EXPECTED SHORTFALL AND VALUE-AT-RISK UNDER A MODEL WITH MARKET RISK AND CREDIT RISK

by

SIU KIN BONG BONNY

A thesis submitted in partial fulfillment of the requirements for

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Abstract of thesis entitled

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A model which takes care of both market and credit risks was presented. A surplus process was proposed which models the credit risk component by a first order Markov chain on Standard and Poor's credit ratings. Under the Markovian regime switching setup, various risk measures have been considered. Risk measures including natural value at risk and expected shortfall were reviewed and adopted. A risk measure called n-period value at risk which is more conservative than the natural value at risk has also been proposed. Recursive equations were developed for these risk measures. In order to deal with some kind of dependency on the credit risks, a weak Markov chain was attempted to model the credit rating dynamics in the surplus process. A weak Markov chain is a Markov chain in which transition depends on two or more transition histories. In particular,

a second order Markov model was considered for the sake of simplicity. Second order transition probabilities, transition matrix, and transition states have been re-estimated and restated to cope with the dependency structure. The surplus process and recursive equations have also been rederived. By assuming that the market risks follow some known distributions, simulations under both Markov chain models were carried out for three scenarios. Under the first order model, the results are consistent with expectation and other research works. The proposed risk measure n-period value at risk works effectively in normal distribution scenario. As a contradiction, although the results for second order model are generally consistent with the use of first order Markov model, there exists some discrepancies between two particular credit state combinations. It was concluded that the estimated second order transition matrix and the length of observation period would be the cause and further investigation is needed to solve the problem.

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Declaration

I declare that this thesis represents my own work, except where due acknowledgements are made, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any other institution for a degree, diploma or other qualification.

Signed ____

SIU Kin Bong Bonny

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Chapter 1

Introduction of Risk Measures and Credit State Transition

1.1 Value at Risk as Risk Measure

In global economy, it have been raised for a long time to have a common method to measure the risk of an investment portfolio. Value at Risk (VaR), being simple to interpret, has become popular in risk management subjects. VaR is generally defined as the possible maximum loss over a given holding period within a pre-defined confidence level (Yamai et al, 2002a). Artzner et al. (1997, 1999), Acerbi et al. (2001) and Tasche (2002) defined VaR mathematically as the lower $100\alpha\%$ percentile of the portfolio return distribution X:

$$VaR_{\alpha}(X) = -inf\{x|P[X \leq x] > \alpha\} \tag{1.1}$$

where X is a random variable denoting the portfolio return.

Remarks: Since loss is defined as negative in the return distribution (profit being positive), we multiply -1 to obtain positive VaR when a portfolio incurs losses within a given confidence level. Hence, VaR can be negative if no losses is incurred, i.e. the $100\alpha\%$ percentile is positive.

Risk managers and regulators have put a lot of focus on VaR in the early years because of the promise it holds for improving risk management (Schachter, 1998). The aim of developing VaR is to provide a quick, single number measure which encapsulates risk information of a portfolio and could be communicated to non-technical senior management for a sense of exposure to changes in market. Due to its conceptual simplicity, VaR is used to calculate economic capital in various aspects. International bank regulators have agreed to allow local banks to adopt VaR models in calculating regulatory capital. US securities regulator also allows corporations to use VaR to express their exposure to market risk in financial reports. Since the capital calculated by VaR at the $100\alpha\%$ confidence level corresponds to the amount required to keep the firm's default probability below $100\alpha\%$, risk managers believe that they can control the firms' default probability through the use of VaR.

There are three practical approaches to compute VaR (Schachter, 1998), including the parametric method, historical simulation and Monte Carlo simulation. Each of them has its strengths and weaknesses. Parametric VaR is expressed as a multiple of the standard deviation of portfolio's return. Although the computation is quick, its quality would be degraded with increasing number of non-linear instruments and decreasing normality of portfolio return. Historical simulation expresses the portfolio's hypothetical return as a chart. Each hypothetical return is calculated as if today's portfolio repeats itself in the history of market rates and prices. VaR is then read directly from the chart. Although this approach is free from distribution assumption, the range of portfolio values is limited by the actual historical data. Monte Carlo approach also expresses the hypothetical return as a chart in which the hypothetical return is obtained by randomly selecting a price distribution and rate estimated from historical data. Therefore, VaR is not limited by the observed price ranges since it is sampled from an estimated price distribution. However, this approach is the most expensive and time consuming.

Although VaR provides risk managers with a quick and readily accessable value on their portfolio, many researchers have criticized the use of VaR. The main critics are: (1) VaR is not a coherent risk measure under certain situation due to its non-subadditivity. (2) VaR fails to eliminate tail risk since it disregards the tail distribution beyond its value. (3) Information given by VaR may mislead rational investors who maximize expected utility. In particular, employing VaR as the only risk measure is more likely to construct perverse positions which are resulted from a trade-off between small VaR and heavy-tailed distribution. Such portfolio usually incurs larger losses beyond the VaR level.

1.1.1 The Coherence of Risk Measures

One of the most important critics on VaR is its incoherence and non-subadditivity in general situation (Acerbi et al, 2002; Artzner et al, 1999; Yamai et al, 2002a). The concept of coherence was first introduced by Artzner et al. (1997, 1999) for the discussion and measurement of risks. They presented four desirable properties of risk measures and regarded those satisfying all four properties as coherent. The four properties are: (1) Translation Invariance (2) Subadditivity (3) Positive Homogeneity and (4) Monotonicity. All of these properties have their own practical significance.

Definition 1.1. (Coherence) *Consider a set V of real-valued random variables on some probability space (Ω, A, P) such that $E[X^-] < \infty$ for all $X \in V$, then $\rho : V \to R$ is a coherent risk measure if it satisfies the Axioms 1.2 to 1.5.*

Remarks: $\rho(.)$ is a risk measure and returns real-values from random variables. The higher the returned real-values, the larger the risk of the random variables set.

Axiom 1.2. (Monotonicity) *With the setup in Definition 1.1, for $X, Y \in V$ and $X \leq Y$, a risk measure ρ is monotonic if*

$$\rho(Y) \leq \rho(X).$$

Intuitively, when the return of a particular investment is always better than the other, the risk related to that particular investment should always be lower. In specific, if the return of an investment is always negative, this investment would always have positive risks.

Axiom 1.3. (Subadditivity) *With the setup in Definition 1.1, for $X, Y, X+Y \in V$, a risk measure ρ is subadditive if the following equation holds*

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

In words, a risk measure ρ is subadditive when the risk of total position is less than or equals to the sum of individual portfolio's risk. Subadditivity requires the risk measure to consider risk reduction by portfolio diversification effect. It also implies that if a merger or an assets pooling exercise is effective, no extra risk would be created.

Axiom 1.4. (Positive Homogeneity) *With the setup in Definition 1.1, for $X \in V, h > 0, hX \in V$, a risk measure ρ is positive homogeneous if*

$$\rho(hX) = h\rho(X).$$

This property is introduced in a reverse sense to Subadditivity for positive h and is used to model a situation in which no diversification can take place.

Axiom 1.5. (Translational Invariance) *With the setup in Definition 1.1, for $X \in V$, $a \in R$, a risk measure ρ is translational invariance if*

$$\rho(X + a) = \rho(X) - a.$$

Axiom 1.5 implies that if we have more capital on hand ($a > 0$), we would subject to less risk within the holding period by an equivalent value.

Although VaR was found satisfying monotonicity, positive homogeneity and translational invariance, Acerbi et al. (2002), Artzner et al. (1999) and Yamai et al. (2002a) have criticized the use of VaR since VaR is not generally subadditive and therefore incoherent. Embrechts et al. (2002) have proved that VaR satisfies subadditivity only when the underlying profit-loss distribution is a member of the elliptical distribution with finite variance. Normal, student t and pareto are examples of elliptical distribution. In particular, Yamai et al. (2002a) have provided the proof for normal distribution by first showing that standard deviation is subadditive, followed by rewriting VaR as a scalar multiple of standard deviation.

Theorem 1.6. *For random variables X and Y having finite standard deviations σ_X, σ_Y and covariance σ_{XY}, the standard deviation of random variable $X + Y$, σ_{X+Y}, satisfies subadditivity as follows:*

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y.$$

Proof. Let $Z = a(X-\mu_X)-(Y-\mu_Y)$, where $a \in \mathbf{R}$, $\mu_X = E(X)$ and $\mu_Y = E(Y)$, then

$$\begin{aligned} E[Z^2] &= a^2E[(X-\mu_X)^2] - 2aE[(X-\mu_X)(Y-\mu_Y)] + E[(Y-\mu_Y)^2] \\ &= a^2\sigma_X^2 - 2a\sigma_{XY} + \sigma_Y^2 \geq 0 \end{aligned}$$

Let $a = \frac{\sigma_{XY}}{\sigma_X^2}$, we have

$$\begin{aligned} & E[Z^2] = \frac{\sigma_X^2\sigma_Y^2 - \sigma_{XY}^2}{\sigma_X^2} \geq 0 \\ \Longleftrightarrow \quad & -\sigma_X\sigma_Y \leq \sigma_{XY} \leq \sigma_X\sigma_Y \end{aligned}$$

Therefore,

$$\sigma_{X+Y} \equiv \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}} \leq \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y} = \sigma_X + \sigma_Y$$

□

Lemma 1.7. *VaR of a portfolio with normally distributed return, $X \sim N(\mu_X, \sigma_X^2)$, is a subadditive risk measure.*

Proof. Since VaR refers to the lower $100\alpha\%$ percentile in the return distribution and under a normal distribution, each quantile can be expressed as a scalar multiple of the standard deviation, VaR can be expressed as a scalar multiple of standard deviation as well. As we have shown in Theorem 1.6 that standard deviation satisfies subadditivity, VaR, being a scalar multiple of standard deviation, also satisfies subadditivity. □

In business, the actual profit-loss seldom behaves such ideally and always comes with heavy tail and assymetry. It makes VaR difficult and impractical to implement when fund managers decide their portfolio strategies. Acerbi et al. (2002) claimed that VaR fails to stimulate diversification and does not account for the severity of loss. Artzner et al. (1997, 1999) concluded that companies using non-subadditive risk measures should break themselves into many small sectors or sub-companies in order to divide their assets into several fractions to reduce the total risk. However, this action is impractical in nature and is intuitively incorrect.

Artzner et al. (1997, 1999) have provided two simple examples to demonstrate the non-subadditivity problem aroused by VaR. The first example involves short positions in two digital options and the second example is about constructing a credit concentrated portfolio. Yamai et al. (2002a) restated these examples with illustrative tables. Tasche (2002) have specified a parametric example in which two pareto distributed random variables and their joint distribution were considered. Parametric expression of VaR was then obtained and compared. All these examples showed that VaR of the diversified portfolio is larger than that of individual positions. In specific, we have rewritten the digital option example in details with graphical illustration.

In Figure 1.1, we assume there are two digital options. These options will pay a fix amount of 1,000 depending on the stock price at time T. Initial premium are required for both options. Three positions are then constructed by these options. Their payoffs and 1% VaR are listed. Since VaR of position 'Short both Option AA and Option BB' is larger than that of individual positions ('Short Option AA' / 'Short Option BB'), VaR is non-subadditive. The payoffs are also displayed graphically to demonstrate the three scenarios with respect to stock price changes. Finally, the payoff for each position is ordered (from loss to profit) and plotted. 1% VaR is then read from the chart.

1.1.2 Tail Risk of VaR

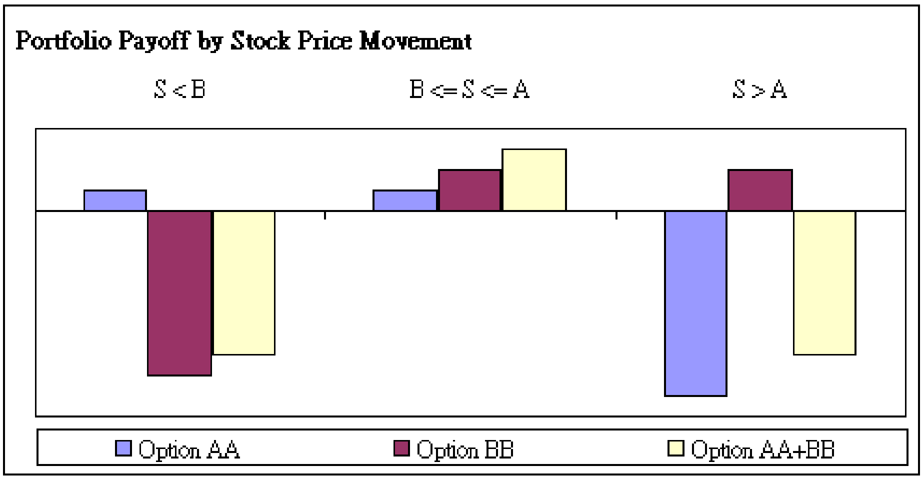
In Yamai et al. (2002c), the tail risk was considered and some properties were discussed. They defined tail risk as the risk when a risk measure disregards tail information and fails to summarize the choice between portfolios. A mathematical definition of tail risk via the concept of stochastic dominance is provided. The tail risk of VaR is also discussed.

Yamai et al. (2002c) started from considering the distribution function which is involved in the derivation of stochastic dominance. Their belief is that different orders of distribution function enclose different tail information.

Definition 1.8. **(N^{th} Order Distribution Function)** *Let X be a random variable and $F(x)$ be the distribution function. The n^{th} order distribution function is*

Available Investment	Initial Premium	Payoff Condition
Option AA	a	Pay 1000 if stock price at time T > A
Option BB	b	Pay 1000 if stock price at time T < B

	Portfolio Payoff			1% VaR
Stock Price at time T (S)	S < B	B <= S <= A	S > A	
Probability	0.80%	98.40%	0.80%	
Short Option AA only	a	a	a-1000	-a
Short Option BB only	b-1000	b	b	-b
Short both Options AA+BB	a+b-1000	a+b	a+b-1000	1000-a-b



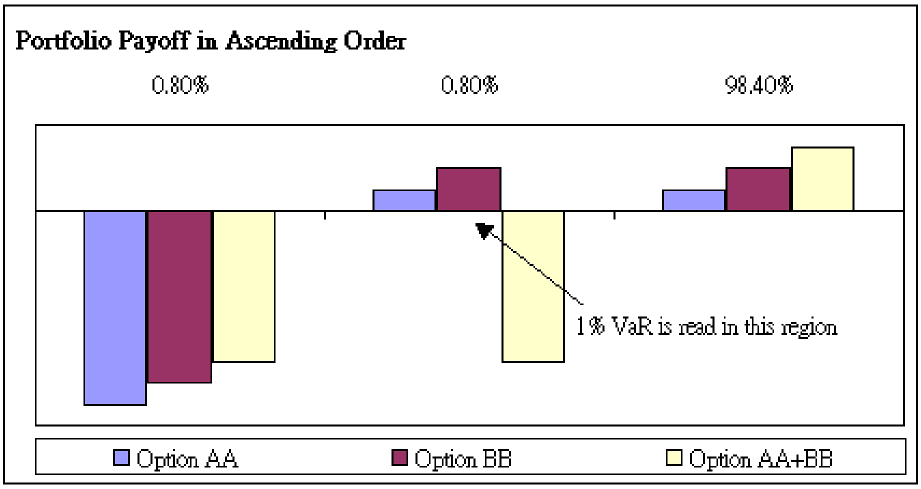


Figure 1.1: Digital Options Example - Payoff and VaR

defined for all $n \geq 2$ *as*

$$F^{(n)}(x) \equiv \int_{-\infty}^{x} F^{(n-1)}(u)du, \ F^{(1)}(x) \equiv F(x).$$

In order to evaluate the n^{th} order distribution function, Yamai et al. (2002c) rewrote it as a scalar multiple of the $(n-1)^{th}$ lower partial moment, denoted by $LPM_{n-1,x}(X)$.

