Static Portfolio Choice

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Motivation

We apply the theory of rational decision making under uncertainty to the financial problem of choosing a portfolio. We start with a two-period setting in which portfolios are chosen in an initial period, after which all uncertainty is resolved and all wealth is consumed in a second period.

1. The choice between a safe asset and a single risky asset.
2. The choice between two risky assets.
3. The general N-asset case, using the classic mean-variance analysis of Markowitz (1952).

Risk Exposure Selection

- Investor with initial wealth W and two assets, safe asset with return R_f and risky asset with return $R_f + X$.
- Wealth evolution after one period $W(1 + R_f) + \theta X = W_0 + \theta X$.
- Investor's optimization problem is $\max_\theta V(\theta) = \mathbb{E}[u(W_0 + \theta X)]$.
- Without being more explicit, $V'(0) = \mathbb{E}[Xu'(W_0)]$ shows that:
  1. There should be **no investment in a risky asset with a zero expected excess return**.
  2. The investment in the risky asset should be **positive if it has a positive expected excess return**, regardless of the level of risk aversion.
- This **principle of participation** tells us that nonparticipation in risky asset markets with positive risk premia cannot be justified by risk aversion alone.

The **principle of participation** tells us the sign of the optimal investment in a risky asset, but we would like to derive an explicit expression for the magnitude.

- Introduce a **small reward** by scaling the mean μ with a small but positive scalar k

$$X = k\mu + Y,$$

  where Y is the zero-mean fixed scale risk component.

- The first-order condition now becomes

$$\mathbb{E}[(k\mu + Y)u'(W_0 + \theta^*(k)(k\mu + Y))] = 0.$$

- Differentiating with respect to k

$$\mu\mathbb{E}[u'(W)] + \theta^*(k)\mu\mathbb{E}[(k\mu + Y)u''(W)] + \theta^{*\prime}(k)\mathbb{E}[(k\mu + Y)^2 u''(W)] = 0.$$

Small Reward for Risk (cont.)

- Evaluating this expression at $k = 0$, and using the fact that $\theta^*(0) = 0$

$$\theta^{*\prime}(0) = \frac{\mu}{\mathbb{E}[Y^2]} \frac{1}{A(W_0)}.$$

- A Taylor expansion for the investment in the risky asset gives

$$\theta^*(k) \approx \theta^*(0) + k\theta^{*\prime}(0) = \frac{\mathbb{E}[X]}{\mathbb{E}[(X - \mathbb{E}[X])^2]} \frac{1}{A(W_0)}.$$

Consequently, for small risks using the Arrow-Pratt methodology, we have that

- The optimal investment is the **mean-variance ratio for the risky excess return, divided by absolute risk aversion**.
- The optimal share or weight of wealth to invest is the **mean-variance ratio for the risky excess return, divided by relative risk aversion**

$$w^*(k) = \frac{\theta^*(k)}{W_0} \approx \frac{\mathbb{E}[X]}{\mathbb{E}[(X - \mathbb{E}[X])^2]} \frac{1}{R(W_0)}.$$

For $X \sim \mathcal{N}(\mu,\sigma^2)$ and (exponential) **CARA utility**, with risk aversion A we have $\max_\theta V(\theta) = \mathbb{E}[-\exp(-A(W_0 + \theta X))]$ or equivalently

$$\min_\theta V(\theta) = \mathbb{E}[\exp(-A(W_0 + \theta X))].$$

Note that for any lognormal random variable

$$\log(\mathbb{E}[Z]) = \mathbb{E}[\log(Z)] + \frac{1}{2}\mathrm{Var}[\log(Z)].$$

Thus,

$$\min_\theta \log(\mathbb{E}[\exp(-A(W_0 + \theta X))]) = -A(W_0 + \theta\mu) + \frac{1}{2}A^2\theta^2\sigma^2,$$

or equivalently $\max_\theta A(W_0 + \theta\mu) - \frac{1}{2}A^2\theta^2\sigma^2$ with the independent of the initial wealth solution:

$$\theta^* = \frac{\mu}{A\sigma^2}.$$

The CARA-Normal Case (cont.)

The CARA-normal framework is widely used because it has numerous tractable features, but also shortcomings:

$+$ Multiple assets can easily be analyzed.

$+$ Additive background risk (random income or nontradable assets).

$+$ Equilibrium with heterogeneous agents.

$\pm$ Wealth irrelevance for risky investment.

\- Bounded utility and unbounded returns.

\- Trending risk premia.

\- Arbitrary time interval.

Problem:

- Normality assumption cannot hold over more than one time interval.

Observation:

- If returns are identically distributed with finite variance in every period, and if we compound them over time, the distribution of returns will converge to a lognormal distribution.

Solution:

- Assume lognormal returns. In a discrete-time model with iid returns, returns over multiple periods will again be lognormal.

The Objective Function in Wealth

- With lognormally distributed wealth and power utility, then

$$\max \ \mathbb{E}_t \left[\frac{W_{t+1}^{1-\gamma}}{1-\gamma} \right].$$

- Assuming $\gamma < 1$ for relative risk aversion, we rewrite the problem as

$$\max \ \log \left(\mathbb{E}_t \left[W_{t+1}^{1-\gamma} \right] \right) = \max \ \left[(1-\gamma)\mathbb{E}_t \left[w_{t+1} \right] + \frac{1}{2}(1-\gamma)^2 \sigma_{wt}^2 \right],$$

  where $w_t = \log(W_t)$ and σ_{wt}^2 is the conditional variance of log wealth.

- With the budget constraint $w_{t+1} = r_{p,t+1} + w_t$, with $r_{p,t+1} = \log(1 + R_{p,t+1})$ the log return on the portfolio,

$$\max \ \left[\mathbb{E}_t \left[r_{p,t+1} \right] + \frac{1}{2}(1-\gamma)\sigma_{pt}^2 \right] = \max \ \left[\mathbb{E}_t \left[\log(1 + R_{p,t+1}) \right] - \frac{\gamma}{2}\sigma_{pt}^2 \right],$$

  where σ_{pt}^2 is the conditional variance of the portfolio return.

