Beyond VaR: Triangular Risk Decomposition

Helmut Mausser and Dan Rosen

This paper describes triangular risk decomposition, which provides a useful, geometric view of the relationship between the risk of a position and that of the portfolio. We review triangular decomposition for the case of the parametric, or delta-normal, Value-at-Risk (VaR), which assumes that changes in a portfolio's value are normally distributed with mean zero. We then generalize it for the case of a non-zero mean and for arbitrary distributions, consistent with the simulation-based approach for calculating the non-parametric VaR. We examine a portfolio of foreign exchange contracts under both the parametric and simulation-based approaches, and discuss the strengths and limitations of triangular decomposition.

This is the second of a series of papers in which we discuss tools for managing, as opposed to simply measuring, a portfolio's Value-at-Risk (VaR). Our initial paper (Mausser and Rosen 1998) introduced a risk management toolkit using the simulation-based approach for calculating VaR, extending the concepts presented by Litterman (1996, 1997) for the parametric, or delta-normal, VaR. In particular, we focused on the calculation of the marginal VaR, marginal risk contribution and trade risk profile for the positions comprising a portfolio. In this paper, we consider triangular risk decomposition, a useful tool for visualizing and interpreting a position's risk contribution relative to the remainder of the portfolio.

Triangular decomposition was introduced for parametric VaR by Litterman (1996, 1997), who showed that it followed directly from the combination of two volatilities into a portfolio volatility. The decomposition provides a simple geometric view of the relationship between a position's risk and that of the portfolio, allowing risk managers to quickly understand the correlations between individual positions and the balance of the portfolio. This correlation information is readily analyzed to obtain VaR-minimizing, or best hedge, positions for each asset, as well as to uncover counter-intuitive implied views that can be exploited to improve expected returns while maintaining, or even reducing, the current level of risk.

In this paper, we first review triangular decomposition as a tool for managing parametric VaR, which assumes that changes in portfolio value are normally distributed with zero mean. The methodology is illustrated on a portfolio of foreign exchange (FX) forward contracts. We then successively relax the assumptions of non-zero means and of normality and construct triangular risk decompositions for arbitrary distributions, using the simulation-based approach for estimating the non-parametric Value-at-Risk (nVaR). The simulation-based techniques are then used to analyze the FX portfolio under Monte Carlo and historical scenarios. We discuss novel interpretations of the resulting triangular decompositions, most notably the concept of a position-dependent *implied correlation*, and identify the limitations of

triangular decomposition under the scenario-based approach. Our conclusions and suggestions for further study complete the paper.

The parametric case

Triangular decomposition provides a useful visualization of how the individual risks of two assets, or more precisely, positions in those assets, combine to form the risk of a portfolio. It illustrates the correlation between the positions and provides insights for managing portfolio risk by manipulating the size of the positions. The decomposition is based on the properties of volatility; since the delta-normal VaR is a constant multiple of volatility, the triangular decomposition can be applied directly to parametric VaR as well.

Triangular decomposition of volatility

Consider a portfolio that contains one unit of each of two assets. Let the returns on the assets be r_1 and r_2, with respective volatilities σ_{r_1} and σ_{r_2} and correlation ρ_{12}. If the current unit values of the assets are v_1 and v_2 (both assumed to be non-negative), then the changes in the unit values, $v_1 r_1$ and $v_2 r_2$, have respective volatilities $\sigma_1 = v_1 \sigma_{r_1}$ and $\sigma_2 = v_2 \sigma_{r_2}$ and correlation ρ_{12}. The variance of the change in portfolio value (hereafter, unless noted otherwise, we refer only to volatilities as related to changes in value rather than returns per se) is

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 \qquad (1)$$

Consider now the triangle in Figure 1. As shown by Litterman (1996, 1997), the two volatilities combine to create a portfolio volatility in the same manner that (the lengths of) two sides of a triangle and the angle between them combine to create (the length of) the third side of a triangle. This derives from the fact that Equation 1 has the same form as the cosine law

$$A^2 = B^2 + C^2 - 2BC\cos(\theta) \qquad (2)$$

if one associates the lengths of sides A, B and C with the volatilities of the portfolio and of the assets, and defines $\cos(\theta) = -\rho_{12}$. That is, if the

lengths of sides B and C correspond to asset volatilities and the angle θ is determined by their correlation, then the portfolio volatility is given by the length of side A. Negative correlation corresponds to an acute angle, $0 \le \theta < 90$, positive correlation to an obtuse angle, $90 < \theta \le 180$ and uncorrelated assets to a right angle, $\theta = 90$. In Figure 1, for example, the assets are positively correlated and this results in a portfolio volatility that is greater than either of the asset volatilities (i.e., $A > B$ and $A > C$). Note that both the cosine and the correlation take on values between –1 and 1 inclusive, which is consistent with the fact that volatility is *sub-additive*: the volatility of a portfolio cannot exceed the sum of the asset volatilities.
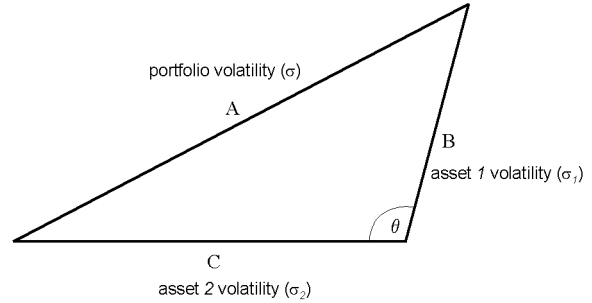

portfolio volatility (σ)

