Mean-variance paradigm

Asset Management: Advanced Investments

Markus Leippold

Spring 2018

University of Zurich
Department of Banking and Finance

Outline

Universität Zürich[UZH]

- Major step in quantitative management of portfolios: Markowitz (1952).
- Awarded the Nobel price in 1990.
- In its simplest form, the mean-variance analysis provides a framework to construct and select portfolios, based on expected performance of investments and risk appetite of investors.
- However, at many firms, portfolio management remains a purely judgemental process based on qualitative, not quantitative assessments. Quantitative efforts are focused on offering support in risk management.
- Note: theory of portfolio selection is a normative theory, compared to asset pricing models such as the CAPM. These are positive theories.
- Markowitz' portfolio theory is based on the diversification principle.

- Assume that returns are uncorrelated $\sigma_{ij} = 0$ for $i \neq j$.
- Consider an equally weighted portfolio $w_i = 1/n$.
- The variance of this equally weighted portfolio is

$$\sigma_p^2 = \sum_{i=1}^{n} \frac{1}{n^2} \sigma_{ii} = \frac{1}{n} \overline{\sigma}_{ii}, \tag{1}$$

where $\overline{\sigma}_{ii}$ is the average variance of an individual return, i.e.,

$$\overline{\sigma}_{ii} = \frac{1}{n} \sum_{i=1}^{n} \sigma_{ii}.$$

- Note that the variance of this portfolio tends to zero as we increase the number of assets in the portfolio (assuming that the variances of the individual returns are uniformly bounded).

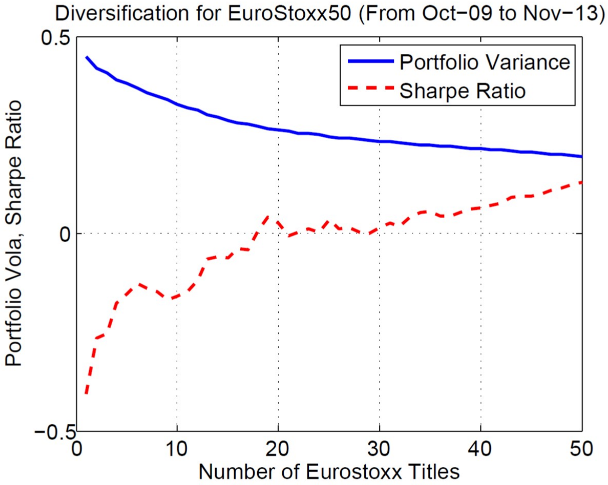
- If, however, returns are correlated, we have

$$\begin{aligned}\sigma_p^2 &= \sum_{i=1}^{n} \frac{1}{n^2}\sigma_{ii} + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{n^2}\sigma_{ij} \\ &= \sum_{i=1}^{n} \frac{1}{n^2}\sigma_{ii} + \frac{n-1}{n}\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)}\sigma_{ij} \\ &= \frac{1}{n}\overline{\sigma}_{ii} + \frac{n-1}{n}\overline{\sigma}_{ij} = \frac{1}{n}\left(\overline{\sigma}_{ii} - \overline{\sigma}_{ij}\right) + \overline{\sigma}_{ij}\end{aligned}$$

- Therefore, when the number of assets increases, the average covariance $\overline{\sigma}_{ij}$ dominates. In the limit, only this non-diversifiable risk matters.
- Realistically, $\overline{\sigma}_{ij}$ is positive.
- We need $\overline{\rho}_{ij} \geq -1/(n-1)$ for the matrix $\Sigma = \{\sigma_{ij}\}$ to be positive semidefinite. Σ is called the covariance matrix.

Diversification (cont.)

- It may take a while until correlation kicks in.
- Diversified portfolios may induce a higher correlation with the benchmark.



Let X and Y be random vectors, and X_i and Y_i denote random scalars. Then the random vector X can be written as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n, \end{bmatrix}, \mathbf{X} \in \mathbb{R}^n$$

where the random variables X_i have all finite variance, then the covariance matrix Σ is the matrix where the (i,j)-th entry is the referred to the covariance between asset i and j:

$$\sigma_{ij} = \mathrm{cov}(X_i, X_j) = \mathbb{E}(X_i - \mu_i)(X_j - \mu_j)$$

where $\mu_i = \mathbb{E}(X_i)$ denotes the expected value of the ith entry of the random vector X

For $\Sigma = \mathbb{E}\left[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^{\top}\right]$ and $\boldsymbol{\mu} = \mathbb{E}(\mathbf{X})$, where X is a random n-dimensional variable and Y a random m-dimensional variable, the following basic properties hold true

1. $\Sigma = \mathbb{E}(\mathbf{X}\mathbf{X}^{\mathrm{T}}) - \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}}$
2. Σ is a positive-semidefinite and symmetric matrix
3. $\mathrm{cov}(\mathbf{A}\mathbf{X} + \mathbf{a}) = \mathbf{A}\,\mathrm{cov}(\mathbf{X})\,\mathbf{A}^{\mathrm{T}}$
4. $\mathrm{cov}(\mathbf{X},\mathbf{Y}) = \mathrm{cov}(\mathbf{Y},\mathbf{X})^{\mathrm{T}}$
5. $\mathrm{cov}(\mathbf{X}_1 + \mathbf{X}_2,\mathbf{Y}) = \mathrm{cov}(\mathbf{X}_1,\mathbf{Y}) + \mathrm{cov}(\mathbf{X}_2,\mathbf{Y})$
6. If $n = m$, then $\mathrm{var}(\mathbf{X} + \mathbf{Y}) = \mathrm{var}(\mathbf{X}) + \mathrm{cov}(\mathbf{X},\mathbf{Y}) + \mathrm{cov}(\mathbf{Y},\mathbf{X}) + \mathrm{var}(\mathbf{Y})$
7. $\mathrm{cov}(\mathbf{A}\mathbf{X} + \mathbf{a}, \mathbf{B}^{\mathrm{T}}\mathbf{Y} + \mathbf{b}) = \mathbf{A}\,\mathrm{cov}(\mathbf{X},\mathbf{Y})\,\mathbf{B}$
8. If $\mathbf{X}$ and $\mathbf{Y}$ are independent or uncorrelated, then $\mathrm{cov}(\mathbf{X},\mathbf{Y}) = \mathbf{0}$

where $\mathbf{X}, \mathbf{X}_1$ and $\mathbf{X}_2$ are random $n \times 1$ vectors, $\mathbf{Y}$ is a random $m \times 1$ vector, $\mathbf{a}$ is a $m \times 1$ vector, $\mathbf{b}$ is a $n \times 1$ vector, and $\mathbf{A}$ and $\mathbf{B}$ are $n \times m$ matrices of constants.

Market:

- There are n risky assets with time-t prices $S_{1,t}, \ldots, S_{n,t}$ and random returns $r_1, \ldots, r_n$,

$$r_{i,t} = \frac{S_{i,t} - S_{i,t-1}}{S_{i,t-1}} = \frac{S_{i,t}}{S_{i,t-1}} - 1, \quad i = 1,2, \ldots ,n.$$

- Gross return on security i as $R_{i,t} = 1 + r_{i,t}$.
- Since we work in a one-period model, we drop the time indices.
- We define as $\mu_i = \mathbb{E}[r_i]$ the expected return on asset i.
- By $cov(r_i,r_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j, i,j = 1, \ldots ,n$ we denote the covariance between asset i and asset j and $\Sigma \in \mathbb{R}^{n\times n}$ denotes the symmetric positive semi-definite covariance matrix. By ρ_{ij} we denote the correlation coefficient.