Theorem 1.9. *If* $F^{(n)}(x)$ *is the* n^{th} *order distribution function of a random variable* X*, then the following equality holds:*

$$F^{(n)}(x) = \frac{1}{(n-1)!}\int_{-\infty}^{x}(x-u)^{n-1}f(u)du \equiv \frac{1}{(n-1)!}LPM_{n-1,x}(X).$$

Since higher order distribution function represents more tail information, Yamai et al. (2002c) proposed the use of distribution functions to rank portfolios' return in terms of tail risk and introduced the concept of stochastic dominance.

Definition 1.10. (Stochastic Dominance) *Let* X_1 *and* X_2 *be random variables denoting portfolios' return.* X_1 *is ranked* n^{th} *order stochastically dominant over* X_2 *(i.e.* $X_1 \geq_{SD(n)} X_2$*) if the following holds for all* $x \in R$.

$$F_1^{(n)}(x) \leq F_2^{(n)}(x)$$

where $F^{(n)}(x)$ *is the* n^{th} *order distribution function as defined in Definition 1.8.*

With the portfolio return ranked by the distribution functions, a risk measure is said to be consistent with stochastic dominance and free of tail risk if it can provide the same ranking.

Definition 1.11. (Consistency with Stochastic Dominance) *For any random variables X_1 and X_2, a risk measure ρ is consistent with n^{th} order stochastic dominance if*

$$X_1 \geq_{SD(n)} X_2 \implies \rho(X_1) \leq \rho(X_2)$$

Definition 1.12. (Freeness of Tail Risk) *For any random variables X_1 and X_2 with density functions f_1 and f_2 such that*

$$\int_{-\infty}^{x} (x-u)^{n-1} f_1(u)du \leq \int_{-\infty}^{x} (x-u)^{n-1} f_2(u)du \text{ for all } x \leq K, K \in R,$$

then a risk measure ρ is free of n^{th} order tail risk with a threshold K if

$$\rho(X_1) < \rho(X_2).$$

By combining Definition 1.10, 1.11 and 1.12, the following theorem becomes available in evaluating the tail risk of risk measures.

Theorem 1.13. *When portfolios are ranked by n^{th} order stochastic dominance, a risk measure consistent with n^{th} order stochastic dominance is free of n^{th} order tail risk with any level of threshold.*

With the above setup, Yamai et al. (2002c) claimed that VaR is free of different orders of tail risk under various situations:

(1) VaR is free of first order tail risk if the underlying portfolios are ranked by first order stochastic dominance.

Since VaR is defined as the lower $100\alpha\%$ percentile of the underlying distribution, VaR is consistent with first order stochastic dominance such that the following holds:

$$X_1 \geq_{SD(1)} X_2 \Longleftrightarrow F_{X_1}(x) \leq F_{X_2}(x) \implies VaR_\alpha(X_1) \leq VaR_\alpha(X_2).$$

Therefore, if the underlying portfolios are ranked by first order stochastic dominance (i.e. $X_1 \geq_{SD(1)} X_2$), then VaR is free of first order tail risk. However in practice, this condition on distribution functions is too strict to hold and greatly reduces its applicability.

(2) VaR is free of second order tail risk if the underlying portfolios follow the same type of elliptical distribution of equal mean and are ranked by second order stochastic dominance.

Under the elliptical distribution condition, Embrechts et al. (2002) showed that for any random variables X, $VaR_\alpha(X)$ can be represented as the sum of the mean and a scalar multiple of the standard deviation, i.e.

$$VaR_\alpha(X) = E(X) + q_\alpha \sigma_X$$

where q_α is the $100\alpha\%$ percentile of the standardized distribution of the same type. With this information, Yamai et al. (2002c) considered the consistency with second order stochastic dominance of standard deviation. They also proved that under equal mean condition, VaR is also consistent with second order stochastic dominance. Therefore, if the underlying portfolios are ranked by second order stochastic dominance (i.e. $X_1 \geq_{SD(2)} X_2$), then VaR is free of second order tail risk.

Yamai et al. (2002a, 2002d) claimed the use of VaR not only disregards the loss beyond the specified quantile but also increases the extreme loss because rational investors employing VaR only are more likely to construct perverse positions. Yamai et al. (2002a) illustrated this problem with three examples, including an option portfolio, a credit portfolio and a continuous-time dynamically traded portfolio. In these examples, they admitted that tail risk leads to practical problems because investors can manipulate the final payoffs either by initial position set-up or by dynamic trading. Investors adopting VaR as the only risk measure can construct their portfolio in a way that VaR is minimized with the increase of possible extreme losses. This problem cannot be solved by raising the confidence level alone. In addition, the use of VaR also enhances credit concentration.

1.2 Expected Shortfall as Risk Measure

In view of VaR's deficiency, a coherent risk measure called Expected Shortfall (also named 'conditional VaR', 'mean excess loss', 'beyond VaR' or 'tail VaR') was suggested by Artzner et al. (1999) as a remedy for VaR. Expected shortfall (ES) aims at measuring the losses beyond VaR. It is defined as the conditional expectation of loss given that this loss is beyond the VaR (Acerbi et al., 2001; Artzner et al, 1999; Rockafellar et al., 2001; Tasche, 2002; Yamai et al., 2002a).

Suppose X is a random variable denoting the return of a given portfolio and $VaR_\alpha(X)$ is the $100\alpha\%$ value at risk, then for continuous distribution X, expected shortfall is defined as:

$$\begin{aligned} ES_\alpha(X) &= E[-X| - X \geq VaR_\alpha(X)] \\ &= \frac{-\int_{-\infty}^{-VaR_\alpha(X)} x f(x) dx}{\alpha} \end{aligned} \tag{1.2}$$

Remarks: Here we assume that the return distribution is continuous. If it is discrete, ES needs to be modified to make it a coherent risk measure, see Definition 1.14 for details. Unless otherwise specified, the continuous version in (1.2) will be adopted in the remaining of this thesis.

Figure 1.2 shows the graphical presentation of VaR and ES. Since ES is defined as the tail mean beyond VaR, ES is therefore always larger than VaR.

A number of comparative analysis on ES and VaR have been carried out

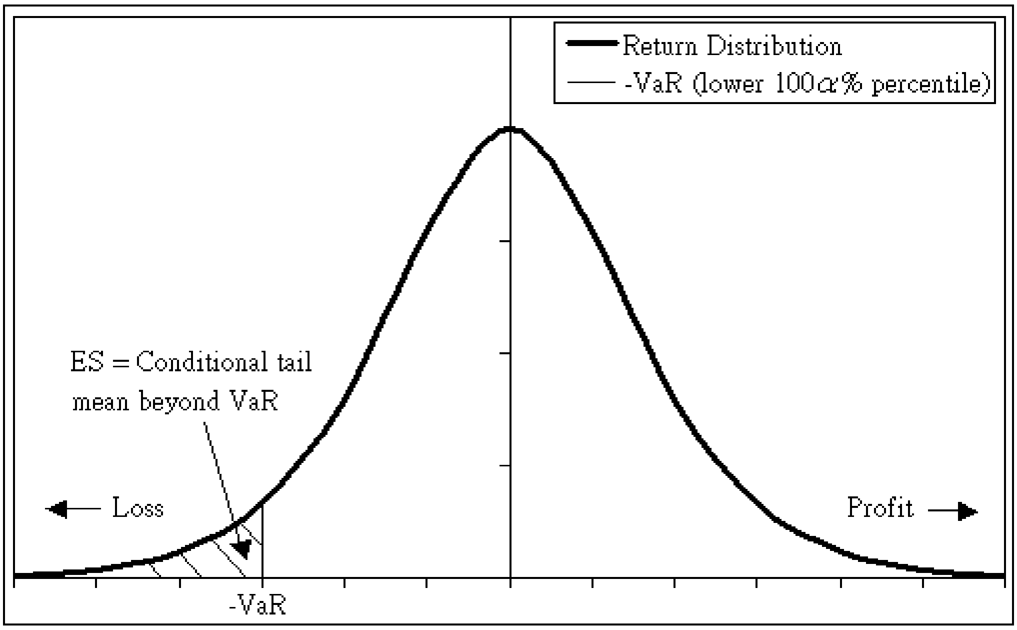


Figure 1.2: Return Distribution, VaR and ES

by various researchers. Advantages of ES over VaR include: (1) ES is subadditive and coherent. (2) ES considers losses beyond VaR and has less tail risks, therefore ES is less likely to suggest perverse portfolio construction. Adopting ES as risk management tool is more conservative as it imposes stricter economic capital requirement.

1.2.1 Subadditivity and Coherence of Expected Shortfall

In this section, we will review the proof of subadditivity for the discontinuous version of ES by Acerbi et al. (2001, 2002) starting from defining the lower q-quantile, higher q-quantile and discontinuous version of ES:

Definition 1.14. *For a random variable X and a probability level α ($\alpha \in (0,1)$), the lower α-quantile ($x_{(\alpha)}$), higher α-quantile ($x^{(\alpha)}$) and expected shortfall ($ES_\alpha(X)$) are defined as:*

$$x_{(\alpha)} = inf\{x|F(x) \geq \alpha\} \tag{1.3}$$

$$x^{(\alpha)} = inf\{x|F(x) > \alpha\} = -VaR_\alpha(X) \tag{1.4}$$

$$ES_\alpha(X) \equiv -\frac{1}{\alpha}(E[X1_{X\leq x_{(\alpha)}}] + x_{(\alpha)}(\alpha - F(x_{(\alpha)}))) \tag{1.5}$$

where $F(x) = P(X \leq x)$ is the cumulative distribution function of X.

First of all, value at risk equals to the negative of higher α-quantile. Secondly, it can be observed that $x^{(\alpha)} \geq x_{(\alpha)}$ and the equality holds if and only if X is continuous. In such case, we also have $P(X = x_{(\alpha)}) = 0$, $F(x_{(\alpha)}) = \alpha$ and $ES_\alpha(X)$ reduced to $-\frac{1}{\alpha}E[X1_{X\leq x_{(\alpha)}}]$ which is equivalent to the expression in equation (1.2).

If X is discontinuous and has jumps at $x_{(\alpha)}$, we have $P(X = x_{(\alpha)}) > 0$ and $F(x_{(\alpha)}) = P(X \leq x_{(\alpha)}) \geq \alpha$.

Theorem 1.15. *Given two random variables X, Y and define $Z = X + Y$, then*

$$ES_\alpha(Z) \leq ES_\alpha(X) + ES_\alpha(Y).$$

Proof. Firstly, we restate the expression of $ES_\alpha(X)$ in equation (1.5) by introducing the term

$$
1^{\alpha}_{X\le x_{(\alpha)}} = \begin{cases} 1_{X\le x_{(\alpha)}}, & \text{if } P[X=x_{(\alpha)}]=0 \\ 1_{X\le x_{(\alpha)}} + \frac{\alpha - F(x_{(\alpha)})}{P(X=x_{(\alpha)})} 1_{X=x_{(\alpha)}}, & \text{if } P[X=x_{(\alpha)}]>0 \end{cases} \tag{1.6}
$$

$$
= \begin{cases} 1+0=1, & \text{if } X < x_{(\alpha)} \\ 1+\frac{\alpha - F(x_{(\alpha)})}{P(X=x_{(\alpha)})} = \frac{\alpha - F(x^-_{(\alpha)})}{P(X=x_{(\alpha)})} \in [0,1], & \text{if } X = x_{(\alpha)} \\ 0+0=0, & \text{if } X > x_{(\alpha)} \end{cases}
$$

where the second term in the sum would become zero if $P(X = x_{(\alpha)}) = 0$ and by the definition of $x_{(\alpha)}$, the equations $F(x^-_{(\alpha)}) \le \alpha \le F(x_{(\alpha)})$ and $P(X = x_{(\alpha)}) = F(x_{(\alpha)}) - F(x^-_{(\alpha)})$ hold.

The below properties of $1^{\alpha}_{X\le x_{(\alpha)}}$ will be applied later in the proof:

$$
E[1^{\alpha}_{X\le x_{(\alpha)}}] = \alpha \tag{1.7}
$$

$$
0 \le 1^{\alpha}_{X\le x_{(\alpha)}} \le 1 \tag{1.8}
$$

To restate $ES_\alpha(X)$, we apply equation (1.6) and consider the expectation of $(X1^{\alpha}_{X\le x_{(\alpha)}})$ which gives

$$
\begin{aligned}
E[X1^{\alpha}_{X\le x_{(\alpha)}}] &= E[X1_{X\le x_{(\alpha)}}] + E\left[\frac{\alpha - F(x_{(\alpha)})}{P(X=x_{(\alpha)})} X1_{X=x_{(\alpha)}}\right] \\
&= E[X1_{X\le x_{(\alpha)}}] + x_{(\alpha)}(\alpha - F(x_{(\alpha)})) = -\alpha ES_\alpha(X)
\end{aligned}
$$

To complete the proof, consider

$$\begin{aligned}
& (-\alpha ES_\alpha(Z)) - (-\alpha ES_\alpha(X)) - (-\alpha ES_\alpha(Y)) \\
= \; & E[Z1^\alpha_{Z\leq z_{(\alpha)}} - X1^\alpha_{X\leq x_{(\alpha)}} - Y1^\alpha_{Y\leq y_{(\alpha)}}] \\
= \; & E[X(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{X\leq x_{(\alpha)}})] + E[Y(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{Y\leq y_{(\alpha)}})]
\end{aligned}$$

By property (1.8), we have

$$\begin{cases}
(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{X\leq x_{(\alpha)}}) \geq 0, & \text{if } X > x_{(\alpha)} \\
(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{X\leq x_{(\alpha)}}) \leq 0, & \text{if } X \leq x_{(\alpha)}
\end{cases}$$

$$\Leftrightarrow \quad X(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{X\leq x_{(\alpha)}}) \geq x_{(\alpha)}(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{X\leq x_{(\alpha)}}) \text{ for all } X \qquad (1.9)$$

Hence, by property (1.7) and (1.9),

$$\begin{aligned}
& E[X(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{X\leq x_{(\alpha)}})] + E[Y(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{Y\leq y_{(\alpha)}})] \\
\geq \; & x_{(\alpha)}E[(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{X\leq x_{(\alpha)}})] + y_{(\alpha)}E[(1^\alpha_{Z\leq z_{(\alpha)}} - 1^\alpha_{Y\leq y_{(\alpha)}})] \\
= \; & x_{(\alpha)}(\alpha - \alpha) + y_{(\alpha)}(\alpha - \alpha) = 0
\end{aligned}$$

and the proof is complete. $\square$

Hence, the subadditivity of ES is given by Theorem 1.15. Since ES also satisfies monotonicity, positive homogeneity and translational invariance, ES is a coherent risk measure (Acerbi et al., 2001; Artzner et al., 1999).

