# Modelling the Term Structure of Interest Rates: a Literature Review

Rosa María Canto Gutiérrez

A thesis submitted in part fulfillment of the degree of Master of Science

HANKEN

Swedish School of Economics and Business Administration

Department of Finance and Statistics

Vasa

2008

**HANKEN Swedish School of Economics and Business Administration**

Modelling the Term Structure of Interest Rates: a Literature Review

Abstract:

The term structure of interest rates is an extremely important element in Finance. It is one of the most important indicators for pricing contingent claims, determining the cost of capital and managing financial risk. As a result; extensive research in this field has been developed. The purpose of this paper is firstly; to present the theories of the term structure of interest rates. Secondly; to describe some of the most important single, volatility and two-factor models of the term structure. Finally; to provide several frameworks in which interest rate models can be nested and compared between each other. Due to complex forecasting dynamics; little is known about the existence of a model able to outperform others. The model chosen will depend on the objective for which it is required. However; the most accurate models are those in which the levels of interest rate are related to fluctuations on volatility. We conclude that the most important elements in an interest rate model are the short term rate and volatility to changes in interest rates; consistent with Longstaff- Schwartz (1992) and Bali (2003). Furthermore; a two-factor model is accurate for modeling the term structure of interest rates.

Keywords:

Term structure of interest rates, single factor models, volatility models, multifactor models, interest rates, forecasts.

**List of Contents**

**List of Tables**

# 1 Introduction

According to Gibson, Lhabitant and Talay (2001); to understand and model the term structure of interest rates represents one of the most challenging and complex topics of Financial Research. However; the benefits provided are worthwhile since interest rates may be used in an extensive variety of applications; such as investments, long term debt, determination of the cost of capital, measurement of credit risk, valuation of contingent claims as well as pricing, hedging and managing the risk of interest rate derivatives. Furthermore; interest rates are used in the establishment of fiscal and monetary policies.

Interest rates can be determined as the price paid for the use of money. However; the interest rates critical to the investment decision are the real interest rates; since this kind of interest rates take inflation into consideration. Interest rates cluster within different levels depending on several factors which include; risk, maturity, size, taxability and market imperfections (McConnell and Brue; 1996; pp. 584). These factors are important because if we include them all in the same set; it is possible to determine the degree of riskiness involved.

Regarding investment opportunities; it is extremely important to be able to determine the level of interest rates; since this may imply important investment's performance. High interest rates reduce the present value of future cash flows; which reduces the attractiveness of investment opportunities. Loans are highly sensitive to the level of interest rates. As a result; the demand for loans is importantly affected when interest rates increase since this implies higher interest payments. Consequently; the cost of capital is seen to be more expensive. (Bodie, Kane and Marcus; 1996; pp. 496)

Interest rates can be classified as the premium required or offered in capital markets in order to allocate capital and investment in the most profitable and productive form between borrowers and lenders. (McConnell et al; 1996; pp. 586) Future outcomes are unpredictable ex ante. Every investment involves certain amount of risk. According to the Capital Asset Pricing Model, expected returns are seen as rewards for assuming a specific amount of risk. Therefore; as risk increases; the expected return to investors should be higher in order to compensate them for bearing excess risk.

Maturity plays also an important role as a determinant on interest rates; since this factor determines the length on a loan. Longer term loans imply higher interest rates than shorter term loans since a longer term implies higher future uncertainty of expected outcomes (McConnell et al; 1996; pp.585). Risk and maturity can be seen as the most important factors determining the level on interest rates. However; loan size, taxability, and market imperfections can be seen as important secondary factors.

If we consider interest rate risk; it is important to take into consideration the risk of short, intermediate and long term interest rates. Within the risk of short interest rates, the risk of changes in LIBOR, risk of changes in Treasury bill rate, risk of changes in commercial paper, and extensive additional risks related to the short term interest rates are included. (Chance; 2004; p.p. 545)

According to Koutmos (1998); rates of return on financial assets behave differently at different frequencies. Financial theory and asset pricing models state that as maturity increases the risk involved or assumed by investors is higher. Therefore; the expected returns should be greater. Consequently; investors would be interested in knowing elasticity estimates according to specific frequencies. As maturity increases the mean increases accordingly suggesting the existence of a term premium built on the term structure.

Furthermore; the term structure of interest rates may be a very important subject not only in the field of Finance but also in the field of Economics. According to Turnovsksy (1989); the term structure of interest rates is an important method for the transmission of macroeconomic policies. Term structure could be seen as a signaling indicator.

In the case where interest rates are high; investors may prefer to invest their money in a bank account. On the other hand; if interest rates are low; investors may prefer to consume. These reactions may lead to economical development or to stagnation. In addition; McConnell et al (1996; pp. 585) state that the term structure of interest rates is an extremely important price since it affects the level and composition of investment goods production.

Monetary policy associated with short term assets on long term interest rates affect the rate of investment and economical growth. It is stated that stochastic disturbances in fiscal policies have been the dominant sources of interest rate fluctuations. As an important consequence of a macroeconomic approach, it is demonstrated that in the presence of risk averse speculators, an increase in the variance of government policy has two possible effects on the variance of interest rates. First; a larger variance in policy will translate to larger variances in rates. Second; by influencing private speculative behavior; the direct effect is strengthen or weakened. (Turnovsky; 1989)

The term structure of interest rate is an administered price due to the fact that monetary authorities in purpose manipulate the money supply to influence interest rates and consequently the levels of output, employment and prices. Within a macro economical framework; low interest rates increase investment and expand the economy while high interest rates reduce investment and constrains the economy. (McConnell et al; 1996; pp. 585) Adequate fiscal and monetary policies may reverse a recession while eliminating or establishing a tariff may alter importantly the demand and revenue of the affected industry. (Campbell et al; 1996; pp. 588)

When a macroeconomic policy is established; governments are able to determine whether to maintain a specific level of interest rates or not. Frequently; there are expectations or forecasts on the behavior of interest rates. However; when expected interest rates differ from the real ones; a government can decide whether apply an adjustment or not in order to fit the differences. Depending on how the long term rate and term premium respond to short rate shocks; next period's expected short term interest rates may increase or decrease. However; a policy of pegging the long term bond rate implies that the long term rate will be constant over time (Hutchinson and Toma; 1991)

Interest rates are frequently used in several additional applications; such as fixed income securities and interest rate derivatives. In both cases interest rates play a key role; since this component is utilized to determine the present value. Within the area of financial risk management, it is important to determine how a change in the level of interest rates will affect a specific asset. Interest rate derivatives allow for the possibility of hedging risk as a result of unpredictable shocks. Financial instruments are widely applied since they are able to explain how risk management maximizes shareholders wealth.

The main reason for managing financial risk is due to volatility on interest rates, changes in exchange rates, commodity prices and stock prices. Enterprises usually are more efficient by solving internal risks than external risks. (Chance; 2004; p.p. 543) Risk management can be considered as a financial decision. It also reduces the probability of bankruptcy. Furthermore; it allows firms to generate the cash flows required to develop their investment projects.

Financial risk management is also observed as a tool for obtaining arbitrage opportunities. To manage financial risk should create value to shareholders; since it allows them for the possibility of obtaining something they could not accomplish by theirselves. (Chance; 2004; p.p. 545)

In addition; interest rate fluctuations create one of the most important determinants on bond prices. As interest rates fluctuate; bondholders are exposed to capital losses and gains. (Bodie et al; 1996; pp.450) However; this exposure is only experienced when bondholders hold bonds only for a determined period of time. In the case when bonds are hold until maturity; there is no exposure to risk. In addition; it has been shown empirically that long term bonds are more sensitive to interest rate movements than short term bonds.

According to Neftci (2000); the price of a bond depends on the stochastic behavior of the current and future spot rates in the economy. As a result; bond prices must be a function of the current and future spot rates. Additionally; interest rates can not be constant, since that would imply predictability on bond prices. Moreover; constant interest rates may imply no volatility on the underlying assets. Consequently; the demand for any interest rate derivatives would reduce completely since all possible risk could be vanished.

Consistent with this opinion; Halkos and Papadamou (2007) state that due to increased volatility on financial markets, deviations of bond market efficiency and term premium behaviour has increased in importance. Furthermore; Telser (1966) argues that risk aversion of borrowers and lenders result into hedging strategies against the risk of changes in interest rates in order to match the timing of payments and receipts. It is argued that it is difficult to predict whether a given term structure reflects a set of beliefs about future conditions that could imply higher future interest rates.

As a consequence of risk aversion; interest rate derivatives have increased significantly in importance, variety and trading volume over the last years. A reason explaining this event could be given due to frequent fluctuations in volatility, which affect the underlying asset. An alternative approach to obtain protection against stochastic changes in volatility is given by pursuing fixed income securities or interest rate derivatives. Derivatives are frequently seen as a tool to hedge interest rate risk; however; occasionally, they are also used for speculative purposes.

Fixed income securities are also called interest rate derivatives because these instruments are continuously affected by changes on interest rate. Therefore; have a high degree of sensitivity against interest rate fluctuations. Interest rate derivatives can be divided into different classifications; such as interest rate futures and forwards, forward rate agreements, caps and floors, interest rate swaps, bond options and swaptions. (Neftci; 2000)

According to a survey realized by International Swaps and Derivatives Association (ISDA) on derivatives used by the world's 500 largest companies in year 2003; 92% of these companies used derivative instruments to manage and hedge their risks more effectively. Firms were situated in 26 countries and represented a broad variety of industries.

Derivatives can be complex tools; however, they are fundamental tools for hedging financial risk. Risk in an economy can be shifted around. Risk has not disappeared; however, it tends to shift towards those who feel they can manage it more efficiently. (Wall Street Journal; 2006) In an essay published by State Street Global Advisors; it is argued that the demand for fixed income derivatives has increased dramatically in the last few years, driven specially by the growth in hedge fund strategies. Hedge fund assets have more than doubled over the last five years. (Zielinski and Mauro; 2006)

The most profound evolution in the application of derivatives to fixed income has been in the separation of risk from liquidity. In a traditional fixed income security, the party wishing to transfer risk needed to provide funding to settle the security in addition to finding an intermediary willing to value and trade that risk. Since most fixed income derivatives are unfunded, risk is transferred without an upfront cash transfer. Cash payment is necessary only when certain predefined events are triggered.

Fixed income securities are used by diverse industrial sectors; such as commercial companies, hedge funds as well as investment banks. Extensive literature has been written regarding fixed income securities; reasons for utilizing fixed income securities, possible benefits implied and valuation approaches.

According to Goltz (2005), the field of fixed income derivative pricing and hedging has created an extensive amount of new valuation techniques due to the complexity in securities' cash flows. Various types of derivative securities are used to shift the risk associated in fixed income securities.

In order to build a risk free portfolio; a mix of bond options and bond futures should be constructed. The purpose of including options is given due to non linear payoffs and allowance to construct funds with skewed or dissymmetric return profiles; with the objective of limiting extreme values of a portfolio return distribution**.**

## *1.1 Motivation*

Interest rates are the most important factor when determining the pricing of contingent claims, the cost of capital and managing financial risk. If we are able to price any contingent claim ex ante, great arbitrage opportunities will arise.

Furthermore; if we are able to determine the cost of capital ex ante; we can determine the most appropriate moment for lending or taking a loan. We could determine the best moment for obtaining the least expensive loan and the most appropriate moment for investing.

Regarding risk management; if we are able to determine future interest rate levels ex ante; we would be able to determine the best hedge strategy in order to minimize any possible risk. Benefits from having accurate interest rate estimations ex ante are countless. By forecasting future interest rates ex ante will bring extensive economical benefits to those able to determine this variable accurately

## *1.2 Purpose*

The purpose of this paper is firstly; to present the theories of the term structure of interest rates. Secondly; to describe some of the most important single, volatility and two-factor models of the term structure. Finally; to provide several frameworks in which interest rate models can be nested and compared between each other.

## *1.3 Structure*

This paper has the following structure. Chapter 2 describes some of the most significant contributions of the term structure of interest rates. Chapter 3 clarifies the theories of the term structure of interest rates. Chapter 4 explains some of the most important interest rate models. Chapter 5 compares the different models between each other. Finally; Chapter 6 concludes.

## 2 Background

Given the importance of the term structure of interest rates; previous research related to this topic is quite extensive; not only because of the considerable amount of research developed on this area but also due to all the variables and applications involved in this field. However; on this section we will present some of the literature review that to our opinion represent the most significant contributions to this topic. Table I located at the end of this Chapter provides a summary from previous research in this field.

Telser (1966) analyzes the importance of empirical research on the term structure of interest rates when choosing between different theories with respect to the determinants of the term structure. Two interesting theories are analyzed: the expectations theory and the liquidity preference theory. In addition; modified versions of both theories are introduced in order to incorporate the effects of other variables that could affect expectations in addition to the forecast error. It is concluded that both theories have advantages and limitations. As a result; no consensus can be achieved when determining the most accurate theory for describing the term structure of interest rates.

Nelson (1972) demonstrates that estimated term premiums on long term interest rates and short term interest rates do not represent return differentials in bonds. Long term interest rates have been higher than short term interest rates due to an additional rate of return by investing in long term bonds. However; an average differential between long and short term interest rates does not represent a differential between realized rates of return. It is concluded that the existence of term premiums appear from a more complex set of maturity preferences than those postulated by the liquidity preference theory.

Elliot and Baier (1979) examine the ability of six different econometric interest rate models to explain and predict interest rates. In order to test the accuracy of the models; each model is fitted to US monthly data over a sample period of 7 years. Results demonstrate that from the models analyzed; four econometric models are able to explain current interest rates movements quite accurately but their ability to forecast future interest rates by applying actual information is seemed to be inaccurate.

Brennan and Schwartz (1982) apply a model for the pricing of US government bonds from 1948 to 1979, with the objective of evaluation the ability of the pricing model to detect underpriced or overpriced bonds. A strong relation between prediction errors and bond returns was observed. Results suggest a lack of significant relation between future values of the short term interest rate and the long term interest rate. Factors do not appear to be associated with uncertainty about expected future interest rates. The valuation model seems to be misspecified since it assumes constant rather than stochastic variance rates. However; the model is adequate over short periods of time.

Cox, Ingersoll and Ross (1985) develop an intertemporal general equilibrium asset pricing model to study the term structure of interest rates. The model takes into consideration key factors for determining the term structure of interest rates; which include anticipation of future events, risk preferences, investment alternatives and preferences about the timing of consumption. In addition; the model is able to eliminate negative interest rates. This model allows for detailed predictions about how changes in a diverse range of underlying variables will affect the term structure.

Turnovsky (1989) examines the behavior of the term structure of interest rates with a stochastic macroeconomic framework; resulting into solutions for long and short term interest rates obtained from future fiscal and monetary policies. The effects of the term structure are analyzed through different scenarios; which include temporary vs. permanent policy changes, unanticipated vs. anticipated changes, long vs. short rates, and real vs. nominal rates. General results indicate that the response of the term structure is highly sensitive to the nature of the underlying shocks involved on the economy, since these involve the sensitivity to risk aversion on investors.

Hutchinson and Toma (1991) demonstrate that a short rate increase was followed by a short term interest rate decrease in the following periods. The binding phase of the support program from April 1942 to 1944 was found to be consistent with mean reversion. It is observed that public's expectations about persistence of short term interest rates changed as the policy changed. In addition; the creation of the bond price support program limited upward movements in future short term interest rates with the objective of controlling inflationary expectations.

Chan, Karolyi, Longstaff and Sanders (1992) compare eight models of short term interest rate with the objective of determining which model best fits the short term Treasury bill yield data. The models are fitted within a same framework in order to allow for a correct comparison within the models. Results indicate that models which best describe the dynamics of interest rates over time are those that allow the conditional volatility of interest rate changes to be highly dependent on the level of the interest rates. It is found that Vasicek and Cox-Ingersoll-Ross Square Root models perform poorly in comparison to Dothan and Cox-Ingersoll-Ross Variable Rate models.

Gustavsson (1992) contributes with an introduction regarding the concept of no arbitrage pricing and probability measures. It is stated that in the case of complete markets; equivalent probability measures should exist leading to martingales. The concept of no arbitrage pricing is seen as an extension of the classical fair game hypothesis and a preferred approach for the pricing of financial assets. The study focuses mainly in the arbitrage free pricing of bonds as well as in the proper understanding in which the role of risk interacts with market prices.

Longstaff and Schwartz (1992) develop a two factor general equilibrium model of the term structure of interest rates. The model is applied to derive closed form expressions for discount bond prices and discount bond option prices. Factors used are the short term interest rate and volatility of short term interest rates. The model is able to determine the value of interest rate contingent claims as well as hedging strategies of interest rate contingent claims. The model demonstrates advantages over two factor models which include endogenously determination of interest rate risk and a simplified version of the term structure of interest rates.

Johansson (1994) models a continuous time stochastic process on short term interest rates. The data set consists of the average interest rate for overnight loans on the interbank market for the five largest Swedish banks. Daily data on interest rates is applied since this is considered to be the best possible approximation to instantaneously risk free rates. The sample period ranges from 1986 to 1991. Results suggest that accuracy on parameters is dependent on sample's time length. Additionally; in the presence of heteroskedasticity and a reduction in time length; it is observed that bias in the estimators increase significantly.

Brenner, Harjes and Kroner (1996) analyze two different interest rate models; LEVELS and GARCH models. It is concluded that LEVELS models over indicate the dependence of volatility on interest rate levels and are unsuccessful in capturing serial correlation in variances. On the other hand; GARCH models depend extensively on serial correlation in variances and do not capture the relationship between interest rates and volatility. An alternative class of models is developed; which are able to capture serial correlation in variances and dependence of variances on levels.

Koedijk, Nissen, Schotman, and Wolff (1997) compare their model against a single factor model, GARCH model, and to a level GARCH model for one month Treasury bill rates. Models are estimated by applying the methods of quasi-maximum likelihood. Results demonstrate that pure GARCH models are non stationary in variance. In addition; log likelihood values demonstrate that GARCH effects and level GARCH effects are important determinants of interest rate volatility.

Géczy, Minton, and Schrand (1997) examine the use of currency derivatives on firms that have exposure to foreign exchange rate risk. Firms with greater growth opportunities and tighter financial constraints are more likely to use currency derivatives. This result is consistent with the notion that firms use derivatives to reduce the variation in cash flows or earnings that might otherwise prevent firms from investing in valuable growth opportunities. Derivatives may provide a valuable benefit to firms that use them rationally.

Koutmos (1998) applied a single factor interest rate model to describe the volatility of interest rates across six different maturities and three different frequencies. Maturities include three and six months, one, two, three and five years. Frequencies include daily, weekly and monthly rates. Results demonstrate that the level of interest rate is an important determinant of interest rate volatility. The sensitivity of volatility to changes in interest rates was found to be lower than one, contradicting the findings obtained by Chan et al (1992). No statistical evidence of mean reversion over the long run mean was observed.

Miller and Stone (1998) develop a multifactor model based on arbitrage pricing theory for short term interest rates. In addition; it is examined whether information contained in the term structure of interest rates can be used to improve forecasts of short term interest rates when comparing them to forecasts using only historical information. Results indicate that only two or three factors explain all the variation in interest rates representing the term structure. Lagged values generate out of sample forecast errors which grow as the forecast horizon increases.

Koski and Pontiff (1999) investigate investment managers' use of derivatives by comparing return distributions for equity mutual funds that use and do not use derivatives. Results demonstrate that derivatives users have risk exposure and return performance similar to non users. It is also analyzed changes in fund risk in response to prior fund performance. Changes in risk are substantially less severe for funds using derivatives; consistent with the explanation that managers use derivatives to reduce the impact of performance and risk.

Bielecki and Rutkowski (2000) develop a new approach to modeling credit risk, valuation of defaultable debt and pricing of credit derivatives by basing on the Heath, Jarrow and Morton methodology. By using available information about credit spreads and recovery rates it was possible to model the intensities of credit migrations between several credit ratings classes; resulting in a conditionally Markovian model of credit risk. Furthermore; credit risk and interest rate risk models were combined in order to obtain an arbitrage free model of defaultable bonds.

Chaudhry, Christie-David, Koch, and Reichert (2000) demonstrate that banks use currency swaps as a hedging tool while currency options are used as a speculative role.

The use of forward contracts and currency commitments contribute mildly to any type of risk. Regarding the application of interest rate derivatives in corporations; it is also questionable whether the main objective is to decrease or increase total risk. The possibility that firms use derivatives to increase their risk exposures recently has been a principal concern guiding regulatory agencies in their considerations of derivatives regulations.

Koutmos (2000) analyze the effect of information shocks and the levels on interest rate volatility on six countries; including Canada, France, Germany, Japan, United Kingdom, and United States. Following Chan et al (1992); short term interest rates are assumed to be mean reverting. Results demonstrate that the level of interest rates is an important source of heteroskedasticity. The best volatility specification requires the application of information shocks and interest rate levels. In addition; it is demonstrated that mean reversion is not an important aspect on short term rates.

Gibson, Lhabitant, and Talay (2001) describe and compare some of the most popular models related to the term structure of interest rates. Advantages and disadvantages of each model are discussed in terms of bonds and interest rate contingent claims, continuous time valuation and hedging parameter estimation Models are classified by general characteristics in order to provide a better understanding of the different aspects involved. Each model presents certain advantages and disadvantages. Therefore; the model's selection will depend on the specific purpose for which the model is required.

Henstchel and Kothari (2001) investigate in a panel of 425 large US firms whether firms systematically reduce or increase their riskiness with derivatives. Empirical findings demonstrate an absence of significant relationship between derivatives use and return volatility. Firms primarily use derivatives to reduce the risks associated with short term contracts. Since the cash flows associated with these contracts typically represent a small fraction of firm value, risk reduction for those contracts is unlikely to have important effects on the overall firm volatility. This finding opposes to the theories of corporate risk management.

Collin-Dufresne and Goldstein (2002) assume that a portfolio consisting of bonds uniquely; can be used to hedge volatility risk. However; it was found that swap rates have limited explanatory power for returns on at-the-money strategies; since portfolios are exposed to volatility risk; determined as unspanned stochastic volatility (USV). USV can be captured within a HJM framework. Nevertheless; it is demonstrated that bivariate models can not exhibit USV. Furthermore, required conditions for trivariate Markov USV affine systems were determined. In addition; it is clarified that for trivariate USV models, bonds alone are not enough for identifying all parameters. Derivatives are required.

Bali (2003) extends the one factor BDT term structure model into a two factor setting in order to determine price implications for discount bonds. Several one factor diffusion and two factor stochastic volatility models are compared in terms of their ability to capture the dynamics of interest rate volatility. Results indicate that level models fail to model the GARCH effects in conditional volatility. However; GARCH models fail to capture the relationship between volatility and interest rate levels. Empirical evidence on one, three and six months Eurodollar deposit rates show that two factor BDT model performs better in forecasting the future volatility of interest rate changes.

According to Glasserman (2003), simply compounded interest rates and their parameters are more directly observable in practice and are the basis of recent research on market models. The paper characterizes the arbitrage free dynamics of interest rates with jumps and diffusion by modeling the term structure through simple forward rates or forward swap rates. It is clearly explained how jump and diffusion risk premium are related to simple forward rates. In addition; models and pricing formulas for some derivative securities, interest rate caps and options on swaps are formulated. Furthermore; the effect of jumps on implied volatilities in interest rate derivatives is described.

Treepongkaruna and Gray (2003) provide an accurate description and several examples of how to use Monte Carlo simulation to value interest rate derivatives when the short rate follows arbitrary time series process. A comparison of various interest rate derivatives is done by applying closed form solutions, a trinomial tree procedure and a Monte Carlo simulation technique. It is demonstrated that a simulation technique can be applied to more complex short rate processes. Advantages and disadvantages of the simulation approach against competing approaches are empirically demonstrated.

Aas (2004) concludes a review of single-factor diffusion, GARCH, regime-switching and jump-diffusion models. It is concluded that interest rates display conditional volatility patterns that are not only a function of past interest rate shocks but also are considered as a function of the lagged level of the series. It is demonstrated by fitting the models to the Norwegian 1-month and 3-month interest rates that GARCH models result into non stationary models and explosive volatility patterns. Additionally; a level GARCH model is developed in which the single factor and the GARCH models are combined. This model seems to capture more accurately the behavior of interest rates since it allows for a better fit of all elements involved.

Borokhovich, Brunarski, Crutchley and Simkins (2004) examine the relation between the composition of a firm's board of directors and the firm's use of interest rate derivatives. Directors' incentives towards the application of derivatives vary depending on whether they are inside directors or outside directors[1]. It is observed a significant and positive relation between the quantity of interest rate derivatives used by firms and the proportion of outside directors on the firm's board. Results suggest that the use of interest rate derivatives in firms with boards dominated by outsiders is likely to be aligned with shareholders' interests.

Goard and Hansen (2004) compare their own model with models developed by Ahn and Gao (1999) and Goard (2000). When fitting time series on interest rates; the most accurate results will be obtained from data sets with longer periods of time. The longer the data set, the more realistic are the short and long term trends in interest rates' behavior. Even for shorter periods of time, Goard and Hansen (2004) model determine the time dependence on mean reversion level. A short rate model with time dependent moving target is more accurate to model the term structure of interest rates than no time dependent models.