- All assets are assumed non-redundant, i.e., expected returns are linearly independent.

Portfolio:

- $\boldsymbol{w} = (w_1, \ldots, w_n)^T$, w_i denotes the proportion of wealth invested in asset i, with $\sum_{i=1}^{n} w_i = 1$.
- The rate of return of the portfolio is $r_P = \sum_{i=1}^{n} w_i r_i$.
- Expected portfolio return:

$$\mu_P = \mathbb{E}[r_P] = \sum_{i=1}^{n} \mathbb{E}[w_i r_i] = \sum_{i=1}^{n} w_i \mu_i$$

where $\mu_i = \mathbb{E}[r_i]$. In vector notation $\mu_P = \boldsymbol{w}^\top \boldsymbol{\mu}, \quad \boldsymbol{w}, \boldsymbol{\mu} \in \mathbb{R}^n$.

- Portfolio Variance:

$$\sigma_P^2 = var(r_P) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov(r_i, r_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \sigma_{ij} w_j \qquad (2)$$

or in matrix notation $\sigma_P^2 = \boldsymbol{w}^T \Sigma \boldsymbol{w}$.

For example when $n = 2$, we can express the portfolio variance as

$$(w_1, w_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1^2 \sigma_1^2 + w_1 w_2 (\sigma_{12} + \sigma_{21}) + w_2^2 \sigma_2^2.$$

or using the correlation coefficient we can write the portfolio variance as follows

$$\begin{aligned} \sigma_P^2 &= (w_1, w_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= w_1^2 \sigma_1^2 + w_1 w_2 (\rho_{12} \sigma_1 \sigma_2 + \rho_{21} \sigma_2 \sigma_1) + w_2^2 \sigma_2^2. \end{aligned}$$

Illustration: Two Risky Assets

The market consists of only two stocks, A and B, and we consider three different levels of correlation:

1. Perfect positive correlation, i.e. $\rho_{AB} = 1$
2. Perfect negative correlation, i.e. $\rho_{AB} = -1$
3. No correlation, i.e. $\rho_{AB} = 0$.

Suppose the stocks have the following characteristics:

	Expected Return	Standard Deviation
Security A	13%	8%
Security B	6%	4%

We only consider long-only strategies with $w_A + w_B = 1$.

w(a) > 0
w(b) > 0

Illustration: Two Risky Assets (cont.)

Case $\rho_{AB} = 1$: = perfect positive correlation

- We obtain

= SD

$$\boxed{\sigma_P} = w_A \sigma_A + w_B \sigma_B \tag{3}$$

and the expected return, independent of correlation, is

$$\mu_P = w_A \mu_A + (1 - w_A) \mu_B \tag{4}$$

- Expected portfolio return and ~~variance~~ SD are **linear** combinations of the return and risk each security.
- All combinations of two securities with perfect correlation have to lie on straight line in the risk and return space.

Problem

Verify, that in the special case when $\rho_{AB} = 1$ that all combinations of two securities with perfect correlation have to lie on straight line in the risk and return space.

Solution

Solving for w_A in Equation (3) gives

$$w_A = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B} \tag{5}$$

and substituting this expression into Equation (4) gives

$$\begin{aligned}
\mu_P &= \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\mu_A + \left(1 - \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\right)\mu_B \\
&= \underbrace{\left(\mu_B - \frac{\mu_A - \mu_B}{\sigma_A - \sigma_B}\sigma_B\right)}_{=:a} + \underbrace{\left(\frac{\mu_A - \mu_B}{\sigma_A - \sigma_B}\right)}_{=:b}\sigma_P \\
&= a + b\sigma_P
\end{aligned} \tag{6}$$

which represents a straight line in the (μ_P, σ_P) space.

Case $\rho_{AB} = -1$: = perfect negative correlation

- Portfolio variance reduces to

$$\sigma_p^2 = (w_A\sigma_A - (1 - w_A)\sigma_B)^2 \tag{7}$$

- Observe that $\sigma_p^2(\rho = 1) \geq \sigma_P^2(\rho = -1)$.
- With $\rho = -1$, we can find a long-only portfolio (w_A,w_B) such that

$$\sigma_p = 0 = w_A\sigma_A - (1 - w_A)\sigma_B \leftrightarrow w_A = \frac{\sigma_B}{\sigma_A + \sigma_B} \tag{8}$$

- Since $w_A \gtrless \frac{\sigma_B}{\sigma_A+\sigma_B}$ implies $\sigma_p = (w_A\sigma_A - (1 - w_A)\sigma_B) \gtrless 0$, we write $|\sigma_p|$ for the volatility.
- For the expected return, we get

$$\mu_P = \underbrace{\left(\mu_B + \frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}\sigma_B\right)}_{=:a} + \underbrace{\left(\frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}\right)}_{=:b}|\sigma_P|$$

$$= a + b|\sigma_P|, \tag{9}$$

Case $\rho_{AB} = 0$:

- The portfolio variance reduces to

$$\sigma_p^2 = w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 \tag{10}$$

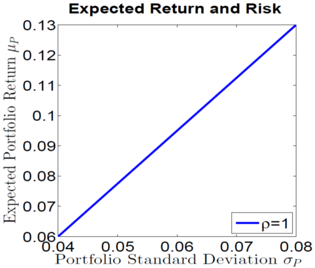
- Equation (10) implies that the expected return can no longer be represented by a **linear** function of the portfolio variance (and standard deviation σ_p). This is true for any $\rho \in (-1,1)$.
- We can obtain a portfolio allocation such that the portfolio standard deviation is minimized, i.e.,

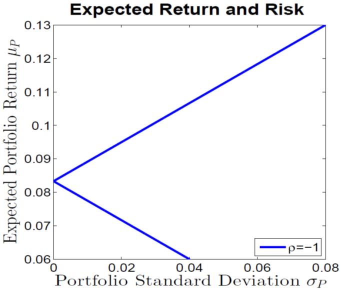
$$\frac{\partial \sigma_p}{\partial w_A} = 0 \leftrightarrow w_A^* = \frac{\sigma_B^2 - \sigma_A \sigma_B \rho_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho_{AB}} \tag{11}$$

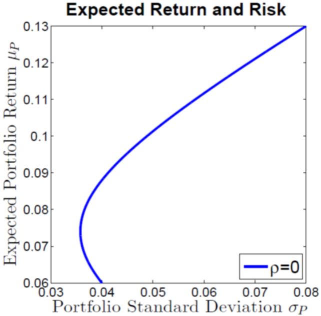
- The solution in Equation (11) is called the **minimum variance portfolio** in the case we have only **two** assets A and B.
- Note that Equation (11) implies that for $\rho_{AB} = 0$ that $0 < w_A < 1$.

