Quantitative Macroeconomics

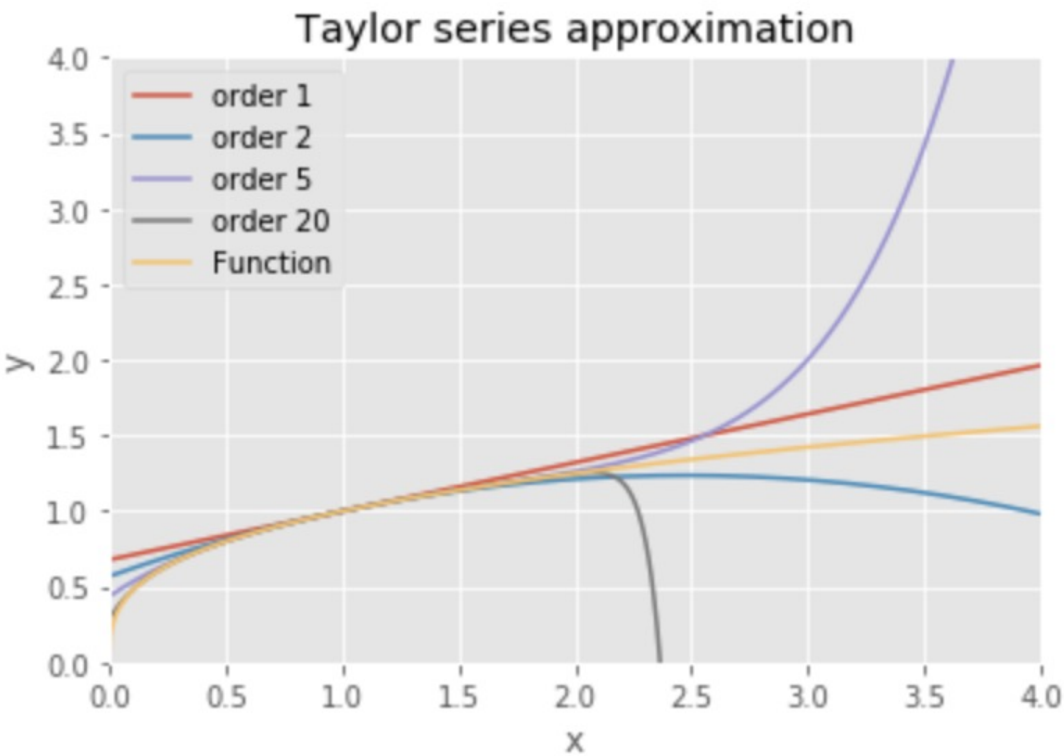
Problem Set 1
September 28, 2018

Mridula Duggal[1]

[1]With help from Pau, Gabriela and Elena.

Taylor Approximations

1. Approximate $f(x) = x^0.321$



The above Taylor Aprroximation is for the function $f(x) = x^{(0.321)}$. It is importat to note that a Taylor approximation is a local approximation. For instance, in this particular example we are approximating $f(x)$ around 1.

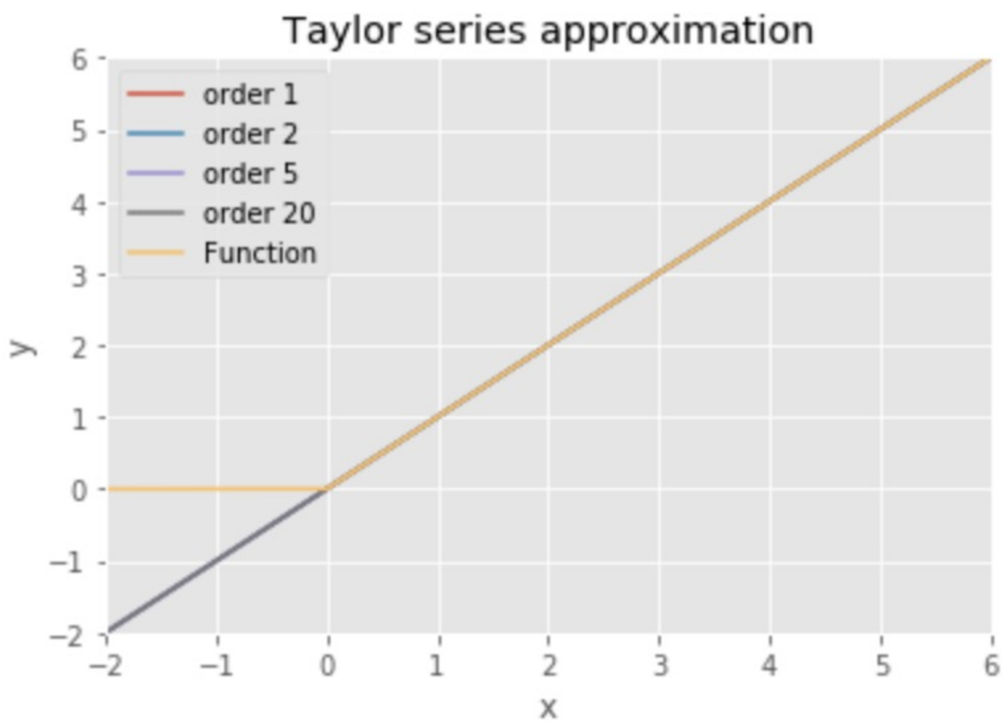
As we see, the higher the order of the Taylor series approximation, the greater the error we realise in approximation. This is because for this particular function, the singularity point is zero. At this point, the function is not well behaved or the derivative does not exist at that point. For $x > 1$ the approximation increases greatly because the radius of convergence for the Taylor series round one is unity. Therefore, the further away we are from the singularity point when approximating the function, the higher the error we face. Moreover, the error in a Taylor series is of the order $n + 1$ where, $n + 1$ is the number of times the function is differentiable.

