

# Effective modeling of wrong way risk, counterparty credit risk capital, and alpha in Basel II

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*One of the critical issues in the Basel II internal ratings based method for counterparty credit risk (CCR) is the calculation of exposure at default, which requires estimation of a parameter called the alpha multiplier. A major challenge in calculating the alpha multiplier is the modeling of wrong way risk (ie, correlation between exposures and defaults in a credit portfolio). We present a computationally efficient approach to modeling wrong way risk and estimating CCR capital and alpha. The methodology fully leverages existing counterparty exposure simulations used for risk management and credit limits, and preserves the joint distribution of counterparty exposures. Although the methodology can be applied with general integrated market-credit risk models, we show that a simplified model to correlate directly the (precomputed) exposures with credit events leads to a parsimonious, computationally tractable approach, which is easy to implement and consistent with the Basel II definition and credit portfolio model. To assess the impact of wrong way risk and for regulatory applications, alpha is defined and*

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*plotted as a function of the correlation between exposures and defaults. This leads to an intuitive numerical solution for the inverse problem of finding the level of market-credit correlation that hits the regulatory floor of 1.2. Several market factors driving counterparty exposures can also be considered to stress the market-credit dependence structure. An analysis of a realistic trading book is used to demonstrate the methodology and its application within the regulatory framework.*

## 1 INTRODUCTION

*Counterparty credit risk (CCR)* is the risk that the counterparty defaults before the final settlement of a transaction's cashflows. An economic loss occurs if the counterparty portfolio has a positive economic value at the time of default. Unlike a loan, where only the lending bank faces the risk of loss, CCR creates a bilateral risk: the market value can be positive or negative to either counterparty and can vary over time with the underlying market factors.

The Basel II Accord (BCBS (2006)) allows banks to use an internal ratings based approach to compute minimum capital requirements for CCR of derivatives portfolios. The capital charges are determined through a risk-weight formula, which uses four quantitative inputs provided by the bank: the *probability of default (PD)*, *exposure at default (EAD)*, *loss given default (LGD)* and *maturity (M)*. For derivatives portfolios, Basel II originally outlined the same treatment of *EADs* as in the Basel I Accord, through the "mark-to-market plus add-on" method (BCBS (1995)). The final Basel II Accord responded to industry comments and allows for internal models for *EADs* and *M*, based on the concepts of *expected positive exposure (EPE)*, *effective EPE*, *effective maturity* and the *alpha multiplier*.

The definition of a portfolio's alpha is below; economic capital is denoted as *EC*:

$$\alpha = \frac{EC^{\text{Total}}}{EC^{\text{EPE}}} \quad (1)$$

where  $EC^{\text{Total}}$  is the economic capital for CCR from a joint simulation of market and credit risk factors and  $EC^{\text{EPE}}$  is the economic capital when counterparty exposures are deterministic and equal to *EPE*.

Alpha provides a means to condition internal *EPE* estimates on a "bad state" of the economy, and to adjust internal *EPEs* for:

- the uncertainty of, and correlation between, counterparty exposures (and *LGDs*);
- the correlation between counterparty exposures and defaults;
- the lack of granularity across the portfolio.

In addition, regulators also view alpha as a means to offset model and estimation errors. For a more detailed discussion of the motivation for, and use of, alpha we refer to Fleck and Schmidt (2005), Picoult (2005) and Wilde (2005).

Industry and supervisors' numerical exercises suggest that alpha may range from 1.1 for large global dealer portfolios to over 2.5 for new users with concentrated exposures and little current exposure (BCBS (2005); Canabarro *et al* (2003); Wilde (2005); Fleck and Schmidt (2005)). Basel II defines a supervisory alpha of 1.4, but gives banks the option to estimate their own alpha, subject to a floor of 1.2. Supervisors originally recognized the arguments why a floor of 1.2 may be too high, but they chose to keep this floor, given the limited experience of validating modeled values of alpha. However, the credit crisis has also resulted in an increased awareness of systemic risk, and of the dangers of wrong way risk, which naturally lead to higher values of alpha.

Canabarro *et al* (2003) present a systematic study of alpha under the assumption of independent exposures and defaults using simulation and analytical techniques. Wilde (2005) and Fleck and Schmidt (2005) also present results using a 20% market-credit correlation, estimated from regressing *PDs* on interest rate levels using Moody's data. In Wilde (2005), the base case alpha increases to 1.15–1.21. On realistic portfolios the authors find a less pronounced effect. Wilde (2005) further suggests that credit default swaps should have some form of wrong way risk (as the exposures are correlated to the credit driver) and therefore can generate higher alphas (comparable alpha increases to 1.25–1.27).

In practice, there are major challenges for estimating the underlying joint market and credit risk factor processes driving counterparty losses, and in particular for modeling accurately the co-dependence between exposures and defaults ("market-credit correlations" or "wrong way and right way risk"). Thus, we require efficient algorithms for computing capital, and methodologies for understanding the limitations of the underlying models used and their impact on the estimated capital. The implementation of integrated market and credit risk simulation models from the bottom up (see, for example, Iscoe *et al* (1999)) is conceptually straightforward but presents significant estimation and validation difficulties and can be computationally intensive. Given the computational complexity, stress-testing of the results and estimation of sensitivities to the model parameters, such as market-credit correlations, are rarely performed.

In this paper, we present a general, computationally efficient approach to estimate CCR capital and alpha, and stress-test the impact of market-credit correlations. The most expensive step in the capital calculation is the simulation of exposures for all counterparties over a large number of scenarios. Typically, a bank may compute counterparty exposure measures for risk management and credit limits using a Monte Carlo simulation. A key objective of the proposed CCR capital methodology is to leverage these "precomputed" counterparty exposures into an effective credit portfolio model. Although the method can be applied with general integrated market-credit risk models, we show that in the case of a single-factor credit model (such as the Basel II model) it leads to a simplified approach, which is computationally efficient as well as easy to implement and validate. Furthermore, it leads to a systematic way to stress-test the effect of market-credit correlations and

the impact of various market factors driving the exposures. From a Basel II regulatory perspective, this allows us to define and plot alpha as a function of a “market-credit correlation”, and solve the inverse problem of finding the level of correlation that hits the regulatory floor of 1.2. This estimate can be compared with internal estimates, industry studies, and regulatory requirements, providing transparency on reasonable values for alpha.

While this paper presents the methodology in the context of calculating the alpha multiplier required in Basel II, it is in fact quite general for modeling wrong way risk, and can be applied in any context where market and credit risk must be correlated (and a pre-existing set of exposure scenarios is available). For example, another important application is the calculation of the *credit value adjustment* (CVA) for over-the-counter derivatives.<sup>1</sup> In order to calculate CVA, the bank must compute the (discounted, risk-neutral) expected credit losses for a counterparty portfolio, accounting for the correlations between the exposures and the (risk-neutral) distribution of the counterparty’s default time (as well its own default time, for bilateral CVA). The algorithm discussed in Section 3 below can use effectively a precomputed counterparty exposure profile (from the bank’s exposure and limits simulations, as mentioned above), and correlate the exposures with the obligors’ defaults. This allows an accurate calculation of CVA, as well as a methodical way to stress-test the impact of wrong way risk (and the market-credit correlations) on CVA.

The remainder of the paper is organized as follows. The next section briefly reviews background material on counterparty credit risk related to the Basel II Accord, including the basic Gaussian copula model of default risk, the definition of counterparty exposure profiles and expected positive exposure, and the calculation of economic capital and alpha. The third section presents our basic method for correlating default risk and counterparty exposures to calculate alpha. Consistent with Basel II, we focus on a single-factor credit risk model, and further discuss extensions to multifactor models. The fourth section presents an example applying the methodology to a real derivatives portfolio of a typical large financial institution. The fifth section presents conclusions. The appendix provides further technical details regarding the basic algorithm and its extensions.

## 2 CREDIT RISK MODEL, COUNTERPARTY EXPOSURES AND ALPHA

In this section we introduce the notation and provide the basic background for computing CCR capital and alpha.

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<sup>1</sup>Unilateral CVA is defined as the difference between the price of a counterparty portfolio when the potential for counterparty default is ignored and the price when the possibility of a default is included in the analysis. It is essentially the market value of counterparty default risk. Bilateral CVA takes into account both the counterparty’s and the bank’s own potential to default.

