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Chapter 16

Economic Credit Capital Allocation and Risk Contributions

Helmut Mausser

Algorithmics Incorporated, Toronto, Canada
E-mail: hmausser@algorithmics.com

Dan Rosen

R^2 Financial Technologies and the Fields Institute, Toronto, Canada
E-mail: drosen@fields.utoronto.ca

Abstract

Economic capital (EC) acts as a buffer for financial institutions to absorb large unexpected losses, thereby protecting depositors and other claim holders and providing confidence to external investors and rating agencies on the financial health of the firm. Once the amount of capital has been determined, it must be allocated equitably among the various components of a portfolio (e.g., activities, business units, obligors or individual transactions). Capital allocation is an important management decision support and business planning tool, required for pricing, profitability assessment and limits, building optimal risk-return portfolios and strategies, performance measurement and risk based compensation.

This chapter provides a practical overview of the measurement of economic credit capital contributions and their application to capital allocation. We discuss the advantages and disadvantages of various risk measures and models, the interpretation of various allocation strategies as well as the numerical issues associated with this task. We stress four key points. First, marginal risk contributions provide a useful basis for allocating EC since they are additive and reflect the benefits of diversification within a portfolio. Second, the choice of the risk measure can have a substantial impact on capital allocation. In particular, Value at Risk (VaR) and expected shortfall (ES) contributions avoid the inconsistencies, and potentially inefficient allocations, associated with the widely-used volatility-based methods. The quantile level chosen for measuring risk can also have a significant impact on the relative amount of capital allocated to portfolio components. Third, VaR and ES contributions can be calculated analytically under certain simple models. These methods provide fast calculations and can be used to understand capital allocation strategies better, but they present important practical limitations, as well. Finally, Monte Carlo methods may be required to compute risk contributions in more realistic credit models. Computing VaR and ES contributions is challenging, especially at the extreme quantiles typically used for credit capital definition. The quality of contribution estimates can be improved by exploiting the conditional independence framework underlying the most common models, through the use of more sophisticated quantile estimators (especially for VaR) and through the use of variance reduction techniques, such as Importance Sampling.

1 Introduction

In financial institutions, *economic capital* (EC) acts as a buffer to absorb large unexpected losses, thereby protecting depositors and other claim holders and providing confidence to external investors and rating agencies on the financial health of the firm. In contrast, regulatory capital refers to the minimum capital requirements which banks are required to hold, based on regulations established by the banking supervisory authorities. From the perspective of the regulator, the objectives of capital adequacy requirements are to safeguard the security of the financial institutions and to ensure their ongoing viability, as well as to create a level playing field. For example, regulations for internationally active banks are given by the *Basel Accord*, the framework created by the Basel Committee on Banking Supervision (BCBS), which is now the basis for banking regulation around the world (BCBS, 1988; BCBS, 2004).

Economic capital covers all the risks (e.g., market, credit, operational and business) that may force a financial institution into insolvency. While most of the concepts and methodologies in this chapter have broader applicability, we focus on *economic credit capital* – the buffer against those risks specifically associated with obligor credit events such as *default*, *credit migrations* (downgrades or upgrades) and *credit spread* changes.

Traditionally, capital is designed to absorb *unexpected* losses, at a specified confidence level, while credit reserves are set aside to cover *expected* losses. Thus, EC is commonly defined as the difference between the portfolio's value-at-risk (VaR) and the expected loss of the portfolio. The VaR level (i.e., quantile) is chosen in a way that trades off providing a high return on capital for shareholders with protecting debt holders and depositors (and maintaining a desired credit rating).

Once the amount of capital has been determined, it must be allocated equitably among the various components of the credit portfolio (e.g., activities, business units, obligors or individual transactions). This is vital for management decision support and business planning, performance measurement and risk based compensation, pricing, profitability assessment and limits, as well as building optimal risk-return portfolios and strategies.

There is no unique method to allocate EC; each methodology has its advantages and disadvantages, and might be appropriate for a given managerial application. In particular, marginal risk contributions yield an additive decomposition of EC that accounts for the effects of diversification within the portfolio. An EC allocation based on the marginal contribution to the volatility (or standard deviation) of the portfolio losses is the most common approach used today by practitioners. However, this allocation scheme is ineffective if the loss distribution is not Normal, as is typical of credit losses. This can produce inconsistent capital charges, and in some cases a loan's capital charge can even exceed its exposure (see Praschnik et al., 2001; Kalkbrener et al., 2004).

Given the definition of EC, it is more natural to allocate capital based on contributions to VaR. However, VaR has several shortcomings since it is not a coherent risk measure (Artzner et al., 1999). Specifically, while VaR is subadditive for Normal distributions, this is not true in general. This limitation is especially relevant for credit loss distributions, which may be far from Normal and not even smooth. Furthermore, VaR refers to one particular loss (i.e., one point in the loss distribution), which makes it difficult to obtain accurate and stable risk contributions with Monte Carlo simulation.

Recently, several authors have proposed using expected shortfall (ES) for allocating EC (see, for example, Kalkbrener et al., 2004). As a coherent risk measure, ES represents a good alternative both for measuring and allocating capital. In particular, Kalkbrener et al. (2004) show that ES yields a linear (or additive), diversifying capital allocation. This requires a slight modification of the standard interpretation of economic capital, namely that it offsets an expected loss conditional on exceeding a certain quantile.

The marginal risk contribution of a position in a portfolio is based on the derivative of the risk measure with respect to the size of that position. While this derivative may not always exist, it has been shown (see Gouriéroux et al., 2000; Tasche, 2000, 2002) that the derivatives of quantile measures (e.g. VaR, or ES) can be expressed as conditional expectations. Several semi-analytical approaches have been proposed for calculating VaR or ES contributions (e.g. Martin et al., 2001; Kurth and Tasche, 2003) and, for certain simple models, risk contributions can be obtained analytically (e.g. Gordy, 2003a; Emmer and Tasche, 2005; Garcia Cespedes et al., 2006).

More realistic credit models typically entail the use of Monte Carlo (MC) simulation, which readily supports diversification through multiple factors and more flexible co-dependence structures, multiple asset classes and default models, as well as stochastic (correlated) modeling of exposures and loss given default. However, computing conditional expectations with simulation is computationally challenging, primarily due to the effects of random noise in the data and the discrete nature of individual credit losses.

Various numerical methods have been proposed to improve the quality of simulation-based risk contributions. For instance, the standard quantile estimator, which considers only a single observation, is known to be unreliable for this purpose (Mausser and Rosen, 1998) and several authors have suggested estimating VaR contributions from multiple observations (Praschnik et al., 2001; Hallerbach, 2003; Mausser, 2003; Mausser and Rosen, 2005). Alternatively, importance sampling generates a greater number of observations from the tail of the loss distribution in order to obtain more stable estimates (Kalkbrener et al., 2004; Merino and Nyfeler, 2004; Glasserman, 2005).

This chapter provides a practical overview of the measurement of economic credit capital contributions and their application to capital allocation. We discuss the advantages and disadvantages of various risk measures and models, the implications of various allocation strategies, as well as the numerical issues associated with measuring risk contributions.

Various methods of calculating VaR and ES contributions are presented, including both analytical and simulation-based approaches. For the latter, we explain the difficulties associated with estimating the required conditional expectations, and describe numerical techniques, such as semi-analytical convolutions, L-estimators and Importance Sampling, that can be used to address these problems. Several examples are provided to illustrate concepts relevant to using marginal risk contributions for allocating EC and, in particular, stress the shortcomings of volatility contributions relative to quantile-based measures such as VaR and ES.

The reader is further referred to Martin (2004) and Glasserman (this volume) for comprehensive presentations on credit portfolio modeling and computational methodologies for calculating portfolio credit risk. For basic presentations on economic capital and regulatory capital see Aziz and Rosen (2004) and Rosen (2004) and the references cited therein. Finally, for credit portfolio optimization see Mausser and Rosen (2000, 2001).

The rest of this chapter is organized as follows. Section 2 briefly reviews the general framework for credit portfolio models and describes the popular Normal-copula model. Section 3 first introduces capital allocation and then focuses on marginal risk contributions as a way of accomplishing this task. Section 4 describes three models where marginal risk contributions can be obtained analytically: the one-factor credit model in Basel II, the granularity adjustment and a multi-factor extension to Basel II. The more general problem of computing risk contributions with simulation is considered in Section 5. Illustrative examples are presented in Section 6, and Section 7 offers concluding remarks and suggestions for further research.

2 Credit portfolio models and general framework

Over the last decade, several credit portfolio models have been developed for measuring economic credit capital. Some popular industry models include CreditMetrics (Gupton et al., 1997), CreditRisk+ (Credite Suisse Financial Products, 1997), Credit Portfolio View (Wilson, 1997a, 1997b), KMV's Portfolio Manager (Crosbie, 1999). Although the models appear quite different on the surface, they all share an underlying mathematical equivalence among them (Koyluoglu and Hickman, 1998; Gordy, 2000; Frey and McNeil, 2003). The models differ in their distributional assumptions, restrictions, calibration and solution, but can be calibrated to yield similar results if the input data is consistent.

All of the above are single-period models and generally assume that market risk factors, such as interest rates, are constant. While these assumptions are appropriate for portfolios of simple loans or bonds, they may lead to significant errors when a portfolio contains derivatives or instruments with embedded optionality (such as credit lines), or when exposures vary over time (and hence the timing of the default affects the portfolio losses). An example of a multi-step integrated market and credit risk model that overcomes these limitations is given in Iscoe et al. (1999).

The credit portfolio modeling framework which encompasses these models is referred to as the *conditional independence framework*. In general terms, it consists of five parts:

1. *Systemic Scenarios* ("*states of the world*"). This models the evolution of the relevant "systemic" or sector-specific credit drivers that affect credit events, as well as market factors driving obligor exposures, over the period of analysis.
2. *Conditional default and credit migration probabilities*. Default and migration probabilities vary as a result of changing economic conditions. At each point in time, an obligor's default/migration probabilities are conditioned on the state of the world. Default correlations among obligors are determined from a correlated default/migration model, which describes how changes in the credit drivers affect the conditional default/migration probabilities.
3. *Conditional obligor exposures, recoveries and losses*. The credit exposure to an obligor is the amount that the institution stands to lose should the obligor default. Recovery rates are generally expressed as the percentage of the exposure that is recovered through such processes as bankruptcy proceedings, the sale of assets or direct sale to default markets. Exposures can be assumed to be constant in all scenarios for banking instruments without optionality as well as bonds, but not for other instruments such as derivatives, lines of credit, collateral or unhedged exposures in various currencies.
4. *Conditional portfolio loss distribution*. Conditional on a scenario, obligor defaults (and migrations) are independent. This property facilitates obtaining the conditional portfolio loss distribution, as the conditional portfolio loss is the sum of independent random variables (i.e., obligor losses).
5. *Unconditional portfolio loss distribution*. The unconditional distribution of portfolio credit losses is obtained by averaging the conditional loss distributions over all scenarios.

We now illustrate the framework with the so-called Normal copula model, originally popularized by the CreditMetrics and KMV portfolio models.

2.1 Multi-factor Normal copula model

Consider a portfolio of N obligors and a single-step model. Without loss of generality, assume that each obligor j has a single loan with loss given default and exposure at default given by LGD_j, EAD_j respectively.

For each obligor, we define a continuous variable, the *creditworthiness index* (*CWI*), that represents its financial health, and assume that it has a standard Normal distribution. The CWI of each obligor j depends on d systemic factors (also assumed to be independent and standard Normal) through the multi-factor model

$$\begin{aligned} Y_j &= \beta_{j1} Z_1 + \beta_{j2} Z_2 + \cdots + \beta_{jd} Z_d + \sigma_j \varepsilon_j, \\ \sigma_j &= \sqrt{1 - (\beta_{j1}^2 + \beta_{j2}^2 + \cdots + \beta_{jd}^2)} \end{aligned} \tag{1}$$

where

- $Z_1, \ldots, Z_d$ are independent, standard Normal distributed systemic risk factors;
- ε_j is a standard Normal distributed specific risk factor for obligor j;
- $\beta_{j1}, \ldots, \beta_{jd}$ are the sensitivities of obligor j to the systemic risk factors.

Suppose that obligors migrate into one of R possible credit states, ordered by increasing credit quality (i.e., in a default-only model, $R = 2$ and $r = 1$ is the default state), and let p_{jr} denote the unconditional probability that obligor j transitions into state r.

Assume that defaults for each obligor are driven by a Merton-type model, so that obligor j defaults when its CWI, Y_j, falls below a given threshold at the horizon. If PD_j denotes the obligor's (unconditional) default probability ($PD_j = p_{j1}$), we can express the default threshold by $\Phi^{-1}(PD_j)$, with $\Phi^{-1}(\cdot)$ the cumulative standard Normal distribution. Also in a given scenario given by the outcomes of the risk factors $\boldsymbol{Z}$, the conditional default probability of obligor j is

$$\begin{aligned} p_{j1}(Z) &= \Pr\left[Y_j < \Phi^{-1}(PD_j) \mid Z\right] \\ &= \Phi^{-1}\left(\frac{\Phi^{-1}(PD_j) - (\beta_{j1} Z_1 + \beta_{j2} Z_2 + \cdots + \beta_{jd} Z_d)}{\sigma_j}\right). \end{aligned} \tag{2}$$

Similar formulae are obtained for the conditional transition probabilities to each credit state r (see, for example, Gupton et al., 1997).[1]

Other credit models can be used to obtain different functional forms similar to Eq. (2). For example in a *logit model* the expression for conditional default

[1] This model is also referred to as an *ordered probit* model.

probabilities is given by[2]

$$p_{j1}(Z) = \left[1 + a \cdot \exp\left(b \cdot \left(\sum_{j=1}^{d} \beta_{ij} Z_j\right)\right)\right]^{-1}.$$

Credit migrations can also be handled similarly in such a model (this is referred to as an *ordered logit* model).