Log Portfolio Return and Log Asset Returns

- The simple return on the portfolio is a linear combination of the simple returns on the risky and riskless assets.
- The log return on the portfolio is the log of this linear combination!
- Over short time intervals, however, we can use a Taylor approximation of the nonlinear function relating log individual-asset returns to log portfolio returns.
- With α_t the portfolio share in the risky asset, which has log return r_{t+1}, and writing $r_{f,t+1}$ for the log return on the riskless asset, we have

$$\begin{aligned} r_{p,t+1} - r_{f,t+1} &= \log(1 + \alpha_t(\exp(r_{t+1} - r_{f,t+1}) - 1)) \\ &\approx \alpha_t(r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1 - \alpha_t)\sigma_t^2, \end{aligned}$$

using the fact that for short time intervals and lognormally distributed returns, $(r_{t+1} - r_{f,t+1})^2 \approx \sigma_t^2$.

Solving the Approximate Problem

- With two assets, the mean excess portfolio return is $\mathbb{E}_t[r_{p,t+1}] - r_{f,t+1} = \alpha_t(\mathbb{E}_t[r_{t+1}] - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2$.
- Substituting,

$$\max_{\alpha_t}(\mathbb{E}_t[r_{t+1}] - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2 + \frac{1}{2}(1-\gamma)\alpha_t^2\sigma_t^2.$$

- The solution is again of the same form as the Arrow-Pratt solution for CARA with small risk, but with a relative risk aversion coefficient that is independent of wealth:

$$\alpha_t = \frac{\mathbb{E}[r_{t+1}] - r_{f,t+1} + \sigma_t^2/2}{\gamma\sigma_t^2} \approx \frac{\mathbb{E}_t[R_{t+1}] - R_{f,t+1}}{\gamma\sigma_t^2}.$$

Growth-Optimal Portfolio

- When $\gamma = 1$, the investor has log utility and chooses the **growth-optimal portfolio** with the maximum expected log return:

$$\max \ \mathbb{E}_t \left[r_{p,t+1} \right].$$

- As the investment horizon increases, the GOP outperforms any other portfolio with increasing probability:
  - The difference between the log return on GOP and the log return on any other portfolio is **normally distributed with a positive mean**.
  - With iid returns, the mean and variance of the excess log return both grow in proportion to the investment horizon.
  - Hence, the Sharpe ratio of the excess log return grows with the square root of the horizon, and **so does the probability of a positive excess return**.
- Markowitz (1976) argues that long-term investors should invest as if they have log utility.

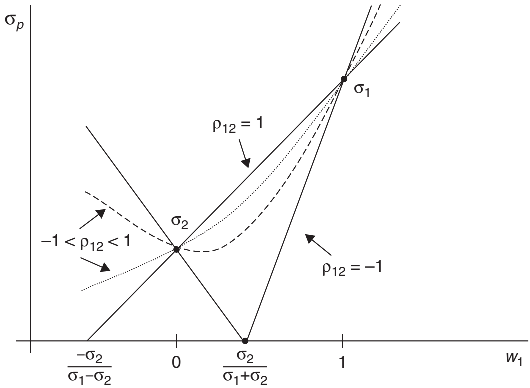
Combining Risky Assets

- To combine multiple risky assets, we start with the classic mean-variance analysis of Markowitz (1952) that judges portfolios by their **first two moments** of returns.
- Hence, investors have quadratic utility or returns have distributions for which the first two moments are sufficient statistics; and short sales are permitted.
- The mean μ_p and variance σ_p^2 of the portfolio return containing two risky assets with returns R_1 and R_2 are

$$\mu_p = \mathbb{E}[w_1 R_1 + w_2 R_2] = w_1 \mu_1 + (1 - w_1)\mu_2,$$
$$\sigma_p^2 = \mathrm{Var}[w_1 R_1 + (1 - w_1) R_2] = w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2 w_1 (1 - w_1)\sigma_{12}.$$

Two Risky Assets (cont.)

- Since $\sigma_{12} = \sigma_1\sigma_2\rho_{12} \leq \sigma_1\sigma_2$, for $\rho_{12} = \text{Cor}[R_1,R_2]$ and $0 \leq w_1 \leq 1$, the portfolio variance is bounded by $\sigma_p^2 \leq (w_1\sigma_1 + (1-w_1)\sigma_2)^2$.



- The Global Minimum-Variance (GMV) portfolio can be derived by setting the partial derivatives of the portfolio variance with respect to the portfolio weights equal to zero ($\frac{\partial \sigma_p^2}{\partial w_1} = \frac{\partial \sigma_p^2}{\partial w_2} = 0$):

$$w_{1,GMV} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}, \quad w_{2,GMV} = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

- Uncorrelated assets with $\sigma_{12} = 0$:

$$w_{1,GMV} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \quad w_{2,GMV} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2},$$

hold each asset in proportion to the variance share of the other asset in total variance.

- With equal variance $\sigma_1^2 = \sigma_2^2$, $w_{1,GMV} = w_{2,GMV} = 0.5$

One Risky and One Safe Asset

- Assume there is a riskless asset with $\sigma_2^2 = 0$ and riskfree interest rate $R_2 = R_f$. Then the mean and variance of the portfolio return can be rewritten as $\mu_p = R_f + w_1(\mu_1 - R_f)$ and $\sigma_p^2 = w_1^2 \sigma_1^2$ which implies $w_1 = \sigma_p \sigma_1$ and thus

$$\mu_p - R_f = \sigma_p \left(\frac{\mu_1 - R_f}{\sigma_1} \right).$$

- This defines a straight line, called the **Capital Allocation Line (CAL)**, on a mean-standard deviation diagram. The slope $SR_1 = \left(\frac{\mu_1 - R_f}{\sigma_1} \right)$ is called the **Sharpe ratio (SR)** or reward-risk ratio of the risky asset.
- As seen before, the standard rule of myopic portfolio choice is

$$w_1 = \frac{\mu_1 - R_f}{\gamma \sigma_1^2} = \frac{SR}{\gamma \sigma_1} \text{ and } \sigma_p = \frac{SR}{\gamma},$$

where γ is the coefficient of relative risk aversion.

