GARCH Modeling

An Application in Portfolio Selection

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ARCH/GARCH
ARCH

- Conditional heteroskedastic models characterize the time evolution of the conditional variance of the asset return.
- ARCH(q)

$$a_t = \sigma_t \epsilon_t, \tag{1}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2, \tag{2}$$

where ϵ_t i.i.d. (0,1), $\alpha_0 > 0$, $\alpha_i \geq 0$, $i > 0$.

However, ARCH model has the following weaknesses

- ARCH model assumes that positive and negative shocks have the same effect on volatility, which depends on the square of previous shocks. In practice, negative shocks have larger impacts;
- ARCH models with Gaussian innovations have limited ability to capture excess kurtosis;
- ARCH model provides no indication about underlying causes of the behavior of the conditional variance;
- ARCH models are likely to over-predict volatility, since they respond slowly to large isolated shocks to the return series.

ARCH/GARCH
GARCH

- GARCH (Bollerslev, 1986), as an extension to ARCH, is also able to capture features such as volatility clustering, leverage effect, etc.
- GARCH includes the past conditional variance into explanation of the future conditional variance. This is particularly useful in forecasting volatility.

- For a log return series r_t, let $a_t = r_t - \mu_t$ be the innovation term at time t. Then a_t follows a GARCH(p,q) model if

$$a_t = \sigma_t \epsilon_t, \tag{3}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i a_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{4}$$

where ϵ_t i.i.d. (0,1), $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{max(p,q)} (\alpha_i + \beta_i) < 1$.

Other GARCH Specifications
IGARCH

- IGARCH is a restricted version of GARCH model. The persistence parameters sum up to 1, and there is a unit root in the GARCH process.
- IGARCH(1,1)

$$a_t = \sigma_t \epsilon_t, \tag{5}$$

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) a_{t-1}^2, \tag{6}$$

where ϵ_t is defined as before, and $0 < \beta_1 < 1$.

Other GARCH Specifications
EGARCH

- Nelson (1991)
- To allow for asymmetric effects between positive and negative asset returns, consider the weighted innovation

$$g(\epsilon_t) = \theta\epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)] \tag{7}$$

- EGARCH(p,q)

$$a_t = \sigma_t\epsilon_t, \tag{8}$$

$$ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 L + ... + \beta_q L^q}{1 - \alpha_1 L - ... - \alpha_p L^p} g(\epsilon_{t-1}), \tag{9}$$

where L is the lag operator. The two polynomials have zeros outside the unit circle, and no common factors.

Other GARCH Specifications
TGARCH

- The model uses zero as its threshold to separate the impacts of pas shocks.
- TGARCH(p,q)

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} (\alpha_i + \gamma_i N_{t-i}) a_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \tag{10}$$

where N_{t-i} is an indicator for negative a_{t-i},

$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases} \tag{11}$$

and α_i, γ_i, and β_j are nonnegative parameters satisfying conditions similar to those of GARCH models.

Data

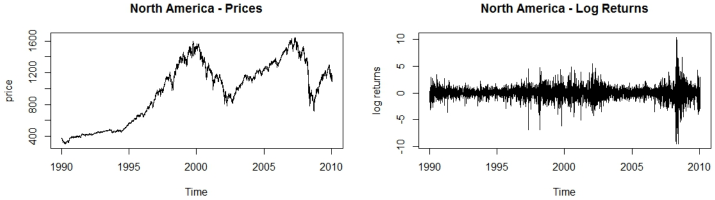
- The data set consists of daily data on dollar-denominated Morgan Stanley Capital International (MSCI) stock market indices.
- It covers three geographical areas: North America, Europe, and Asia. The first two are developed market indices, whereas the last one represents an emerging market.
- The MSCI indices represent a broad aggregation of national equity markets.

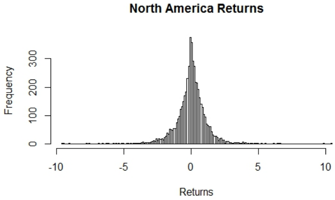
- The point of view adopted here is that of a U.S. investor managing an international portfolio, but does not hedge against any currency risk.
- The sample period is from July 16, 1990 to July 16, 2010.
- Will perform analysis on pre-crisis sample in subsequent sections.