Furthermore, Yamai et al. (2002a) proved that under normal return distribution with zero mean, ES provides equivalent risk management information to that of VaR since they are scalar multiple of each other. Rockafellar et al. (2001) extended this result to the whole elliptical distribution family. The proof for normal distribution is as follows:

Proposition 1.16. *Suppose $X \sim N(0, \sigma_X^2)$, expected shortfall ($ES_\alpha(X)$) is a scalar multiple of value at risk ($VaR_\alpha(X)$) and provides equivalent information for risk management.*

Proof.

$$
\begin{aligned}
ES_\alpha(X) &= E[-X| - X \geq VaR_\alpha(X)] = \frac{E[-X \cdot 1_{X \leq -VaR_\alpha(X)}]}{\alpha} \\
&= -\frac{1}{\alpha\sigma_X\sqrt{2\pi}} \int_{-\infty}^{-VaR_\alpha(X)} te^{-\frac{t^2}{2\sigma_X^2}} dt \\
&= -\frac{1}{\alpha\sigma_X\sqrt{2\pi}} \{-\sigma_X^2 e^{-\frac{t^2}{2\sigma_X^2}}\}_{-\infty}^{-VaR_\alpha(X)} \\
&= \frac{\sigma_X}{\alpha\sqrt{2\pi}} e^{-\frac{VaR_\alpha(X)^2}{2\sigma_X^2}} \\
&= \frac{\sigma_X}{\alpha\sqrt{2\pi}} e^{-\frac{q_\alpha^2\sigma_X^2}{2\sigma_X^2}} \text{ (by Lemma 1.7)} \\
&= \frac{e^{-\frac{q_\alpha^2}{2}}}{\alpha\sqrt{2\pi}} \sigma_X
\end{aligned}
$$

Under the assumption of $X \sim N(0, \sigma_X^2)$, expected shortfall can be expressed as a scalar multiple of standard deviation. Hence, ES and VaR are scalar multiple to each other and provide equivalent information for risk management. □

1.2.2 Tail Risk of ES

Under the same setup as in Section 1.1.2, Yamai et al. (2002c) have also discussed the tail risk of ES. They showed that ES is consistent with second order stochastic dominance by expressing ES into a α-quantile ($q(\alpha)$) integral which is also consistent with second order stochastic dominance.

$$\begin{aligned}
ES_\alpha(X) &= E(-X|-X \geq VaR_\alpha(X)) \\
&= \frac{1}{\alpha}\int_{-\infty}^{q(\alpha)} (-x)f(x)dx \\
&= -\frac{1}{\alpha}\int_0^{\alpha} F^{-1}(t)dt \quad \text{(let F(x) = t)} \\
&= -\frac{1}{\alpha}\int_0^{\alpha} q(t)dt
\end{aligned}$$

Therefore we have:

$$X_1 \geq_{SD(2)} X_2 \Longleftrightarrow \int_0^{\alpha} q_1(t)dt \geq \int_0^{\alpha} q_2(t)dt,\ \alpha \in [0,1] \implies ES_\alpha(X_1) \leq ES_\alpha(X_2)$$

As a result, ES is free of second order tail risk if the underlying portfolios are ranked by second order stochastic dominance.

Yamai et al. (2002c) stated that ES puts a heavier weight on tail loss than VaR and therefore subjects to less tail-risk. Although ES is a better risk measure than VaR in terms of tail risk and market stress, both measures may underestimate the losses when the return distribution has fat tails (Yamai et al., 2002d). As a result, alternative risk measures were introduced which involve less tail risk. Yamai et al. (2002c, 2002d) introduced copula based risk measures, including the Gumbel, Gaussian and Frank copulas, for extreme value distribution under market stress. They also proposed the n^{th} lower partial moment which eliminates tail risk more effectively. Based on these observations, they criticized the use of both VaR and ES while admitting that ES is still a better choice over VaR. However in this dissertation, we only focus on VaR and ES because they are simple to apply.

1.3 The Markovian Regime Switching

Thus far we have reviewed the risk measures that will be used in this thesis, we will move on to review the basics of the credit risk component of our suggested model in this section. From time to time, analysts have shown many concerns in corporations' credibility and therefore built lots of ranking and rating systems to quantify each corporation's specific credit risk. Studies were carried out to monitor the credit rating paths in terms of Markovian regime switching model in which credit states and transition probabilities were defined. Morgan (1997) initialized the study in this subject while Jarrow et al. (1997) proposed the use of Markov model to incorporate credit ratings in debt valuation. Based on this idea, Kijima and Komoribayashi (1998) have made some further studies while Arvanitis et al. (1999) and Yang (2000, 2003) have built credit spread models and ruin theory models respectively.

In their studies, various credit ratings from Moody's or Standard and Poor's (S&P) were incorporated into a transition matrix. Transition probabilities between credit states were then estimated by historical data or simulated. The same Markov model will be applied in this thesis and the setup is as follows.

Let I_t be a time-homogeneous Markov chain of order r, $r \geq 1$, with finite state space $N = (1, 2, ..., k)$ representing k different credit states for $t \geq r - 1$. For $i_0, i_1, \ldots, i_{n+1} \in N$, we have

$$P\left[I_{n+1} = i_{n+1} | I_0 = i_0, I_1 = i_1, ..., I_{n-1} = i_{n-1}, I_n = i_n\right] \tag{1.10}$$

$$= P\left[I_{n+1} = i_{n+1} | I_{n-r+1} = i_{n-r+1}, ..., I_{n-1} = i_{n-1}, I_n = i_n\right]$$

The expression (1.10) denotes the probability of moving to state i_{n+1} given full histories of credit states is equivalent to the transition probability given only past r periods of histories.

Since I_t is time-homogeneous, we can write

$$P\left[I_{n+1} = i_{n+1} | I_{n-r+1} = i_{n-r+1}, ..., I_{n-1} = i_{n-1}, I_n = i_n\right] = q_{i_{n-r+1} i_{n-r+2} \ldots i_{n-1} i_n i_{n+1}}$$

where $i_{n-r+1}, i_{n-r+2}, \ldots, i_{n+1} \in N$. Then we can construct the $k^r \times k$ transition matrix

$$
\begin{bmatrix}
q_{11\ldots111} & q_{11\ldots112} & \cdots & q_{11\ldots11(k-1)} & q_{11\ldots11k} \\
q_{11\ldots121} & q_{11\ldots122} & \cdots & q_{11\ldots12(k-1)} & q_{11\ldots12k} \\
\vdots & \vdots & & \vdots & \vdots \\
q_{11\ldots1k1} & q_{11\ldots1k2} & \cdots & q_{11\ldots1k(k-1)} & q_{11\ldots1kk} \\
q_{11\ldots211} & q_{11\ldots212} & \cdots & q_{11\ldots21(k-1)} & q_{11\ldots21k} \\
\vdots & \vdots & & \vdots & \vdots \\
q_{11\ldots2k1} & q_{11\ldots2k2} & \cdots & q_{11\ldots2k(k-1)} & q_{11\ldots2kk} \\
\vdots & \vdots & & \vdots & \vdots \\
q_{kk\ldots kk1} & q_{kk\ldots kk2} & \cdots & q_{kk\ldots kk(k-1)} & q_{kk\ldots kkk}
\end{bmatrix} \tag{1.11}
$$

In Yang's (2000, 2003) study, a model involving both market risks and credit risks was built. The credit risk was modelled by a first order ($r = 1$) Markov chain as defined in (1.10) and (1.11) and a surplus process was constructed as a function of portfolio returns, credit ratings and time. In addition, recursive equations on VaR calculation, ruin probability, distribution of ruin time and severity of ruin were obtained. For various portfolio return assumptions like normal distribution and shifted t distribution, numerical illustrations were also provided. In next chapter, we will review the setup and recursive equations for the first order Markov chain. The weak Markov chain (i.e. $r \geq 2$) will also be considered in later chapters. Numerical results for both cases will be provided.

1.4 Schema of this Thesis

We end this chapter by outlining the organization of the remaining chapters in this thesis. In Chapter 2, we will discuss Yang's (2000, 2003) surplus process under first order Markov chain setup. Recursive equations for VaR, ES and a newly introduced risk measure, n-period VaR will be derived. Simulations for these risk measures will be carried out across various initial credit states. Three underlying return distributions (including normal, shifted gamma and shifted pareto distributions) will be simulated at two confidence levels. Detailed parameters setup and simulation steps of the risk measures will also be elicited. The simulation results for each distribution assumption will be shown in Chapter 3. The implication of the results and the performance of risk measures will be reviewed and discussed.

In Chapter 4, weak Markov Chain, in particular the second order, will be considered as an alternative to model credit state transition. Transition states will be restated and transition probabilities will be re-estimated by actual S&P ratings data. Recursive equations will be rederived. Simulation results of portfolio return and risk measures will be provided and compared to that of first order Markov chain assumption. The simulation details, results and discussion will be displayed in Chapter 5.

In the last chapter, conclusions on the Markov and weak Markov models' formulation, simulation results, model limitations and further developments will be covered. This thesis has reviewed Yang's (2000, 2003) model with market and credit risks and initiated the incorporation of weak Markov models into the surplus process. We are also conveyed by the simulation results that the expected shortfall would be more conservative than VaR in heavy tail return scenario. Final remarks including the summaries of our findings and suggestions of directions for further investigations will also be given in the last chapter.

Chapter 2

The Surplus Process with Markovian Regime Switching and Return Scenario Consideration

The model proposed in this chapter inherits from Yang's (2000, 2003) integrated method for managing both market and credit risks. Model definition, formulation and the derivation of recursive formulas will be reviewed. Three risk measures will be considered, including the VaR, ES and the newly introduced n-period VaR. Simulations for these risk measures will be carried out. The simulation parameters of the return scenarios will be specified later in this chapter and the respective simulation results will be provided in Chapter 3.

2.1 Model Specification

In this model, we will apply the first order Markov chain to represent the credit rating dynamics. Estimates of credit rating transition probabilities are obtained from JP Morgan (1997) and are applied to the transitional matrix under non-default condition.

To start with, let I_t be a time-homogeneous first order Markov chain ($r = 1$) with finite state space $N = (1, 2, ..., 7)$ representing seven different credit states.

By (1.10) and (1.11), we have the transitional probability for $r = 1$

$$P[I_{n+1} = i_{n+1} | I_0 = i_0, I_1 = i_1, ..., I_{n-1} = i_{n-1}, I_n = i_n] \tag{2.1}$$

$$= P[I_{n+1} = i_{n+1} | I_n = i_n] = q_{i_n i_{n+1}}$$

and the transitional matrix of dimension 7 x 7

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{16} & q_{17} \\ q_{21} & q_{22} & \cdots & q_{26} & q_{27} \\ \vdots & \vdots & & \vdots & \vdots \\ q_{61} & q_{62} & \cdots & q_{66} & q_{67} \\ q_{71} & q_{72} & \cdots & q_{76} & q_{77} \end{bmatrix} \tag{2.2}$$

where $\sum_{j=1}^{7} q_{ij} = 1$ for $i = (1, 2, ..., 7)$.

Originally, there are over 20 credit ratings in S&P credit rating system. JP Morgan (1997) has grouped them into seven key categories under non-default condition. Among the seven categories, state 1 is regarded as the highest credit class which corresponds to S&P's AAA rating (or Moody's Aaa) while state 7 is regarded as the lowest class, corresponding to S&P's CCC grade (or Moody's Caa). The same grouping will be applied in this thesis and it is shown in Table 2.1. These seven credit states constitute the state space of the Markov chain. The transition probability estimates are obtained from JP Morgan (1997) and shown in equation (2.3).

Credit State (1 to 7)	S&P Credit Rating	Categorized Credit Rating
1	AAA	AAA
2	AA+ AA AA-	AA
3	A+ A A-	A
4	BBB+ BBB BBB-	BBB
5	BB+ BB BB-	BB
6	B+ B B-	B
7	CCC or below (exclude default)	CCC

Table 2.1: Grouping of S&P Credit Ratings

$$Q = \begin{bmatrix} .9081 & .0833 & .0068 & .0006 & .0012 & .0000 & .0000 \\ .0070 & .9065 & .0779 & .0064 & .0006 & .0014 & .0002 \\ .0009 & .0227 & .9111 & .0552 & .0074 & .0026 & .0001 \\ .0002 & .0033 & .0596 & .8709 & .0531 & .0117 & .0012 \\ .0003 & .0014 & .0068 & .0781 & .8140 & .0893 & .0101 \\ .0000 & .0012 & .0025 & .0045 & .0684 & .8805 & .0429 \\ .0027 & .0000 & .0028 & .0162 & .0296 & .1401 & .8086 \end{bmatrix} \qquad (2.3)$$

With the above setup, the credit state dependent surplus process at time t is defined as:

$$U_t = u + \sum_{m=1}^{t} \Delta X_m^{I_{m-1}} = u + \Delta Y_t \qquad (2.4)$$

where u is the initial surplus of the firm and ΔX_m^i is the return in the m^{th} time interval given that the firm's credit rating at time $m-1$ is of state i. ΔY_t refers to the aggregate return over the entire period.

In equation (2.4), we assume that ΔX_m^i ($i = 1, 2, ..., k$; $m = 1, 2, ...$) are independent random variables. We further assume that for any fix $i = 1, 2, ..., k$, ΔX_m^i ($m = 1, 2, ...$) are identically distributed and $\Delta X^1, ..., \Delta X^k$ are independent but not necessarily follow the same distribution. Therefore, the portfolio return of a firm in each time interval depends only on the credit state at the start of each period.

2.1.1 Recursive Equations of VaR at Maturity - the Natural Form

In this section, we will define the VaR of a portfolio at its maturity (i.e. time n). We regard this form of VaR as natural (named as VaR thereafter) because it is the most intuitive way and also a common practice to observe the spread of portfolio return at maturity, with the ignorance of interim distribution. We will focus on the interim distribution in the next section.