## 2.1 Basic credit portfolio model

Consider a derivatives portfolio with  $M$  counterparties indexed by  $j = 1, \dots, M$ . We focus on single-period losses due to counterparty default over a fixed horizon (eg, one year). Each counterparty has a probability of default  $PD_j$ , and the portfolio's credit loss can be written as:

$$L = \sum_{j=1}^M l_j \cdot 1_{D_j} \quad (2)$$

where  $1_{D_j}$  is the *default indicator* of obligor  $j$ , which is one if default occurs and zero otherwise, and  $l_j$  is the *realized loss at default*,  $l_j = LGD_j \cdot EAD_j$ , where  $LGD_j$  is the *loss given default* for the  $j$ th counterparty, and  $EAD_j$  is its *exposure at default*. In general,  $EAD$ s and  $LGD$ s can be either deterministic or stochastic (indeed, the ratio  $\alpha$  in Equation (1) is defined as the economic capital calculated using stochastic  $EAD$ s divided by the economic capital calculated using deterministic  $EAD$ s).

We define the credit portfolio model in the Merton–Vasicek Gaussian copula framework that underlies the Basel Accord (see, for example, Vasicek (2002); Gordy (2003)). In the general case, joint counterparty defaults are driven by a set of systematic factors, denoted by  $Z_i$ ,  $i = 1, \dots, n$ , as well as an idiosyncratic factor for each counterparty,  $\varepsilon_j$ ,  $j = 1, \dots, M$ . The factors  $Z_i$  and  $\varepsilon_j$  are independent standard normal random variables. Corresponding to a given counterparty is a random variable, its *default or creditworthiness index*:

$$Y_j = \sum_{i=1}^n \beta_{ij} \cdot Z_i + \sigma_j \cdot \varepsilon_j \quad (3)$$

where  $\beta_{ij}$  is the sensitivity of obligor  $j$  to systematic factor  $i$  and:

$$\sigma_j = \sqrt{1 - \sum_{i=1}^n \beta_{ij}^2}$$

This specification ensures that  $Y_j$  has a standard normal distribution. Counterparty  $j$  has default probability  $PD_j$  and defaults if  $Y_j \leq \Phi^{-1}(PD_j)$ , where  $\Phi$  is the standard cumulative normal distribution function, and  $\Phi^{-1}$  is its inverse. The *default indicator* of counterparty  $j$  is given by:

$$1_{D_j} = \begin{cases} 1 & \text{if } Y_j \leq \Phi^{-1}(PD_j) \\ 0 & \text{otherwise} \end{cases}$$

Note that the assumed independence of the systematic factors is not a restriction on the model. A model employing correlated factors (representing, eg, gross domestic product, interest rates, as well as sector and geographic factors) can always be transformed into one using independent factors by a simple linear transformation (see, for example, McNeil *et al* (2005)).

Conditional on the systematic factors  $Z$ , defaults are independent, and the conditional default probability for counterparty  $j$  is given by:

$$PD_j(Z) = E[1_{D_j} | Z] = \Phi\left(\frac{\Phi^{-1}(PD_j) - \sum_{i=1}^n \beta_{ij} \cdot Z_i}{\sigma_j}\right) \quad (4)$$

where  $PD_j$  denotes the unconditional default probability of obligor  $j$ . The model underlying the Basel II minimum capital requirements is a single-factor credit model ( $n = 1$ ), and in this case we employ the notation  $Z = Z_1$ ,  $\beta_{ij} = \beta_j$ .

## 2.2 Counterparty potential future exposure profiles and PFEs

In this section, we review basic exposure measures used in the Basel II Accord. For a more detailed discussion, see De Prisco and Rosen (2005) and Fleck and Schmidt (2005). Define the *counterparty exposure* as the economic loss, incurred on all outstanding transactions if the counterparty defaults, accounting for netting and collateral but unadjusted by possible recoveries. It is essentially the cost of replacing the set of contracts at the time of default. Thus, for a single position or a set of transactions within a netting agreement,  $Exposure = \max(0, \text{mark-to-market})$ . More generally, exposure can account for the complexity of netting hierarchies within a given counterparty portfolio as well as the details of mitigation techniques such as collateral and margin calls.

We are interested in both the *current exposure* and the *potential future exposure* (PFE), ie, the future changes in exposures during the contracts' lives. This is important for derivatives since their values can change substantially over time according to the state of the market. *Potential future exposures* take into account the aging of the portfolio and market factor movements, which affect contracts' future values.

The future is described by discrete sets of times and scenarios. The set of times is denoted by  $\{t_0, t_1, \dots, t_N = T\}$ , with  $t_0$  denoting today and  $T$  the capital horizon date (for simplicity, and consistent with the Basel requirements, we set  $T = 1$  year). A *scenario* is a path containing all the market information up to  $T$ . Denote the  $s$ th scenario by  $\omega_s$  and its probability by  $p_s$ ,  $s = 1, \dots, S$ . For counterparty  $j$ , the PFE at time  $t_k$  in scenario  $\omega_s$  is denoted by  $PFE_j(\omega_s, t_k)$ ,  $j = 1, \dots, M$ . We refer to the matrix of PFE values for every scenario  $\omega_s$  and time  $t_k$  as the *counterparty PFE profile*. Typically, counterparty PFE profiles are computed over 1,000–5,000 scenarios and 12–60 time steps (depending also on the simulation horizon).<sup>2</sup> Once this PFE profile is calculated, various statistical measures can be defined, such as average exposures (over time and scenarios) and peak exposures. In particular,

<sup>2</sup>Note that banks typically carry out exposure simulations for limit management, CVA and other purposes beyond regulatory capital calculation. In these applications it is usual to simulate exposures beyond one year and until the maturity of the longest transaction in the portfolio, which can often be 30–40 years hence.

the following *PFE* measures are relevant for the calculation of CCR capital and alpha: *expected exposure* (over all scenarios) at  $t_k$ :

$$EE_j(t_k) = \sum_{s=1}^S PFE_j(\omega_s, t_k) \cdot p_s \quad (5)$$

*time-averaged exposure* (for scenario  $\omega_s$  up to  $t_k$ ):

$$\mu_j^{t_k}(\omega_s) = \frac{1}{t_k} \int_0^{t_k} PFE_j(\omega_s, t) dt \quad (6)$$

*expected positive exposure*:

$$EPE_j(t_k) = \frac{1}{t_k} \int_0^{t_k} EE_j(t) dt = \sum_{s=1}^S \mu_j^{t_k}(\omega_s) \cdot p_s \quad (7)$$

*effective expected exposure*:

$$\mu_j^E(t_k) = \max_{0 \leq i \leq k} [EE_j(t_i)] = \max[\mu_j^E(t_{k-1}), EE(t_k)] \quad (8)$$

and *effective EPE*:

$$\text{Effective } EPE_j(t_k) = \frac{1}{t_k} \int_0^{t_k} \mu_j^E(t) dt \quad (9)$$

For convenience, we employ the abbreviation  $EPE_j = EPE_j(T)$ . The *EPE* is the average of the *PFE* over time and scenarios. In contrast, the *effective EPE* first computes the maximum expected exposures at a given date or any prior date, and then averages these over the capital horizon. While *EPE* is generally considered as the relevant exposure measure for economic capital, the Basel II *EAD* measures are more conservatively defined in terms of *effective EPE* over a one-year horizon. For the rest of the paper, we focus on *EPE*.

## 2.3 Regulatory CCR capital charge in Basel II

The Basel II formula for minimum capital requirements is based on a single-factor credit model as described earlier in this section, and assuming an asymptotic fine-grained portfolio (essentially a very large portfolio). The capital associated to a given counterparty portfolio is given by:

$$\begin{aligned} \text{Capital}(j) = & EAD_j \cdot LGD_j \cdot \left[ \Phi \left( \frac{\phi^{-1}(PD_j) + \beta_j \Phi^{-1}(0.999)}{\sqrt{1 - \beta_j^2}} \right) - PD_j \right] \\ & \cdot \frac{1 + (M_j - 2.5)b_j}{1 - 1.5b_j} \end{aligned} \quad (10)$$



where:

- $PD_j$  and  $LGD_j$  are the bank's internal estimates of the counterparty's default probability and the loss given default, respectively;
- the asset correlation  $\beta_j^2$  and maturity adjustment  $b$  are parameters defined in the accord;<sup>3</sup>
- $M_j$  is the *effective maturity* for the  $j$ th counterparty portfolio, given by:

$$M_j = \frac{\sum_{k=1}^{t_k \leq 1 \text{ year}} \mu_j^E(t_k) \cdot \Delta t_k \cdot D_k + \sum_{t_k > 1 \text{ year}}^{\text{maturity}} EE_j(t_k)}{\sum_{k=1}^{t_k \leq 1 \text{ year}} \mu_j^E(t_k) \cdot \Delta t_k \cdot D_k} \quad (11)$$

- $EAD_j$  is the exposure at default of the counterparty – within the final version of the accord a bank can use its own  $EAD$  internal estimates given by:

$$EAD = \alpha \cdot \text{Effective } EPE \quad (12)$$

The use of  $EPE$  as a basis for  $EAD$  is justified as follows. For very large portfolios (infinitely granular) and where  $PFEs$  are independent of each other and of default events, it can be shown that the portfolio's economic capital can be computed by a model that assumes deterministic exposures given by the  $EPEs$  ( $\alpha = 1$ ). Alpha is defined by Equation (1) and represents the deviation from this ideal case (in terms of the correlation between exposures, the correlations of exposures and credit events, and portfolio granularity).