Let c_{jr} denote the loss that is incurred if obligor j transitions into state r.[3] Since an integrated market/credit model recognizes that credit losses can be dependent on the systemic factors, one can denote by $c_{jr}(\boldsymbol{Z})$ the loss that results if obligor j transitions into state r, given the factors $\boldsymbol{Z}$ (e.g. Iscoe et al., 1999). However, for simplicity, we ignore such dependence in the sequel.

To understand conditional independence models, consider sampling randomly from the systemic risk factors, $\boldsymbol{Z}$. We refer to each sample Z^m, $m = 1, \ldots, M$, as a scenario. For ease of notation, define the random variable L_j^m to be the loss of obligor j conditional on Z^m, and denote the conditional transition probability and (known) loss of obligor j in state r as p_{jr}^m and c_{jr}^m, respectively. For each obligor j, we have a discrete conditional credit loss distribution

$$L_j^m \sim D_j^m = \left\{\left(c_{jr}^m, p_{jr}^m\right) \mid r = 1, \ldots, R\right\}. \tag{3}$$

Since obligors are conditionally independent, the portfolio loss in scenario m, denoted L^m, is a sum of independent random variables and its distribution is the convolution of the obligor loss distributions

$$L^m \equiv \sum_{j=1}^{N} L_j^m \sim D^m = D_1^m \otimes \cdots \otimes D_N^m \tag{4}$$

D^m has finite support C^m, comprising up to r^N elements (in practice, this number is often less because various combinations of obligor losses sum to the same portfolio loss).

Computing the conditional portfolio loss distribution (Eq. (4)) exactly is computationally challenging if the number of possible portfolio losses (i.e., C^m) is large. However, any method for obtaining the distribution of a sum of independent random variables can be applied in this case. In practice, various techniques that have been used to approximate this convolution efficiently include[4]:

- Fast Fourier Transform methods (in conjunction with a discretization scheme, where losses are assigned to "buckets" in a common grid).

[2] See Wilson (1997a, 1997b), Bucay and Rosen (2000).

[3] Losses are net of recovery in the case of default, $r = 1$.

[4] See, for example, Finger (1999), Martin (2004), Glasserman (this volume), and the references cited therein.

- Saddle Point methods, based on analytical approximations of the distribution around the quantile of interest.
- Analytical approximations of the conditional portfolio loss distributions by simpler distributions. For example, if the portfolio is large and granular, the Law of Large Numbers suggests that the conditional portfolio loss distribution is effectively a point mass at the sum of the mean obligor losses (all other moments vanish). Alternatively, the Central Limit Theorem can be applied under the assumption that all conditional losses are Normally distributed.
- Direct sampling from each obligor's conditional loss distribution (D_j^m) and aggregation to get a sample observation from D^m.

The unconditional loss distribution requires integrating the conditional loss distributions across all scenarios (the joint distribution of the systemic factors). Thus, it is the average of the M conditional portfolio loss distributions (i.e., its support is the union of the C^m and all probabilities are multiplied by $1/M$). In the case of a one-factor model, the integration might be done analytically, but with multi-factor models, simulation is typically required (although some semi-analytical approximations are available, e.g. Pykhtin, 2004).

Credit capital is commonly defined in terms of a high quantile (e.g. in Basel II it is defined at the 99.9% level). Therefore, in practice, it might be difficult to get numerically stable and accurate VaR or expected shortfall estimates for realistic credit portfolios using standard MC methods. This problem is further amplified when calculating risk contributions (see next section). Variance reduction techniques, such as Importance Sampling (IS) and Control Variates, can be applied which reduce significantly the variance of Monte Carlo simulation. For example, IS increases the number of extreme observations, thereby improving accuracy in the tail of the portfolio loss distribution (e.g. Glasserman and Li, 2005). Tchistiakov (2004) applies a control variate technique to estimate portfolio risk, where the control variable is derived from the limiting distribution of a homogeneous portfolio (Vasicek loss distribution) that approximates the portfolio.

3 Capital allocation and risk contributions

In addition to computing the total EC for a portfolio, it is important to develop methodologies to *attribute* this capital *a posteriori* to various sub-portfolios such as the firm's activities, business units, counterparties or even individual transactions, and to *allocate* it *a priori* in an optimal fashion, to maximize risk-adjusted returns.

In general, the sum of the stand-alone EC for each sub-portfolio or asset is higher than the total portfolio EC due to the benefits of diversification. There is no unique method to allocate EC down a portfolio, and we can classify the capital allocation methodologies that are currently used in practice into three broad categories[5]:

- *Stand-alone* capital contributions – a sub-portfolio is assigned the amount of capital that it would consume on a stand-alone basis. As such, it does not reflect the beneficial effect of diversification; the sum of stand-alone capital for the individual sub-portfolios may exceed the total EC for the portfolio.
- *Incremental* capital contributions (or *discrete marginal* capital contributions) – calculated by taking the EC for the entire portfolio and subtracting from it the EC for the portfolio without the sub-portfolio. This method captures the amount of capital that would be released if the sub-portfolio were sold or added. Thus it is a natural measure for evaluating the risk of acquisitions or divestitures. A disadvantage of this method is that it does not yield an additive risk decomposition.
- *Marginal* capital contributions (or *diversified* capital contributions) – measures of risk contributions that are additive. By construction, the sum of diversified capital for all the sub-portfolios is equal to total EC for the portfolio. Marginal contributions are specifically designed to allocate the diversification benefit among the sub-portfolios, by capturing the amount of the portfolio's capital that should be allocated to each sub-portfolio, on a marginal basis, when viewed as part of the portfolio.

Several alternative methods for additive risk contributions have been proposed from game theory (see Denault, 2001; Koyluoglu and Stoker, 2002). For example, the Shapley method is based on the formation of coalitions so that a group of units benefits more as a group than if each works separately. This method is computationally intensive, and may be impractical for problems with even a small number of sub-portfolios. A variant called the Aumann–Shapley method requires less computation and is, thus, potentially more practical. Under most (but not all) conditions, these methods yield similar results to marginal contributions. While these methods are today receiving some academic attention, they are mostly not yet used in practice by financial institutions.

We now describe the methodology for capital allocation based on marginal risk contributions, leading to an additive decomposition of a portfolio's risk.

3.1 Definitions

Consider a credit portfolio that contains positions in N obligors (while our discussion assumes that obligors are the components of interest, one could alternatively consider individual loans or transactions). For each obligor j, define x_j to be the size of the position (number of units), and let the random variable

[5] There is no unique terminology for risk contributions, and here we follow Aziz and Rosen (2004).

l_j denote the credit loss per unit position. Credit losses can arise from default events, credit migration or more general spread movements.

If L_j denotes the credit loss due to the jth obligor then the loss of the portfolio is

$$L = \sum_{j=1}^{N} L_j = \sum_{j=1}^{N} l_j x_j. \tag{5}$$

Let F denote the portfolio loss probability distribution, which may or may not be available in closed form (e.g., F may be defined empirically by the results of a Monte Carlo simulation). Let $\rho(L)$ denote a measure of the risk of the portfolio, as implied by the distribution F. An additive decomposition of the risk $\rho(L)$ satisfies

$$\rho(L) = \sum_{j=1}^{N} C_j^{\rho} \tag{6}$$

where C_j^{ρ} represents the risk contribution of obligor j. The relative risk contribution of obligor j is defined to be its proportion of the total risk:

$$R_j^{\rho} = \frac{C_j^{\rho}}{\rho(L)}.$$

If $\rho(L)$ is homogeneous of degree one and differentiable, then Euler's Theorem implies that

$$C_j^{\rho} = x_j \frac{\partial \rho}{\partial x_j}. \tag{7}$$

From Eq. (7), the marginal risk contribution of an entity can be loosely interpreted as the rate of change of the portfolio due to a 1% change in the positions of that entity.

3.2 Risk measures and coherent capital allocation

Economic capital is designed in practice to absorb unexpected losses up to a certain confidence level, α, while credit reserves are set aside to absorb expected losses (EL). Thus, economic capital is typically estimated as the α-quantile of the portfolio loss distribution (VaR_α) minus the expected losses over a specified time horizon[6]:

$$EC_\alpha = VaR_\alpha - EL. \tag{8}$$

[6] The regulatory proposal in Basel II is based on the 99.9% VaR (BCBS, 2004).

This is the approach commonly taken by practitioners, and generally leads to conservative estimates of EC. More formally Eq. (8) represents only a simplifying approximation to the true EC (see Kupiec, 2002; Aziz and Rosen, 2004). The rationale for subtracting EL is that credit products are already priced such that net interest margins less non-interest expenses are sufficient to cover estimated EL (and also a desired return to capital). More precisely, the credit VaR measure appropriate for EC should consider losses relative to the portfolio's initial mark-to-market (MtM) value and not relative to the EL in its end-of-period distribution. Also, credit VaR normally ignores the interest payments that must be made on the funding debt. These payments must be added explicitly to the EC.

Three risk measures are often used for allocating economic capital among a portfolio's constituent positions: *volatility*, *VaR* and *expected shortfall* (sometimes called CVaR or conditional tail expectation). All three measures are homogeneous and hence lead to additive marginal risk contributions.

In current practice, the most common approach for assigning capital on a diversified basis computes a component's marginal contribution to the volatility of the portfolio loss distribution and scales it to correspond to the economic capital (e.g. Smithson, 2003). Specifically, if $\rho(L) \equiv \sigma(L)$ then Eq. (7) leads to the well-known formula

$$C_j^{\sigma} = \frac{\mathrm{cov}(L_j, L)}{\sigma(L)} \tag{9}$$

and the capital charged to obligor j is

$$C_j^{EC_\alpha} = R_j^{\sigma} \times EC_\alpha.$$

This approach works well if losses are normally distributed, since quantiles are constant multiples of the volatility in this case.[7] Due to the non-normality of credit loss distributions, however, volatility allocation often produces inconsistent capital charges (see Praschnik et al., 2001). In particular, Kalkbrener et al. (2004) show that a loan's capital charge can exceed its exposure.

The VaR contribution of obligor j is

$$C_j^{VaR_\alpha} = E[L_j \mid L = VaR_\alpha]. \tag{10}$$

This follows from the relation between partial derivatives and conditional expectations (e.g., Tasche, 1999; Gouriéroux et al., 2000). The capital charged to obligor j is then

$$C_j^{EC_\alpha} = C_j^{VaR_\alpha} - E[L_j]. \tag{11}$$

An obligor's contribution to EL is simply its expected loss, which is easy to compute analytically. Thus, capital allocation essentially reduces to the more difficult task of measuring VaR contributions (i.e., Eq. (10)).

[7] More precisely, this is the case for elliptic distributions in general.

It is widely recognized that VaR has several shortcomings since it is not a coherent risk measure (in the sense of Artzner et al., 1999). Specifically, while VaR is sub-additive (or diversifying) for normal distributions, this is not true in general. This limitation is relevant for credit loss distributions, which may be far from normal and not even smooth. In particular, the discreteness of individual credit losses leads to non-smooth profiles and marginal contributions.

Expected shortfall (ES) is a coherent risk measure and presents a good alternative to VaR and volatility both for measuring and allocating capital. As with VaR, ES contributions represent conditional expectations (Tasche, 2002; Scaillet, 2004)

$$C_j^{ES_\alpha} = E[L_j \mid L \geqslant VaR_\alpha]. \tag{12}$$

Since ES acts as a buffer for an expected loss conditional on exceeding a certain quantile, its use in allocating economic capital requires a rescaling similar to that of volatility. That is, the capital charged to obligor j equals

$$C_j^{EC_\alpha} = R_j^{ES_\alpha} \times VaR_\alpha - E[L_j]. \tag{13}$$

Although VaR and ES may not be differentiable in some cases,[8] it is reasonable to define the risk contributions by Eqs. (10) and (12) in general (e.g., Kurth and Tasche, 2003; Hallerbach, 2003).

Kalkbrener et al. (2004) formally introduce an axiomatic approach to define the concept of a *coherent capital allocation*. The three axioms can be summarized as follows:

- *Linear (or additive) allocation*: the capital allocated to a union of sub-portfolios is equal to the sum of the capital amounts allocated to the individual sub-portfolios.
- *Diversifying allocation*: the capital allocated to a sub-portfolio X of a larger portfolio Y never exceeds the risk capital of X considered as a stand-alone portfolio.
- *Continuous allocation*: a small increase in a position only has a small effect on the risk capital allocated to that position.

The authors show that these three axioms uniquely determine a capital allocation scheme – which is essentially a marginal capital allocation. Also, they show that any allocation satisfying these axioms is associated with a coherent risk measure. Notably, ES yields a linear (or additive), diversifying and continuous capital allocation, while VaR yields an additive but not a diversifying allocation.

[8] Laurent (2003) discusses the differentiability of risk measures when the loss distribution is discrete.

4 Credit risk contributions in analytical models

In the presence of diversification, the marginal capital required for a counterparty or loan may depend on the overall portfolio composition. If capital charges are based on marginal portfolio contributions, these charges are not, in general, *portfolio-invariant*, and are different from their stand-alone capital. Thus, an interesting question is: under what circumstances do portfolio models yield portfolio-invariant capital contributions?