For the defined surplus process (2.4), let $T = inf\{n; U_n \leq 0\}$ be the stopping time (also named default time) of the surplus process. Then, we define the 100α% VaR, given that default does not occur before time n, by considering the followings:

For $y \leq u$ (or $-y \geq -u$),

$$\begin{aligned}
& P\{\Delta Y_n \leq -y, T \geq n | I_0 = i_0, U_0 = u\} \qquad (2.5) \\
= \; & P\{\Delta X_1^{i_0} + \cdots + \Delta X_n^{I_{n-1}} \leq -y, \\
& \Delta X_1^{i_0} > -u, \ldots, \Delta X_1^{i_0} + \cdots + \Delta X_{n-1}^{I_{n-2}} > -u\} \\
= \; & \alpha
\end{aligned}$$

where y is the natural VaR, $VaR_\alpha(\Delta Y_n)$.

Denote the last expression by $d_n^{i_0}(u, y)$, then we can calculate it recursively:

For $n \geq 2$,

$$\begin{aligned}
d_n^{i_0}(u,y) &= P\{\Delta X_1^{i_0} + \cdots + \Delta X_n^{I_{n-1}} \leq -y, \\
&\quad \Delta X_1^{i_0} > -u, \ldots, \Delta X_1^{i_0} + \cdots + \Delta X_{n-1}^{I_{n-2}} > -u\} \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} P\{\Delta X_2^{i} + \cdots + \Delta X_n^{I_{n-1}} \leq -(y+x), \\
&\quad \Delta X_2^{i} > -(u+x), \ldots, \Delta X_2^{i} + \cdots + \Delta X_{n-1}^{I_{n-2}} > -(u+x)\} f^{i_0}(x)dx \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} d_{n-1}^{i}(u+x, y+x) f^{i_0}(x)dx
\end{aligned}$$

where $f^{i_0}(x)$ is the density function of ΔX^{i_0},

$$d_1^{i_0}(u,y) = P\{\Delta X_1^{i_0} \leq -y\}$$

and

$$d_2^{i_0}(u,y) = P\{\Delta X_1^{i_0} + \Delta X_2^{I_1} \leq -y, \Delta X_1^{i_0} > -u\}$$

Suppose that $\Delta X^{I_t = i} \sim N(\mu_i, \sigma_i^2)$ and $n = 2$, we have

$$d_2^{i_0}(u,y) = \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} d_1^{i}(u+x, y+x) \frac{1}{\sqrt{2\pi}\sigma_{i_0}} e^{\frac{-(x-\mu_{i_0})^2}{2\sigma_{i_0}^2}} dx$$

where

$$d_1^{i}(u+x, y+x) = \int_{-\infty}^{-(y+x)} \frac{1}{\sqrt{2\pi}\sigma_i} e^{\frac{-(x-\mu_i)^2}{2\sigma_i^2}} dx$$

2.1.2 Recursive Equations of n-Period VaR

In this section, we define the term n-period VaR as an alternative form of VaR, which takes into account the interim distribution over the time horizon.

The n-period VaR tracks the whole portfolio return path (i.e. ΔY_1, ΔY_2, $\ldots$, ΔY_n) instead of the aggregate return at maturity only. Therefore, we expect the n-period VaR would be more conservative than the natural VaR and predicts larger losses. We will compare the simulation results and review this again in later chapters.

For $y \leq u$ (or $-y \geq -u$) and let T be the stopping time, we define the $100\alpha\%$ n-period VaR under non-default condition as:

$$P\{\Delta Y_j \leq -y, \exists j \leq n, T \geq n | I_0 = i_0, U_0 = u\} = \alpha \tag{2.6}$$

where y is the n-period VaR, $VaR_\alpha^n(\Delta Y_n)$.

Then, we calculate the probability on the left hand side recursively by considering the followings:

$$P\{\Delta Y_j \leq -y, \exists j \leq n, T \geq n | I_0 = i_0, U_0 = u\} = \sum_{j=1}^{n} h_{n,j}^{i_0}(u, y)$$

where

$$\begin{aligned} h_{n,j}^{i_0}(u, y) &= P\{\Delta X_1^{i_0} > -y, \ldots, \Delta X_1^{i_0} + \cdots + \Delta X_{j-1}^{I_{j-2}} > -y, \\ &\quad \Delta X_1^{i_0} + \cdots + \Delta X_j^{I_{j-1}} \leq -y, \\ &\quad \Delta X_1^{i_0} > -u, \ldots, \Delta X_1^{i_0} + \cdots + \Delta X_{n-1}^{I_{n-2}} > -u\} \end{aligned}$$

and for $1 \leq j \leq n$, $h_{n,j}^{i_0}(u, y)$ are mutually exclusive.

To obtain $h_{n,j}^{i_0}(u, y)$, we start by varying n and j.

For $n = 1$, $j = 1$,

$$h^{i_0}_{1,1}(u, y) \;=\; P\{\Delta X^{i_0}_1 \leq -y\}$$

For $n = 2$, $j = 0$,

$$h^{i_0}_{2,0}(u, y) = P\{\Delta X^{i_0}_1 > -u\}$$

For $n = 2$, $j = 1$,

$$\begin{aligned} h^{i_0}_{2,1}(u, y) &= P\{\Delta X^{i_0}_1 \leq -y, \Delta X^{i_0}_1 > -u\} \\ &= P\{-u < \Delta X^{i_0}_1 \leq -y\} \end{aligned}$$

For $n = 2$, $j = 2$,

$$\begin{aligned} h^{i_0}_{2,2}(u, y) &= P\{\Delta X^{i_0}_1 > -y, \Delta X^{i_0}_1 + \Delta X^{I_1}_2 \leq -y, \Delta X^{i_0}_1 > -u\} \\ &= P\{\Delta X^{i_0}_1 > -y, \Delta X^{i_0}_1 + \Delta X^{I_1}_2 \leq -y\} \end{aligned}$$

Therefore, consider $n \geq 3$, $j = 0$,

$$\begin{aligned} h^{i_0}_{n,0}(u, y) &= P\{\Delta X^{i_0}_1 > -u, \Delta X^{i_0}_1 + \Delta X^{I_1}_2 > -u, \ldots, \\ &\quad \Delta X^{i_0}_1 + \Delta X^{I_1}_2 + \cdots + \Delta X^{I_{n-2}}_{n-1} > -u\} \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} P\{\Delta X^{i}_2 > -(u+x), \Delta X^{i}_2 + \Delta X^{I_2}_3 > -(u+x), \ldots, \\ &\quad \Delta X^{i}_2 + \Delta X^{I_2}_3 + \cdots + \Delta X^{I_{n-2}}_{n-1} > -(u+x)\} f^{i_0}(x)dx \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} h^{i}_{n-1,0}(u+x, y) f^{i_0}(x)dx \end{aligned}$$

For $n \geq 3$, $j = 1$,

$$\begin{aligned}
h^{i_0}_{n,1}(u,y) &= P\{\Delta X^{i_0}_1 \leq -y, \Delta X^{i_0}_1 > -u, \Delta X^{i_0}_1 + \Delta X^{I_1}_2 > -u, \ldots, \\
&\quad \Delta X^{i_0}_1 + \Delta X^{I_1}_2 + \cdots + \Delta X^{I_{n-2}}_{n-1} > -u\} \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{-y} P\{\Delta X^i_2 > -(u+x), \Delta X^i_2 + \Delta X^{I_2}_3 > -(u+x), \ldots, \\
&\quad \Delta X^i_2 + \Delta X^{I_2}_3 + \cdots + \Delta X^{I_{n-2}}_{n-1} > -(u+x)\} f^{i_0}(x) dx \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{-y} h^i_{n-1,0}(u+x, y) f^{i_0}(x) dx
\end{aligned}$$

And finally, for $n \geq 2$, $n \geq j \geq 2$,

$$\begin{aligned}
h^{i_0}_{n,j}(u,y) &= P\{\Delta X^{i_0}_1 > -y, \ldots, \Delta X^{i_0}_1 + \cdots + \Delta X^{I_{j-2}}_{j-1} > -y, \\
&\quad \Delta X^{i_0}_1 + \cdots + \Delta X^{I_{j-1}}_j \leq -y, \\
&\quad \Delta X^{i_0}_1 > -u, \ldots, \Delta X^{i_0}_1 + \cdots + \Delta X^{I_{n-2}}_{n-1} > -u\} \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-y}^{\infty} P\{\Delta X^i_2 > -(y+x), \ldots, \Delta X^i_2 + \cdots + \Delta X^{I_{j-2}}_{j-1} > -(y+x), \\
&\quad \Delta X^i_2 + \cdots + \Delta X^{I_{j-1}}_j \leq -(y+x), \\
&\quad \Delta X^i_2 > -(u+x), \ldots, \Delta X^i_2 + \cdots + \Delta X^{I_{n-2}}_{n-1} > -(u+x)\} f^{i_0}(x) dx \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-y}^{\infty} h^i_{n-1,j-1}(u+x, y+x) f^{i_0}(x) dx
\end{aligned}$$

For example, when $n = 2$, we have

$$
\begin{aligned}
& P\{\Delta Y_j \leq -y, \exists j \leq 2, T \geq 2 | I_0 = i_0, U_0 = u\} \\
= \ & h_{2,1}^{i_0}(u, y) + h_{2,2}^{i_0}(u, y) \\
= \ & P\{-u < \Delta X_1^{i_0} \leq -y\} + \\
& P\{\Delta X_1^{i_0} > -y, \Delta X_1^{i_0} + \Delta X_2^{I_1} \leq -y\}
\end{aligned}
$$

2.1.3 Recursive Equations of n-Period VaR under Default

For the case when default occurs before maturity (time n), we construct the respective recursive equations for n-period VaR by considering:

For $y > u$ (or $-y < -u$),

$$
P\{\Delta Y_T \leq -y, T \leq n | I_0 = i_0, U_0 = u\} = \sum_{m=1}^{n} h_m^{i_0}(u, y) = \alpha
$$

where

$$
\begin{aligned}
h_m^{i_0}(u, y) = \ & P\{\Delta X_1^{i_0} > -u, \ldots, \Delta X_1^{i_0} + \cdots + \Delta X_{m-1}^{I_{m-2}} > -u, \\
& \Delta X_1^{i_0} + \cdots + \Delta X_m^{I_{m-1}} \leq -y\}
\end{aligned}
$$

and for $1 \leq m \leq n$, $h_m^{i_0}(u, y)$ are mutually exclusive.

To obtain $h_m^{i_0}(u, y)$, we start by varying m.

For $m = 1$,

$$
h_1^{i_0}(u, y) = P\{\Delta X_1^{i_0} \leq -y\} = F^{i_0}(-y)
$$

For $m = 2$,

$$\begin{aligned} h_2^{i_0}(u,y) &= P\{\Delta X_1^{i_0} > -u, \Delta X_1^{i_0} + \Delta X_2^{I_1} \leq -y\} \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} P\{\Delta X_2^i \leq -(x+y)\} f^{i_0}(x) dx \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} h_1^i(u+x, y+x) f^{i_0}(x) dx \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} F^i(-(y+x)) f^{i_0}(x) dx \end{aligned}$$

For $m = 3$,

$$\begin{aligned} h_3^{i_0}(u,y) &= P\{\Delta X_1^{i_0} > -u, \Delta X_1^{i_0} + \Delta X_2^{I_1} > -u, \\ &\quad \Delta X_1^{i_0} + \Delta X_2^{I_1} + \Delta X_3^{I_2} \leq -y\} \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} P\{\Delta X_2^i > -(u+x), \Delta X_2^i + \Delta X_3^{I_2} \leq -(y+x)\} f^{i_0}(x) dx \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} h_2^i(u+x, y+x) f^{i_0}(x) dx \end{aligned}$$

Therefore, for $m \geq 2$,

$$h_m^{i_0}(u,y) = \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} h_{m-1}^i(u+x, y+x) f^{i_0}(x) dx$$

2.1.4 Distribution Function and Expected Shortfall

We observe that the distribution of ΔX^i is involved in all the above recursive equations. In addition, knowing the distribution of ΔX^i also enables the derivation of the distribution function of ΔY_n and thus the expected shortfall. We define the distribution function of ΔY_n for $n \geq 2$ as:

$$\begin{aligned}
F_{\Delta Y_n}(-y|I_0 = i_0) &= P\{\Delta Y_n \leq -y|I_0 = i_0\} \\
&= P\{\Delta X_1^{i_0} + \cdots + \Delta X_n^{I_{n-1}} \leq -y\} \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-\infty}^{\infty} P\{\Delta X_2^{i} + \cdots + \Delta X_n^{I_{n-1}} \leq -(y+x)\} f^{i_0}(x)dx \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-\infty}^{\infty} F_{\Delta Y_{n-1}}(-(y+x)|I_0 = i) f^{i_0}(x)dx
\end{aligned}$$

To define expected shortfall, the natural VaR ($VaR_\alpha(\Delta Y_n)$) will be adopted in the calculation of conditional expectation:

$$\begin{aligned}
ES_\alpha(\Delta Y_n) &= E[-\Delta Y_n|\Delta Y_n \leq -VaR_\alpha(\Delta Y_n), I_0 = i_0] \\
&= -\frac{\int_{-\infty}^{-VaR_\alpha(\Delta Y_n)} y \cdot f_{\Delta Y_n}(y|I_0 = i_0)dy}{\alpha}
\end{aligned} \tag{2.7}$$

where $f_{\Delta Y_n}(y|I_0 = i_0)$ is the density function of ΔY_n given $I_0 = i_0$.

It is also possible to define the n-period expected shortfall, $ES_\alpha^n(\Delta Y_n)$, using n-period VaR by considering:

$$ES_\alpha^n(\Delta Y_n) = E[-min(\Delta Y_j; j \leq n)|min(\Delta Y_j; j \leq n) \leq -VaR_\alpha^n(\Delta Y_n), I_0 = i_0]$$

where the distribution function of $min(\Delta Y_j; j \leq n)$, denoted by G_n, is required.

Remarks: Since the derivation of G_n is difficult, we will focus on the expected shortfall for natural VaR in the remaining thesis for simplicity.

2.2 Simulation Logic and Parameters Setup

Instead of applying numerical integration on the derived recursive formulas in the last section, we will adopt the Monte Carlo simulation as an alternative to evaluate the risk measures including natural VaR, n-period VaR and ES. Monte Carlo method is a standard numerical tool in finance nowadays. A total of three return scenarios are specified, each with seven ($k = 7$) sets of parameters designed for each initial credit rating. The three scenarios are: (1) Normal Distribution, (2) Shifted Gamma Distribution and (3) Shifted Pareto Distribution. For each scenario, the three risk measures will be simulated using the logic specified in Figure 2.1. The results will be listed and compared in next chapter.