## 2.4 Economic capital and alpha calculation

The denominator of alpha (Equation (1)) is given by the economic capital assuming that  $EADs$  are deterministic and given by the  $EPEs$  (for  $T = 1$  year). It can thus be computed by standard analytical or simulation methods. The calculation of the numerator requires a joint simulation of market and credit risk factors, which incorporates:

- the uncertainty of, and correlation between, counterparty exposures, and
- the correlation between exposures and defaults.

Let us first explain this calculation in the context of a single-step credit portfolio model as described in Section 2.1 (we later extend the model to account for defaults

<sup>3</sup>They are given by:

$$\beta_j^2 = 0.12 \left( \frac{1 - \exp(-50 \cdot PD_j)}{1 - \exp(-50)} \right) + 0.24 \cdot \left( 1 - \frac{1 - \exp(-50 \cdot PD_j)}{1 - \exp(-50)} \right)$$

$$b_j = (0.11852 - 0.05478 \cdot \ln(PD_j))^2$$



at any time  $0 \leq t \leq T$ ). Consistent with the use of  $EPE$  in the numerator of alpha, we use the *time-averaged exposures* (Equation (6)) as the single-step stochastic exposure equivalent in the loss variable. Assume a one-year horizon, and define two credit loss variables:

$$L^{EPE} = \sum_{j=1}^M EPE_j \cdot 1_{D_j} \quad (13)$$

$$L^T = \sum_{j=1}^M \mu_j^T \cdot 1_{D_j} \quad (14)$$

where  $L^{EPE}$  denotes credit losses with deterministic exposures while  $L^T$  accounts for random exposures.<sup>4</sup> For simplicity, we also assume that the  $PFE$  profiles are already adjusted by their  $LGD$ . Using (10) and (11), we can write alpha in Equation (1) as:

$$\alpha = \frac{EC(L^T)}{EC(L^{EPE})} \quad (15)$$

where  $EC(L)$  denotes the economic capital for a loss variable  $L$ .<sup>5</sup>

There are various ways of structuring a Monte Carlo simulation to compute  $EC$ . A basic algorithm proceeds as follows:

- 1) Simulate a set of joint scenarios of the market factors (affecting  $PFEs$ ), and systematic credit drivers and idiosyncratic factors (Equation (2)).
- 2) In each scenario:
  - calculate  $PFEs$  and time-averaged exposures (Equation (6)) for each counterparty;
  - determine counterparty defaults;<sup>6</sup>
  - calculate portfolio losses.
- 3) Obtain the portfolio loss distribution from all the scenarios, and compute  $EC$ .

There are several simple variants of this simulation algorithm.<sup>7</sup> However, efficient simulation algorithms for capital calculation with stochastic exposures must address

<sup>4</sup>For notational convenience, we suppress the dependence of  $\mu_j^T$  and  $L^T$  on the scenario  $\omega_s$ .

<sup>5</sup>In practice, the specification of alpha depends on the definition of  $EC$ . Typically  $EC$  is given by the  $q$ -percentile loss minus the expected loss,  $EC(L) = \text{VaR}_q(L) - EL$  (eg, Basel II for the banking book) but some definitions use VaR directly (eg, Canabarro *et al* (2003)). The  $EC$  measures may cover systematic risk only (as in the Basel II formula), or both systematic and idiosyncratic risk.

<sup>6</sup>It is also possible to calculate the exposures only for counterparties that default in a given scenario. This results in a more efficient simulation, which avoids the computation of exposures for counterparties who do not default.

<sup>7</sup>The conditional independence in the credit model can be further exploited as follows. First, the credit and market factors are simulated. Under each scenario, the distribution of conditional portfolio losses is computed using simulation or convolution techniques such as Fourier or saddle-point methods. The (unconditional) portfolio losses are obtained by averaging the conditional losses over all scenarios.

several fundamental issues:

- First, the number of Monte Carlo scenarios required to capture the 99.9% losses is very large (hundreds of thousands or millions). Standard variance reduction techniques, such as importance sampling, may be available (cf, Glasserman (2004); Bluhm *et al* (2003)), but they might not be fully effective without further considerations.
- The most expensive step, by far, is the calculation of counterparty exposures (over all scenarios and multiple time steps). This must be minimized.
- Many institutions have implemented *PFE* simulation engines for risk management and limits. Given their significant computational cost, it would be ideal to reuse these precomputed simulations.

Based on these observations, in the next section, we introduce a new methodology for computing alpha and stress-testing the model assumptions.

### 3 METHODOLOGY FOR COUNTERPARTY CREDIT RISK CAPITAL AND ALPHA

We seek a general framework for computing economic capital with stochastic exposures and alpha, which has the following properties:

- It should be parsimonious, simple to implement, calibrate, and understand.
- It should leverage, where available, existing *PFE* engines and credit portfolio models used in an institution.
- It should be computationally efficient.
- It should provide transparency for business users and regulators on the impact of parameters that are difficult to estimate (eg, market-credit correlations), and lead to a better understanding of wrong way risk.

A capital calculation that leverages precomputed *PFE* profiles is greatly advantageous. First, it minimizes the work carried out during the most computationally intensive step. These precomputed *PFEs* can be further reused for multiple capital calculations under various credit model assumptions for the purpose of model validation, sensitivities and stress-testing. As mentioned earlier, correlations between exposures and credit events can be difficult to estimate and validate. They can also vary considerably with portfolios and market conditions, and their impact on capital is difficult to assess. Thus it is important to have the ability to stress-test capital estimates and to understand the model risk.

In this section, we present a general methodology that has the following key components:

- non-parametric sampling of *EADs* from precomputed *PFE* profiles – this can be done using a single-factor or a multifactor model;

- flexible modeling of co-dependence structure of *EADs* and systematic credit factors using specified copulae;
- stress-testing of the market factors and market-credit correlations.

To provide some intuition, consider the case where exposures are stochastic, but uncorrelated to defaults. Assume that the *PFEs* have been precomputed over a set of scenarios. A simple algorithm can be implemented by computing portfolio losses in each scenario by randomly sampling a precomputed *PFE* scenario and independently sampling the systematic credit factors,  $Z$ , and idiosyncratic factors leading to defaults. Substantial computational savings can thus be achieved by sampling non-parametrically from presimulated *PFEs*. This strategy can be generalized to simulate correlated exposures and defaults, as described below.

In what follows, we first describe the methodology for the case of a single-factor credit model (such as the one in the Basel II formula). The methodology is then extended to multifactor models for both the exposures and credit factors. This section focuses on a single-step credit portfolio model. However, the methodology can also be implemented with random default times and precomputed multistep *PFE* profiles as shown in Appendix A.<sup>8</sup>

### 3.1 Correlated market-credit alpha under a single-factor credit model

Consider a single-step credit portfolio model, where defaults are driven by a single systematic factor  $Z$ . Assume that the time-averaged *PFEs* (Equation (6)) have been precomputed over a finite set of scenarios:

$$\mu_j^T(\omega_s) = \frac{1}{T} \int_0^T PFE_j(\omega_s, t) dt \quad s = 1, \dots, S$$

Define the matrix of exposures  $A$  with entries  $a_{sj} = \mu_j^T(\omega_s)$ . Thus,  $a_{sj}$  is the exposure at default of instrument  $j$  under scenario  $s$ , and the matrix  $A$  contains all of the relevant exposure information required for a simulation of portfolio credit losses. In some instances we may be interested in correlating the systematic credit factors to a set of market factors. Market factors can include interest rates, spreads, foreign exchange rates, equity prices, market indices, etc. In this case, we assume that the exposures are driven by  $K$  market factors  $X_k$ ,  $k = 1, \dots, K$ , and define the augmented matrix  $A^*$ , where each row  $s$  defines

<sup>8</sup>Generally, the additional volatility owing to time-dependent exposures tends to increase capital. This impact is usually small compared to the other factors discussed in this paper (particularly the strength of the market-credit correlation). An example showing this relatively modest effect is given in Rosen and Saunders (2010). However, note that results are portfolio-specific, and it is possible to find extreme exposure profiles for which the time variation may make a significant difference for a given risk measure.

the outcome in a scenario  $\omega_s$  of both the exposures and the market factors  $(\mu_1^T(\omega_s), \dots, \mu_M^T(\omega_s), X_1(\omega_s), \dots, X_K(\omega_s))$ .<sup>9</sup> The matrix  $A^*$  contains the exposures at default of all instruments in the portfolio under all scenarios, as well as the values of all market factors. The rows of  $A^*$  give the values of all observables (instrument exposures and market factors) under a particular scenario  $s$ , while the columns of  $A^*$  give the values of a given observable under all possible scenarios. Knowing which scenario  $\omega_s$  has occurred (equivalently, which row of  $A^*$  is being considered) is sufficient to specify the values of the exposures of all instruments in the portfolio, as well as all the market factors.

