Interest Rate Derivative Modeling Using LIBOR Market Models

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Abstract

This paper reports the work of final project for Fixed Income Securities and Derivative Modeling. In this project, a LIBOR Market Model (LMM), also known as the BGM Model (Brace Gatarek Musiela Model), is established for forward rate dynamics modeling. Pricing formulas for options are derived based on Black's formula. A Monte Carlo simulation framework is designed to generate the stochastic processes. Then a binomial tree and a trinomial tree are set up to numerically solve the system respectively. The model is calibrated to the U.S. market swaption volatilities.

Keywords: LIBOR market model; derivative pricing; Monte Carlo simulation; binomial/trinomial tree, calibration.

Final project for Fixed Income (MATH 626)

1. Introduction

Interest rate models are brought up for many reasons, including financial forecasting, risk analysis, trading, and fixed income securities pricing. From the perspective of forward rates modeling, the Heath-Jarrow-Morton (HJM) model [1] is a general framework that contains any arbitrage-free interest rate model adapted to a finite set of Brownian motions.

However, working directly with instantaneous forward rates is not particularly attractive in applications. First, instantaneous forward rates are never quoted in the market, nor do they specify directly in the payoff of any traded derivative contract. In addition, an infinite set of instantaneous forward rates cannot be solved numerically in a computer.

To overcome the difficulties that HJM model faces, a formulation of the forward rates in terms of a non-overlapping set of simply compounded LIBOR rates is proposed by Brace et al. (1997) [2], Jamshidian (1997) [3], and Miltersen et al (1997) [4]. Not only do we then conveniently work with a finite set of directly observable rates that can be represented on a computer, but an explosion-free log-normal forward rate model also becomes possible. However, we should emphasize that LIBOR market model will still be a special case of an HJM model.

This paper is organized as follow. The stochastic differential equations for LIBOR market models are derived based on Brace, Gatarek and Musiela's approach in Section 2. Pricing formula for call options are derived in Section 3. The Monte Carlo simulation framework is introduced in Section 4. The binomial tree and trinomial tree solution procedures are explained in Section 5. The method of calibration to U.S. market price is illustrated in Section 6.

2. The Formulation for LIBOR Market Models

In this section, we first provide a rigorous construction of a log-normal model for forward LIBORs. Then considering the scenario where we have a collection of reset dates, we formulate a system of stochastic differential equations under the forward probability measure.

2.1. Forward LIBOR Modeling

Assume we are given a family $B(t,T)$ of zero coupon bond prices, and thus also the collection $F_B(t,T,U)$ of forward process, where forward process is defined as:

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}, \quad \forall t \in [0, T \wedge U] \tag{1}$$

where $T, U \in [0, T^*]$ are any two maturities and T^* is a fixed horizon date.

Also assume a fixed strictly positive real number $\delta < T^*$ representing the length of the accrual period. Then the forward δ-LIBOR rate $L(t,T)$ for the future date $T \leq T^* - \delta$ prevailing at time t is given by the market formula:

$$1 + \delta L(t,T) = F_B(t, T, T+\delta), \quad \forall t \in [0,T] \tag{2}$$

We can then re-express $L(t,T)$ directly in terms of bond prices:

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)}, \quad \forall t \in [0,T] \tag{3}$$

with the initial term structure of forward LIBOR satisfying:

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)} \tag{4}$$

Suppose the family $F_B(t,T,T^*)$ of forward processes satisfying the stochastic differential equations under the probability space $(\Omega, (\mathcal{F}_t)_{t \in [0,T^*]}, \mathbb{P}_{T^*})$:

$$dF_B(t,T,T^*) = F_B(t,T,T^*)\gamma(t,T,T^*) \cdot dW_t^{T^*} \tag{5}$$

where $W_t^{T^*}$ is a d-dimensional Brownian motion under $\mathbb{P}_{T^*}$. Then according to Girsanov theorem [5], one can change the measure into the forward measure $\mathbb{P}_{T+\delta}$:

$$dL(t,T) = \delta^{-1} F_B(t,T,T+\delta)\gamma(t,T,T+\delta) \cdot dW_t^{T+\delta} \tag{6}$$

where $W_t^{T+\delta}$ is defined by:

$$W_t^{T+\delta} = W_t^{T^*} - \int_0^t \gamma(u, T+\delta, T^*)du \tag{7}$$

This implies that $L(t,T)$ solves the equation:

$$dL(t,T) = \delta^{-1}(1 + \delta L(t,T))\gamma(t,T,T+\delta) \cdot dW_t^{T+\delta} \tag{8}$$

which can be rewritten as:

$$dL(t,T) = L(t,T)\lambda(t,T) \cdot dW_t^{T+\delta} \tag{9}$$

where for every $t \in [0,T]$ we have:

$$\lambda(t,T) = \frac{1+\delta L(t,T)}{\delta L(t,T)}\gamma(t,T,T+\delta) \tag{10}$$

Thus, the collection of forward processes uniquely specifies the family of forward LIBORs.

As mentioned earlier, LIBOR market modeling is still a special case of HJM framework. Thus, to construct the dynamics of $L(t,T)$ under the martingale measure (risk-neutral measure) $\mathbb{P}^*$, we can postulate it is governed by the following SDE:

$$dL(t,T) = \mu(t,T)dt + L(t,T)\lambda(t,T)\cdot dW_t^* \tag{11}$$

where λ is known, but the drift μ is unspecified. Recall that in HJM framework, the arbitrage-free dynamics of the instantaneous froward rate $f(t,T)$ has the form:

$$df(t,T) = \sigma(t,T)\cdot\sigma^*(t,T)dt + \sigma(t,T)\cdot dW_t^* \tag{12}$$

where $\sigma^*(t,T)$ is defined as:

$$\sigma^*(t,T) = \int_t^T \sigma(t,u)du \tag{13}$$

Remark: Equation (13) is a very important result derived from HJM framework. One can find a detailed proof in [6].

In addition, from Equation (3) we have the following relationship:

$$1+\delta L(t,T) = \exp\,(\int_T^{T+\delta} f(t,u)du) \tag{14}$$

Apply Ito's rule to Equation (14) and compare the terms with SDE (11), we can find that:

$$\sigma^*(t,T+\delta) - \sigma^*(t,T) = \int_T^{T+\delta}\sigma(t,u)du = \frac{\delta L(t,T)}{1+\delta L(t,T)}\lambda(t,T) \tag{15}$$

By setting the initial condition $\sigma(t,T) = 0$ for $0 \le t \le T \le t+\delta$, we obtain the following relationship:

$$b(t,T) = -\sigma^*(t,T) = -\sum_{k=1}^{[\delta^{-1}T]} \frac{\delta L(t,T-k\delta)}{1+\delta L(t,T-k\delta)}\lambda(t,T-k\delta) \tag{16}$$

In conclusion, under the martingale measure $\mathbb{P}^*$, the process $L(t,T)$ satisfies:

$$dL(t,T) = L(t,T)\sigma^*(t,T)\cdot\lambda(t,T)dt + L(t,T)\lambda(t,T)\cdot W_t^* \tag{17}$$

Notice that this result is equivalent to Equation (9) under the forward measure $\mathbb{P}_{T+\delta}$.

In the generic model of forward LIBOR, for any maturity $x > 0$, and for a fixed δ, at time t, the process $\{L(t,x); t, x \geq 0\}$ satisfies:

$$\begin{aligned} dL(t,x) = &\left(\frac{\partial}{\partial x} L(t,x) + L(t,x)\lambda(t,x) \cdot \sigma(t,x) + \frac{\delta L^2(t,x)}{1+\delta L(t,x)} |\lambda(t,x)|^2 \right) dt \\ &+ L(t,x)\lambda(t,x) \cdot dW_t^* \end{aligned} \tag{18}$$

To summarize, LIBOR market model assumes a lognormal volatility structure on the LIBOR rate $L(t,T)$ which is defined by Equation (3), for a given maturity $T \geq 0$ and a fixed $\delta > 0$. This leads to the volatility model given in Equation (16). $L(t,T)$ is then given by Equation (17). Finally, if the maturity is not fixed, the general model for $\{L(t,x); t, x \geq 0\}$ is given by Equation (18). The proof of the existence and uniqueness of the solution can be found in [1].