2.2.1 Normal Distribution Return Scenario

In this section, we use the setup in Yang (2000, 2003) by assuming that the portfolio return for all seven credit states follow normal distribution with different parameters and are time-independent. The mean (μ) and standard deviation (σ) for each credit state are specified and summarized in Table 2.2. Under this setup, we assume that companies with good credit ratings should have better return with less costs of credit risks and investment volatility, therefore having a positive mean and small standard deviation.

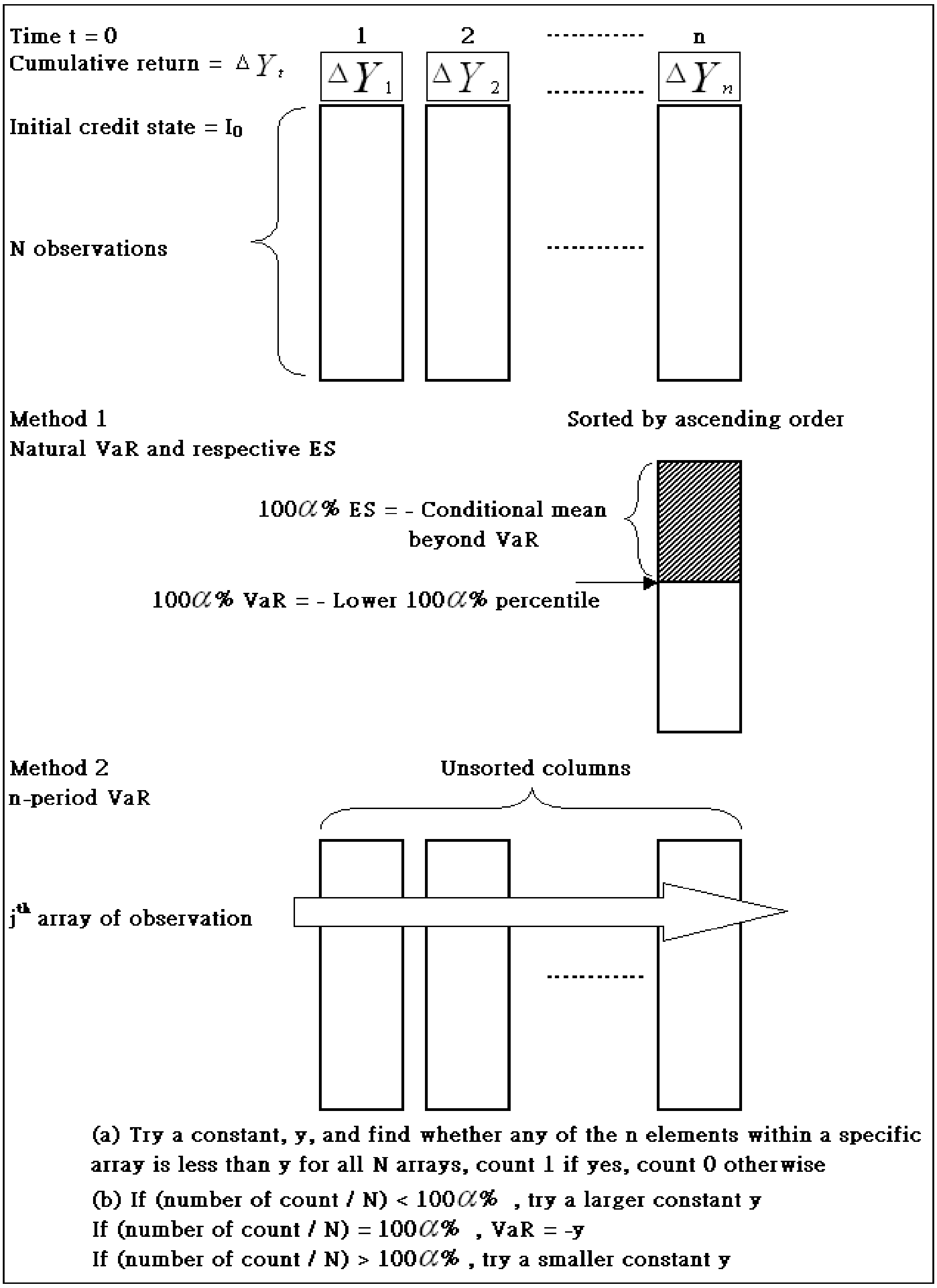


Figure 2.1: Simulation Logic for Natural VaR, n-Period VaR and ES

Mathematically, with transition matrix Q defined in equation (2.3), we want to estimate the natural VaR ($VaR_\alpha(\Delta Y_n)$), n-period VaR ($VaR^n_\alpha(\Delta Y_n)$) and ES ($ES_\alpha(\Delta Y_n)$) respectively in

$$\begin{aligned} & P\{\Delta Y_n \leq -VaR_\alpha(\Delta Y_n), T \geq n | I_0 = i_0, U_0 = u, Q\} \\ = \ & P\{\sum_{m=1}^{n} \Delta X_m^{I_{m-1}} \leq -VaR_\alpha(\Delta Y_n), T \geq n | I_0 = i_0, U_0 = u, Q\} = \alpha, \end{aligned}$$

$$\begin{aligned} & P\{\Delta Y_j \leq -VaR^n_\alpha(\Delta Y_n), \exists j \leq n, T \geq n | I_0 = i_0, U_0 = u, Q\} \\ = \ & P\{\sum_{m=1}^{j} \Delta X_m^{I_{m-1}} \leq -VaR^n_\alpha(\Delta Y_n), \exists j \leq n, T \geq n | I_0 = i_0, U_0 = u, Q\} = \alpha, \end{aligned}$$

and

$$ES_\alpha(\Delta Y_n) = E(-\sum_{m=1}^{n} \Delta X_m^{I_{m-1}} | -\sum_{m=1}^{n} \Delta X_m^{I_{m-1}} \geq VaR_\alpha(\Delta Y_n))$$

where $\Delta X^{I_m = i} \sim N(\mu_i, \sigma_i^2)$ and is time independent.

The estimated values, respective standard deviations and confidence intervals for each initial credit state are shown for $\alpha = 1\%$ and $n = 3$ in Table 3.1. Table 3.2 shows the numerical results for $\alpha = 5\%$ and $n = 3$.

2.2.2 Shifted Gamma Distribution Return Scenario

Similar to normal distribution, the model parameters set in the shifted gamma distribution reflect both credit risk and market risk. The density function

Initial Credit State	Mean	Standard Deviation
1	5	0.5
2	3	1
3	2	2
4	1	3
5	0	4
6	-1	5
7	-2	6

Table 2.2: Parameters for Normal Distribution Return Scenario

of gamma distribution is

$$f(x) = \frac{(x/\theta)^{\alpha} e^{-(x/\theta)}}{x\Gamma(\alpha)}.$$

We assume that $\Delta X^{I_m=i} \sim (Ga(\alpha_i, \theta_i) - 4)$. Large values of α are selected for higher credit states to reflect their relatively low tail risk. For simplicity, we fix $\theta = 1$. The parameter values are provided in Table 2.3. The estimated values, respective standard deviations and confidence intervals for each initial credit state are shown for $\alpha = 1\%$ and $n = 3$ in Table 3.3. Table 3.4 shows the numerical results for $\alpha = 5\%$ and $n = 3$.

Remarks: Since $Ga(\alpha_i, \theta_i)$ returns positive values only, the term '-4' is added to adjust the whole distribution to possible losses and reflect tail risk.

2.2.3 Shifted Pareto Distribution Return Scenario

Similar to the above, the parameters set in the shifted pareto distribution reflect both credit risk and market risk. The density function of pareto distribu-

Initial Credit State	Alpha α	Theta θ
1	7	
2	6	
3	5	
4	4	1
5	3	
6	2	
7	1	

Table 2.3: Parameters for Shifted Gamma Distribution Return Scenario

tion is

$$f(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}.$$

We assume $\Delta X^{I_m=i} \sim (Pareto(\alpha_i, \theta_i) - 0.5)$. Small values of α are selected for higher credit states to reflect their relatively low tail risk. The parameter values are provided in Table 2.4. The estimated values, respective standard deviations and confidence intervals for each initial credit state are shown for $\alpha = 1\%$ and $n = 3$ in Table 3.5. Table 3.6 shows the numerical results for $\alpha = 5\%$ and $n = 3$.

Remarks: Like gamma scenario, the term '-0.5' is added to apply losses on the positive pareto return distribution.

Initial Credit State	Alpha α	Theta θ
1	5	1
2	10	
3	15	
4	20	
5	25	
6	30	
7	35	

Table 2.4: Parameters for Shifted Pareto Distribution Return Scenario

Chapter 3

Simulation Results and Discussion for the Surplus Process with Markovian Regime Switching

Simulations are performed for the three assumed scenarios under non-default condition by using the transition matrix defined in (2.3). Simulated values of natural VaR, n-period VaR and ES are generated at two confidence levels, $\alpha = 0.1$ and 0.5. We further assume that the time period under investigation, n, equals to 3. For each return distribution and level of significance, 500 sets of simulations are run with a sample size of 10,000 each time. The average, standard deviation and 95% confidence interval for the estimates are computed. The results are listed in Tables 3.1 to 3.6. The initial credit states, estimated values, standard deviations and confidence intervals of respective risk measures are shown for each combination of return scenarios and confidence levels.

3.1 Findings and Discussion

3.1.1 Normal Scenario

Tables 3.1 and 3.2 show the results for normal distribution scenario.

(1) For both 1% and 5% cases and each initial credit state, natural VaR is always less than ES. This satisfies our assumption and is intuitively correct.

(2) For both 1% and 5% cases, standard deviation (SD) of ES is larger than that of VaR. This observation matches with our expectation because we use 10,000 samples to locate the VaR (lower $100\alpha\%$ percentile) but only $10{,}000\alpha$ samples beyond VaR are used to compute the average as ES. ES is therefore more fluctuating and has larger variance.

(3) At 5% significance and for each initial credit state, SD of both ES and VaR are less than that of 1% level. For ES, this result is consistent with the last observation because switching from 5% to 1% level implies that less samples are used to deduce the conditional mean beyond the lower $100\alpha\%$ percentile and results in higher variance. For VaR, since the domain of normal distribution starts from negative infinity to positive infinity, we believe that sampling at the tails results in more varied samples. Therefore, the smaller the α, the closer the VaR to the tail and the higher the variance it has.

(4) At 1% significant level and across the initial credit state from state 1 to 7, both VaR and ES have a higher SD at the ends and lower SD in the middle (e.g. SD of VaR estimates start from 0.2538 at state 1, move to 0.2318 at state 3 and reach 0.3944 at state 7).

(5) Contradicting to observation (4), at 5% significant level, both SD of VaR and ES show a monotonic trend with SD attaining the lowest values at state 1 and increase towards state 7.

(6) For both 1% and 5% levels, the 3-period VaR is always greater than the respective natural VaR for each initial credit state. This observation implies that n-period VaR is a more conservative risk measure which is consistent with our claim in last chapter.

3.1.2 Shifted Gamma Scenario

Tables 3.3 and 3.4 show the results for shifted gamma distribution scenario. Since gamma distribution, especially with small α, carries a stronger tail characteristic than normal distribution, we expect the simulated results will show up different observations.

(1) Like normal scenario, VaR is less than ES for each initial credit state and both significant levels. This satisfies our assumption and is intuitively correct.

(2) For both 1% and 5% cases, both SD of VaR and ES show a monotonic trend with highest SD at state 1 and decrease towards state 7. It is different from what we have observed in the normal scenario. One of the possible

reasons is the different tail characteristic possessed by normal and gamma distribution. For small α, since gamma is highly skewed and has a heavy tail, many of the samples are drawn from that high density region and are applied to evaluate the VaR and ES which results in small variance.

(3) Contradicting to normal scenario, for both 1% and 5% cases, SD of ES is not always larger than that of VaR. Across the seven initial credit states, SD of ES is higher at state 1, becomes smaller in the middle and eventually smaller than the SD of VaR at state 7. This observation can be explained by the fact that the domain of gamma distribution is bounded below by zero instead of negative infinity. Therefore, ES is calculated by values bounded by $[-12, VaR_{\alpha}(\Delta Y_3)]$ (the number '-12' comes from a shift of -4 for 3 times) and results in smaller variance.

(4) Difference between the ES and VaR estimates decreases across the initial credit states at both significant levels.

(5) For both 1% and 5% cases, the 3-period VaR is in general greater than the respective natural VaR for each initial credit state, although the difference is not as large as in the normal scenario for some initial credit states. However, this observation can still imply that n-period VaR is a more conservative risk measure than VaR.

3.1.3 Shifted Pareto Scenario

Tables 3.5 and 3.6 show the results for shifted pareto distribution scenario. Since pareto distribution carries even stronger tail characteristic than normal and gamma distribution, more interesting results are expected.

(1) Again for both 1% and 5% cases, VaR is always less than ES for each initial credit state. This satisfies our assumption and is intuitively correct.

(2) Similar to gamma scenario, for both 1% and 5% cases, both SD of VaR and ES show a monotonic trend with highest SD at state 1 and decrease towards state 7. The reason listed in gamma scenario is also applicable here because both distributions possess heavy tail characteristic.

(3) The SD of both VaR and ES at both levels are very small (all of them are less than 0.0030). This is resulted from an extremely heavy-tailed distribution.

(4) Like gamma scenario, the spread between the ES and VaR values decreases across the initial credit states at both significant levels. However, the greatest spread between them is less than 3%. It is very low when compared to the gamma scenario in which the maximum spread is over 50%. This observation can also be explained by the heavy-tail characteristic of pareto distribution.

(5) At both 1% and 5% levels, the 3-period VaR is generally equal to or greater than the respective natural VaR for each initial credit state. In fact, the difference between both risk measures is very small. However, it still implies that n-period VaR is a more conservative risk measure although the improvement may not be significant and cost effective enough to justify.

To summarize, we find that the simulation results match with our expectation. First of all, the results are consistent with the definition of value at risk and expected shortfall (Artzner et al., 1997; Yamai et al., 2002a). Secondly, the proposed risk measure, n-period VaR, demonstrates its applicability in the suggested surplus process and is more conservative than the natural VaR. The n-period VaR works particularly well when the domain of the underlying distribution is not bounded by below. Although the computation of n-period VaR is intensive and time consuming, we hope that it will not be a big problem with today's powerful computer. Thirdly, we observe that SD of ES is greater than that of VaR under normal scenario, this observation is consistent with the simulation results listed by Yamai et al. (2002b). Yamai et al. (2002b) claimed that the estimation error of ES can be reduced if the sample size is increased. Last but not least, one important remark is that if ES is found larger than the initial surplus u, it implies that the investment is too risky and the firm has to do something with its portfolio to improve the situation.