A

B

asset *1* volatility (σ_1)

θ

C

asset *2* volatility (σ_2)

Figure 1: Triangular decomposition of volatility

Since our objective is to consider the effects of position size on the portfolio risk, let us now generalize the above relationship to allow for an arbitrary number of units, x_1 and x_2, in each of the assets. The volatilities of the resulting positions are $|x_1|\sigma_1$ and $|x_2|\sigma_2$ and their correlation is $\hat{\rho}_{12} = \rho_{12}\text{sgn}(x_1 x_2)$. To recognize the latter relationship, note that shorting (i.e., taking a negative position in) one of the assets results in a correlation between the positions that is the negative of the correlation between the assets. Consistent with Equation 1, the variance of the portfolio is

$$\sigma^2(x) = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2\hat{\rho}_{12}|x_1|\sigma_1|x_2|\sigma_2 \qquad (3)$$

which again satisfies the cosine law (Equation 2) when the lengths of the sides correspond to position volatilities and the cosine of the angle θ is the negative of the correlation between the positions.

Triangular decomposition of VaR

The above decomposition of volatility applies also to parametric VaR since the latter is a constant multiple of volatility. A portfolio's $100(1 - \alpha)\%$ VaR (which we denote $VaR(x)$, implicitly recognizing its dependence on α) is

$$VaR(x) = Z_\alpha \sigma(x) \qquad (4)$$

where Z_α denotes the standard normal z-value that delimits a probability of α in the right tail (typically $\alpha = 0.01$ or 0.05). A more detailed derivation of Equation 4 is presented in the Appendix.

Recognizing that the VaR of a position x_i in a single asset i is

$$VaR(x_i) = Z_\alpha |x_i| \sigma_i$$

we can multiply both sides of Equation 3 by Z_α^2 to obtain

$$(VaR(x))^2 = (VaR(x_1))^2 + (VaR(x_2))^2$$

$$+ 2\hat{\rho}_{12}(VaR(x_1))(VaR(x_2)) \qquad (5)$$

Equation 5 has the same form as the cosine law (Equation 2) and thus establishes the existence of a triangular decomposition for VaR. It follows that, like volatility, VaR is sub-additive.

Thus far, we have considered portfolios with positions in only two different assets. While the above procedure can be extended to include three assets (the resulting decomposition is a tetrahedron, rather than a triangle), portfolios typically contain hundreds or thousands of different positions. Fortunately, it remains possible to use the triangular decomposition for visualizing the risk of an individual position in relation to the remainder of the portfolio.

Consider an arbitrary portfolio and consolidate all positions in assets other than i into a single position called the **base portfolio**. Given the Values-at-Risk(VaRs) of the portfolio, the position x_i and the base portfolio, we can construct a triangle that relates the position's VaR to the overall portfolio VaR in the same manner that we did for the volatilities of the two

positions in Figure 1. As the size of the position in asset i varies, so too does the position VaR (i.e., the length of side B) and the portfolio VaR (i.e., the length of side A). Hence, if the positions in all assets other than i remain fixed at their current values, then we can infer the effects on the portfolio VaR due to changing x_i by simply extending side B.

For example, Figure 2 shows a (long) position that is negatively correlated with the base portfolio; its presence in the portfolio currently has the effect of reducing the portfolio VaR (i.e., $A < C$). As the size of the position is increased from x_i to x_i^*, there is a further reduction in the portfolio VaR (i.e., $A^* < A$). In fact, x_i^* (where A^* is perpendicular to B) is the VaR-minimizing, or **best hedge**, position, denoted x_i^{bh}. As the size of the position is increased beyond x_i^*, however, the portfolio VaR begins to increase. Generally, enlarging a position that is negatively correlated with the base portfolio can reduce the overall risk, but only up to a point (in contrast, increasing the size of a position that is positively correlated with the base portfolio can never reduce risk).
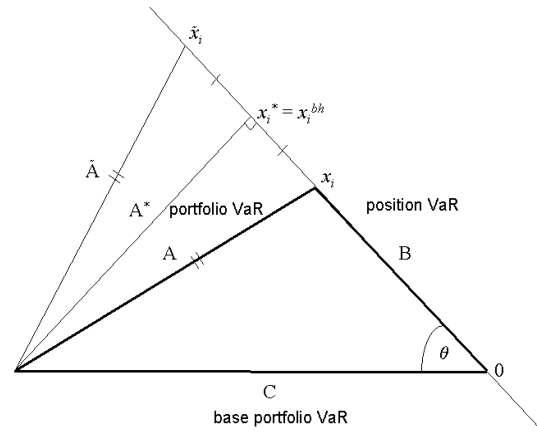

Figure 2: Triangular risk decomposition showing counter-intuitive implied view

Triangular decomposition can also uncover counter-intuitive implied views. In Figure 2, for example, an investor who is bullish on asset i can achieve a greater expected return without incurring additional risk by increasing the

Instrument	Currency	Days to Maturity	Strike Price (USD)	Position (x 10^6)	Value (x 10^3 USD)
CAD/USD .73 100d	CAD	100	0.73	0.5	2.5
CAD/USD .74 30d	CAD	30	0.74	1.0	–8.3
DEM/USD .57 60d	DEM	60	0.57	6.0	73.2
DEM/USD .59 120d	DEM	120	0.59	5.0	–28.2
FRF/USD .16 40d	FRF	40	0.16	8.0	83.3
JPY/USD .0091 11d	JPY	11	0.0091	10.0	–0.9