When the credit model is a single-factor model, the systematic factor,  $Z$ , can only be correlated with one “aggregate” factor driving exposures.<sup>10</sup> We refer to this aggregate factor as the *systematic exposure factor* and denote it by  $W$ . The essence of the single-factor algorithm for correlating market and credit risk is as follows:

*The value of the random variable  $W$  serves as a market indicator that determines which scenario  $\omega_s$  from the discrete simulation has occurred. The dependence between market risk and systematic credit risk is modeled by specifying the copula of the random vector  $(Z, W)$ .*

The co-dependence between exposures and  $Z$  can be introduced in a straightforward way through a Gaussian copula.<sup>11</sup> Thus, we map the factor  $W$  to a standard normal random variable:

$$W^N = \Phi^{-1}(F(W)) \quad (16)$$

where  $F$  denotes the cumulative distribution of  $W$ . We then assume that  $(W^N, Z)$  has a bivariate normal distribution with correlation  $\rho$ .

To specify fully the model, we define how the value of the market indicator  $W$  determines which of the exposure scenarios  $\omega_s$  occurs. This is achieved in two steps:

- First, choose  $W$  in such a way that it orders the scenarios in a financially meaningful way.

<sup>9</sup>We also require single-step “time-compressed” values for the market factors  $X$ , in a manner analogous to Equation (6) for exposures, since we are using here a single-step credit portfolio model, and the simulation provides multistep paths for the exposures and market factors. The algorithm to obtain these single-step values depends on the type of factor. For example, total returns might be relevant for equity factors, but average values are more appropriate for interest rates or exchange rates. See Rosen and Saunders (2010) for further discussion and examples of the use of market factors.

<sup>10</sup>If  $X$  is a multivariate normal vector with standard normal marginals and non-singular correlation matrix  $\Sigma$ , and  $Y$  is standard normal such that  $(X, Y)$  is multivariate normal and  $E[YX_j] = \rho_j$ , then  $Y$  may be written as  $Y = v \cdot X + \sqrt{1 - \|v\|_2^2} \cdot \eta$ , where  $\eta$  is a standard normal random variable independent of  $X$ , and  $v = \Sigma^{-1}\rho$ . Furthermore, this choice of  $w$  is unique. We refer to  $v \cdot X$  as the aggregate factor.

<sup>11</sup>Note that the use of a Gaussian copula is simply for ease of exposition and any other copula can be used.

- Then select the thresholds  $C_s$  so that scenario  $\omega_s$  occurs if the market indicator variable  $W^N$  satisfies  $C_{s-1} < W^N \leq C_s$ .

There are several practical ways to order the scenarios, and these are discussed below in Section 3.2. To understand the methodology better, consider the simple case where the scenarios are sorted by the total portfolio exposure. Thus, define  $W$  as the total portfolio exposure, with its value in each scenario given by:

$$W(\omega_s) = \sum_{j=1}^M \mu_j^T(\omega_s) \quad (17)$$

We begin by sorting the exposure scenarios by the value of  $W$ . Denote the sorted scenarios by  $\tilde{\omega}_s, s = 1, \dots, S$ , so that we have:

$$\sum_{j=1}^M \mu_j^T(\tilde{\omega}_s) \leq \sum_{j=1}^M \mu_j^T(\tilde{\omega}_{s+1}), \quad s = 1, \dots, S-1 \quad (18)$$

The exposure factor  $W$  is mapped to a standard normal random variable  $W^N$  as follows. The scenario  $\tilde{\omega}_s$  occurs if  $W^N$  lies in a given interval:

$$\omega = \tilde{\omega}_s \quad \text{if } C_{s-1} < W^N \leq C_s, \quad s = 1, \dots, S \quad (19)$$

where  $-\infty = C_0 < C_1 < C_2 < \dots < C_{S-1} < C_S = \infty$ , and the thresholds  $C_s, s = 1, \dots, S-1$  are calibrated to match the scenario probabilities. For equally probable Monte Carlo scenarios  $C_s = \Phi^{-1}(s/S), s = 1, \dots, S-1$ .

Based on this model, we can now perform an efficient Monte Carlo simulation to compute CCR capital as follows:

- 1) Simulate a set of scenarios on the jointly normally distributed random variables  $(W^N, Z_k, \varepsilon_j)$  using standard procedures.
- 2) In each scenario:
  - determine the exposure scenario from the matrix  $A$  corresponding to the outcome of  $W^N$ , and from this the exposures for every counterparty;
  - determine defaults for the counterparties;
  - calculate portfolio losses.
- 3) Obtain the portfolio loss distribution from all the scenarios, and compute  $EC$ .

Note that the algorithm preserves both the marginal distribution of  $PFEs$ , as well as their exact co-dependence structure (as determined by the scenario set). Furthermore, while we only correlate the systematic credit factor,  $Z$ , with one “aggregate” factor driving exposures, this does not imply that the exposures themselves are driven by a single factor.

An important feature of the methodology is its natural ability to understand the impact of wrong way risk in the portfolio by explicitly stress-testing the exposure

factors (the direction), as well as the market-credit correlation  $\rho$ . This is particularly important given the difficulty in estimating accurately a market-credit co-dependence structure. The next section briefly discusses the various choices for exposure factors, as well as their advantages and disadvantages.

### 3.2 Choice of the systematic exposure factor

In the example above, the systematic exposure factor  $W$  is chosen as the total portfolio exposure. Wrong way risk is introduced through the (negative) correlation between the exposure factor and  $Z$ . Higher values of  $W$  imply higher exposures, and lower values for the systematic factor  $Z$  imply greater likelihood of defaults. While this choice seems conservative, any other factor  $W$  can be used to order the exposure scenarios, leaving unchanged the rest of the algorithm.

In practice, the systematic exposure factor  $W$  can be defined as either a function of the counterparty exposures,  $\mu_j^T$ , or the market factors,  $X$ , which drive these exposures.<sup>12</sup> Just as varying the parameter  $\rho$  allows alpha to be stress-tested with respect to the *strength* of market-credit correlation, varying the choice of the systematic factor  $W$  allows the *direction* of market-credit correlation to be stress-tested.

#### 3.2.1 Systematic exposure factor based on portfolio exposures

As described earlier in Section 3.1, we can define a market-credit correlation structure directly between the *EADs* and the systematic credit factor. In the general case, the exposure factor  $W$  is given by a linear combination (or weighted average) of the counterparty exposures, and we can write it as:

$$W(\omega_s) = \sum_{j=1}^M v_j \cdot \mu_j^T(\omega_s) \quad (20)$$

Different factors are obtained from the choice of the weight vector  $v$ .

- *Total portfolio exposure.* This is the factor introduced earlier by Equation (17):

$$W(\omega_s) = \sum_{j=1}^M \mu_j^T(\omega_s)$$

As  $W$  and  $Z$  become more (negatively) correlated, a larger number of defaults tend to occur with larger exposures. Note that while it can be viewed as a

<sup>12</sup>We focus on *systematic wrong way exposures*. More generally, the co-dependence between the exposures and the default indicators can be defined. This might be practically (and empirically) difficult. The specific correlation generally has a small impact on a bank's portfolio (it may be important for some large counterparties). Since it generally arises from collateral, it can be also dealt with by conservative collateral assumptions (Fleck and Schmidt (2005)).

conservative approach, this may not always be the case, since large exposures are not the only drivers for credit losses. For example, the total portfolio exposure may be driven by a small number of counterparties with large exposures but (very) small  $PDs$ .