If economic capital is defined in terms of a VaR measure, Gordy (2003a) shows that two conditions are necessary and sufficient to guarantee portfolio-invariant contributions:

- The portfolio must be asymptotically fine-grained; i.e. no single exposure can account for more than an arbitrarily small share of total portfolio exposure.
- There must be only a single systematic risk factor.

The "single-factor, asymptotically-fine-grained" portfolio model is at the heart of the new Basel II banking credit regulation (BCBS, 2004). In this context, capital only covers *systemic credit risk*; it does not account for the idiosyncratic risk that exists in non-granular portfolios, leading to counterparty (or name) concentrations. Gordy (2003a, 2003b) and Martin and Wilde (2002) further present an asymptotic approximation for the idiosyncratic credit risk when portfolios are not sufficiently granular (the so called "granularity" adjustment). Of course, in the presence of idiosyncratic credit risk, marginal capital contributions are dependent on the portfolio composition (specifically, the level of name concentration risk in the portfolio).[9]

We now briefly describe the analytical formulae for credit risk contributions in the one-factor, Basel II model and its extension to account for idiosyncratic risk (the so-called granularity adjustment). Finally, we discuss capital allocations in the context of a simple multi-factor extension of the Basel II model. By introducing explicitly the concept of a *diversification factor* at both the portfolio and obligor or sector levels, the multi-factor extended model provides useful intuition on capital contributions, and their sources.

4.1 Capital contributions in the Basel II model

Consider a portfolio with N obligors and a single-step model. Without loss of generality, assume that each obligor j has (unconditional) default probability PD_j, and a single loan with loss given default and exposure at default given by LGD_j, EAD_j respectively.[10]

[9] See for example Emmer and Tasche (2005) for a discussion of risk contributions in a one factor model with the granularity adjustment.

[10] We use here the notation commonly used in the Basel accord, where now the product $EAD_j \cdot LGD_j = c_{j1}$, as given in Section 2. As they are assumed deterministic, they are the same in each scenario m.

For each obligor j, the credit losses at the end of the horizon (e.g., one year) are driven by a Merton model, as given in Section 2, but in this case with one, single, systemic factor. Obligor j defaults when its creditworthiness index falls below a given threshold, given by $\Phi^{-1}(PD_j)$.

The creditworthiness of obligor j is driven by a single systemic factor:

$$Y_j = b_j Z + \sqrt{1 - b_j^2}\varepsilon_j \tag{14}$$

with Z is a standard Normal variable representing the single systemic, economy-wide factor, and the ε_j are independent standard Normal variables representing the idiosyncratic movement of obligors' creditworthiness. We commonly refer to b_j^2 as the asset correlation of obligor j.

Gordy (2003a) shows that the α-percentile systemic portfolio loss (i.e. the loss assuming the portfolio is asymptotically fine-grained), VaR_α, is given by the sum of individual obligor losses, when an α-percentile move occurs in the systemic sector factor Z:

$$VaR_\alpha = \sum_j LGD_j \cdot EAD_j \cdot \Phi\left(\frac{\Phi^{-1}(PD_j) - b_j z^\alpha}{\sqrt{1 - b_j^2}}\right) \tag{15}$$

where z^α denotes the α-percentile of a standard normal variable.

EC is defined to cover only the *unexpected losses* (i.e., Eq. (8)), where the expected losses are $E[L] = \sum_{j=1}^N LGD_j \cdot EAD_j \cdot PD_j$.[11] Thus, the capital for the portfolio can be written as

$$EC_\alpha = \sum_{j=1}^N C_j^{EC_\alpha} \tag{16}$$

where $C_j^{EC_\alpha}$ denotes the capital contribution of counterparty j:

$$C_j^{EC_\alpha} = LGD_j \cdot EAD_j \cdot \left[\Phi\left(\frac{\Phi^{-1}(PD_j) - b_j z^\alpha}{\sqrt{1 - b_j^2}}\right) - PD_j\right]. \tag{17}$$

The capital contribution in Eq. (17) does not depend on the composition of the rest of the portfolio. In Section 4 we present an example of the allocation produced by this model and discuss the impact of the quantile chosen on the capital allocation.

[11] The following discussion still holds if capital is defined by VaR, by simply adding back the EL at the end of the analysis.

4.2 Capital contributions with idiosyncratic risk (the granularity adjustment)

When there is one systemic factor and the portfolio is (infinitely) granular, the credit portfolio loss distribution is obtained analytically and risk contributions are portfolio-invariant. Idiosyncratic risk arises when the portfolio is of finite size and not homogeneous (i.e. with some counterparty or name concentrations). In this case, even with a one-factor model, a general analytical solution might not be available, and various methods can be used to approximate the loss distribution.

Risk measures (including VaR and ES) can be decomposed into their systemic risk and idiosyncratic contributions. For example, the variance of portfolio losses can be written as the sum of the variance of the conditional expected losses and the expected conditional variance of losses:

$$V[L] = V[E[L|Z]] + E\big[V[L|Z]\big].$$

The first term is the contribution of systemic risk, and the second can be interpreted as the idiosyncratic risk, which vanishes as the number of obligors in a portfolio goes to infinity (and the idiosyncratic risk is diversified away). In a moderately large portfolio, the systemic component may be much larger than the idiosyncratic risk, but the latter may be too large to be neglected.

VaR and ES can be decomposed in a similar manner, although their idiosyncratic components do not have general closed-form expressions. When the conditional variance of losses is small, we can obtain analytical approximations of VaR and ES for non-granular portfolios as "small adjustments" to the infinitely granular portfolio. This *granularity adjustment* method is essentially a second order Taylor series expansion of the quantile (around the "infinitely granular" portfolio).[12]

Denote by VaR_α and $VaR_\alpha^S = VaR_\alpha(E[L|Z])$ the VaR of the (non-granular) portfolio and the systemic VaR of the portfolio (the VaR of the portfolio assuming it is infinitely granular), respectively. The VaR of the portfolio is approximated by:

$$VaR_\alpha \approx VaR_\alpha^S + GA_\alpha. \tag{18}$$

The general formula for the granularity adjustment is

$$GA_\alpha = -\frac{1}{2f(y)}\frac{\partial}{\partial y}\big[\sigma^2\big(z^\alpha\big)f(y)\big]\big|_{y=VaR_\alpha^S} \tag{19}$$

where $f(y)$ denotes the density function of the infinitely granular portfolio's loss, and $\sigma^2(z^\alpha)$ is the (idiosyncratic) variance of the portfolio losses conditional on the systemic factor level corresponding to the systemic portfolio losses being equal to VaR_α^S. A similar expression is available for ES.

[12] Gordy (2003a, 2003b) presented this approach first and has then been refined (see for example Martin and Wilde, 2002). Pykhtin (2004) further extended the method to multiple factors.

By applying Eqs. (18) and (19) directly to a given (one-factor) portfolio model, we can obtain closed form expressions to approximate the portfolio VaR and the risk contributions. For the one-factor Merton model, the systemic VaR, VaR_α^S, is given by expression (15) and the granularity adjustment is[13]:

$$GA_\alpha = \sum_{j=1}^{N} C_j^{GA_\alpha} \tag{20}$$

with

$$C_j^{GA_\alpha} = \frac{EAD_j^2 LGD_j^2}{2(VaR_\alpha^S)'}\Bigg[\left(\sqrt{\frac{b_j^2}{1-b_j^2}} \cdot \phi\big(PD_j^\alpha\big) \cdot \big(1 - 2\Phi\big(PD_j^\alpha\big)\big)\right) + \left(z^\alpha + \frac{(VaR_\alpha^S)''}{(VaR_\alpha^S)'}\right) \cdot \big(\Phi\big(PD_j^\alpha\big) - \Phi\big(PD_j^\alpha\big)^2\big)\Bigg] \tag{21}$$

where $PD_j^\alpha = (\Phi^{-1}(PD_j) - b_j z^\alpha)/\sqrt{1-b_j^2}$, and $(VaR_\alpha^S)'$ and $(VaR_\alpha^S)''$ denote the first and second derivatives of expression (15).

The terms $C_j^{GA_\alpha}$ can be interpreted as the idiosyncratic risk contribution of each obligor j. In this case, contributions are not portfolio invariant, since they depend on the total portfolio composition through the terms $(VaR_\alpha^S)'$ and $(VaR_\alpha^S)''$.

4.3 Credit risk contributions in an extended multi-factor model

A model that yields portfolio-invariant capital contributions is desirable for regulatory purposes, management transparency and computational tractability. However, such a model does not fully recognize diversification and may not be useful for capital allocation. We thus require also tools to understand and measure diversification in a multi-factor portfolio setting. Pykhtin (2004) recently obtains an elegant, analytical multi-factor adjustment to the Basel II one-factor model. This method can also be used effectively to compute capital contributions numerically (given its closed form solution to compute portfolio capital). However, closed-form expressions for capital contributions are quite intricate.

Garcia Cespedes et al. (2006) present a simple model that recognizes the diversification obtained from a multi-factor credit setting. The authors introduce the concept of a *diversification factor* at the portfolio level and also at the obligor or sub-portfolio level to account for diversification contributions to the portfolio (*marginal diversification factors*). Tasche (2006) further presents a

[13] See Emmer and Tasche (2005).

mathematical foundation for the diversification factor and analytical formulae for computing diversification contributions.[14]

To illustrate the Garcia Cespedes et al. model, consider a single-step model with K homogeneous sectors (each of these sectors can represent an asset class or geography, etc.). Similar to the Basel II model, for each obligor j in a given sector k, the credit losses at the end of the horizon are driven by a single-factor Merton model.[15] In this case, however, the creditworthiness of obligor j, in sector k, is driven by a single systemic factor:

$$Y_j = b_k Z_k + \sqrt{1 - b_k^2}\varepsilon_j \tag{22}$$

where Z_k is a standard Normal variable representing the systemic factor for sector k, and the ε_j are independent standard Normal variables representing the idiosyncratic movement of an obligor's creditworthiness. While in the Basel II model all sectors are driven by the same systemic factor Z, here each sector can be driven by a different factor.

Assume further that the systemic factors are correlated through a single macro-factor, Z,

$$Z_k = B_k Z + \sqrt{1 - B_k^2}\eta_k, \quad k = 1, \dots, K \tag{23}$$

where η_k are independent standard Normals, and each sector has a different correlation level B_k to the systemic, economy-wide factor, Z.

Assume, as before, that each obligor j has a single loan with loss given default and exposure at default given by LGD_j and EAD_j, respectively. Since credit losses within each sector are driven by a one-factor model, for asymptotically fine-grained sector portfolios, the stand-alone α-percentile capital for a given sector k, $EC_{\alpha,k}$, is given by

$$EC_{\alpha,k} = \sum_{j \in Sector\, k} LGD_j \cdot EAD_j \cdot \left[\Phi\left(\frac{\Phi^{-1}(PD_j) - b_k z^\alpha}{\sqrt{1 - b_k^2}}\right) - PD_j\right]. \tag{24}$$

Under Basel II, or equivalently assuming perfect correlation between all the sectors, the overall capital is simply the sum of the stand-alone capital for all individual sectors (for simplicity, we omit the parameter α hereafter from the notation):

$$EC^{1f} = \sum_{k=1}^{K} EC_k. \tag{25}$$

[14] The paper presents a two-dimensional example which has an analytical solution. Problems of dimension N require numerical integration of dimension $N - 1$.

[15] We focus the discussion on a one-period Merton model for default losses. The methodology and results are quite general and can be used with other credit models, and can also incorporate losses due to credit migration, in addition to default.

We define the *capital diversification factor*, *DF*, as the ratio of the actual capital computed using the multi-factor model and the stand-alone capital, $DF \leqslant 1$. This allows us to express the (diversified) economic capital as:

$$EC = DF \cdot EC^{1f}. \tag{26}$$

Economic capital is thus a function of

- the "additive" bottom-up capital from the one-factor (Basel II) model, EC^{1f}, and
- *DF*, a "factor adjustment" which represents the diversification of the portfolio.

The basic idea behind the model is to approximate *DF*, by a scalar function of a small number of parameters, which leads to a reasonable approximation of the true, multi-factor, economic credit capital, and which can be tabulated.

We can think of diversification basically being a result of two sources:

- The relative size of various sector portfolios; clearly a portfolio with one dominating, very large, sector results in high concentration risk and limited diversification. So we seek a parameter representing essentially an "effective number of sectors" accounting for their sizes.
- The cross-sector correlations. Hence a natural choice for a parameter in our model is some form of average cross-sector correlation.

Ideally, the "concentration index" representing the first source of diversification should account for the size of the exposures and also the differences in credit characteristics as they affect capital. Thus, a sector with a very large exposure on highly rated obligors, might not necessarily represent a large contribution from a capital perspective.

The Garcia Cespedes et al. model expresses the economic capital in Eq. (26), for a given confidence level, as

$$EC = DF\big(CDI, \bar{B}^2\big) \cdot \sum_{k=1}^{K} EC_k \tag{27}$$

where the two parameters in the diversification factor are:

- The *capital diversification index*, *CDI*, given by the sum of squares of the *capital weights* in each sector

$$CDI = \frac{\sum_k EC_k^2}{(EC^{1f})^2} = \sum_k w_k^2 \tag{28}$$

  with $w_k = EC_k / EC^{1f}$ the contribution to one-factor capital of sector k.
- The (capital weighted) average cross-sector correlation: $\bar{B}^2$.