Table 1: FX portfolio

position up to a maximum level of $\tilde{x}_i$. Thus, assuming that it is feasible to increase x_i, the current long position in fact implies a bearish view on the part of the investor.
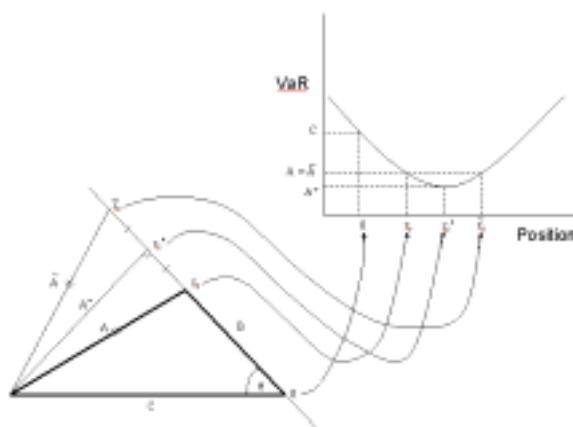

Figure 3: Triangular decomposition and trade risk profile

If we plot the portfolio VaR against the size of the position in instrument i, we obtain the **trade risk profile** (TRP). Figure 3 shows the relationship between the TRP and the triangular decomposition. The TRP has a characteristic shape and a unique minimum at the best hedge position. Note that while both the TRP and the triangular decomposition indicate whether changes in position increase or decrease the portfolio VaR, the latter provides a useful visualization of the correlation between the position and the rest of the portfolio that is difficult to infer from the TRP.

Example: FX portfolio

Table 1 shows a portfolio of foreign exchange (FX) forward contracts as of July 1, 1997. The exchange rates, in USD, are 0.73 (CAD), 0.58 (DEM), 0.17 (FRF) and 0.0090 (JPY). The total value of the portfolio is 122,000 USD and its one-day 99% VaR, calculated from the RiskMetrics risk factor dataset, is 78,000 USD. Mausser and Rosen (1998) analyzed the marginal VaRs and VaR contributions, and constructed sample TRPs for this portfolio. We now extend the analysis to include triangular decompositions.

Table 2 displays each position's correlation with the base portfolio, its best hedge position, the size of the trade required to attain the best hedge position and the reduction in VaR that can be achieved. Note that a larger correlation generally corresponds to a greater risk-reduction potential for a given size of trade (see Litterman 1996). For example, the DEM positions (with correlations of approximately 0.98) can each reduce risk by almost 88% with a trade of 13 million units, while the JPY position (with a correlation of 0.478) can reduce risk by only 13% when 219 million units are traded. Since the size of the correlation increases as the angle θ approaches 0° or 180°, we can associate greater risk reduction potential with the position side of the triangle being "more horizontal" in the triangular decomposition.

An examination of the triangular risk decomposition for DEM/USD .57 60d (Figure 4) shows that this position is positively correlated (0.975) with the base portfolio and, therefore,

Instrument	Correlation with Rest of Portfolio	Best Hedge Position (x 10^6)	Size of Trade to Best Hedge (x 10^6)	VaR Reduction (%)
CAD/USD .73 100d	–0.102	1.4	0.9	0.2
CAD/USD .74 30d	–0.140	1.9	0.9	0.2
DEM/USD .57 60d	0.975	–7.0	13.0	87.7
DEM/USD .59 120d	0.980	–8.0	13.0	87.6
FRF/USD .16 40d	0.969	–38.3	46.3	79.2
JPY/USD .0091 11d	0.478	–209.2	219.2	13.2

Table 2: FX portfolio correlations, best hedges and potential VaR reductions

acts to increase the overall risk. The portfolio VaR can be reduced by selling DEM/USD .57 60d contracts and the best hedge requires taking a short position of 7 million units in these contracts. In contrast, the CAD/USD .74 30d position (Figure 5) is negatively correlated (–0.140) with the rest of the portfolio and is currently reducing risk. A further reduction in VaR can be achieved by purchasing additional CAD/USD .74 30d contracts, up to the best hedge position of 1.9 million units. As noted previously, the greater risk reduction potential of DEM/USD .57 60d relative to CAD/USD .74 30d is reflected by the fact that the position side of the triangle is more horizontal in Figure 4 ($\theta = 167°$) than in Figure 5 ($\theta = 82°$).

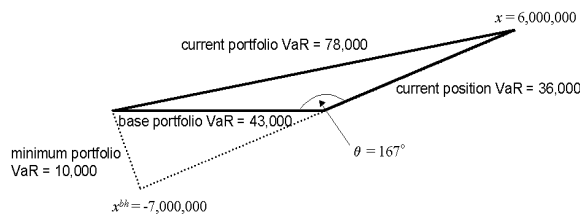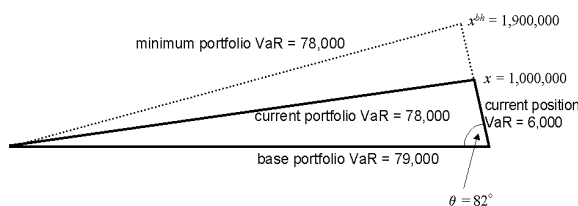**Figure 4:** Triangular risk decomposition for DEM/USD .57 60d

Figure 5: Triangular risk decomposition for CAD/USD .74 30d

The non-parametric case

When portfolios contain options or other non-linear instruments, or when the investment horizon is longer (say, one month or one year), the assumptions underlying the parametric VaR can be difficult to justify. We now generalize triangular decomposition for use with arbitary distributions by first allowing non-zero means and then relaxing the normality assumption.