Therefore, using this method to approximate a function may not be the best since, fucntions such as the utility fuction have to satisfy the Inada conditions, points of singilarity.

2. Approximate $f(x) = \frac{x+|x|}{2}$

Similar to the function in section, we approximate the *Ramp Function*, $f(x) = \frac{x+|x|}{2}$ using a Taylor series approximation.

The following graph provides the approximation.

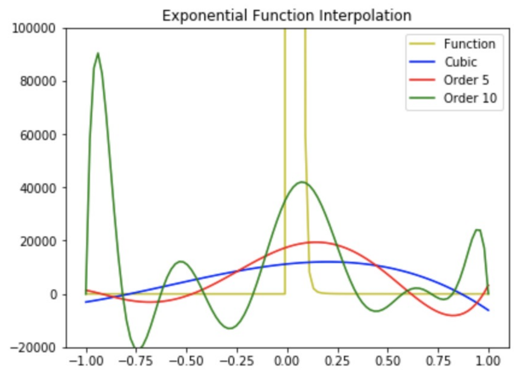


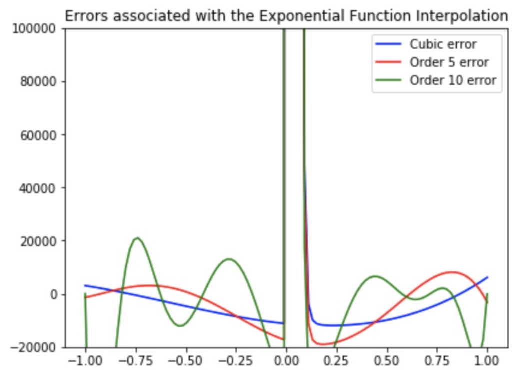
As we can see in the graph above, the true function has a kink at zero since the function has an absolute value involved. We also find, that when we approximate the function using a Taylor series, we cannot capture the kink. Therefore, the approximations are straight lines. This occurs due to the fact that at zero the function is not differentiable. Again, leading to the problem of singularity.

Interpolation of the Exponenetial, Ramp and Runge

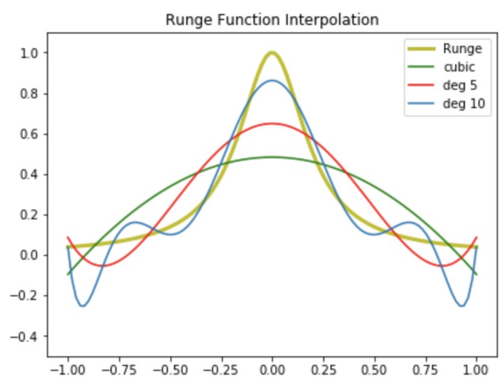
3. Interpolation using evenly spaced nodes

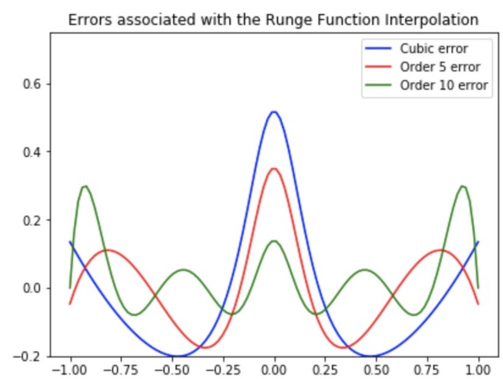
Exponential Function



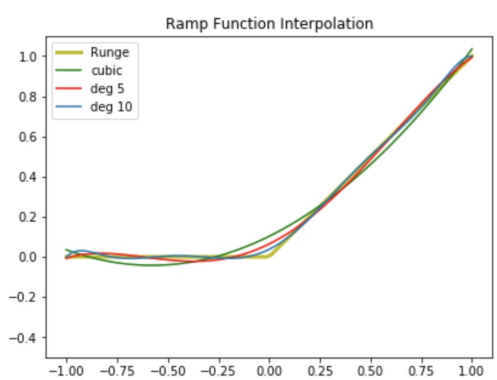


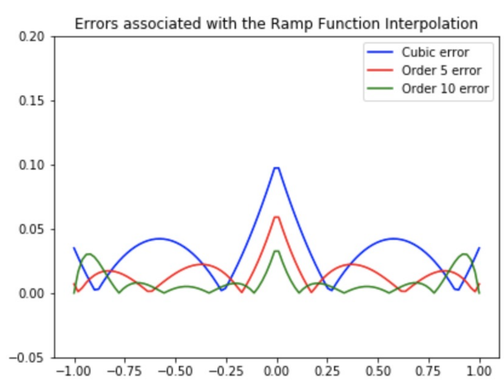
Runge Function





Ramp Function





The graphs above, show the interpolation of the Exponential, Runge and Ramp function, respectively. First, the difference between using an interpolation method and approximation method is that when doing interpolation we know the true function.Therefore when interpolating we can force the data to go through certain chosen points. To do so, we choose a family of functions from which the interpolated function must be drawn. Moreover, can be written as a linear combination of a set of n known *basis functions.*