Buraschi and Corielli (2005) analyze advantages and disadvantages of model recalibration for pricing and hedging objectives. Implementation of asset pricing models is based on a periodic adaptation of its parameters and initial conditions in order to eliminate conflicts between models and market prices. Results demonstrate that this practice violates the assumptions in which models are developed. In addition; it is observed that model updating could have serious consequences on risk management strategies. As a consequence; the background behind model updating must be carefully considered.

Cassasus, Collin-Dufresne and Goldstein (2005) propose a parsimonious unspanned stochastic volatility (USV) model of the term structure and study its implications for fixed income option prices. Results suggest that the drift and quadratic variation of the short rate are affine in three state variables; including short rate, long term mean and variance, following a joint Markov process. Bond prices are exponential affine functions of only two state variables, independent of the current interest rate volatility level. A process of this kind can be adjusted to fixed income derivative prices and extended to fit any arbitrary term structure. An application using data on caps and floors

[1] Inside directors are directors who are also managers of the firm. Outside directors are directors that have no affiliation with the firm.

shows that the model can capture very well the implied Black spot volatility surface and simultaneously fit the observed term structure.

Goltz (2005) realizes a simulation on interest rates and interest rate volatilities by applying the Longstaff-Schwartz interest rate model. A Protective Put Buying (PPB) strategy consisting of a long position in the underlying asset and a long position in a put option rolled over as the option expires, was created. The objective is to determine the additional benefits investors could receive from a PPB strategy with respect to investing only in the bond market. Results show that a PPB strategy dominates the bond futures strategy; since the PPB strategy has lower downside risk and higher returns.

Mato (2005) states that interest rate risk immunization is one of the key aspects of fixed income portfolio management. This is obtained as a result of risk aversion's increments due to fluctuations on interest rates. The relation between classic measures and more recent risk measures is analyzed by performing an empirical study from mid 1990s and early 2000s on the US Treasury bonds market with portfolios with different maturities and structures. Empirical findings demonstrate an absence of relationships between portfolios optimized by classic measures and modern measures, leading to considerable different portfolios.

Faff and Gray (2006) develop simulation experiments to determine bias and accuracy of GMM estimates of short rate parameters. After the paper developed by Chan et al (1992); the GMM technique has become very popular among researchers. This paper analyzes the different techniques available. Results indicate that GMM avoids estimation difficulties which are present under alternative approaches. However; it presents difficulties in estimating the speed of mean reversion on interest rates. This problem is not unique from GMM; it also appears in Ordinary Least Squares, Maximum Likelihood, and Bayesian Approach. As a result; caution regarding mean reversion should be applied.

Platen (2006) derives a unified framework for portfolio optimization, derivative pricing, financial modeling and risk measurement. Since the dynamics of the market portfolio with stochastic volatility are difficult to determine and model; the objective of the paper is to identify the growth optimal portfolio as a unifying measure. For the minimal market model described, no equivalent risk neutral martingale measure exists.

Reno and Roma (2006) estimate the diffusion coefficient of single factor models for the short rate by applying non parametric methods. Models analyze include those proposed by Ait-Sahalia (1996), Stanton (1997) and Bandi-Phillips (2003). By applying a Monte Carlo simulation of Vasicek and Cox-Ingersoll-Ross model; it is observed that

Aït-Sahalia estimator is not applicable for values of mean reversion displayed by interest rate data. Stanton and Bandi-Phillips models perform better. However; estimators depend on the choice of bandwidth selected. Short term data demonstrate characteristics which are inconsistent with diffusion.

Sanford and Martin (2006) analyze several single factor continuous time models for the Australian short rate. Models are nested in a general single factor diffusion process for the short term interest rate. Models are a discretized version from continuous time models. By applying a Bayesian approach with a Markov Chain Monte Carlo Algorithm, posterior distributions of parameters were iterated. Iterations are applied to estimate Bayes factors for each one of the models. According to Bayes factors obtained from each one of the models, it is concluded that Cox-Ingersoll-Ross Square Root (1985) has the greatest support from all models.

Beltrattti and Colla (2007) focus on affine term structure models as tools for active bond portfolio management. Essentially affine class of models performs better than completely affine class of models in forecasting bond yields since they allow for positive and negative values. As a result; models are able to unify small unconditional sample means of bond excess returns with high variance. It is concluded that in the case of the term structure of interest rates it is not possible to determine a best model. The model applied to solve a specific situation depends on the objective of the decision maker.

Halkos and Papadamou (2007) examine the significance of risk modeling and asymmetries when testing different theories regarding the term structure of interest rates. The study tests a modified version of the expectation hypothesis which allows for time varying risks and asymmetries. Government bond data from Germany, France, United Kingdom, United States of America and Canada was applied. Empirical evidence suggests the presence of non linear effects of spreads on excess holding period yields in all maturity structures with the exception of the short term maturity. Evidence of mean reversion process of returns on large spread effects in international bond markets was observed. Longer maturity bonds can be very sensitive to the modeling process of risk.

Koutmos and Philippatos (2007) investigate short term mean reversion in major European countries; including, France, Germany and United Kingdom by applying Longstaff & Schwartz (1992) and Bali (2000) interest rate models. Results indicate the presence of asymmetric mean reversion in all European interest rates analyzed. Findings demonstrate that interest rates are non stationary following interest rate increases but are mean reverting following interest rate decreases. Non stationary is offset due to stronger mean reversion. Volatility is asymmetric; rising more with positive innovations. A two factor model describes interest rate dynamics accurately.

Mahdavi (2008) demonstrates the behavior of short term rates in United States, United Kingdom, Canada, Japan, Australia, Denmark, Sweden and the Euro zone under the no arbitrage condition. The restriction relates the rate of change in the short term rate to the slope of the forward rate. A one factor model able to nest several models is estimated using GMM approach of Hansen (1982). Short term rates do not follow a simple mean reversion process. Results demonstrate that no single model can explain the short term process for all countries. Market price of interest rate risk is highly non linear but continuously increasing in the level of the short term rate.

**Table I.** *Summary Table*
This table summarizes previous research mentioned through Chapter 2.

| Author | Year | Objective | Findings |
|---|---|---|---|
| Telser | 1966 | To analyze different theories of the term structure of interest rates. | Both theories analyzed present advantages and limitations. No consensus has arised for determining the most accurate theory. |
| Nelson | 1972 | To demonstrate that estimated term premiums on long and short term interest rates do not represent return differentials in bonds. | The existence of term premiums appear from a more complex set of maturity preferences than those obtained from the liquidity preference theory. |
| Elliot and Baier | 1979 | To examine the ability of six different econometric models for explaining and forecasting interest rates. | Four econometric models are able to explain current interest rates movements accurately. However; their ability to forecast future interest rates is inaccurate. |
| Brennan and Schwartz | 1982 | To propose a two factor model for pricing governmental bonds. Factors include the long and short term interest rates. | A lack of significant relation between future values of the long and short term interest rates is observed. The valuation model is misspecified due to constant variance rates. |
| Cox-Ingersoll-Ross | 1985 | To develop an intertemporal general equilibrium asset pricing model to study the term structure. | The model predicts how changes in several variables will affect the term structure. The model eliminates negative interest rates. The variance is proportional to the level of interest rates. |
| Turnovsky | 1989 | To examine the behavior of the term structure of interest rates with a stochastic macroeconomic framework. | The response of the term structure is highly sensitive to underlying shocks involved in any economy. This response will denote investors' sensitivity towards risk aversion. |
| Hutchinson and Toma | 1991 | To demonstrate mean reversion in the binding phase of US support program from 1942 to 1944. | Public's expectations changed as the policy changed. The support program limited interest rates upward movements with the purpose of controlling inflationary expectations. |
| Chan-Karolyi-Longstaff-Sanders | 1992 | To create a common framework for comparing eight different models of the short term interest rate with the objective of selecting the best model | The best models able to describe the dynamics of interest rates through time are those that allow the conditional volatility to be highly dependent on the level of interest rates. |
| Longstaff-Schwartz | 1992 | To develop a two factor general equilibrium model of the term structure. Factors include short | The model presents advantages over two factor models since it is able to determine interest rate risk endogenously and presents a |

| | | | |
|---|---|---|---|
| | | term interest rates and volatility. | simplified version of the term structure of interest rates. |
| Johansson | 1994 | To model a continuous time stochastic process on short term interest rates. | Accuracy on models' parameters is dependent on sample's time length. Given heteroskedasticity and a reduction on time length; bias in the estimators increase importantly. |
| Brenner-Harjes-Kroner | 1996 | To analyze single factor models and GARCH models. To develop a new class of models able to capture serial correlation and dependence of variance. | Single factor models over indicate the dependence of volatility and fail to capture serial correlation in variances. GARCH models do not capture the relation between interest rates and volatility. |
| Géczy-Minton-Schrand | 1997 | To examine the use of currency derivatives on firms that have exposure to foreign exchange rate risk. | Firms with greater growth opportunities and tighter financial constraints are more likely to use currency derivatives. |
| Koutmos | 1998 | To apply a single factor interest rate model to describe volatility of interest rates across different maturities and frequencies. | The level of interest rate is an important determinant of interest rate volatility. No statistical evidence of mean reversion over the long run was observed. |
| Miller-Stone | 1998 | To develop a multifactor model based on arbitrage pricing theory for short term interest rates. | Only two or three factors explain the entire variation in interest rates representing the term structure. |
| Koski-Pontiff | 1999 | To investigate possible benefits from applying derivatives by comparing return distributions for equity mutual funds. | Derivatives users have risk exposure and return performance similar to non users. However; changes in risk are less severe for funds that apply derivatives. |
| Bielecki-Rutkowsky | 2000 | To develop a new approach for modeling credit risk, valuation of defaultable debt and pricing of credit derivatives based on Heath-Jarrow-Morton. | Intensities of credit migrations between several classes of credit ratings were modeled. An arbitrage free model of defaultable bonds was obtained by considering interest rate and credit risk. |
| Chaudhry-Christie-David-Koch-Reichert | 2000 | To demonstrate the objectives for applying interest rate derivatives in corporations and financial institutions. | Banks use currency swaps as hedging tools and currency options as speculative tools. The objective of applying derivatives in corporations is still unclear. |
| Koutmos | 2000 | To analyze the effect of information shocks and levels of interest rate volatility on six countries. | Levels on interest rates are an important source of heteroskedasticity. The best volatility specification requires the application of information shocks and interest rate levels. |
| Gibson-Lhabitant-Talay | 2001 | To describe and compare some of the most popular models of the term structure of interest | Each model presents advantages and limitations. The selection of the model will depend on the specific purpose for which the model |

| | | | |
|---|---|---|---|
| | | rates. | is required. |
| Henstchel-Kothari | 2001 | To investigate whether firms systematically reduce or increase their riskiness with the application of derivatives | Firms use derivatives to reduce risks associated with short term contracts. However; risk reduction does not represent important effects on overall firm volatility. |
| Collin-Dufresne-Goldstein | 2002 | To demonstrate that a portfolio consisting of bonds exclusively can be applied to hedge volatility risk. | Two factor models can not exhibit Unspanned Stochastic Volatility. In three factor USV models, bonds are not enough for identifying all parameters. |
| Bali | 2003 | To extend the single factor BDT term structure model into a two factor model. To compare single and two factor stochastic volatility models. | Level models fail to model GARCH effects in volatility. GARCH models do not capture the relation between volatility and interest rates. Two factor BDT model outperforms. |
| Glaserman | 2003 | To describe the arbitrage free dynamics of interest rates with jumps and diffusion by modeling the term structure through forward rates and swaps. | Models and pricing formulas for contingent claims are formulated. The effect of jumps on implied volatilities on interest rate derivatives is described. |
| Trepongkaruna-Gray | 2003 | To provide an accurate description and examples for applying Monte Carlo simulation to value interest rate derivatives. | Simulation techniques can be applied to more complex short rate processes. Advantages and disadvantages of the simulation approach against competing approaches are demonstrated. |
| Aas | 2004 | To review single factor diffusion, GARCH, regime-switching and jump-diffusion interest rate models. To develop a level-GARCH model. | GARCH models result into non stationary models and explosive volatility behavior. Level-GARCH combines single factor and GARCH. It captures more accurately the behavior of interest rates. |
| Borokhovich-Brunarsky-Crutchley-Simkins | 2004 | To examine the relation between the composition of a firm's board of directors and the firm's use of interest rate derivatives. | A significant positive relation is observed between the amount of interest rate derivatives applied by firms and the proportion of outside directors on firm's board of directors. |
| Goard-Hansen | 2004 | To develop a time dependent short term interest rate model and compare it with models developed by Ahn-Gao (1999) and Goard (2000). | Goard-Hansen model is able to determine the time dependence on mean reversion level even for shorter periods of time. A time dependent model determines more accurately the term structure. |
| Burashi-Corielli | 2005 | To analyze advantages and disadvantages of model recalibration for pricing and hedging objectives. | Model recalibration violates the assumptions in which models are developed. Model updating could have serious consequences on risk management strategies. |
| Cassasus-Collin-Dufresne- | 2005 | To propose a parsimonious unspanned stochastic volatility model of the term structure and analyze | Drift and quadratic variation of the short rate are affine in three state variables; including short rate, long term mean and variance. |

| | | | |
|---|---|---|---|
| Goldstein | | its implications for fixed income option prices. | Bond prices are affine functions of only two state variables. |
| Goltz | 2005 | To create a Protective Put Buying strategy by applying Longstaff-Schwartz model with the goal of determining additional benefits to investors. | A Protective Put Buying strategy dominates the bond futures strategy since PPB strategy results into lower downside risk and higher returns. |
| Mato | 2005 | To analyze the relation between classic measures and modern risk measures by examining portfolios with different maturities and structures. | An absence of relationships between portfolios optimized by classic measures and modern measures is observed. These measures lead to considerable different portfolios. |
| Faff-Gray | 2006 | To develop simulation experiments for determining bias and accuracy of GMM estimates of short rate parameters. | GMM avoids estimation difficulties observed in alternative approaches. It presents difficulties for estimating the speed of mean reversion. This problem is not exclusively of GMM method. |
| Platen | 2006 | To obtain a unified framework for portfolio optimization, derivative pricing, financial modeling and risk measurement. | The growth optimal portfolio as a unifying measure is obtained. For the minimal market model described; there is no equivalent risk neutral martingale measure. |
| Reno-Roma | 2006 | To estimate the diffusion coefficient of single factor models for the short rate by applying non parametric methods. | Aït-Sahalia estimator is not applicable for values of mean reversion. Stanton and Bandi-Phillips models perform better. Estimators depend on the choice of bandwidth selected. |
| Sanford-Martin | 2006 | To analyze several single factor continuous time models for the Australian short rate. | According to Bayes factors; it is concluded that Cox-Ingersoll-Ross Square Root model has the greatest support from all models. |
| Beltratti-Colla | 2007 | To focus on essentially affine and completely affine term structure models as tools for active bond portfolio management. | In the case of the term structure of interest rates; it is not possible to determine a best model. The model applied for solving a situation will depend on the objective of the decision maker. |
| Halkos-Papadamou | 2007 | To examine the significance of risk modeling and asymmetries when testing different theories for the term structure of interest rates. | Non linear effects of spreads on excess holding period yields are observed in all maturities being the short term an exception. Mean reversion in international bond markets is observed. |
| Koutmos-Philippatos | 2007 | To investigate short term mean reversion in major European countries by applying Longstaff-Schwartz and Bali interest rate models. | Asymmetric mean reversion is observed in all European interest rates analyzed. Interest rates are non stationary following interest rate increases and mean reverting following interest rate decreases. |
| Mahdavi | 2008 | To demonstrate the behavior of short term interest rates in seven countries and the Euro zone under the no arbitrage condition. | A single factor model able to nest several models is estimated. Short term rates do not follow simple mean reversion process. No single model can explain the short term process for all countries. |

# 3 Theories of the term structure of interest rates

In order to be able to evaluate and understand correctly the term structure of interest rates; it is important to comprehend the different theories involved. It is required to understand how the spot rates or discount factors are determined; as well as to comprehend the explanations that determine the shape of the term structure of interest rates. According to Nelson (1972); the term structure of interest rates is determined mainly by two different theories, which are the liquidity preference theory and the preferred habitat theory.

Moreover; Gibson, Lhabitant and Talay (2001) agree that the term structure of interest rates is mainly explained by three theories which analyze the relationship between interest rates of various maturities and the value of the term premium. These theories include the expectations hypothesis, the liquidity preference theory and the preferred habitat theory.

However; according to Cox, Ingersoll and Ross (1985), there are mainly four tendencies with respect to the term structure of interest rates. These include expectations hypothesis, liquidity preference theory, market segmentation hypothesis and preferred habitat theory. In the following subsections, the different theories of the term structure of interest rates will be briefly described.

## *3.1 Liquidity Preference Theory*

This theory was developed by Hicks. It predicts that a term premium may be obtained by capital invested in long term bonds because bond holders will require compensation for exposure to capital fluctuations. (Nelson; 1972) According to the liquidity preference theory, investors are risk averse, prefer short term maturities and will require a premium in order to commit in long term securities. (Gibson et al; 2001)

Liquidity Preference Theory admits the importance of expected future spot rates but gives more importance to the effects of the risk preferences of market participants. This hypothesis states that risk aversion will cause forward rates to be greater than expected spot rates by an amount which increases with maturity. The term premium is the increment given to investors in order to hold longer term securities since those imply higher risk. (Cox et al; 1985)

Telser (1966) rejects that the actual term structure provides unbiased forecasts of future interest rates. Risk averse lenders are more concerned towards the stability of principal rather than the stability of income. In addition; it is stated that universal risk aversion of borrowers and lenders, does not constrain the term structure in the way described by the liquidity preference theory.

The term structure is determined by,

$$R(t,T) = \frac{1}{T-t}\left[\int_t^T E_t(r(s))ds + \int_t^T L(s,T)ds\right], \tag{1}$$

where $L(t,T) > 0$ denotes the instantaneous term premium at time $t$ for a bond maturing at time $T$. (Gibson et al; 2001)

### *3.2 Preferred Habitat Theory*

This theory was developed by Modigliani and Sutch (1966). It states that market participants are assumed to have preferred maturity ranges but will decide to change their selected habitat if a enough term premium is offered. (Nelson; 1972) According to the preferred habitat theory, investors and borrowers have different specific time horizons. (Gibson et al; 2001)

This theory uses some arguments similar to those of the market segmentation theory. The purpose of this theory is to have a credible logic for term premiums which does not restrict them in sign. (Cox et al; 1985)

The term structure is determined by,

$$R(t,T) = \frac{1}{T-t}\left[\int_t^T E_t(r(s))ds + \int_t^T L(s,T)ds\right], \tag{2}$$

where $L(s,T)$ refers to the risk premium which can take positive, negative or zero values depending on the offer and demand of bonds. Therefore; the term structure of interest rates can take any form. (Gibson et al; 2001)

### *3.3 Expectations theory*

This theory gives increasing importance on the expected values of future spot rates. This theory states that bonds are priced so that the implied forward rates are equal to the expected spot rates. This hypothesis is characterized by two propositions. The first proposition states that the return on holding a long term bond to maturity is equal to the expected return on repeated investment in a series of short term bonds. On the second proposition; it is mentioned that the expected rate of return over the next holding period is the same for bonds of all maturities. (Cox et al; 1985)

Telser (1966) states that spot interest rates of loans of different maturities depend uniquely on market expectations of future interest rates. The current term structure forecasts the future term structure.

According to the expectations theory, it is argued that the term structure of interest rates is driven by the investors' expectations on future spot rates; where the forward rate is an unbiased estimator of the future spot rate. The rate of return on a bond maturing at time $T$ should be equivalent to the geometric average of the expected short term rate from $t$ to $T$.

The term structure is determined by,

$$R(t,T) = \frac{1}{T-t}\int_t^T E_t(r(s))ds, \tag{3}$$

where $t$ denotes the starting point, $T$ represents maturity and $r(s)$ represents the short term rate. However; this theory can be divided into four different subcategories; which represent four different interpretations of the expectation hypothesis. These interpretations include the naïve expectation hypothesis, the local expectation hypothesis, the return to maturity expectations hypothesis and the unbiased expectation hypothesis. (Gibson et al; 2001)

### *3.3.1 Naïve expectation hypothesis*

This hypothesis states that the expected return on any strategy for any holding period is the same. The expected return on any investment should be equivalent between investing in a long term bond and rolling over a short term period. Therefore; the investor should be indifferent between these two investment's strategies. (Gibson et al; 2001)

The term structure is determined by,

$$-\frac{\ln B(t,T)}{T-t} = E\left[\frac{1}{T-t}\int_t^T r(s)ds\right]. \tag{4}$$

### *3.3.2 Local expectation hypothesis*

This hypothesis is similar to the naïve expectation hypothesis; since it suggests that the expected returns on bonds with different maturities should be the same over a short term investment horizon. (Gibson et al; 2001)

The term structure is determined by,

$$B(t,T) = E\left[ e^{e - \int_t^T r(s)ds} \middle| r(t) \right]. \tag{5}$$

*3.3.3 Return to maturity expectations hypothesis*

This hypothesis states that the expected return on holding any bond until maturity will have the same expected return as rolling over a set of short term bonds. This hypothesis is also known as the Lutz hypothesis. (Gibson et al; 2001)

The term structure is determined by,

$$\frac{1}{B(t,T)} = E\left[ \exp \int_t^T r(s)ds \middle| r(t) \right]. \tag{6}$$

*3.3.4 Unbiased expectation hypothesis*

This hypothesis assumes that the forward rate is equal to the future expected spot rate. The unbiased expectation hypothesis is also known as the Malkiel hypothesis. (Gibson et al; 2001)

The term structure is determined by,

$$\frac{\partial B(t,T)/\partial T}{B(t,T)} = E[r(T)]; \tag{7}$$

which is also equivalent to,

$$-\ln B(t,T) = \int_t^T E(r(s)ds). \tag{8}$$

According to Gibson et al (2001), by applying the Jensen inequality[2]; it is demonstrated that these theories are mutually inconsistent. However; there is an exception, which includes the unbiased expectation hypothesis and the naïve expectation hypothesis.

[2] Jensen inequality relates the value of a convex function of an integral to the integral of the convex function. For a real convex function $\varphi$, with $x_i$ numbers, and positive weights denoted by $\alpha_i$; Jensen's inequality can be determined by $\varphi\left(\frac{\sum \alpha_i x_i}{\sum \alpha_i}\right) \le \frac{\sum \alpha_i \varphi(x_i)}{\sum \alpha_i}$, where the inequality is reversed if $\varphi$ is concave.

### *3.4 Market segmentation hypothesis*

This theory is developed by Culberston. Under the market segmentation hypothesis; individuals have strong maturity preferences. Bonds of different maturities trade in separate and different markets. Demand and supply of bonds of a particular maturity are affected by the prices of bonds of other maturities. (Cox et al; 1985)

However; these theories present several difficulties. First, better understanding of the determinants involved in the term premium is required. Second, all theories are developed in ex ante terms and they must be related to ex post realizations in order to be testable. (Cox et al; 1985)

Cox et al (1985) consider the problem of determining the term structure as being a problem in general equilibrium theory, which include elements of all previous theories. The model takes into consideration anticipations of future events, risk preferences and consumption timing preferences or preferred habitat. This model allows for detailed predictions about how changes in a diverse range of underlying variables will affect the term structure.

# 4 Interest Rate Models

The term structure of interest rates measures the relationship among the yields on default free securities that differ only in their term to maturity. Term structure demonstrates market anticipations of future events. By understanding the term structure; it is possible to predict how changes in the underlying asset will affect the yield curve. (Cox et al; 1985)

Risk free interest rates are a major element in Finance. It is assumed that managers, investors, bankers and policy makers would be very interested in obtaining the most accurate indicator of future interest rate levels. As a result; several models and techniques have been developed with the objective of modeling, estimating and forecasting the term structure of interest rates.

According to Gibson et al (2001); continuous time models provide a more adequate framework to produce more accurate solutions and more exact hypothesis. Many interest rate models are only models of the stochastic evolution of a given interest rate; which is usually determined by the short term rate. This interest rate is often defined as Markovian since its future evolution only depends on its current value and not on historical information.

Factor models assume that the term structure of interest rates is driven mainly by three terms; which include the shift of the term structure, the twist and the butterfly effect. The shift of the term structure is a parallel movement of all rates and it accounts for around 80 to 90 percent of the total variance. The twist is the situation in which long rates and short rates move in opposite directions. It accounts for around 5 to 10 percent of the total variance. Finally; the third effect is the butterfly effect in which the intermediate rate moves in opposite direction of the short and long term rate. It accounts for approximately 1 to 2 percent of the total variance. (Gibson et al; 2001)

Since the shift of the term structure explains a large fraction of the yield curve; it could be possible to reduce the model into a one factor model. However; some securities are very sensitive to the shape of the term structure and to volatility. Therefore; in these cases more complicated models are required. (Gibson et al; 2001) i.e. two-factor models, multifactor models, volatility models between others.