The *CDI* is the well-known Herfindahl concentration index applied to the *stand-alone capital* of each sector (instead of the exposures, *EADs*, as is commonly used). Intuitively, it gives an indication of the portfolio diversification across sectors (not accounting for the correlation between them). For example, in the two-factor case, the *CDI* ranges between 0.5 (maximum diversification) and one (maximum concentration). The inverse of the *CDI* can be interpreted as an "effective number of sectors" in the portfolio, from a capital perspective.

In a similar way, the average correlation parameter is also capital weighted, to account better for the "contributions" of each sector (and accounting for the credit quality in addition to size). From the various possible definitions for an average sector correlation, we choose the following one. Assume a general sector factor correlation matrix, Q (which can be more general than that resulting from Eq. (23), where $Q_{ij} = \beta_i \beta_j,\ j \neq i$), and a vector of portfolio weights $W = (w_1 \dots w_S)^T$. We define the average sector factor correlation as

$$\bar{B}^2 = \frac{\sum_i \sum_{j \neq i} Q_{ij} w_i w_j}{\sum_i \sum_{j \neq i} w_i w_j} = \frac{\sigma^2 - \delta^2}{\vartheta^2 - \delta^2}$$

where $\sigma^2 = W^T Q W$ is the variance of the random variable given by the weighted sum of the factors, $\delta^2 = \sum_i w_i^2$ and $\vartheta^2 = (\sum_i w_i)^2$. $\bar{B}^2$ is an average correlation in the sense that $W^T B W = W^T Q W = \sigma^2$, with B the correlation matrix with all the non-diagonal entries equal to $\bar{B}^2$. For our specific case, we chose the portfolio weights to be the stand alone capital for each sector. Therefore, $\delta^2 = \sum_i EC_i^2$ and $\vartheta^2 = (\sum_i EC_i)^2 = (EC^{sf})^2$.

Garcia Cespedes et al. (2006) calibrate the model (27) through Monte Carlo simulation, and tabulate the diversification factor for several levels of correlation and *CDI* (see Fig. 1).

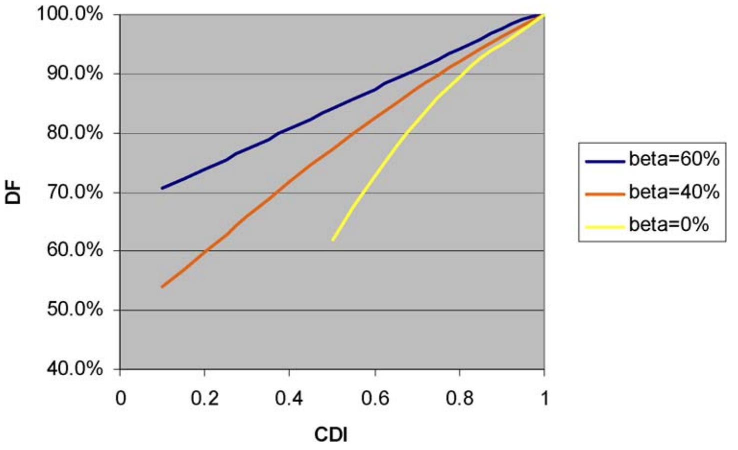

Fig. 1. Calibrated *DF* model in Garcia Cespedes et al. (2006).

Using the diversification factor model, one might be tempted simply to allocate back the diversification effect evenly across sectors, so that the total capital contributed by a given sector is $DF \cdot C_k$. We refer to these as the *unadjusted capital contributions*. This does not account, however, for the fact that each sector contributes differently to the overall portfolio diversification. Instead, we seek a capital decomposition of the form

$$EC^{mf} = \sum_{k=1}^{K} DF_k \cdot EC_k. \tag{29}$$

We refer to the factors DF_k in (29) as the *marginal sector diversification factors*.

For a general model of the form (26), if DF a homogeneous function of degree zero in the EC_k's, Euler's theorem leads to the additive marginal capital decomposition (29) with

$$DF_k = \frac{\partial EC^{mf}}{\partial EC_k}, \quad k = 1, \ldots, K. \tag{30}$$

In the specific model (27), the DF only depends on CDI and $\bar{\beta}$, which are both homogeneous of degree zero. By solving for the derivatives in expression (30), the marginal sector diversification factors are given by

$$\begin{aligned} DF_k = DF &+ 2\frac{\partial DF}{\partial CDI} \cdot \left[\frac{EC_k}{EC^{sf}} - CDI\right] \\ &+ 2\frac{\partial DF}{\partial \bar{B}^2} \cdot \frac{1 - (EC_k/EC^{sf})}{1 - CDI} \cdot [\bar{Q}_k - \bar{B}^2] \end{aligned} \tag{31}$$

where

$$\bar{Q}_k = \frac{\sum_{j \neq k} Q_{kj} EC_j}{\sum_{j \neq k} EC_j}$$

can be interpreted as the average correlation of sector factor k to the rest of the systemic sector factors in the portfolio.

The marginal capital allocation resulting from the model leads to an intuitive decomposition of diversification effects (or concentration risk) into three components: overall portfolio diversification, sector size and sector correlation:

$$DF_k = DF + \Delta DF_{\text{Size}} + \Delta DF_{\text{Corr}}.$$

The three components represent:

- The overall portfolio DF;
- An adjustment due to the "*relative size*" of the sector to the overall portfolio. Intuitively, for $DF > 0$ and all sectors having the same correlation B^2, a sector with small stand-alone capital ($w_k < CDI$) contributes, on the margin, less to the overall portfolio capital; thus, it gets a higher diversification benefit DF_k;

- An adjustment due to cross-sector correlations. Sectors with lower than average correlation get a higher diversification benefit, as one would expect.

5 Numerical methods to compute risk contributions

Simulation may be required to obtain portfolio loss distributions and calculate risk contributions when the underlying credit model presents a richer co-dependence structure described by multiple systemic factors, when the portfolio contains name concentrations (i.e. it is not granular), when credit losses account for migration and spread risk, or when exposures and LGDs are stochastic (and correlated). We can divide the simulation methods for calculating risk contributions into two broad classes:

- Full Monte Carlo (MC) simulation, with direct sampling of credit events and losses. In this case, the output of the simulation is an independent, identically-distributed sample of size M, where each observation comprises losses for all obligors (and the portfolio loss which is the sum of obligor losses). We make no assumptions about the model that underlies the sample.
- Two stage numerical solution based on the conditional independence framework for credit portfolio models (Section 2). In this case, it is possible to simulate first the systemic factors and then employ various numerical methods to obtain the unconditional portfolio loss distribution. Each systemic scenario comprises the conditional loss distributions for all obligors, with the conditional portfolio loss distribution as the convolution of these losses. As noted earlier, conditional portfolio loss distributions may be obtained using various techniques.

We now briefly summarize the application of these methods to compute credit risk contributions for VaR and ES.

5.1 Monte Carlo simulation with direct sampling of credit events

In a direct simulation approach, VaR and ES are estimated from the order statistics of the sampled portfolio losses.[16] Given the extreme quantiles typically used to measure credit risk, obtaining accurate risk contributions is a challenging task since the conditional expectations (Eqs. (10) and (12)) depend on rare events. This is of particular concern for VaR since the contribution is conditional on a single level of loss, while the ES contribution is conditional on a range of losses. Thus the accuracy of the VaR contributions depends critically on the chosen quantile estimator. In particular, the sample quantile, which is

[16] Order statistic k is the kth smallest loss.

frequently used in practice, is poorly suited for this task since it relies on a single order statistic. In contrast, L-estimators yield more robust estimates of VaR contributions.

5.1.1 Sample quantile estimators

Consider estimating a portfolio's VaR and ES at the 95% level, from an independent MC sample of size 100. In practice, the 95% VaR is often taken to equal the sample quantile (i.e., the 96th order statistic $L^{(96)}$), since $\mathrm{P}(L \geqslant L^{(96)}) = 0.05$ for the sample. A corresponding estimate of the 95% ES is given by the arithmetic average of order statistics 96 through 100.

The sample quantile, as defined in the example above, corresponds to an estimator known as the upper empirical cumulative distribution value (UECV). More generally, in a sample of size M, the UECV estimator estimates the α-quantile of the loss distribution by

$$\overline{VaR}_\alpha = L^{(\lfloor M\alpha \rfloor + 1)},$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, and the ES at level α by

$$\overline{ES}_\alpha = \frac{1}{M(1-\alpha)} \left[\big(\lfloor M\alpha \rfloor + 1 - M\alpha\big) L^{(\lfloor M\alpha \rfloor + 1)} + \sum_{k=\lfloor M\alpha \rfloor + 2}^{M} L^{(k)} \right].$$

Since the portfolio loss in any particular scenario is given by the sum of the obligor losses, in the example above, the contribution of obligor j to the 95% VaR is estimated by $L_j^{(96)}$ (i.e., the loss of obligor j that occurs with the 5th largest portfolio loss). Its contribution to the 95% ES is estimated by averaging the losses of obligor j that occur with the five largest portfolio losses.

From Eq. (10), the VaR contribution of obligor j equals the average loss of obligor j when the sampled portfolio loss equals the VaR estimate. Formally, if $L^{(k)} = \overline{VaR}_\alpha$ for $k_\alpha^{\min} \leqslant k \leqslant k_\alpha^{\max}$ then

$$\overline{C}_j^{VaR_\alpha} = E\big[L_j \mid L = \overline{VaR}_\alpha\big] = \frac{1}{k_\alpha^{\max} - k_\alpha^{\min} + 1} \sum_{k=k_\alpha^{\min}}^{k_\alpha^{\max}} L_j^{(k)}.$$

If only one sampled portfolio loss equals the estimated VaR (as is typically the case in a MC simulation) then $k_\alpha^{\min} = k_\alpha^{\max} = \lfloor M\alpha \rfloor + 1$ and the VaR contribution is estimated from a single observation, namely

$$\overline{C}_j^{VaR_\alpha} = L_j^{(\lfloor M\alpha \rfloor + 1)}.$$

Since a given portfolio loss may result from numerous combinations of obligor losses, estimates of VaR contributions provided by the sample quantile are often unreliable (e.g., Mausser and Rosen, 1998; Mausser, 2003). Two approaches for improving these estimates are:

- use quantile estimators that average multiple order statistics (e.g., L-estimators);
- use sampling strategies that increase the number of observations with a portfolio loss of $\overline{VaR}_\alpha$ (e.g., importance sampling).[17]

In contrast, the estimated ES contributions

$$\overline{C}_j^{ES_\alpha} = \frac{1}{M(1-\alpha)}\left[\left(k_\alpha^{\max} - M\alpha\right)\overline{C}_j^{VaR_\alpha} + \sum_{k=k_\alpha^{\max}+1}^{M} L_j^{(k)}\right]$$

tend to be more robust because ES is, by definition, an average of multiple order statistics. Nevertheless, both L-estimators and importance sampling can be applied to refine estimated ES contributions.

5.1.2 L-estimators

To improve the quality of the VaR contributions, several authors have proposed computing a weighted average of the losses over a range of order statistics around the sample quantile (e.g., Praschnik et al., 2001; Hallerbach, 2003). Conceptually, this is consistent with a more general class of quantile estimators known as L-estimators.

An L-estimator (e.g. Sheather and Marron, 1990) computes a quantile estimate as a weighted average of multiple order statistics.[18] Specifically, in a sample of size M, VaR_α is estimated as

$$\overline{VaR}_\alpha = \sum_{k=1}^{M} w_{\alpha,M,k}^{VaR} L^{(k)}, \tag{32}$$

where the weights depend only on the VaR level and the sample size. The VaR contribution for obligor j is then estimated as[19]

$$\overline{C}_j^{VaR_\alpha} = \sum_{k=1}^{M} w_{\alpha,M,k}^{VaR} L_j^{(k)}. \tag{33}$$

To estimate ES_α, observe that

$$ES_\alpha = E\left[F^{-1}(p) \mid p \geqslant \alpha\right]$$

[17] As pointed out in Glasserman (2005), it may be necessary to define a small "window" around $\overline{VaR}_\alpha$ in order to obtain a sufficiently large sample for estimating the conditional expectation.

[18] It is interesting to note the similarities between L-estimators and spectral risk measures (Acerbi, 2002). The quantity computed by the L-estimator is a spectral risk measure (i.e., coherent) if the weights are non-negative, non-decreasing (with respect to loss size) and sum to one.

[19] If a loss occurs multiple times in the sample, then its total weight is distributed equally among all relevant order statistics. For example, if $L_{(j)} = L_{(j+1)}$ for some j, then we set the weights for both order statistics equal to $\frac{1}{2}(w_{\alpha,S,j}^{VaR} + w_{\alpha,S,j+1}^{VaR})$ in Eq. (33).

$$= \frac{1}{1-\alpha} \int\limits_{\alpha}^{1} F^{-1}(p)\, \mathrm{d}p. \tag{34}$$

Replacing $F^{-1}(p)$ in Eq. (34) by its estimate from Eq. (32) yields

$$\begin{aligned}
\overline{ES}_{\alpha} &= \frac{1}{1-\alpha} \int\limits_{\alpha}^{1} \left(\sum_{k=1}^{M} w_{p,M,k}^{VaR} L^{(k)} \right) \mathrm{d}p \\
&= \sum_{k=1}^{M} \left(\frac{1}{1-\alpha} \int\limits_{\alpha}^{1} w_{p,M,k}^{VaR}\, \mathrm{d}p \right) L^{(k)}.
\end{aligned} \tag{35}$$

Equation (35) defines an L-estimator for expected shortfall with weights

$$w_{\alpha,M,k}^{ES} = \frac{1}{1-\alpha} \int_{\alpha}^{1} w_{p,M,k}^{VaR}\, \mathrm{d}p.$$

The ES contribution for obligor j is estimated as

$$\overline{C}_{j}^{ES_{\alpha}} = \sum_{k=1}^{M} w_{\alpha,M,k}^{ES} L_{j}^{(k)},$$

with the weights again adjusted for duplicate losses, if any (see footnote 19).