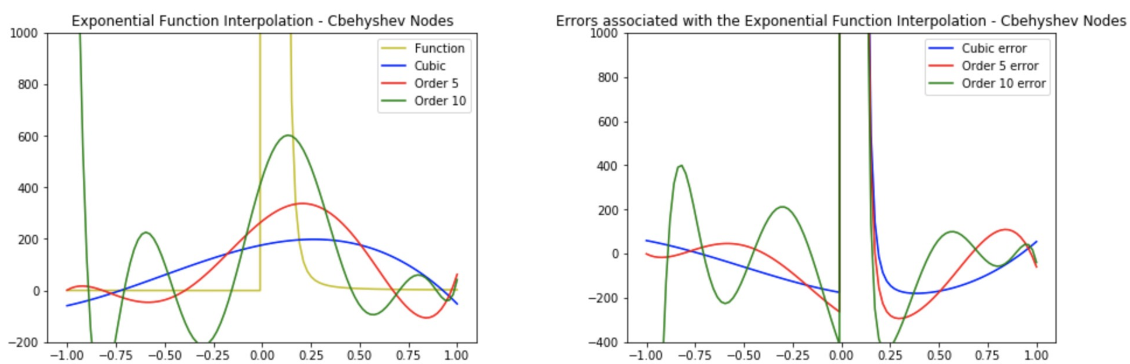
$$\tilde{f}(x) = \sum_{j=1}^{n} \theta_j \psi_j(x)$$

Where, ψ_j are the basis functions and θ_j are the coefficients that need to be determined. Finally, we can use different points or constraints that we would like the function to satisfy. For instance, we could use higher oder derivatives or the slope of $\tilde{f}(x)$ as a constraint when interpolating.

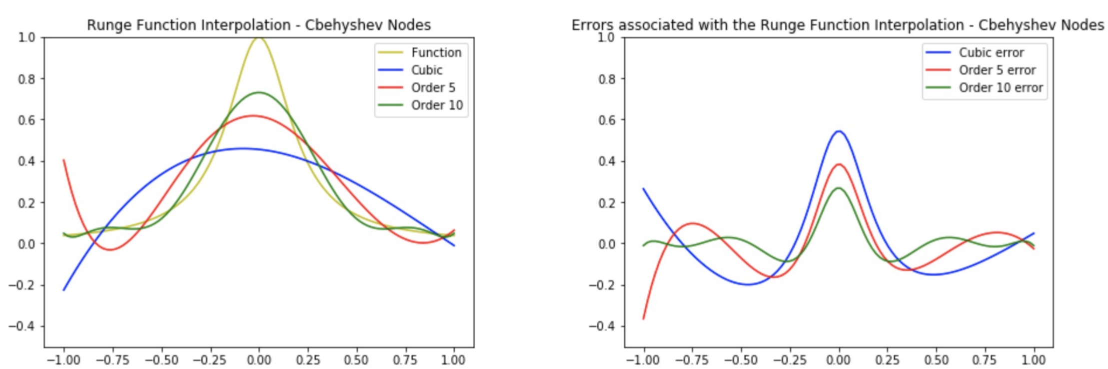
When using evenly spaced interpolation methods we find that there is high variability at the tails. This is because when we use interpolation methods with evenly spaced nodes, we get different errors across the each part of the domain. Therefore, we find that in the exponential function and the Runge function the tails have high errors. For the Ramp function just like the case of the Approximations, the interpolation method misses the kink. While the polynomial of order 10 comes very close to the true ramp function. We cannot capture the kink using this method.

3. Interpolation using Chebyshev nodes

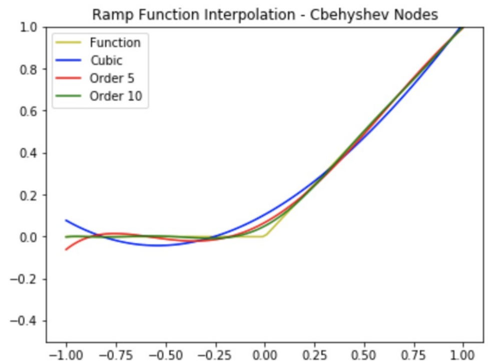
Exponential Function

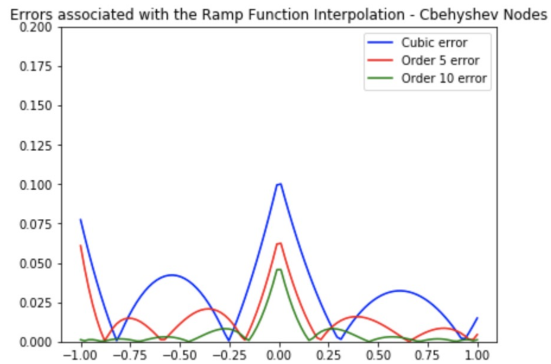


Runge Function



Ramp Function





The above graphs approximate the same functions. However, we now use Chebyshev nodes instead of evenly spaced nodes. Chebyshev nodes are given by,

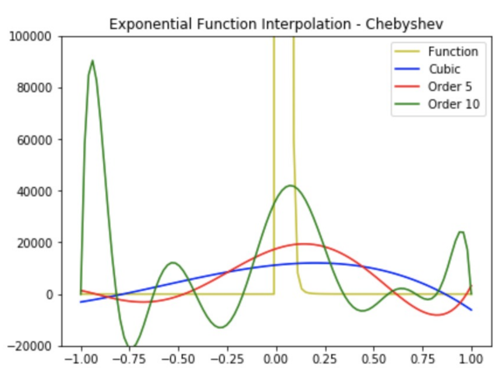
$$z_k = -cos\left(\frac{2K-1}{2m}\pi\right),\ k = 1, ..., m$$

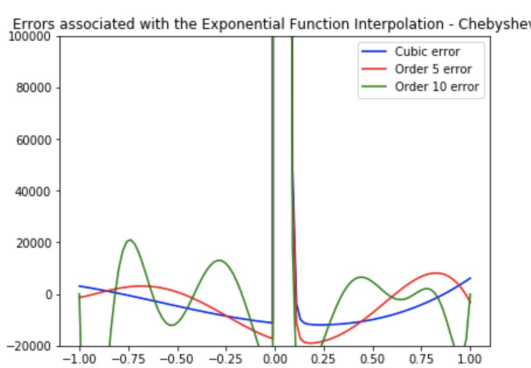
In using these nodes with cubic polynomials and monomials we find similar results as before. However, it is important to note that when working with higher order monomials, the Chebyshev nodes produce a numerical error. This could have to do with the precision of the computer. The current are up to order 10. However, after order 8 the programme generates a warning about the numerical error[2].