Aas (2004) argues that due to the complex dynamics of possible forecasting abilities; no consensus has emerged for modeling interest rates and its volatility. Consistent with this opinion; Brenner et al. (1996), affirm that given the importance of the short term riskless interest rate; no consensus has emerged on the dynamics of level or volatility of interest rates.

Interest rates display conditional volatility patterns that are not only a function of past interest shocks but also are considered as some function of the lagged level of the series. (Aas; 2004) According to Chan et al (1992); several models underperform because of their restrictions on modeling adequately the term structure volatility. It is stated that short term riskless interest rates are one of the most fundamental and important prices determined in financial markets.

Brenner et al. (1996); state that the choice of a model for determining short term rates is crucial for the pricing of bonds, interest rate derivatives, and hedging interest rate risk. This is a consequence of current level and stochastic properties of volatility, since these elements affect the dynamic hedge ratios and the distribution of future interest rate levels; which determine a derivative's price. It is clarified that the use of an incorrect model could lead to incorrect inferences, mishedged or unhedged risks or pricing errors.

Due to the importance of determining the level of interest rates ex ante; the list of models which try to describe and forecast interest rates is rather extensive. Chan et al (1992) state that more models have been developed for explaining interest rate behavior than for any other issue in Finance within a continuous time setting.

Some of these models include the ones developed by Merton (1973), Brennan and Schwartz (1979, 1980, 1982), Vasicek (1977), Dothan (1978), Hull (1993), Cox. Ingersoll and Ross (1980, 1985), Schaefer and Schwartz (1984), Longstaff (1989), Hull and White (1990, 1994), Black and Karasinski (1991), Longstaff and Schwartz (1992), Black, Derman and Toy (1987, 1990), Sandmann and Sondermann (1993), Miltersen, Sandman and Sondermann (1997), Rendleman and Bartter (1980), Cox (1975), Cox and Ross (1976), Black and Scholes (1973), Ball and Torous (1983), Langetieg (1980), Duffie and Kan (1993), Richard (1978), Fong and Vasicek (1991, 1992, 1993), Chen (1996), Ho and Lee (1986), Heath, Jarrow and Merton (1992), Goldstein (1997) and Kennedy (1997).

The following table describes more in detail the interest rate models previously developed by the above mentioned authors. Parameters of the different models are specified, a brief description is provided as well as general results[3].

[3] For a complete mathematical description of each one of these models, please refer to Gibson, Lhabitant and Talay (2001).

**Table II.** *Model descriptions*

Several single factor, lognormal, affine, multifactor, multiperiod binomial and infinite factor interest rate models are described with the objective of providing a general overview of the particular specifications of each one of these models as well as to provide advantages and disadvantages from the models previously developed in the Financial Literature.

| Category | Model | Year | Specification | Description | Findings |
|---|---|---|---|---|---|
| Single Factor Model | Merton | 1973 | $dr(t) = \mu_r dt + \sigma_r dW(t)$ | This model is the first one developed for modeling the term structure of interest rates. This model is relevant to the Financial literature because it explains the behavior of interest rates. Even that this model presents bias; it is important because it is the starting point from the vast diversity of short term interest rate models developed. | This model assumes a constant risk premium and allows for negative values of interest rates. |
| Single Factor Model | Black-Scholes | 1973 | $\frac{dB(t,T_B)}{B(t,T_B)} = \mu dt + \sigma dW(t)$ | Both authors developed the complete framework for option valuation assets in which the price follows a Geometric Brownian | The model could be considered as rather simple. However; it presents disadvantages; such as, the volatility parameter decreases as time approaches |

| | | | | | |
|---|---|---|---|---|---|
| | Merton | 1973 | | Motion. This model is used extensively for short dated options on long term bonds. | to maturity, the short term rate is normally distributed and can have negative values. The model does not restrict arbitrage opportunities. |
| Single Factor Models | Cox<br>Cox-Ross | 1975<br>1976 | $dr(t) = \mu r(t)dt + \sigma_r r^{\gamma}(t)dW_t$ | The short term rates dynamics is modelled by applying a constant-elasticity of variance diffusion. | The model nests models developed by Dothan (1978), Brennan-Schwartz (1980) and Cox-Ingersoll-Ross (1980). |
| | Vasicek | 1977 | $dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$ | Vasicek model defines an elastic random walk around a trend with a mean reverting characteristic. Short term interest rates are modeled following an Ornstein-Uhlenbeck process. | Similar to Merton's model; Vasicek postulates a constant risk premium. |
| | Dothan | 1978 | $dr(t) = \sigma_r r(t)dW(t)$ | The short term rate is lognormally distributed.<br>The model follows a Geometric Brownian Motion. This model is also known as a Geometric Random Walk. | The term structure is a monotonically decreasing function of time to maturity, an increasing function of interest rates and a decreasing convex function of volatility. |

| | | | | | |
|---|---|---|---|---|---|
| Single Factor Models | Brennan-Schwartz | 1980 | $dr(t) = \kappa(\theta - r(t))dt\sigma + r(t)dW(t)$ | Brennan and Schwartz extend Dothan's model by adding a mean reverting term. | The distribution for $r(t)$ is unknown and contingent claim prices must be computed using numerical methods. |
| | Rendleman-Bartter | 1980 | $dr(t) = \mu_r r(t)dt + \sigma_r r(t)dW(t)$ | This model assumes that $r(t)$ follows a Geometric Brownian Motion with constant drift and diffusion parameters. | This model is studied by Marsh and Rosenfeld (1983). |
| | Cox-Ingersoll-Ross | 1980 | $dr(t) = \sigma_r r(t)^{3/2} dW_t$ | This model is proposed to study the behavior of variable rate securities. | The model is also applied by Constantinides and Ingersoll (1984) to value bonds in the presence of taxes. |
| | Ball-Torous | 1983 | $\frac{dB(t,T_B)}{B(t,T_B)} = \mu dt + \sigma dW(t)$ | The model allows the constraint that bond prices approach to its face value at maturity by assuming a Brownian bridge process instead of a Geometric Brownian Motion. | The model presents several disadvantages. These include that the model is incompatible with the initial term structure, volatility is homoskedastic, and the model allows for arbitrage opportunities. |

| | | | | | |
|---|---|---|---|---|---|
| Single Factor Models | Cox-Ingersoll-Ross | 1985 | $dr(t)=\kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$ | Cox-Ingersoll-Ross developed an equilibrium model in which interest rates are determined by the supply and demand of individuals having a logarithmic utility function. | The short rate process developed is known as the Square Root Process. It is similar to Vasicek's model with the exception that the volatility is assumed to be heteroskedastic. |
| | Longstaff | 1989 | $dr(t)=\kappa\left(\theta - \sqrt{r(t)}\right)dt + \sigma\sqrt{r(t)}dW(t)$ | This model is a modification from the model developed by Cox-Ingersoll-Ross. This model is also known as the Double Square Root Model. | This model provides a closed form solution for the pricing zero coupon bonds. Empirical evidence suggests that this model outperforms Cox-Ingersoll-Ross's model. |
| Single Factor Time Varying Process | Hull | 1993 | $dr(t)=(\theta(t) - \kappa(t)r(t))dt + \sigma(t)r^{\beta}(t)dW(t)$ | This model differs from the previous ones because it allows for an exogenous determination of the risk premium; determined as, $\lambda(r,t)=\lambda r^{\gamma}$. The model allows for time varying parameters. However; the model should be restricted to only one time varying parameter. | The functions $\theta(t)$, $\kappa(t)$, $\sigma(t)$, are time varying and can replicate the model to current market prices. Furthermore; the volatility parameter may be non stationary. The model allows for negative interest rate values. |

| | | | | | |
|---|---|---|---|---|---|
| Lognormal Models | Black-Derman-Toy | 1987 1990 | $d\ln(r(t)) = (\theta(t) - a\ln(r(t)))dt + \sigma_r dW(t)$ | This model incorporates the mean reversion behavior of interest rates. By assuming a log normal process, the possibility of negative values disappears. | This model can price exactly any set of discount bonds and current market information can be represented. However; the model lacks of analytical properties and its implications are unknown. |
| | Black-Karasinski | 1991 | $d\ln(r(t)) = (\theta(t) - \kappa(t)\ln(r(t)))dt + \sigma_r(t)dW(t)$ | This model is a binomial tree approach with time steps of different lengths. It is an extension of Black-Derman-Toy's model (1990),<br>This model allows for time varying reversion speed $(\kappa(t))$. | The model is able to fit the yield curve, volatility curve and cap curve.<br>Rates explode with positive probability, which implies infinite expected roll over returns and arbitrage opportunities. |
| | Sandmann-Sondermann | 1993 | $\frac{dr^*(t)}{r^*(t)} = \mu_{r^*}(t)dt + \sigma_{r^*}(t)dW(t)$ | This model tries to correct the explosion results obtained in Black-Karasinski model by specifying the instantaneous rate as a simple interest rate over a fixed finite period of time. | The model implies that the continuously compounded rate follows a diffusion that is neither normal nor lognormal but a dynamic combination of both. |

| | | | | | |
|---|---|---|---|---|---|
| Lognormal Model | Miltersen-Sandman-Sondermann | 1997 | $\frac{df^*(t,T_1,T_2)}{f^*(t,T_1,T_2)} = \mu_{f^*}(s,T_1,T_2) + \sigma_f(s,T_1,T_2)dW(t)$ | This model is a lognormal term structure model for simple annual forward rates. The initial term structure is used to develop the model. | This model is very similar to the one developed by Heath-Jarrow-Morton (1992) with the exception that it applies a simple forward rate instead of a continuous forward rate. |
| Affine Model | Duffie-Kan | 1993 | $r(t) = \delta Y(t)$ <br> $\sigma_r^2(t, r(t)) = a(t)b(t)r(t)$ | In this model; the drift and volatility coefficient of the state variable processes are affine functions; implying that both factors are interrelated. | This framework allows for the decomposition of term structure models into their primary factors; including the curvature, twist and slope. |
| Multifactor Model | Langetieg | 1980 | $dx_i(t) = \kappa_i(\theta_i - x_i(t))dt + \sigma_i dW_i(t)$ | This model is an extension of Vasicek's model by assuming that the short term rate is a sum of *n* factors. | A single factor model can always be extended to a multifactor setup by defining the short term rate as a function of stochastic factors with the purpose of adding economical signification. |
| Multifactor Model | Richard <br> Cox-Ingersoll-Ross | 1978 <br> 1985 | $dq(t) = \mu_q(t)dt + \sigma_q(t)dW_q(t)$ <br> $d\pi(t) = \mu_\pi(t)dt + \sigma_\pi(t)dW_\pi(t)$ | In this model, the term structure of interest rates are determined by two factors; which are the real short term rate and the expected instantaneous inflation rate | This model allows for the development of analytical solutions for the pricing of zero coupon bonds. |

| | | | | | |
|---|---|---|---|---|---|
| | | | | determined by two independent Brownian motions. | |
| Multifactor Models | Brennan-Schwartz | 1979 1982 | $dr(t) = \mu_r(\ )dt + \sigma_r(\ )dW_r(t)$ $dl(t) = \mu_l(\ )dt + \sigma_l(\ )dW_l(t)$ | A two factor model is developed, in which the term structure of interest rates depend on the short term rate and the long term rate. | This specification allows the model to reflect the assumption that long term rates contain some information about the future value of the short rate. |
| | Schaefer-Schwartz | 1984 | $ds(t) = m(\mu - s(t))dt + \gamma dW_1(t)$ $dl(t) = \beta(s(t), l(t), t), + \sigma\sqrt{l(t)}dW_2(t)$ | This model is a two factor model in which the term structure of interest rates is determined by the long term rate and the spread between the long and the short rate. | The selection of these variables is based on orthogonality conditions. The market price of risk is assumed to be constant. |
| | Longstaff-Schwartz | 1992 | $dX(t) = (a - bX(t))dt + c\sqrt{X(t)}dW_2(t)$ $dY(t) = (d - eY(t))dt + f\sqrt{Y(t)dW_2(t)}$ | This model is a two factor model in which the term structure model is determined by the short term rate and volatility. The model takes into consideration the investment and consumption decisions and can be considered as an affine two factor model. | The market price of risk is endogenously determined by the model instead of exogenously imposed. This allows that the risk premium is consistent with the absence of arbitrage. The model allows for analytical expressions of options on zero coupon bonds. |

| | | | | | |
|---|---|---|---|---|---|
| Multifactor Models | Fong-Vasicek | 1991 1992a 1992b | $dr(t)=\alpha(\bar{r}-r(t))dt+\sqrt{v(t)}dW_1(t)$ | This is a two factor model in which the term structure of interest rates is derived by the short term rate and the variance of changes in the short term rate. | This process is very similar to Vasicek's model. However; it includes an additional uncertain parameter determined by a stochastic variance which evolves under a risk neutral measure. |
| | Hull-White | 1994 | $dr(t)=(\theta(t)+u-r(t))dt+\sigma_1 dW_1(t)$<br>$du(t)=-bu(t)dt+\sigma_2 dW_2(t)$ | This model is an extension of their previous model. This model is similar to the one factor model but with a stochastic drift. In addition; the model belongs to the affine class of models. | In this model, the short term rate is mean reverting, *u* is a component of the mean reversion level and it is mean reverting to 0 at a rate of *b*. |
| | Chen | 1996 | $dr(t)=\kappa(\theta(t)-r(t))dt+\sqrt{\sigma(t)}\sqrt{r(t)}dW_2(t)$<br>$d\theta=\upsilon(\bar{\theta}-\theta(t))dt+\xi\sqrt{\theta}dW_1(t)$<br>$d\sigma(t)=\mu(\bar{\sigma}-\sigma(t))dt+\eta\sqrt{\sigma}dW_3(t)$ | This is a three factor model in which the term structure of interest rates depends on the current short rate, the stochastic mean and the stochastic volatility of the short term rate. | This model allows for obtaining closed form solutions for discount bonds and interest rate derivatives in very specific cases. |

| | | | | | |
|---|---|---|---|---|---|
| Multiperiod Binomial Model | Ho-Lee | 1986 | $r(t)=r(t-1)+(f(0,t)-f(0,t-1)) + \log\left(\frac{\pi+(1-\pi)\delta^{t}}{\pi+(1-\pi)\delta^{t-1}}\right) -(1-\pi l)\log(\delta)+\varepsilon_t$ | The initial term structure is exogenously obtained. It follows a lattice model with upward and downward movements, tending towards a binomial tree. The probabilities of upward and downward movements are determined by risk neutral probability measures. | The model will always generate upward sloping term structures, it does not incorporate any mean reversion feature leading to explosive or negative values. The model assumes homoskedasticity and it is not arbitrage free in all cases. |
| | Heath-Jarrow-Morton | 1992 | $df(t,T)=\mu_f(t,T)dt+\sigma_f(s,T)dW(s)$ | This model is an extension of Ho-Lee model in three directions. This model takes into consideration forward rates; allows for continuous trading, and it is extended for allowing multiple factors. | This model explains the whole term structure dynamics in an arbitrage free framework and it is completely compatible with an equilibrium model. |
| Infinite Factor Model | Goldstein Kennedy | 1997 1997 | $df(t,T)=\mu_f(t,T)dt+\sigma_f(t,T)dW_T(t)$ | This is a new class of models for modeling the term structure of interest rates. It is based on random fields. In this model, forward rates follow a diffusion process in which for each maturity date there is a corresponding Brownian motion. | Consistent with Heath-Jarrow-Morton model, the drift coefficient is determined by the volatility structure and the correlation structure.However; the model lacks of PDE for interest rate contingent claims. |

However; even though there is an extensive amount of models available; little is known about how these models compare in their ability to capture the behavior of the short term interest rate. The main reason could be the absence of a common parameter or framework in which different models could be classified and analyzed; since without a common framework it is impossible to realize a correct comparison.

As a result; Chan et al (1992) grouped several models into a stochastic differential equation in order to compare the most important differences between models. It is concluded that the main aspect that differentiate the models is their ability to capture the time varying volatility of the short term interest rate.

Aas (2004) compares different models which could be used for determining the term structure of interest rates. These models include single factor diffusion, GARCH (General Autoregressive Conditional Heteroskedasticity), regime switching and jump diffusion models. Each model is described according to its advantages and disadvantages.

In the following sections, different models are presented and clarified in order to understand more in detail their structures, descriptions, advantages and disadvantages. Models are grouped according to the families of models in which they belong. Furthermore; Chapter 5 illustrates more in detail comparisons between different models by nesting them into common frameworks.

### *4.1 Single factor models*

Single factor models assume that all information about term structure at any point in time can be summarized by one single specific factor; specified as the short term interest rate. Therefore; only the short term interest rate and time to maturity will affect the price of an interest rate contingent claim. (Gibson et al; 2001)

According to Aas (2004), single factor models present bias in modeling accurately the complex dynamics of interest rates; since this kind of models parameterize the volatility only as a function of interest rate levels but fail to capture adequately the serial correlation in conditional volatility. However; single factor models are broadly used in practice because of their ability to fit the dynamics of short term interest rates.

Brennan et al (1996) state that the strongest criticism on single factor models is that it restricts volatility to be a function of interest rate levels only and not to the news arrival process. As an alternative approach; GARCH models arise, since in these models, actual volatility is a function of last period's unexpected news.

However; GARCH models also present bias in the parameters when modeling short term interest rate volatility. GARCH models do not permit volatility to be a function of interest rate levels. Empirical applications find that $\hat{a}_1 + \hat{b} \approx 1$. This means that current shocks affect volatility infinitely far into the future so shocks will never die away. In addition; GARCH models allow for negative interest rates.

In the following subsections, some important single factor, volatility and two-factor models will be presented.

### *4.1.1 Merton Model*

Merton (1973) was the first one to propose a single factor model for the term structure of interest rates. Merton's model is very important because after the development of his model; several additional modifications and improvements have been made in order to model and forecast the term structure of interest rates in the most accurate form. As a result; several additional models have been developed.

Merton's model presents bias since it allows for negative interest rates and also it assumes that volatility is constant. These assumptions do not hold in reality. Therefore; the model is seem as inaccurate, since it is not able to describe the behavior of interest rates. According to Gibson et al (2001), the lack of boundary conditions on the first and second moments of the distribution (mean and variance), allow for negative and infinite interest rates. Therefore: the model is seen as insufficient since it lacks of stability.

The short term rate model developed by Merton can be described as,

$$dr(t) = \mu_r dt + \sigma_r dW(t), \tag{9}$$

where $\mu_r$ and $\sigma_r$ are constant and $W(t)$ is a standard Brownian motion. In addition; in this model, a constant risk premium, $\lambda$, is assumed.

### *4.1.2 Vasicek Model*

Vasicek (1977) proposes to model the short term interest rate as an Ornstein-Uhlenbeck[4] process. As a result; a model which defines elastic random movements around a trend with a mean reverting characteristic when $r(t)$ moves over or under $\theta$ is developed. The expected variation of $r(t)$ becomes negative or positive and $r(t)$ tends to return to

---

[4] Ornstein-Uhlenbeck process is also known as a mean reverting process. It is a stochastic process denoted by the stochastic differential equation, $dr_t = -\theta(r_t - \mu)dt + \sigma dW_t$, where $\theta$, $\mu$, $\sigma$ are parameters and $W_t$, is a Wiener process. This process is the continuous time analogue to the discrete time AR (1) process. It has bounded variance and allows a stationary probability distribution.

its average long term level, $\theta$, at an adjustment speed, $\kappa$. In addition; a constant risk premium is established.

The short term interest rate model developed by Vasicek can be described as,

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t), \quad (10)$$

where $\kappa, \theta, \sigma$ are positive constants and $W(t)$ is a Wiener process. The mean reversion process reduces the probability of unreasonable large or low interest rates. The term structure can have a positive, negative or convex shape. In addition; the volatility term structure of the yields is a decreasing function of time to maturity with a boundary restriction value of zero. (Gibson et al; 2001)

### *4.1.3 Cox, Ingersoll, Ross Model*

Cox et al (1985) develop a single factor model of the term structure of interest rates determined by the supply and demand of individuals having a logarithmic utility function. It is assumed that the state of technology can be represented by a single state variable. Additionally; it is stated that changes in production opportunities over time are described by a single state variable. Means and variances of rates of returns on the production processes are proportional to future production opportunities. The objective is that neither the means nor the variances could dominate the portfolio decision for large values of future production opportunities. This model is also known as the Square Root model.

The model developed by Cox, Ingersoll and Ross (1985) can be described as,

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{rdz_1}, \quad (11)$$

where $\kappa$ determines the speed of adjustment, $\theta > 0$ is a continuous time first order autoregressive process and $z_1$ is a Wiener process.

This model has important implications with respect to the term structure of interest rates. It allows for several properties, which include: elimination of negative interest rates; if the interest rate reaches zero, it can become positive but will never become negative; the absolute value of interest rate increases when interest rate increases. Finally; there is a steady state distribution of interest rates. (Cox et al; 1985) This model has an important improvement with respect to Vasicek's model since the variance is proportional to the short term rate and it is not constant. (Gibson et al; 2001)

Under this model; anticipations, risk aversion, investment alternatives and preferences about the timing of consumption play a key role in determining the term structure. The model includes the main factors consistent with maximizing behavior and rational expectations. (Cox et al; 1985)

### *4.1.4 Chan, Karolyi, Longstaff, Sanders Model*

Chan et al (1992) propose one of the most general single factor models. Their model provides a basic description of the stochastic nature of interest rates consistent with empirical evidence that states that interest rates tend to be mean reverting.

The most general single factor model is described as,

$$r_t = \alpha_0 + (1+\alpha_1) r_{t-1} + \varepsilon_t ,$$

$$E(\varepsilon_t) = 0 ,$$

$$Var(\varepsilon_t) = \sigma^2 r_{t-1}^{2\gamma} , \tag{12}$$

where $\alpha_1$ determines the speed of mean reversion towards a stationary level, $r_{t-1}$ is the variable lagged one day, $\varepsilon_t$ is the error term and $\gamma \geq 0$ is the elasticity of the volatility against the level of interest rate.

Chan et al (1992) applied their model as a common framework in order to provide empirical evidence with respect to the factors that could affect the term structure of interest rates. It was found that when models are fitted into a common framework; models could be nested together in order to compare their performance in capturing the stochastic behavior of the short term interest rate.

According to Brenner et al (1996), the strongest criticism of single factor models is that it restricts volatility to be a function of interest rate levels and not of the news arrival process. As an alternative; GARCH models are proposed since in GARCH models, this period’s volatility is a function of last period’s unexpected news.

## *4.2 Volatility Models*

According to Brooks (2005); to model and forecast volatility has been the topic of extensive empirical and theoretical research over the last couple of years. Volatility is observed as one of the most important concepts in the field of Finance. It is measured as the standard deviation or variance of returns. It is often used as a measure of total risk on financial assets. Several models have been developed with the objective of determining and forecasting volatility levels. Models able to capture appropriately the dynamics of volatility include Exponentially Weighted Moving Average models

(EWMA), autoregressive volatility models, Autoregressive Conditional Heteroskedastic models (ARCH) and Generalized ARCH models (GARCH). In the following sections, several volatility models adapted to the term structure of interest rates will be briefly explained.

### *4.2.1 GARCH Models*

General Autoregressive Conditional Heteroskedasticity (GARCH) model was developed independently by Bollerslev (1986) and Taylor (1986). This kind of models allows for volatility clustering and volatility persistence. According to Aas (2004), returns from financial markets such as interest rates measured over short time intervals are characterized by volatility clustering; which indicates that small changes in price are followed by small changes and large changes are followed by large changes. GARCH models have succeeded in capturing volatility clustering for equity prices and interest rates.

GARCH model allows the conditional variance to be dependent on previous lags, so the conditional variance equation in its simplest case can be seen as,

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2. \tag{13}$$

Equation (13) is a GARCH (1,1) model, where $\sigma_t^2$ is known as the conditional variance since it is a one period ahead estimate for the variance. By applying a GARCH model, it is possible to interpret the current fitted variance as a weighted function of a long term average value, information about volatility during the previous period and the fitted variance from the model during the previous period. (Brooks; 2005; pp.453)

GARCH models are better and more widely used since this kind of models are more parsimonious and avoid overfitting. Therefore; the model is less probable to violate non-negativity constraints. It is observed that the GARCH (1,1) model containing only three elements in the conditional variance equation is a very parsimonious model that allows an infinite number of past squared errors to influence the current conditional variance. (Brooks; 2005; pp. 455)

It is possible to extend GARCH (1, 1) model into a GARCH (p, q) formulation. However; any higher order GARCH model is rarely used within the academic financial literature since it is considered that a GARCH (1, 1) model will be enough to capture volatility clustering in the data. (Brooks; 2005; pp. 455)

According to Bali (2003); GARCH models that incorporate volatility only as a function of interest rate shocks fail to capture the relationship between interest rate levels and volatility. Results are consistent across different periods, maturities and short rate drifts.