Given the vector of mean returns $\boldsymbol{\mu}$ on risky assets with covariance matrix Σ, we want to derive the portfolio $\boldsymbol{w}$ that solves:

$$\min_{\boldsymbol{w}} \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \quad \text{s.t. } \boldsymbol{\mu}^\top \boldsymbol{w} = \mu_p \text{ and } \mathbf{1}^\top \boldsymbol{w} = 1,$$

with Lagrangian

$$\begin{aligned}
&\mathcal{L}(\boldsymbol{w},\lambda_1,\lambda_2) = \frac{1}{2}\boldsymbol{w}\top\Sigma\boldsymbol{w} + \lambda_1(\mu_p - \boldsymbol{\mu}^\top \boldsymbol{w}) + \lambda_2(1 - \mathbf{1}^\top \boldsymbol{w}), \\
&\text{FOC}: \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \Sigma \boldsymbol{w}^* - \lambda_1 \boldsymbol{\mu} - \lambda_2 \mathbf{1} = 0.
\end{aligned}$$

As shown in Merton (1972), the optimal portfolio holdings are

$$\boldsymbol{w}^* = \lambda_1 \Sigma^{-1} \boldsymbol{\mu} + \lambda_2 \Sigma^{-1} \mathbf{1}$$

with $A := \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}$, $B := \mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}$, $C := \mathbf{1}^\top \Sigma^{-1} \mathbf{1}$ and $D := AC - B^2$:

$$\lambda_1 = \frac{C\mu_p - B}{D}, \quad \lambda_2 = \frac{A - B\mu_p}{D}.$$

The minimized portfolio variance is

$$\begin{aligned}
\sigma_p^2 &= \boldsymbol{w}^\top \Sigma \boldsymbol{w} \\
&= \boldsymbol{w}^\top \Sigma (\lambda_1 \Sigma^{-1} \boldsymbol{\mu} + \lambda_2 \Sigma^{-1} \mathbf{1}) \\
&= \lambda_1 \boldsymbol{w}^\top \boldsymbol{\mu} + \lambda_2 \boldsymbol{w}^\top \mathbf{1} \\
&= \lambda_1 \mu_p + \lambda_2 \\
&= \frac{A - 2B\mu_p + C\mu_p^2}{D}.
\end{aligned}$$

Covariance of MVPs

The covariance of a MVP w_p with **any** asset or portfolio w_q, i.e., not necessarily on the MV frontier is

$$\mathrm{Cov}(R_p, R_q) = \frac{C}{D}\left(\mu_p - \frac{B}{C}\right)\left(\mu_q - \frac{B}{C}\right) + \frac{1}{C}.$$

Mean Variance Frontier

Given the above result, we can derive the efficient frontier in μ_p, σ_p-space. Since $\mathrm{Var}(R_p) = \sigma_p^2 = \frac{C}{D}(\mu_p - \frac{B}{C})^2 + \frac{1}{C}$, we can solve for μ_p to get

$$\mu_p = \frac{B}{C} + \sqrt{\frac{D}{C}\left(\sigma_p^2 - \frac{1}{C}\right)}.$$

The Global Minimum-Variance Portfolio

- For the GMV portfolio the first constraint (on the mean) is dropped ($\lambda_1 = 0$) and we get $\boldsymbol{w}_{GMV} = \lambda_2 \Sigma^{-1}\mathbf{1}$.
- The remaining constraint implies that $1 = \mathbf{1}^\top \boldsymbol{w}_{GMV} = \lambda_2 \mathbf{1}^\top \Sigma^{-1}\mathbf{1}$ and thus

$$\boldsymbol{w}_{GMV} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}.$$

- The variance of the global minimum-variance portfolio return is

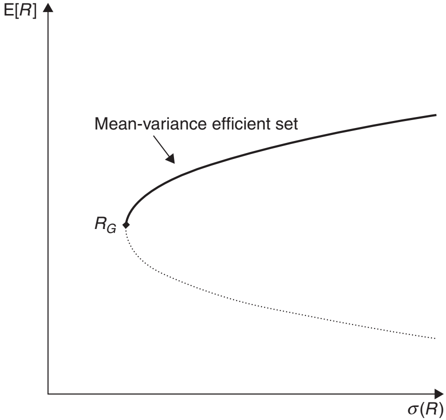
$$\sigma^2_{GMV} = \boldsymbol{w}_{GMV}^\top \Sigma \boldsymbol{w}_{GMV} = \frac{1}{C} = \frac{1}{\mathbf{1}^\top \Sigma \mathbf{1}}.$$

- For the special case where all assets are symmetrical, having the same variance and the same correlation ρ with each other, we get an equally weighted GMV portfolio $w_{i,GMV} = 1/N$ with

$$\sigma^2_{GMV} = \boldsymbol{w}_{GMV}^\top \Sigma \boldsymbol{w}_{GMV} = \frac{\mathbf{1}^\top \Sigma \mathbf{1}}{N^2} = \rho\sigma^2 + \frac{(1-\rho)\sigma^2}{N}.$$

Mean-Variance Efficient Set

The set of minimum-variance portfolios that satisfy $\mu_p > B/C$ is called the mean-variance efficient set. When $\mu_p < B/C$, then the Lagrange multiplier $\lambda_1 < 0$, implying that a higher mean return is consistent with a lower variance.



Mutual fund theorem of Tobin (1958)

All mean-variance optimal investors hold combinations of just two underlying portfolios or "mutual funds":

$$\begin{aligned}
\boldsymbol{w} &= \lambda_1 \Sigma^{-1}\boldsymbol{\mu} + \lambda_2 \Sigma^{-1}\mathbf{1} \\
&= \lambda_1 \mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}\left(\frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}}\right) + \lambda_2 \mathbf{1}^\top \Sigma^{-1}\mathbf{1}\left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}\right) \\
&= \omega_1\left(\frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}}\right) + \omega_2 \boldsymbol{w}_{GMV},
\end{aligned}$$

with $\omega_1 + \omega_2 = \lambda_1 B + \lambda_2 C = 1$.