An L-estimator that has been found to perform well in practice is due to Harrell and Davis (1982). Empirical evidence suggests that the Harrell–Davis (HD) estimator outperforms the sample quantile for VaR contributions (Mausser, 2003; Mausser and Rosen, 2005). Appendix A derives the HD estimator weights for VaR and ES.

It is useful to compare Eq. (10), which defines a VaR contribution as a conditional expectation, and Eq. (33), which expresses it as weighted sum of ordered statistics. This suggests an intuitive interpretation of the weights in the L-estimator: for a desired quantile, they reflect the estimated probabilities that each order statistic equals the actual VaR, i.e.,

$$\overline{C}_{j}^{VaR_{\alpha}} = \sum_{k=1}^{M} \Pr\left[L^{(k)} = VaR_{\alpha}\right] E\left[L_j \mid L = L^{(k)}\right].$$

In fact, Sheather and Marron (1990) point out that the HD estimator is actually the bootstrap estimator of $\mathrm{E}[L^{((M+1)\alpha)}]$, where the expectation is computed analytically rather than by resampling.

For extreme quantiles of the portfolio loss distribution, standard MC may not generate enough observations in the tail to estimate VaR or ES contributions accurately, regardless of the quantile estimator used. Further improvements in the accuracy of risk contributions can be achieved by a combination of

- Exploiting the structure of the model and taking advantage of the underlying conditional independence framework;
- Applying variance reduction techniques such as Importance Sampling, Control Variates or Quasi MC methods.

5.2 Risk contributions in conditional independence models

Consider the conditional independence framework described in Section 2. We can express risk contributions as follows. Let the random variable

$$\bar{L}_j^m = L^m - L_j^m$$

denote the combined loss of all obligors other than j in scenario m. Since L_j^m and $\bar{L}_j^m$ are independent, the VaR contribution of obligor j in scenario m is

$$E\big[L_j^m \mid L^m = VaR_\alpha\big] = \frac{\sum_{r=1}^{R} c_{jr}^m p_{jr}^m \Pr[\bar{L}_j^m = VaR_\alpha - c_{jr}^m]}{\Pr[L^m = VaR_\alpha]}. \tag{36}$$

The unconditional contribution of obligor j is then computed as follows

$$E[L_j \mid L = VaR_\alpha] = \sum_{m=1}^{M} E\big[L_j^m \bigm| L^m = VaR_\alpha\big] \Pr\big[Z^m \bigm| L = VaR_\alpha\big]. \tag{37}$$

Expected shortfall contributions are obtained similarly by conditioning on the loss being greater than or equal to the VaR in Eqs. (36) and (37). That is, the probabilities in the numerator and denominator of Eq. (36) are substituted by

$$\Pr\big[\bar{L}_j^m \geqslant VaR_\alpha - c_{jr}^m\big] \quad \text{and} \quad \Pr\big[L^m \geqslant VaR_\alpha\big]. \tag{38}$$

Equation (36) shows that the conditional contribution for obligor j essentially entails convoluting the distributions of L_j^m and $\bar{L}_j^m$. As with the computation of the conditional portfolio loss distribution (Eq. (4)), various numerical methods may be used to approximate this convolution efficiently. For example, Saddle Point methods provide semi-analytical expressions for VaR and ES contributions (see Martin et al., 2001). Also, by assuming that conditional portfolio losses are roughly Normal and applying the Central Limit Theorem, we can obtain analytical formulae for risk contributions. Appendix A further presents the analytical expressions for VaR and ES contributions under the CLT.

It is important to emphasize that the objective of these methods is essentially to capture the idiosyncratic risk (which arises from the conditionally independent obligor losses in each scenario). On their own, these methods are generally not effective for risk contributions of very large and granular portfolios (where the systemic scenarios are largely driving the portfolio losses) or when capital is calculated at high quantiles in the tail. This requires a greater emphasis on generating "relevant" scenarios on the systemic factors, and hence MC variance reduction methods can provide significant improvements.

5.3 Variance reduction techniques

Variance reduction techniques can be used to improve significantly the quality of risk contribution estimates, particularly for extreme quantiles. In particular, Importance Sampling can be used for simulating both systemic and specific risk factors to have more relevant scenarios in the tail of the distribution. The following list represents several examples of its application to estimating risk contributions:

- Merino and Nyfeler (2004) compute ES contributions in a default-only model, using importance sampling to estimate the probabilities in Eq. (38). Their approach requires first obtaining VaR_α for the desired α-quantile of the unconditional portfolio loss distribution. Then, if VaR_α exceeds $E[L^m]$, they adjust each conditional default probability p_{j1}^m (by means of a so-called "exponential twisting") so that the expected value of L^m under the adjusted probability measure equals VaR_α.[20] Thus, importance sampling is applied to the specific risk factors in each systemic scenario.
- Kalkbrener et al. (2004) also compute ES contributions in a default-only model, but apply importance sampling to the systemic risk factors. That is, they sample $Z_1, \ldots, Z_d$ from Normal distributions whose means are shifted to increase the likelihood of an extreme loss. The conditional expectation (see Eq. (36)) is effectively estimated based on a single sample from the conditional portfolio loss distribution associated with each systemic scenario (i.e., the entire simulation consists of M samples, where M is the number of systemic scenarios).
- Glasserman (2005) applies IS jointly to the systemic factors (shifting both their means and covariances) and specific factors (exponential tilting) to compute VaR and ES contributions. He also derives an analytical approximation that, instead of sampling from the shifted distributions, computes the conditional expectation directly.

6 Case studies

We now present several examples that demonstrate the use of marginal risk contributions for capital allocation and highlight some of the practical issues involved. Specifically, the intent is to illustrate the key properties of risk contributions, the management implications of using various risk measures, and the related numerical issues. We consider the following cases:

[20] Although they consider a default only model, the exponential twisting can be generalized for credit migration losses as well.

- The first example shows the behavior of VaR and ES contributions obtained using a one-factor model and a granular portfolio. It demonstrates that the choice of the quantile can have a significant impact on the capital allocation.
- The second example analyzes the impact of diversification from a multi-factor model on the portfolio capital and the capital contributions. It shows the sensitivity of the marginal allocations to the size of their components and the level of diversification.
- The third example analyzes an international bond portfolio with simulation. It shows how the discrete nature of the issuer credit losses makes it difficult to compute risk contributions and uses L-estimators to mitigate these effects.
- The final example compares the risk contributions of the bond portfolio based on volatility, VaR and ES measures. We show how volatility-based contributions can lead to an inefficient allocation of capital and discuss management implications.

6.1 Risk contributions in a one-factor credit model – impact of quantile

The risk measure used to define capital and measure risk contributions can have a significant impact on capital allocation decisions. We now illustrate the sensitivity of the capital allocation to the confidence level (quantile) when using VaR-type measures. Similar effects are observed when using expected shortfall.

Consider, as first example, the credit portfolio described in Table 1. It consists of ten homogeneous pools or sectors, each containing a very large number of obligors. The portfolio weights are uniform, with each sector contributing to 10% of the total exposure. Without loss of generality, we apply 100% LGD to all sectors. We model portfolio losses using a one-factor Merton model, and assign uniformly an asset correlation of 15% (this is consistent, for example, with mortgage portfolios in Basel II). For modeling purposes we assume that the portfolio is infinitely granular, and only susceptible to systemic risk.

The expected losses in the portfolio are 3.5% of the total exposure and, given that sectors are all the same size, their EL contributions are proportional to their PD. The VaR losses of this portfolio are obtained through the closed form expression (15) and the sector contributions are portfolio invariant. The 99.9% portfolio losses are over 19% and the total capital is just short of 16%. The first four sectors contribute to almost 80% of the VaR.[21] Three sectors contribute to almost 86% of EL and 72% of VaR.

Figure 2 shows the portfolio losses in the tail of the distribution. The maximum loss at a confidence level of 100% is the total exposure of 100. The figure

[21] We focus hereon on the VaR contributions, which also include EL contributions, but similar conclusions can be drawn for capital (as defined by unexpected losses only) contributions.

Table 1.
Portfolio Description (Uniform Exposures)

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	10	100%	11.00%	0.15	31.3%	25.2%
2	10	100%	10.00%	0.15	28.4%	24.0%
3	10	100%	9.00%	0.15	25.6%	22.7%
4	10	100%	2.00%	0.15	5.7%	9.1%
5	10	100%	1.50%	0.15	4.3%	7.5%
6	10	100%	1.00%	0.15	2.8%	5.7%
7	10	100%	0.30%	0.15	0.9%	2.4%
8	10	100%	0.20%	0.15	0.6%	1.8%
9	10	100%	0.10%	0.15	0.3%	1.0%
10	10	100%	0.05%	0.15	0.1%	0.6%
Total	100				3.5	19.3

Capital = 15.8.

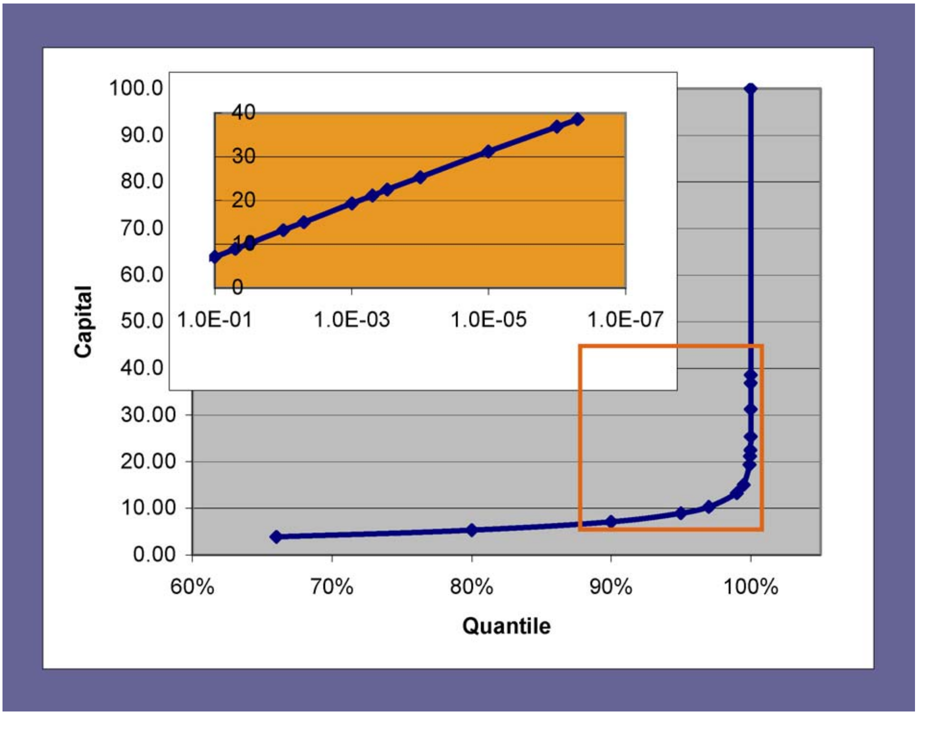

Fig. 2. Tail of Portfolio Loss Distribution (uniform exposures).

further zooms in on the tail from the 90%–99.99995%, on a log scale (where an exponential law describes well the losses in that range of the tail).

The quantile chosen can have a substantial impact on the capital allocation. This is illustrated in Fig. 3, which gives the VaR contributions as functions of the quantile (and tabulates these contributions for several quantiles). We can make the following observations from Fig. 3:

- Since all counterparties have equal exposure, at a confidence level of 100%, every sector contributes to one tenth of the losses, regardless of their credit quality. PD influences loss exposures however at all other confidence levels.

	Quantile					
	90%	99%	99.9%	99.99%	99.999%	100%
1	30.1%	27.4%	25.2%	23.3%	21.7%	10%
2	27.7%	25.7%	24.0%	22.4%	21.0%	10%
3	25.3%	24.0%	22.7%	21.4%	20.2%	10%
4	6.4%	8.0%	9.1%	10.0%	10.6%	10%
5	4.9%	6.4%	7.5%	8.5%	9.2%	10%
6	3.3%	4.6%	5.7%	6.6%	7.4%	10%
7	1.0%	1.7%	2.4%	3.1%	3.8%	10%
8	0.7%	1.2%	1.8%	2.3%	2.9%	10%
9	0.3%	0.7%	1.0%	1.5%	1.9%	10%
10	0.2%	0.4%	0.6%	0.9%	1.2%	10%
1/CDI	4.2	4.7	5.2	5.8	6.2	10.0

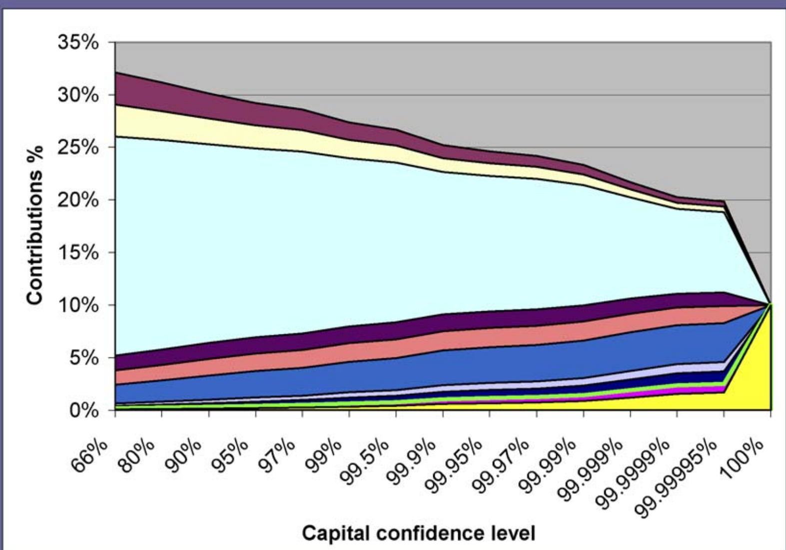

Fig. 3. Risk Contributions in the Tail of the Distribution (uniform exposures).