Summary Statistics

- For each index, we compute a series of corresponding log-returns. Multiply them by 100 so they can be read as percetage returns.
- The following summary statistics will be presented for each region
  - Time series plot of price series and return series
  - Histogram of return series
  - Sample mean, minimum value, maximum value, variance, skewness, and kurtosis
- We also show the sample covariance matrix and the sample correlation matrix of these three returns.

Summary Statistics
North America

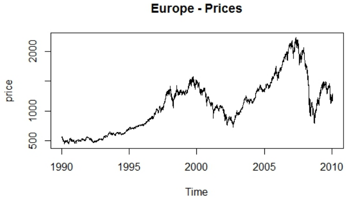


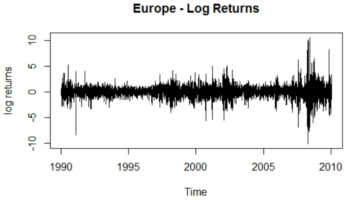


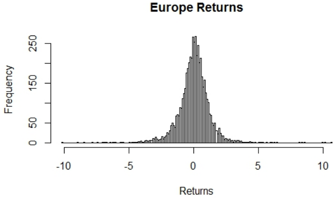
- Seems to be symmetric.
- Skewness and kurtosis values suggest otherwise.

Mean	Min	Max	Var	Skew	Kurt
0.0212	-9.5045	10.4277	1.3188	-0.2857	12.3076

Summary Statistics
Europe



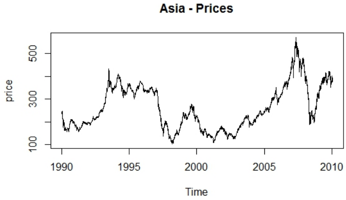


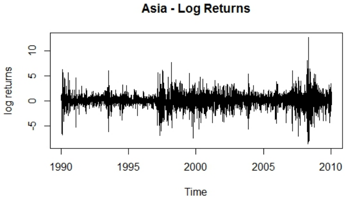


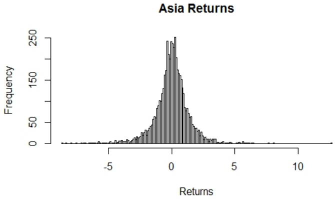
- Graph shows asymmetry of return series.
- Sample summary statistics verifies this result.

Mean	Min	Max	Var	Skew	Kurt
0.0157	-10.1783	10.6980	1.5140	-0.1596	12.0893

Summary Statistics
Asia







- Similarly, we can see evidence of asymmetry from the graph.
- Likewise, calculation indicates the same.

Mean	Min	Max	Var	Skew	Kurt
0.0095	-8.6206	12.6517	1.8826	-0.2279	8.8195

Summary Statistics
Sample Covariance & Correlation Matrix

- Sample Covariance Matrix

$$\Omega = \begin{pmatrix} 1.3188 & 0.6845 & 0.2771 \\ 0.6845 & 1.5140 & 0.6316 \\ 0.2771 & 0.6316 & 1.8826 \end{pmatrix}$$

- Sample Correlation Matrix

$$\rho = \begin{pmatrix} 1.0000 & 0.4844 & 0.1758 \\ 0.4844 & 1.0000 & 0.3741 \\ 0.1758 & 0.3741 & 1.0000 \end{pmatrix}$$

- North America and Europe are most highly correlated, followed by Europe and Asia. North American and Asian returns are least correlated.
- Possible Explanations
  - Components of systematic risk
  - Non-synchronous trading

Unconditional Portfolio Selection
Minimum Variance Frontier

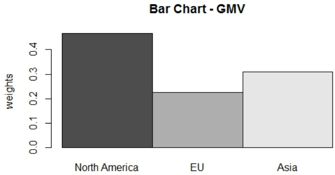
The minimum variance frontier can be obtained through the following steps