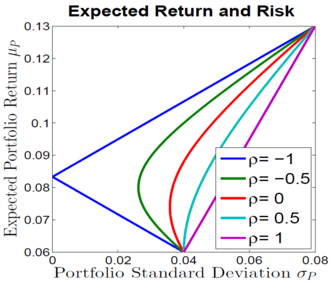
long-only PF

Two Risky Assets: $\rho \in [-1,1]$





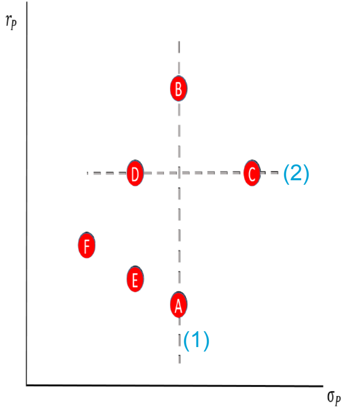




Universität Zürich[UZH]

Key take-aways

- Portfolio risk, as measured by σ_P, decreases as the correlation coefficient in the returns of two securities decreases
- Risk reduction is greatest when the securities are perfectly negatively correlated. In this case it is possible to construct a portfolio which is risk free, i.e., $\sigma_P = 0$.
- If the securities are perfectly positively correlated, there is no portfolio risk reduction regardless of how much we invest in each asset.



Clearly, we prefer a portfolio that

1. offers a higher return for the same risk
2. offers lower risk for the same return

- B $\succ$ A because it offers higher expected return for the same amount of risk
- Likewise, D $\succ$ E.
- D $\succ$ C because this portfolio offers lower risk for the same return
- Portfolio F offers the lowest possible risk of any portfolio
- Thus, the portfolios A, E and C can be eliminated because they are dominated by other portfolios
- Therefore, only the portfolios F, D and B are **efficient**.

Universität Zürich[UZH]

The **efficient frontier** connects all the portfolios that lie between the **global minimum portfolio** and **maximum return portfolio**.

- In the two asset case and assuming that $\rho_{AB} < 1$, we can find the global minimum variance portfolio by setting

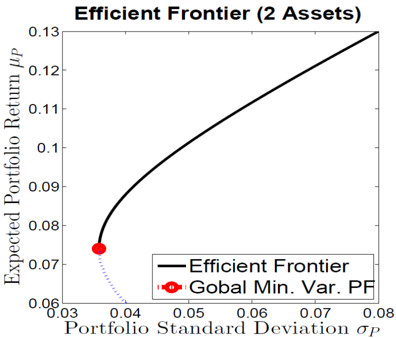
$$\frac{\partial \sigma_p}{\partial w_A} = 0 \leftrightarrow w_A^{MV} = \frac{\sigma_B^2 - \sigma_A \sigma_B \rho_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho_{AB}} \tag{12}$$

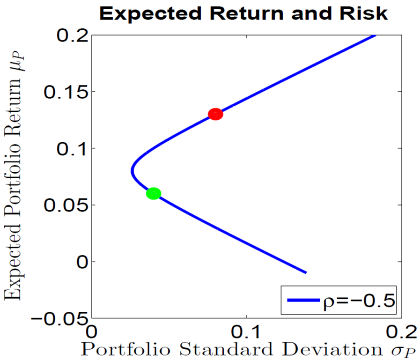
  which is exactly Equation (11).

- Thus the expected return and variance of the minimum variance portfolio are

$$\mu_P^{MV} = w_A^{MV} \mu_A + \left(1 - w_A^{MV}\right) \mu_B$$

$$\sigma_P^{MV} = {w_A^{MV}}^2 \sigma_A^2 + 2 w_A^{MV} (1 - w_A^{MV}) \rho_{AB} \sigma_A \sigma_B + \boxed{\left(1 - w_A^{MV}\right)^2} \sigma_B^2$$





left: Two asset case when $\rho_{AB} = 0$. With no short selling allowed, the expected portfolio return is bounded between μ_B and μ_A. This is also true for the portfolio ~~variance~~ SD σ_P.

right: Two asset case when $\rho_{AB} = -0.5$. With short selling allowed, the expected portfolio return is no longer bounded between μ_B and μ_A. This is also true for the portfolio ~~variance~~ SD σ_P.

Suppose $\sigma_B = 0$ offering $\mu_B = r_B = r_f$. How does our portfolio analysis change if we have one risky asset (A) and one risk free asset?

- The expected portfolio return and variance are given by

$$\mu_P = w_A \mu_A + (1 - w_A) r_f, \quad \sigma_P^2 = w_A^2 \sigma_A^2. \tag{13}$$

correlation is zero

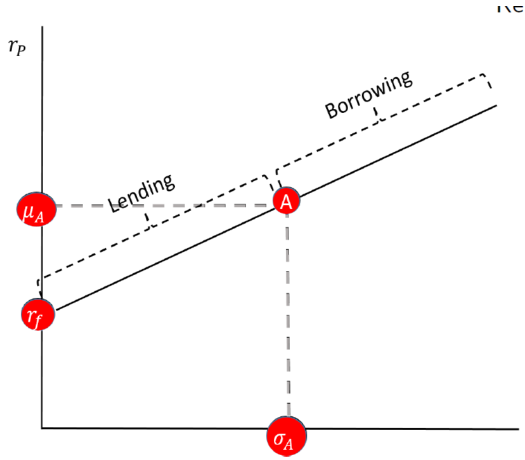
Hence,

$$\mu_P = \left(1 - \frac{\sigma_P}{\sigma_A}\right) r_f + \frac{\sigma_P}{\sigma_A} \mu_A = r_f + \frac{\mu_A - r_f}{\sigma_A} \sigma_P \tag{14}$$

Equation (14) implies the following

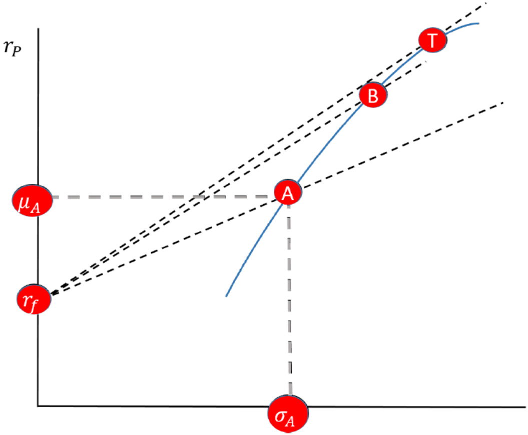
- All possible combination of risk less borrowing and lending with asset A lie on a **straight** line in a (μ_P, σ_P) diagram.
- r_f is the intercept, we obtain when we invest everything into the risk free asset.
- The slope of this line is $\frac{\mu_A - r_f}{\sigma_A}$.

Introducing a risk-free asset (cont.)



Portfolios left of A are combination of **lending** at r_f and holding portfolio A.

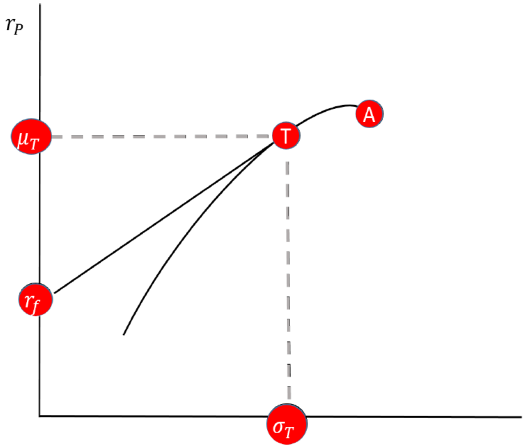
- Portfolios right to A are combination of **borrowing** r_f and holding portfolio A.
- Note that any combination of a risky security or portfolio and a risk free asset will lie along a straight line in the (μ_P, σ_P) diagram.
- Ideally, we would like to make the slope of (14) as **steep** as possible, i.e., we would like to find the portfolio (or security) that offers us the **highest excess return per unit of risk**.