Thus, any optimal portfolio with target μ_p consists of two mutual funds:

1. A portfolio $\boldsymbol{w}_1$ with $\mu_P = A/B$, such that $\boldsymbol{w}_1 = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}}$.
2. The GMV portfolio $\boldsymbol{w}_{GMV}$.

When choosing weights w in the risky assets, where can now lend or borrow at the riskless rate R_f.

- Optimization problem:

$$\min_{\boldsymbol{w}} \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \quad \text{s.t. } \mu_p - R_f = (\boldsymbol{\mu} - R_f\mathbf{1})^\top \boldsymbol{w},$$

- Lagrangian $\mathcal{L}(\boldsymbol{w},\lambda_1,\lambda_2) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \lambda_1(\mu_p - R_f - (\boldsymbol{\mu} - R_f\mathbf{1})^\top \boldsymbol{w})$ gives FOC

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \Sigma \boldsymbol{w}^* - \lambda_1(\boldsymbol{\mu} - R_f\mathbf{1}) = 0.$$

- Hence, we get

$$\boldsymbol{w}^* = \lambda_1 \Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1}).$$

- Plugging in $\boldsymbol{w}^*$ into the constraint gives $\lambda_1 = \frac{\mu_p - R_f}{E}$ with $E = (\boldsymbol{\mu} - R_f\mathbf{1})^\top \Sigma^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})$.

- The minimized portfolio variance depends on the Lagrange multiplier and on E:

$$\sigma_p^2 = \boldsymbol{w}^\top \Sigma \boldsymbol{w} = \lambda_1^2 (\boldsymbol{\mu} - R_f \mathbf{1})^\top \Sigma^{-1} \Sigma \Sigma^{-1} (\boldsymbol{\mu} - R_f \mathbf{1}) = \lambda_1^2 E.$$

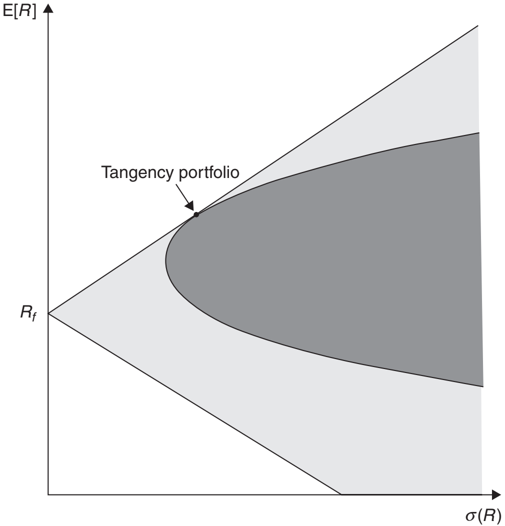
If we now substitute out the Lagrange multiplier, we find that

$$\sigma_p^2 = (\mu_p - R_f)^2 / E \qquad \text{or} \qquad |\mu_p - R_f| = \sqrt{E} \sigma_p.$$

<u>Interpretation:</u>

- We obtain a V-shaped line on a plot of mean against standard deviation.
- The upper branch is also known as the **Capital Allocation Line (CAL)**
- There is a unique portfolio, the **Tangency portfolio**, that is on the CAL but that contains only risky assets, and therefore is also on the mean-variance efficient of risky assets.

- The slope of the CAL $\sqrt{E}$ is the Sharpe ratio of the tangency portfolio and the **maximum possible Sharpe ratio**.



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Covariances and Risk Premia

- ☞ First note that $\boldsymbol{w}\Sigma = \mathbb{E}[(\boldsymbol{R} - \boldsymbol{\mu})(\boldsymbol{w}^\top \boldsymbol{R} - \boldsymbol{w}^\top \boldsymbol{\mu})]$ is the vector of covariances of the portfolio $\boldsymbol{w}$ with the risky assets.
- From the FOC we have $\Sigma \boldsymbol{w}^* = \lambda_1(\boldsymbol{\mu} - R_f \mathbf{1})$. Hence, the vector of covariances should be proportional to the vector of risk premia.
- Multiplying the FOC by $\boldsymbol{w}$, solving for λ_1 and plugging it back the the FOC yields:

$$\boldsymbol{\mu} - R_f \mathbf{1} = (\mu_p - R_f)\frac{\Sigma \boldsymbol{w}}{\boldsymbol{w}^\top \Sigma \boldsymbol{w}} = (\mu_p - R_f)\boldsymbol{\beta}.$$

- ☞ We conclude that the first-order condition for mean-variance optimization in the presence of a risk-free asset is equivalent to the statement that **asset risk premia equal the products of asset betas with the portfolio risk premium**.

Tobin (1958)'s Separation Theorem

The relative portfolio fraction is independent of the choice of the targeted portfolio return μ_p.

Implications for Portfolio Delegation

- The separation theorem implies that any investor's portfolio decision is the same.
- The only difference between investors is the relative portion between the risky portfolio and the risk-free interest rate R_f.
- This portion depends on the investor's risk aversion.

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Advisor and investor type	Percent of portfolio: Cash	Bonds	Stocks	Ratio of bonds to stocks
A. Fidelity				
Conservative	50	30	20	1.50
Moderate	20	40	40	1.00
Aggressive	5	30	65	0.46
B. Merrill Lynch				
Conservative	20	35	45	0.78
Moderate	5	40	55	0.73
Aggressive	5	20	75	0.27
C. Jane Bryant Quinn				
Conservative	50	30	20	1.50
Moderate	10	40	50	0.80
Aggressive	0	0	100	0.00
D. *The New York Times*				
Conservative	20	40	40	1.00
Moderate	10	30	60	0.50
Aggressive	0	20	80	0.25

- The zero beta portfolio was introduced by Black (1972) to derive a CAPM without riskless interest rate.
- Why no risk-free rate? E.g., we might have inflation uncertainty or credit rationing.
- Black (1972) showed that the major results of the CAPM do not require the existence of a risk-free asset.
- Without access to a risk free asset, investors instead use a zero-beta portfolio, i.e., a portfolio of risky assets with zero covariance with the market portfolio.
- The zero-beta CAPM implies that beta is still the correct measure of systematic risk and the model still has a linear specification.