2.2. Scenario Generation

Now consider the case that a collection of reset/settlement dates $0 < T_0 < T_1 < \cdots < T_n$ is given. Under the actual probability measure $\mathbb{P}$, we assume that each LIBOR $L_i(t) = L(t, T_i), i = 0, \ldots, n-1$, solves the the stochastic differential equation that we present in Section 2.1:

$$dL_i(t) = L_i(t)(\mu_t^i dt + \lambda_i(L_i(t), t) dW_t^i) \tag{19}$$

Assume the Brownian motions $W^0, W^1, \ldots, W^{n-1}$ have the instantaneous correlations given by:

$$d\left\langle W^i, W^j \right\rangle_t = \rho_t^{i,j} dt \tag{20}$$

Recall that the forward measure $\mathbb{P}_{T_n}$ corresponds to the choice of the zero coupon bond maturing at T_n as a numeraire. Using this new numeraire, we define the relative bond prices:

$$U_i(t) := U_i(t, T_i) = \frac{B(t, T_i)}{B(t, T_n)} \tag{21}$$

Rewrite as:

$$U_i(t) = \Pi_{j=i}^{n-1} (1 + \delta_{j+1} L_j(t)) \tag{22}$$

Then we have the following results:

Proposition Under the forward measure $\mathbb{P}_{T_n}$:

$$dL_i(t) = L_i(t)(\hat{\mu}_i(t) + \lambda_i(L_i(t), t)d\hat{W}_t^i) \tag{23}$$

where

$$\hat{\mu}_i(t) = -\sum_{j=i+1}^{n-1} \frac{\delta_{j+1}L_j(t)}{1+\delta_{j+1}L_j(t)}\lambda_j(L_j(t), t)\lambda_i(L_i(t), t)\rho_t^{i,j} \tag{24}$$

This result is a direct application of the construction in Section 2.1 and it can also be proved through Girsanov's theorem [5].

3. Derivatives Pricing

The main motivation for the construction of LIBOR model is the market practice of pricing caps (and swaptions) by means of Black-Sholes-Merton type formulas. Thus in this section, we present the formulation of interest rate driven derivatives pricing, mainly on caplet, which is one leg of a cap.

The pricing formula can be derived based on Black's formula [7], which is widely used in practice to price options on future contracts, bond options, interest rate caps and floors, and swaptions [4].

Assume that the forward LIBOR follows a geometric Brownian motion with no drift term, which is specified in Equation (9).We rewrite it here:

$$dL(t, T) = L(t, T)\lambda(t, T) \cdot dW_t^{T+\delta}$$

Notice that the above equation has unique solution:

$$L(t, T) = L(0, T)\exp(\lambda(t, T) \cdot W_t^{T+\delta} - \frac{1}{2}|\lambda(t, T)|^2 t), \quad \forall t \in [0, T] \tag{25}$$

Also assume $L(t, T)$ is a lognormal stochastic variable. The payoff at time T for a call option (caplet) is: $\max(L(T, T) - K, 0)$ where K is the strike.

Then the market price at time t of a caplet with expiry date T is given by:

$$\begin{aligned}\text{Caplet}(t) &= \delta B(t, T+\delta)E_{T+\delta}[(L(T, T) - K)^+|\mathcal{F}_t] \\ &= \delta B(t, T+\delta)(L(t, T)N(\hat{e}_1(t, T)) - KN(\hat{e}_2(t, T)))\end{aligned} \tag{26}$$

where

$$
\begin{aligned}
\hat{e}_{1,2} &= \frac{\ln(L(t,T)/K \pm \frac{1}{2}\hat{v}_0^2(t,T)}{\hat{v}_0(t,T)} \\
\hat{v}_0^2(t,T) &= \int_t^T |\lambda(s, T-s)|^2 ds
\end{aligned}
\tag{27}
$$

Furthermore, a cap settled in arrears at times $T_j, j = 1, \ldots, n$, where $T_j - T_{j-1} = \delta_j, T_0 = T$, is priced by the formula:

$$
\mathrm{Cap}(t) = \sum_{j=1}^{n} \delta_j B(t, T_j)(L(t, T_{j-1})N(\hat{e}_1(t, T_{j-1})) - KN(\hat{e}_2(t, T_{j-1}))) \tag{28}
$$

4. Monte Carlo Simulation Framework

Given the complexity of the processes in the LIBOR market model, it is not possible to obtain a closed-form solution for all the forward rates. However, in practice, one can always set up a Monte Carlo simulation to get the forward LIBORs. In this section, we propose a framework to do so.