The blue curve represents the efficient frontier.

- Combinations of B and r_f are superior to combinations of A and r_f.
- Ideally, we would like to make the line combining r_f and any portfolio as steep as possible
- The steepest slope we can attain is reached when we combine the risk free rate with portfolio T. tangent to efficient curve
- Any portfolio allocation on that line, offers the highest possible return risk trade off.

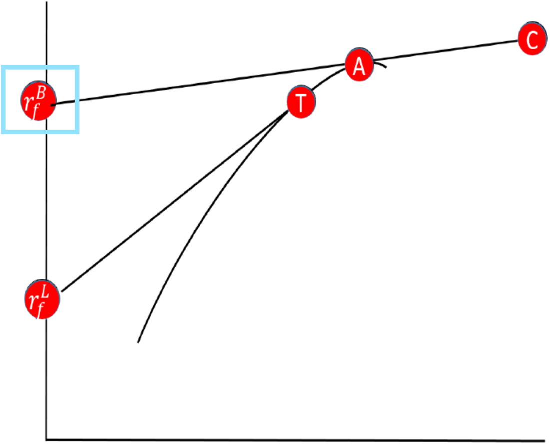
Introducing a risk-free asset (cont.)



In reality, it is always the case that the investor can invest into government bonds earning the risk free rate r_f.

- However, borrowing at r_f is not necessarily always possible.
- The efficient frontier is a straight line combining the risk free rate r_f and the portfolio T plus the curve from portfolio T to portfolio A.
- Thus, investors holding portfolios between T and A, hold only portfolios of risky assets.
- Note that any investor who holds at least some risk free assets will invest all the remaining funds in the risky portfolio T.

Borrowing and lending at different rates



- Adding further realism, we can assume that the investor faces two different rates depending whether he/she is a borrower or a lender.
- Since borrowing is usually more costly, we assume $\boxed{r_f^B > r_f^L.}$
- In this case, the efficient frontier consists of three parts. Two lines and a curve.
- Hence, the efficient frontier combines r_f^L and portfolio T, the curve between portfolio T and portfolio A, and finally the line after portfolio A originated at r_f^B.

Problem

Suppose you have the following two assets you can invest in

$$\mu_A = 11.8\%, \quad \sigma_A = 20.3\%, \quad \rho_{AB} = 0.66$$
$$\mu_B = 9.2\%, \quad \sigma_B = 18.4\%$$

and no short selling is allowed.

Questions:

(a) What is the expected return μ_P^{MV} and the standard deviation σ_P^{MV} of the global minimum variance portfolio? How does σ_P^{MV} compare to the individual asset volatilities σ_A and σ_B?

(b) Is it efficient to hold only asset B?

Universität Zürich[UZH]

Solution

(a) We obtain the optimal minimum variance portfolio weight w_A^{MV} by solving

$$\frac{\partial \sigma_p}{\partial w_A} = 0 \leftrightarrow w_A^{MV} = \frac{\sigma_B^2 - \sigma_A \sigma_B \rho_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho_{AB}}$$

for which we obtain $w_A = 36\%$. Thus we invest 36% in asset A and 64% in asset B. The expected return and variance of the global minimum variance portfolio are

$$\mu_P^{MV} = w_A^{MV} \mu_A + \left(1 - w_A^{MV}\right) \mu_B = 10.136\%$$

$$\sigma_P^{MV} = {w_A^{MV}}^2 \sigma_A^2 + 2w_A^{MV}(1 - w_A^{MV})\rho_{AB}\sigma_A\sigma_B + \left(1 - w_A^{MV}\right)^2 \sigma_B^2 = 17.48\%$$

(b) No the portfolio consisting of only stock B is inefficient. Take $w_A = 0.72$, then we get a portfolio volatility $\sigma_P = 18.4\%$ and an expected return of $\mu_P = 11.07\%$. Thus as the risk is the same (same as asset B's risk), the portfolio consisting of investing 72% in asset A and the remainder in asset B delivers superior return for the same amount of risk.

Also, note that the global minimum variance portfolio offers both higher expected return as well as lower variance compared to asset B.

Outline

Definition

Given n risky assets with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ, the portfolio P is the minimum-variance portfolio (MVP) of all portfolios with mean return μ_p, if its portfolio weight vector $\boldsymbol{w}$ satisfies $\boldsymbol{w}_p^\top \Sigma \boldsymbol{w}_p = \min_{\boldsymbol{w}} \boldsymbol{w}^\top \Sigma \boldsymbol{w}$ under the restrictions $\boldsymbol{w}_p^\top \boldsymbol{\mu} = \mu_p$ and $\boldsymbol{w}_p^\top \mathbf{1} = 1$ where $\mathbf{1}$ is an n-vector of ones.

- To obtain an MVP, we have to find the portfolio that minimizes σ_p^2 for a given μ_p:

$$\min_{\boldsymbol{w}} \sigma_p^2 \text{ s.t. } \mathbb{E}(R_p) = \sum_{i=1}^n w_i \mu_i \text{ and } \sum_{i=1}^n w_i = 1,$$

  with portfolio variance $\sigma_p^2 = \mathrm{Var}\left[\sum_{i=1}^n w_i R_i\right] = \boldsymbol{w}^\top \Sigma \boldsymbol{w}$.
- The **mean-variance (efficient) frontier** (MVF) collects all portfolios that solve the above problem for different μ_p.

[1] For a refresher on Matrix notation, check the matrix cookbook.

Proposition (Merton (1972))

With no restrictions on the portfolio weights, the weights of the efficient frontier for a required return μ_p (assuming $\boldsymbol{\mu} \neq k\mathbf{1}$ for all $k \in \mathbb{R}$) are

$$\boldsymbol{w}_p = \Sigma^{-1}\left(\mu_p \boldsymbol{k}_1 + \boldsymbol{k}_2\right), \quad \boldsymbol{k}_1 = \frac{c\boldsymbol{\mu} - b\mathbf{1}}{d}, \boldsymbol{k}_2 = \frac{a\mathbf{1} - b\boldsymbol{\mu}}{d},$$

and

$$\begin{aligned} a &= \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}, \quad b = \boldsymbol{\mu}^\top \Sigma^{-1} \mathbf{1} \\ c &= \mathbf{1}^\top \Sigma^{-1} \mathbf{1}, \quad d = ac - b^2. \end{aligned}$$

Corollary (Black's separation theorem)

Any portfolio of MVPs is also an MVP and the minimum variance frontier can be generated by any two distinct MVPs.