We can see from the graph with the errors that there are significant changes in the tails upon using the Chebyshev nodes, compared to evenly spaced nodes. This is because the Chebychev nodes try to minimise the error at the tails compared to the error in the middle of the function.

Interpolation using Chebyshev Nodes and Polynomials

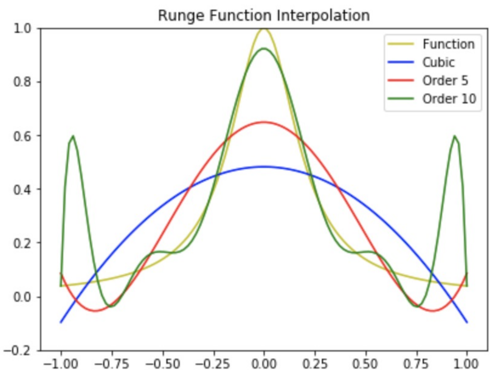
Exponential Function

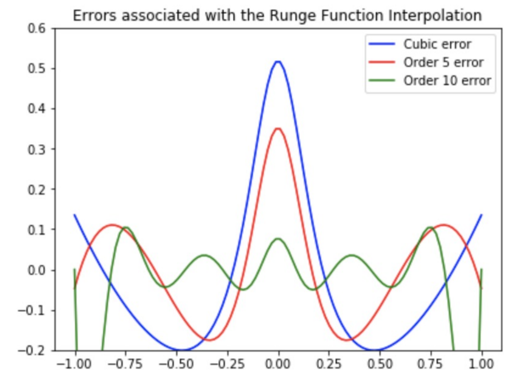




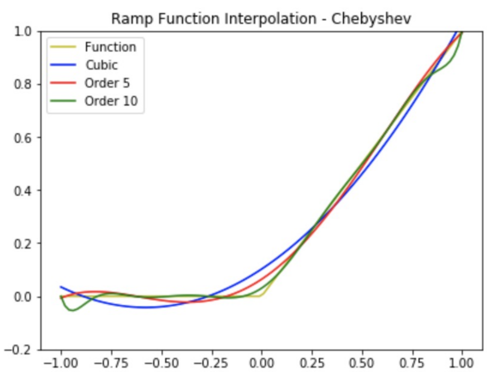
Runge Function

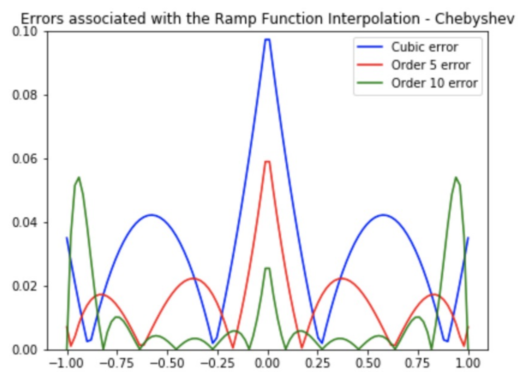
[2]For the purpose of this exercise I have ignored this warning





Ramp Function

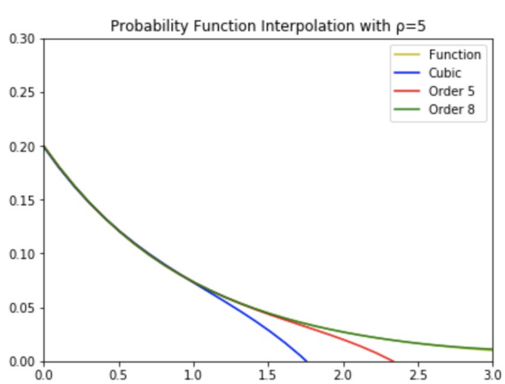


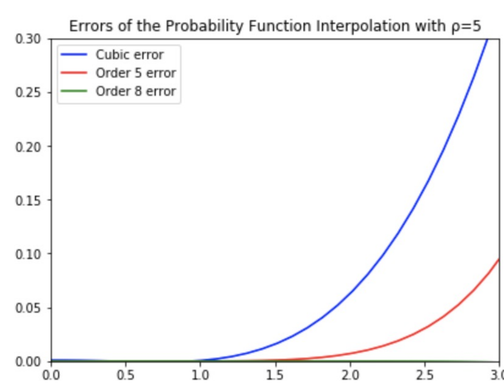


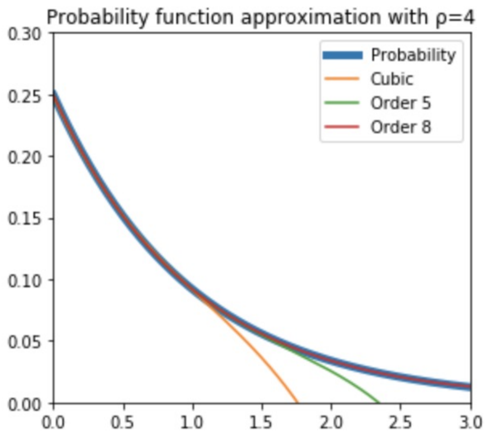
The graphs for the approximations using Chebyshev nodes and polynomials do not differ significantly for the graphs with evenly spaced interpolation nodes. However, as above the errors resulting from this interpolation are relatively "well behaved"

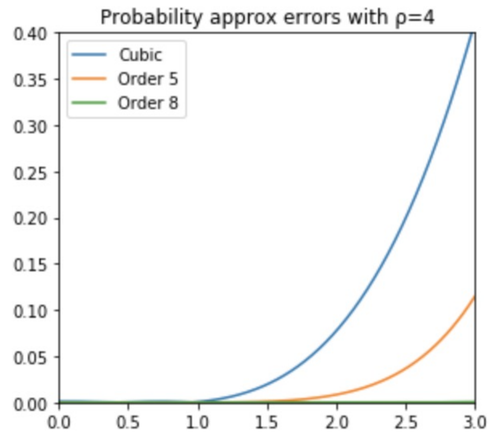
This implied Chebyshev is a better method for interpolation since it puts more weight on the nodes which are further away from the centre.