Initial Credit State	Risk Measure	Average	S.D.	95% CI
1	VaR	-6.7070	0.2538	(-7.2045 , -6.2095)
	3-Period VaR	-3.6664	0.0297	(-3.7246 , -3.6082)
	ES	-3.1555	0.3801	(-3.9005 , -2.4105)
2	VaR	-0.8060	0.2536	(-1.3031 , -0.3088)
	3-Period VaR	0.3273	0.1751	(-0.0158 , 0.6704)
	ES	2.8282	0.4075	(2.0295 , 3.6268)
3	VaR	3.9809	0.2318	(3.5266 , 4.4353)
	3-Period VaR	4.6369	0.1554	(4.3324 , 4.9414)
	ES	6.7047	0.3730	(5.9737 , 7.4357)
4	VaR	10.2319	0.2526	(9.7368 , 10.7269)
	3-Period VaR	10.5018	0.2424	(10.0268 , 10.9768)
	ES	12.7051	0.3664	(11.987 , 13.4231)
5	VaR	16.9933	0.2858	(16.4331 , 17.5535)
	3-Period VaR	17.1518	0.2734	(16.6159 , 17.6877)
	ES	19.7333	0.3531	(19.0413 , 20.4252)
6	VaR	23.1503	0.3242	(22.5149 , 23.7858)
	3-Period VaR	23.1979	0.3022	(22.6055 , 23.7903)
	ES	26.1826	0.4064	(25.386 , 26.9791)
7	VaR	29.0302	0.3944	(28.2571 , 29.8032)
	3-Period VaR	29.1776	0.4057	(28.3825 , 29.9727)
	ES	32.5379	0.4384	(31.6787 , 33.3971)

Table 3.1: Simulated Natural VaR, 3-Period VaR and ES - Normal, $\alpha = 1\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
1	VaR	-10.3765	0.0657	(-10.5052 , -10.2478)
	3-Period VaR	-4.1389	0.0106	(-4.1597 , -4.1181)
	ES	-7.9778	0.1116	(-8.1966 , -7.759)
2	VaR	-4.8690	0.0724	(-5.0109 , -4.7272)
	3-Period VaR	-1.1824	0.0209	(-1.2234 , -1.1414)
	ES	-2.2472	0.1329	(-2.5077 , -1.9866)
3	VaR	0.4525	0.0877	(0.2806 , 0.6245)
	3-Period VaR	2.1002	0.0605	(1.9816 , 2.2188)
	ES	2.7268	0.1423	(2.4479 , 3.0058)
4	VaR	6.0031	0.1169	(5.774 , 6.2322)
	3-Period VaR	6.7344	0.0967	(6.5449 , 6.9239)
	ES	8.6311	0.1565	(8.3243 , 8.9379)
5	VaR	11.7812	0.1493	(11.4885 , 12.0739)
	3-Period VaR	12.2212	0.1401	(11.9465 , 12.4959)
	ES	14.9957	0.1814	(14.6402 , 15.3511)
6	VaR	17.1023	0.1826	(16.7445 , 17.4601)
	3-Period VaR	17.4296	0.1878	(17.0615 , 17.7977)
	ES	20.8160	0.2158	(20.393 , 21.239)
7	VaR	21.8997	0.2091	(21.4898 , 22.3097)
	3-Period VaR	22.2779	0.2052	(21.8756 , 22.6802)
	ES	26.2510	0.2414	(25.7778 , 26.7241)

Table 3.2: Simulated Natural VaR, 3-Period VaR and ES - Normal, $\alpha = 5\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
1	VaR	0.9551	0.1276	(0.7051 , 1.2051)
	3-Period VaR	2.0994	0.0635	(1.9749 , 2.2239)
	ES	2.1600	0.1560	(1.8543 , 2.4658)
2	VaR	3.2341	0.1174	(3.004 , 3.4643)
	3-Period VaR	3.5127	0.0951	(3.3264 , 3.6991)
	ES	4.4041	0.1431	(4.1236 , 4.6847)
3	VaR	5.0710	0.0973	(4.8803 , 5.2617)
	3-Period VaR	5.1337	0.0893	(4.9586 , 5.3087)
	ES	5.9769	0.1120	(5.7574 , 6.1964)
4	VaR	6.9780	0.0819	(6.8175 , 7.1384)
	3-Period VaR	6.9782	0.0759	(6.8295 , 7.127)
	ES	7.6975	0.0916	(7.518 , 7.877)
5	VaR	8.8903	0.0625	(8.7679 , 9.0128)
	3-Period VaR	8.8978	0.0618	(8.7767 , 9.0189)
	ES	9.4183	0.0697	(9.2818 , 9.5549)
6	VaR	10.3271	0.0369	(10.2548 , 10.3995)
	3-Period VaR	10.3265	0.0372	(10.2536 , 10.3994)
	ES	10.6585	0.0388	(10.5824 , 10.7346)
7	VaR	11.4906	0.0195	(11.4525 , 11.5287)
	3-Period VaR	11.4923	0.0187	(11.4557 , 11.5289)
	ES	11.6260	0.0183	(11.5902 , 11.6618)

Table 3.3: Simulated Natural VaR, 3-Period VaR and ES - Shifted Gamma, $\alpha = 1\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
1	VaR	-1.5177	0.0748	(-1.6642 , -1.3711)
	3-Period VaR	0.9589	0.0366	(0.8872 , 1.0306)
	ES	-0.0015	0.0819	(-0.162 , 0.159)
2	VaR	0.9284	0.0682	(0.7947 , 1.0621)
	3-Period VaR	1.9977	0.0353	(1.9286 , 2.0669)
	ES	2.3505	0.0779	(2.1978 , 2.5033)
3	VaR	3.1323	0.0593	(3.0162 , 3.2485)
	3-Period VaR	3.4466	0.0500	(3.3487 , 3.5445)
	ES	4.3205	0.0637	(4.1956 , 4.4453)
4	VaR	5.3439	0.0519	(5.2422 , 5.4457)
	3-Period VaR	5.3991	0.0494	(5.3022 , 5.4959)
	ES	6.3411	0.0553	(6.2326 , 6.4495)
5	VaR	7.5977	0.0398	(7.5198 , 7.6756)
	3-Period VaR	7.6020	0.0488	(7.5064 , 7.6976)
	ES	8.3840	0.0418	(8.3021 , 8.4659)
6	VaR	9.4460	0.0297	(9.3879 , 9.5042)
	3-Period VaR	9.4415	0.0293	(9.3841 , 9.4988)
	ES	9.9813	0.0270	(9.9284 , 10.0341)
7	VaR	11.0358	0.0187	(10.9992 , 11.0725)
	3-Period VaR	11.0358	0.0187	(10.9991 , 11.0725)
	ES	11.3108	0.0157	(11.28 , 11.3416)

Table 3.4: Simulated Natural VaR, 3-Period VaR and ES - Shifted Gamma, $\alpha = 5\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
1	VaR	1.4225	0.0029	(1.4169 , 1.4282)
	3-Period VaR	1.4224	0.0028	(1.4169 , 1.4279)
	ES	1.4437	0.0025	(1.4389 , 1.4485)
2	VaR	1.4586	0.0015	(1.4556 , 1.4616)
	3-Period VaR	1.4587	0.0016	(1.4557 , 1.4618)
	ES	1.4697	0.0014	(1.4671 , 1.4724)
3	VaR	1.4714	0.0010	(1.4695 , 1.4734)
	3-Period VaR	1.4716	0.0010	(1.4696 , 1.4736)
	ES	1.4790	0.0009	(1.4772 , 1.4808)
4	VaR	1.4782	0.0008	(1.4766 , 1.4797)
	3-Period VaR	1.4782	0.0007	(1.4769 , 1.4795)
	ES	1.4840	0.0007	(1.4826 , 1.4853)
5	VaR	1.4825	0.0006	(1.4813 , 1.4837)
	3-Period VaR	1.4825	0.0006	(1.4813 , 1.4837)
	ES	1.4871	0.0006	(1.486 , 1.4882)
6	VaR	1.4852	0.0006	(1.4841 , 1.4863)
	3-Period VaR	1.4853	0.0006	(1.4841 , 1.4864)
	ES	1.4891	0.0005	(1.4882 , 1.4901)
7	VaR	1.4870	0.0005	(1.486 , 1.488)
	3-Period VaR	1.4871	0.0005	(1.4862 , 1.488)
	ES	1.4905	0.0004	(1.4896 , 1.4913)

Table 3.5: Simulated Natural VaR, 3-Period VaR and ES - Shifted Pareto, $\alpha = 1\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
1	VaR	1.3488	0.0029	(1.3432 , 1.3544)
	3-Period VaR	1.3489	0.0029	(1.3432 , 1.3546)
	ES	1.3934	0.0022	(1.389 , 1.3978)
2	VaR	1.4214	0.0014	(1.4186 , 1.4241)
	3-Period VaR	1.4212	0.0013	(1.4186 , 1.4239)
	ES	1.4440	0.0012	(1.4417 , 1.4463)
3	VaR	1.4459	0.0009	(1.4442 , 1.4477)
	3-Period VaR	1.4462	0.0009	(1.4444 , 1.448)
	ES	1.4614	0.0008	(1.4599 , 1.4629)
4	VaR	1.4588	0.0008	(1.4573 , 1.4603)
	3-Period VaR	1.4588	0.0008	(1.4572 , 1.4604)
	ES	1.4706	0.0006	(1.4693 , 1.4718)
5	VaR	1.4670	0.0006	(1.4658 , 1.4681)
	3-Period VaR	1.4670	0.0006	(1.4659 , 1.4682)
	ES	1.4764	0.0005	(1.4754 , 1.4773)
6	VaR	1.4722	0.0005	(1.4712 , 1.4732)
	3-Period VaR	1.4722	0.0005	(1.4713 , 1.4732)
	ES	1.4801	0.0004	(1.4793 , 1.4809)
7	VaR	1.4755	0.0004	(1.4746 , 1.4764)
	3-Period VaR	1.4755	0.0005	(1.4746 , 1.4764)
	ES	1.4824	0.0004	(1.4817 , 1.4832)

Table 3.6: Simulated Natural VaR, 3-Period VaR and ES - Shifted Pareto, $\alpha = 5\%$, $n = 3$

Chapter 4

The Surplus Process with Weak Markovian Regime Switching

4.1 Introduction

In this chapter, we construct a model of market and credit risk riding on the surplus process defined in (2.4), with credit transition being described by a weak Markov chain. A weak Markov chain is a Markov chain with order greater than or equals two (Tsoi and Luo, submitted paper). For a r^{th} order weak Markov chain, transition probabilities between transition states depend on the past r periods of histories. As a result, the model development starts with constructing new transition probabilities and a transition matrix as defined in equation (1.10) and (1.11) respectively. Since it is well known that dependent structure is a common phenomenon in finance and is difficult to deal with, we illustrate the weak Markov idea by first exploring the second order case ($r = 2$) in this chapter. Equation (1.10) and (1.11) will be restated. The formulation as well as the recursive equations will be reviewed. Simulations will be carried out, with the parameters specified at the end of this chapter and the results displayed in next chapter.

4.2 Derivation of the Second Order Transition Matrix

In this section, we are going to estimate the transition matrix of a weak Markov chain of order two. Let I_t be a time-homogeneous Markov chain of order two, with finite state space $N = (1, 2, ..., k)$ representing k different credit states (non-default).

Therefore, the transitional probability for $r = 2$ becomes

$$P\left[I_{n+1} = i_{n+1} | I_0 = i_0, I_1 = i_1, ..., I_{n-1} = i_{n-1}, I_n = i_n\right] \tag{4.1}$$

$$= P\left[I_{n+1} = i_{n+1} | I_{n-1} = i_{n-1}, I_n = i_n\right] = q_{i_{n-1} i_n i_{n+1}}$$

and the transitional matrix reduces to

$$Q = \begin{bmatrix} q_{111} & q_{112} & \cdots & q_{11(k-1)} & q_{11k} \\ q_{121} & q_{122} & \cdots & q_{12(k-1)} & q_{12k} \\ \vdots & \vdots & & \vdots & \vdots \\ q_{1k1} & q_{1k2} & \cdots & q_{1k(k-1)} & q_{1kk} \\ q_{211} & q_{212} & \cdots & q_{21(k-1)} & q_{21k} \\ \vdots & \vdots & & \vdots & \vdots \\ q_{k(k-1)1} & q_{k(k-1)2} & \cdots & q_{k(k-1)(k-1)} & q_{k(k-1)k} \\ q_{kk1} & q_{kk2} & \cdots & q_{kk(k-1)} & q_{kkk} \end{bmatrix} \tag{4.2}$$

where $\sum_{j=1}^{k} q_{hij} = 1$ for $h, i = (1, 2, ..., k)$.

For the transition matrix defined in (4.2), the transition probabilities are then estimated by using the S&P Long Term Domestic Issuer Credit Rating - Current (SPDRC) data obtained from Compustat North America Industrial Database of Wharton Research Data Services (University of Pennsylvania, 2005). The Standard & Poor's issuer credit rating is a current opinion of an issuer's overall creditworthiness, apart from its ability to repay individual obligation. This opinion focuses on the obligor's capacity and willingness to meet its long-term financial commitments (those with maturities of more than one year) as they come due. The long term issuer credit rating ranges from AAA (extremely strong capacity to meet financial obligations) to CC (highly vulnerable). In order to provide more detailed indications of credit quality, S&P may modify ratings from AA to CCC with the addition of a plus sign (+) or minus sign (-) to show the relative standing within the major debt rating categories.

A total of 176 sets of quarterly data covering 1962 1^{st} quarter to 2005 4^{th} quarter are obtained from the Wharton database. Table 4.1 shows the credit rating mapping and descriptions for the variable SPDRC. As a first attempt, we define the same finite state space as in last chapters, i.e. $N = (1, 2, ..., 7)$. Data preprocessing has been done to re-group the credit ratings from raw data (variable SPDRC) into the defined groupings in Table 2.1 under non-default condition.

Remarks: Credit ratings 'CC', 'C' and 'CI' are categorized as 'CCC or below'.

Credit ratings 'D' and 'SD' refer to 'Default' which is excluded together with 'Unassigned', 'Not Meaningful', 'Suspended' and 'Missing' throughout the following estimation.

In order to simplify the interpretation and implementation of the surplus process, only non-default states will be considered. Therefore, in our weak Markovian regime switching development, if there exists one or more default state within a consecutive of three observing periods (i.e. quarters), this observation will be regarded as invalid and excluded. To proceed, three consecutive quarterly ratings of a particular corporation will be extracted each time throughout the 176 available periods. A maximum of 174 observations are extracted for a specific firm. The frequentist approach is applied (Schuermann and Jafry, 2003) in which all of the observations are summed up to deduce the second order transition matrix based on their appearing frequency. A total number of 127,087 valid second order transitions are extracted. The breakdown and the transition probabilities estimates are listed from Tables 4.2 to 4.8.