MV Frontier with Risky Assets (con'd)

Universität Zürich[UZH]

Proposition

The covariance of a MVP $\boldsymbol{w_p}$ with **any** asset or portfolio $\boldsymbol{w_q}$, i.e., not necessarily on the MV frontier is

$$\text{Cov}(R_p, R_q) = \frac{c}{d}\left(\mu_p - \frac{b}{c}\right)\left(\mu_q - \frac{b}{c}\right) + \frac{1}{c}.$$

Mean Variance Frontier

With this proposition, we can derive the efficient frontier in μ,σ-space. Since $\text{Var}(R_p) = \sigma_p^2 = \frac{c}{d}\left(\mu_p - \frac{b}{c}\right)^2 + \frac{1}{c}$, we can solve for μ_p to get

$$\mu_p = \frac{b}{c} + \sqrt{\frac{d}{c}\left(\sigma_p^2 - \frac{1}{c}\right)}.$$

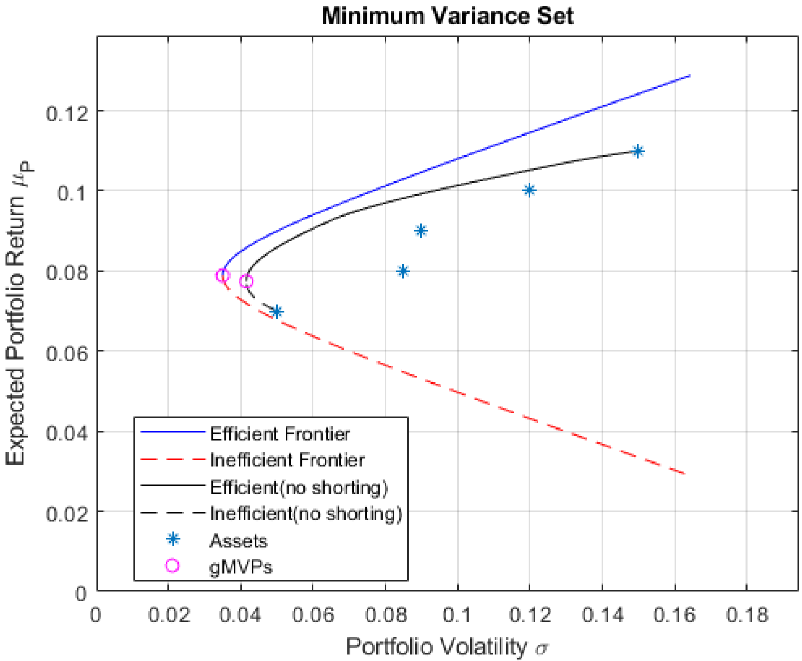
Implications

- Covariance of two efficient MVPs at least $\frac{1}{c} > 0$.
- For a given MVP $\boldsymbol{w}_p$, an uncorrelated or negatively correlated MVP $\boldsymbol{w}_q$ always lies on the lower part of the MV frontier.
- The covariance between two inefficient MVP is always positive.
- Holding a μ_p fixed, the covariance with another μ_q decreases with increasing "distance", when moving along the frontier towards the global MVP, and increases when moving away from the global MVP.

Corollary

The global MVP is characterized by $\boldsymbol{w}_g = \frac{1}{c}\Sigma^{-1}\mathbf{1}$ *with* $\mu_g = b/c$ *and* $\sigma_g^2 = 1/c$. *Furthermore, the covariance of any asset or portfolio return* R_p *with the global MVP is* $Cov(R_g, R_p) = \frac{1}{c}$.

Illustration



a) Prove the Merton (1972) proposition on slide 32. What are the signs of the scalars a, b, c and d?

b) Prove Black's separation theorem on slide 32.

c) Show that the covariance of a mean-variance portfolio $\boldsymbol{w}_p$ with any asset or portfolio $\boldsymbol{w}_q$, i.e. not necessarily on the MV frontier, is

$$\mathrm{Cov}(R_p, R_q) = \frac{c}{d}(\mu_p - \frac{b}{c})(\mu_q - \frac{b}{c}) + \frac{1}{c} \quad \text{if } \boldsymbol{\mu} \neq k\mathbf{1} \text{ for all } k \in \mathbb{R}.$$

What is the result for the covariance if $\boldsymbol{\mu} = k\mathbf{1}$ for some $k \in \mathbb{R}$?

d) Compute the variance of the minimum variance portfolios, the mean return μ_g, variance σ_g^2 and weights $\boldsymbol{w}_g$ of the global minimum variance portfolio.

e) Show that the portfolio weights derived in exercise a) can be rewritten as

$$\boldsymbol{w}_p = \boldsymbol{w}_g + (\mu_p - \mu_g)\boldsymbol{z}^\star,$$

where $\boldsymbol{z}^\star = \Sigma^{-1}\boldsymbol{k}_1$ is a pure hedge portfolio in the sense that $\mathbf{1}^T\boldsymbol{z}^\star = 0$.

Outline

Zero Beta Portfolio

- The zero beta portfolio was introduced by Black (1972) to derive a CAPM without riskless interest rate.
- Why no risk-free rate? E.g., we might have inflation uncertainty or credit rationing.
- Black (1972) showed that the major results of the CAPM do not require the existence of a risk-free asset.
- Without access to a risk free asset, investors instead use a zero-beta portfolio, i.e.. a portfolio of risky assets with zero covariance with the market portfolio.
- The zero-beta CAPM implies that beta is still the correct measure of systematic risk and the model still has a linear specification.

Corollary

For each MVP portfolio $\boldsymbol{w}_p$, except for the global MVP, there exists a unique MVP, **the zero-beta MVP with respect to $\boldsymbol{w}_p$**, *that has zero covariance with $\boldsymbol{w}_p$.*

- Note that in the absence of the risk-free rate, the efficient frontier consists of those MVPs with expected return higher than or equal to μ_g.
- Also note that the return of the global MVP must be equal or higher than the return of any zero-beta portfolio.

Proposition

Let R_q denote the return of **any** asset or portfolio, let R_p be any MVP except the global MVP, and R_{0p} the return of an MVP zero-beta portfolio w.r.t. p. Then, for the regression

$$R_q = \beta_0 + \beta_1 R_{0p} + \beta_2 R_p + \epsilon, \quad \mathbb{E}\left(\epsilon | R_p, R_{0p}\right) = 0,$$

it holds that $\beta_0 = 0$, $\beta_2 = \beta_{pq}$, $\beta_1 = 1 - \beta_{pq}$, where $\beta_{pq} = \frac{\mathrm{Cov}(R_p, R_q)}{\sigma_p^2}$. Therefore,

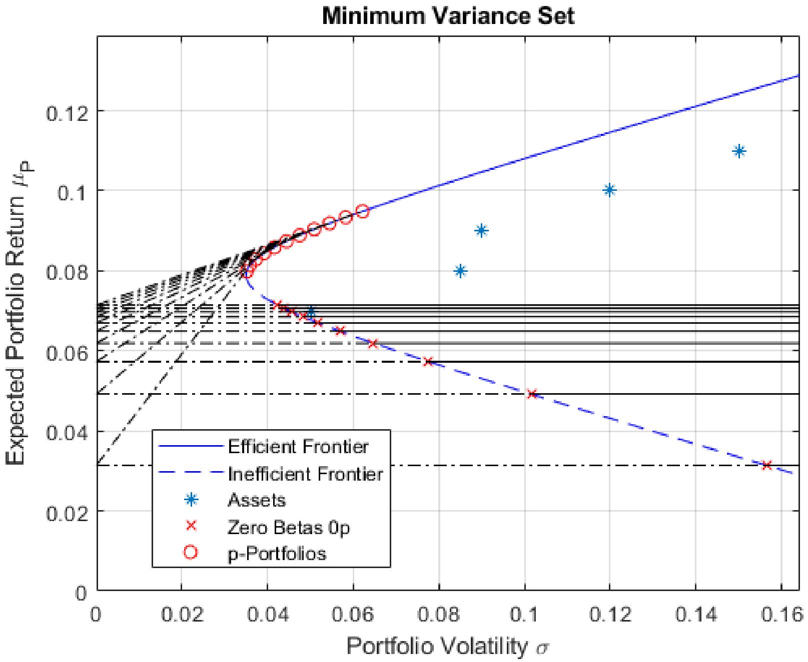
$$\mu_q = \mu_{0p} + \beta_{pq}\left(\mu_p - \mu_{0p}\right),$$

i.e., every portfolio has a beta-representation in terms of an MVP and a portfolio orthogonal to the MVP.