- $A = \mu'\Omega^{-1}\iota$, $B = \mu'\Omega^{-1}\mu$, $C = \iota'\Omega^{-1}\iota$, $D = BC - A^2$, where μ is the expected returns vector of assets, and ι is a column vector of ones.
- $g = (B\Omega^{-1}\iota - A\Omega^{-1}\mu)/D$, $h = (C\Omega^{-1}\mu - A\Omega^{-1}\iota)/D$
  Set portfolio return level μ_p, $w_{\mu_p} = g + \mu_p h$.
- Calculate expected return and variance for the portfolio
  $\mu_p = w'_{\mu_p}\mu$, $\sigma_p^2 = w'_{\mu_p}\Omega w_{\mu_p}$.
- Plot (σ_p, μ_p).

Unconditional Portfolio Selection

GMV Portfolio

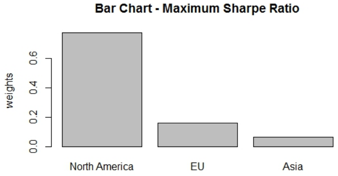
- For the global minimum variance portfolio $\mu_{GMV} = -g'\Omega h / h'\Omega h$, $w_{GMV} = g + \mu_{GMV} h$.



Unconditional Portfolio Selection

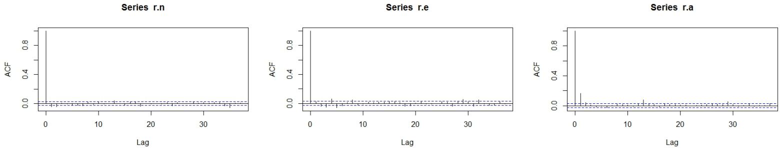
Maximum Sharpe Ratio Portfolio

- Assume zero risk-free rate
  Sharpe ratio = μ_p/σ_p

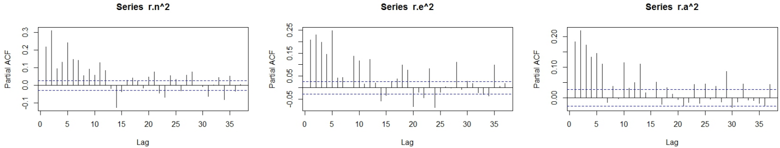


Order Selection
ACF & PACF

- ACF of Log-Return Series



- PACF of Squared Log-Return Series



Order Selection
Ljung-Box Test

- Square of standardized residuals from GARCH(1,1)

Region	χ^2	df	p-value
North America	11.1442	12	0.5166
	20.0467	24	0.6941
	21.9329	30	0.8565
Europe	5.4786	12	0.9401
	12.9652	24	0.9667
	16.2135	30	0.9809
Asia	7.5157	12	0.8217
	21.8657	24	0.5873
	30.6466	30	0.4329

- In all cases, p-values are large enough. Therefore the null hypothesis of no autocorrelation cannot be rejected. The model is well-fitted.

Order Selection

GARCH Order - North America

Region	Order	AIC	BIC
North America	(1,1)	2.6624	2.6675
	(1,2)	2.6628	2.6691
	(1,3)	2.6632	2.6707
	(2,1)	2.6607	2.6669
	(2,2)	2.6600	2.6676
	(2,3)	2.6602	2.6690
	(3,1)	2.6610	2.6686
	(3,2)	2.6604	2.6692
	(3,3)	2.6605	2.6706

GARCH Order

GARCH Order - Europe

Region	Order	AIC	BIC
Europe	(1,1)	2.8319	2.8369
	(1,2)	2.8319	2.8382
	(1,3)	2.8323	2.8399
	(2,1)	2.8323	2.8386
	(2,2)	2.8323	2.8399
	(2,3)	2.8327	2.8415
	(3,1)	2.8327	2.8402
	(3,2)	2.8327	2.8415
	(3,3)	2.8331	2.8431

GARCH Order

GARCH Order - Asia

Region	Order	AIC	BIC
Asia	(1,1)	3.1001	3.1051
	(1,2)	3.0995	3.1058
	(1,3)	3.0996	3.1072
	(2,1)	3.1003	3.1066
	(2,2)	3.0998	3.1074
	(2,3)	3.1000	3.1088
	(3,1)	3.1007	3.1082
	(3,2)	3.1001	3.1089
	(3,3)	3.1004	3.1104

GARCH Order

- For all three return series, GARCH(1,1) achieves the smallest BIC.
- Generally, GARCH(1,1) also achieves a small AIC value. While the AIC value may not be the smallest, we also notice that AIC values across models are not distinctively different.
- These two points should serve as sufficient justification to using GARCH(1,1) model in estimation.