An univariate GARCH (1, 1) model for interest rates can be determined as,

$$
\begin{aligned}
&r_t = \alpha_0 + (\alpha_1 + 1) r_{t-1} + \varepsilon_t, \\
&E(\varepsilon_t) = 0, \\
&Var(\varepsilon_t) = \sigma_t^2, \\
&\sigma_t^2 = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 \sigma_{t-1}^2,
\end{aligned}
\tag{14}
$$

where $\varepsilon_t$ is serially independent, $\alpha_1$ determines the speed of mean reversion and $\beta_1 + \beta_2$ determines whether the process is stationary or not.

A process is said to be stationary if $\beta_1 + \beta_2 < 1$; otherwise the process is said to be non stationary and possible shocks will never die away. When fitting a GARCH (1, 1) model to the short term interest rates; estimates corresponding to non stationarity are often observed. (Koedjk et al; 1997) However; if we assume non stationarity; persistence of shocks will extend indefinitely into the future and it would not be possible to determine or forecast future volatility behavior since shocks will tend to explosive patterns. As a result; future levels of interest rates will not be possible to forecast accurately ex ante. (Knif; 2007)

Brenner et al. (1996) suggests the existence of three problems when applying GARCH models to determine short term interest rate volatility. The first problem states that GARCH models do not allow volatility to be a function of interest rate levels. The second problem demonstrates the existence of empirical evidence where $\hat{a}_1 + b \approx 1$, implying explosive volatility effects after a given shock. The third problem clarifies that GARCH models allow for negative interest rates.

According to Bali (2003) and Rodrigues and Rubia (2003) this behavior is explained due to a model misspecification in GARCH (1, 1). It is said that GARCH (1, 1) does not capture the relationship between interest rates and volatility. Aas (2004) argues that level and GARCH effects need to be incorporated in the conditional volatility process in order to determine a correct specification.

Brooks (2005) suggests some models which are an improved version to the GARCH family of models. This improvement consists on the ability to better fit the behavior of short term interest rates; presenting a more accurate performance in comparison to GARCH (1, 1). The dominance of these models is mainly due to the ability to represent accurately asymmetric effects. Some of these models include EGARCH (Exponential GARCH) proposed by Nelson (1991) and the GJR model named after its authors Glosten, Jagannathan and Runkle (1993)

The EGARCH conditional variance equation is determined by,

$$\ln\left(\sigma_t^2\right) = \omega + \beta \ln\left(\sigma_{t-1}^2\right) + \gamma \frac{u_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \alpha \left[ \frac{|u_{t-1}|}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right]. \tag{15}$$

This model has several advantages over the GARCH specification. First, since the $\ln(\sigma_t^2)$ is modeled, then even if the parameters are negative, $\sigma_t^2$ will be positive. There is no need to impose non-negativity constraints on the model parameters. Second, asymmetries are allowed under the EGARCH formulation, since if the relationship between volatility and returns is negative, $\gamma$, will be negative. (Brooks; 2005; pp. 470)

On the other hand; the GJR model is a simple extension of GARCH model with an additional term added to account for possible asymmetries. The conditional variance is determined by,

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma u_{t-1}^2 I_{t-1} \tag{16}$$

where $I_{t-1} = 1$ if $u_{t-1} < 0$ and 0 otherwise.

Several researchers assume that one factor as well as GARCH models do not model correctly the term structure of interest rates and present bias in the estimations. According to Brenner, Harjes and Kroner (1996), GARCH models fail in capturing adequately the relationship between interest rates and volatility As a result, an extensive amount of models have been developed or adjusted with the objective of obtaining a better description with respect to future behavior on interest rates. Some of this modifications or new approaches have been developed by Brenner et al (1996), Aas (2004) between others.

### *4.2.2 Combined Level GARCH Models*

One factor models overemphasize the sensitivity of volatility to interest rate levels and fail to capture adequately the serial correlation in conditional variance. GARCH models that parameterize the volatility only as a function of unexpected interest rate shocks are not able to capture the relationship between interest rates and volatility. Models combining one factor models and GARCH models have been developed; creating as a result combined level GARCH models, since these models allow for a better fit of all elements involved.(Aas; 2004)

Some of these approaches were developed by Longstaff and Schwartz (1992), Brenner, Harjes, and Kroner (1996), Koedijk, Nissen, Schotman, and Wolff ( 1997), Bali (2003), between others.

Longstaff and Schwartz (1992) present a two factor model determined by,

$$r_t = \alpha_0 + (1+\alpha_1)r_{t-1} + \alpha_2\sigma_{t-1}^2 + \varepsilon_t ,$$
$$E(\varepsilon_t) = 0 ,$$
$$Var(\varepsilon_t) = \sigma_t^2 ,$$
$$\sigma_t^2 = \beta_0 + \beta_1\varepsilon_{t-1}^2 + \beta_2\sigma_{t-1}^2 + \beta_3 r_{t-1} , \tag{17}$$

where $\alpha_1$ determines the speed of mean reversion, $r_{t-1}$ describes the level of interest rates lagged one day and $\beta_0 + \beta_1\varepsilon_{t-1}^2 + \beta_2\sigma_{t-1}^2$ defines the GARCH in Mean[5] model developed by Engle, Lilien and Robins (1987).

The model developed by Brenner et al (1996) can be described as,

$$r_t = \alpha_0 + (1+\alpha_1)r_{t-1} + \alpha_2\sigma_{t-1}^2 + \varepsilon_t ,$$
$$E(\varepsilon_t) = 0 ,$$
$$Var(\varepsilon_t) = \sigma_t^2 ,$$
$$\sigma_t^2 = \beta_0 + \beta_1\left(\frac{\varepsilon_{t-1}}{\sigma_{t-1} r_{t-2}^{\gamma}}\right)^2 + \beta_2\sigma_{t-1}^2 + \beta_3\left(\frac{\varepsilon_{t-1}}{\sigma_{t-1} r_{t-2}^{\gamma}}\right) I_{t-1} , \tag{18}$$

where $\alpha_1$ is the speed of mean reversion, $\gamma$ is the elasticity of volatility against the level of interest rates, $r_{t-1}$ describes the level of interest rates lagged one day and $I_{t-1}$ is a dummy variable[6] that takes the value of 1 when $\varepsilon_{t-1}$ is negative and 0 otherwise in

[5] GARCH in Mean allows the return of a security to be partly determined by its risk. Most models applied in Finance assume that investors should be rewarded for taking additional risk by obtaining a higher return. GARCH in Mean allows for the relation between risk and return. The model is denoted by $Y_t = \mu + \delta\sigma_{t-1} + u_t$, $u_t \sim N(0, \sigma_t^2)$ and $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2$. If $\delta$ is positive and statistically significant; increased risk given by an increase in the conditional variance will lead to a rise in the mean return. As a result; $\delta$ can be interpreted as a risk premium.

[6] Dummy variables can also be defined as qualitative variables because they are often used to numerically represent a qualitative entity. Dummy variables are usually specified to take the values of 1 if a specific statement is observed and 0 otherwise. The value of 1 can be considered as a true statement and 0 as a false statement. This kind of variables is usually applied on cross-sectional and time series regressions.

order to allow for asymmetric volatility effects. Parameter $\left(\frac{\varepsilon_{t-1}}{\sigma_{t-1} r_{t-2}^{\gamma}}\right)^2$ is an unscaled prediction error applied in GARCH equations. It indicates that volatility does not follow an ordinary GARCH (1,1) model; creating difficulties on determining the stationary conditions.

As a result; Koedijk et al (1997) propose the following model,

$$r_t = \alpha_0 + (1+\alpha_1) r_{t-1} + \alpha_2 r_{t-1}^2 + \sigma_t r_{t-1}^{\gamma} \varepsilon_t ,$$

$$E(\varepsilon_t) = 0 ,$$

$$Var(\varepsilon_t) = \sigma_t^2 r_{t-1}^{2\gamma} ,$$

$$\sigma_t^2 = \beta_0 + \beta_1 \left(\frac{\varepsilon_{t-1}}{r_{t-2}^{\gamma}}\right)^2 + \beta_2 \sigma_{t-1}^2 , \tag{19}$$

where $\alpha_1$ is the speed of mean reversion $\gamma$ is the elasticity of volatility against the level of interest rates and $r_{t-1}$ describes the level of interest rates lagged one day.

Bali (2003) develops an additional model which is described as,

$$r_t = (1+\alpha_1) r_{t-1} + \alpha_2 r_{t-1} \log(r_{t-1}) + \frac{1}{2} \sigma_{t-1} r_{t-1} + \varepsilon_t$$

$$E(\varepsilon_t) = 0 ,$$

$$Var(\varepsilon_t) = \sigma_t^2 r_{t-1}^{2\gamma} ,$$

$$\sigma_t^2 = \beta_0 + \beta_1 \left(\frac{\varepsilon_{t-1}}{r_{t-2}^{\gamma}}\right)^2 + \beta_2 \sigma_{t-1}^2 , \tag{20}$$

where $\alpha_1$ is the speed of mean reversion, $\gamma$ is the elasticity of volatility against the level of interest rates and $r_{t-1}$ describes the level of interest rates lagged one day.

In GARCH models; $\beta_1 + \beta_2 > 1$ implies that shocks follow an exponential path. Therefore; current shocks affect volatility levels indefinitely far in the future. However; in level GARCH models; volatility persistence can not be measured only as a function of $\beta_1 + \beta_2$, since volatility is now a function of the stochastic volatility factor and interest rate levels. As a result; persistence will be determined by the level of volatility and interest rates lagged one day. (Bali; 2003)

### *4.2.3 TVP- Levels Models*

This kind of models was developed by Brenner, Harjes and Kroner (1996). The models can be interpreted as a time varying parameter version of the levels or single factor models. These models have important aspects; the sensitivity of volatility to levels is a function of information flows. In addition; volatility clustering is presented; it means that the level of volatility will be higher during high information periods and lower during low information periods. Furthermore; the model nests both levels model and GARCH model.

The TVP-Levels model can be described as,

$$\varphi_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + b\varphi_{t-1}^2, \tag{21}$$

where $\varepsilon_{t-1}$ is the residual, and $a_0, a_1, b$ are positive. Parameter $\varphi_t^2$ defines last period's forecast error. It has the ability to vary through time as new information arrives, following an autoregressive process. Large interest rate shocks measured by large residuals cause increases in rate volatility through their effect on $\varphi_t^2$.

### *4.2.4 Asymmetric TVP- Levels Models*

Nevertheless; a potential weakness of the TVP-Levels model is that it does not allow for asymmetric effects. Therefore; positive and negative shocks will have the same impact on volatility. However; if we impose some restrictions to TVP-Levels model, it is possible to create an Asymmetric TVP-Levels model, described as,

$$\varphi_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\eta_{t-1}^2 + b\varphi_{t-1}^2, \tag{22}$$

where $a_2$ determines the impact on volatility. In the case where $a_2 > 0$, bad news or negative shocks will have a larger impact on volatility than good news or positive shocks.

Asymmetric TVP-Levels implies that negative shocks determined by bad news, will have a larger impact on volatility than positive shocks, which are determined by good news. The time varying parameter model allows volatility to depend on levels and information. An important advantage of this model over TVP-Levels is that it allows for asymmetric effects on information shocks. (Brenner et al; 1996)

#### *4.2.5 GARCH-X Model*

This type of model adds a level term to the GARCH (1, 1) model, where volatility depends on information flows. This model nests Level and GARCH models. An important difference between this model and TVP Levels model is that rate levels have a bigger impact when information flows are higher and a smaller impact when information flows are lower.

The model can be described as,

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + b\sigma_{t-1}^2 + a_3 r_{t-1}^{2\gamma}. \tag{23}$$

#### *4.2.6 Asymmetric GARCH-X Model*

This model is an asymmetric extension of the GARCH-X model. The only difference between GARCH-X model and Asymmetric GARCH-X model is the existence of an asymmetric term; denoted by $a_2\eta_{t-1}^2$. If $a_2 > 0$, then negative shocks have a larger impact on volatility than positive shocks. The model can be described as,

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\eta_{t-1}^2 + b\sigma_{t-1}^2 + a_3 r_{t-1}^{2\gamma}. \tag{24}$$

Brenner et al. (1996) analyze and compare five improvements to the GARCH family of models. These models include the ones described in sections 4.3.2 through 4.3.5. It is demonstrated that distributions between the different models are very similar. Therefore; any of the above mentioned models will yield similar derivative prices.

This result; however; is restricted to values ranging between .07 and .14 in the level of interest rates. However; if the level of interest rates lies outside these constrained levels, it is observed that GARCH-X models have wider confidence intervals. This implies that the error distribution could reach higher levels which will reduce the accuracy of model's estimations. The model selected is determinant when interest rates show volatility clustering[7]. (Brenner et al; 1996)

Empirical evidence demonstrates that by allowing volatility to be a function of interest rate levels and shocks to interest rate market, a larger impact on the pricing of long dated interest rate based derivative securities could be observed. Volatility persists for short periods of time. In addition; no simple analytical solution exists to the pricing of derivatives from the above mentioned models. (Brenner et al; 1996)

---

[7] Volatility clustering describes the tendency of large positive or negative changes in asset prices to follow large changes and small positive or negative changes to follow small changes. This means that the current level of volatility tends to be positively correlated with its level during the immediately preceding periods.

Brenner et al (1996) conclude that the sensitivity of interest rate volatility to levels is exaggerated in the literature. In addition; they state that the existing theoretical models of interest rates present bias in the way volatility is modeled. A disagreement with the findings presented by Chan et al (1992) arises[8]. This disagreement could lead to further research with the objective of developing a model which is able to determine the relation between interest rate levels and volatility and interest rate shocks and volatility.

### *4.3 Multifactor Models*

In single factor models, the short term rate, $r(t)$, is the only explanatory variable. Single factor models have received several critiques due to a perfect correlation between bonds with different maturities. As a result, the long rate is a deterministic function of the short term rate. Models may be seen as lacking of accuracy when determining prices since differences between estimated and real prices do exist. In addition; looking from a macroeconomic perspective, it is unrealistic to think that the term structure is only determined by the short term interest rate. Furthermore; it is difficult to obtain accurate volatility structures for the forward rates. (Gibson et al; 2001)

As a result to the critiques given to single factor models; several models have been developed with the objective of improving and correcting possible bias presented in single factor models. Models are based on the no arbitrage conditions and equilibrium. This new class of models belongs to the family of multifactor models; which include two-factor, three-factor models and n-factor models. (Gibson et al; 2001)

It is expected that by moving from one factor to several factors the results will be more precise and an improved fit will be obtained. However; several negative consequences when incorporating additional factors could exist. These include loss of tractability, partial differential equations of a higher dimensionality, and slower results. In addition; the choice of the factors selected plays a determinant role. (Gibson et al; 2001)

Empirical evidence shows that a two factor model is more accurate than the single factor in forecasting future volatility of interest rates. Econometric results show that one factor diffusion models that incorporate volatility only as a function of interest rate levels overestimate the sensitivity of volatility to interest rate levels and are inaccurate

---

[8] Chan et al (1992) conclude that the relation between interest rate volatility and the level of interest rates is the most important aspect of any model of the short term riskless rate. By contrast; Brenner et al (1996) confirm that the relation between interest rate volatility and the level of interest rates is equally important when modelling the volatility parameter as a function of unexpected news.

in capturing the serial correlation in conditional volatility. (Bali; 2003)

Some of the multifactor models are those developed by Cox, Ingersoll and Ross (1985), Richard (1978), Brennan and Schwartz (1979, 1982), Longstaff and Schwartz (1992), Duffie and Khan (1993), Schaefer and Schwartz (1987), Fong and Vasicek (1991), Das and Foresi (1996), Koedijk, Nissen, Schotman and Wolff (1997), Bali (2003), between others. In all those models the spot rate is compared with respect to an additional variable in order to determine the term structure of the short term rate. Some of these proxies include inflation rate, volatility, mean, the spread between long and short term rates as well as the yields on a fixed set of bonds.

A single factor model can be extended to a multifactor model by defining the short term rate as a function of a set of several stochastic factors. However; some single factor models do not obtain any benefits by being extended to a multifactor setup. Anyhow; in some cases, extensions of single factor models could provide useful results. Nevertheless; it is important to mention that by adding an additional factor coherent economic signification must be added. (Gibson et al; 2001)

In the following subsections; a brief description of some multifactor models will be introduced with the objective of providing a general overview of the models. However; the models developed by Longstaff and Schwartz (1992) and Bali (2003) are described in a more detailed framework since these models include volatility as one determinant factor in order to model the term structure of interest rates. Consistent with previous research; we consider that volatility is an extremely important factor in order to determine and forecast future levels of interest rates.

### *4.3.1 Brennan and Schwartz Model*

Brennan and Schwartz (1982) propose a two factor model in which the term structure of interest rates is determined by two factors which include the short term rate and the long term rate. The model reflects the assumption that long term rates contain some information about the future value of the short term rate.

The model is fitted for the pricing of US government bonds for a period ranging from 1958 to 1979; with the objective of evaluating the ability of the pricing model to detect underpriced or overpriced bonds. A strong relation between prediction errors and bond returns was found.

The prices of a default free bond at any point in time are expressed in terms of two stochastic processes; the short and long term rates. Processes lead to a partial differential equation (PDE) in which all default free bonds must satisfy equilibrium values. The short rate is defined as the yield on the currently maturing discount bond with stochastic factors; whether the long rate is defined as the yield on a bond whose maturity is infinite. The price changes on bonds over any short interval of time will depend on the rates of return of the corresponding changes in the proxies applied.

Parameters involved must be estimated before the model can be implemented. These parameters include the utility function and the market price of risk. It is assumed that market inefficiencies and profit opportunities will depend in the adequately application of the equilibrium model and accuracy of the data.

The model is described as,

$$dr = \beta_1(r,\ell,t)dt + \eta_1(r,\ell,t)dZ_1,$$

$$d\ell = \beta_2(r,\ell,t)dt + \eta_2(r,\ell,t)dZ_2, \tag{25}$$

where $t$ is defined as the calendar time, $dZ_1$ and $dZ_2$ are correlated increments to a standard Wiener process; where $E[dZ_1] = E[dZ_2] = 0$, $E[dZ_1 \cdot dZ_2] = \rho dt$ and $E[dZ_1^2] = E[dZ_2^2] = dt$ follow a stochastic process, $r$ and $\ell$ are proxies with respect to the underlying assets which are denoted as the short and long term interest rates. The model reflects the assumption that long term rates contain some information about the future value of the short term rate.

The coefficients of a Partial Differential Equation (PDE) depend on four functions which are derived from stochastic process for the short and long term interest rates. These variables are denoted by $\beta_1, \eta_1, \eta_2, \rho$. In addition; they also depend on the market price of the short term interest rate risk.

An unanticipated increment in each interest rate is proportional to the current value of that rate. Coefficient $dt$ in the short rate equation reflects the expectations based theories of the term structure; which states that long rates are based upon expectations about future short rates. It is observed that if $a_1 < 0$, interest rates may become negative. However; this happening is theoretically unacceptable under any macro economical policy. By combining the short and long term rates with a negative short term interest rate; the output will be positive as far as a positive long rate is greater than a negative short term rate.

Brennan et al (1982); demonstrate that the short term rate will tend to regress towards the current value of the long term rate. These outcomes are consistent with the ones obtained by Shiller (1979); where mean reversion on long rates with respect to short rates was observed. In the case when long rates are higher relative to short rates, interest rates will tend to move down in the following periods.

There is considerable intertemporal variation in the predictive ability of the model. The model error lacks of tendency towards systematically growth as time evolves. Results suggest lack of a significant relation between future values of the short term interest rate and the long term interest rates. Factors do not appear to be associated with uncertainty about expected future interest rates.

Results provide evidence towards null hypothesis's rejection since it is demonstrated that variance rates are dependent of factor scores. Therefore; the valuation model is misspecified since it assumes constant rather than stochastic variance rates. However; the two factor model presented is adequate over short periods of time.

### *4.3.2 Longstaff and Schwartz Model*

Longstaff and Schwartz (1992) describe two approaches for modeling the term structure of interest rates in a continuous setup. The equilibrium approach was first developed by Cox-Ingersoll-Ross (1985). This approach relates the underlying economy with exogenous factors or economical variables and investors' preferences. In contrast; the arbitrage approach proposed by Vasicek (1977), assumes a stochastic evolution of interest rates and the prices of all contingent claims are obtained by imposing no arbitrage opportunities in the economy.

Advantages from the equilibrium approach over the arbitrage approach are clarified. The main difference between these approaches lies on the dynamics of interest rate volatility. In the equilibrium approach, the term structure and dynamics are endogenously determined and the market price of risk is obtained as part of equilibrium. However; this approach does not provide any guidance with respect to the risk premium. Therefore; arbitrage opportunities could arise.

Longstaff et al.(1992) develop a two factor general equilibrium model of the term structure of interest rates using the Cox-Ingersoll-Ross (CIR) framework. The model is applied to the valuation of discount bonds and interest rate sensitive securities. The factors include the short term interest rates and the instantaneous volatility of changes in the short term interest rates.

These factors are considered to be the most important ones to explain the behavior of interest rates. The model seems to present advantages since contingent claims are able to reflect the level of interest rate and volatility. Furthermore; the dynamics of interest rate volatility are endogenously determined.

The model developed by Longstaff and Schwartz (1992) is described as,

$$\frac{dQ}{Q} = (\mu X + \theta Y)dt + \sigma\sqrt{Y}\,dZ_1, \tag{26}$$

where $X$ and $Y$ are state variables, $X$ represents the component of expected returns unrelated to production uncertainty, $\mu, \theta, \sigma$ are positive constants and $Z_1$ is a scalar Wiener process. Equation (27) does not require expected returns and production volatility to be perfectly correlated.

The dynamics of the short term interest rate and volatility are described as,

$$dX = (a - bX)dt + c\sqrt{X}\,dZ_2$$

$$dY = (d - eY)dt + f\sqrt{Y}\,dZ_3, \tag{27}$$

where $X$ determines the short term interest rate, $Y$ describes the volatility term, $a, b, c, d, e, f > 0$, $Z_2$ and $Z_3$ are scalar Wiener processes, $\Delta X$ represents technological changes unrelated to production uncertainty, $Z_2$ must be uncorrelated with $Z_1$ and $Z_3$ and $\theta > \sigma^2$ in order to guarantee a positive risk free rate.

This model includes several assumptions which are required to be taken into consideration. These include a fixed number of identical individuals with time additive preferences, and perfectly competitive continuous markets for riskless borrowing and lending. The investment decision is subject to maximizing the conditional expectation operator subject to the budget constraint; where investor's derivative utility of wealth function is partially separable.

An important advantage of this model is that the risk premium is endogenously determined and not exogenously imposed. This implication ensures that the risk premium is consistent with the absence of arbitrage. As a result; contingent claims can reflect both factors when obtaining the adequate price.

Volatility of interest rates is a key determinant for the pricing of contingent claims. Therefore; Longstaff and Schwartz model leads to valuation expressions that are more consistent with real prices than models that exclude interest rate volatility. The short term interest rate and volatility are interdependent; where the stochastic evolution on the short rate depends on volatility and viceversa. Consequently; if we mix together the two processes; we obtain a joint Markov process[9].

Short term interest rate tends to stationarity over the long run. Changes in interest rates and volatility are positively correlated. Values of volatility can range between $[0,\infty)$, which explains an important advantage of this model over single factor models; since in single factor models the variance is completely determined by the short term interest rate. However; in this model, the variance can depend in additional factors besides the short term interest rate.

Discount bond prices depend on the short term interest rate and volatility of the interest rates. Therefore; the yield curve can have a greater variety of forms than the ones it could have on a single factor model. The slope of the yield curve can vary depending on the specific maturity modeled. Changes in the level of interest rate volatility have significant effects on the slope, volatility of the yield curve and shape of the term structure.

Longstaff and Schwartz model is able to capture the dynamics of interest rates and volatility and the cross sectional structure of yield changes. In addition; the model demonstrates that volatility is an increasing function on maturity or duration of the bond and a fundamental determinant in valuation of contingent claims. These results introduce important implications to contingent claims; since their respective prices will depend on these variables.

Volatility of the riskless interest rate is estimated by applying a GARCH (Generalized Autoregressive Conditional Heteroskedasticity) framework developed by Bollerslev (1986). ARCH and GARCH processes have been used to model an extensive variety of economic time series such as inflation, foreign exchange rates, stock returns and interest rates.

[9] A Markov process is a stochastic process in which the probability distribution of the current state is conditionally independent of the path of past states. Within a mathematical notation; a Markov process is expressed for any $n$ and $t_1 < t_2 < \ldots < t_n$, where $P[X(t_n) \leq X_n | X(t) \forall t \leq t_{n-1}] = P[X(t_n) \leq X_n | X(t_{n-1})]$.

The model can be described as,

$$r_{t-1} - r_t = \alpha_0 + \alpha_1 r_t + \alpha_2 V_t + \varepsilon_{t+1}$$
$$\varepsilon_{t+1} \sim N(0, V_t)$$
$$V_t = \beta_0 + \beta_1 r_t + \beta_2 V_{t-1} + \beta_3 \varepsilon_t^2 . \tag{28}$$

Equation (28) is a discrete time approximation of the continuous time framework. It allows for heteroskedasticity[10] on levels of interest rates as time evolves. Volatility follows an autoregressive process since its current value depends on its lagged value.