Normal losses with non-zero mean

Suppose that the changes in value for a unit position in instrument i are normally distributed according to $\mathbf{N}(\mu_i, \sigma_i)$, where μ_i and σ_i are the expected (mean) loss and the volatility, respectively. It follows that the losses for a portfolio consisting of positions x_i and x_l in instruments i and l are distributed $\mathbf{N}(\mu(x), \sigma(x))$, where $\mu(x) = \mu_i x_i + \mu_l x_l$ and $\sigma(x)$ is given by the square root of Equation 3. Since we define VaR to be the entire potential loss, rather than just the unexpected loss, the VaR of the portfolio is

$$VaR(x) = \mu(x) + Z_\alpha \sigma(x) \qquad (6)$$

Equation 6 is identical to Equation 4 except for the additional expected loss term $\mu(x)$, which is simply the sum of the expected losses for the individual positions. We construct the triangular decomposition based only on the unexpected losses (UL), where UL = $Z_\alpha \sigma(x)$. To reflect the expected loss components, we simply adjust the lengths of the sides of the triangle (i.e., extend

the side for a positive mean loss and shorten the side for a negative mean loss).

In Figure 6, for example, triangle XYZ decomposes the unexpected losses at the current position. Because UL is a constant multiple of volatility, $-cos(\theta)$ equals the correlation between the position and the base portfolio, as in the parametric case. In this example, the base portfolio has a positive expected loss while the position and the portfolio have negative expected losses, as indicated by segments ZC, YB and YA, respectively. Thus, the VaRs (i.e., the sums of the expected and unexpected components) correspond to the lengths of segments XC, ZB and XA. Note that since the expected loss components of VaR are additive, we have YA = |ZC – YB|.
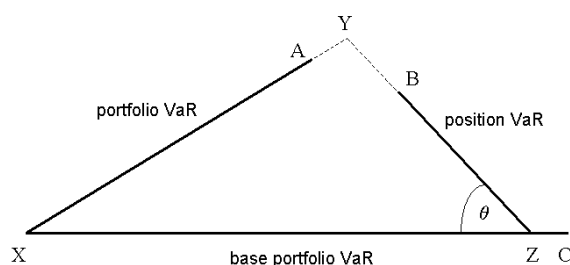

Figure 6: Triangular decomposition for normally-distributed losses with non-zero mean

Recall that in the parametric case, extending segment YZ shows how the portfolio VaR is affected by changes in the position size (e.g., Figure 2). When expected losses are non-zero, however, segment YZ reflects only the unexpected losses, and therefore, we refer to the line obtained by extending YZ as the **UL profile**. To illustrate the effects of position size on the VaR itself, it is necessary to construct a **VaR profile** that traces the endpoint of the portfolio VaR segment as the size of the position is varied, as shown in Figure 7. Note that the VaR profile passes through the points A and C, corresponding to the VaR at the current and zero positions, respectively, but that it is not necessarily a straight line.

Figure 7: Triangular decomposition with profiles for normally-distributed losses with non-zero mean

Arbitrary loss distributions

Portfolios that contain non-linearities generally do not satisfy the assumption that losses are normally distributed. For such portfolios, risk is typically measured using non-parametric Value-at-Risk (nVaR), an empirical estimate of the VaR that is derived from a complete valuation of the portfolio under a set of historical or Monte Carlo scenarios. For this reason, we often refer to nVaR as the simulation-based VaR. Mausser and Rosen (1998) describe the common procedure for calculating nVaR and derive a set of tools for managing risk under the simulation-based approach. An attractive feature of these tools is the fact that they require only a single, initial simulation of the portfolio to obtain mark-to-future values for all instruments; the marginal nVaR, contribution and the non-parametric trade risk profile (nTRP) can be obtained through manipulation of these values. This is true for the triangular decomposition as well. Furthermore, we show that the triangular decomposition for nVaR leads to the interesting concept of a position-dependent *implied correlation*, and also exposes the limitations of nVaR as a coherent measure of risk (see Artzner et al. 1998).

Basic concepts

We first provide an intuitive explanation of the ideas underlying the triangular decomposition for nVaR. In the ensuing discussion, we refer to Figure 8, which constructs the triangular decomposition and the UL profile for one position in a portfolio.
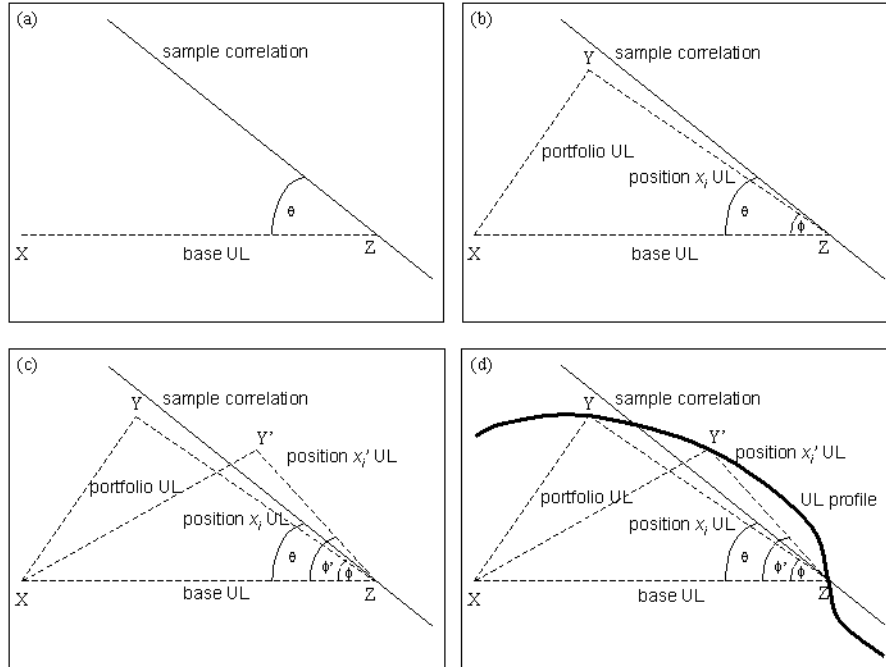
Figure 8: Construction of nVaR triangular decomposition and UL profile

