Lambda^{-1}\right]_{ii} = \frac{1}{\lambda_i}_{,}$$

showing that also the inverse of A is positive definite.

Therefore, as $\boldsymbol{\mu} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ it follows that $\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu} > 0$ and $\mathbf{1}^\top \Sigma^{-1} \mathbf{1} > 0$. Note that b can either be positive or negative depending on $\boldsymbol{\mu}$.

For d, if $\boldsymbol{x} = \left(\Sigma^{-1/2}\right)^T \mathbf{1}$ ($\Sigma^{-1/2}$ relates to the unique Cholesky decomposition of Σ^{-1}) and $\boldsymbol{y} = \left(\Sigma^{-1/2}\right)^T \boldsymbol{\mu}$, we have $a = \|\boldsymbol{y}\|^2$, $c = \|\boldsymbol{x}\|^2$ and $b = |\boldsymbol{x} \cdot \boldsymbol{y}|$. Hence the expression $d = a \cdot c - b^2$ becomes $\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2 - |\boldsymbol{x} \cdot \boldsymbol{y}|^2 \geq 0$, by the Cauchy-Schwarz inequality. Note that unless $\boldsymbol{\mu} \neq k\mathbf{1}$ for some constant $k \in \mathbb{R}$ the inequality is strict, $d > 0$.

b) We may assume w.l.o.g. that $\boldsymbol{\mu} \neq k\mathbf{1}, k \in \mathbb{R}$, as otherwise there exists only one MVP.

Let there be n assets and let $\boldsymbol{m} = (m_1, \ldots, m_l)^T$ be a set of MVP's with expected returns $\boldsymbol{\mu}_m \in \mathbb{R}^l$. Each portfolio m_i has a portfolio weight vector $\boldsymbol{w}_{m_i} \in \mathbb{R}^n$. Next, let $\boldsymbol{a} \in \mathbb{R}^l, \sum_{i=1}^l a_i = 1$, be a vector of portfolio weights. Then, $\boldsymbol{w}_b = \sum_{i=1}^l a_i \boldsymbol{w}_{m_i}$ is a portfolio of MVP's which can be rewritten as

$$
\begin{aligned}
\boldsymbol{w}_b &= a_1 \boldsymbol{w}_{m_1} + \ldots + a_l \boldsymbol{w}_{m_l} \\
&= a_1 \Sigma^{-1} \left(\mu_{m_1} \boldsymbol{k}_1 + \boldsymbol{k}_2\right) + \ldots + a_l \Sigma^{-1} \left(\mu_{m_l} \boldsymbol{k}_1 + \boldsymbol{k}_2\right) \\
&= \Sigma^{-1} \left(\mu_h \boldsymbol{k}_1 + \boldsymbol{k}_2\right),
\end{aligned}
$$

where $\mu_h = \boldsymbol{a}^T \boldsymbol{\mu}_m$ and where we have used that $\sum_{i=1}^{l} a_i = 1$. Thus $\boldsymbol{w}_b$ is of the form of an MVP.

Given two distinct MVP's $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ with mean returns μ_1 and μ_2, we can construct a portfolio $\lambda w_1 + (1-\lambda) w_2$ with arbitrary return μ_p which lies on the mean-variance frontier by solving the equation $\lambda \mu_1 + (1-\lambda)\mu_2 = \mu_p \Leftrightarrow \lambda = \frac{\mu_p - \mu_2}{\mu_1 - \mu_2}$.