The GARCH system is estimated by using the Berndt-Hall-Hall-Hausman (BHHH) numerical algorithm to estimate the maximum likelihood parameters, as well as cross sectional restrictions imposed by the model. This algorithm is known as the Generalized Methods of Moments (GMM). A variety of different starting values are applied in order to ensure that the volatility estimates are robust and the algorithm converges to the global maximum.

This approach presents several advantages; such as a normal distribution on yield changes. GMM procedure requires stationarity[11], ergodicity[12] and the existence of relevant expectations. In addition; GMM estimators are consistent even if the error terms in the moment equations are conditionally heteroskedastic, serially correlated, or correlated across maturities. In order to provide diagnostics, the ability of the model to fit the term structure of interest rate volatility is examined. This is realized by comparing the unconditional standard deviations of yield changes.

It is observed that GARCH estimates are positive and successful in capturing volatility patterns ex post. Results indicate that changes in volatility and interest rates have explanatory power for yield changes and for the predicting abilities of the model. Additionally; results provide evidence for supporting the two factor model. However; additional two factor models can not be rejected since the information obtained is not

---

[10] Heteroskedasticity is defined as a sequence or a vector of random variables in which variables present different variances at different points in time.

[11] A time series is considered to be stationary if the distribution function of $\{x_t, t \in M + \mu\}$ is the same as the distribution function $\{x_t, t \in M\}$. If the distribution of its values remains the same as time progresses; a series is considered stationary. Therefore, the probability that $X$ falls in a particular interval is the same at any point in time.

[12] Ergodicity can be described as covariance stationarity. A time series $\{x_t\}$ is said to be covariance stationary if its first and second moments are finite and $\mathrm{cov}(x_t, x_{t+s})$ depends only on s. Time averages of functions of observations on time series at T points converge in mean square to the corresponding population expectations as $T \to \infty$.

enough for supporting such a statement.

### *4.3.3 Affine Models*

Duffie and Kan (1993) introduce models in which drift and volatility coefficients of the state variable processes are affine functions. This means that both coefficients are interrelated. One factor causes the other and vice versa. Within this framework; it is possible to decompose the term structure movements into their primary factors. Affine models are important because they are able to nest several models within the single factor models class. In addition; the product between volatility matrix, price of risk vector and volatility of the log state price deflator are linear in the state variables.

Duffie (2002) extended the affine class of models in the way that the variance of the log state price deflator is not affine in the state variables. This allows the possibility of modeling time variation in risk price which was not associated with time variation on volatility matrix. Affine class of models perform better in forecasting bond yields; since this class of models are able to unify small unconditional sample means of bond excess returns with high volatility.

According to Beltratti et al (2007), bond prices at each point in time are a function of the current state vector portfolio. Models require forecasts of future prices. In order to predict future bond prices; it is required to forecast the future state vector conditional on information available at time $t$. Affine models allow for the creation of closed form formulas; which determine the first two conditional moments of the probability distribution. The ability to predict excess returns and risks is crucial for risk averse investors and explains differences between statistical and financial evaluations. It is argued that affine term structure models are useful since this kind of models can be applied as tools for active bond portfolio management.

However; since affine models are considered as accurate determinants of future state vector's forecasts; it could be possible to describe this kind of models as the best predictors of future interest rates. However; according to Beltratti et al (2007); in the case of the term structure of interest rates there is no best model. The model to be used to solve specific situations depends on the objective of the decision maker in which the model is required. Therefore; the goal is set as a first objective and then the decision about the model selection should be obtained.

Affine models show that in order for the forward rate to be affine in the spot rate; the volatility of the short term interest rate must be restricted to the following form,

$$\sigma_r^2\left(t, r(t)\right) = a(t) + b(t)r(t). \tag{29}$$

### 4.3.4 Koedijk, Nissen, Schotman, Wolff Model

Koedjk, Nissen, Schotman and Wolff (1997) develop a model of the short term interest rate volatility which takes into consideration the level effect of Chan, Karolyi, Longstaff and Sander's model (1992) and the conditional heteroskedasticity effect of the GARCH family of models. (Bollerslev; 1986) This combination creates a model which allows different effects to dominate as the level of interest rate changes. This model is known as KNSW; named after its creators.

CKLS compare 8 known continuous time models of the short term interest rate in terms of their ability to capture the behavior of the short term interest rate. Results indicate that volatility is a critical element for determining the best or most efficient model of the term structure. It is concluded that the models which best describe the behavior of interest rates over time are those that allow the volatility of interest rates to be dependent on the level of interest rates.

An alternative class of models which is able to capture the effects on volatility of interest rates is the family of ARCH (Engle; 1982) and GARCH (Bollerslev; 1986) models; where the main factors are volatility clustering and volatility persistence. According to financial literature and consistent with previous research; volatility is seen as one of the most important elements for forecasting the level of interest rates and determining the prices of contingent claims.

Results indicate that the incorporation of a volatility effect in the model specification is relevant for the pricing of shorter term options on long term bonds. This relevance is determined by a strong dependence between magnitudes of price changes and the level of interest rates. In addition; the model allows for different interest rate sensitivities in volatility by combining the high interest rate sensitivity of CKLS with GARCH type volatility clustering.

The conditional volatility process is described as,

$$h_t^2 = \beta_1 r_{t-1}^{2\gamma} + \left(\frac{r_{t-1}}{r_{t-2}}\right)^{2\gamma} \left(\beta_2 e_{t-1}^2 + \beta_3 e_{t-1}^2\right). \tag{30}$$

Equation (30) allows for time varying persistence of shocks which depends on interest rate levels. If we set $\beta_2 = \beta_3 = 0$, we obtain the CKLS model. If we set $\gamma = 0$, we have the GARCH model. In contrast; if we set $\alpha_2 = 0$ and $\gamma = \frac{1}{2}$, we obtain the Longstaff and Schwartz model.

Unconditional distribution is not available in closed form. However; it can be modeled if we specify a value for the parameters. In discrete time, negative values are possible for the short term interest rates. However; if $h_t^2$ is small enough with respect to its mean and drift; the probability of short term interest rates to be negative is very small.

The discrete time model on short term interest rates can be described as,

$$r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \alpha_2 r_{t-1}^2 + e_t. \tag{31}$$

When $\alpha_2 < 0$, the mean reverting drift is able to return high interest rate levels to its mean in the case of explosive variance or high levels of heteroskedasticity; reaching its limit when $r = 0$. An exact continuous time limit of the discrete time process in equation (31) is not available. However; it is possible to develop a process which is a similar approximation to the model developed by Longstaff and Schwartz.

The KNSW model is described as,

$$\begin{aligned} dr_t &= \kappa(\mu - r_t)dt + \sigma r_t^{\gamma} dW_{1t} \\ d\sigma_t^2 &= \theta(\omega - \sigma_t^2)dt + \varphi\sigma_t dW_{2t}, \end{aligned} \tag{32}$$

where $dr_t$ denotes the level on interest rates, $d\sigma_t^2$ determines the volatility level, $\sigma$ follows a diffusion process and $\gamma$ is not restricted to a specific value.

### *4.3.5 Bali Model*

Bali (2003) extends the Black, Derman and Toy (1990) single factor arbitrage free model into a BDT two factor term structure model. BDT model is able to model the volatilities of interest rates and the term structure of interest rates. The model is a single factor discrete time lattice model.

Factors included in Bali's model are the short term interest rate and volatility since it has been demonstrated that current interest rate level and interest rate volatility are two of the most important factors in explaining movements on the term structure. Volatility is a key factor for the pricing of interest rate sensitive securities.

The model for the term structure of interest rates developed by Bali extends the two factor model of BDT by including a volatility component. The model in a continuous setup can be described as,

$$d\ln r_t = \phi_1(\mu_1 - \ln r_t)dt + \sigma_t dW_{1,t}\,. \tag{33}$$

Stochastic volatility models can be described as,

$$d\sigma_t^2 = \phi_2(\mu_2 - \sigma_t^2)dt + \xi_2\sigma_t^2 dW_{2,t}$$

$$d\sigma_t = \phi_3(\mu_3 - \sigma_t)dt + \xi_3\sigma_t dW_{3,t}\,, \tag{34}$$

where $W_{1,t}$, $W_{2,t}$, and $W_{3,t}$, are independent standard Brownian motion processes with zero mean and constant variance, $d\sigma_t^2$ denotes a GARCH process in a continuous setup and determines the stochastic volatility while $d\sigma_t$ denotes a TS-GARCH process in a continuous setup and measures the stochastic standard deviation. The above described models imply mean reversion of the long term interest rate level and mean reversion of the volatility and standard deviation of log interest rate changes.

However; in a discrete time setup; the above models diverge slightly. It has been demonstrated that the short term structure of interest rates present heteroskedasticity. Therefore; models should be adjusted accordingly. GARCH and TS-GARCH effects are incorporated in the diffusion function by allowing conditional variance and standard deviation to accommodate volatility clustering and dependence on the level of interest rates. In order to develop a two factor model on BDT model; two volatility models were used. These include GARCH and TS-GARCH models.

The model developed by Bali (2003) in a discrete time setup can be described as,

$$r_t - r_{t-1} = \left(\alpha_1 r_{t-1} + \alpha_2 r_{t-1}\ln r_{t-1} + \frac{1}{2}h_{t-1}r_{t-1}\right) + r_{t-1}^{\gamma}\sqrt{h_t z_t} \tag{35}$$

$$\varepsilon_t = \sqrt{h_t z_t}\,,\ z_t \sim iid\ N(0,1)$$

where $h_t$ can follow a GARCH or TS-GARCH process. GARCH effects in the diffusion function are denoted by $h_t = \beta_0 + \beta_1\varepsilon_{t-1}^2 + \beta_2 h_{t-1}$, while TS-GARCH effects are denoted by $\sqrt{h_t} = \beta_0 + \beta_1|\varepsilon_{t-1}| + \beta_2\sqrt{h_{t-1}}$. In both models; $\Delta = 1$, $\alpha_1 = \phi_1\mu_1$, $\alpha_2 = -\phi_1$, $\beta_0 > 0$, $0 \le \beta_1 < 1$, and $0 \le \beta_2 < 1$.

GARCH model captures next period volatility by squaring current period's shocks. However; for large shocks the volatility increases dramatically. TS-GARCH model specifies the conditional standard deviation as a moving average of absolute residuals.

Ito's lemma is applied in order to determine the accuracy of the models for capturing the volatility of interest rates within a single and two factor models.

Findings suggest strong evidence of mean reversion in the one and three month Eurodollar. Strong GARCH and TS-GARCH effects in the diffusion function are observed. This finding suggests a strong contribution on last period's volatility to capture the behavior of short term rates in the BDT model. Within the GARCH model; it was found that $\beta_1 + \beta_2 > 1$; which implies that current shocks affect volatility forecasts infinitively long into the future.

According to the maximized log-likelihood on one, three and six Eurodollar deposit rates; it is observed that the two factor BDT model dominates the original one factor BDT model in its forecasting ability for determining future volatility of interest rates. Furthermore; evidence obtained from log-likelihood values suggests an outperformance of level GARCH models with respect to GARCH and level models.

Findings demonstrate that the stochastic volatility process provides a better representation of actual volatility than GARCH, diffusion and constant volatility models within the BDT framework. In addition; it is observed that level models that parameterize diffusion processes only as the level of interest rates fail to model the GARCH effects in conditional volatility and exaggerate the sensitivity of volatility to interest rates. However; GARCH models that specify conditional volatility only as a function of news arrival process fail to capture the relationship between volatility and interest rate levels.

However; it is interesting to observe a divergence from results presented by Chan et al. (1992). Results presented by Bali demonstrate that the value of elasticity of variance to interest rates is lower. In addition; it is also observed that volatility is more sensitive to the stochastic volatility factor than to the level of riskless rate and that the level model overestimates the sensitivity of volatility to interest rates. A significant reduction in sensitivity from level model to level GARCH model was found. Findings confirm the presence of strong nonlinearity in the drift of the spot rate process.

**Table III.** *Models summary and description*

Models analyzed are summarized in the following table with the objective of providing a deeper understanding of the different models described within this chapter.

| Category | Model | Year | Specification | Description | Findings |
|---|---|---|---|---|---|
| Single Factor Model | Merton | 1973 | $dr(t) = \mu_r dt + \sigma_r dW(t)$ | This is the first model proposed to describe and model the term structure of interest rates. After this model, several improvements have been introduced. | The model presents bias since it allows for negative values of interest rates and assumes homoskedasticity. The model is seem as inaccurate for describing the term structure. |
| Single Factor Model | Vasicek | 1977 | $dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$ | The short term interest rates is modelled as an Ornstein- Uhlenbeck process. The model presents elastic random movements around a trend demonstrating mean reversion. | The model applies a constant risk premium. Interest rate can follow any direction. However; it is observed that within a *n* period length, interest rates will return to its mean value. Volatility is a decreasing function of time. |
| Single Factor Model | Cox-Ingersoll-Ross | 1985 | $dr = \kappa(\theta - r)dt + \sigma\sqrt{rdz_1}$ | The single factor model is determined by the supply and demand of individuals having a logarithmic utility. Means and | Several improvements to the term structure are included. Elimination of negative interest rate values, absolute value of interest rates increase when |

| | | | | | |
|---|---|---|---|---|---|
| | | | | variances are proportional to future production opportunities. | interest rates increase, and variance is proportional to the short term rate. |
| Single Factor Model | Chan-Karolyi-Longstaff-Schwartz | 1992 | $r_t = \alpha_0 + (1+\alpha_1) r_{t-1} + \varepsilon_t$<br>$E(\varepsilon_t) = 0$<br>$Var(\varepsilon_t) = \sigma^2 r_{t-1}^{2\gamma}$ | This model is one of the most general single factor models. It provides a basic description of the stochastic movements of interest rates consistent with mean reverting evidence. | This model is applied as a common framework to providing empirical evidence for comparing different models to each other. The framework allows determining the existence of a superior model able to forecast the term structure of interest rates. |
| Volatility Model | GARCH | 1986 | $r_t = \alpha_0 + (\alpha_1 + 1) r_{t-1} + \varepsilon_t$<br>$E(\varepsilon_t) = 0$<br>$Var(\varepsilon_t) = \sigma_t^2$<br>$\sigma_t^2 = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 \sigma_{t-1}^2$ | This family of models allows for volatility clustering and persistence. GARCH allows the conditional variance to be dependent in previous lags; where most recent innovations receive greater weights. | GARCH models have succeeded in capturing volatility clustering for equity prices and interest rates. The model is able to determine mean reversion and the existence of stationarity in the series. Some authors state that GARCH models do not capture the relationship between interest rate and volatility. |
| | Combined | 1992 | $r_t = \alpha_0 + (1+\alpha_1) r_{t-1}$<br>$+ \alpha_2 \sigma_{t-1}^2 + \varepsilon_t$ | This family of models is obtained as a result from the combination of | This family of models is able to determine the speed of mean |

| | | | | | |
|---|---|---|---|---|---|
| Volatility Model | Level GARCH | 1996 1997 2003 | $E(\varepsilon_t)=0, \quad Var(\varepsilon_t)=\sigma_t^2$ $\sigma_t^2=\beta_0+\beta_1\varepsilon_{t-1}^2+\beta_2\sigma_{t-1}^2+\beta_3 r_{t-1}$ | single factor models with GARCH models in order to obtain a better fit of all elements involved. | reversion, the elasticity of volatility against the level of interest rates, persistence and stationarity conditions |
| Volatility Model | TVP Levels | 1996 | $\varphi_t^2=a_0+a_1\varepsilon_{t-1}^2+b\varphi_{t-1}^2$ | This model is a time varying parameter version of single factor models. | This model determines the sensitivity of volatility as a function of information flows, and volatility clustering The model is able to nest single factor models and GARCH model. |
| Volatility Model | Asymmetric TVP Levels | 1996 | $\varphi_t^2=a_0+a_1\varepsilon_{t-1}^2+a_2\eta_{t-1}^2+b\varphi_{t-1}^2$ | This model is a time varying parameter version of single factor models allowing for asymmetric effects. | The model is an improved version of TVP Levels Model since it allows for volatility's impact determination from positive and negative shocks. Negative shocks have a larger impact. |
| Volatility Model | GARCH-X | 1996 | $\sigma_t^2=a_0+a_1\varepsilon_{t-1}^2+b\sigma_{t-1}^2+a_3 r_{t-1}^{2\gamma}$ | This model adds a level term to GARCH (1,1) model, by allowing volatility to depend on information flows. | The model nests level and GARCH models. Rate levels have a greater impact when information flows are higher and a smaller impact when information flows are smaller. |

| | | | | | |
|---|---|---|---|---|---|
| Volatility Model | Asymmetric GARCH-X | 1996 | $\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\eta_{t-1}^2 + b\sigma_{t-1}^2 + a_3 r_{t-1}^{2\gamma}.$ | This model is an asymmetric extension of GARCH-X model. The difference between GARCH-X models lies in parameter $a_2\eta_{t-1}^2$. | This model allows for asymmetric effects. If $a_2 > 0$, negative shocks will have a larger impact on volatility than positive shocks. |
| Multifactor Model | Brennan-Schwartz | 1982 | $dr = \beta_1(r,\ell,t)dt + \eta_1(r,\ell,t)dZ_1$ $d\ell = \beta_2(r,\ell,t)dt + \eta_2(r,\ell,t)dZ_2$ | This is a two factor model in which the term structure is determined by the short term rate and the long term rate. The model assumes that long term rates contain information about the behavior of short term rates. | This model leads towards Partial Differential Equations depending on functions derived from stochastic processes. The short term rate will tend to regress towards the long term rate value. Variance rates are dependent on factor scores. |
| Multifactor Model | Longstaff-Schwartz | 1992 | $dX = (a - bX)dt + c\sqrt{X}\,dZ_2$ $dY = (d - eY)dt + f\sqrt{Y}\,dZ_3$ | This is a two factor general equilibrium model for the term structure using Cox-Ingersoll-Ross framework. Factors include short term interest rate and instantaneous volatility. The factors are considered to be the most important ones. | The risk premium is endogenously determined and not exogenously imposed. Risk premium is consistent with the absence of arbitrage. The model leads to valuation expressions more consistent with real prices. Both factors are interdependent. |

| | | | | | |
|---|---|---|---|---|---|
| Affine Model | Duffie-Kan | 1993 | $\sigma_r^2(t, r(t)) = a(t) + b(t)r(t)$ | In this kind of models, drift and volatility coefficients are affine variables, creating intercorrelation between coefficients. The term structure can be decomposed into their primary factors. Affine models are able to nest several models within the single factor models. | The product between volatility matrix, price of risk vector and volatility of the log state price deflator are linear in the state variables. Affine models perform better in forecasting bond yields by unifying small unconditional means of bond excess returns with high volatility. This allows for active bond portfolio management. |
| Multifactor Model | Koedijk-Nissen-Schotmann-Wolff | 1997 | $dr_t = \kappa(\mu - r_t)dt + \sigma r_t^{\gamma} dW_{1t}$ <br> $d\sigma_t^2 = \theta(\omega - \sigma_t^2)dt + \varphi\sigma_t dW_{2t}$ | This model takes into consideration the model developed by Chan et al. with conditional heteroskedasticity effect . This combination results into different effects as the level on interest rates changes. | Results indicate that volatility is a critical element for determining the most efficient model and pricing of contingent claims. Models which best describe the behavior of interest rates are those that allow the volatility of interest rates to be dependent on the level of interest rates. |

| | | | | | |
|---|---|---|---|---|---|
| Multifactor Model | Bali | 2003 | $d\ln r_t = \phi_1(\mu_1 - \ln r_t)dt + \sigma_t dW_{1,t}$<br><br>$d\sigma_t^2 = \phi_2(\mu_2 - \sigma_t^2)dt + \xi_2\sigma_t^2 dW_{2,t}$<br><br>$d\sigma_t = \phi_3(\mu_3 - \sigma_t)dt + \xi_3\sigma_t dW_{3,t}$ | Bali's model is an extension of Black, Derman, Toy (1990) single factor model. The outcome is a two factor model able to determine the volatilities of interest rates.<br>Parameters include the short term interest rates and volatility. | The model implies mean reversion of the long term interest rate level and mean reversion of volatility and standard deviation of changes on log interest rates. Volatility is assumed to be heteroskedastic. Mean reversion on the short term rate is observed. |

## 5 Comparison between models

The term structure of interest rates is considered as one of the most important elements in Finance. Several researchers state that given the increasing importance; the term structure of interest rates has received more attention than any other issue in Finance. As a result; an extensive amount of models have been developed and improved with the objective of describing and forecasting future interest rate behavior taking into consideration different proxies.

However; little is known about the existence of a model which represents superiority in its forecasting abilities. In addition; given the extensive amount of models available; it is difficult to compare them with respect to each other. This difficulty arises due to the fact that each particular model consists of specific factors or variables, which differ from model to model. In addition; the proxies applied to determine the behavior of the term structure of interest rates and its forecasting abilities will also differ from model to model.

Nevertheless; some researchers have succeed in creating common frameworks, which are able to nest several models and therefore; compare their forecasting abilities. It is important to mention; however; that there is no common framework which could nest all available models. Several frameworks do exist; however; each framework is able to nest and compare a limited amount of models. From the vast variety of existent models, it is not possible to determine with absolutely accuracy which is the best model for determining the behavior of interest rates and its volatility.

When modeling the term structure of short term interest rates two parameters play a determinant role for obtaining accurate estimations on the behavior of interest rates. These are the speed of mean reversion, $\beta$, and the level of heteroskedasticity on changes on interest rates, $\gamma$. Chan et al (1992) were the first ones in realizing an accurate comparison of different short term models by applying the Generalized Methods of Moments[13]. (Faff and Gray; 2006)

[13] In all studies where GMM was applied; no statistically significant mean reversion was found. As a result; the ability to detect mean reversion became doubtful. In order to determine the bias and precision of the GMM technique, Faff and Gray (2006) develop a simulation approach. Interest rate data is simulated by applying CKLS model within a discrete time setup. By applying weekly observations, parameters allow a long run equilibrium rate of 10% and five years for reverting towards its mean value. GMM has been widely employed by short rate models. GMM avoids estimation difficulties which are present under alternative approaches. In addition; accurate goodness of fit and comparison of nested models are provided. GMM has difficulties in estimating the speed of mean reversion on interest rates.

In the following subsections, some existent approaches will be discussed with the objective of explaining more in detail the common frameworks in which models have been nested in order to allow the possibility of comparing them with respect to each other. The different frameworks are listed chronologically, starting from the oldest framework developed.

### *5.1 Elliot and Baier Framework*

Elliot and Baier (1979) compare six different econometric interest rate models with the objective of determining models' competences for explaining and forecasting interest rate levels. Models are divided in three categories. The first category includes models which apply distributed lags of past rates to explain current rates. The second category includes models derived from some version of liquidity preference theory. Finally; the third category includes models which are reduced forms of multisector econometric models.

From each category, two models are chosen and analyzed. Within the first category; models analyzed include those developed by Modigliani and Sutch (1966) and Modigliani and Shiller (1973). The second category includes models developed by Feldstein and Eckstein (1970) and Feldstein and Chamberlain (1973). Finally; the third category includes models developed by Sargent (1969) and Echols and Elliot (1976).

Modigliani and Sutch (1966) state that past patterns of short rates and inflation rates contain useful information in setting the current long rate. The model is described as,

$$R_t = a + b_0 B_t + \sum_{n=1}^{23} b_n B_{t-n} + K_t + e_t; \qquad \sum_{i=1}^{23} b_i > 0, \tag{36}$$

where $R_t$ is defined as the nominal rate of interest on long term securities, $a, b_0, b_n$ are fixed coefficients, $B_t, B_{t-n}$ are nominal rates of interest on short term securities, $K_t$ is the effect of government supplies of bonds with long and short maturities and $e_t$ is the error term.

Modigliani and Shiller (1973) develop a more generalized model in which past patterns of short rates and inflation rates contain useful information in order to establish the actual long rate.

---

Even with a large sample, the magnitude of mean reversion is severely overestimated. The bias can be caused due to the difficulty of estimating the autoregressive coefficient for nearly stationary time series. However; this problem is not unique from GMM; it also appears in Ordinary Least Squares, Maximum Likelihood, and Bayesian Approaches. As a result; caution regarding the existence of interest rate mean reversion should be applied.

The model is described as,

$$R_t = a + b_0 B_t + \sum_{n=1}^{23} b_n B_{t-n} + c_0 \dot{P}_t + \sum_{n=1}^{23} c_n \dot{P}_{t-n} + dL_t + e_t; \qquad \sum_{i=1}^{23} b_i > 0,\ d > 0, \tag{37}$$

where $R_t$ is defined as the nominal rate of interest on long term securities, $a, b_0, b_n, c_n$ are fixed coefficients, $B_t, B_{t-n}$ are nominal rates of interest on short term securities, $\dot{P}_t, \dot{P}_{t-n}$ are the rate of changes in general prices, $L_t$ is a liquidity premium variable and $e_t$ is the error term.