- At the 66% level, the three sectors with the highest PD (lowest credit quality) contribute 87% of the VaR. This goes down to 72% at a 99.9% level (the one used in Basel II) and to less than 63% for a 99.999% level.
- In general, as the quantile increases, the capital attributed to the low quality sectors is shifted to high quality sectors. For example, the shift in quantile from 99% to 99.99% results almost an almost 10% capital reallocation from low to high quality sectors.

The table also gives the inverse of the CDI (as defined in Eq. (28)), which essentially gives an "effective number of sectors" as accounted for their capital contributions. This is a simple summary measure that allows us to see the overall impact of the quantile on the capital allocation. At a 90% level the portfolio shows 4.2 effective sectors. This increases to 5.2 at the 99.9% level, 6.2 at 9.999% and 10 at 100% (which is the number of effective sectors as seen from an exposure perspective).

In this example, only the credit quality (PD) was varied across sectors. While the risk attributed to different sectors changed with the quantile, the ranking of sectors by risk contribution remains the same under all quantiles. This is not the case, however, when sectors vary across other dimensions as well. Consider now the portfolio in Table 2.

The total exposure and the distribution of PDs are the same as in the previous case (as are the losses at 100% level). However, both EL and capital at the 99.9% level are much smaller, since the exposures are in this case distributed proportionately to the credit quality (as is often the case in balanced portfolios).

The impact of the quantile chosen on the capital allocation is more complex in this case, due to the opposing effects of the distributions of credit quality and

Table 2.
Portfolio Description (Non-uniform Exposures)

Sector	EAD	LGD	PD	Corr	EL	VaR (99.9%)
1	2	100%	11.00%	0.15	25.9%	16.2%
2	2	100%	10.00%	0.15	23.6%	15.4%
3	2	100%	9.00%	0.15	21.2%	14.6%
4	2	100%	2.00%	0.15	4.7%	5.9%
5	5	100%	1.50%	0.15	8.8%	12.1%
6	5	100%	1.00%	0.15	5.9%	9.2%
7	5	100%	0.30%	0.15	1.8%	3.8%
8	10	100%	0.20%	0.15	2.4%	5.7%
9	30	100%	0.10%	0.15	3.5%	10.0%
10	37	100%	0.05%	0.15	2.2%	7.1%
Total	100				0.85	6.0

Capital = 5.2.

	Quantile						
	90%	99%	99.9%	99.99%	99.999%	99.9999%	100%
1	24.2%	19.6%	16.2%	13.5%	11.4%	9.7%	2%
2	22.3%	18.4%	15.4%	13.0%	11.1%	9.5%	2%
3	20.3%	17.2%	14.6%	12.4%	10.6%	9.2%	2%
4	5.2%	5.7%	5.9%	5.8%	5.6%	5.3%	2%
5	9.8%	11.4%	12.1%	12.3%	12.1%	11.7%	5%
6	6.7%	8.3%	9.2%	9.6%	9.8%	9.7%	5%
7	2.1%	3.1%	3.8%	4.5%	4.9%	5.3%	5%
8	2.8%	4.3%	5.7%	6.8%	7.7%	8.5%	10%
9	4.2%	7.1%	10.0%	12.6%	15.0%	17.1%	30%
10	2.6%	4.8%	7.1%	9.5%	11.8%	13.9%	37%
1/CDI	5.9	7.4	8.5	9.1	9.2	9.0	4.1

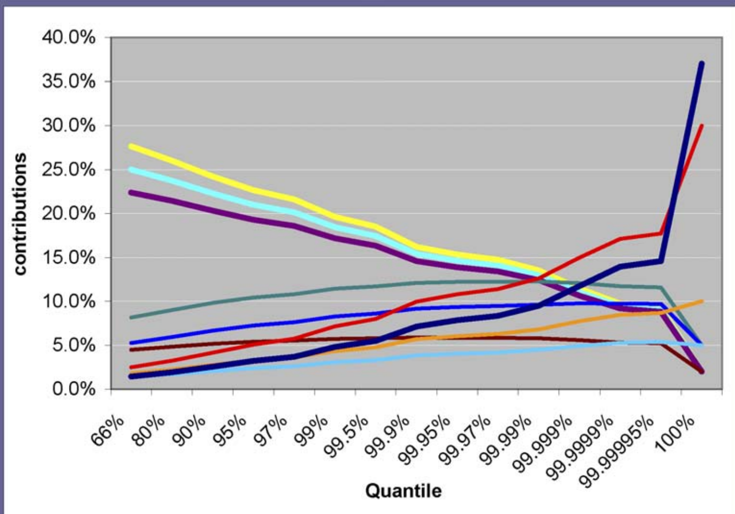

Fig. 4. Risk Contributions in the Tail of the Distribution (non-uniform exposures).

exposure sizes. This is shown in Fig. 4. We can make the following observations on this portfolio:

- The three sectors with the lowest creditworthiness are the biggest consumers of capital at lower confidence levels. At a 66% level they account for 75% of losses. This number goes down to 55% and then to 39%, at 99% and 99.99% levels respectively. At high confidence levels, these sectors are not even the highest contributors. At a 100% level, the three biggest sectors (which are also the best credits) account for 77% of the capital.

- The ranking of the sectors based on their capital consumption changes with the quantile. This can be seen by the lines intersecting in the graph in Fig. 4 and also from the adjacent table.
- The effective number of sectors (inverse of the CDI) is not a monotonic function of the quantile. The effective number of sectors reaches its peak around the 99.999% level and then goes down again to slightly over four at the 100% level.

We can think of the CDI as a measure of the "dispersion" of the risk contribution profiles in the plot at a given quantile. Thus the minimum occurs at the quantile where the dispersion is the smallest. This is basically the quantile at which the portfolio looks the most diversified. If all the capital contributions are the same for each sector, the effective number of sectors is the maximum and coincides with the actual number of sectors (in this case 10).

6.2 The diversification factor and capital contributions

Consider again the portfolio in Table 1, in the previous section, consisting of ten homogeneous, granular, sectors with uniform weights (each contributing 10% of the total exposure). Using a one-factor credit model, the first three sectors (with high PDs) contribute over 72% of the losses at the 99.9% level, and almost 69% of the capital (given their high EL contributions).

Now, assume that the portfolio is driven by the multi-factor model given by Eqs. (22) and (23). Each sector is driven by a single factor, with inter-sector correlation of 15% as in the previous example. Furthermore, assume that all sectors have the same correlation level to the single systemic factor (Eq. (23)). Figure 5 summarizes the capital and allocations, assuming intra-sector correlation levels of 100%, 60% and 40%.

The capital diversification index (CDI) for the portfolio is 0.18, which implies 5.6 effective sectors (the Herfindahl index on the exposures is 0.1). The one factor model corresponds to the case when all the sectors have correlation of 100% and hence the diversification factor (DF) is 100%. Correlations of 60% and 40%, results in 27% and 60% lower capital, respectively (DFs of 73.2% and 40%).

The last three columns (and the graph) give the capital allocations for the different correlation values. In the one factor model (100% correlation), each sector contributes its stand-alone capital. In the presence of diversification, Eq. (27) shows that each sector's *marginal diversification factor*, DF_k, depends also on the relative size of the sector (in terms of its stand-alone capital) and the relative intra-sector correlation (which in this example is the same for all sectors). Smaller sectors contribute more to the overall diversification and get percentage capital allocations smaller than their corresponding stand-alone contributions. The bigger portfolios, in contrast, get bigger contributions (percent-wise). This effect grows with the level of diversification, as can be seen from the figure (i.e. the smaller the correlation the higher this effect). Thus,

				Capital %		
Positions	**EAD**	**EL%**	**VaR %**	**Corr=100%**	**Corr=60%**	**Corr=40%**
1	10%	31.3%	25.2%	23.9%	25.1%	26.7%
2	10%	28.4%	24.0%	23.0%	24.0%	25.3%
3	10%	25.6%	22.7%	22.0%	22.8%	23.8%
4	10%	5.7%	9.1%	9.9%	9.2%	8.3%
5	10%	4.3%	7.5%	8.3%	7.5%	6.6%
6	10%	2.8%	5.7%	6.3%	5.7%	4.8%
7	10%	0.9%	2.4%	2.7%	2.4%	1.9%
8	10%	0.6%	1.8%	2.0%	1.7%	1.4%
9	10%	0.3%	1.0%	1.2%	1.0%	0.8%
10	10%	0.1%	0.6%	0.7%	0.6%	0.5%
Total	100	3.52	19.33	15.81	11.58	9.28

CDI	**0.180**
Sectors	5.6

DF	**100%**	**73.2%**	**40.0%**
Slope		0.34	0.59

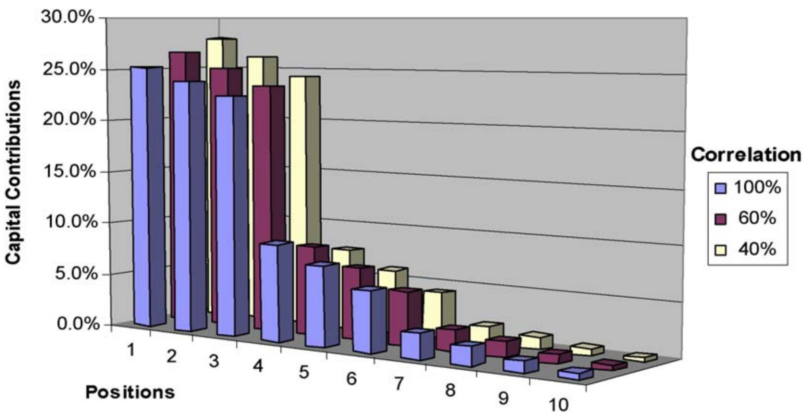

Fig. 5. Capital Contributions in a Multi-Factor Model (uniform exposures portfolio).

while the three largest portfolios contribute 69% of the capital in a one-factor model, their percent contribution grows to almost 76%.

6.3 VaR contributions, discreteness of loss distributions and L-estimators

We now analyze the credit risk of a portfolio of emerging markets debt under a multi-factor model. The portfolio, comprising 197 long-dated corporate and sovereign bonds issued by 86 obligors, has a mark-to-market value of 8.3 billion USD and a duration of approximately five years. The credit model considers both default and MtM losses. Credit migrations for each obligor occur among eight possible credit states, including a terminal default state, with transition probabilities specified by a Standard & Poor's transition matrix. The co-dependence structure is defined by a multi-factor model of asset returns (for details, see Bucay and Rosen, 1999). For illustration purposes only, and to keep the example simple, we compute the portfolio credit loss distribution using a Monte Carlo simulation with 20,000 scenarios.[22]

Figure 6 shows the VaR and ES for a range of quantile levels, as obtained by the UECV and HD estimators.[23] Both estimators yield virtually identical results.

When applied to risk contributions, the UECV and HD estimators give similar results for ES but not for VaR. For example, Fig. 7 shows the risk attributed to Brazilian debt. While the HD estimator consistently identifies Brazil as a significant source of risk under both measures, the VaR contributions produced by the UECV estimator are erratic and, in fact, frequently fail to attribute any risk to Brazil.

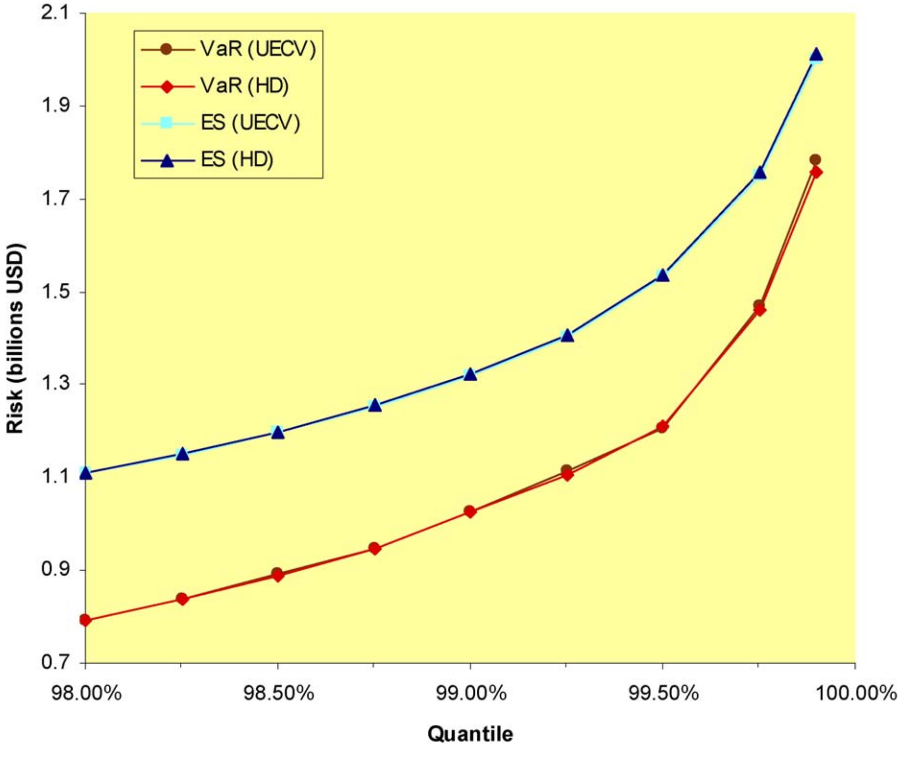

Fig. 6. Risk of Bond Portfolio.