Recall that a key characteristic of the triangular decomposition in the case of normally-distributed losses is the correlation between a position and the base portfolio; the correlation determines the angle between the respective sides of the triangle. Given a simulated set of portfolio values, for any two positions, x_i and x_l, one can calculate a sample correlation, $\hat{r}_{il}$, (i.e., an estimate of the true correlation, $\hat{\rho}_{il}$) based on the unexpected losses in the sample. (We provide the mathematical details shortly). In Figure 8(a) we plot the sample correlation line, which makes an angle θ with the base of the triangle. Consistent with the previous cases, $cos(\theta) = -\hat{r}_{il}$. Note that a triangular decomposition of (sample) volatility (e.g., Figure 1) would overlay the base and the sample correlation lines.

At the current position x_i, we calculate the unexpected losses for the position, the base portfolio and the overall portfolio, and we construct a triangle with sides of corresponding lengths. Unlike the normal case, however, this triangle does not necessarily align with the sample correlation line. In Figure 8(b), for example, the resulting triangle contains an angle $\phi < \theta$ between the position side and the base, indicating that the current position implies a

correlation ($-cos(\phi)$) that is less than the sample correlation. In other words, holding this position results in an unexpected loss for the portfolio that is less than what would be observed if losses were normally distributed. Specifically, the value $-cos(\phi)$, which we call the **implied correlation**, equals the correlation between the current position and the base portfolio that is necessary to produce the observed UL decomposition under normally-distributed losses. The implied correlation more accurately reflects a position's risk reduction potential (in terms of UL) than the sample correlation when losses are non-normal.

Figure 8(c) constructs a UL decomposition for a different position size x'_i. In this case, the implied correlation, $-cos(\phi')$, exceeds the sample correlation, suggesting that the portfolio's UL is larger than what would be observed under a normal distribution.

If we construct such triangles for a range of position sizes, then the path traced by the top vertex of the triangle defines the UL profile. When the underlying loss distribution is non-normal, one obtains a position-dependent implied correlation and the UL profile is not a straight line (Figure 8(d)). In contrast, if the

underlying loss distribution is in fact normal, then given a sufficiently large sample, the UL profile will overlay the sample correlation line and the implied correlation will be effectively constant across the range of position sizes.

The expected losses can now be incorporated as before, namely by adjusting the lengths of the sides. Figure 9 shows the resulting triangular decomposition with both the UL profile and the nVaR profile.
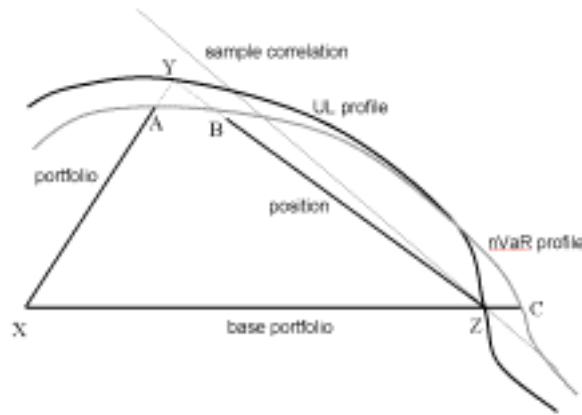

Figure 9: Simulation-based triangular decomposition

Numerical computations

We now provide the mathematical details for constructing triangular decompositions for nVaR and calculating the sample correlation.

To construct the simulation-based triangular risk decomposition for instrument i, let us fix the positions in all instruments other than i to their current values. The loss incurred by the portfolio in scenario j is

$$L_j(x_i) = \Delta V_{ij} + x_i \, \Delta v_{ij}$$

where ΔV_{ij} includes the losses due to all instruments other than i, Δv_{ij} is the per unit loss of instrument i in scenario j and x_i is the size of the position in instrument i.

Let p_j denote the probability of scenario j. Define

$$\mu(x_i) = \sum_{j=1}^{M} p_j \, L_j(x_i)$$

and

$$UL_j(x_i) = L_j(x_i) - \mu(x_i)$$

to be the expected (mean) and unexpected losses, respectively, of position x_i in scenario j. Let us denote the threshold scenario for the position x_i as s_i^o, so that the nVaR is

$$nVaR(x_i) = \mu(x_i) + UL_{s_i^o}(x_i)$$

It follows that the nVaR of a portfolio consisting of positions x_i and x_l is

$$nVaR(x) = \mu(x) + UL_{s^o}(x) \tag{7}$$

where s^o is the threshold scenario for the portfolio,

$$\mu(x) = \mu(x_i) + \mu(x_l)$$

and

$$UL_{s^o}(x) = UL_{s^o}(x_i) + UL_{s^o}(x_l)$$

Notice the similarity between Equations 7 and 6; in both cases, the Value-at-Risk is decomposed into expected and unexpected losses.