c) Using a), we compute for $\boldsymbol{\mu} \neq k\mathbf{1}, k \in \mathbb{R}$,

$$
\begin{aligned}
\mathrm{Cov}(R_p, R_q) &= \mathrm{Cov}(\boldsymbol{w}_p^T \boldsymbol{R}, \boldsymbol{w}_q^T \boldsymbol{R}) = \boldsymbol{w}_p^T \Sigma \boldsymbol{w}_q \\
&= (\mu_p \boldsymbol{k}_1 + \boldsymbol{k}_2)^T \boldsymbol{w}_q = \mu_p \frac{c\mu_q - b}{d} + \frac{a - b\mu_q}{d} \\
&= \frac{c}{d}\mu_p \mu_q - \frac{b}{d}\mu_p - \frac{b}{d}\mu_q + \frac{b^2 + d}{cd} = \frac{c}{d}(\mu_p - \frac{b}{c})(\mu_q - \frac{b}{c}) + \frac{1}{c}.
\end{aligned}
$$

For $\boldsymbol{\mu} = k\mathbf{1}$ we easily obtain $\mathrm{Cov}(R_p, R_q) = \boldsymbol{w}_p^T \Sigma \boldsymbol{w}_q = \frac{1}{c}\mathbf{1}^\top \Sigma^{-1} \Sigma \boldsymbol{w}_q = \frac{1}{c}$.

d) From a) we know that

$$
\boldsymbol{w} = \delta \Sigma^{-1} \mathbf{1} + \lambda \Sigma^{-1} \boldsymbol{\mu} \tag{1}
$$

and using the definitions for δ and λ we find for a portfolio on the efficient frontier with $\boldsymbol{\mu} \neq k\mathbf{1}$

$$
\sigma_p^2(\mu_p) = \boldsymbol{w}^T \Sigma \boldsymbol{w} = \delta^2 c + 2\delta\lambda b + \lambda^2 a = \frac{c\mu_p^2 - 2b\mu_p + a}{d}. \tag{2}
$$

For minimizing the variance, we compute

$$\frac{\partial \sigma_p^2}{\partial \mu_p} = \frac{2c\mu_p - 2b}{d} = 0, \qquad \frac{\partial^2 \sigma_p^2}{\partial \mu_p^2} = \frac{2c}{d} > 0 \quad \Rightarrow \mu_g = \frac{b}{c} \tag{3}$$

and plug this into (2) and obtain

$$\sigma_g^2 = \frac{-b^2/c + a}{d} = \frac{1}{c}.$$

Using (3) we find that $\lambda = 0, \delta = \frac{1}{c}$ and hence with (1)

$$\boldsymbol{w}_g = \frac{1}{c}\Sigma^{-1}\mathbf{1}.$$

We notice that we get the same result for σ_g^2 and $\boldsymbol{w}_g$ if $\boldsymbol{\mu} = k\mathbf{1}$ holds for some $k \in \mathbb{R}$.

e) We rewrite

$$\begin{aligned}\boldsymbol{w}_p &= \Sigma^{-1}\left[(\mu_p - \mu_g + \mu_g)\boldsymbol{k}_1 + \boldsymbol{k}_2\right] = \Sigma^{-1}(\mu_g\boldsymbol{k}_1 + \boldsymbol{k}_2) + (\mu_p - \mu_g)\Sigma^{-1}\boldsymbol{k}_1 \\ &= \boldsymbol{w}_g + (\mu_p - \mu_g)\boldsymbol{z}^\star.\end{aligned}$$

Moreover, notice that $\mathbf{1}^T\Sigma^{-1}\boldsymbol{k}_1 = \frac{1}{d}(cb - bc) = 0$.

2 Exercises Slide 42, Set 1: Zero Beta Portfolio

a) We first note $\mathrm{Cov}(R_q, R_p) = \mathrm{Cov}(\beta_0 + \beta_1 R_{0p} + \beta_2 R_p + \epsilon, R_p) = \beta_2\sigma_p^2$. Thus, $\beta_2 = \beta_{pq}$. Similarly, $\beta_1 = \mathrm{Cov}(R_q, R_{0p})/\sigma_{0p}^2$. Taking expectations on both sides of the regression equation gives $\beta_0 = \mu_q - \beta_1\mu_{0p} - \beta_2\mu_p$. Using the proposition on Slide 8, write $\mathrm{Cov}(R_p, R_p) = e\left(\mu_p - f\right)^2 + g$ with $e = c/d, f = b/c, g = 1/c$, and $\mathrm{Cov}(R_{0p}, R_p) = e\left(\mu_p - f\right)\left(\mu_{0p} - f\right) + g = 0$ to get $\mu_{0p} = f - \frac{g}{e(\mu_p - f)}$. Using the latter equation, we get