By including a liquidity preference approach, the second category examines the effect on interest rates caused by the interrelationship between supply and demand for money and bonds. Feldstein and Eckstein (1970) focus on the relationship between bonds and money. Demand for money is more influenced by income changes and negatively influenced by changes in real interest rates.

The model is denoted as,

$$\begin{gathered} R_t = a + b_0 \ln\left(M_t^R\right) + b_1 \ln\left(Y_t^c\right) + \sum_{n=0}^{23} b_{n+2} \dot{P}_{t-n} + b_{24}\left(R_{t-1} - R_{t-2}\right) + e_t; \\ b_0 < 0,\ b_1 > 0,\ \sum_{n=0}^{23} b_{n+2} > 0,\ b_{24} > 0, \end{gathered} \tag{38}$$

where $R_t$ is the nominal rate of interest, $a, b_0, b_1, b_{n+2}, b_{24}$ are fixed coefficients, $M_t^R$ is the real monetary base per capita, $Y_t^c$ is the real personal income per capita. $\dot{P}_{t-n}$ is the rate of changes in general prices, $R_{t-1,t-2}$ is the rate of interest lagged $t-n$ periods, and $e_t$ is the error term.

The model developed by Feldstein and Chamberlain (1973) is considered as a second generation model of the previous model developed by Feldstein and Eckstein (1970). It includes four financial assets; which are money, bonds, equities and short term bills.

The model is described as,

$$R_t = a + b_0 \ln\left(M_t^R\right) + b_1 \ln\left(Y_t^c\right) + L_1\left(\dot{P}_{t-n}\right) + b_2\left(R_{t-1} - R_{t-2}\right) + L_2\left(S\dot{P}_{t-n}\right) + L_3\left(M_{t-n}^N\right) + b_3 \cdot Q_t + e_t;$$

$$b_0,\ L_3 < 0,\ \ b_1, b_2, L_1, L_2 > 0, \tag{39}$$

where $R_t$ is the nominal rate of interest, $a, b_0, b_1, b_2, b_3$ are fixed coefficients, $M_t^R$ is the real monetary base per capita, $Y_t^c$ is the real personal income per capita. $\dot{P}_{t-n}$ is the rate of changes in general prices, $R_{t-1}, R_{t-2}$ is the rate of interest lagged $t-n$ periods, $S\dot{P}_{t-n}$ is the percentage change on the S&P 500 index, $M_{t-n}^{N}$ is the rate of change in nominal monetary base per capita, $Q_t$ is a dummy variable[14] for the excess rate of the certificates of deposit over the commercial paper rate, $L_1, L_2, L_3$ are lag operators and $e_t$ is the error term.

The third category includes a multifactor approach to interest rate movements. This approach emphasizes the interrelated impact of commodity market influences to real rates of interest and monetary influences to nominal rates of interest due to inflationary expectations.

Sargent (1970) describes investment as a function of real rates of interest and the current change in real income. In addition; the model takes into consideration changes in the nominal money supply by applying lag operators.

The model is described as,

$$R_t = a + b_0 \Delta Y_t + b_2 M_t + L_1 \dot{P}_t + e_t; \qquad b_0, L_1 > 0, b_1, b_2 < 0, \tag{40}$$

where $R_t$ is the nominal rate of interest, $a, b_0, b_2$ are fixed coefficients, $\Delta Y_t$ is the change in real income, $M_t$ is the rate of change in the nominal money supply, $\dot{P}_t$ is the rate of change in general prices, $L_1$ is the lagged operator and $e_t$ is the error term.

Echols and Elliot (1976) extended Sargent's model by including several variables which include government expenditure, taxing sector, private investment, saving sector and the international sector. The interest rate model is described as,

$$R_t = a + L_1 \Delta Y_t + b_0 Y_t + b_1 E_t^N + b_2 G_t^N + L_2 \dot{P}_t + b_4 M_t + e_t;$$
$$L_1, L_2 > 0; \quad b_0, b_1, b_2, b_4 > 0, \tag{41}$$

where $R_t$ is the nominal rate of interest, $a, b_0, b_1, b_2, b_4$ are fixed coefficients, $\Delta Y_t$ is the change in real income, $Y_t$ is the real personal income, $E_t^N$ is the net export balance, $G_t^N$ is the net government surplus or deficit, $\dot{P}_t$ is the rate of changes in general prices, $M_t$ is the rate of change in nominal money supply, $L_1, L_2$ are lagged operators and $e_t$ is the error term. Table IV located at the end of this section; provides a general overview of the different models analyzed.

Models are fitted to data for the USA. By applying different exogenous variables into

[14] In this case; the dummy variable takes the value of 1 if the statement is positive and 0 otherwise.

the models; results indicate that from the six econometric models analyzed; only four econometric models are able to explain current interest rate movements. These models include the ones developed by Modigliani and Sutch, Modigliani and Shiller, Feldstein and Chamberlain, and Echols and Elliot. Even that these econometric models explain accurately the current rate; they are inaccurate to forecast future rates.

Elliot and Baier demonstrate that even the best econometric model will not be able to predict next month's rate as accurate as the hypothesis that next month's rate will not change from its current value. Econometric models provide several alternative subsets of exogenous variables which are strongly associated with fluctuations on interest rates. The information on these subsets relevant to current rates has little relation with future fluctuations.

Results indicate that accurate predictions of future interest rates by any econometric model are only possible having future information. Moreover, results suggest that interest rates adjust rapidly to changes in underlying economic and capital markets' influences. If fluctuations in current economic variables apply all influence on interest rates immediately; it is expected to observe a weak relationship between current economic variables on future interest rates.

Findings obtained by Elliot and Baier are consistent with martingale theory. According to martingale theory[15]; it is not possible to predict any future movement because all information is already priced at $t = 0$. As a result; it would not be possible to forecast or predict any future movements in asset prices. (Neftci; 2000)

In order to have a measurable interest rate model able to accurately forecast, predict or determine actual and future interest rate levels; it is extremely important to maintain volatility levels into measurable parameters. If interest rates expand into an immeasurable size in the future, no interest rate model will be accurate enough to model and forecast future interest rate movements. Therefore; it is concluded that volatility parameter is one of the most important parameters on any interest rate model. As a result; required attention should be given to this parameter, since any bias in the volatility parameter will imply bias in interest rate estimations and consequently bias in any possible application.

---

[15] A process $S_t$ is a martingale if its future movements are completely unpredictable given a family of information sets.

**Table IV.** *Model descriptions*

Six econometric interest rate models are compared with the objective of explaining and forecasting interest rates. Models are divided in three categories: 1) Models which apply distributed lags of past rates to explain current rates, 2) Models derived from some version of liquidity preference theory and 3) Models which are reduced forms of multisector econometric models.

| Category | Model | Year | Specification | Description |
|---|---|---|---|---|
| Models that focus on distributed lags of past rates for explaining current rates. | Modigliani-Sutch | 1966 | $R_t = a + b_0 B_t + \sum_{n=1}^{23} b_n B_{t-n} + K_t + e_t;$ $\sum_{i=1}^{23} b_i > 0,$ | The model states that past patterns of short rates and inflation rates contain useful information in setting the current long rate. |
| | Modigliani-Shiller | 1973 | $R_t = a + b_0 B_t + \sum_{n=1}^{23} b_n B_{t-n} + c_0 \dot{P}_t + \sum_{n=1}^{23} c_n \dot{P}_{t-n} + dL_t + e_t;$ $\sum_{i=1}^{23} b_i > 0,\ d > 0$ | The model applies short rates and inflation rates as indicators for determining the actual long rate. |
| Models derived from some version of liquidity preference theory | Feldstein-Eckstein | 1970 | $R_t = a + b_0 \ln\left(M_t^R\right) + b_1 \ln\left(Y_t^c\right) + \sum_{n=0}^{23} b_{n+2} \dot{P}_{t-n} + b_{24}\left(R_{t-1} - R_{t-2}\right) + e_t;$ $b_0 < 0, b_1 > 0, \sum_{n=0}^{23} b_{n+2} > 0, b_{24} > 0$ | The model focuses on the relationship between bonds and money. Demand for money is more influenced by income changes and negatively influenced by changes in real interest rates. |

| | | | | |
|---|---|---|---|---|
| Models derived from some version of liquidity preference theory | Feldstein-Chamberlain | 1973 | $R_t = a + b_0 \ln\left(M_t^R\right) + b_1 \ln\left(Y_t^c\right) + L_1\left(\dot{P}_{t-n}\right) + b_2\left(R_{t-1} - R_{t-2}\right)$ $+ L_2\left(S\dot{P}_{t-n}\right) + L_3\left(M_{t-n}^N\right) + b_3 \cdot Q_t + e_t;$ $b_0, L_3 < 0,\ b_1, b_2, L_1, L_2 > 0$ | The model is considered as a second generation model of the previous model developed by Feldstein and Eckstein (1970). It includes four financial assets; which are money, bonds, equities and short term bills. |
| Models that are reduced forms of multisector econometric models | Sargent | 1970 | $R_t = a + b_0 \Delta Y_t + b_2 M_t + L_1 \dot{P}_t + e_t;$ $b_0, L_1 > 0, b_1, b_2 < 0,$ | The model describes investment as a function of real rates of interest and the current change in real income. It takes into consideration changes in the nominal money supply by applying lag operators. |
| | Echols-Elliot | 1976 | $R_t = a + L_1 \Delta Y_t + b_0 Y_t + b_1 E_t^N + b_2 G_t^N + L_2 \dot{P}_t + b_4 M_t + e_t;$ $L_1, L_2 > 0;\quad b_0, b_1, b_2, b_4 > 0$ | This model is an extension of Sargent's model (1970). It includes additional variables such as government expenditure, taxing sector, private investment, saving sector and the international sector. |

## 5.2 *Chan- Karolyi-Longstaff-Sanders Framework*

Chan et al (1992) analyzed eight different interest rate models by applying a common framework. It is possible to group the models within a common framework by restricting or modifying the values of $\alpha, \beta, \sigma, \gamma$. The models analyzed include Merton (1973), Vasicek (1977), Cox-Ingersoll-Ross Square Root (1985), Dothan (1978), Geometric Brownian Motion (1973), Brennan-Schwartz (1980), Cox-Ingersoll-Ross Variable Rate (1980), and Constant Elasticity of Variance (1975).

The common framework applied is determined by a stochastic differential equation (SDE), which can be described as,

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dZ \, , \tag{42}$$

where $r$ denotes the interest rate level, $t$ defines time, $\gamma$ determines the level of interest rate elasticity, $Z$ denotes a Brownian motion and $\alpha, \beta, \sigma, \gamma$ are parameters. Parameter $(\alpha + \beta r)$ denotes the drift and $\sigma r^{\gamma} dZ$ determines the variance of unexpected interest rate changes. It is clarified that by restricting the value of the parameters, different models can be replicated.

If we rewrite $(\alpha + \beta r)$ as $\beta(r - \alpha^{*})$, it is observed that $\beta$ determines the speed of mean reversion. The more negative the value of $\beta$, the faster interest rates will respond to deviations of $\alpha^{*}$. Parameter $\gamma$ allows volatility to depend on the level of interest rates. At higher levels of $\gamma$; volatility will be more sensitive to interest rate levels. The following table describes the different models nested within the same framework.

**Table V.** *Model descriptions and parameter restrictions*

Models of the short term interest rate can be nested within a common framework by restricting values of $\alpha, \beta, \sigma, \gamma$. The common framework is denoted as

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dZ$$

| Model | Year | Specification | Description | Restrictions |
|---|---|---|---|---|
| Merton | 1973 | $dr = \alpha dt + \sigma dZ$ | The model is applied for pricing discount bonds. It is a stochastic process for the riskless rate denoted by a Brownian motion with drift. | $\beta = 0$<br>$\gamma = 0$ |
| Vasicek | 1977 | $dr = (\alpha + \beta r)dt + \sigma dZ.$ | This model is extensively applied for the valuation of contingent claims. It derives an equilibrium approach of discount bond prices by applying an Ornstein-Uhlenbeck[16] process. | $\gamma = 0$ |
| Cox-Ingersoll-Ross Square Root | 1985 | $dr = (\alpha + \beta r)dt + \sigma r^{\frac{1}{2}} dZ.$ | The model is a single factor general equilibrium term structure model applied in the valuation of interest rate sensitive contingent claims. It implies that conditional volatility is proportional to the levels of interest rates. | $\gamma = \frac{1}{2}$ |

[16] Ibid 3

| | | | | |
|---|---|---|---|---|
| Dothan | 1978 | $dr = \sigma r dZ$ | This model is extensively used for valuation of discount bonds. | $\alpha = 0$<br>$\beta = 0$<br>$\gamma = 1$ |
| Geometric Brownian Motion | 1973 | $dr = \beta r dt + \sigma r dZ.$ | This model is developed by Black and Scholes. | $\alpha = 0$<br>$\gamma = 1$ |
| Brennan-Schwartz | 1980 | $dr = (\alpha + \beta r)dt + \sigma r dZ.$ | This model is applied in deriving a numerical model for convertible bond prices. | $\gamma = 1$ |
| Cox-Ingersoll-Ross Variable Rate | 1980 | $dr = \sigma r^{\frac{3}{2}} dZ.$ | This model is utilized in the study of variable rate securities. | $\alpha = 0$<br>$\beta = 0$<br>$\gamma = \frac{3}{2}$ |
| Constant Elasticity of Variance | 1975<br>1976 | $dr = \beta r dt + \sigma r^{\gamma} dZ.$ | This model is introduced by Cox (1975) and further developed by Cox and Ross (1976). | $\alpha = 0$ |

The majority of the interest rate processes are introduced in the context of a single factor model of the term structure. However; the analysis is not limited to a single factor model. If we set $\beta = 0$ by assuming that the conditional volatility of changes in the riskless rate is constant; it is possible to nest Merton and Vasicek models into Vasicek model. Geometric Brownian Motion (GBM) and Dothan models can be nested together if we set $\beta = 0$. Furthermore; GBM and Brennan-Schwartz are nested together if we set $\alpha = 0$. In addition; Dothan, Brennan-Schwartz and GBM imply that the conditional volatility of changes in the riskless rate is proportional to $r^2$.

Models are tested using the Generalized Method of Moments of Hansen (1982)[17]. The criteria chosen for determining the best model is its ability to capture volatility on changes in interest rates. Volatility is considered as an important determinant on applications of term structure models, such as valuation of contingent claims and hedging of interest rate risk. The ability of a term structure model to capture interest rate volatility is a direct measure of its hedging usefulness.

By comparing the models under a same framework it is possible to determine which model describes more accurately the behavior of interest rates. Nested models for the short term interest rates indicate that the main aspect that differentiates the models is their ability to capture the time varying volatility of the short term interest rates.

Results suggest that the relation between interest rate volatility and the level of interest rates is the most important aspect of any dynamic model of the short term riskless rate. This effect is significant because extensive attention has been given to other aspects. For example Vasicek and Merton models have been criticized for allowing negative values to interest rates. However; a more dangerous defect of these models is the assumption that interest rate changes are homoskedastic[18].

Volatility of riskless rate is a key variable governing the value of interest rate contingent claims. In order to determine the rates of one month US Treasury bill,

---

[17] Generalized Method of Moments (GMM) has important advantages since it does not require a normal distribution of interest rate changes. This technique only requires stationarity and ergodicity on the distribution of interest rates. These conditions are very important when testing the term structure models in a continuous time setup. GMM estimators and standard errors are consistent even when error term disturbances n-periods ahead are conditionally heteroskedastic.

[18] Homoskedasticity is defined as the assumption in which the variance of the error term is constant at different points in time; where $\mathrm{var}(u_t) = \sigma^2$.

equation (42) is modified by applying a discrete time setup. Levels of interest rates are determined by,

$$r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \sigma r_{t-1}^{\gamma} \varepsilon_t, \qquad \varepsilon_t \sim D(0,1). \tag{43}$$

Empirical evidence obtained from one month Treasury Bills suggests that the value of $\gamma$ is the most important determinant for differentiating interest rate models. It was demonstrated that models with $\gamma \geq 1$ capture the dynamics of the short term interest rate better than those with $\gamma < 1$, since the volatility of the process is highly sensitive to the level of interest rates.

As a final statement it is concluded that interest rate models demonstrate different implications for capturing the dynamics of the short term interest rate and valuation of interest rate contingent claims. As a result; the ability to obtain accurate volatility estimations is fundamental for modeling and forecasting a correct term structure of interest rates.

### *5.3 Koedijk-Nissen-Schotman-Wolff Framework*

Koedijk et al (1997) develop a model of the short term interest rate volatility which takes into consideration the level effect of Chan et al. (1992) with the conditional heteroskedasticity effect of the GARCH models framework (1986). Koedijk-Nissen-Schotman-Wolff (KNSW) model is analyzed with respect to three different models by applying a common volatility framework in order to measure interest rate volatility. It is possible to group the analyzed models within a common framework by restricting the values of $\beta_2, \beta_3, \gamma, \alpha_2$. These models include GARCH (1,1) by Bollerslev (1986), Longstaff-Schwartz (1993), Koedjk-Nissen-Schotman-Wolff (1997) and Chan-Karolyi- Longstaff- Sanders (1992).

The common volatility framework able to nest the different models is described as:

$$h_t^2 = \beta_0 + \beta_1 r_{t-1}^{2\gamma} + \left(\frac{r_t}{r_{t-1}}\right)^{2\gamma} \left(\beta_2 \varepsilon_{t-1}^2 + \beta_3 h_{t-1}^2\right) + \beta_4 r_{t-1} \tag{44}$$

where $\beta_1$ denotes the intercept, $\gamma$ describes the sensitivity to changes in interest rates, $r_t$ is the level of interest rates at current time, $r_{t-1}$ is the level of interest rates lagged one period, and $\varepsilon_t$ describes the error term. The following table describes the models.

**Table VI.** *Model descriptions*

Models of the short term interest rate can be nested within a common volatility framework by restricting values of $\alpha, \beta, \gamma$. The common volatility framework is denoted as

$$h_t^2 = \beta_0 + \beta_1 r_{t-1}^{2\gamma} + \left(\frac{r_t}{r_{t-1}}\right)^{2\gamma} \left(\beta_2 \varepsilon_{t-1}^2 + \beta_3 h_{t-1}^2\right) + \beta_4 r_{t-1}.$$

| Model | Year | Specification | Description |
|---|---|---|---|
| GARCH (1,1) | 1986 | $r_t = \alpha_0 + (\alpha_1 + 1) r_{t-1} + \varepsilon_t$<br>$\sigma_t^2 = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 \sigma_{t-1}^2$<br>$\mathrm{E}(\varepsilon_t) = 0$<br>$Var(\varepsilon_t) = \sigma_t^2$ | This model is developed by Bollerslev. It allows for volatility clustering and volatility persistence. |
| Chan-Karolyi-Longstaff-Sanders | 1992 | $r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \sigma r_{t-1}^{\gamma} \varepsilon_t$<br>$\varepsilon_t \sim D(0,1)$ | This model is a discrete approximation of the stochastic differential equation (SDE). |
| Longstaff - Schwartz | 1993 | $r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \alpha_2 h_{t-1}^2 + \varepsilon_t$<br>$h_t^2 = \beta_1 + \beta_2 \varepsilon_{t-1}^2 + \beta_3 h_{t-1}^2 + \beta_4 r_{t-1}$ | This model is a two factor model for the term structure. The factors applied are the short term interest rate and conditional variance of changes in interest rates. |
| Koedjk-Nissen-Schotman-Wolff | 1997 | $h_t^2 = \beta_1 r_{t-1}^{2\gamma} + \left(\frac{r_{t-1}}{r_{t-2}}\right)^{2\gamma} \left(\beta_2 \varepsilon_{t-1}^2 + \beta_3 h_{t-1}^2\right)$ | This is a two factor model which combines the high interest rate sensitivity of CKLS with GARCH type volatility clustering. The model allows for persistence of shocks which depend on interest rate level. |

If we set $\beta_2 = \beta_3 = 0$, the CKLS model is obtained. In addition; if $\gamma = 0$, the GARCH model is replicated. Furthermore; if $\alpha_2 = 0$ and $\gamma = \frac{1}{2}$, the Longstaff and Schwartz model is obtained.

Models are estimated by applying the method of quasi maximum likelihood (QML), since this method is consistent and asymptotically normal for any distribution of the error term, $\varepsilon_t$. However; severe multicollinearity[19] problems do not allow to estimate models within the nesting model. In Longstaff-Schwartz model, $\beta_0$ can not be estimated if $\gamma$ is a free parameter.

It is observed that for moderate interest rate levels, the mean reversion is relatively small. However; when interest rates are greater than 15%, the drift decreases importantly but the standard error remains large.

In order to be able to focus on the dynamics of volatility; interest rate models are compared to each other by restricting them to a conditional mean. If $\gamma > 1$ models are said to be non stationary. As a result; models are restricted to $\beta_2 + \beta_3 < 1$ in order to obtain stationarity. However; empirical evidence demonstrates that GARCH models are highly significant and show strong persistence of variance shocks since $\beta_2 + \beta_3 = 1.03$.

The main difference between models is their ability to determine the elasticity of volatility to the level of interest rates. In highly volatile periods of time, normality is strongly rejected for all models. Therefore; cautious regarding the interpretations of the models are required.

Regarding asymmetric effects; it has been empirically demonstrated that different volatility levels follow unsimilar patterns. These patterns can be demonstrated through the news impact curve developed by Engle and Ng (1993). The news impact curve is able to demonstrate the effects of last shocks, $\varepsilon_t$, on conditional volatility, $h_t^2$.

---

[19] Multicollinearity indicates serial correlation between different parameters. If the correlation between two or more variables is equal to 1, then it is said that a perfect multicollinearity exists. However; the existence of perfect multicollinearity is rather rare. In mathematical notation; multicollinearity can be defined as, $\lambda_1 X_{1i} + \lambda_2 X_{2i} + \ldots + \lambda_k X_{ki} = 0$

Models such as EGARCH[20] and GJR[21] are able to describe the effects of positive and negative shocks affecting the variables. According to the news impact curve, negative shocks present higher variances than positive shocks.

Depending on the level of interest rates; asymmetric effects obtained from the models analyzed may diverge importantly. News impact curve of KNSW model depends on interest rate levels, last period's shock, and last period's conditional volatility. Moreover; on GARCH model; the news impact curve depends on last period's innovation and last period's conditional volatility. Furthermore; the news impact curve of CKLS model depends only on the last period interest rate level.

At a low interest rate level, $r = 4\%$, KNSW presents the highest asymmetric effects following by CKLS. GARCH model shows the least asymmetric effect on the presence of negative shocks. At a moderate interest rate level, $r = 8\%$, GARCH and KNSW behave in a very similar form. The least asymmetric is GARCH model since KNSW model is slightly more asymmetric than GARCH model in the presence of a negative shock, while CKLS demonstrates to be the most asymmetric model. At a high level, $r = 12\%$, models diverge importantly. The behavior of the models is quite different on the presence of negative shocks. GARCH model is almost insensitive to positive and negative shocks, while KNSW is slightly more sensitive to negative shocks. CKLS model presents the highest asymmetric effects as well as the highest sensitivity to negative shocks.

It is observed that KNSW model is able to capture skewness[22] of the data. Skewness of CKLS model is small; while GARCH model is symmetric but presents excess kurtosis[23]. In addition; it is possible to observe that CKLS and KNSW model behave in a very similar form; which are slightly skewed to the right.

---

[20] EGARCH denotes Exponential General Autoregressive Conditional Heteroskedasticity. This model is developed by Nelson (1991). It is able to describe the asymmetric effects obtained as a result from positive and negative innovations.

[21] GJR model is named after its creators; Glosten, Jagannathan, and Runkle (1993).

[22] Skewness is the third moment of a normal distribution. Skewness measures the extent to which a distribution is not symmetric about its mean value. A normal distribution is symmetric and will have coefficient of skewness equal to 0. Any value higher or lower than 0 will indicate the presence of skewness in the distribution.

[23] Kurtosis is defined as the fourth moment of a normal distribution. Kurtosis measures the degree of fatness in the tails of the distribution. A normal distribution is defined to have a coefficient of kurtosis equal to 3 and will have a coefficient of excess kurtosis equal to 0. As a result; any value above or below zero in the excess kurtosis coefficient will indicate the presence of kurtosis in the distribution.

Mean reversion is almost imperceptible over one period horizon. Volatility function is a quadratic function for the GARCH specification but can be very asymmetric for the KNSW model. It is demonstrated that adding a GARCH effect into the levels effect is relevant for the pricing of short term discount bond options. As a final statement; it is concluded that as the level of interest rates increase; a change in the value of volatility implies a larger change on the price of an option.

### *5.4 Koutmos Framework*

Koutmos (2000) models interest rate volatility in international markets in order to determine the effects of volatility due to interest rate levels versus information shocks. Markets analyzed include Canada, France, Germany, Japan, United Kingdom and United States. Possible benefits include evaluation of market interdependence as well as similarities and differences on interest rates from different markets. In addition; international interest rate contingent claim users can have an accurate measure of interest rate volatility for pricing objectives.