The sample correlation between positions x_i and x_l, can be calculated as

$$\hat{r}_{il} = \frac{\sum\limits_{j=1}^{M} p_j (UL_j(x_i))(UL_j(x_l))}{\sqrt{\sum\limits_{j=1}^{M} p_j (UL_j(x_i))^2} \sqrt{\sum\limits_{j=1}^{M} p_j (UL_j(x_l))^2}} \tag{8}$$

Some difficulties with nVaR

The sub-additivity of the parametric VaR ensures that triangles can be constructed for all position sizes in the normal case. However, this does not always hold for nVaR. As a simple example, Table 3 shows the losses for a portfolio, consisting of positions x_1 and x_2, under a set of five scenarios. From Equation 8, the sample correlation between the two positions is 0.88.

Scenario j	$L_j(x_1)$	$L_j(x_2)$	Portfolio Loss $L_j(x_1,x_2)$
1	7	4	11
2	3	5	8
3	0	1	1
4	–1	0	–1
5	–4	–5	–9
Mean	1	1	2

Table 3: Losses for a two-position portfolio

Suppose that the nVaR is given by the second-largest loss. Table 4 shows the nVaR and the corresponding unexpected loss component for the two-position portfolio. Note that for both measures, the portfolio loss exceeds the sum of the instrument losses (i.e., neither the nVaR nor the unexpected losses are sub-additive). This suggests that combining positions 1 and 2 produces a total risk that is greater than the sum of their individual risks, which implies a correlation greater than one. Since it is not possible to construct a triangle when the magnitude of the implied correlation exceeds one, the UL and nVaR profiles may in fact contain gaps.

	Instrument 1	Instrument 2	Portfolio
nVaR	3	4	8
Unexpected loss	2	3	6

Table 4: nVaR and unexpected loss for a two-position portfolio

Example: FX portfolio – Monte Carlo scenarios

To compare the parametric and simulation-based triangular decompositions, the FX portfolio was simulated over a set of 1,000 Monte Carlo scenarios. The one-day 99% nVaR is 77,000 USD (with 95% confidence, the nVaR is between 70,000 USD and 89,000 USD), which differs from the parametric value by less than 2%. Since the FX contracts are linear instruments and the Monte Carlo approach assumes that changes in the risk factors are normally distributed, the resulting loss histogram is approximated well by a normal distribution (Figure 10). Thus, one expects the triangular decompositions to be consistent with those obtained using the delta-normal approach.

Table 5 summarizes the correlations, best hedge positions and potential nVaR reductions obtained from the simulation. As expected, the results are consistent with those of the parametric approach (Table 2).

Instrument	Sample Correlation with Rest of Portfolio	Implied Correlation	Best Hedge Position (x 10^6)	Size of Trade to Best Hedge (x 10^6)	nVaR Reduction (%)
CAD/USD .73 100d	–0.071	–0.196	1.0	0.5	0.8
CAD/USD .74 30d	–0.108	–0.202	1.6	0.6	0.8
DEM/USD .57 60d	0.974	0.946	–7.4	13.4	87.0
DEM/USD .59 120d	0.979	0.945	–8.2	13.2	86.9
FRF/USD .16 40d	0.969	0.917	–38.0	46.0	78.6
JPY/USD .0091 11d	0.507	0.607	–258.6	268.6	18.8

Table 5: FX portfolio correlations, best hedges and potential nVaR reductions

Figure 10: Simulated distribution of losses

Figure 11 shows that the UL profile for DEM/USD .57 60d is generally aligned with the sample correlation line. The nVaR and UL profiles correspond to position sizes between –8.9 million and 8.1 million units. The "jaggedness" of the profiles is due to the finite number of scenarios in the scenario set (note that the non-parametric TRP is piecewise linear, as discussed in Mausser and Rosen (1998)); increasing the number of scenarios generally results in smoother profiles. The angle made by the base and position sides of the triangle (161°) is comparable to that obtained under the parametric approach (167°), consistent with a strong positive correlation between the long DEM/USD .57 60d position and the base portfolio.
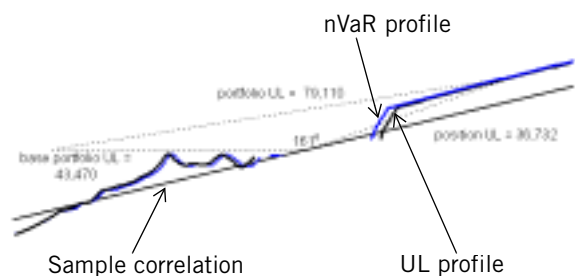

Figure 11: Triangular decomposition for DEM/USD .57 60d (1,000 scenarios)

Similarly, the triangular decomposition for CAD/USD .74 30d (Figure 12) shows good agreement between the UL profile and the sample correlation line for position sizes between –2.4 million and 4.3 million units. Again, the

angle made by the base and position sides of the triangle (78°) is close to that obtained under the parametric approach (82°), suggesting that the long CAD/USD .74 30d position and the base portfolio are negatively correlated. In both Figures 11 and 12, the nVaR profile is slightly to the left of the UL profile, indicating a small negative expected loss component for the range of positions considered.
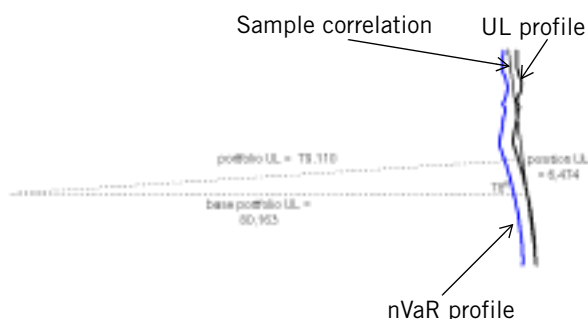