$$\begin{aligned}\mathrm{Cov}(R_{0p}, R_q) &= e\left(\mu_q - f\right)\left(\mu_{0p} - f\right) + g = g\frac{\mu_p - \mu_q}{\mu_p - f}, \\ \sigma_{0p}^2 &= e\left(\mu_{0p} - f\right)^2 + g = g\left(\frac{g}{e(\mu_p - f)^2} + 1\right).\end{aligned}$$

Using these formulas for β_1 gives

$$\begin{aligned}
\beta_1 &= \frac{g\frac{\mu_p-\mu_q}{\mu_p-f}}{g\left(\frac{g}{e(\mu_p-f)^2}+1\right)} = \frac{e\left(\mu_p-f\right)\left(\mu_p-\mu_q\right)}{\sigma_p^2} = \frac{e\left(\mu_p-f\right)\left(\mu_p-f-\left(\mu_q-f\right)\right)+g-g}{\sigma_p^2} \\
&= \frac{e\left(\mu_p-f\right)^2+g-\left(e\left(\mu_p-f\right)\left(\mu_q-f\right)+g\right)}{\sigma_p^2} = \frac{\sigma_p^2-\mathrm{Cov}(R_p,R_q)}{\sigma_p^2} = 1-\beta_{pq}.
\end{aligned}$$

Finally, $\beta_0 = \mu_q - (1-\beta_{pq})\mu_{0p} - \beta_{pq}\mu_p = \mu_q - \mu_{0p} - \beta_{pq}\left(\mu_p-\mu_{0p}\right)$. Consider

$$\begin{aligned}
\beta_{pq}\left(\mu_p-\mu_{0p}\right) &= \frac{\mathrm{Cov}(R_p,R_q)}{\sigma_p^2}\left(\mu_p-f+\frac{g}{e(\mu_p-f)}\right) \\
&= \frac{\mathrm{Cov}(R_p,R_q)}{e(\mu_p-f)^2+g}\,\frac{e(\mu_p-f)^2+g}{e(\mu_p-f)} = \frac{\mathrm{Cov}(R_p,R_q)}{e(\mu_p-f)} \\
&= \frac{e(\mu_p-f)(\mu_q-f)+g}{e(\mu_p-f)} = \mu_q-f+\frac{g}{e(\mu_p-f)} = \mu_q-\mu_{0p}.
\end{aligned}$$

Plugging in gives $\beta_0 = 0$ and hence $\mu_q = \mu_{0p} + \beta_{pq}(\mu_p - \mu_{0p})$.

b) Notice that the formula depends on R_{0p} only via its mean. Moreover, 1d) shows that some portfolio $\boldsymbol{w}_q$ is a zero-beta portfolio if and only if $\frac{c}{d}(\mu_p-\frac{b}{c})(\mu_q-\frac{b}{c}) = -\frac{1}{c}$, meaning that all zero-beta portfolios have the same expected return. Consequently, the formula holds for any of them.

c) First, by 1d) it does not exist for the global MVP.

Moreover, an important assumption behind the Zero-Beta portfolio is that short-sales are possible. To obtain Zero-Beta portfolios we typically would have to short sell some assets. If there are short-sales constraints the Zero-Beta portfolio generally fails to exist.

3 Exercises Slide 47, Set 1: MV Frontier with a Risk-Free Asset

a) We want to derive the weights in the risky assets for a portfolio on the frontier with return μ_p, if there exists a risk-free asset.