Among the 49 possible combinations of credit histories, we notice that seven of them do not have any valid observations in the source data and transition probabilities cannot be estimated. Even though there are valid observations for the remaining 42 combinations, some of them are too rare to be observed and the estimation may be distorted. As a result, re-grouping of credit states is preferred

Standard & Poor's Long-Term Domestic Issuer Credit Rating - Current (SPDRC)		
Code	Rating	Description
1	Unassigned	Unassigned
2	AAA	"AAA" indicates the highest rating assigned by Standard & Poor's. Capacity to pay interest and repy principle is extremely strong.
3	Unassigned	Unassigned
4	AA+	"AA" indicates a very strong capacity to pay interest and repay principal. There is only a small degree of difference between "AAA" and "AA" ratings.
5	AA	
6	AA-	
7	A+	"A" indicates a strong capacity to pay interest and repay principle. The are, however, somewhat more sysceptible to adverse effects of changes in circumstances and economic conditions that "AAA" or "AA" debt issues.
8	A	
9	A-	
10	BBB+	"BBB" indicates an adequate capacity to pay interest and repay principle. Although it normally exhibits adequate protection parameters, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity to pay interest and repay principle than debt issues with higher ratings.
11	BBB	
12	BBB-	
13	BB+	"BB" indicates less near-term vulnerability to default that other speculative issues. However, they face major ongoing uncertainties or exposures to adverse business, financial, or economic conditions that could lead to inadequate capacity to meet timely interest and principle payments. S&P also uses the "BB" rating for debt subordinated to senior debt that is assigned an actual or implied "BBB-" rating.
14	BB	
15	BB-	
16	B+	"B" indicates a greater vulnerability to default but currently have the capacity to meet interest payments and principle payments. Adverse business, financial, or economic conditions will likely impair capacity or willingness to pay interest and repay principle. S&P also assigns the "B" rating to debt that is subordinated to senior debt that is assigned an actual or implied "BB" or "BB-" rating.
17	B	
18	B-	
19	CCC+	"CCC" indicates an identifiable current vulnerability to default and is dependent upon favorable business, financial, and economic conditions to meet timely payment of interest and repayment of principle. In the event of adverse, business, financial, or economic conditions, "CCC" issues are not likely to have the capacity to pay interest and repay principle. S&P also assigns the "CCC" rating to debt subordinated to senior debt that is assigned an actual or implied "B" or "B-" rating.
20	CCC	
21	CCC-	
22	Unassigned	Unassigned
23	CC	"CC" is typically applied to debt suboridated to senior debt that is assigned an acutal or implied "CCC" rating.
24	C	"C" is typically applied to debt subordianted to senior debt that is assigned an actual or implied "CCC-" rating. S&P also assigns the "C" rating for situations in which a bankruptcy petition has been filled, but debt service payments continue.
25	Unassigned	Unassigned
26	CI	"CI" is reserved for income bonds on which no interest is being paid.
27	D	"D" indicates that payment is in default. S&P assigns the "D" rating when interest payments or principal payments are not made on the date due even if the applicable grace period has not expired, unless S&P believes that such payments will be made during such grace periods. S&P also assigns the "D" rating upon the filing of a bankruptcy petition if debt service payments are jeopardized.
28	Not Meaningful	Not Meaningful
29	SD	"SD" (Selective Default) is assigned when Standard & Poor's believes that the obligor has selectively defaulted on a specific issue or class of obligations in a timely manner.
90	Suspended	S&P suspended the bond rating on a class of debt.

Table 4.1: Possible Values of S&P Long Term Domestic Issuer Credit Rating (SPDRC)

and the details are shown in Table 4.9.

After the credit state re-grouping, the 127,087 samples are assessed and assigned again to the new transition state space, $N = (1, 2, 3)$. There are a total of nine new combinations and 27 transitions after re-grouping. The estimated probabilities are listed from Tables 4.10 to 4.12. These estimates will be applied in the simulation later on.

Previous State	Current State	Next State							Total Number of Samples
		1	2	3	4	5	6	7	
1	1	0.9805	0.0174	0.0010	0.0003	0.0007	0.0000	0.0000	2,878
1	2	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	49
1	3	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	3
1	4	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	1
1	5	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	2
1	6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
1	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0

Table 4.2: Estimated Second Order Transition Matrix for Credit States (1,1) to (1,7)

Previous State	Current State	Next State 1	2	3	4	5	6	7	Total Number of Samples
2	1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	20
2	2	0.0017	0.9709	0.0255	0.0015	0.0000	0.0003	0.0000	11,754
2	3	0.0000	0.0000	0.9900	0.0100	0.0000	0.0000	0.0000	299
2	4	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	18
2	5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
2	6	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	4
2	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0

Table 4.3: Estimated Second Order Transition Matrix for Credit States (2,1) to (2,7)

Previous State	Current State	Next State 1	2	3	4	5	6	7	Total Number of Samples
3	1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3
3	2	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	99
3	3	0.0001	0.0034	0.9773	0.0179	0.0008	0.0004	0.0001	31,025
3	4	0.0000	0.0000	0.0018	0.9766	0.0144	0.0072	0.0000	556
3	5	0.0000	0.0000	0.0000	0.0000	0.9600	0.0400	0.0000	25
3	6	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	12
3	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	2

Table 4.4: Estimated Second Order Transition Matrix for Credit States (3,1) to (3,7)

Previous State	Current State	Next State 1	2	3	4	5	6	7	Total Number of Samples
4	1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	2
4	2	0.1250	0.8750	0.0000	0.0000	0.0000	0.0000	0.0000	8
4	3	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	273
4	4	0.0001	0.0004	0.0099	0.9736	0.0146	0.0011	0.0002	31,393
4	5	0.0000	0.0000	0.0000	0.0022	0.9195	0.0626	0.0157	447
4	6	0.0000	0.0000	0.0000	0.0256	0.0000	0.8462	0.1282	39
4	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	2

Table 4.5: Estimated Second Order Transition Matrix for Credit States (4,1) to (4,7)

Previous State	Current State	Next State 1	2	3	4	5	6	7	Total Number of Samples
5	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
5	2	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1
5	3	0.0000	0.0000	0.8750	0.1250	0.0000	0.0000	0.0000	8
5	4	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	342
5	5	0.0001	0.0001	0.0004	0.0161	0.9594	0.0217	0.0021	22,454
5	6	0.0000	0.0000	0.0000	0.0000	0.0101	0.9256	0.0644	497
5	7	0.0000	0.0000	0.0000	0.0000	0.0000	0.1143	0.8857	35

Table 4.6: Estimated Second Order Transition Matrix for Credit States (5,1) to (5,7)

Previous State	Current State	Next State 1	2	3	4	5	6	7	Total Number of Samples
6	1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1
6	2	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	2
6	3	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	6
6	4	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	20
6	5	0.0000	0.0000	0.0000	0.0055	0.9863	0.0082	0.0000	364
6	6	0.0000	0.0002	0.0006	0.0011	0.0177	0.9586	0.0218	21,531
6	7	0.0000	0.0000	0.0028	0.0000	0.0000	0.0387	0.9586	362

Table 4.7: Estimated Second Order Transition Matrix for Credit States (6,1) to (6,7)

Previous State	Current State	Next State 1	2	3	4	5	6	7	Total Number of Samples
7	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
7	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
7	3	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	2
7	4	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	1
7	5	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	12
7	6	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	113
7	7	0.0000	0.0000	0.0012	0.0008	0.0054	0.0400	0.9525	2,422

Table 4.8: Estimated Second Order Transition Matrix for Credit States (7,1) to (7,7)

Credit State for Second Order Markov Chain (1 to 3)	Credit State (1 to 7)	S&P Credit Rating	Categorized Credit Rating
1	1	AAA	AAA
	2	AA+	AA
		AA	
		AA-	
	3	A+	A
		A	
		A-	
2	4	BBB+	BBB
		BBB	
		BBB-	
	5	BB+	BB
		BB	
		BB-	
3	6	B+	B
		B	
		B-	
	7	CCC or below (exclude default)	CCC

Table 4.9: Re-Grouping of S&P Credit Ratings for Second Order Markov Model

Previous State	Current State	Next State			Total Number of Samples
		1	2	3	
1	1	0.9777	0.0219	0.0004	14,701
1	2	0.0000	1.0000	0.0000	321
1	3	0.0000	0.0000	1.0000	6

Table 4.10: Estimated Second Order Transition Matrix for Credit States (1,1) to (1,3) after Credit State Re-Grouping

Previous State	Current State	Next State 1	2	3	Total Number of Samples
2	1	1.0000	0.0000	0.0000	112
2	2	0.0020	0.9893	0.0087	63,247
2	3	0.0000	0.0038	0.9962	527

Table 4.11: Estimated Second Order Transition Matrix for Credit States (2,1) to (2,3) after Credit State Re-Grouping

Previous State	Current State	Next State 1	2	3	Total Number of Samples
3	1	1.0000	0.0000	0.0000	4
3	2	0.0000	1.0000	0.0000	379
3	3	0.0002	0.0087	0.9911	47,790

Table 4.12: Estimated Second Order Transition Matrix for Credit States (3,1) to (3,3) after Credit State Re-Grouping

4.3 Formulation of the Surplus Process and Recursive Equations

Similar to Chapter 2, we attempt to formulate the surplus process again when the first order Markov model is replaced by a second order one. By applying equation (4.2) and restating Yang's (2000, 2003) model, we define a two-state dependent surplus process as:

$$U_n = u + \sum_{m=1}^{n} \Delta X_m^{I_{m-2} I_{m-1}} = u + \Delta Y_n \tag{4.3}$$

In addition, if we let T be the stopping time (i.e. $T = inf\{n; U_n \leq 0\}$), $U_0 = u$ be the initial surplus and further assume that $I_0 = i_0$ and $I_{-1} = i_{-1}$ are known credit ratings in current and last periods respectively, then we can obtain iterative formulas for natural VaR, n-period VaR and ES respectively in the following sections.

4.3.1 Recursive Equations of Natural VaR

For $y \leq u$, we define natural VaR by considering:

$$\begin{aligned} & P\{\Delta Y_n \leq -y, T \geq n | I_0 = i_0, I_{-1} = i_{-1}, U_0 = u\} \qquad (4.4) \\ = \; & P\{\Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_n^{I_{n-2}I_{n-1}} \leq -y, \\ & \Delta X_1^{i_{-1}i_0} > -u, \ldots, \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -u\} = \alpha \end{aligned}$$

We denote the last expression by $d_n^{i_{-1}i_0}(u, y)$ and calculate it recursively.

For $n \geq 2$,

$$\begin{aligned} d_n^{i_{-1}i_0}(u, y) &= P\{\Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_n^{I_{n-2}I_{n-1}} \leq -y, \\ & \quad \Delta X_1^{i_{-1}i_0} > -u, \ldots, \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -u\} \\ &= \sum_{i=1}^{k} q_{i_{-1}i_0i} \int_{-u}^{\infty} P\{\Delta X_2^{i_0i} + \cdots + \Delta X_n^{I_{n-2}I_{n-1}} \leq -(y + x), \\ & \quad \Delta X_2^{i_0i} > -(u + x), \ldots, \Delta X_2^{i_0i} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -(u + x)\} f^{i_{-1}i_0}(x) dx \\ &= \sum_{i=1}^{k} q_{i_{-1}i_0i} \int_{-u}^{\infty} d_{n-1}^{i_0i}(u + x, y + x) f^{i_{-1}i_0}(x) dx \end{aligned}$$

where

$$d_1^{i_{-1}i_0}(u,y) = P\{\Delta X_1^{i_{-1}i_0} \le -y\}$$

and

$$d_2^{i_{-1}i_0}(u,y) = P\{\Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} \le -y, \Delta X_1^{i_{-1}i_0} > -u\}$$

4.3.2 Recursive Equations of n-Period VaR

For the n-period VaR, we consider the following equation with $y \le u$:

$$P\{\Delta Y_j \le -y, \exists j \le n, T \ge n | I_0 = i_0, I_{-1} = i_{-1}, U_0 = u\} = \alpha \qquad (4.5)$$

Then by varying n, j and constructing mutually exclusive probabilities $h_{n,j}^{i_{-1}i_0}(u,y)$, we restate the last expression as

$$P\{\Delta Y_j \le -y, \exists j \le n, T \ge n | I_0 = i_0, I_{-1} = i_{-1}, U_0 = u\} \;=\; \sum_{j=1}^{n} h_{n,j}^{i_{-1}i_0}(u,y)$$

where for $1 \le j \le n$

$$\begin{aligned} h_{n,j}^{i_{-1}i_0}(u,y) \;=\; & P\{\Delta X_1^{i_{-1}i_0} > -y, \ldots, \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_{j-1}^{I_{j-3}I_{j-2}} > -y, \\ & \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_j^{I_{j-2}I_{j-1}} \le -y, \\ & \Delta X_1^{i_{-1}i_0} > -u, \ldots, \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -u\} \end{aligned}$$

For $n = 1$, $j = 1$,

$$h_{1,1}^{i_{-1}i_0}(u,y) \;=\; P\{\Delta X_1^{i_{-1}i_0} \le -y\}$$

For $n = 2$, $j = 0$,

$$h_{2,0}^{i_{-1}i_0}(u, y) = P\{\Delta X_1^{i_{-1}i_0} > -u\}$$

For $n = 2$, $j = 1$,

$$\begin{aligned} h_{2,1}^{i_{-1}i_0}(u, y) &= P\{\Delta X_1^{i_{-1}i_0} \leq -y, \Delta X_1^{i_{-1}i_0} > -u\} \\ &= P\{-u < \Delta X_1^{i_{-1}i_0} \leq -y\} \end{aligned}$$

For $n = 2$, $j = 2$,

$$\begin{aligned} h_{2,2}^{i_{-1}i_0}(u, y) &= P\{\Delta X_1^{i_{-1}i_0} > -y, \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} \leq -y, \Delta X_1^{i_{-1}i_0} > -u\} \\ &= P\{\Delta X_1^{i_{-1}i_0} > -y, \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} \leq -y\} \end{aligned}$$

Therefore, consider $n \geq 3$, $j = 0$,

$$\begin{aligned} h_{n,0}^{i_{-1}i_0}(u, y) &= P\{\Delta X_1^{i_{-1}i_0} > -u, \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} > -u, \ldots, \\ &\quad \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -u\} \\ &= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} P\{\Delta X_2^{i_0 i} > -(u+x), \Delta X_2^{i_0 i} + \Delta X_3^{i I_2} > -(u+x), \ldots, \\ &\quad \Delta X_2^{i_0 i} + \Delta X_3^{i I_2} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -(u+x)\} f^{i_{-1}i_0}(x) dx \\ &= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} h_{n-1,0}^{i_0 i}(u+x, y) f^{i_{-1}i_0}(x) dx \end{aligned}$$