Figure 12: Triangular decomposition for CAD/USD .74 30d (1,000 scenarios)

Example: FX portfolio - historical scenarios

The previous example shows that the simulation-based and parametric triangular decompositions give consistent results when losses are normally distributed. We now value the FX portfolio under a set of 359 historical scenarios, thereby eliminating the normality assumptions inherent in the Monte Carlo simulation. The resulting loss histogram (Figure 13) is negatively skewed and the one-day 99% nVaR is 97,000 USD (with a 95% confidence interval of [79,000, 137,000]), compared to a VaR of 108,000 USD obtained from the best normal approximation. Since the historical scenarios do not coincide with the time period used for the previous analysis, it is not meaningful to compare these results directly with those of the Monte Carlo simulation. Instead, we simply examine the effects of non-normality on the triangular decomposition.

Figure 14 shows the triangular decomposition for CAD/USD .74 30d (the profiles represent position sizes between –5.7 million and 4.3 million units). Unlike Figure 12, there is a marked discrepancy between the UL profile and the sample correlation line. Note that at the current long position, the implied correlation is

negative (–0.161) while the sample correlation is positive (0.099). Since both the UL and the nVaR of the portfolio at the current position (99,366 USD and 97,288 USD, respectively) are less than those of the base portfolio (100,343 USD and 98,051 USD, respectively), the long CAD/USD .74 30d position is in fact reducing the overall portfolio risk, which is consistent with the negative implied correlation. For short positions (i.e., those below the base of the triangle), the UL profile, having a lesser slope than the sample correlation line, indicates that the implied correlation is larger (more negative) than the sample correlation. Thus, the latter value underestimates the risk reduction potential of shorting CAD/USD .74 30d contracts.
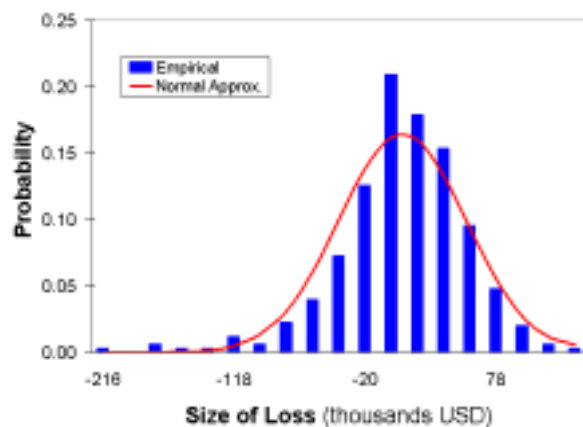

Figure 13: Historical distribution of losses

The UL profile for the DEM/USD .57 60d contract (Figure 15) displays an even larger deviation from the sample correlation line. Note that the triangular decomposition cannot be constructed at the current position; the implied correlation is 1.092, which reflects the fact that the portfolio's UL (99,366 USD) exceeds the sum of the base portfolio's UL (53,232 USD) and the position's UL (43,940 USD). In fact, the profiles in Figure 15 correspond only to position sizes between –5.7 million and zero units; triangular decompositions cannot be constructed outside this range.
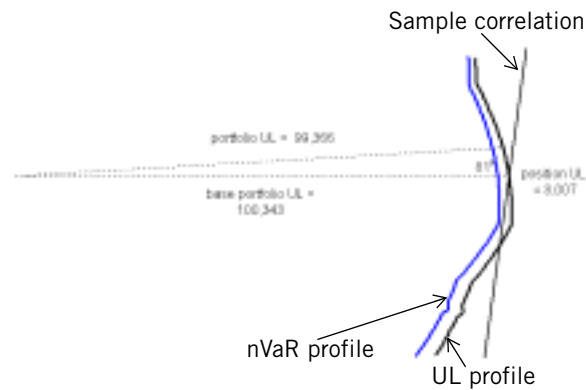

Figure 14: Triangular decomposition for CAD/USD .74 30d (historical scenarios)

It is interesting to note that the UL and nVaR profiles are curved in Figure 15. The sample correlation of 0.986 (recall that this is for long positions in DEM/USD .57 60d) means that the portfolio risk can be significantly reduced by shorting these contracts. The curved UL profile indicates that the implied correlation increases as more contracts are shorted, and eventually exceeds one (at a position of –5.7 million units); beyond this point, the portfolio's UL is less than the difference between the base portfolio's UL and the position's UL.
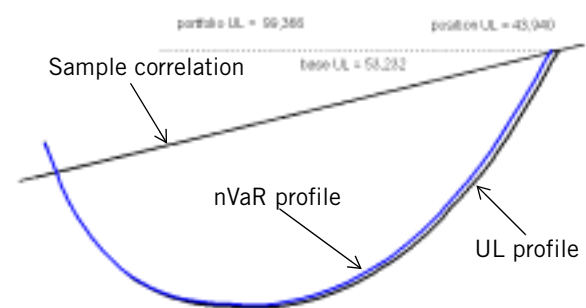

Figure 15: Triangular decomposition for DEM/USD .57 60d (historical scenarios)

Triangular decomposition for more complex portfolios

Constructing and interpreting the triangular risk decomposition can be difficult when one allows arbitrary distributions for portfolio losses. This is not altogether surprising since triangular decomposition derives from combining

volatilities, and this extends directly to VaR only if losses are normally distributed with zero mean.