The short term interest rate is one of the most important financial variables since it is one of the most important determinants of the term structure. Following Chan, Karolyi, Longstaff and Sanders (1992) where several short term interest rate models are evaluated; the single factor diffusion model applied is determined by,

$$dr = \mu(r)dt + \sigma(r)dZ\ , \tag{45}$$

where $r$ is the instantaneous spot rate, $Z$ is a standard Brownian motion.

Drift and diffusion depend on the level of interest rate following,

$$\mu(r) = (\alpha + \beta r)dt\ , \tag{46}$$

$$\sigma(r) = \sigma r^{\gamma}\ , \tag{47}$$

where $\alpha, \beta, \sigma, \gamma$ are fixed parameters. Short term rate is mean reverting process with $\beta$ as the mean reversion parameter and $\gamma$ as the elasticity of variance with respect to the level of interest rates.

The econometric specification is determined by a discrete approximation from the continuous time models described in equations (45) through (47). The discrete approximation is described as,

$$r_t - r_{t-1} = \alpha + \beta r_{t-1} + \varepsilon_t \tag{48}$$

$$\mathrm{E}_{t-1}(\varepsilon_t) = 0$$

$$\mathrm{E}_{t-1}(\varepsilon_t^2) = (h_t^2),$$

where $\varepsilon_t$ is the innovation or information shock at time $t$, and $\mathrm{E}_{t-1}$ is the conditional expectations error. The discretized interest rate model described in equation (48) can be estimated by applying three different models for the conditional variance. The following table describes the models for the conditional variance and provides a brief description of each one of these models.

**Table VII.** *Model descriptions and parameter restrictions*
Models of the short term interest rate are nested within the volatility specification denoted as $h_t^2 = \sigma^2 r_{t-1}^{2\gamma}$, in order to determine the relevance of volatility effects due to interest rate levels versus information shocks.

| Model | Notation | Description | Restrictions |
|---|---|---|---|
| Chan-Karolyi-Longstaff-Sanders (1992) | $h_t^2 = \sigma^2 r_{t-1}^{2\gamma}$ | Changes in interest rates are conditionally heteroskedastic, since the conditional variance at time $t$ depends on interest rates at $t-1$. | None |
| GARCH (1, 1) augmented for asymmetries (1987, 1990, 1993) | $h_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2 u_{t-1}^2 + bh_{t-1}^2$ | The term $u_{t-1}^2$ determines asymmetric effects. The model allows the conditional variance to depend on innovations but not on the level of interest rates. | $u_{t-1} = \min(0, \varepsilon_{t-1})$ |
| Koutmos (2000) | $h_t^2 = a_0 + \sigma^2 r_{t-1}^{2\gamma} + a_1\varepsilon_{t-1}^2 + a_2 u_{t-1}^2 + bh_{t-1}^2$ | This model nests the previous two models. It provides an excellent framework for evaluating alternative hypotheses. The importance of the level of interest rates and information shocks are evaluated by applying restrictions on the parameters. | $a_0 = a_1 = a_2 = b = 0$ CKLS model is replicated; $\sigma = 0$ and/or $\gamma = 0$, GARCH (1, 1) model augmented for asymmetries is obtained. |

Data applied are weekly three month interest rates for Canada, France, Germany, Japan, United Kingdom and United States of America. To estimate $\gamma$ and $\sigma^2$ at the same time, results in high standard errors due that both parameters are highly collinear. As a result; the approach applied is to fix the values of, $\gamma$, elasticity of interest rates. This parameter is fixed at values ranging from .5, 1.0, and 1.5. Models are estimated conditionally to these three values. By imposing these parameters, some of the most popular continuous time models are obtained.

Mean reversion does not seem to be relevant for any interest rate model analyzed. This finding is important since most models place special attention to mean reversion. However; from an economical point of view, mean reversion is reasonable. Nevertheless; empirical evidence rejects this idea.

Chan et al (1992) conclusion regarding that models where $\gamma > 1$ perform better than models where $\gamma < 1$, does not hold across national interest rates; specially in the presence of information shocks. The importance of using interest rate levels as well as information shocks in the conditional variance is obtained by calculating gains in the likelihood value.

According to Koutmos (2000); modeling volatility as a function of past information shocks is more important. Superiority of GARCH (1,1) model is determined by Ljung-Box statistics. Empirical evidence provided by Chan et al (1992) and GARCH (1,1) models indicate that both sources of variation (interest rate levels and information shocks), should be included when specifying the conditional volatility of short term interest rates; as applied in Koutmos model.

With exception of Japan; GARCH (1,1) model provides a better fit than CKLS model. If it is required to choose a model for the pricing of contingent claims and pricing of interest rate derivative securities; GARCH (1,1) model would be preferred since it demonstrates to have more accurate estimations. This empirical evidence provides strong international evidence against models that rely uniquely on interest rate levels. The best volatility specification requires the use of both factors; information shocks and interest rate levels. It is observed that mean reversion is not an important aspect of short term interest rates.

*5.5 Goard-Hansen Framework*

Goard et al (2004) compare their model with models of the short term interest rate developed by Goard (2000) and Ahn-Gao (1999). The short term riskless interest rate is a key variable of great importance in Finance. A popular group of these models is the single factor spot rate models determined by the following stochastic differential equation,

$$dr = u(r,t)dt + w(r,t)dX \ , \tag{49}$$

where $u(r,t)dt$ denotes the drift, $w(r,t)dX$ describes the variance and $dX$ denotes a Brownian motion.

Ahn and Gao (1999) present a one factor model determined by equation (50) and compare it with the models analyzed by Chan et al (1992). It is concluded that the linearity in the drift seems to be the main source of misspecification. As a result; Ahn-Gao model (1999) outperforms those models developed by Chan et al (1992).

The model developed by Ahn and Gao (1999) is described as,

$$dr = \kappa(\theta - r)r\, dt + \sigma r^{3/2} dX. \tag{50}$$

Furthermore; Goard (2000) develops a model based on a more general version of the model developed by Ahn and Gao (1999). Goard model is denoted as,

$$dr = \left[c^2 r(a(t) - qr)\right]dt + c^{3/2} dX \ , \tag{51}$$

where $c,q$ are independent constants, $a$ is an arbitrary function of time. Since $c,q$ are independent; it implies that volatility and mean reversion are also independent.

Chan et al. (1992) and Ahn et al. (1999) used the Generalized Method of Moments of Hansen (1982) to test nested models. Equations (50) and (51) are also compared to each other by applying GMM and several different datasets.

The model developed by Goard and Hansen (2004) is described as,

$$dr = \{\beta(\mathrm{t})r + \alpha r^2\}dt + \sigma r^{3/2} dX \ ,$$

$$\beta(t) = \beta_1 + \beta_2 \sin(h\pi t) + \beta_3 \cos(h\pi t) + \beta_4 \sin(2h\pi t) + \beta_5 \cos(2h\pi t) \tag{52}$$

where $\alpha, \sigma, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ denote the parameters to be estimated and $h$ is a set constant.

The drift term is non linear and the basis level of interest rates at which drift is zero is a function of time. In addition; the drift term is a quadratic function of the interest rates. Diffusion parameter in equations (50) and (51) is equal to the parameter obtained in Chan et al. (1992); which is defined as the power-law volatility parameter. It implies that the sensitivity of variation in the spot rate is highly sensitive to the spot rate. Consistent with Ait-Sahalia (1996) and Stanton (1977) which demonstrate that a sharp decline in the drift function for high interest rates is necessary to prevent interest rates to explode.

A single factor model for the short term interest rate is developed described as,

$$dr = \{\alpha_1 + \beta(t)r + \alpha_2 r^2\}dt + \alpha_3 r^{3/2} dX \ ,$$

$$\beta(t) = \beta_1 + \beta_2 \sin(h\pi t) + \beta_3 \cos(h\pi t) + \beta_4 \sin(2h\pi t) + \beta_5 \cos(2h\pi t) \qquad (53)$$

Equation (53) is able to nest models developed by Chan et al (1992), Ahn et al (1999) and by Goard et al (2004). In order to allow for estimation and comparison within the different models; a discretization from the models is required. Parameters are adjusted for serial correlation and heteroskedasticity. For each model $H_0$ is evaluated against $H_1$. $H_0$ states that the nested model does not impose over identification restrictions and is not misspecified; while $H_1$ denotes that the nested model impose over identifying restriction and is misspecified.

Equation (53) is applied to several time series from USA, Australia, Thailand, and United Kingdom. The range of time applied on these series may diverge. USA, Australia, Thailand and UK utilize daily data; while USA and Australia also apply monthly data of Treasury bill yields. It is observed that the first differenced time series for each data set is stationary.

Findings demonstrate that even with relatively short rate data sets; evidence of time variation in the drift term does exist. Several single factor models of the short interest rates do exist. However; many of them perform poorly in their ability to capture the actual behavior of the spot rate. Goard (2000) outperform models that assume long term reversion to its mean. A second order Fourier series to model the free function of time is developed which included the realistic $cr^{3/2}$ in the volatility.

When fitting time series on interest rates; the more accurate and realistic results will be determined on data sets with longer periods of time. The longer the data set, the more obvious are the short and long term trends in interest rates' behavior. Even for shorter periods of time, Goard and Hansen (2004) model determine the time dependence on mean reversion level. A short rate model with time dependent moving target is more accurate to model the term structure of interest rates than no time dependent models. It is found that time dependent model developed by Goard, outperforms Ahn-Gao model.

## *5.6 Sanford-Martin Framework*

Sanford et al (2006) analyze several single factor continuous time models for the Australian short term rate with the objective of determining the extent of the level effect. Models are nested in a general single factor diffusion process for the short rate interest rate. Models analyzed include Vasicek (1977), Cox-Ingersoll-Ross Square Root (1985), a variation of Courtadon (1982) short rate model, Chan, Karolyi, Longstaff, Sanders (1992) and Sanford and Martin (2006).

To model correctly the instantaneous short rate is extremely important in Finance since it is fundamental for the pricing of fixed income securities. Many alternative models for the short rate process exist. However; it is still unclear which one from all the models is the one that performs the best. Findings regarding the extent of the level effect on interest rates are inconclusive. This demonstrates doubts on the accuracy of methods related to derivative pricing that assume a particular value for the level effect parameter.

As a result; the objective of Sanford-Martin Framework is to provide empirical evidence related to the outperformance of a specific model of the term structure of interest rates. Alternative nested models are indexed by different values for the level effect parameter, $\delta$. A Bayesian approach is applied in order to determine uncertainty regarding unknown quantities through posterior probabilities.

Alternative models are nested in a general single factor diffusion process for the short rate. Each alternative model is indexed by the volatility parameter. The following equation is the general model in which several single continuous time models for the short rate at time $t$ are nested,

$$dr_t = (\theta + \kappa r_t)dt + \sigma r_t^{\delta} dW_t \tag{54}$$

where $\kappa$ represents mean reversion, $\sigma$ determines volatility, $\delta$ describes the level effect parameter of the short rate process, and $dW_t$ describe independent increments of a Wiener process. In the following table; the relationship between Equation (54) and models previously developed in the Financial literature is demonstrated.

**Table VIII.** *Model specifications and restrictions*

Models of the short term interest rate can be nested within a common framework by restricting the values of the level effect parameter, $\delta$. The common framework is described as, $dr_t = (\theta + \kappa r_t)dt + \sigma r_t^{\delta} dW_t$.

| Model | Notation | Year | Specifications | Restrictions |
|---|---|---|---|---|
| Sanford and Martin | $M_0$ | 2006 | $dr_t = (\theta + \kappa r_t)dt + \sigma r_t^{\delta} dW_t$ | $\delta =$ free parameter |
| Vasicek | $M_1$ | 1977 | $dr_t = \kappa(\theta - r)dt + \sigma dW_t$ | $\delta = 0$ |
| Cox-Ingersoll-Ross Square Root | $M_2$ | 1985 | $dr_t = \kappa(\theta - r)dt + \sigma\sqrt{rdW_t}$ | $\delta = 0.5$ |
| Courtadon | $M_3$ | 1982 | $dr_t = (\theta + \kappa r_t)dt + \sigma r_t^{\delta} dW_t$ | $\delta = 1.0$ |
| Chan-Karolyi-Longstaff-Sanders | $M_4$ | 1992 | $dr_t = \kappa(\theta - r)dt + \sigma r^{\delta} dW_t$ | $\delta = 1.5$ |

Equation (1) has to be discretized in order to be employed.[24] The discretized equation is denoted as

$$r_{t+\Delta t} - r_t = (\theta + kr_t)\Delta t + \sigma r_t^{\delta} \sqrt{\Delta t\, \varepsilon_t}\,, \tag{55}$$

where $\varepsilon_t \sim i.i.d.N(0,1)$ and $\Delta t$ should be as small as possible in order to reduce possible bias by applying the discrete time approximation of the continuous time model. Bias is reduced by increasing the data set with higher frequency in order to reduce the size of $\Delta t$. As a result; equation (55) will be a more accurate approximation of Equation (54).

Bayesian approach is applied for analyzing the models, $M_j$, $j = 0,1,2,3,4$, with an emphasis on the production of Bayes factors and posterior model probabilities, for selecting the most accurate model[25]. All Bayes factors are expressed relative to the model developed by Sanford et al, where the parameter $\delta$ is left unrestricted.

By assuming equal probabilities for all models and $\sum 1$, models are ranked according to their probabilities and selected depending on the highest probability obtained. The method applied simulates augmented data points between the observed

[24] Discretization is realized by applying an Euler scheme which results into a discrete time version of Equation (54).

[25] Bayes Theorem is applied to determine the posterior probability distribution of the parameters on the j$^{th}$ model conditioning on the observed data. Posterior probabilities are then assigned to each model analyzed. Models are assumed to have equal probabilities. As a result; the decision of the optimal model selected will be determined by the posterior probabilities, since these probabilities measure the support in the data. Consequently; the model selected will be the one with the highest probability.

short rate data; since this technique reduces time between observations. As a result; the discrete time approximation to the continuous time model becomes more accurate.

Markov chain Monte Carlo Algorithm is applied to estimate parameters and Bayes factors associated to Equation (55). Monte Carlo Standard Errors are calculated for each parameter in order to determine the accuracy of the simulations taking correlation into consideration. The reference for all models is denoted as $M_0$ where $\delta$ is an unrestricted parameter. Parameter $M_0$ is used as a reference point in order to compare and evaluate the performance of the nested models.

Empirical research is based on weekly observations on the Australian 90 day rate from 1990 to 2000. The selection of the data is determined due to the presence of high liquidity. This period of time is selected since it represents historically high interest rates in the early nineties and low interest rates in the second part of the sample period. In addition; there is a tendency for the volatility in interest rates to be positively correlated with the current level of rates. Skewness in the data is observed due to a leverage effect. Furthermore; unexpected news might be important in understanding the volatility of interest rates.

Empirical results demonstrate that estimates of mean reversion imply a high persistence parameter of $\approx .99$ for weekly short rate data; corresponding to a near presence of unit root in the data. Chan et al. (1992) and Brenner et al. (1996) estimate values of $\delta = 1.5$ and $\delta = 1.559$ for US data. However; estimates of $\delta$ for Australian data was found to range between values of .929 to 1.704, depending on the data set. Results differ from the ones obtained in previous research since the time series analyzed consists of a shorter period of time, reducing the periods which present extreme volatility.

Bayes factors for each one of the models are applied. Factors support the Square Root model developed by Cox-Ingersoll-Ross (1985) for all levels of augmentation. It is observed that when $h = 0$, Cox-Ingersoll-Ross Square Root has an important dominance over Sanford and Martin model. However; as augmentation increases and bias related to the approximation of a continuous time model to a discrete time process reduces; Sanford and Martin model increases in accuracy with respect to Cox-Ingersoll-Ross Square Root.

Results demonstrate that only Cox-Ingersoll-Ross Square Root and Sanford and Martin model have positive posterior probability. Therefore; results provide support for Cox-Ingersoll-Ross Square Root (1985) diffusion models and no support for any of the additional restricted models, including the model developed by Chan et al. (1992). Since Cox-Ingersoll-Ross Square Root (1985) provides the highest support to the data; it is observed that their pricing equations can be applied to the Australian context accurately.

### *5.7 Reno-Roma-Schaefer Framework*

Reno et al. (2006) estimate the diffusion coefficient of several single factor models for the short term interest rate by applying a non parametric approach introduced by Florens-Zmirou (1993) and Aït-Sahalia (1996). The advantage of this approach for interest rate data is its flexibility. Non parametric estimators produce $\mu(r_t)$ and $\sigma(r_t)$ functions which seem to be nonlinear in the rates and deviate from benchmark models.

Models analyzed include those proposed by Aït-Sahalia (1996), Stanton (1997) and Bandi and Phillips (2003) by focusing on Aït-Sahalia non parametric diffusion coefficient. Continuous time models of the term structure are based on diffusion dynamics for the state variables. In single factor models this state variable is determined by the risk free interest rate.

The diffusion process for the instantaneous interest rate is denoted by a stochastic differential equation described as,

$$dr_t = \mu(r_t)dt + \sigma(r_t)dW_t\,, \tag{56}$$

where $\mu(r_t)dt$ denotes the drift, $\sigma(r_t)dW$ clarifies the diffusion process and $W(t)$ is a standard Brownian motion. Parameters $\mu(r)$ and $\sigma(r)$ allow for a unique solution of a stochastic differential equation. As a result, the volatility parameter can be easily determined.

Aït-Sahalia estimator (1996) following Cox-Ingersoll-Ross process is described as,

$$dr(t) = \kappa(\alpha - r(t))dt + \sigma\sqrt{r(t)}dW(t). \tag{57}$$

By applying Cox-Ingersoll-Ross process; it is observed that Aït-Sahalia estimator performs poorly except in the case when the short rate presents a very high degree of mean reversion. However; in the presence of low mean reversion; the estimator does not produce reliable values of the variance and can suggest the existence of a non linear relation between the variance of interest rates and interest rates. Consequently; low mean reversion implies important bias.

Stanton estimator (1997) differs from Aït-Sahalia since the drift is not required to be known. Stanton demonstrates than given a discrete sample data, the drift and diffusion are able to be represented.

The following equations describe the drift and diffusion coefficients on Stanton's model,

$$\mu(r_t) = \frac{1}{\Delta} \mathrm{E}_t\left[r_{t+\Delta} - r_t\right] + O(\Delta) \tag{58}$$

$$\sigma(r_t) = \sqrt{\frac{1}{\Delta} \mathrm{E}_t\left[(r_{t+\Delta} - r_t)^2\right]} + O(\Delta) \tag{59}$$

where $\mu(r_t)$ denotes the drift while $\sigma(r_t)$ clarifies the variance. These equations can be approximated as conditional expectations. Expected values can be obtained through the non parametric estimate of the conditional density.

Stanton estimator obtains better variance estimators than Aït-Sahalia. Due to the non parametric estimator; any possible error arising from this parameter will be eliminated in some extent. In small samples; the bias is in the same direction; so it disappears completely. As a result; the variance is better estimated that in the case of the Ait-Sahalia estimator. Even in Stanton estimator, $h$, has an important role. As a result; its accuracy is determinant.

The estimator proposed in Bandi-Phillips (2003) is denoted as,

$$\hat{\sigma}^2(r) = \frac{\sum_{i=1}^{T} K\left(\frac{r - \hat{r}_i}{h_s}\right)\left(\frac{1}{m_i}\sum_{j=0}^{m_i}\left[\hat{r}_{t_{i,j+1}} - \hat{r}_{t_{i,j}}\right]^2\right)}{\sum_{i=1}^{T} K\left(\frac{r - \hat{r}_i}{h_s}\right)}, \tag{60}$$

where $K$ is a Kernel estimator[26] and $t_{i,j}$ is a subset of indices such that,

$$t_{i,0} = \inf\{t \geq 0 : \left|\hat{r}_t - \hat{r}_i\right| \leq \varepsilon_s\},$$

$$t_{i,j+1} = \inf\{t \geq t_{i,j} + 1 : \left|\hat{r}_t - \hat{r}_i\right| \leq \varepsilon_s\},$$

[26] Kernel density estimation is a form of estimating the probability density function of a random variable. Given specific data on a sample size, kernel density estimation allows to extrapolate the data to the entire population. In the case where $X_1, \ldots, X_N \sim f$ is an i.i.d. random variable; the mathematical notation can be defined as, $\hat{f}_h(x) = \frac{1}{Nh}\sum_{i=1}^{N} K\left(\frac{x - x_i}{h}\right)$, where $K$ is some kernel and $h$ is the bandwidth parameter. Frequently, $K$ is considered to be a standard Gaussian function with zero mean and constant variance. Parameter $K$ is denoted as $K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

where $m_i$ is the number of times that $\left|\hat{r}_t - \hat{r}_i\right| \le \varepsilon_s$ and $\varepsilon_s$ is a parameter to be selected.

The main difference between estimators developed by Stanton and Bandi-Phillips remains in interest rate parameter. It is observed that Stanton estimator weights interest rates by the quadratic variation at time $t$, while Bandi-Phillips estimator weights interest rates with the average quadratic variation of all observations close to interest rates. The difference is obtained from the proxy applied.

By comparing figures representing Bandi-Phillips method on simulated paths of Cox-Ingersoll-Ross and Vasicek model; it is observed that the performance of both estimators is almost identical. However; the errors from Bandi-Phillips are slightly lower. In addition; the data applied from Aït-Sahalia (1996) and Stanton (1997) differs. Aït-Sahalia data consists of seven day Eurodollar deposit rates while Stanton model a longer time series consisting of daily yields on three month US Treasury Bill rates.

Different values for the bandwidth parameter estimated variances denote important differences. Estimates from Stanton dataset show instability in Ait-Sahalia estimator especially when mean reversion is low. However; with larger values of $\mathrm{h}$; all estimators seem to be more stable. The two data sets give estimates of variance that almost differ by a unit root. This could be caused by the maturity of the two rates. However; a more reliable reason is due to the presence of peaks in Ait-Sahalia dataset and the absence of those peaks in Stanton dataset.

A Monte Carlo simulation of the Vasicek and Cox-Ingersoll-Ross model is applied with the data set used in Aït-Sahalia model. It is observed that Aït-Sahalia estimator is not applicable for values of mean reversion displayed by interest rate data. However; Stanton and Bandi-Phillips models perform better, since its estimators seem to be reasonable accurate. Estimators will depend on the bandwidth parameter chosen. This selection plays an important role since it will affect the results.

Stanton and Bandi-Phillips estimators provide very similar estimators of the variance function. It is demonstrated that Aït-Sahalia parameter is problematic since it only provides reasonable estimates for unrealistically high mean reversion speed. It is also demonstrated that by applying positive interest rate data, the outcomes regarding variance estimates are biased since they result into negative values.

Sample size could represent problematic situations in the case of the mean parameter, $\alpha$, particularly in the case of two factor models where the mean changes are denoted by stochastic movements. This misspecification could result in a potential problem for non parametric estimations of the single factor models. An additional parameter that is affected due to sample size is the degree of mean reversion.

Results demonstrate that estimates of the unconditional density of $r(t)$ vary with sample size and degree of mean reversion. It is observed that the rate of convergence of the sample estimates can be very slow when mean reversion is low. With a high degree of mean reversion, the estimates are unbiased. However; with a small sample size; mean and variance are poorly estimated, since the bias is larger as the sample size reduces. In the case of the average mean, estimates are unbiased.

The following table clarifies mean reversion estimations obtained from previous research on the term structure of interest rates. It is observed that the mean reversion parameter should lie between .1 and .2; being Aït-Sahalia model an exception with a mean reversion value of .978.

**Table IX.** *Description of models, estimation of mean reversion parameter and method employed as noted from previous literature*.

| Author | Year | Model | Method | Mean Reversion |
|---|---|---|---|---|
| Chan-Karolyi-Longstaff-Sanders | 1992 | $dr = (\alpha + kr)dt + \sigma r^{\gamma} dW(t)$ | GMM[27] | 0.1779 |
| Aït-Sahalia | 1996 | $dr = (\alpha + kr)dt + \sigma(r)dW(t)$ | FGLS[28] | 0.978 |
| Andersen-Lund | 1997 | Stochastic Volatility | EMM[29] | 0.173 |
| Durham | 2003 | $dr = (\alpha + kr)dt + \sqrt{\beta_1 + \beta_2 r}dW(t)$ | MLE[30] | 0.1875 |
| Durham | 2003 | $dr = (\alpha + kr)dt + \beta_1 r^{\beta_2} dW(t)$ | MLE[31] | 0.1049 |
| Durham | 2003 | $dr = (\alpha + kr)dt + \sqrt{\beta_1 + \beta_2 + \beta_3 r^{\beta_4}} dW(t)$ | MLE[32] | 0.1056 |

Results demonstrate that mean reversion estimates, $\kappa$, should lie between a range of values from .1 and .2; with the exception of Aït-Sahalia model. This demonstrates that even with the smallest sample sizes; a reliable estimation of the unconditional density is almost impossible to obtain.

[27] GMM denotes Generalized Method of Moments.

[28] FGLS denotes Feasible Generalized Least Squares. This technique is very similar to Generalized Least Squares technique with the difference that it applies an estimated variance-covariance matrix since the true matrix is not known directly.