We relax the assumption $\boldsymbol{w}_p^\top \mathbf{1} = 1$ and denote the fraction invested in the risk-free asset by $1 - \boldsymbol{w}_p^\top \mathbf{1}$.

The optimization problem then reads

$$\min_{\boldsymbol{w}_p} \frac{1}{2} \boldsymbol{w}_p^\top \Sigma \boldsymbol{w}_p, \quad \text{s.t.} \quad \boldsymbol{w}_p^\top \boldsymbol{\mu} + (1 - \boldsymbol{w}_p^\top \mathbf{1}) R_f = \mu_p,$$

such that the Lagrangian reads $L = \frac{1}{2} \boldsymbol{w}_p^\top \Sigma \boldsymbol{w}_p + \delta(\boldsymbol{w}_p^\top \boldsymbol{\mu} + (1 - \boldsymbol{w}_p^\top \mathbf{1}) R_f - \mu_p)$. Then the first order conditions are

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{w}_p} &= \Sigma \boldsymbol{w}_p + \delta(\boldsymbol{\mu} - R_f \mathbf{1}) = 0 \\ \frac{\partial L}{\partial \delta} &= \boldsymbol{w}_p^\top \boldsymbol{\mu} + (1 - \boldsymbol{w}_p^\top \mathbf{1}) R_f - \mu_p = 0. \end{aligned}$$

Solving for $\boldsymbol{w}_p$ yields

$$\boldsymbol{w}_p = -\Sigma^{-1} \delta(\boldsymbol{\mu} - \mathbf{1} R_f). \tag{4}$$

Plugging this expression into the second FOC using the assumption $\boldsymbol{\mu} \neq R_f \mathbf{1}$, we get

$$\delta = -\frac{\mu_p - R_f}{(\boldsymbol{\mu} - R_f \mathbf{1})^\top \Sigma^{-1} (\boldsymbol{\mu} - R_f \mathbf{1})}.$$

Using this expression for δ and insert it into (4), the risky portfolio weights $\boldsymbol{w}_p$, we get

$$\boldsymbol{w}_p = \frac{\mu_p - R_f}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \Sigma^{-1} \boldsymbol{\mu}^e.$$

The second derivative of the Lagrangian is $\frac{\partial^2 L}{\partial \boldsymbol{w}_p^\top \partial \boldsymbol{w}_p} = \Sigma \geq 0$, showing that the function is convex, which means that we have indeed found a minimum.

b) The weighted portfolio is given by $\boldsymbol{w}_p^w = \frac{\boldsymbol{w}_p}{\mathbf{1}^T \boldsymbol{w}_p} = \frac{\Sigma^{-1} \boldsymbol{\mu}^e}{\mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}^e}$, which is independent of μ_p.

c) Notice that for a tangency portfolio with weights $\boldsymbol{w}_T$ we have $\mathbf{1}^\top \boldsymbol{w}_T = 1$, and, as it lies on the efficient frontier, (4) gives us (with $\boldsymbol{\mu} \neq R_f \mathbf{1}$)

$$1 = \mathbf{1}^\top \boldsymbol{w}_T = -\delta \mathbf{1}^\top \Sigma^{-1} \left(\boldsymbol{\mu} - \mathbf{1} R_f\right) = -\delta \left(b - c R_f\right) \quad \Leftrightarrow \quad \delta = -\frac{1}{\left(b - c R_f\right)}$$

$$\Rightarrow \boldsymbol{w}_T = \frac{\Sigma^{-1} \left(\boldsymbol{\mu} - R_f \mathbf{1}\right)}{\left(b - c R_f\right)}. \tag{5}$$

d) Taking the riskless asset position into account, the overall expected return of a portfolio is given by

$$\mu_p = \boldsymbol{\mu}^\top \boldsymbol{w} + (1 - \boldsymbol{w}^\top \mathbf{1}) R_f = R_f + (\boldsymbol{\mu} - \mathbf{1} R_f)^\top \boldsymbol{w}.$$