The examples we have considered are reasonably well-behaved in the sense that the underlying distributions are either normal or close to normal (e.g., Figures 10 and 13) and the expected losses are relatively small. As the deviations from the normal-distribution with zero mean become larger, the triangular decomposition can become increasingly complex. As an example, consider Figure 16, which shows the triangular risk decomposition for a put option in a well-hedged portfolio that has both a non-normal loss distribution as well as a significant expected loss component. Certainly, the interpretation of Figure 16 is less straightforward than that of the parametric triangular decompositions in Figure 4 or 5, for instance.
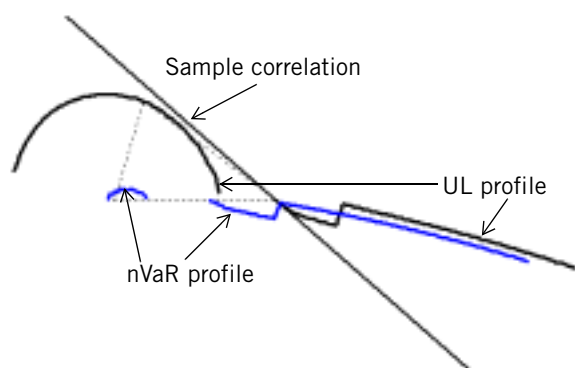

Figure 16: Triangular risk decomposition from a well-hedged portfolio

Conclusions

Triangular risk decomposition is a tool for visualizing and interpreting a position's risk contribution relative to the remainder of the portfolio. The geometric representation of risk allows risk managers to quickly understand the correlations between individual positions and the balance of the portfolio, and to exploit this information by finding best hedges and counter-intuitive implied views.

Triangular decomposition of parametric VaR follows directly from the decomposition of volatility, as demonstrated by Litterman (1996, 1997), due to the assumption that changes in a

portfolio's value are normally distributed with zero mean. We relaxed this assumption and derived a triangular decomposition first for the case of normally-distributed changes in value with non-zero mean and then for arbitrary distributions, which rely on the simulation-based approach to obtain the non-parametric VaR. In both cases, the decomposition is based on separating the VaR into expected and unexpected loss components. The triangular decomposition using a simulation-based approach gives rise to the novel concept of an implied correlation. While the correlation between a position and the remainder of the portfolio remains constant under the normality assumption, it typically varies with the size of the position when this assumption is relaxed. The magnitude of the implied correlation can exceed one at times, in which case it is not possible to construct a triangular risk decomposition for the corresponding position size. This is due to one of the limitations of the non-parametric VaR as a coherent risk measure, namely its violation of sub-additivity.

In our initial paper, we noted that sampling a finite number of scenarios resulted in the piecewise linearity of the nTRP and we proposed using a smooth approximation to obtain more robust risk analytics. One might consider a similar approach for constructing the triangular decomposition; using the smooth approximation to the nTRP to obtain the portfolio nVaR may result in better agreement between the implied correlation and the sample correlation values.

We have considered triangular decomposition only in terms of the portfolio risk. Zerolis (1996) describes a geometric representation of risk and return in which risk is measured by volatility. The incorporation of return in the decompositions presented in this paper represents another opportunity for future research.

While triangular decomposition is closely related to the trade risk profile, which plots the portfolio VaR as a function of position size, it provides a useful visualization of correlation that is lacking in the trade risk profile. Like other simulation-based risk management tools—marginal nVaR, risk contributions, trade risk profiles—the

triangular decomposition requires only a single, initial simulation of the portfolio to value each instrument under every scenario. Thus, it represents a valuable and practical addition to the risk manager's toolkit.

References

Artzner, P., F. Delbaen, J.M. Eber and D. Heath, 1998, "Coherent measures of risk," Working Paper, Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S.

Litterman, R., 1996, "Hot spots and hedges," Risk Management Series, Goldman Sachs.

Litterman, R., 1997, "Hot spots and hedges (I)," *Risk,* 10(3): 42-45.

Mausser, H. and D. Rosen, 1998, "Beyond VaR: from measuring risk to managing risk," *Algo Research Quarterly,* 1(2): 5-20.

Zerolis, J., 1996, "Triangulating risk," *Risk,* 9(12): 36-38.

Appendix

The parametric, or delta-normal, method for calculating VaR generally assumes that the log price changes of the market risk factors are joint normally distributed with zero mean; that is, if r_k is the log return on risk factor k, then $r \sim N(0, Q^*)$, where Q^* is the covariance matrix of risk factor returns.

Consider a portfolio with N holdings that is exposed to W market risk factors. Each instrument in the portfolio is decomposed into a set of risk factor positions so that the change in

the instrument's value, Δv_i, can be expressed linearly in terms of the risk factor returns:

$$\Delta v_i = \sum_{k=1}^{W} m_k^i r_k \qquad (A1)$$

The vector $\boldsymbol{m}^i$, which gives the exposure of one unit of instrument i to each risk factor, is the VaR map of instrument i. We can express the change in the value of the portfolio as the sum of the changes in the values of its holdings:

$$\Delta V(x) = \sum_{i=1}^{N} x_i \sum_{k=1}^{W} m_k^i r_k$$

Equation A1 can be written more compactly as

$$\Delta V(x) = m(x)^T r$$

where

$$m(x) = \sum_{i=1}^{N} m^i x_i$$

is the VaR map of the portfolio.

It follows that $\Delta V(\boldsymbol{x})$ is normally distributed with mean zero and volatility

$$\sigma(x) = \sqrt{m(x)^T Q^* m(x)}$$

The portfolio's $100(1 - \alpha)\%$ VaR is

$$VaR(x) = z_\alpha \sqrt{m(x)^T Q^* m(x)}$$

which is equivalent to Equation 4.