- *Expected loss.* The market factor given by:

$$W(\omega_s) = EL(\omega_s) = \sum_{j=1}^M PD_j \cdot \mu_j^T(\omega_s) \quad (21)$$

This factor gives higher weights to counterparties of less quality. A similar approach can be based on the “Basel II capital” for each exposure scenario:<sup>13</sup>

$$W(\omega_s) = \sum_{j=1}^M N\left(\frac{N^{-1}(PD_j) + \beta_j Z_{0.999}}{\sqrt{1 - \beta_j^2}}\right) \cdot \mu_j^T(\omega_s) \quad (22)$$

- *Principal components of the exposure matrix.* We may specify the market factor based on the principal components of the matrix  $A$ .<sup>14</sup> Any principal component or combination of them may be used as an exposure factor. For example, choosing the first principal component is equivalent to selecting the direction of greatest variation in the exposures. To the extent that maximizing variance implies maximizing risk, this assumption can be seen as conservative. In this case, the correlation  $\rho$  is formally the correlation of the Gaussian factor driving the first principal component of  $A$  and the systematic credit factor. The other principal components are uncorrelated by construction. This gives roughly the “maximal” systematic correlation that can be built into the model – any other aggregate factor will likely lead to lower credit losses in the tail.

Note that as in the case of total portfolio exposure factors, we can also use the principal components of the matrix of expected losses in place of exposures to underweight counterparties with highly variable exposures, but low  $PDs$ .

### 3.2.2 Systematic exposure factor based on market risk factors

The exposure factor  $W$  can be also defined directly as a function of the market factors  $X$ , which drive exposures. In the general case, it can be written as a weighted

<sup>13</sup>This is equivalent to an expected loss conditional on a 99.9% credit scenario.

<sup>14</sup>The principal components of  $A$  are the eigenvalues of the covariance matrix of the exposure scenarios,  $\Sigma = S^{-1} \cdot (A - \mu)^T (A - \mu)$ , where  $\mu$  is a vector containing the average exposure of each counterparty, ie,  $\mu_j = EPE_j(T)$ . All the principal components can be computed using the singular value decomposition of  $A - \mu$ . The computation of the first principal component (or a small number of them) may also be done more efficiently with specialized algorithms, which do not require a full singular value decomposition or even the estimation of the covariance matrix. Thus, problems that occur in practice with a large number of counterparties (larger than the number of scenarios) do not cause difficulties for the method.



combination of the market factors, with its value in each scenario given by:

$$W(\omega_s) = \sum_{k=1}^K v_k \cdot X_k(\omega_s) \quad (23)$$

In this case, the parameter  $\rho$  represents the correlation between an aggregate market factor or “market index” and the credit factor  $Z$ .

As an example, consider a simple fixed income portfolio (eg, swaps) in one currency. The market factors are the general interest rate levels, and the weights can be chosen to represent a parallel shift in the curve. The market-credit correlation  $\rho$  is the correlation between changes in interest rate levels and a systematic factor driving defaults. Fleck and Schmidt (2005) estimated a 20% market-credit correlation from regressing *PDs* on interest rates using Moody’s data.

### 3.2.3 Choosing the exposure factors and the correlations

There are various advantages and disadvantages when we consider the range of exposure factor choices.

- Working with a co-dependence structure defined directly between *EADs* and the credit factor (Equation (20)) presents several conceptual and computational benefits. This model specification is very simple and leads to a natural stress-testing framework for exploring the impact of wrong way risk. Since the market-credit correlation  $\rho$  directly determines the co-dependence of the exposures and the defaults, stress-testing and parameter sensitivity is well defined and practical. Conservative correlation assumptions can also be set based on the portfolio composition and industry studies. However, a disadvantage is that the correlation of the exposures and credit factors cannot be directly estimated from historical data alone, since the exposures are portfolio-dependent.
- In contrast, when a market risk factor is used (Equation (23)), the impact on capital of a given portfolio due to the “increased” market-credit correlations may not be obvious or easy to assess, making it difficult to determine the errors in the original model on a portfolio’s alpha. However, it leads to a more straightforward statistical estimation of this market-credit correlation from historical data, and seems more natural for traditional credit portfolio models.

Basic statistical techniques can be applied to obtain estimates of the correlation between the market and credit factors (see, for example, Fleck and Schmidt (2005) or Rosen and Saunders (2010)). We stress, however, that there are substantial challenges in estimating this correlation accurately in practice, and any such an analysis will be prone to many assumptions, data limitations and estimation errors. First, there are potentially substantial model errors due to misspecification (single-factor models, multifactor models based on aggregate data, etc.). Second, the resulting

parameters are generally subject to significant estimation errors, owing to issues such as the small number of default observations and the short time series available of overlapping market and credit information. While various proxies can also be used (spreads, equity returns, etc.) to complement the estimation, their empirical relevance has not been demonstrated. In addition, as the credit crisis has further shown, the estimation of market-credit co-dependence based purely on past historical information seems to have poor predictive properties in practice, and particularly when a crisis emerges.

One of the strengths of our methodology is that it allows a bank to compute economic capital and alpha efficiently across a broad spectrum of possible levels of market-credit correlation (as well as possible directions for market-credit correlation, governed by the choice of exposure factor), which permits effective and efficient stress-testing of counterparty credit risk capital. Finally, we note that, given the correlation of  $Z$  and a specific exposure factor, it is possible to derive an analytic formula for the correlation between  $Z$  and any random variable defined on the finite set of scenarios  $\omega_s$ .<sup>15</sup> Therefore, it is easy to map from one factor to another. For example, if an empirical correlation between, say, a market index and the systematic credit factor  $Z$  is estimated, the correlation between an exposure factor in the portfolio (say, total exposure) and  $Z$  can be obtained.

### 3.3 Multifactor extensions

In this section, we briefly discuss extensions of the methodology to the multifactor case. First, note that the algorithm in Section 3.1 still applies for multifactor credit models with a single “market” factor. In this case, we can correlate the market factor (and hence the instrument exposures) to both systematic and idiosyncratic credit factors.<sup>16</sup>

The methodology can also be extended for use within a general multifactor market and credit risk framework. However, we note that any multifactor extension must sacrifice some of the advantageous properties of the single-factor case. In particular, the extension discussed below can modify slightly the joint distribution of the portfolio exposures.

The joint distribution of counterparty exposures (and market factors) is fully defined (within simulation errors) by the matrix  $A^*$ . When correlating multiple credit factors to exposures (or market factors), one approach is to preserve the marginal distributions of all individual counterparty exposures, given by the simulated scenarios,

<sup>15</sup>This formula and its proof are given in a technical appendix that is available from the authors upon request.

<sup>16</sup>This is accomplished, for example, by defining the “market factor” as  $W = \rho_0 \xi + \sum_{i=1}^N \rho_i Z_i + \sum_{j=1}^N \sigma_j \varepsilon_j$  where  $\xi$  is a standard normal random variable, independent of all credit drivers and  $\sum_{i=0}^N \rho_i^2 + \sum_{j=1}^N \sigma_j^2 = 1$ .

but to alter their joint distribution by applying a different copula.<sup>17</sup> This is described in more detail in Appendix B. For example, we can assume that the joint distribution of counterparty exposures and credit factors has a multifactor Gaussian copula (or, in principle, any other parametric copula). When working with a joint model of multiple credit factors and market factors that drive instrument exposures (eg, Iscoe *et al* (1999)), the correlations between the market factors and the instrument exposures may first be calculated, and then a Gaussian copula is applied to the market, credit and exposure factors. While this definition creates some distortions of the original co-dependence structure of the exposures, these differences are generally small in comparison to model specification or estimation errors. For example, when the simulated market factors are Gaussian (which is generally the case in exposure simulations) and the portfolio is fairly linear, then the Gaussian copula specification has a minor impact beyond the standard sampling numerical errors.

Note that while a multifactor credit model generally produces a richer co-dependence structure of counterparty defaults, it also presents several drawbacks. In addition to increased computational complexity, by introducing a multifactor model we lose the ability to parameterize alpha directly as a function of a single correlation parameter for stress-testing. In this case, a practical, single-dimension parameterization can be done, for example, in terms of an average or an implied correlation (see Cespedes *et al* (2006)). The computational efficiency of the algorithm presented in this paper still enables us to perform the stress-testing exercise in a timely manner and plot alpha as a function of this parameter.

## 4 EXAMPLE

We now illustrate the application of the methodology for a real derivatives portfolio typical of a large financial institution, as of February 2006. We focus on the “regulatory counterparty risk perimeter”, ie, the derivatives transactions in the trading book, which are subject to CCR capital.