Then using (5) with $b - cR_f = \mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}^e$, we find for the expected return of the tangency portfolio

$$\mu_T = R_f + \frac{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}^e}.$$

To find the tangency portfolio's standard deviation we use $\sqrt{\boldsymbol{w}_T^\top \Sigma \boldsymbol{w}_T}$ which leads to

$$\sigma_T = \frac{\sqrt{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}^e}.$$

4 To Recall: Some Conceptual Questions

a) Explain the difference between the Markowitz Model and the CAPM.

b) What are the assumptions underlying the Capital Asset Pricing Model?

c) Explain Tobin's Seperation theorem and its implications.

d) Standard deviation and beta both measure risk. Which risk do they measure within the CAPM?

e) What are the pros and cons when measuring performance with alpha, i.e., the regression slope from the Capital Asset Pricing Model (or a factor model)?

Solution: Conceptual Questions

a) CAPM is used in finance to determine a theoretically appropriate required rate of return (and thus the price if expected cash flows can be estimated) of an asset, if that asset is to be added to an already well-diversified portfolio, given that asset's non-diversifiable risk. The CAPM formula takes into account the asset's sensitivity to non-diversifiable risk (also known as systematic risk or market risk), as well as the expected return of the market and the expected return of a theoretical risk-free asset. It builds on the work of Markowitz on portfolio diversification in a mean-variance framework. Markowitz only looks at the optimization problem of a single investor, but not on the impact of this optimization on an aggregate market level by using equilibrium arguments.

b)
- The model assumes that the variance of returns is an adequate measurement of risk. This might be justified under the assumption of a probability distribution completely described by the first two moments (e.g. normally distributed returns) or also when a quadratic utility function is applied.
- It assumes that investors are rational and risk-averse.
- Moreover, investors are price-takers and can lend and borrow any amount under the same risk-free rate.
- The model assumes that all investors have access to the same information and agree about the risk and expected return of all assets. (Homogeneous expectations assumption)
- No taxes or transaction costs.
- No preference between markets and assets for individual investors, and investors choose assets solely as a function of their risk-return profile.
- It also assumes that all assets are infinitely divisible.

c) If there is a risk-free asset with return R_f, then the weights for a portfolio on the frontier with return μ_p are

$$\boldsymbol{w}_p = \frac{\mu_p - R_f}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \Sigma^{-1} \boldsymbol{\mu}^e,$$

where $\boldsymbol{\mu}^e$ denotes the vector of excess return of the assets over the risk-free rate (see Exercise 3). Tobin's Separation Theorem states that the relative portfolio fraction is independent of the choice of the target return μ_p. The implications for portfolio delegation are:

- The separation theorem implies that any investor's portfolio decision is the same.
- The only difference between investors is the relative portion between the risky portfolio and the risk-free interest rate R_f.
- This portion depends on the investor's risk aversion.

d) Standard deviation measures 'risk' arising from both systematic and unsystematic sources. The beta only measures the risk with respect to the variance from the market portfolio.

e) CAPM linearly links the return of each portfolio and asset to the market return. Systematic deviations give rise to a statistically significant intercept. This intercept is commonly referred to as "alpha" or skill. The advantage of this approach is that it is intuitive. However, using the CAPM we implicitly assume that this is the right model to describe returns. This assumption is a strong one. Even if we have multiple betas, we might fail to correctly assess alpha, i.e., the extra performance. If not all betas are accounted for, then there will be a bias:

$$\begin{aligned} r_0(t) &= \underbrace{\alpha(t)}_{\text{Correct Skill}} + \sum_{i=1}^{m} \beta_i(t) r_i(t) + \sum_{j=m+1}^{M} \beta_j(t) r_j(t) + \epsilon_t \\ &= \underbrace{\widetilde{\alpha}(t)}_{\text{False Skill}} + \sum_{i=1}^{m} \beta_i(t) r_i(t) + \epsilon_t \end{aligned}$$