GARCH Estimation
Estimation

- garchFit function in fGarch package - QMLE
  Default initial values as specified in the function.
- Self-written function - MLE
  Three set of initial values in order to explore the sensitivity to initial values of this estimation.
  $\omega_0 = 0.1$, $(\alpha_0, \beta_0) = (0.1, 0.8), (0.15, 0.8), (0.05, 0.9)$.

GARCH Estimation

Parameters - North America

- garchFit

	Coefficient	t-value	p-value
μ	11.1442	12	0.5166
ω	5.4786	12	0.9401
α	7.5157	12	0.8217
β	21.8657	24	0.5873

- Specified initial values

Initial Value	μ	ω	α	β
(0.1,0.1,0.8)	11.1442	12	0.5166	
(0.1,0.15,0.8)	20.0467	24	0.6941	
(0.1,0.05,0.9)	21.9329	30	0.8565	
()	5.4786	12	0.9401	
()	12.9652	24	0.9667	

GARCH Estimation
Estimated Variances - North America

GARCH Estimation
Parameters - Europe

- garchFit

	Coefficient	t-value	p-value
μ	11.1442	12	0.5166
ω	5.4786	12	0.9401
α	7.5157	12	0.8217
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- Specified initial values

Initial Value	μ	ω	α	β
(0.1,0.1,0.8)	11.1442	12	0.5166	
(0.1,0.15,0.8)	20.0467	24	0.6941	
(0.1,0.05,0.9)	21.9329	30	0.8565	
()	5.4786	12	0.9401	
()	12.9652	24	0.9667	

GARCH Estimation
Estimated Variances - Europe

GARCH Estimation

Parameters - Asia

- garchFit

	Coefficient	t-value	p-value
μ	11.1442	12	0.5166
ω	5.4786	12	0.9401
α	7.5157	12	0.8217
β	21.8657	24	0.5873

- Specified initial values

Initial Value	μ	ω	α	β
(0.1,0.1,0.8)	11.1442	12	0.5166	
(0.1,0.15,0.8)	20.0467	24	0.6941	
(0.1,0.05,0.9)	21.9329	30	0.8565	
()	5.4786	12	0.9401	
()	12.9652	24	0.9667	

GARCH Estimation
Estimated Variances - Asia

GARCH Estimation
”de-GARCHed” Return

- Let $(r_t - \hat{\mu})/\sqrt{(\hat{\sigma}_t^2)}$ denote a ”de-GARCHed” return.
- Sample correlation matrix of these returns

$$\rho_{de-GARCHed} = \begin{pmatrix} 1.0000 & 0.4844 & 0.1758 \\ 0.4844 & 1.0000 & 0.3741 \\ 0.1758 & 0.3741 & 1.0000 \end{pmatrix}$$

- Correlation matrix has smaller values now. This is reasonable since the ”de-GARCH” process eliminates persistent heteroskedastic component of returns.

Conditional Portfolio Selection

Re-Estimation

- Re-estimate GARCH(1,1) model for each asset using the data only up to September 15, 2006.
  Use full-sample estimates as the initial values.

Region	μ	ω	α	β
North America	0.0467	0.0057	0.0631	0.9320
Europe	0.0181	0.0170	0.0949	0.8923
Asia	0.0145	0.0127	0.0856	0.9106

Conditional Portfolio Selection

Portfolio Weight

- Assume for simplicity that the 3 assets are mutually independent.
- Consider the following day-t portfolio weights

$$\omega_{it} = \frac{(1/\hat{\sigma}^2_{i,t+1})}{\sum_{i=1}^{3}(1/\hat{\sigma}^2_{i,t+1})}, i = 1, 2, 3, \tag{12}$$

  where $\sigma^2_{i,t+1}$ denotes a forecast made on day t of the next day's return on asset i.
- Here the investor follows a simple volatility timing strategy, i.e. allocating \$1 each day based solely on predicted changes in the relative volatilities of the risky assets, and holding the chosen portfolio for one day.