Interpolation of the Probability function using Chebyshev Nodes and Polynomials









The two graphs above show the probability function at $\rho_1 = 5$ and $\rho_1 = 4$, respectively.

CES Function

5. Elasticity of Subsitution and Labour Share

$$f(k,h) = \left[(1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

Let $\dfrac{\sigma - 1}{\sigma} \equiv \rho$

$$f(k,h) = [(1-\alpha)k^{\rho} + \alpha h^{\rho}]^{\frac{1}{\rho}}$$

The MPL and MPK are given by,

$$MPL = f_h(k,h) = [\alpha\rho h^{\rho-1}]\frac{1}{\rho}((1-\alpha)k^{\rho} + \alpha h^{\rho})^{\frac{1}{\rho}-1}$$

$$MPK = f_k(k,h) = [(1-\alpha)\rho k^{\rho-1}]\frac{1}{\rho}((1-\alpha)k^{\rho} + \alpha h^{\rho})^{\frac{1}{\rho}-1}$$

Where f is a differentiable real-valued function of a single variable. We define the elasticity of $f(x)$ with respect to x as,

$$\sigma(x) = \frac{xf'(x)}{f(x)} \equiv \frac{\delta d(x)/f(x)}{\delta x/x}$$

Let $\dfrac{\delta ln(f(x))}{\delta ln(x)} = \dfrac{\delta v}{\delta u}$.

By chain rule, $\dfrac{\delta v}{\delta u} = \dfrac{\delta v}{\delta x}\dfrac{\delta x}{\delta u} = \dfrac{f'(x)}{f(x)}x$

Therefore,

$$ln\left(\frac{f_k}{f_h}\right) = ln\left[\frac{\frac{1}{\rho}((1-\alpha)k^\rho + \alpha h^\rho)^{\frac{1}{\rho}-1}\alpha\rho h^{\rho-1}}{\frac{1}{\rho}((1-\alpha)k^\rho + \alpha h^\rho)^{\frac{1}{\rho}-1}(1-\alpha)\rho k^{\rho-1}}\right]$$

$$= ln\left(\frac{\alpha}{(1-\alpha)}\left(\frac{h}{k}\right)^{\rho-1}\right)$$

$$= \frac{\alpha}{(1-\alpha)}(1-\rho)ln\left(\frac{k}{h}\right)$$

$$ln\left(\frac{k}{h}\right) = \frac{1}{(1-\rho)}\frac{(1-\alpha)}{\alpha}ln\left(\frac{f_k}{f_h}\right)$$

The term $\frac{1}{(1-\rho)}\frac{(1-\alpha)}{\alpha}$ is the elasticity of substitution. We know $\alpha = 0.5$ substituting this we get, we get that substitution of elasticity is equal to,

$$\frac{1}{1-\frac{\sigma-1}{\sigma}} = \sigma$$

Labour Share

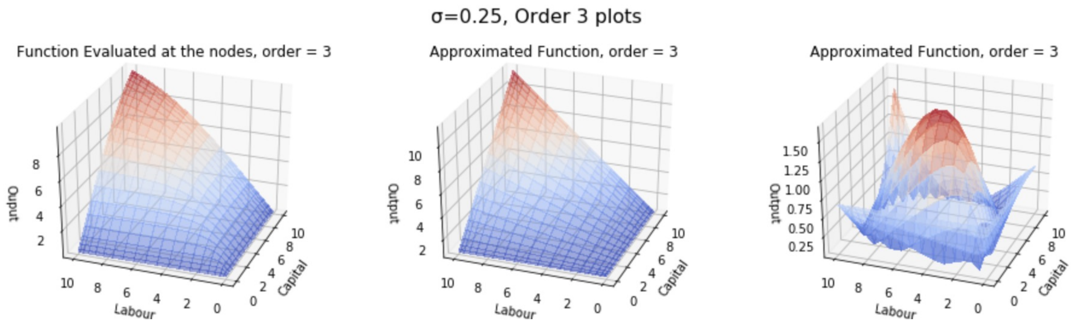
Assuming competitive markets,

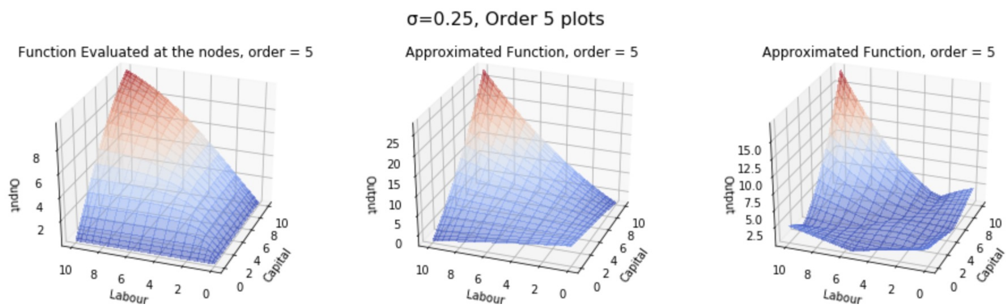
$$f_h = \alpha h^{-\frac{1}{\sigma}} f(k,h)^{-\frac{1}{\sigma}}$$