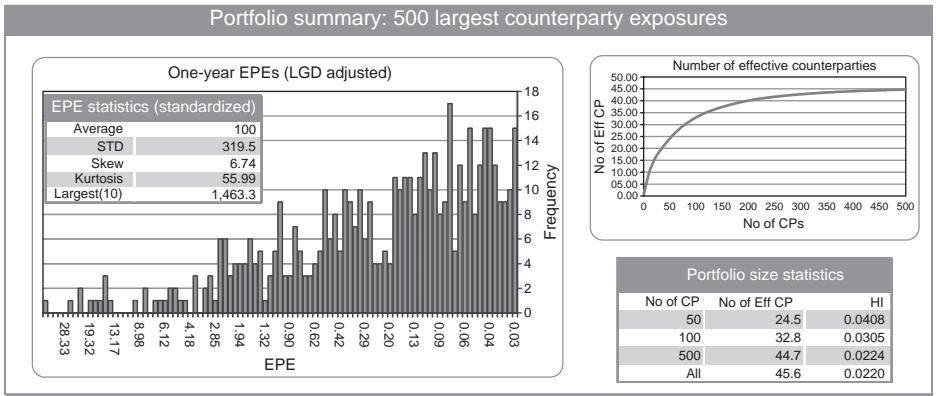
### 4.1 Exposures summary

The portfolio contains over 1,500 counterparties. *Potential future exposure* profiles are simulated over a one-year horizon with 2,000 market scenarios and 12 monthly time steps using a multifactor simulation model.<sup>18</sup> Underlying transactions include fixed income, foreign exchange, equity and credit derivatives in multiple currencies. A summary of the exposures for the 500 largest counterparties is presented in Figures 1 and 2 (see page 89); the results are expressed in terms of exposures

<sup>17</sup>The algorithm from Section 3.1 may be regarded as applying the empirical copula (ie, the one implied by the simulated scenarios in  $A^*$ ) to generate the joint distribution of the instrument exposures.

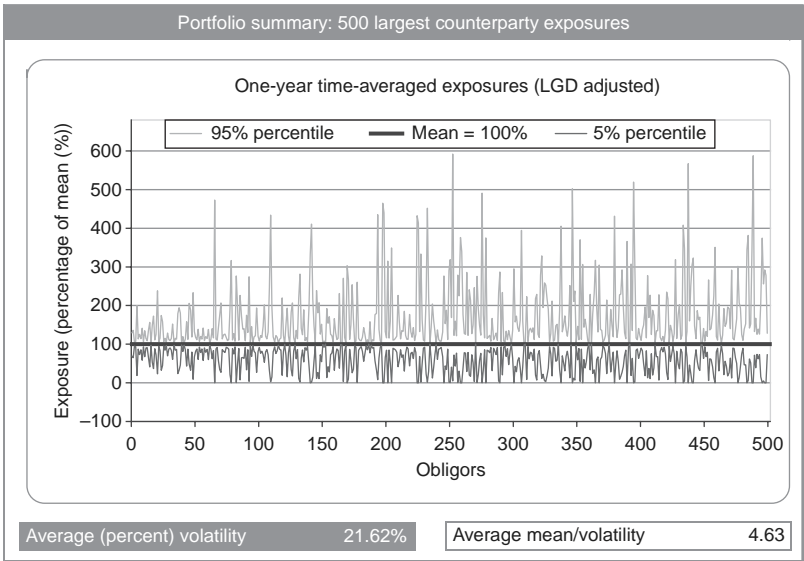
<sup>18</sup>The model generally assumes mean reversion for relevant parameters such as interest rates and is estimated using historical data at the time of the simulation.

**FIGURE 1** Portfolio summary: exposure distribution and numbers of effective counterparties.



Note that the horizontal axis is increasing to the left.  
Source:  $R^2$  Financial Technologies Inc.

**FIGURE 2** Exposures summary (volatilities).



Source:  $R^2$  Financial Technologies Inc.

already multiplied by *LGD*. Figure 1 gives the histogram of the mean exposures for all counterparties as well as various statistics (standardized to give a mean of 100). To understand name concentrations, we also plot the effective number of counterparties (given by the inverse of the Herfindahl index) as a function of the actual number of counterparties (in descending order). Figure 2 plots the exposures for each counterparty at 95% and 5% levels, as a percentage of the mean exposures; counterparties are in descending order of their mean exposure.

Two points are particularly important for the capital calculation:

- While the number of counterparties is large, the portfolio is concentrated in a small number, with an effective number of about 46. The largest 400 counterparties already account for 44 of these.
- The average volatility of exposures is 22%.<sup>19</sup> This can be explained by noticing that the largest counterparties tend to be mostly “in the money”, and thus have relatively low volatilities compared to their mean exposures.

Based on these observations, in particular, the small number of effective counterparties, we expect a substantial idiosyncratic contribution to portfolio capital. Due to the relatively low exposure volatility, the impact of the stochastic exposures on economic credit capital (as captured by  $\alpha$ ) can be expected to be modest, even for high market-credit correlations.

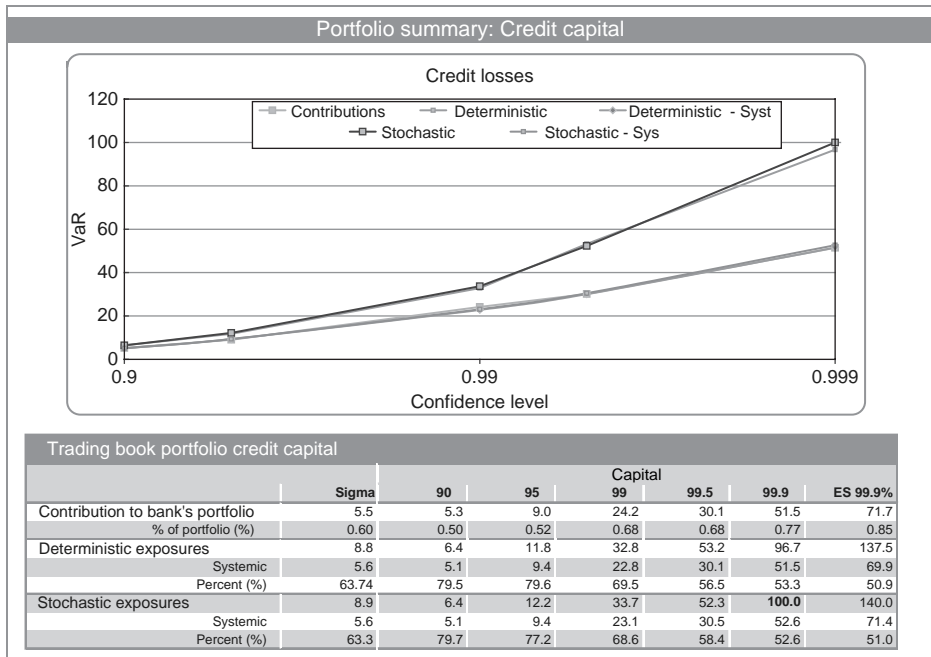
## 4.2 CCR capital and alpha for uncorrelated exposures and credit events

A summary of the (standardized) credit capital for the derivatives portfolio is given in Figure 3 (see page 91). It shows the loss volatility, capital and expected shortfall at various confidence levels with both stochastic exposures and deterministic *EPEs*. The risk measures are presented on a stand-alone basis and as contributions to the bank’s total portfolio. In addition, Figure 3 decomposes the stand-alone capital into its systematic and idiosyncratic components.

Since the banking book portfolio is much larger, the derivatives portfolio’s incremental capital contribution to the overall capital is less than 1%. This contribution represents almost the same amount as the “pure systematic” credit capital, which is computed under the assumption that the portfolio is infinitely granular. Thus, the derivatives portfolio achieves a diversification benefit close to the maximum possible by being part of the bank’s portfolio.

Recall that  $\alpha$  is the ratio of economic capital computed using stochastic exposures to that computed when exposures are replaced by deterministic *EPEs*. The economic capital in the numerator and denominator can be calculated using only systematic risk (systematic  $\alpha$ ), or both systematic and idiosyncratic risk (total

<sup>19</sup>Volatility is defined as the standard deviation of the time averaged exposure (see Equation (3)) divided by the mean of the time averaged exposure.

**FIGURE 3** Credit capital and risk contributions.

Source:  $R^2$  Financial Technologies Inc.

alpha).<sup>20</sup> Considering both alpha measures together helps clarify the role idiosyncratic risk plays in the portfolio. When market and credit risks are uncorrelated, the values of these alphas are 1.02 (systematic alpha) and 1.03 (total alpha).