- Make the first set of these one-day-ahead forecasts on September 15, 2006. After that, use the returns data available up that specific point in time to update the forecasts of the next day's returns.
- Re-estimate the model parameters every 50 days. Use the previous set of estimates as initial values for optimization.

w =

$$
1\text{-}7 \begin{pmatrix} 0.1995 & 0.6814 & 0.1192 \\ 0.4934 & 0.3396 & 0.1670 \\ 0.4036 & 0.3747 & 0.2216 \\ 0.5196 & 0.4443 & 0.0361 \\ 0.4603 & 0.3149 & 0.2249 \\ 0.4941 & 0.2357 & 0.2702 \\ 0.4880 & 0.3605 & 0.1515 \end{pmatrix} \quad 8\text{-}14 \begin{pmatrix} 0.3638 & 0.3377 & 0.2984 \\ 0.2245 & 0.5893 & 0.1862 \\ 0.4313 & 0.4333 & 0.1354 \\ 0.4837 & 0.1020 & 0.4143 \\ 0.4416 & 0.3959 & 0.1624 \\ 0.4850 & 0.3177 & 0.1973 \\ 0.6152 & 0.1289 & 0.2559 \end{pmatrix}
$$

$$
15\text{-}21 \begin{pmatrix} 0.5715 & 0.2222 & 0.2063 \\ 0.7610 & 0.1395 & 0.0995 \\ 0.6280 & 0.0585 & 0.3135 \\ 0.7091 & 0.0970 & 0.1939 \\ 0.4975 & 0.1462 & 0.3563 \\ 0.5432 & 0.0871 & 0.3697 \\ 0.6266 & 0.1534 & 0.2199 \end{pmatrix}
$$

North America index gets the largest weight.

Conditional Portfolio Selection

Out-of-Sample Sharpe Ratios

- Assume that risk-free rate is zero.
- Sharpe ratio = μ_p/σ_p
- $\mu_p = w_1 E(r_1) + w_2 E(r_2) + w_3 E(r_3)$
  $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2$

- S =

$$
1\text{-}7 \begin{pmatrix} 0.0321 \\ 0.0528 \\ 0.0529 \\ 0.0695 \\ 0.0448 \\ 0.0470 \\ 0.0536 \end{pmatrix}
\quad 8\text{-}14 \begin{pmatrix} 0.0436 \\ 0.0559 \\ 0.0519 \\ 0.0324 \\ 0.0272 \\ 0.0292 \\ 0.0177 \end{pmatrix}
\quad 15\text{-}21 \begin{pmatrix} 0.0280 \\ 0.0257 \\ 0.0301 \\ 0.0324 \\ 0.0318 \\ 0.0323 \\ 0.0381 \end{pmatrix}
$$

- These Sharpe ratios are quite low, possibly owing to the financial crisis.

Extension

- Update data sets
- Repeat the same analysis using IGARCH/TGARCH (code readily available for estimation part)
- Consider ARMA+GARCH class
  - e.g. Estimates from ARMA(1,1)+GARCH(1,1), North America

μ	*AR*	*MA*	ω	α	β
0.0034	0.9292	-0.9451	0.0070	0.0630	0.9317
(1.597)	(21.635)	(-25.734)	(5.027)	(10.636)	(150.118)

t-values are in parenthesis. Coefficient estimates for AR and MA terms are also statistically significant in this case.

Conclusion

- GARCH(1,1) is an appropriate specification for these return series.
- The ”de-GARCH” process takes out persistent heteroskedasticity components of returns, hence ”de-GARCHed” returns series have lower correlations.
- Both unconditional and conditional portfolio selection suggest investing heavily in North America index.
- Out-of-sample forecasts of Sharpe ratios are generally low. The lowest value occurs during the financial crisis.