[29] EMM denotes Efficient Method of Moments. This technique is developed by Gallant and Tauchen (1991). EMM approach has been applied to discrete time stochastic volatility models and continuous time stock return and interest rate models. EMM is a two step procedure. It presents the same asymptotic efficiency as the Maximum Likelihood method. However; it has the advantage of being computationally controllable.

[30] MLE denotes Maximum Likelihood Estimator. This approach is an statistical method which tests the accuracy of a mathematical model for describing some particular data. Given a normal distribution with unknown mean and variance; MLE describes the parameter which is most likely to fit the model.

[31] Ibid

[32] Ibid

*5.8 Mahdavi Framework*

Mahdavi (2008) analyze the stochastic processes of short term interest rates of seven countries, including United States, United Kingdom, Canada, Japan, Australia, Denmark, Sweden and the Euro zone by applying an arbitrage free framework. A no arbitrage condition is the minimum restriction imposed on interest rate models and it must be satisfied in efficient financial markets. By applying no arbitrage conditions it is observed that a change in the short term interest rate is equal to the slope of the forward curve plus a risk premium related to the sign and size of the market price of interest rate risk.

Recent developments on interest rate models are based on equilibrium modeling of financial markets (Cox, Ingersoll, Ross; 1985) or no arbitrage conditions (Heath-Jarrow-Morton; 1992). Duffie (2007) states that all no arbitrage models can be supported by an equilibrium model and all equilibrium models rule out arbitrage opportunities in financial markets. Monthly data on short term interest rate is applied for estimating the parameters of short term rates and the market price of interest rate risk. The method utilized is the Generalized Method of Moments (Hansen; 1982)

The process of the short term rate is described by a stochastic differential equation denoted as,

$$dr(t) = \mu(r)dt + \sigma(r)dZ_t . \tag{61}$$

where $\mu(r)dt$ denotes the drift, $\sigma(r)dZ$ states the variance and $dZ$ is a standard Brownian motion. Restrictions on the model should be applied in order to be able to estimate equation (61) An alternative specification of the volatility is applied in which several previous short term interest rate models are nested. In addition; the database included is larger allowing for the description of short term rates after the introduction of the Euro.

The following equation is the basis of Mahdavi Framework since it shows that in the absence of arbitrage opportunities; the expected change in the riskless rate at time $t$ is equal to the current slope of the forward curve $f_T(t,t)$ minus a risk premium. $\lambda(t)\sigma(t,t)$

The equation developed by Mahdavi (2008) can be denoted as,

$$dr(t) = [f_T(t,t) - \lambda(t)\sigma(t,t)]dt + \sigma(t,t)dZ(t), \tag{62}$$

where $f_T(t,t)$ denotes the slope of the curve at the origin and $\lambda(t)\sigma(t,t)$ describes the variance. Equation (62) presents the advantage that the slope of the curve is observable at time $t$.

However; in order to allow for empirical tests, the model has to be parameterized; resulting into the following equation,

$$dr(t) = \left(f_T(t,t) + \alpha_1 + \alpha_2 r(t) + \alpha_3 r(t)^2\right)dt + \sqrt{\left|\alpha_4 + \alpha_5 r(t) + \alpha_6 r(t)^2 + \alpha_7 r(t)^3\right|}\,dZ(t), \quad (63)$$

where $\alpha_i$, for $i = 1,\dots,7$ are the parameters to be estimated. Volatility is different from previous studies since it nests many of them. In addition; volatility and the market price of risk are correctly defined for all positive values. In the following table; the relationship between Equation (63) and models previously developed is demonstrated.

**Table X.** *Model specifications and parameter restrictions*
Models of the short term interest rate are nested within a common framework by restricting the values of $\alpha$.
The relationship between Mahdavi Model (2008) $dr(t) = \left(f_T(t,t) + \alpha_1 + \alpha_2 r(t) + \alpha_3 r(t)^2\right)dt + \sqrt{\left|\alpha_4 + \alpha_5 r(t) + \alpha_6 r(t)^2 + \alpha_7 r(t)^3\right|}\,dZ(t)$ and models previously developed can be observed.

| **Model** | **Year** | **Specification** | **Restrictions** |
|---|---|---|---|
| Vasicek | (1977) | $dr = \kappa(\theta - r)dt + \sigma dZ$ | $\alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = 0$ |
| Brennan and Schwartz | (1979) | $dr = \kappa(\theta - r)dt + \sigma r dZ$ | $\alpha_3 = \alpha_4 = \alpha_7 = 0$ |
| Cox-Ingersoll-Ross | (1985) | $dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dZ$ | $\alpha_3 = \alpha_4 = \alpha_6 = \alpha_7 = 0$ |
| Chan, Karolyi, Longstaff, Sanders | (1992) | $dr = \kappa(\theta - r)dt + \sigma r^{1.5} dZ$ | $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ |
| Duffie and Khan | (1996) | $dr = \kappa(\theta - r)dt + \sqrt{\alpha + \beta r}dZ$ | $\alpha_3 = \alpha_6 = \alpha_7 = 0$ |
| Ahn and Gao | (1999) | $dr = \kappa(\theta - r)r dt + \sigma r^{1.5} dZ$ | $\alpha_1 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ |

The methodology applied for testing the models is to discretize the short rate process described in Equation (63), allowing for the applications of GMM technique afterwards; similar to Chan et al (1992) procedure.

Empirical evidence demonstrates that short rate changes display significant kurtosis, implying the presence of fatter tails than a normally distributed random variable. For all data sets; it is observed that the average lagged forward is greater than the short term rate which indicates a positive average risk premium for every interest rate process.

Results are consistent with previous studies which suggest that no single model can explain the short term process in all countries. However; some common features among the parameters are observed. In the case of United States; volatility function indicates that the variance increases and then decreases in the level of interest rates; demonstrating a mean reversion process. For United Kingdom, short rate follows a mean reverting process. Canada shows a short rate process similar to the process followed by United States short term rate. Australia demonstrates a non linear function of the level of short term rate and does not follow a mean reverting process. Japan presents a constant drift of the short term rate and volatility indicates that the short term rate follows a log normal process. Denmark presents similarities with the model developed by Chan et al. for the US short term rate. Sweden presents similarities to results observed in UK. Volatility function is highly non linear in the short term rate but the drift is constant. Finally; the Euro Zone presents a volatility function generally increasing in the short term rate. The drift of the short term implies that the short term rate will be expected to decline if the slope of the forward rate is lower than the value of $.009r^2$.

Results from the eight markets analyzed demonstrate that the market price of interest rate risk is below zero for all countries with the exception of Denmark. Equilibrium market price of risk is expected to be negative because an increase on interest rates reduces bond prices. Japan and UK have the highest market prices of risk in absolute terms. The absolute value of the price of risk increases as the short term rate increases. However; for Australia and the Euro zone; the price of risk is flat and very close to zero.

The value of $\gamma$ for the US on the unrestricted model is very close to the value obtained from well behaving short term rate process and closed form solutions for the short term rate. For Canada, Australia and Sweden, the estimated value of $\gamma$ is not significantly different from zero; following a Gaussian process. Results indicate that within the markets studied mean reversion is observed in all cases with the Euro zone as exception. By imposing the no arbitrage condition, significant impacts on the estimated values of $\gamma$ are implied; specifically in the cases that the short term rate presents a higher degree of volatility while presenting smaller impacts on the estimates of the drift coefficient.

# 6 Summary and Conclusions

The term structure of interest rates is an extremely important element in Finance. It is the most important element for pricing contingent claims, determining the cost of capital and managing financial risk. To be able to forecast accurately interest rates ex ante provides considerable economical rewards. As a result; extensive literature within this field has been developed. According to Gibson, Lhabitant and Talay (2001); to understand and model the term structure of interest rates represents one of the most challenging and complex topics of Financial Research.

Models of the term structure of interest rates are classified in several categories. The ones described within this paper include single factor, volatility and multi factor models. The first single factor model of the term structure of interest rates was developed by Merton (1973). The model presents bias since it allows for negative interest rates and assumes homoskedasticity. However; this model is important within the Financial literature because it is the starting point for model improvements and further developments.

Following Merton; several improvements in the models have been developed. Vasicek (1977) improves the model by restricting possible negative values in interest rates. Nevertheless; this single factor model presents bias since homoskedasticity is still allowed. Cox-Ingersoll-Ross (1985) develops a model in which previous bias are omitted. In his model the possibility of negative values on interest rates are eliminated. In addition; this is the first model that assumes heteroskedasticity on the levels of interest rates.

Chan-Karolyi-Longstaff-Sanders (1992) develop a single factor model which restricts negative values of interest rates and assumes heteroskedasticity. This development is significant within the Financial literature since they are the first ones to provide a common framework in which different models can be nested and compared between each other. This framework allows for determining the most accurate model able to forecast the dynamics on the term structure. It is demonstrated that volatility parameter is the single most important factor for determining the accuracy on any interest rate model.

In single factor models, the short term rate is the only explanatory variable. However; looking from a macroeconomic perspective, it is unrealistic to think that the term structure is only determined by the short term interest rate. As a result to critiques given to single factor models; multifactor and volatility models are developed with the objective of correcting possible bias.(Gibson et al; 2001)

Volatility family of models described in this paper includes GARCH, Combined Level GARCH, TVP-Levels, Asymmetric TVP-Levels, GARCH-X, and Asymmetric GARCH-X models. Volatility models allow for volatility clustering and persistence. The conditional variance is dependent on previous lags; giving higher weights to the most recent innovations. In addition; these models are able to determine persistence, the speed of mean reversion and the existence of stationarity in the series.

Asymmetric volatility models allow for volatility determination as a consequence of information arrivals. Volatility is more accurately determined since it is possible to obtain volatility effects as a result from positive and negative innovations. It is observed a greater volatility effect after negative shocks. However; this class of models fails to capture the relationship between interest rates and volatility.

In order to obtain the most accurate forecasts on the term structure of interest rates; multifactor models are developed. This family of models has a common variable which is determined by the spot rate. According to specific frameworks, different variables are taken into consideration in order to determine the most accurate forecast of interest rates. Some of these proxies include inflation rate, volatility, mean, spread between long and short term rates as well as the yields on a fixed set of bonds.

In Brennan-Schwartz (1982) model, the term structure is determined by the short and long term rate. Longstaff-Schwartz (1992) include the short term interest rate and volatility; where both factors are interrelated. These models determine the risk premium endogenously, consistent with the absence of arbitrage. Duffie-Kan (1993) develop a new class of models; known as affine models. This class of models seems to perform better in forecasting abilities since small unconditional means are unified with high volatility, allowing for active portfolio management.

Furthermore; Koedijk-Nissen-Schotmann-Wolff (1997) take into consideration the model developed by Chan et al. (1992) with conditional heteroskedasticity effect allowing for different effects as the levels on interest rates change. Bali (2003) develops Black-Derman-Toy model (1990) into a two factor model where variables are the short term interest rate and volatility. The models imply mean reversion on interest rates and heteroskedastic volatility.

Given the significant benefits of modeling and forecasting accurately the term structure of interest rates; an extensive amount of models have been developed.

However; little is known about the existence of a model which represents superiority in its forecasting abilities. It is difficult to compare models with respect to each other because each particular model consists of specific factors or variables, which differ from model to model.

Nevertheless; some researchers have succeed in creating common frameworks able to nest several models and therefore; compare their forecasting abilities. It is important to mention; however; that there is no common framework which could nest all available models. Several frameworks do exist; however; each framework is able to nest and compare a limited amount of models.

Some of these frameworks include the ones developed by Elliot-Baier (1979), Chan-Karolyi-Longstaff-Sanders (1992), Koedijk-Nissen-Schotman-Wolff (1997), Koutmos (2000), Goard-Hansen (2004), Sanford-Martin (2006), Reno-Roma-Schaefer (2006), and Mahdavi (2008). These frameworks are mainly based on a Stochastic Differential Equation (SDE) in which the short term is determined by the drift and diffusion following a Brownian Motion. However; a SDE is not an exclusively framework. Alternative frameworks; such as a common volatility framework are developed. Comparisons are obtained by nesting the models analyzed within a common framework and restricting some of the parameters.

Findings are consistent across the different frameworks analyzed. It is assumed that volatility is a determinant factor on any interest rate model. Volatility is the most important factor that differentiates models between each other. Models which best describe the behavior of interest rates are those that allow the volatility of interest rates to be dependent on the level of interest rates. Volatility estimations are determinant for modeling and forecasting interest rates accurately. Findings are consistent with Longstaff-Schwartz (1992) and Bali (2003) two factor model in which variables include the short term rate and volatility to changes in interest rates.

According to Koutmos (2000); the best volatility specification requires the application of information shocks and interest rate levels. Mean reversion is not an important aspect on short term rates.

Frameworks analyzed within this paper converge in their findings. Volatility is the single most important factor on any interest rate model. To have an accurate estimation on the volatility parameter will create as a result; accurate estimations on future interest rate levels. However; it is not mentioned which one from the models analyzed is the best model or the one that performs the best.

According to Gibson et al. (2001); each model presents certain advantages and disadvantages. As a result; the selection of the model will depend on the specific purpose for which the model is required. Beltratti and Colla (2007) state that in the case of the term structure of interest rates it is not possible to determine a best model. The model applied to solve a specific situation depends on the objective of the decision maker. Furthermore; Mahdavi (2008) demonstrates that no single model can explain the short term processes in the countries analyzed.

Consistent with Longstaff-Schwartz (1992) and Bali (2003) models, we consider that the most important factors for modeling the term structure of interest rates are the short term rate and the sensitivity to changes in interest rates. This statement has strong acceptance within the Financial literature. According to all frameworks analyzed in this paper; volatility is the single determinant factor influencing the accuracy on any interest rate model.

To our consideration; we believe that future research within this field should focus on developing multifactor models which take volatility and the spot rate as the main two factors. However; additional factors could be included in order to determine the most accurate interest rates possible. By estimating accurate future interest rates ex ante will bring extensive economical benefits to those able to determine this variable accurately.

It has been demonstrated that a two factor model where variables include short interest rate and volatility result into quite accurate estimations. However; if we move towards a three factor model; it would be very interesting to analyze whether estimations improve from the ones obtained in a two factor model.

The term structure of interest rates has been extensively investigated. It is considered as the topic which has received the greatest attention within Financial research. Since the introduction of the first model in 1973 by Merton, an extreme amount of models and improvements to existent models have been developed. To have an accurate interest rate estimation ex ante implies an extreme amount of benefits.

Interest rates are the most important factor when determining the pricing of contingent claims, the cost of capital and managing financial risk. If we are able to price any contingent claim ex ante, great arbitrage opportunities will arise. Furthermore; if we are able to determine the cost of capital ex ante; we can determine the most appropriate moment for lending or taking a loan. We could determine the best moment for obtaining the least expensive loan and the most appropriate moment for investing. Regarding risk management; if we are able to determine future interest rate levels ex ante; we would be able to determine the best hedge strategy in order to minimize any possible risk. Benefits from having accurate interest rate estimations ex ante are countless.

The purpose of this paper was firstly; to present the theories of the term structure of interest rates. Secondly; to describe some of the most important single, volatility and two-factor models of the term structure. Finally; to provide several frameworks in which interest rate models can be nested and compared between each other.

This paper had the following structure. Chapter 2 described some of the most significant contributions of the term structure of interest rates. Chapter 3 clarified the theories of the term structure of interest rates. Chapter 4 explained some of the most important interest rate models. Chapter 5 compared different models between each other.

We chose this topic as our research topic because we have a particular interest towards time series. It is really fascinating to be able to understand and forecast a time series ex ante and to predict future behavior. The term structure of interest rates was chosen since it is a determinant variable within Finance and Economics. The riskless rate of interest is a factor applied in all asset pricing models and a determinant factor in any fiscal and monetary policies.

We consider that the benefits obtained from understanding this topic in detail could be immeasurable. In addition; we have a special preference towards challenges and difficult tasks. When determining the topic's selection; we wanted to have a topic which would be very difficult to develop and which would imply a mental challenge for the author of this paper. In addition; we wanted to choose a topic which is seldom developed by Master's degree students due to its level of difficulty. We consider that these goals were successfully achieved and are completely satisfied with the outcome obtained.

## References


Aas K., 2004, Modelling the stochastic behaviour of short-term interest rates: A survey, Norwegian Computing Center, *Applied Research and Development,* Vol. 21, Issue 4, pp.1-10

Bali T. G., 2003, Modeling the stochastic behavior of short-term interest rates: Pricing implications for discount bonds, *Journal of Banking and Finance,* Vol. 27, Issue 2, p.p. 201-228

Beltratti A., Colla P., 2007, A portfolio-based theory of affine term structure models, *Annals of Operations Research*, Vol. 151, Issue 1, p.p. 193-222

Bielecki T., 2000, Multiple Ratings Model of Defaultable Term Structure, *Mathematical Finance*, Vol. 10, Issue 2, p.p. 125-139

Bodie Z., Kane A., Marcus A., 1996, *Investments*, United States of America, Mc Graw Hill

Borokhovich K., Brunarski K., Crutchley C., Simkins B., 2004, Board Composition and Corporate Use of Interest Rate Derivatives, *Journal of Financial Research*, Vol. 27, Issue 2, p.p. 199-216

Brennan M., Schwartz E., 1982, An Equilibrium Model of Bond Pricing and a Test of Market Efficiency, *Journal of Financial and Quantitative Analysis*, Vol. 17, Issue 3, p.p.301-329

Brenner R., Harjes R., Kroner K., 1996, Another Look at Models of the Short-Term Interest Rate, *The Journal of Financial and Quantitative Analysis*, Vol. 31, Issue 1, pp.85-107

Brooks C., 2005, *Introductory econometrics for finance,* United Kingdom, University Press, Cambridge

Buraschi A., Corielli F., 2005, Risk Management implications of time-inconsistency: Model updating and recalibration of no-arbitrage models, *Journal of Banking and Finance*, Vol. 29, Issue 11, p.p. 2883-2907

Casassus J., Collin-Dufresne P., Goldstein B., 2005, Unspanned stochastic volatility and fixed income derivatives pricing, *Journal of Banking & Finance*, Vol. 29, Issue 11, p.p. 2723-2749

Chan K., Karolyi A., Longstaff F., Sanders A., 1992, An Empirical Comparison of Alternative Models of Short-Term Interest Rate, *The Journal of Finance* Vol. 47, Issue 3, pp. 1209-1227

Chance D., 2004, *An Introduction to Derivatives & Risk Management*, International Student Edition, United States of America, Thomson

Chaudry M., Christie-David R., Koch T., Reichert, A., 2000, The Risk of Foreign Currency Contingent Claims at US Commercial Banks, *Journal of Banking and Finance*, Vol. 2, Issue 9, p.p. 1399-1417

Cox J., Ingersoll J., Ross S., 1985, A Theory of the Term Structure of Interest Rates, *Econometrica*, Vol. 53, Issue 2, pp. 385-407

Collin-Dufresne P., Goldstein R., 2002, Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility, *Journal of Finance*, Vol. 57, Issue 4, p.p. 1685-1730

Elliott J., Baier J., 1979, Econometric Models and Current Interest Rates: How Well do They Predict Future Rates?, *The Journal of Finance*, Vol. 34, Issue 4, pp. 975-986

Faff R., Gray P., 2006, On the estimation and comparison of short-rate models using the generalized method of moments, *Journal of Banking and Finance*, Vol. 30, Issue 11, p.p. 3131-3146

Géczy C., Minton B., Schrand C., 1997, Why Firms Use Currency Derivatives, *Journal of Finance*, Vol. 52, Issue 4, p.p. 1323-1354

Gibson R., Lhabitant F.-S., Talay D., 2001, *Modeling the term structure of interest rates: a review of the literature*, Social Science Research Network

Glasserman P., 2003, The Term Structure of Simple Forward Rates with Jump Risk, *Mathematical Finance*, Vol. 13, Issue 3, p.p. 383-410

Goard J., Hansen N., 2004, Comparison of the performance of a time-dependent short interest rate model with time-independent models. *Applied Mathematical Finance*, Vol.11, Issue 2, p.p. 147-164

Goltz F., *Fixed Income Derivatives: Strategies for Portfolio Managers*, EDHEC Risk and Asset Management Research Centre

Gustavsson T., 1992, *No Arbitrage Pricing and the Term Structure of Interest Rates*, Department of Economics, Uppsala University, Ph.D Thesis.

Halkos G., Papadamou S., 2007, Significance of risk modelling in the term structure of interest rates, *Applied Financial Economics*, Vol. 17, Issue 3, pp. 237-247

Hentschel L., Kothari S.P., 2001, Are Corporations Reducing or Taking Risks with Derivatives?, *Journal of Financial and Quantitative Analysis*, Vol. 36, Issue 1, p.p. 93-116

Hutchinson W., Toma M., 1991, The Bond Price Support Program as a Change in Policy Regimes: Evidence from the Term Structure of Interest Rates, *Journal of Money, Credit and Banking*, Vol. 23, Issue 3, p.p.367-382

International Swaps and Derivatives Association, *Over 90% of the World's 500 largest companies use derivatives to help manage their risks, according to new ISDA survey*, Last visited 15.01.2007 http://www.isda.org

Johansson A., 1994, *Models of a short term interest rate*, Nationalekonomiska Institutionen, Handelshögskolan vid Göteborgs Universitet, Ph.D Thesis.

Knif J., 2006, Lecture notes on Mathematics of Financial Derivatives I, HANKEN Svenska Handelshögskolan

Koedijk K., Nissen F., Schotman P., Wolff C., 1997, The dynamics of Short-Term Interest Rate Volatility Reconsidered, *European Financial Review*, Vol. 1, Issue 1, pp. 105-130

Koski J.L., Pontiff J., 1999, How are derivatives used? Evidence from the mutual fund industry, *Journal of Finance*, Vol. 54, Issue 2, p.p. 791-816

Koutmos G., 1998, The Volatility of Interest Rates Across Maturities and Frequencies, *The Journal of Fixed Income*, Vol. 8, Issue 3, p.p. 27-31

Koutmos G., 2000, Modeling Short-Term Interest Rate Volatility: Information Shocks versus Interest Rate Levels, *The Journal of Fixed Income*, Vol. 9, Issue 4, p.p.19-26

Koutmos G., Philippatos G., 2007, Asymmetric Mean Reversion in European Interest Rates: A Two Factor Model, *The European Journal of Finance*, Vol. 13, Issue 8, p.p. 741-750

Longstaff F., Schwartz E., 1992, Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model, *The Journal of Finance*, Vol. 47, Issue 4, p.p. 1259-1282

Mahdavi M., 2008, A comparison of international short-term rates under no arbitrage condition, *Global Finance Journal*, Vol. 18, Issue 3, p.p. 303-318

Makridakis S., Wheelwright S., McEgee V., 1983, *Forecasting: Methods and Applications*, Singapore, John Wiley & Sons

Mato M.A.M., 2005, Classic and modern measures of risk in fixed-income portfolio optimization, *Journal of Risk Finance*, Vol. 6, Issue 5, p.p. 416-423

McConell C., Brue S., 1996, *Economics -principles, problems and policies*, United States of America, Mc Graw Hill

Miller T., Stone B., Silver H., 1998, Predictability of Short-term Interest Rates: A Multifactor Model for the Term Structure, *Managerial Finance*, Vol. 24, Issue 9, pp. 20-71

Neftci S., 2000, *An Introduction to the Mathematics of Financial Derivatives*, United States of America, Academic Press

Nelson C., 1972, Estimation of Term Premiums from Average Yield Differentials in the Term Structure of Interest Rates, *Econometrica*, Vol. 40, Issue 2, pp. 277-287

Platen E., 2006, A Benchmark Approach to Finance, *Mathematical Finance*, Vol. 16, Issue 1, p.p. 131-151

Reno R., Roma A., Schaefer S., 2006, A Comparison of Alternative Non-parametric Estimators of the Short Rate Diffusion Coefficient, *Economic Notes by Banca Monte dei Paschi di Siena*, Vol. 35, Issue 3, p.p. 227-252

Sanford A., Martin G., 2006, Bayesian Comparison of several continuous time models of the Australian short rate, *Accounting and Finance*, Vol. 46, Issue 2, p.p. 309-326

Smith C., Stulz R., The Determinants of Firms' Hedging Policies, *Journal of Financial and Quantitative Analysis*, Vol. 20, Issue 4, p.p. 391-405

Telser G.L., 1966, A Critique of Some Recent Empirical Research on the Explanation of the Term Structure of Interest Rates, *The Journal of Political Economy*, Vol. 75, Issue 4, pp. 546-561

Treepongkaruna S., Gray T., 2003, Short-term interest rate models: valuing interest rate derivatives using a Monte-Carlo approach, *Accounting and Finance*, Vol. 43, p.p. 231-259

Turnovsky S., 1989, The Term Structure of Interest Rates and the Effects of Macroeconomic Policy, *Journal of Money, Credit and Banking*, Vol. 21, Issue 3, pp. 321-347

Wall Street Journal, 2006, Hot Topic: The Risk Business, *Proquest Database*

Zielinski D., Mauro J., 2006, Growth in Fixed Income Derivatives Require Strict Risk Control, *State Street Global Advisors SSGA*