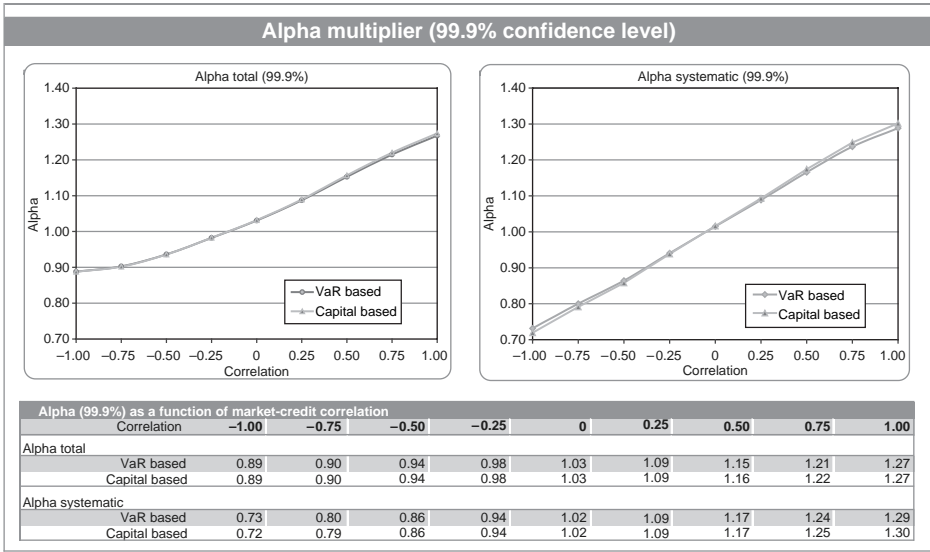
### 4.3 Analysis of alpha for correlated exposures and credit events

Figure 4 (see page 92) depicts the dependence of both total and systematic alpha on the market-credit correlation  $\rho$ .<sup>21</sup> Alpha remains below the regulatory floor of 1.2 for values of  $\rho$  of almost 75%. At the benchmark correlation of 20%, alpha is below 1.1. For this portfolio, alpha increases with correlation. Indeed, negative correlation levels generate “right way exposures” with alphas below 1.0. In practice it is difficult to predict the qualitative relationship between the correlation level  $\rho$  and

<sup>20</sup>Systematic alpha is the ratio of the economic capital of  $E[L^T | Z]$  to that of  $E[L^{EPE} | Z]$ . Total alpha is the ratio of the economic capital of  $L^T$  to that of  $L^{EPE}$ .

<sup>21</sup>Results are computed using one million Monte Carlo scenarios and a kernel estimator for the quantile with equal weights. The examples in this section are calculated by sorting the exposure scenarios by the value of the first principal component of the exposures.

**FIGURE 4** Alpha as a function of market-credit correlation.



Source:  $R^2$  Financial Technologies Inc.

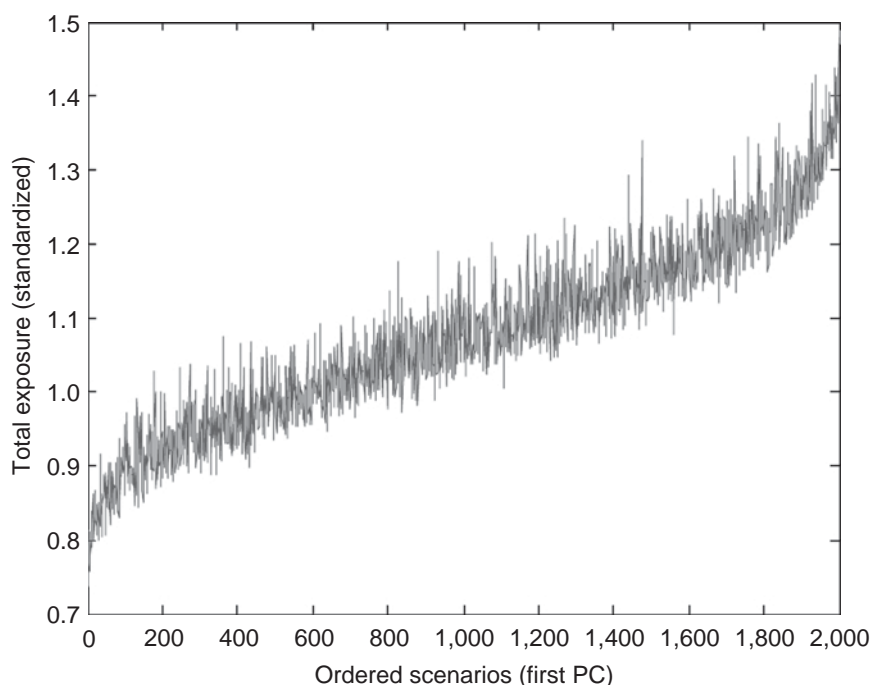
the portfolio alpha before running the simulation. Furthermore, this relationship may change over time, depending on the individual desks’ strategies and market levels. Note that the systematic alpha measure is more sensitive to market-credit correlation than the total alpha. This makes intuitive sense, as the simulation methodology introduces correlation between the systematic factor and the exposures. The total alpha adds idiosyncratic risks, which are by definition uncorrelated to systematic risks and exposures, to both the numerator and the denominator of systematic alpha.

Figure 5 (see page 93) plots the value of the total portfolio exposure for each scenario, where scenarios are ordered by the value of the first principal component. Exposures have been normalized in the scenario set. A similar plot is obtained for expected losses. This figure illustrates the explanatory power of the market factor on the counterparty exposures (the first principal component explains over 60% of the exposure variability). Furthermore, the significant correlation between the first principal component and the total portfolio exposure indicates that capital results will be generally similar whichever of these variables is used as the exposure factor for the purposes of sorting the market scenarios.<sup>22</sup> In this portfolio, the general trend of the exposures is monotonic with respect to the factor. However, we have encountered in practice portfolios with a “smile effect”, where some counterparties are positively

<sup>22</sup>For a further discussion of this point, as well as examples comparing results with different choices of the sorting factor, see Rosen and Saunders (2010).



**FIGURE 5** Ordered counterparty exposures as a function of the first principal component.



correlated and some negatively correlated to the factor. This is depicted in Figure 6 (see page 94).

The impact of the quantile for the alphas with market-credit correlation is depicted in Figure 7 (see page 94). We show the level of alpha for quantiles ranging from 90% to 99.9% as well as for expected shortfall at 99.9%. As expected, higher quantiles are generally more sensitive to the correlation parameter  $\rho$ . While the alpha results for this portfolio look quite stable and “simple” for all quantiles, this is not always the case and anomalies can appear for portfolios with high name concentrations, given the discrete nature and long tail of the loss distributions.

## 5 CONCLUDING REMARKS

This paper presents a practical, computationally tractable and robust methodology for calculating CCR capital and the alpha multiplier and stress-testing wrong way risk. The methodology effectively leverages underlying counterparty *PFE* simulations also used for credit limits and risk management. We demonstrate its application to a realistic bank trading portfolio, and contrast the results with industry studies. For

FIGURE 6 Smile effect of exposures as a function of the first principal component.