[22] The scenario set might be relatively small for estimating accurately extreme tails (e.g. 99.9%). In practice, one would use a larger number of scenarios, or enhanced techniques such as Quasi-MC methods or importance sampling.

[23] Weights less than 10^{-6} are set to zero in our analysis.

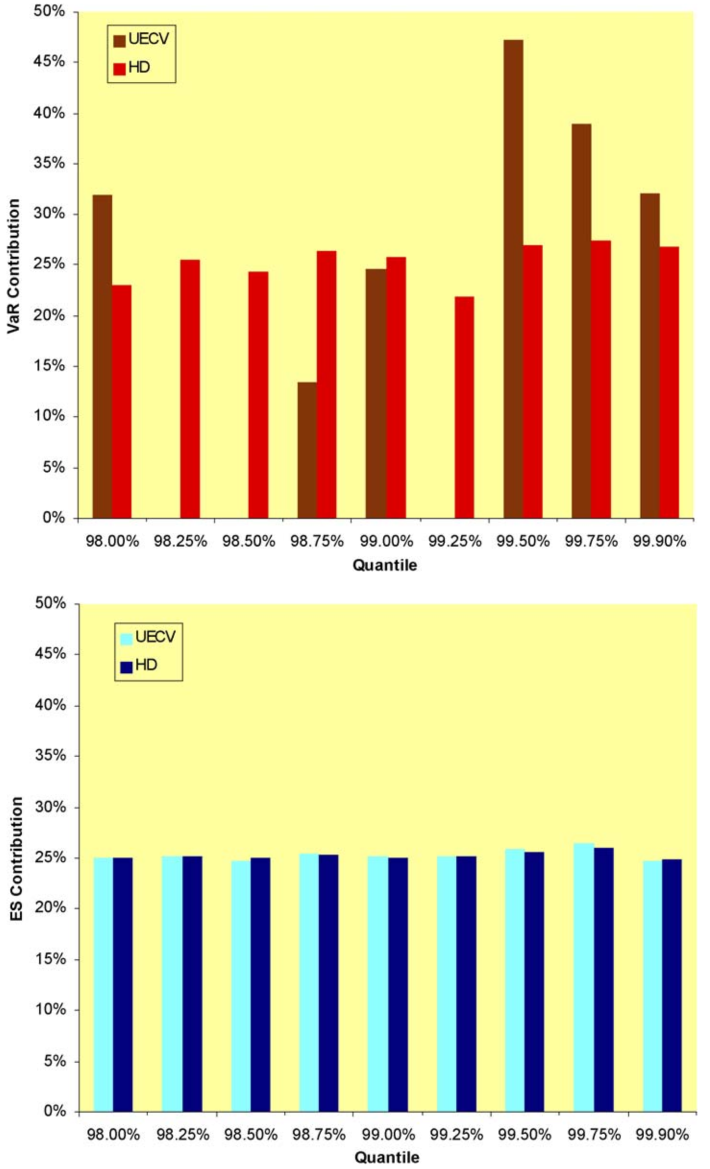

Fig. 7. Risk Contributions for Brazil.

Figure 8 shows the 400 largest portfolio losses in the sample (i.e., the tail of the empirical portfolio loss distribution beyond the 98% quantile) and the component losses due to Brazilian bonds. While the portfolio loss profile is relatively smooth, the Brazilian losses often change drastically from one order statistic to the next. (Note that under the chosen model, an obligor incurs one

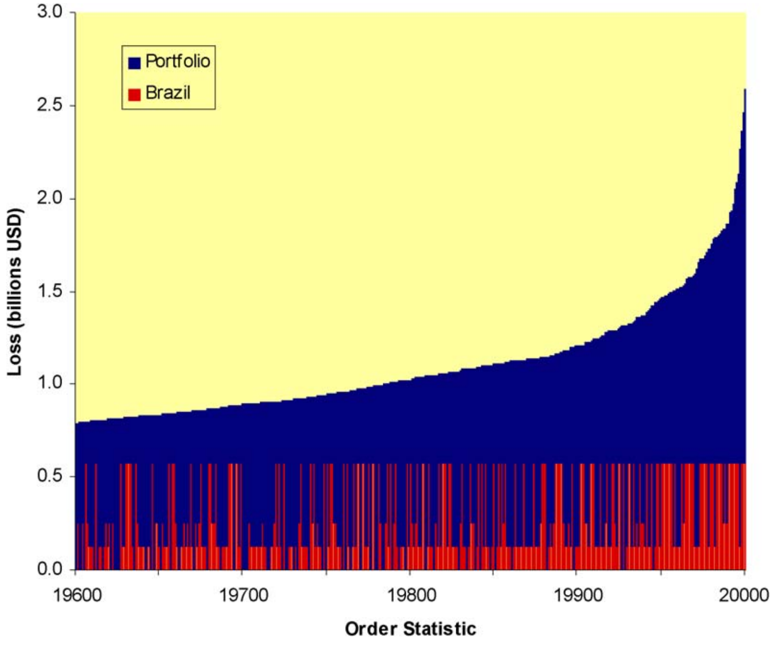

Fig. 8. Brazilian Losses Relative to Portfolio Losses.

of eight possible losses, corresponding to the eight possible credit states, in each scenario.)

In practice, given a sufficiently large sample, a quantile estimator that is based on a single order statistic (e.g., UECV) produces a robust portfolio VaR estimate. Moreover, since the smoothness of the portfolio loss distribution increases with sample size, the accuracy of the VaR estimate improves accordingly. In contrast, UECV estimates of the VaR contributions are unreliable and, since the smoothness of the obligor loss profile is unaffected by sample size, their accuracy cannot necessarily be improved by increasing the number of scenarios.

This property has significant implications for typical credit risk models, in which a loss is triggered by an obligor's default or, more generally, by its transition to a lower credit grade. Since there is a relatively high probability of an individual obligor retaining its original credit rating, many obligors do not incur a loss in a given scenario. As a result, the UECV estimator tends to report an excessive number of zero VaR contributions.

In contrast, ES is defined as an average of the losses spanning a range of order statistics. Since averaging has the effect of smoothing the obligor loss profile, the UECV estimator is generally reliable for ES contributions.

The HD estimator applies a similar averaging approach to VaR estimation. While this improves the stability of the HD VaR contributions, they still show greater variability than the ES contributions (e.g., the dip at the 99.25%

level in Fig. 7). This is due to the different weighting schemes: the ES weights are distributed more or less equally among order statistics beyond the sample quantile, while the VaR weights are focused on a smaller "window" of order statistics around the sample quantile. Since larger samples result in the weights extending over a greater number of order statistics, increasing the sample size produces more reliable estimates of both VaR and ES contributions.

Figure 9 illustrates the estimation of Brazil's contribution to ES and VaR, respectively, at the 99.25% level (the contribution is denoted by the large icon above order statistic 19,850 in each case). The graphs show the relative contribution of Brazil to each of the 400 largest portfolio losses (i.e., the Brazilian loss component divided by the total portfolio loss), shaded to reflect the size of its corresponding estimator weight. The ES contribution is essentially a weighted average of the loss contributions associated with the 200 largest portfolio losses. In contrast, the VaR estimation weights are distributed among order statistics 19,790 through 19,900, with the largest weights surrounding order statistic 19,850.

6.4 Comparing quantile-based and volatility contributions

Assigning capital based on volatility allocation is problematic since an obligor's contribution to volatility may fail to represent the tail of the distribution (e.g., Praschnik et al., 2001; Kurth and Tasche, 2003; Kalkbrener et al., 2004). It is important to understand that risk contributions vary across measures, and also across different confidence levels. This is apparent in Fig. 10, which plots, for the top six obligors, the ranges of the tail-based (VaR and ES) risk contributions, for quantile levels between 98 and 99.9%, against volatility contributions. For example, Brazil contributes between 21.9 and 27.4% of the VaR and between 24.8 and 26.0% of the ES, but accounts for only 20.6% of the volatility.

In this case, the tail-based risk contributions consistently exceed the volatility contribution for the two largest contributors, Brazil and Russia. Moreover, the rankings of Russia and Venezuela are reversed when based on volatility. Also, the range of VaR contributions is typically larger than that for ES. As discussed previously, this might also reflect the relative lack of precision (i.e., a greater sensitivity to random error) on the part of the HD estimator in the former case.

7 Summary and further research

Capital allocation is an important management decision support and business planning tool for financial institutions, which is required for pricing, profitability assessment and limits, building optimal risk-return portfolios and strategies, performance measurement and risk based compensation. This chapter provides a practical overview of the measurement of economic credit capital contributions and their application to capital allocation. We discuss the

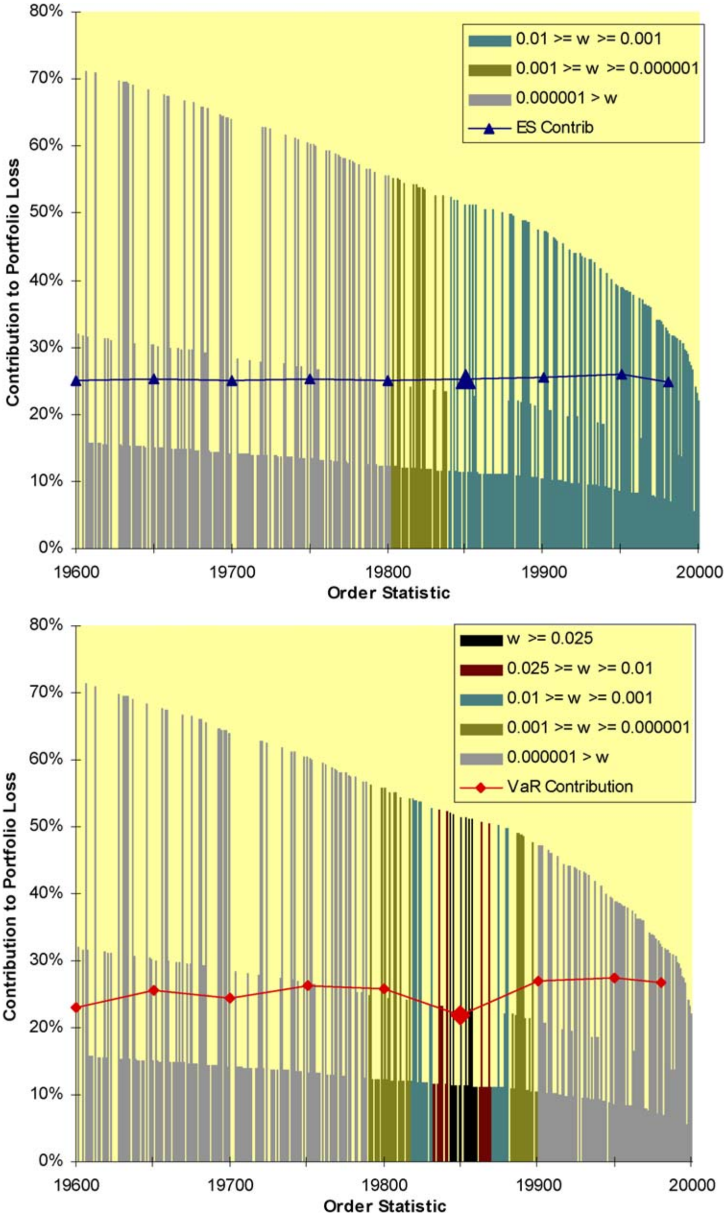

Fig. 9. Estimation of ES and VaR Contributions for Brazil.

advantages and disadvantages of various risk measures and models, the interpretation of various allocation strategies as well as the numerical issues associated with this task.

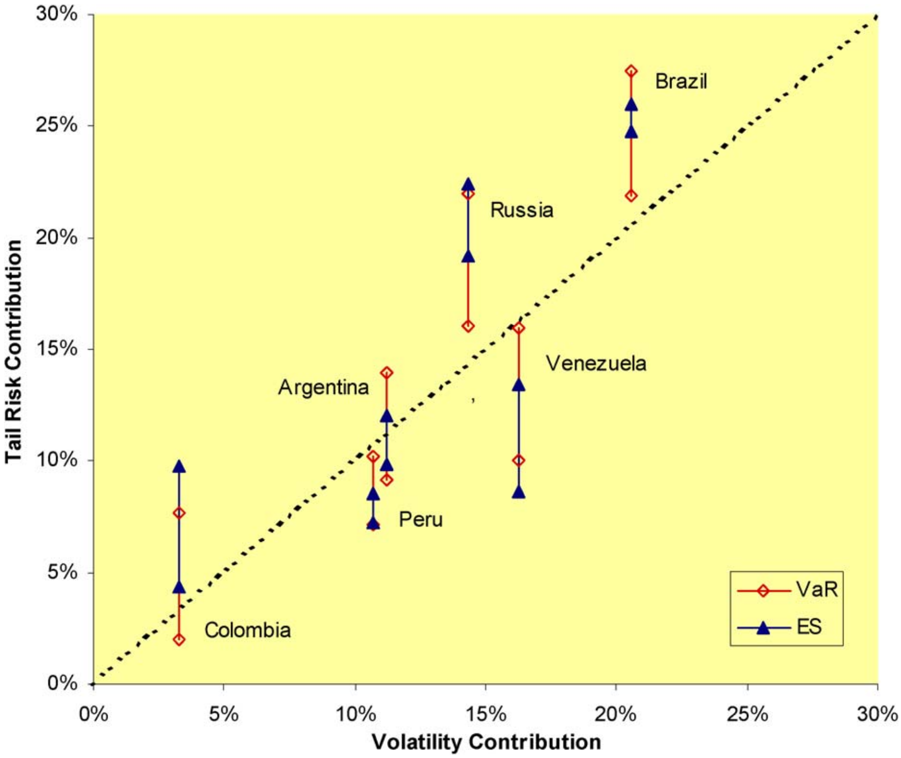

Fig. 10. Most Significant Risk Contributors.