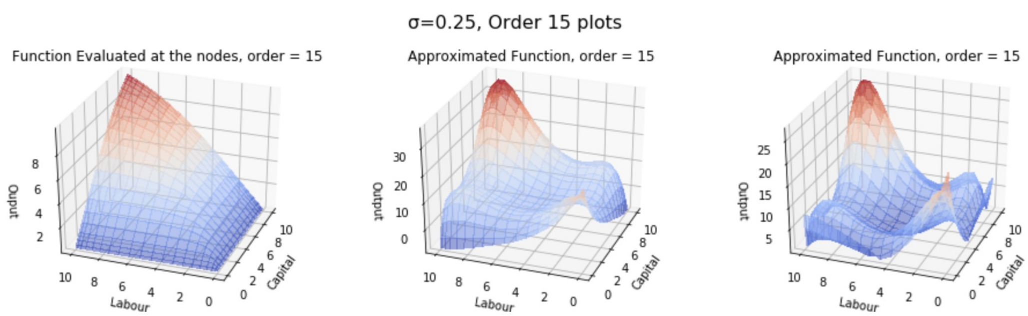
Labor share is equal to

$$f_h \frac{h}{f(k,h)} = \alpha\left(\frac{h}{f(k,h)}\right)^{1-\frac{1}{\sigma}}$$

CES Function with $\sigma = 0.25$

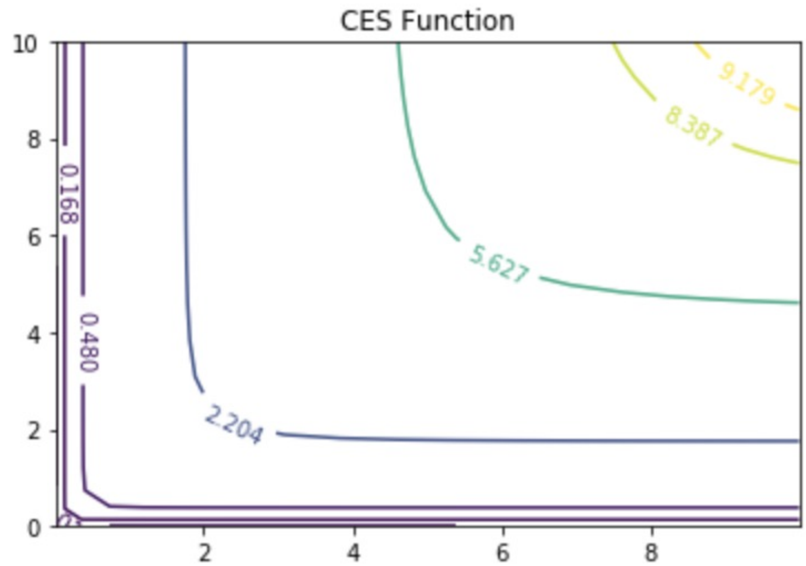


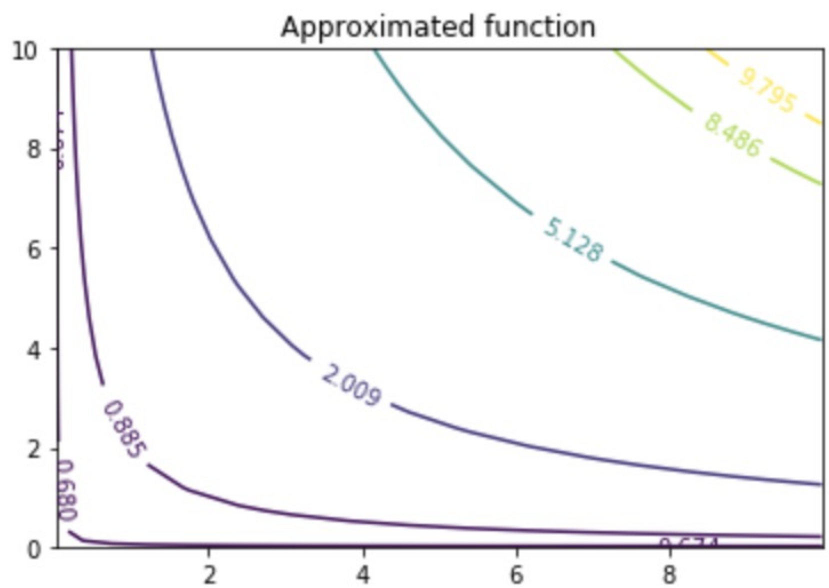




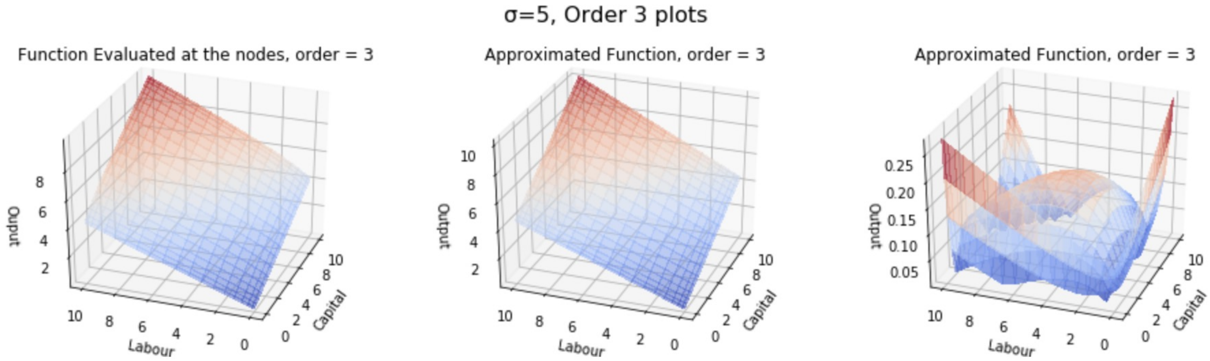
There are negligible differences between orders that are close together. However, for orders that are distant there are significant differences. One possibility for this is we have more nodes than coefficients, making our system over-identified. Thus creating significant differences between the true function and the approximated function.