Consider the scenario given in Section 2.2. Assume the only randomness comes from the Brownian motion in Equation (23). Thus we can first generate the simulated Brownian motion path by:

$$
\hat{W}_t^i(t_{n+1}) = \hat{W}_t^i(t_n) + \epsilon_n \sqrt{t_{n+1} - t_n} \tag{29}
$$

where t_n is the discretized time point. ϵ_n are drawings from a standard normal distribution, which can be realized by some random number generators [8].

Then Equation (24) can be solved by the iteration:

$$
\begin{aligned}
L_i(t_{n+1}) =& L_i(t_n) - \sum_{j=i+1}^{n-1} \frac{\delta_{j+1} L_j(t)}{1 + \delta_{j+1} L_j(t)} \lambda_j(L_j(t), t)\lambda_i(L_i(t), t)\rho_t^{i,j}(t_{n+1} - t_n) \\
&+ \lambda_i(L_i(t_n), t_n)(\hat{W}_t^i(t_{n+1}) - \hat{W}_t^i(t_n))
\end{aligned}
\tag{30}
$$

A pseudo-code for this framework is given below:

```
for (j = 0; j < numOfPaths; j++)
{
    Lt = L0;
    for (i = 0; i < timeSteps; i++)
        {
            e = getOneRandom();
            Lt = Lt + drift + volatility*e*sqrt(dt);
        }
    result += Lt;
}
mean = result / numOfPaths;
return mean;
```

5. Interest Rate Trees

In this section, we introduce how to set up a binomial tree and a trinomial tree to simulate the forward LIBORs under the Monte Carlo Simulation framework that we form in the previous section.

5.1. Binomial Tree for LMM

Binomial trees have been widely used in the option pricing field since it was first proposed by Cox, Ross and Rubinstein in 1979 [9]. However, it can also be used to model interest rate as a simplified version of Monte Carlo simulation.

In Section 4 we assume the only randomness comes from Brownian motion. Here in binomial tree model, we can specify the randomness as:

$$\hat{W}(m) = \begin{cases} \hat{W}(m-1) + \sqrt{\Delta t} & \textit{probability} = 0.5 \\ \hat{W}(m-1) - \sqrt{\Delta t} & \textit{probability} = 0.5 \end{cases} \qquad (31)$$

$$\hat{W}(m,l) = l\sqrt{\Delta t}$$

The number m and l are horizontal and vertical coordinates of the tree respectively, as is shown in the following figure:

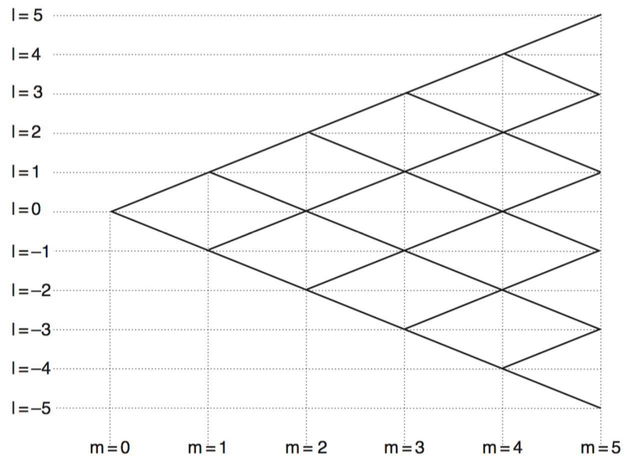

Fig. 1. Binomial Tree

Notice that the drift term in LIBOR market model is not constant, which makes it hard to implement binomial tree model. However, a so called "freezing the drift" trick introduced in [10] can approximate the forward LIBOR pretty well. Then the probability for each coordinate in Figure 1 can be calculated numerically using the recursion:

$$\begin{aligned} P(1,-1) = P(1,1) = 0.5 \\ P(m,l) = 0.5 \cdot P(m-1,l-1) + 0.5 \cdot P(m-1,l+1) \end{aligned} \tag{32}$$

5.2. Trinomial Tree for LMM

In the interest rate driven world, people always prefer trinomial tree instead of the classical binomial tree model. The main advantage of a trinomial tree is that it provides an extra degree of freedom, making it easier for the tree to represent features of the interest rate process such as mean reversion.