- Given only $\boldsymbol{w}_{0p}$ and $\boldsymbol{w}_p$, we obtain a curve inside the MV set touching the efficient frontier in P.

Zero Beta Portfolio (cont.)

- All zero-beta portfolios lie on a horizontal line.
- From any point on that line, we can span a curve which combines $\boldsymbol{w}_p$ with some portfolio $\boldsymbol{w}_{0p}$ orthogonal to $\boldsymbol{w}_p$. The collection of all these curves spans the whole interior of the MV set.
- Hence, any feasible mean variance combination can be constructed from any minimum-variance portfolio $\boldsymbol{w}_p$ and some $\boldsymbol{w}_{0p}$, which is orthogonal to $\boldsymbol{w}_p$ but not necessarily a MVP!


Minimum Variance Set
Expected Portfolio Return μ_P
Portfolio Volatility σ
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
Efficient Frontier
Inefficient Frontier
Assets
Zero Betas 0p
p-Portfolios

a) Proof the proposition on slide 40, i.e., show that

$$\mu_q = \mu_{0p} + \beta_{pq}(\mu_p - \mu_{0p}). \tag{15}$$

b) Show that (15) holds for any Zero-Beta portfolio w.r.t. p (not necessarily MVP).

c) Does the Zero Beta portfolio always exist? If not give an example under which condition(s) it fails to exist.

Outline

Proposition (MV Frontier)

If there is a risk-free asset with return R_f, then the weights in the risky assets for a portfolio on the frontier with return μ_p are

$$\boldsymbol{w}_p = \frac{\mu_p - R_f}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \Sigma^{-1} \boldsymbol{\mu}^e,$$

where $\boldsymbol{\mu}^e$ denotes the vector of excess return of the assets over the risk-free rate, i.e., $\boldsymbol{\mu}^e \equiv \boldsymbol{\mu} - \mathbf{1} R_f$. The fraction $1 - \mathbf{1}^\top \boldsymbol{w}_p$ is invested in the riskless asset.

Proposition (Tobin (1958)'s Separation Theorem)

The relative portfolio fraction is independent of the choice of the targeted portfolio return μ_p.

Implications for Portfolio Delegation

- The separation theorem implies that any investor's portfolio decision is the same.
- The only difference between investors is the relative portion between the risky portfolio and the risk-free interest rate R_f.
- This portion depends on the investor's risk aversion.

Proposition

The tangency portfolio $\boldsymbol{w}_T$ (with maximal Sharpe ratio) is characterized by $\boldsymbol{w}_T = \frac{1}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}^e} \Sigma^{-1} \boldsymbol{\mu}^e$, and

$$\begin{aligned} \mu_T &= R_f + (\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e / \mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}^e, \\ \sigma_T &= \sqrt{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} / \mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}^e. \end{aligned}$$

- It is also easy to verify that the market portfolio can be calculated directly from the maximal Sharpe ratio optimization problem:

$$\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^\top \boldsymbol{\mu}^e}{\sqrt{\boldsymbol{w}^\top \Sigma \boldsymbol{w}}}, \qquad \text{s.t. } \mathbf{1}^\top \boldsymbol{w} = 1.$$

- Fama (1970) demonstrated that under certain assumptions the tangency portfolio must consist of all assets available to investors, and each asset must be held in proportion to its market value relative to the total market value of all assets.
- Therefore, the tangency portfolio is often referred to as the market portfolio.

a) Prove the proposition on slide 45 on the MV frontier with a risk-free asset.
b) Prove Tobin's separation theorem.
c) If we have a risk-free investment, the search for an optimal portfolio is greatly reduced. Due to the two - fund separation principle, we only need to find one more portfolio, the so called tangency portfolio, in order to generate the efficient frontier. Calculate the tangency portfolio weights $\boldsymbol{w}_T$.
d) Calculate the expected return μ_T and the standard deviation σ_T of an investment in the tangency portfolio.

An asset allocation puzzle (Canner et al. 1997)[2]

Advisor and Investor type	Percent of Portfolio Cash	Bond	Stocks	Ratio Bond to Stocks
A. Fidelity				
Conservative	50	30	20	1.5
Moderate	20	40	40	1
Aggressive	5	30	65	0.46
B. Merrill Lynch				
Conservative	20	35	45	0.78
Moderate	5	40	55	0.73
Aggressive	5	20	75	0.27
C. Jane Bryant Quinn				
Conservative	50	30	20	1.5
Moderate	10	40	50	0.8
Aggressive	0	0	100	0
D. The New York Times				
Conservative	20	40	40	1
Moderate	10	30	60	0.5
Aggressive	0	20	80	0.25

[2] Also see the comments in Bajeux-Besnainou et al. (2001) and Shalit and Yitzhaki (2003).

Outline

- The previous MV analysis was focused **only** on the **asset** side of the optimization problem
- However, for many financial firms, not only assets the risk&return trade-off has to be managed, but also the risk&return of **liabilities**
- The most prominent examples: **Pension funds**
  - Pension plan experiences investment risk from both sides of the balance sheet. $\rightarrow$ Pension plan needs to conduct **asset-liability management** to manage both asset and liability risks
  - Estimation of future liabilities is not straightforward as they depend on a number of factors (salary, years of service etc.)
  - Value of the liabilities **fluctuates** even if payments are fixed (discount and inflation effect)
  - The pension plan's liabilities can be viewed as a **stream of future cash flows** whose main driver of its value will be the level of interest rates.
  - In essence, the pension plan is **short a long-term bond**.

Pension Funds: Some Key Statistics

Universität Zürich[UZH]

Assets under Management by pension funds (2015):[3]

- Total pension assets were estimated at **USD 38 Trillions**
- Pension funds were the main investors worldwide (USD 26 trillion, 68% of the total), followed by[4]
  - banks and investment companies (USD 7.7 trillion, 20.2%),
  - insurance companies (USD 4.3 trillion, 11.3%)
  - and employers through their book reserves (USD 0.2 trillion, 0.5%).
- Pension funds' assets relative to GDP stands at about 80%
- The largest pension markets are the US, UK and Japan with 61.5%, 9% and 7.7% of total pension funds assets

[3] If not otherwise stated: Source: Wilson Tower Watson 2016: Global Pension Asset Study 2016

[4] Source: OECD: Pension Markets in Focus 2016.