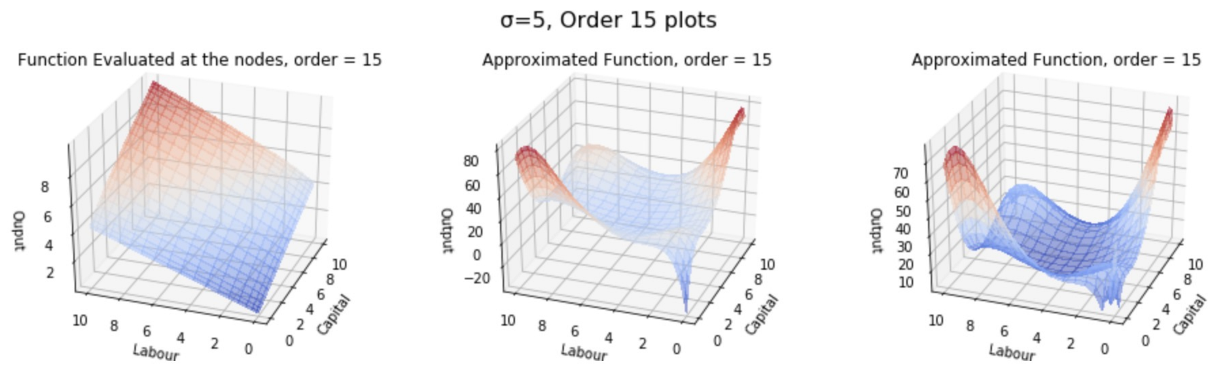
The following graphs show the Isoquants for the same σ.