Zero Beta Portfolio (cont.)

Result

For each MVP portfolio w_p, except for the global MVP, there exists a unique MVP, **the zero-beta MVP with respect to w_p**, that has zero covariance with w_p.

- Note that in the absence of the risk-free rate, the efficient frontier consists of those MVPs with expected return higher than or equal to μ_g.
- Also note that the return of the global MVP must be equal or higher than the return of any zero-beta portfolio.

Zero Beta Portfolio

Let R_q denote the return of **any** asset or portfolio, let R_p be any MVP except the global MVP, and R_{0p} the return of an MVP zero-beta portfolio w.r.t. p. Then, for the regression

$$R_q = \beta_0 + \beta_1 R_{0p} + \beta_2 R_p + \epsilon, \quad \mathbb{E}\left(\epsilon | R_p, R_{0p}\right) = 0,$$

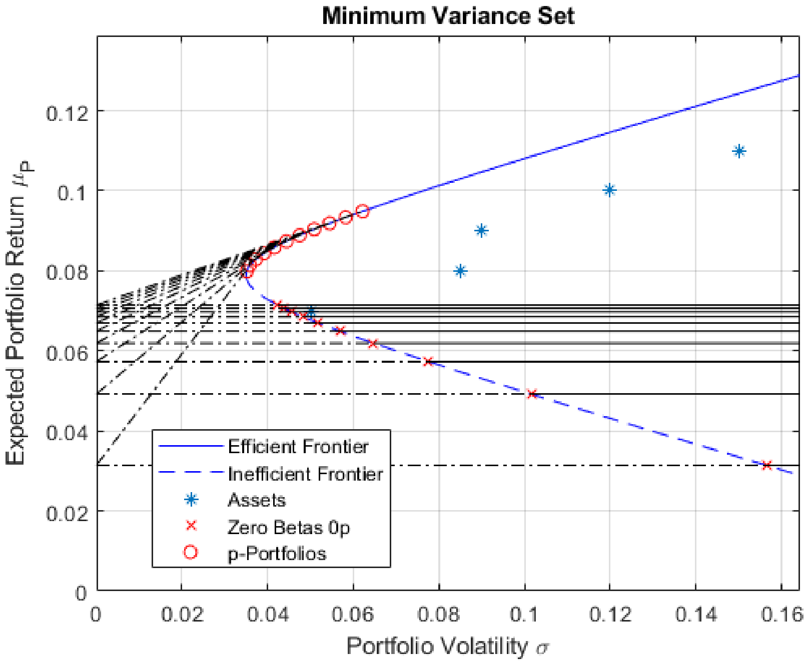
it holds that $\beta_0 = 0$, $\beta_2 = \beta_{pq}$, $\beta_1 = 1 - \beta_{pq}$, where $\beta_{pq} = \frac{\mathrm{Cov}(R_p, R_q)}{\sigma_p^2}$. Therefore,

$$\mu_q = \mu_{0p} + \beta_{pq}\left(\mu_p - \mu_{0p}\right),$$

i.e., every portfolio has a beta-representation in terms of an MVP and a portfolio orthogonal to the MVP.

Zero Beta Portfolio (cont.)

- Given only $\boldsymbol{w}_{0p}$ and $\boldsymbol{w}_p$, we obtain a curve inside the MV set touching the efficient frontier in P.
- All zero-beta portfolios lie on a horizontal line.
- From any point on that line, we can span a curve which combines $\boldsymbol{w}_p$ with some portfolio $\boldsymbol{w}_{0p}$ orthogonal to $\boldsymbol{w}_p$. The collection of all these curves spans the whole interior of the MV set.
- Hence, any feasible mean variance combination can be constructed from any minimum-variance portfolio $\boldsymbol{w}_p$ and some $\boldsymbol{w}_{0p}$, which is orthogonal to $\boldsymbol{w}_p$ but not necessarily a MVP!


Minimum Variance Set
Expected Portfolio Return μ_P
Portfolio Volatility σ
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
Efficient Frontier
Inefficient Frontier
Assets
Zero Betas 0p
p-Portfolios

Proof of Zero Beta Portfolio

- We first note $\mathrm{Cov}(R_q,R_p) = \mathrm{Cov}(\beta_0 + \beta_1 R_{0p} + \beta_2 R_p + \epsilon, R_p) = \beta_2\sigma_p^2$. Thus, $\beta_2 = \beta_{pq}$. Similarly, $\beta_1 = \mathrm{Cov}(R_q,R_{0p})/\sigma_{0p}^2$. Taking expectations on both sides of the regression equation gives $\beta_0 = \mu_q - \beta_1\mu_{0p} - \beta_2\mu_p$. Using the proposition on Slide 8, write $\mathrm{Cov}(R_p,R_p) = e\,(\mu_p - f)^2 + g$ with $e = C/D, f = B/C, g = 1/C$, and $\mathrm{Cov}(R_{0p},R_p) = e\,(\mu_p - f)\,(\mu_{0p} - f) + g = 0$ to get $\mu_{0p} = f - \frac{g}{e(\mu_p - f)}$. Using the latter equation, we get

$$\begin{aligned} \mathrm{Cov}(R_{0p},R_q) &= e\,(\mu_q - f)\,(\mu_{0p} - f) + g = g\frac{\mu_p - \mu_q}{\mu_p - f}, \\ \sigma_{0p}^2 &= e\,(\mu_{0p} - f)^2 + g = g\left(\frac{g}{e(\mu_p - f)^2} + 1\right). \end{aligned}$$

Proof of Zero Beta Portfolio (cont.)