A trinomial tree model can be represented by the figure below:

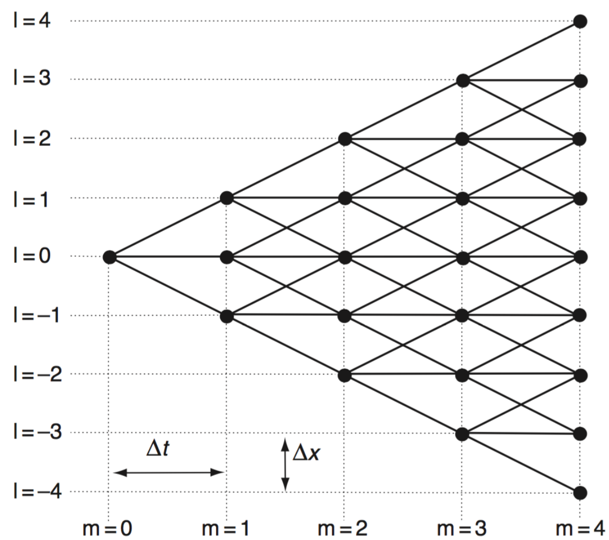

Fig. 2. Trinomial Tree

For any martingale process $M_N(t)$, its approximation $\hat{M}_N(m)$ is given by:

$$\hat{M}_N(m) = \begin{cases} \hat{M}_N(m-1) + \Delta x, & p_u = \frac{\hat{\gamma}^2(m-1)\Delta t}{2(\Delta x)^2} \\ \hat{M}_N(m-1), & p_m = 1 - \frac{\hat{\gamma}^2(m-1)\Delta t}{(\Delta x)^2} \\ \hat{M}_N(m-1) - \Delta x, & p_d = \frac{\hat{\gamma}^2(m-1)\Delta t}{2(\Delta x)^2} \end{cases} \tag{33}$$

where $\hat{\gamma}(m)$ is the approximation of the volatility of forward LIBOR $\gamma(t)$ and it is given by:

$$\hat{\gamma}(m) = \gamma(m \cdot \Delta t) \tag{34}$$

Then the formulation under Monte Carlo framework is similar to binomial tree model in Section 5.1 except now we have three cases for each step. Notice that trinomial tree model is a special case of explicit finite difference method. Then the recursive formulas for the unconditional probability that node (m, l) will be reached are given by:

$$\begin{aligned} P(0,0) =& 1 \\ P(m,l) =& p_u(m-1,l-1) \cdot P(m-1,l-1) + p_m(m-1,l) \cdot P(m-1,l) \\ &+ p_d(m-1,l+1) \cdot P(m-1,l+1) \end{aligned} \tag{35}$$

6. Model Calibration

Model calibration can be defined as the process of adjustment of the model parameters and forcing within the margins of the uncertainties (in model parameters and/or model forcing) to obtain a model representation of the processes of interest that satisfies pre-agreed criteria (Goodness-of-Fit or Cost Function).

In LIBOR market model framework, the parameters that are chosen for calibration are usually cap and swaption volatilities, together with the historically estimated correlation between the forward rates [2]. The market data on actively traded options are always referred as the *calibrating instruments*.

Suppose there are n calibrating instruments. A popular Goodness-of-Fit measure is:

$$\sum_{i=1}^{n}(U_i - V_i)^2 \tag{36}$$

where U_i is the market price of the ith calibrating instrument and V_i is the price given by the model for this instrument. The object of calibration is then to choose the model parameters so that this Goodness-of-Fit measure is minimized.

In this project, we use U.S. historical swaption volatilities as the calibrating instruments. One of the popular approaches of LIBOR market model calibration is called the *separated approach* [10]. Following the algorithm in [11], we reproduce a simplified version of object function only based on minimizing the error. Test experiments are conducted in MATLAB in a flexible manner so that readers can fit in their own data.

7. Conclusion

In this project, a LIBOR Market Model (LMM) is derived based on Brace, Gatarek, and Musiela's approach. Then derivatives pricing formulas are obtained from Black's formula. Since LMM does not have closed-form solution, we set up a Monte Carlo Simulation framework where a binomial and a trinomial model are introduced. A separated approach based on optimization is implemented to calibrate the model.

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