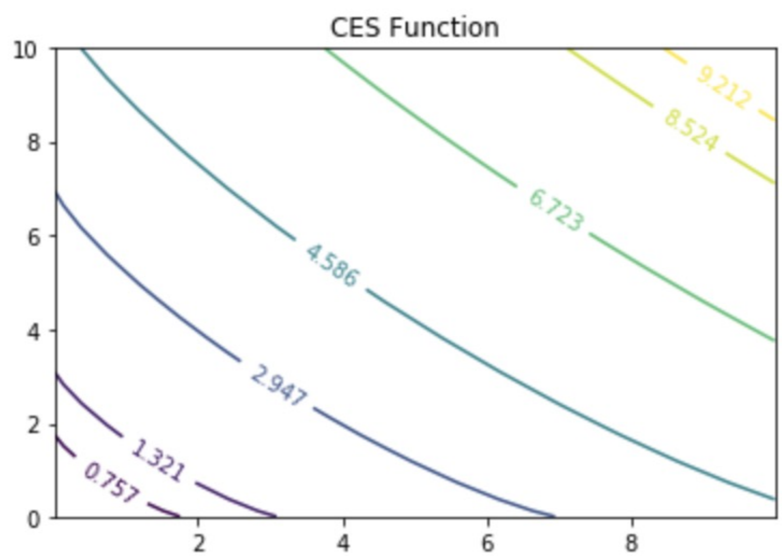


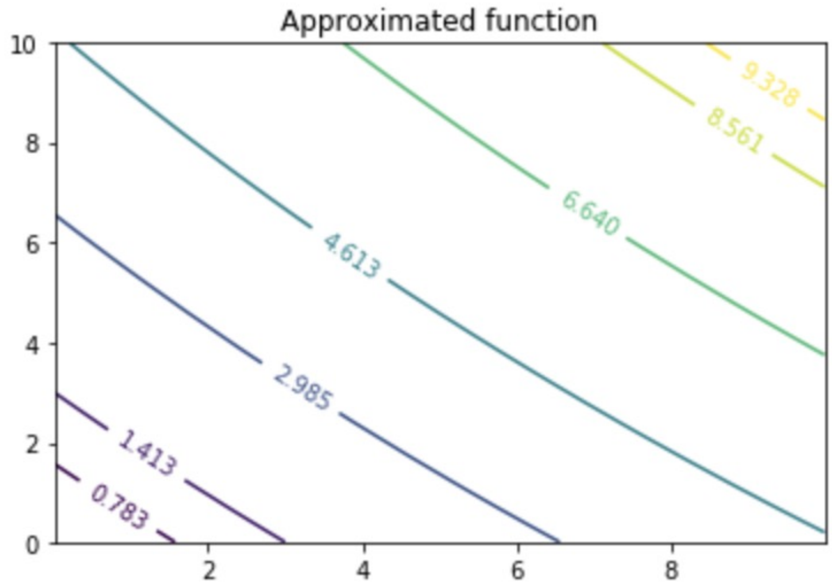


CES Function with $\sigma = 5$



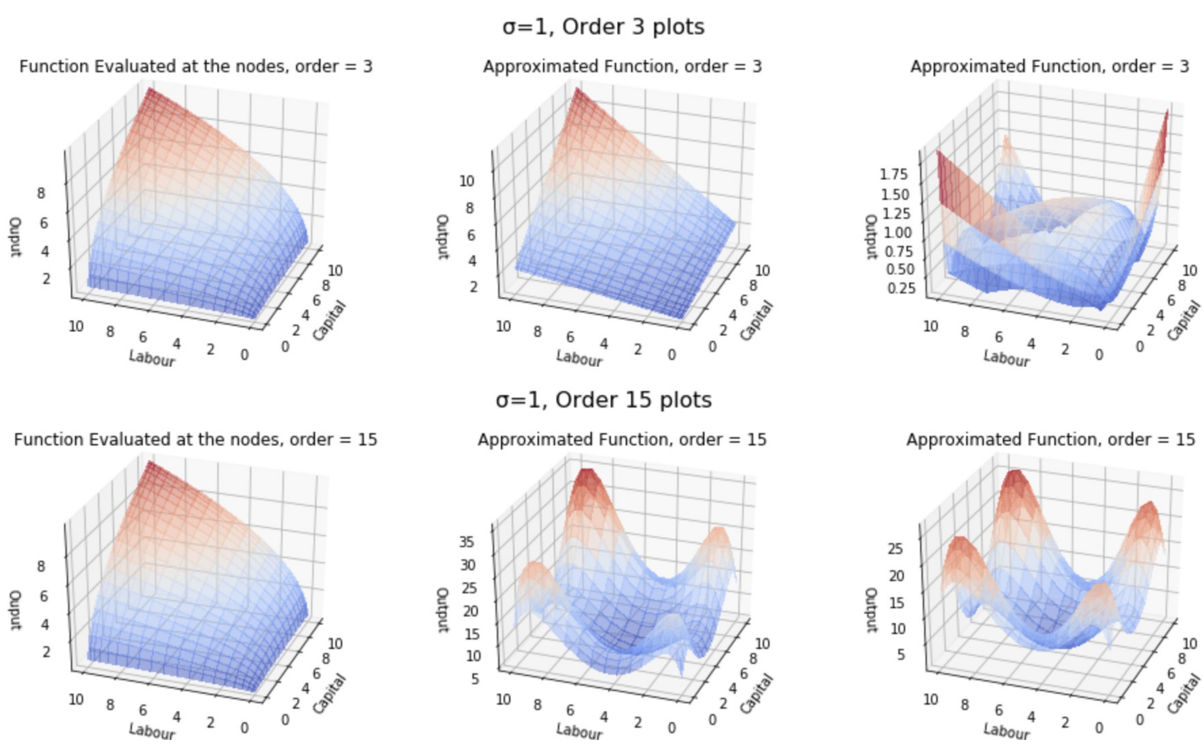


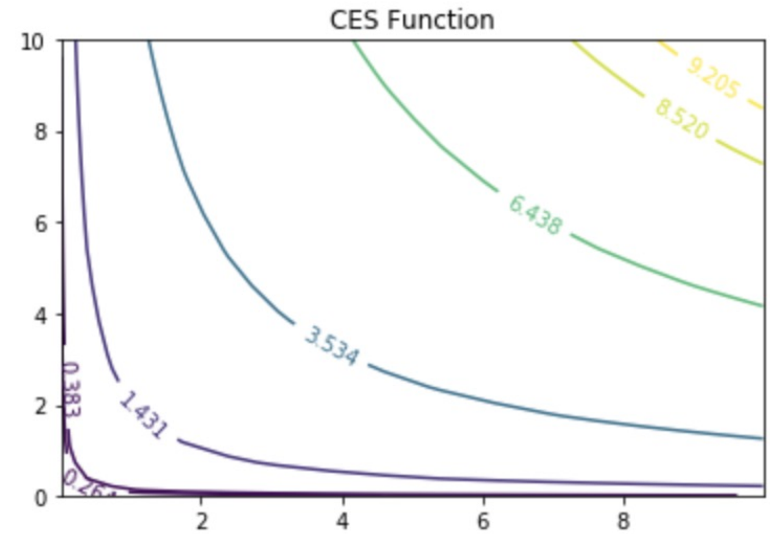


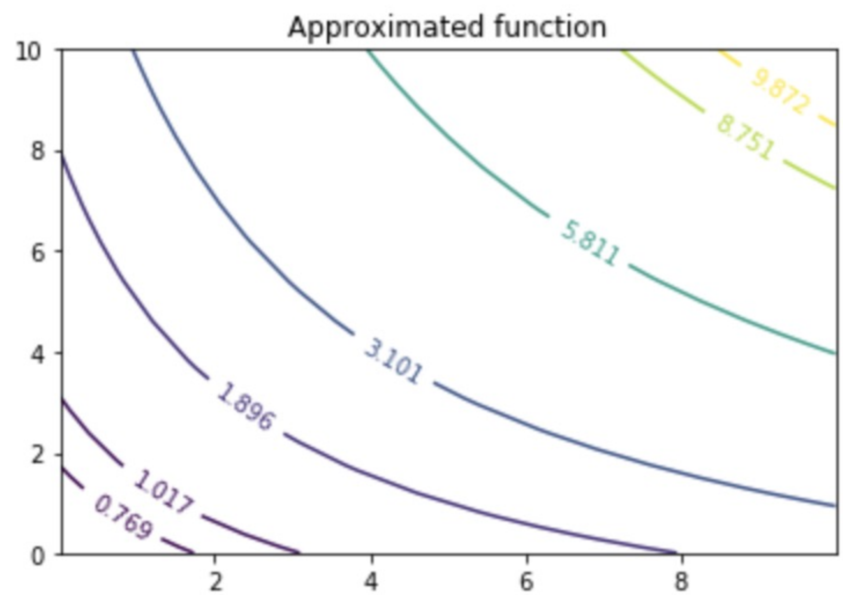


With $\sigma = 5$ we find that the true function and the approximated function are flat. Thus leading to flatter isoquants as well.

CES Function with $\sigma = 1$




CES Function
10
8
6
4
2
0
2
4
6
8
9.205
8.520
6.438
3.534
1.431
0.383


Approximated function
10
8
6
4
2
0
2
4
6
8
9.872
8.751
5.811
3.101
1.896
1.017
0.769