Using these formulas for β_1 gives

$$\begin{aligned}
\beta_1 &= \frac{g\frac{\mu_p-\mu_q}{\mu_p-f}}{g\left(\frac{g}{e(\mu_p-f)^2}+1\right)} \\
&= \frac{e\left(\mu_p-f\right)\left(\mu_p-\mu_q\right)}{\sigma_p^2} \\
&= \frac{e\left(\mu_p-f\right)\left(\mu_p-f-\left(\mu_q-f\right)\right)+g-g}{\sigma_p^2} \\
&= \frac{e\left(\mu_p-f\right)^2+g-\left(e\left(\mu_p-f\right)\left(\mu_q-f\right)+g\right)}{\sigma_p^2} \\
&= \frac{\sigma_p^2-\mathrm{Cov}(R_p,R_q)}{\sigma_p^2} \\
&= 1-\beta_{pq}.
\end{aligned}$$

Proof of Zero Beta Portfolio (cont.)

Finally, $\beta_0 = \mu_q - (1-\beta_{pq})\mu_{0p} - \beta_{pq}\mu_p = \mu_q - \mu_{0p} - \beta_{pq}\left(\mu_p - \mu_{0p}\right)$.
Consider

$$\begin{aligned}
\beta_{pq}\left(\mu_p - \mu_{0p}\right) &= \frac{\mathrm{Cov}(R_p,R_q)}{\sigma_p^2}\left(\mu_p - f + \frac{g}{e(\mu_p - f)}\right) \\
&= \frac{\mathrm{Cov}(R_p,R_q)}{e(\mu_p - f)^2 + g}\frac{e(\mu_p - f)^2 + g}{e(\mu_p - f)} = \frac{\mathrm{Cov}(R_p,R_q)}{e(\mu_p - f)} \\
&= \frac{e(\mu_p - f)(\mu_q - f) + g}{e(\mu_p - f)} \\
&= \mu_q - f + \frac{g}{e(\mu_p - f)} = \mu_q - \mu_{0p}.
\end{aligned}$$

Plugging in gives $\beta_0 = 0$ and hence $\mu_q = \mu_{0p} + \beta_{pq}(\mu_p - \mu_{0p})$.

Proof of Zero Beta Portfolio (cont.)

- Notice that the zero beta formula depends on R_{0p} only via its mean. Moreover, some portfolio $\boldsymbol{w}_q$ is a zero-beta portfolio if and only if $\frac{C}{D}(\mu_p - \frac{B}{C})(\mu_q - \frac{B}{C}) = -\frac{1}{C}$, meaning that all zero-beta portfolios have the same expected return. Consequently, the formula holds for any of them.
- Does a zero-beta portfolio always exist? First, it does not exist for the global MVP. Moreover, an important assumption behind the Zero-Beta portfolio is that short-sales are possible. To obtain Zero-Beta portfolios we typically would have to short sell some assets. If there are short-sales constraints the Zero-Beta portfolio generally fails to exist.

- Modern implementations of this analysis use mean-variance optimization software to find optimal portfolios numerically, taking into account additional constraints such as short-sales constraints or limits on the maximum positions in individual assets.
- Estimates of means are imprecise over short periods, but means may not be constant over long periods of time.
- The problem of estimating the variance-covariance matrix with N assets ($N(N+1)/2$ parameters to estimate).
- Large dimensions: $N \geq T$?
- High-frequency
- Highly leveraged portfolios
- Equally-weighted portfolio?

Mean-Variance with Liabilities

Mean-Variance Optimization with Liabilities

- The previous MV analysis was focused **only** on the **asset** side of the optimization problem
- However, for many financial firms, not only assets the risk&return trade-off has to be managed, but also the risk&return of **liabilities**
- The most prominent examples: **Pension funds**
  - Pension plan experiences investment risk from both sides of the balance sheet. $\rightarrow$ Pension plan needs to conduct **asset-liability management** to manage both asset and liability risks
  - Estimation of future liabilities is not straightforward as they depend on a number of factors (salary, years of service etc.)
  - Value of the liabilities **fluctuates** even if payments are fixed (discount and inflation effect)
  - The pension plan's liabilities can be viewed as a **stream of future cash flows** whose main driver of its value will be the level of interest rates.
  - In essence, the pension plan is **short a long-term bond**.

Assets under Management by pension funds (2015):[1]

- Total pension assets were estimated at **USD 38 Trillions**
- Pension funds were the main investors worldwide (USD 26 trillion, 68% of the total), followed by[2]
  - banks and investment companies (USD 7.7 trillion, 20.2%),
  - insurance companies (USD 4.3 trillion, 11.3%)
  - and employers through their book reserves (USD 0.2 trillion, 0.5%).
- Pension funds' assets relative to GDP stands at about 80%
- The largest pension markets are the US, UK and Japan with 61.5%, 9% and 7.7% of total pension funds assets

[1] If not otherwise stated: Source: Wilson Tower Watson 2016: Global Pension Asset Study 2016

[2] Source: OECD: Pension Markets in Focus 2016.

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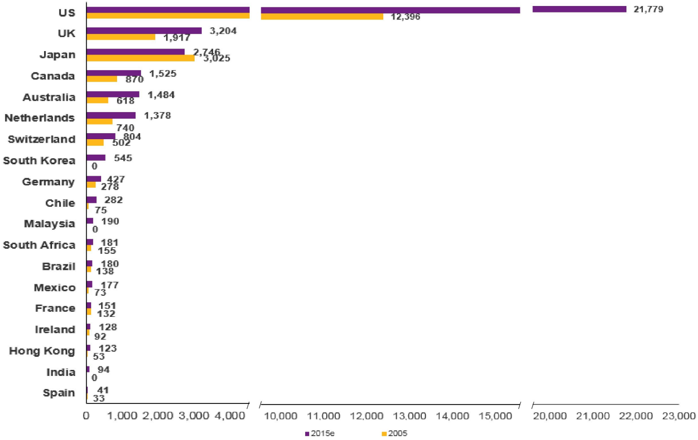
Global Pension Assets