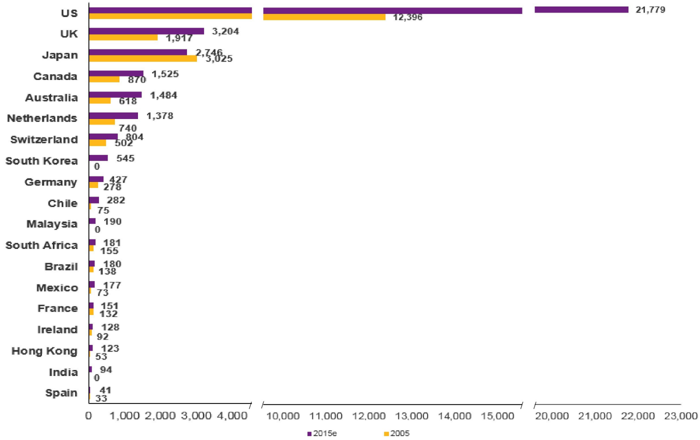
Pension Funds: Some Key Statistics

Universität Zürich[UZH]

P19

Global Pension Assets

Evolution 2005-2015 – USD billion



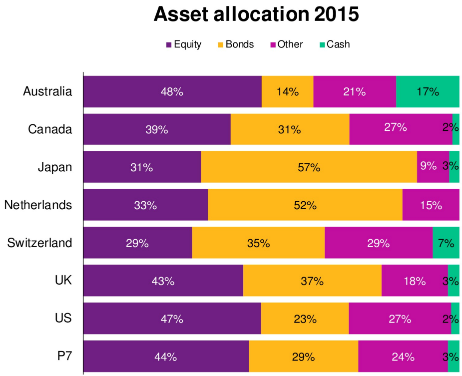
Source: Willis Towers Watson and secondary sources

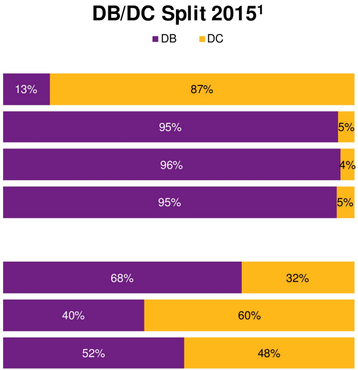
WillisTowersWatson 12

Universität Zürich[UZH]

Global Pension Assets Study 2016

Key findings - Figures





[1] In Switzerland DC stands for cash balance, where the plan sponsor shares the investment risk and all assets are pooled. There are almost no pure DC assets where members make an investment choice and receive market returns on their funds. Therefore, Switzerland is excluded from this analysis..
Source: Willis Towers Watson and secondary sources

When we introduce liabilities to the portfolio allocation problem, the perspective on risk changes:
What is the true risk of the fund?

- It is **NOT** the standard deviation of the asset portfolio
- It's the risk that you **won't be able to pay your liabilities**!
- Moreover, correlation between assets and liabilities is crucial for obtaining optimal portfolio allocations

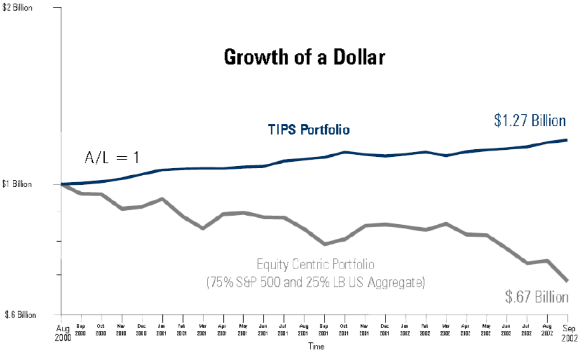
A key (new) variable:

- The **funding ratio** defined as $f_t := A_t/L_t$, where A_t are the assets and L_t the liabilities of the fund at time t.
  - If $f > 1$ the funds is said to be **over**-funded which implies that it can (currently) meet its obligations
  - If $f < 1$, the funds is said to be **under**-funded and thus is not able to pay its liabilities

Funding Ratio over Time

Source: Morning Star Associates LLC, "Liability Relative Optimization: Begin with the End in Mind".

Matching Liabilities not always easy



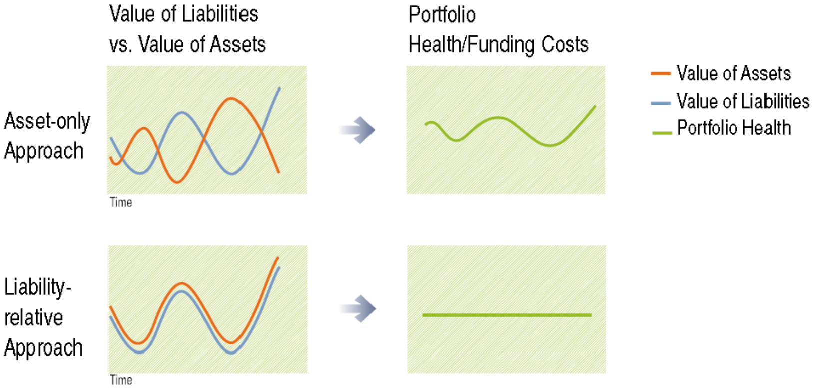
Source: Morning Star Associates LLC, "Liability Relative Optimization: Begin with the End in Mind".

- Treasury Inflation Protected Securities (TIPS) offer protection against inflation risk.
- S&P 500 as primary benchmark for stock-oriented funds, Lehman Brothers (LB) Aggregate Bond index for bond funds.

Our main objective: How do we adopt the asset-only MV optimization problem to incorporate liabilities?

- Solution: Adding liabilities to the objective function of the MV problem
- Also known as surplus, liability-relative investing or asset-liability management (ALM)
- Thus, we focus on both assets and liabilities and how they correlate with each other.

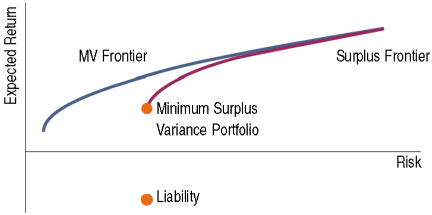
MV Optimization with Liabilities



Remarks:

- Liability-relative MV-optimized portfolio is more **stable**.
- Fluctuations of liabilities will affect portfolio value.

EF with Liabilities



Geometric Interpretation:

- Including liabilities leads to a parallel shift of the efficient frontier:
- a vertical shift plus a side wise shift,
- but no change in the curvature of the efficient frontier.

Universität Zürich[UZH]

Mathematical formulation of the problem:

- Suppose that the benchmark is a risky liability and not cash (see, e.g., Keel and Müller (1995))
- Consider changes in the surplus:

$$\Delta S = A \times R_p - L \times R_l$$

and rewrite:

$$\frac{\Delta S}{A} = R_p - 1/f \times R_l,$$

with $f = A/L$, the funding ratio.

- L could be driven by inflation, wage growth, longevity, etc.. (restricted to exogenous variables!)