Our key points may be summarized as follows:

- Marginal risk contributions provide a useful basis for allocating EC since they are additive and reflect the benefits of diversification within a portfolio.
- The choice of the risk measure can have a substantial impact on capital allocation. VaR and ES contributions avoid the inconsistencies, and potentially inefficient allocations, associated with the widely-used volatility-based method for EC allocation. The quantile level chosen for measuring risk can also have a significant impact on the relative amount of capital allocated to portfolio components.
- VaR and ES contributions can be calculated analytically under certain simple models (e.g. the Basel II model and several extensions). In addition to providing fast calculations, these models can be used effectively to get a deeper understanding of the behavior of risk contributions. However, these models may present important practical limitations. For example, one-factor, systemic risk models result in linear allocation strategies which are portfolio invariant. Modeling in more detail non-granular portfolios and diversification through multi-factor mod-

els provides a richer picture of diversification and results in more realistic capital allocation strategies.
- Sophisticated credit models that capture the behavior of portfolios in a more realistic manner may entail the use of Monte Carlo simulation for assessing risk. Computing VaR and ES contributions is challenging in this case, specially at the extreme quantiles typically used for credit capital definition. The quality of contribution estimates can be improved in three ways. First, the conditional independence framework can be exploited by using advanced methods to perform the convolution of independent random variables. Second, the use of L-estimators provides more stable contributions, especially for VaR. Finally, the use of variance reduction techniques, such as Importance Sampling can be used effectively to get more accurate contributions at high quantiles in the tails.

Several practical problems relevant to capital allocation are currently the focus of research. These include:

- Statistical estimation of multi-factor models as well as their impact on capital allocation and concentration risk. In addition to requiring accurate PDs, exposures and LGDs, capital allocation methods are very sensitive to the correlations of credit events built into the model. Empirical work is important to understand the relative impact of systematic and idiosyncratic risk, as well as the relationship between economic factors and credit events (e.g. Wendin and McNeil, 2005)
- Consistent capital contributions in large portfolios (and small positions). How can risk contributions be measured accurately when contributions are very small (e.g. for very large portfolios)? In some cases, the size of the risk contribution may be smaller than the error range of the estimator (and the parameters used in model). Examples include retail portfolio with millions of transactions, large corporate portfolios or enterprise portfolios. A practical solution may include the application of a simpler (calibrated) analytical model or the use of a consistent hierarchical methodology to allocate contributions through a large portfolio (e.g. using basic properties of granular, homogeneous portfolios).
- Real-time marginal capital calculations. How can marginal capital be computed consistently for a new loan or transaction (accounting properly for diversification) in "real-time"? In this case, full simulation is typically not an option, although performance might be improved with some semi-analytical models. Some practical solutions may include the application of a simpler analytical model (calibrated to a full economic capital model) as given in Garcia Cespedes et al. (2006). In order to gain acceptance, such a model should be intuitive, based on a small number of parameters and recalibrated frequently over time.

- Contributions of systemic factors. While this chapter has considered the risk contributions of portfolio components, a practitioner may also want to know the contribution to economic capital of the various systemic factors (credit drivers) that are at the heart of a multi-factor economic capital model (such as KMV or CreditMetrics). Such factors explain only the systemic portion of the portfolio's total risk.[24] In addition, the standard theory of marginal capital contributions does not work well since the total capital is not a homogeneous function of these factors. Finally, the most interesting cases, in practice, require simulation of the multi-factor models. For further discussion of systemic risk factor contributions and hedging techniques, see Rosen and Saunders (2006a, 2006b).

Appendix A

A.1 The Harrell–Davis estimator

The Harrell–Davis (HD) estimator derives from the fact that, for $0 < \alpha < 1$, the expected value of order statistic $(M+1)\alpha$ converges to $F^{-1}(\alpha)$ as the sample size increases.[25] Thus, the HD estimator computes VaR_α as $\mathrm{E}[L^{((M+1)\alpha)}]$, regardless of the integrality of $(M+1)\alpha$. The resulting weights are

$$\begin{aligned} w^{VaR}_{\alpha,M,k} &= \frac{1}{\beta[(M+1)\alpha, (M+1)(1-\alpha)]} \\ &\quad \times \int\limits_{(k-1)/M}^{k/M} y^{(M+1)\alpha-1}(1-y)^{(M+1)(1-\alpha)-1}\,\mathrm{d}y \\ &= I_{k/M}\big[(M+1)\alpha, (M+1)(1-\alpha)\big] \\ &\quad - I_{(k-1)/M}\big[(M+1)\alpha, (M+1)(1-\alpha)\big] \end{aligned} \tag{39}$$

where $I_X(a,b)$ is the incomplete beta function. Figure 11 compares the weights of the HD and UECV estimators for computing the 95% VaR from a sample of size 100. A similar comparison for the 95% ES is shown in Fig. 12.

[24] Furthermore, a linear combination of these factors may only explain a portion of the systemic risk (see Rosen and Saunders, 2006a).

[25] In practice, $L^{((M+1)\alpha)}$ is computed as a weighted average of order statistics $L^{(\lfloor (M+1)\alpha \rfloor)}$ and $L^{(\lceil (M+1)\alpha \rceil)}$.

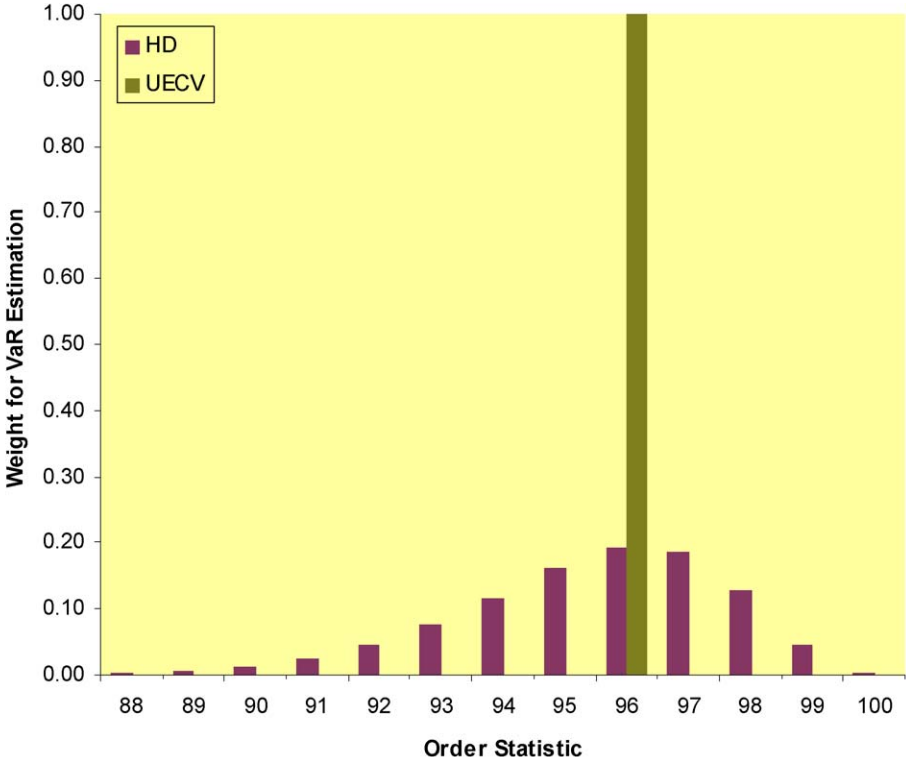

Fig. 11. Weights for Estimating 95% VaR when $S = 100$.

A.2 VaR and ES contributions with CLT in conditional independence models

If conditional losses are Normal, the tail probability of portfolio losses is given by

$$\Pr(L < y) = \frac{1}{M} \sum_{m=1}^{M} \Phi\left(\frac{y - \mu^m}{\sigma^m}\right) \tag{40}$$

where $\Phi(\cdot)$ denotes the cumulative standard Normal probability function,

$$\mu^m = \sum_{j=1}^{N} \mu_j^m \quad \text{and} \quad (\sigma^m)^2 = \sum_{j=1}^{N} (\sigma_j^m)^2$$

with the mean and variance of the individual obligor losses L_i^m (see Eq. (3))

$$\mu_j^m = \sum_{r=1}^{R} c_{jr}^m p_{jr}^m \quad \text{and} \quad (\sigma_j^m)^2 = \sum_{r=1}^{R} p_{jr}^m (c_{jr}^m - \mu_j^m)^2.$$

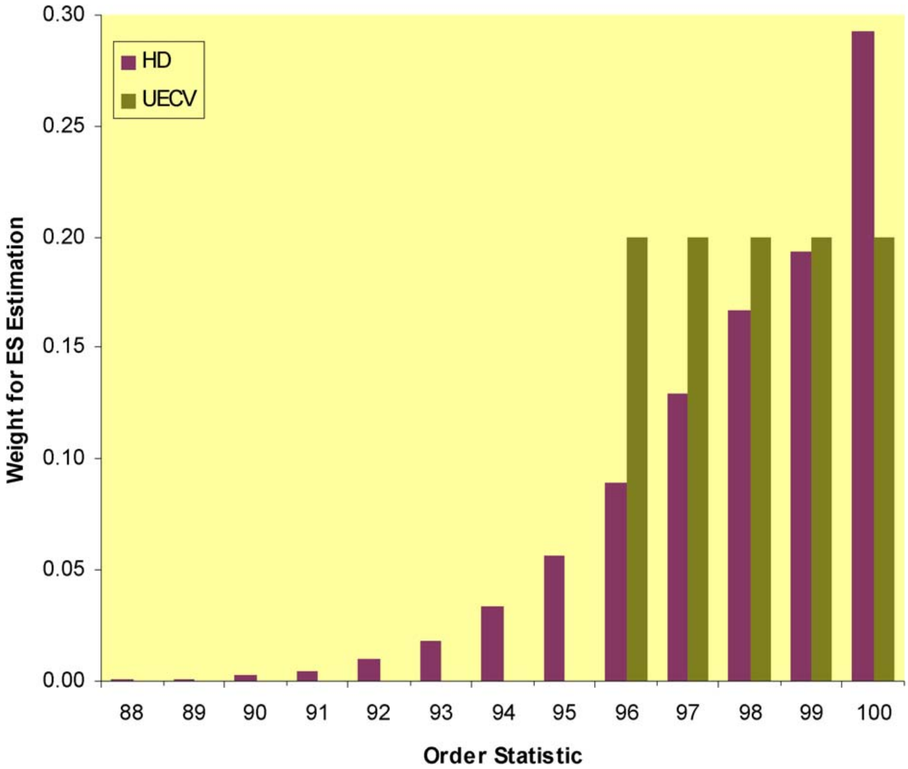

Fig. 12. Weights for Estimating 95% ES when $S = 100$.

Equivalently, if we denote by $\Phi_{\mu,\sigma}$, the $N(\mu, \sigma^2)$ cumulative distribution (with $\varphi_{\mu,\sigma}$, the density functions), we can write Eq. (40) in terms of the estimated $VaR_\alpha(L)$ as

$$\frac{1}{M}\sum_{m=1}^{M}\Phi_{\mu^m,\sigma^m}\bigl(\overline{VaR}_\alpha(L)\bigr) = \alpha. \tag{41}$$

Analytical expressions are obtained in this case for the VaR and ES contributions of a given obligor by computing the conditional expectations or taking the derivative of VaR from Eq. (41) (see, for example, Kreinin and Mausser, 2003; Martin, 2004):

$$\begin{aligned} &E\bigl[L_j \mid L = \overline{VaR}_\alpha\bigr] \\ &\quad = \frac{1}{\varphi_{\mu,\sigma}(\overline{VaR}_\alpha)}\sum_{m=1}^{M}\frac{\varphi_{\mu^m,\sigma^m}(\overline{VaR}_\alpha)}{M}\left(\mu_j^m + \frac{(\sigma_j^m)^2}{\sigma^m}Z_\alpha^m\right) \end{aligned} \tag{42}$$

and

$$E\left[L_j \mid L \geqslant \overline{VaR}_\alpha\right] = \frac{1}{1-\alpha}\sum_{m=1}^{M}\frac{1}{M}\left[\mu_j^m\left(1-\Phi_{0,1}\left(Z_\alpha^m\right)\right)+\frac{(\sigma_j^m)^2}{\sigma^m}\varphi_{0,1}\left(Z_\alpha^m\right)\right] \tag{43}$$

where

$$Z_\alpha^m = \frac{\overline{VaR}_\alpha - \mu^m}{\sigma^m}.$$

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