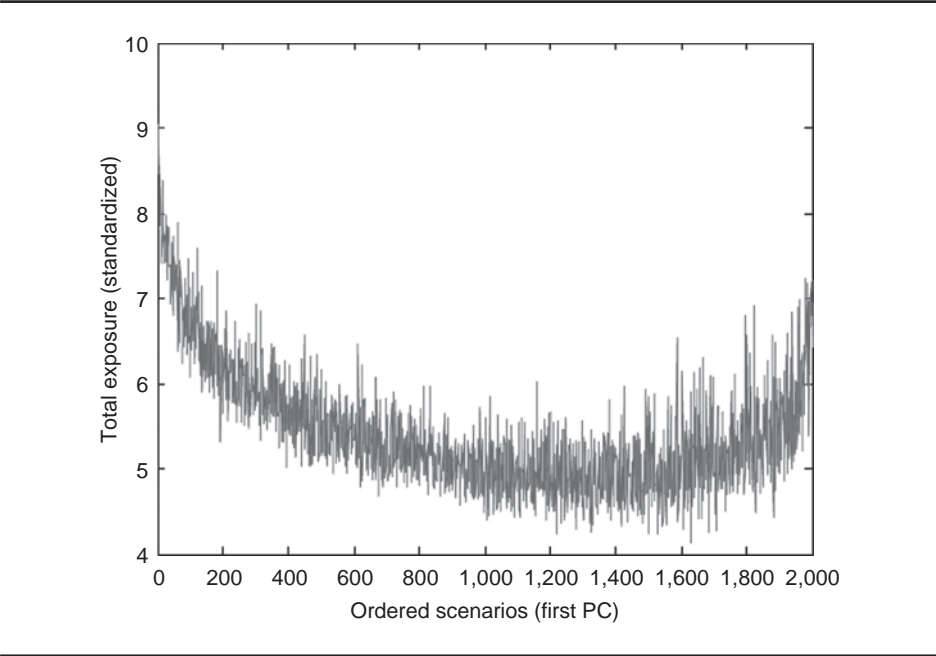
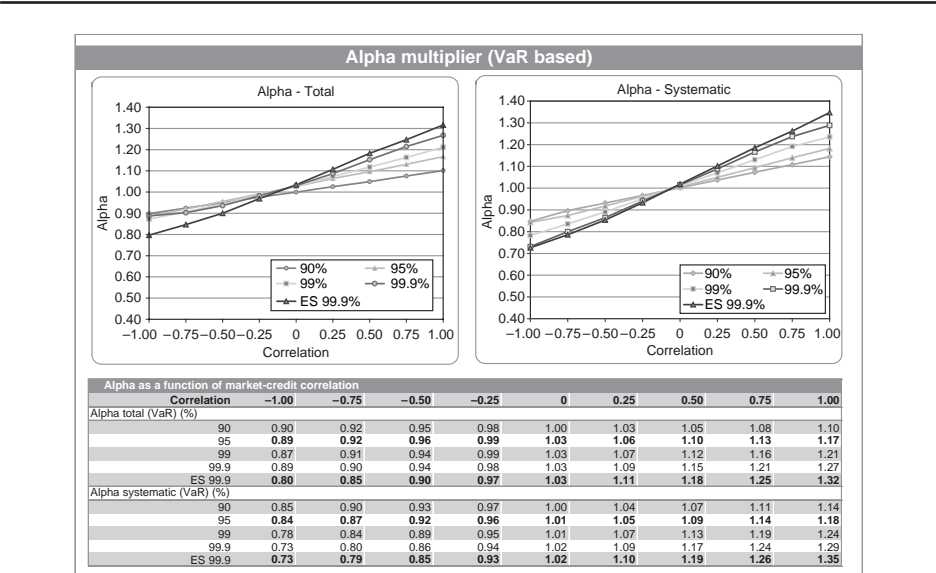


FIGURE 7 Correlated alpha for various quantile levels.



Source:  $R^2$  Financial Technologies Inc.

the example portfolio, alpha remains at or below 1.2 even for conservative correlation assumptions. While the example presents the methodology for a single-factor Gaussian copula, it can be extended in practice to other copulas and a multifactor credit setting, which captures diversification accurately (see Cespedes *et al* (2006)).

The methodology can be applied within general integrated market-credit risk multifactor models. In the specific case of a single-factor credit model (such as the one embedded in the Basel II credit formula), the approach leads to some practical simplifications and can be implemented and validated easily. Furthermore, it provides a natural stress-testing approach for wrong way risk, where the impact of market factors and correlations on economic capital and alpha can be assessed.

## APPENDIX A METHODOLOGY FOR RANDOM DEFAULT TIMES WITH MULTISTEP EXPOSURES

The basic CCR capital methodology in Section 3 assumes a single-step default model. Exposures at default for derivatives portfolios can vary over time across the capital horizon (one year). Below, we briefly outline a simple and effective multistep extension.

The basic ingredients for the multistep credit model are the same as in the single-step case. Each counterparty has a creditworthiness indicator (Equation (3)):

$$Y_j = \sum_{i=1}^n \beta_{ij} \cdot Z_i + \sigma_j \cdot \varepsilon_j$$

We assume that the marginal distribution of the default time of each counterparty is known and given by:

$$F_j(t) = \Pr[\text{Counterparty } j \text{ defaults before } t] \quad (\text{A.1})$$

The default time of counterparty  $j$  is then defined to be:

$$\tau_j = F_j^{-1}(\Phi(Y_j)) \quad (\text{A.2})$$

In practice, the complete marginal distribution  $F_j$  is not observable. At best, it may be possible to observe default probabilities at a finite set of times  $F_j(t_k)$ , and some interpolation assumptions must then be employed in performing the multistep default simulation. In running the simulation, there are a finite number of times at which counterparty credit exposures are available, given by the potential future exposure “cube”  $PFE_j(\omega_s, t_k)$ , and we must select one of the simulation times  $t_k$  as the default time. With  $t_0 = 0$ , the current time, the default times are specified simply as:

$$\tau_j = t_k \quad \text{if } t_{k-1} < F_j^{-1}(Y_j) \leq t_k, \quad k = 1, \dots, N \quad (\text{A.3})$$

and no default otherwise.

Up to this point, the default model has been the basic Gaussian copula model. We employ the same technique to introduce correlation between exposures and systematic credit risk for the multistep case as we did in the single-step case. In particular, a market factor  $W^N$  is defined to have a standard normal distribution such that the correlation between  $W^N$  and  $Z$  is  $\rho$ . Once  $W^N$  is simulated, the exposure scenario is determined by:

$$\omega = \tilde{\omega}_s \quad \text{if } C_{s-1} < W^N \leq C_s \quad (\text{A.4})$$

exactly as in Section 3.1. The key difference, then, between the single-step and the multistep simulation for the exposures is that exposure simulations in the multistep mode result in an entire sample path  $PFE_j(\omega_s, t_k)$  for the exposure of each counterparty rather than a single number.

Multistep scenario sets can be ordered through their time compressed values. That is, to determine an ordering of the scenario set corresponding to the potential future exposure profiles  $PFE_j(\omega_s, t_k)$ , we order the corresponding scenario set in which the path for a given counterparty (for each scenario) is replaced by the time averaged potential future exposure over that path:

$$\mu_j^{t_k}(\omega_s) = \frac{1}{t_k} \int_0^{t_k} PFE_j(\omega_s, t) dt$$

Once the time averages have been computed, the scenarios can be ordered using an arbitrary ordering factor, as described in Sections 3.1 and 3.2 above.

## APPENDIX B GAUSSIAN COPULA OF EXPOSURES AND CREDIT FACTORS

We can define a multifactor Gaussian copula to model the joint distribution of the exposures and credit factors,  $Z$ , as follows (while we focus on a Gaussian copula, other copulas can be similarly defined as well). For ease of exposition, assume that the  $EAD$ s are continuous, and for each counterparty  $j$ , define a Gaussian factor  $X_j \sim N(0, 1)$  by:

$$X_j = \Phi^{-1}[F_j(EAD_j)] \quad (\text{B.1})$$

where  $EAD_j$  denotes the exposure at default of counterparty  $j$  (in this case,  $EAD_j = \mu_j^T$ ) and  $F_j$  is the (marginal) cumulative distribution of  $EAD_j$ . Note that in practice the  $F_j$ 's are defined in non-parametric form by sorting the exposure scenarios for the counterparty in matrix  $A$  (and thus are not continuous). A Gaussian copula, which approximates the joint distributions of the  $EAD$ s, is given by:

$$\begin{aligned} GC &= \Phi_M[X_1, \dots, X_M; C] \\ &= \Phi_M[\Phi^{-1}(F_1(EAD_1)), \dots, \Phi^{-1}(F_M(EAD_M)); C] \end{aligned} \quad (\text{B.2})$$

where  $C$  denotes the correlation matrix of the  $X_j$  (ie, it is the normal rank correlations of the  $EAD$ s). While the copula function (B.2) is driven by  $M$  factors,

a copula with  $m < M$  factors can also be constructed by choosing, for example, the first  $m$  principal components of the correlation matrix  $C$ .

As a second step, a correlated model of exposures and defaults is constructed by specifying the joint distribution of  $(X, Z)$  – in a Gaussian framework. This requires the definition of a correlation matrix between all the market factors and the credit factors. The algorithm for computing economic capital is then similar to the one described in Section 3.1. The simulation step in this case is as follows:

- For a given scenario  $\omega$ , simulate jointly distributed random variables  $(X(\omega), Z(\omega))$  from the multifactor Gaussian copula.
- Map the exposure factor scenario  $X$  into the given exposures using the transformations:

$$EAD_j(\omega) = F_j^{-1}[\Phi(X_j(\omega))], \quad j = 1, \dots, M \quad (\text{B.3})$$

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