- To optimize over assets only, we define Σ as the covariance matrix of the assets and $\boldsymbol{\gamma}$ as the vector of covariances between the assets and the liability return R_l, i.e., $\gamma_i = \mathrm{Cov}(R_i, R_l)$. The mean of asset returns is simply denoted as $\boldsymbol{\mu}$.
- The mean-variance optimization leads to the optimization problem

$$\min_{\boldsymbol{w}} \mathrm{Var}\left(R_p - \frac{1}{f} R_l\right)$$

subject to $\boldsymbol{w}^\top \mathbf{1} = 1$ and

$$\mathbb{E}\left(R_p - \frac{1}{f} R_l\right) \geq \bar{r},$$

where $\bar{r}$ has to be chosen in accordance with the risk tolerance (of the pension fund).

MV Optimization with Liabilities (cont.)

- The optimization problem $\min_{\boldsymbol{w}} \mathrm{Var}\left(R_p - \frac{1}{f} R_l\right)$ can be reformulated as

$$\min_{\boldsymbol{w}} \left(\frac{1}{2} \boldsymbol{w}^\top \Sigma \boldsymbol{w} - \frac{1}{f} \gamma^\top \boldsymbol{w} \right),$$

subject to

$$\begin{aligned} \boldsymbol{\mu}^\top \boldsymbol{w} &\geq \bar{r} + \frac{1}{f} \mathbb{E}(R_l) =: \hat{r} \\ \boldsymbol{w}^\top \mathbf{1} &= 1. \end{aligned}$$

- To solve the optimization problem, we need to solve the first-order conditions:

$$\begin{aligned} \Sigma \boldsymbol{w} - \frac{1}{f} \gamma - \lambda \boldsymbol{\mu} - \delta \mathbf{1} &= 0, \\ \boldsymbol{\mu}^\top \boldsymbol{w} &\geq \hat{r}, \\ \mathbf{1}^\top \boldsymbol{w} &= 1. \end{aligned}$$

- We first look at the minimum variance portfolio, which is obtained by omitting the constraint $\boldsymbol{\mu}^\top \boldsymbol{w} \geq \hat{r}$, i.e., setting $\lambda = 0$ in the above FOCs. Plug

$$\boldsymbol{w} = \frac{1}{f}\Sigma^{-1}\gamma + \delta\Sigma^{-1}\mathbf{1}$$

into $\mathbf{1}^\top \boldsymbol{w} = 1$ and solve for δ to obtain

$$\delta = \frac{1 - \frac{1}{f}\mathbf{1}^\top\Sigma^{-1}\gamma}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}.$$

- Plugging δ into the FOC for $\boldsymbol{w}$, we get for the global minimum variance portfolio

$$\boldsymbol{w}_{min} = \underbrace{\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}}_{\equiv\hat{\boldsymbol{w}}_{min}} + \underbrace{\frac{1}{f}\left(\Sigma^{-1}\gamma - \frac{\gamma^\top\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\right)}_{\equiv\boldsymbol{z}_{min}}.$$

- The term $\hat{\boldsymbol{w}}_{min}$ corresponds to the minimum variance portfolio in the absence of liabilities.
- The term $\boldsymbol{z}_{min}$ is the correction term stemming from liabilities and is linear in γ.
- Note that $\mathbf{1}^\top \boldsymbol{z}_{min} = 0$, so we can interpret $\boldsymbol{z}_{min}$ as a pure hedge portfolio in the presence of liabilities.
- The solution for all efficient portfolios is given as

$$\boldsymbol{w}^\star = \boldsymbol{w}_{min} + \lambda \boldsymbol{z}^\star,$$

where λ is the Lagrangian multiplier of the constraint $\boldsymbol{w}^\top \boldsymbol{\mu} \geq \hat{r}$ and

$$\boldsymbol{z}^* = \Sigma^{-1}\boldsymbol{\mu} - \frac{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}} \Sigma^{-1}\mathbf{1}.$$

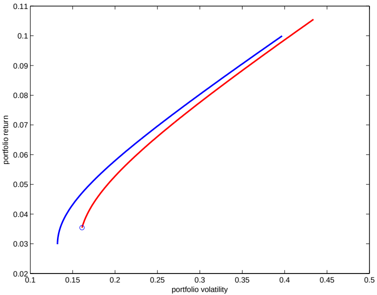
- Obviously, $\boldsymbol{z}^\star$ is not related to the liability. Also note $\mathbf{1}^\top \boldsymbol{z}^\star = 0$.

MV Optimization with Liabilities (cont.)

- Summarizing, we can express the efficient portfolios as a combination of three portfolios, a global minimum variance portfolio, a liability hedging portfolio, and a speculative portfolio:

$$\boldsymbol{w}^{\star} = \hat{\boldsymbol{w}}_{min} + \boldsymbol{z}_{min} + \lambda \boldsymbol{z}^{\star}.$$

Geometrically, the occurrence of liabilities leads to parallel shift of the efficient set by $\boldsymbol{z}_{min}$.



Geometric Interpretation

- Including liabilities leads to a parallel shift of the efficient frontier: a vertical shift plus a sidewise shift, but no change in the curvature of the efficient frontier.

- However, note we are still in the one-liability and one-period case. Multi-liabilities and multiperiod cases become quite complicated (see, Leippold et al. (2004)).
- In addition, liabilities may be endogenous (see, Leippold et al. (2011)).

MV-optimization with Liabilities

Consider changes in the surplus:

$$\frac{\Delta S}{A} = r_p - 1/f \times r_l,$$

with $f = A/L$, the funding ratio.

- To **optimize over assets only**, we define $\mathbf{\Sigma}$ as the covariance matrix of the assets and
- ζ as the vector of covariances between the assets and the liability return r_l, i.e., $\zeta_i = \text{Cov}(r_i, r_l)$.
- The mean of asset returns is simply denoted as μ.

References

Bajeux-Besnainou, I., Jordan, J. V. and Portait, R.: 2001, An asset allocation puzzle: comment, *The American Economic Review* **91**(4), 1170–1179.

Black, F.: 1972, Capital market equilibrium with restricted borrowingf, *The Journal of Business* **45**(3), 444–455.

Canner, N., Mankiw, N. G., Weil, D. N. et al.: 1997, An asset allocation puzzle, *American Economic Review* **87**(1), 181–91.

Fama, E. F.: 1970, Efficient capital markets: A review of theory and empirical work, *Journal of Finance* **25**(2), 383–417.

Keel, A. and Müller, H. H.: 1995, Efficient portfolios in the asset liability context, *ASTIN Bulletin: The Journal of the International Actuarial Association* **25**(01), 33–48.

Leippold, M., Trojani, F. and Vanini, P.: 2004, A geometric approach to multiperiod mean variance optimization of assets and liabilities, *Journal of Economic Dynamics and Control* **28**(6), 1079–1113.

Leippold, M., Trojani, F. and Vanini, P.: 2011, Multiperiod mean-variance efficient portfolios with endogenous liabilities, *Quantitative Finance* **11**(10), 1535–1546.

Markowitz, H.: 1952, Portfolio selection, *The Journal of Finance* **7**(1), 77–91.

Merton, R. C.: 1972, An analytic derivation of the efficient portfolio frontier, *Journal of financial and quantitative analysis* **7**(4), 1851–1872.

Shalit, H. and Yitzhaki, S.: 2003, An asset allocation puzzle: Comment, *The American Economic Review* **93**(3), 1002–1008.

Tobin, J.: 1958, Liquidity preference as behavior towards risk, *The Review of Economic Studies* **25**(2), 65–86.