Evolution 2005-2015 – USD billion



Source: Willis Towers Watson and secondary sources

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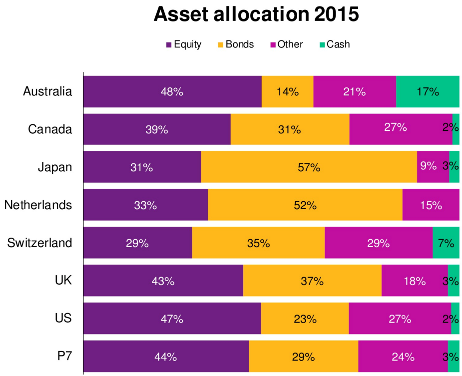
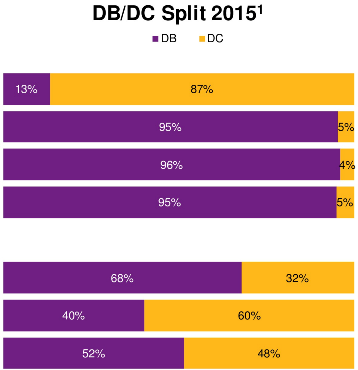
Global Pension Assets Study 2016

Key findings - Figures





[1] In Switzerland DC stands for cash balance, where the plan sponsor shares the investment risk and all assets are pooled. There are almost no pure DC assets where members make an investment choice and receive market returns on their funds. Therefore, Switzerland is excluded from this analysis..
Source: Willis Towers Watson and secondary sources

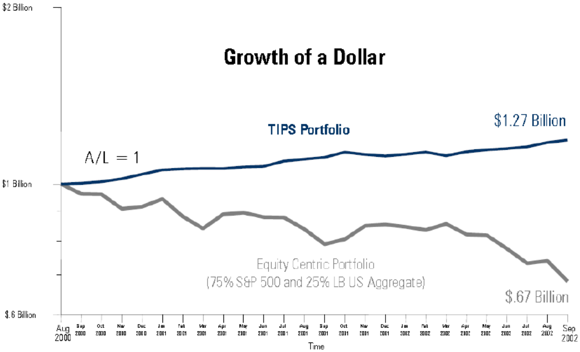
When we introduce liabilities to the portfolio allocation problem, the perspective on risk changes:
What is the true risk of the fund?

- It is **NOT** the standard deviation of the asset portfolio
- It's the risk that you **won't be able to pay your liabilities**!
- Moreover, correlation between assets and liabilities is crucial for obtaining optimal portfolio allocations

A key (new) variable:

- The **funding ratio** defined as $f_t := A_t/L_t$, where A_t are the assets and L_t the liabilities of the fund at time t.
  - If $f > 1$ the funds is said to be **over**-funded which implies that it can (currently) meet its obligations
  - If $f < 1$, the funds is said to be **under**-funded and thus is not able to pay its liabilities

Matching Liabilities not always easy



Source: Morning Star Associates LLC, "Liability Relative Optimization: Begin with the End in Mind".

- Treasury Inflation Protected Securities (TIPS) offer protection against inflation risk.
- S&P 500 as primary benchmark for stock-oriented funds, Lehman Brothers (LB) Aggregate Bond index for bond funds.

Our main objective: How do we adopt the asset-only MV optimization problem to incorporate liabilities?

- Solution: Adding liabilities to the objective function of the MV problem
- Also known as surplus, liability-relative investing or asset-liability management (ALM)
- Thus, we focus on both assets and liabilities and how they correlate with each other.

Mathematical formulation of the problem:

- Suppose that the benchmark is a risky liability and not cash (see, e.g., Keel and Müller (1995))
- Consider changes in the surplus:

$$\Delta S = A \times R_p - L \times R_l$$

  and rewrite:

$$\frac{\Delta S}{A} = R_p - 1/f \times R_l,$$

  with $f = A/L$, the funding ratio.
- L could be driven by inflation, wage growth, longevity, etc.. (restricted to exogenous variables!)

MV Optimization with Liabilities (cont.)

- To optimize over assets only, we define Σ as the covariance matrix of the assets and $\boldsymbol{\zeta}$ as the vector of covariances between the assets and the liability return R_l, i.e., $\zeta_i = \mathrm{Cov}(R_i, R_l)$. The mean of asset returns is simply denoted as $\boldsymbol{\mu}$.
- The mean-variance optimization leads to the optimization problem

$$\min_{\boldsymbol{w}} \mathrm{Var}\left(R_p - \frac{1}{f}R_l\right)$$

subject to $\boldsymbol{w}^\top \mathbf{1} = 1$ and

$$\mathbb{E}\left(R_p - \frac{1}{f}R_l\right) \geq \bar{r},$$

where $\bar{r}$ has to be chosen in accordance with the risk tolerance (of the pension fund).

- The optimization problem $\min_{\boldsymbol{w}} \operatorname{Var}\left(R_p - \frac{1}{f} R_l\right)$ can be reformulated as

$$\min_{\boldsymbol{w}} \left(\frac{1}{2} \boldsymbol{w}^\top \Sigma \boldsymbol{w} - \frac{1}{f} \boldsymbol{\zeta}^\top \boldsymbol{w} \right),$$

subject to

$$\begin{aligned} \boldsymbol{\mu}^\top \boldsymbol{w} &\geq \bar{r} + \frac{1}{f} \mathbb{E}(R_l) =: \hat{r} \\ \boldsymbol{w}^\top \mathbf{1} &= 1. \end{aligned}$$

- To solve the optimization problem, we need to solve the first-order conditions:

$$\begin{aligned} \Sigma \boldsymbol{w} - \frac{1}{f} \boldsymbol{\zeta} - \lambda \boldsymbol{\mu} - \delta \mathbf{1} &= 0, \\ \boldsymbol{\mu}^\top \boldsymbol{w} &\geq \hat{r}, \\ \mathbf{1}^\top \boldsymbol{w} &= 1. \end{aligned}$$

MV Optimization with Liabilities (cont.)

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- We first look at the minimum variance portfolio, which is obtained by omitting the constraint $\boldsymbol{\mu}^\top \boldsymbol{w} \geq \hat{r}$, i.e., setting $\lambda = 0$ in the above FOCs. Plug

$$\boldsymbol{w} = \frac{1}{f}\Sigma^{-1}\boldsymbol{\zeta} + \delta\Sigma^{-1}\mathbf{1}$$

into $\mathbf{1}^\top \boldsymbol{w} = 1$ and solve for δ to obtain

$$\delta = \frac{1 - \frac{1}{f}\mathbf{1}^\top\Sigma^{-1}\boldsymbol{\zeta}}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}.$$

- Plugging δ into the FOC for w, we get for the global minimum variance portfolio

$$\boldsymbol{w}_{min} = \underbrace{\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}}_{\equiv \hat{\boldsymbol{w}}_{min}} + \underbrace{\frac{1}{f}\left(\Sigma^{-1}\boldsymbol{\zeta} - \frac{\boldsymbol{\zeta}^\top\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\right)}_{\equiv \boldsymbol{z}_{min}}.$$

- The term $\hat{\boldsymbol{w}}_{min}$ corresponds to the minimum variance portfolio in the absence of liabilities.
- The term $\boldsymbol{z}_{min}$ is the correction term stemming from liabilities and is linear in $\boldsymbol{\zeta}$.
- Note that $\mathbf{1}^\top \boldsymbol{z}_{min} = 0$, so we can interpret $\boldsymbol{z}_{min}$ as a pure hedge portfolio in the presence of liabilities.
- The solution for all efficient portfolios is given as

$$\boldsymbol{w}^\star = \boldsymbol{w}_{min} + \lambda \boldsymbol{z}^\star,$$

where λ is the Lagrangian multiplier of the constraint $\boldsymbol{w}^\top \boldsymbol{\mu} \geq \hat{r}$ and

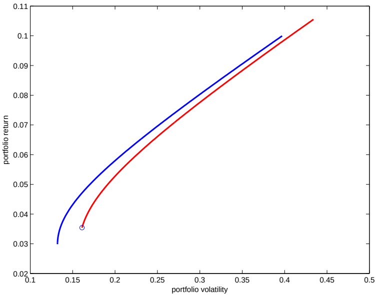
$$\boldsymbol{z}^* = \Sigma^{-1}\boldsymbol{\mu} - \frac{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \Sigma^{-1}\mathbf{1}.$$

- Obviously, $\boldsymbol{z}^\star$ is not related to the liability. Also note $\mathbf{1}^\top \boldsymbol{z}^\star = 0$.

- Summarizing, we can express the efficient portfolios as a combination of three portfolios, a global minimum variance portfolio, a liability hedging portfolio, and a speculative portfolio:

$$\boldsymbol{w}^\star = \hat{\boldsymbol{w}}_{min} + \boldsymbol{z}_{min} + \lambda \boldsymbol{z}^\star.$$

Geometrically, the occurrence of liabilities leads to parallel shift of the efficient set by $\boldsymbol{z}_{min}$.



Geometric Interpretation

- Including liabilities leads to a parallel shift of the efficient frontier: a vertical shift plus a sidewise shift, but no change in the curvature of the efficient frontier.

MV Optimization with Liabilities (cont.)

- However, note we are still in the one-liability and one-period case. Multi-liabilities and multiperiod cases become quite complicated (see, Leippold et al. (2004)).
- In addition, liabilities may be endogenous (see, Leippold et al. (2011)).

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a) Prove the Merton (1972) proposition. What are the signs of the scalars A, B, C, and D?

b) Prove Tobins's separation theorem.

c) Prove that the covariance of a mean-variance portfolio w_p with any asset or portfolio w_q, i.e. not necessarily on the MV frontier, is

$$\mathrm{Cov}(R_p, R_q) = \frac{C}{D}(\mu_p - \frac{B}{C})(\mu_q - \frac{B}{C}) + \frac{1}{C} \quad \text{if} \boldsymbol{\mu} \neq k\mathbf{1} \text{ for all } k \in \mathbb{R}.$$

What is the result for the covariance if $\boldsymbol{\mu} = k\mathbf{1}$ for some $k \in \mathbb{R}$?

d) Compute the variance of the minimum variance portfolios, the mean return μ_g, variance σ_g^2 and weights $\boldsymbol{w}_g$ of the global minimum variance portfolio.

e) Show that the portfolio weights derived in exercise a) can be rewritten as

$$\boldsymbol{w}_p = \boldsymbol{w}_g + (\mu_p - \mu_g)\boldsymbol{z}^\star,$$

where $\boldsymbol{z}^\star = \Sigma^{-1}\boldsymbol{k}_1$ is a pure hedge portfolio in the sense that $\mathbf{1}^T\boldsymbol{z}^\star = 0$ and $\boldsymbol{k}_1 = \frac{C\boldsymbol{\mu} - B\mathbf{1}}{D}$.

f) Proof the zero-beta portfolio, i.e., show that

$$\mu_q = \mu_{0p} + \beta_{pq}(\mu_p - \mu_{0p}). \tag{1}$$

g) Show that equation (1) holds for any Zero-Beta portfolio w.r.t. portfolio $\boldsymbol{w}_p$ (not necessarily MVP).

h) Does the Zero Beta portfolio always exist? If not give an example under which condition(s) it fails to exist.

i) Consider changes in the surplus $\frac{\Delta S}{A} = r_p - 1/f \times r_l$, with $f = A/L$, the funding ratio. Optimize over assets only, with Σ as the covariance matrix of the assets and $\boldsymbol{\zeta}$ as the vector of covariances between the assets and the liability return r_l, i.e., $\zeta_i = \mathrm{Cov}(r_i, r_l)$. The mean of asset returns is simply denoted as $\boldsymbol{\mu}$.

Questions:

1. Solve the mean-variance optimization problem

$$\min_{\boldsymbol{w} \in \mathbb{R}^n} \mathrm{Var}\left(r_p - \frac{1}{f} r_l\right)$$

   subject to $\boldsymbol{w}^\top \mathbf{1} = 1$ and $\mathbb{E}\left(r_p - \frac{1}{f} r_l\right) = \bar{r}$, where $\bar{r}$ is the target surplus return that has to be chosen in accordance with the risk tolerance.
2. Find the expected return $\mu_{g,l}$, variance $\sigma_{g,l}$ as well as portfolio weights $\boldsymbol{w}_{g,l}$ of the global minimum variance portfolio with liabilities.