For $n \geq 3$, $j = 1$,

$$\begin{aligned}
h_{n,1}^{i_{-1}i_0}(u,y) &= P\{\Delta X_1^{i_{-1}i_0} \leq -y, \Delta X_1^{i_{-1}i_0} > -u, \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} > -u, \ldots, \\
& \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -u\} \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{-y} P\{\Delta X_2^{i_0 i} > -(u+x), \Delta X_2^{i_0 i} + \Delta X_3^{i I_2} > -(u+x), \ldots, \\
& \Delta X_2^{i_0 i} + \Delta X_3^{i I_2} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -(u+x)\} f^{i_{-1}i_0}(x)dx \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{-y} h_{n-1,0}^{i_0 i}(u+x,y) f^{i_{-1}i_0}(x)dx
\end{aligned}$$

And finally, for $n \geq 2$, $n \geq j \geq 2$,

$$\begin{aligned}
h_{n,j}^{i_{-1}i_0}(u,y) &= P\{\Delta X_1^{i_{-1}i_0} > -y, \ldots, \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_{j-1}^{I_{j-3}I_{j-2}} > -y, \\
& \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_j^{I_{j-2}I_{j-1}} \leq -y, \\
& \Delta X_1^{i_{-1}i_0} > -u, \ldots, \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -u\} \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-y}^{\infty} P\{\Delta X_2^{i_0 i} + \cdots + \Delta X_j^{I_{j-2}I_{j-1}} \leq -(y+x), \\
& \Delta X_2^{i_0 i} > -(y+x), \ldots, \Delta X_2^{i_0 i} + \cdots + \Delta X_{j-1}^{I_{j-3}I_{j-2}} > -(y+x), \\
& \Delta X_2^{i_0 i} > -(u+x), \ldots, \Delta X_2^{i_0 i} + \cdots + \Delta X_{n-1}^{I_{n-3}I_{n-2}} > -(u+x)\} f^{i_{-1}i_0}(x)dx \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-y}^{\infty} h_{n-1,j-1}^{i_0 i}(u+x, y+x) f^{i_{-1}i_0}(x)dx
\end{aligned}$$

4.3.3 Recursive Equations of n-Period VaR under Default, Expected Shortfall and Distribution Function

The two-state dependent n-period VaR under default for $y > u$ (or $-y < -u$) is defined as:

$$P\{\Delta Y_T \leq -y, T \leq n | I_0 = i_0, I_{-1} = i_{-1}, U_0 = u\} = \sum_{m=1}^{n} h_m^{i_{-1}i_0}(u, y) = \alpha$$

where for $1 \leq m \leq n$,

$$\begin{aligned} h_m^{i_{-1}i_0}(u, y) &= P\{\Delta X_1^{i_{-1}i_0} > -u, \ldots, \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_{m-1}^{I_{m-3}I_{m-2}} > -u, \\ &\quad \Delta X_1^{i_{-1}i_0} + \cdots + \Delta X_m^{I_{m-2}I_{m-1}} \leq -y\} \end{aligned}$$

are mutually exclusive.

We obtain $h_m^{i_{-1}i_0}(u, y)$ by varying m.

For $m = 1$,

$$h_1^{i_{-1}i_0}(u, y) = P\{\Delta X_1^{i_{-1}i_0} \leq -y\} = F^{i_{-1}i_0}(-y)$$

For $m = 2$,

$$\begin{aligned} h_2^{i_{-1}i_0}(u, y) &= P\{\Delta X_1^{i_{-1}i_0} > -u, \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} \leq -y\} \\ &= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} P\{\Delta X_2^{i_0 i} \leq -(x + y)\} f^{i_{-1}i_0}(x) dx \\ &= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} h_1^{i_0 i}(u + x, y + x) f^{i_{-1}i_0}(x) dx \\ &= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} F^{i_0 i}(-(y + x)) f^{i_{-1}i_0}(x) dx \end{aligned}$$

For $m = 3$,

$$\begin{aligned}
h_3^{i_{-1}i_0}(u,y) &= P\{\Delta X_1^{i_{-1}i_0} > -u, \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} > -u, \\
&\quad \Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} + \Delta X_3^{I_1 I_2} \le -y\} \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} P\{\Delta X_2^{i_0 i} > -(u+x), \\
&\quad \Delta X_2^{i_0 i} + \Delta X_3^{i I_2} \le -(y+x)\} f^{i_{-1}i_0}(x)dx \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} h_2^{i_0 i}(u+x, y+x) f^{i_{-1}i_0}(x)dx
\end{aligned}$$

Therefore, for $m \ge 2$,

$$h_m^{i_{-1}i_0}(u,y) = \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-u}^{\infty} h_{m-1}^{i_0 i}(u+x, y+x) f^{i_{-1}i_0}(x)dx$$

Similar to equation (2.7), we have the following expression for the expected shortfall which involves natural VaR defined in Section 4.3.1.

$$\begin{aligned}
ES &= E[-\Delta Y_n | \Delta Y_n \le -VaR] \\
&= -\frac{\int_{-\infty}^{-VaR} y \cdot f_{\Delta Y_n}(y | I_{-1} = i_{-1}, I_0 = i_0)dy}{\alpha}
\end{aligned} \tag{4.6}$$

where $f_{\Delta Y_n}(y|I_{-1} = i_{-1}, I_0 = i_0)$ is the density function of ΔY_n given $I_{-1} = i_{-1}$ and $I_0 = i_0$. It can be obtained by considering the distribution function of ΔY_n

for $n \geq 2$.

$$\begin{aligned}
F_{\Delta Y_n}(-y|I_{-1}=i_{-1}, I_0=i_0) &= P\{\Delta Y_n \leq -y | I_{-1}=i_{-1}, I_0=i_0\} \\
&= P\{\Delta X_1^{i_{-1}i_0} + \Delta X_2^{i_0 I_1} + \cdots + \Delta X_n^{I_{n-2}I_{n-1}} \leq -y\} \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-\infty}^{\infty} P\{\Delta X_2^{i_0 i} + \cdots + \Delta X_n^{I_{n-2}I_{n-1}} \\
&\quad \leq -(y+x)\} f^{i_{-1}i_0}(x)dx \\
&= \sum_{i=1}^{k} q_{i_{-1}i_0 i} \int_{-\infty}^{\infty} F_{\Delta Y_{n-1}}(-(y+x)|I_{-1}=i_0, I_0=i) \\
&\quad f^{i_{-1}i_0}(x)dx
\end{aligned}$$

Remarks: Besides the above recursive formulas, the distribution function can also be obtained by reconstructing a larger state space through reducing the second order Markov chain into a first order one such that formulas in Chapter 2 can be applied. The concept of reconstructing is to treat various combinations of states, for example,

$$\{(1,1),(1,2),\ldots,(1,k),(2,1),\ldots,(2,k),\ldots,(k,1),\ldots,(k,k)\}$$

as the new state space. The Markov chain is then simplified to first order with the number of possible states increased to $k \times k$. The computation will therefore become more complex and it implies that the above recursive formulas are useful.

4.4 Simulation Parameters Setup

In this section, the estimated transition probabilities in Tables 4.10 to 4.12 will be adopted. Under the second order Markov model with three elements in the state space, there are a total of nine possible credit state combinations. The normal, shifted gamma and shifted pareto distribution scenarios will be applied for each of the nine combinations. The parameters are listed in Tables 4.13 to 4.15.

Initial Credit State	Mean	Standard Deviation
(1 , 1)	7	0.25
(1 , 2)	5	0.5
(1 , 3)	3	1
(2 , 1)	2	2
(2 , 2)	1	3
(2 , 3)	0	4
(3 , 1)	-1	5
(3 , 2)	-2	6
(3 , 3)	-3	7

Table 4.13: Parameters for Second Order Markov Model - Normal Distribution Scenario

Initial Credit State	Alpha α	Theta θ
(1,1)	9	1
(1,2)	8	
(1,3)	7	
(2,1)	6	
(2,2)	5	
(2,3)	4	
(3,1)	3	
(3,2)	2	
(3,3)	1	

Table 4.14: Parameters for Second Order Markov Model - Shifted Gamma Distribution Scenario

Initial Credit State	Alpha α	Theta θ
(1,1)	1	1
(1,2)	2.5	
(1,3)	5	
(2,1)	10	
(2,2)	15	
(2,3)	20	
(3,1)	25	
(3,2)	30	
(3,3)	35	

Table 4.15: Parameters for Second Order Markov Model - Shifted Pareto Distribution Scenario

Chapter 5

Simulation Results and Discussion for the Weak Markovian Regime Switching Surplus Process

In this chapter, we attempt to obtain VaR, n-period VaR and ES estimates under weak Markovian regime switching. With the derived second order transition matrix in the last chapter, a total of three return scenarios are assumed and simulated. 500 simulations are run with a sample size of 10,000 in each run. The averages, standard deviations and 95% confidence intervals are also computed and listed in Tables 5.1 to 5.6.

5.1 Findings and Discussion

5.1.1 Normal Scenario

Tables 5.1 and 5.2 show the results for normal distribution scenario.

(1) For both 1% and 5% cases, VaR is always less than ES for each credit state combination. Our assumption is satisfied.

(2) For both 1% and 5% cases, SD of ES is larger than that of VaR. In addition,

SD of both ES and VaR at 5% level are less than that of 1%. These observations are consistent with the use of first order Markov model.

(3) At both 1% and 5% levels, the 3-period VaR is always greater than the respective natural VaR. The difference between these risk measures is large for the combinations $(i, 1), i \in [1, 2, 3]$ and small for $(j, 3), j \in [1, 2, 3]$.

(4) VaR of credit state combination (i, j) increases with i for fix j and increases with j for fix i at both levels of significance.

(5) At 1% level, we discover that the difference of VaR between $(1, 1)$ and $(2, 1)$ is exceptionally small. In addition, ES of $(1, 1)$ is unexpectedly larger than that of $(2, 1)$. Possible reasons for this observation would be inaccurate estimation of transition matrix or inappropriate parameters assumption. We will try to remedy it by reassigning parameter values and simulate the process again. Details will be discussed in later section.

5.1.2 Shifted Gamma Scenario

Tables 5.3 and 5.4 show the results for shifted gamma distribution scenario.

(1) Similar to normal scenario, VaR is less than ES for each credit state combination at both significant levels.

(2) For both 1% and 5% cases, SD of both VaR and ES show a monotonic trend with highest SD at state $(1, 1)$ and decrease towards $(3, 3)$. SD of VaR is generally less than that of ES except at state $(3, 3)$.

(3) The difference between VaR and ES diminishes as we move from $(1, 1)$ to $(3, 3)$ at both levels.

(4) At both 1% and 5% levels, the 3-period VaR is always greater than the respective natural VaR. The difference between these risk measures is particularly small for $(j, 3), j \in [1, 2, 3]$.

5.1.3 Shifted Pareto Scenario

Tables 5.5 and 5.6 show the results for shifted pareto distribution scenario.

(1) Again for both 1% and 5% cases, VaR is always less than ES for each initial credit state combination.

(2) Similar to gamma scenario, for both 1% and 5% cases, SD of both VaR and ES show a monotonic trend with highest SD at state $(1, 1)$ and decrease towards $(3, 3)$. SD of VaR is in general less than that of ES except at state $(3, 3)$.

(3) SD of both VaR and ES at 1% and 5% levels are small since pareto is a heavy-tailed distribution.

(4) Like gamma scenario, the spread between ES and VaR estimates decreases across the initial credit states for both significant levels.

(5) For both 1% and 5% levels, the 3-period VaR is generally equal to or greater than the respective natural VaR for each initial credit state. However, the difference between these risk measures is so small that it may not be cost effective to adopt the n-period VaR instead of natural VaR.

Among the three simulation scenarios, we notice that most of the results are consistent with the first order Markov model assumption. The only exception is found for $(1,1)$ and $(2,1)$ under normal scenario in which the difference of VaR between $(1,1)$ and $(2,1)$ is exceptionally small and ES of $(1,1)$ is unexpectedly larger than that of $(2,1)$. As an attempt to remedy this problem, the simulation parameters are reassigned. The results and respective discussion are listed in next section.

5.1.4 A Remedy for Normal Distribution Scenario

In the last section, three sets of simulation parameters are assumed and simulations are carried for each return scenario. Under each scenario, there are nine possible combinations of 2-period credit state histories. Parameters are assumed for each credit state combination based on the intuition that the first state has a higher priority in determining the portfolio return than the second state (e.g. State (1,2) is better than (2,1); (1,1) is better than (1,2)). Since we observed some unusual VaR and ES estimates under normal distribution scenario, we attempt to reassign the parameters based on the idea that the second state (more recent) has a higher priority in determining the portfolio return than the first state. The new parameters are listed in Figure 5.7. Simulations are performed and the results are shown in Tables 5.8 and 5.9.

We focus on the estimates of combination (1,1) and (2,1). The same problem is discovered at 1% significance in which both natural VaR and ES estimates of (1,1) are less negative than that of (2,1). We believe that the estimated second order transition matrix is the cause. Since real data is used for estimation, insufficient samples may lead to estimation error which distorts the applicability of the transition matrix. In specific, we observe that the probability estimates for combination (2,1) moving to state 1, $\hat{q}_{211} = 1$ is greater than that of combination (1,1) moving to state 1, $\hat{q}_{111} = 0.9777$. This observation contradicts with our intuition and leads to unexpected simulation results. Therefore, re-estimation of probability transition matrix, credit state regrouping and the extension of observation period are possible rectifications to the problem.

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1 , 1)	VaR	-12.4973	0.2676	(-13.0219 , -11.9728)
	3-Period VaR	-6.4065	0.0110	(-6.428 , -6.385)
	ES	-9.9777	0.2842	(-10.5347 , -9.4207)
(1 , 2)	VaR	2.9583	0.1653	(2.6344 , 3.2822)
	3-Period VaR	3.0835	0.1743	(2.7419 , 3.4251)
	ES	4.4121	0.1999	(4.0203 , 4.804)
(1 , 3)	VaR	26.1192	0.3778	(25.3788 , 26.8597)
	3-Period VaR	26.2300	0.3832	(25.4789 , 26.9811)
	ES	29.4747	0.4608	(28.5716 , 30.3779)
(2 , 1)	VaR	-11.1317	0.0807	(-11.29 , -10.9735)
	3-Period VaR	2.6517	0.0748	(2.505 , 2.7984)
	ES	-10.3993	0.1007	(-10.5968 , -10.2019)
(2 , 2)	VaR	9.4527	0.2117	(9.0378 , 9.8676)
	3-Period VaR	9.7668	0.1641	(9.4451 , 10.0885)
	ES	11.6892	0.3199	(11.0622 , 12.3163)
(2 , 3)	VaR	30.7825	0.3983	(30.0019 , 31.5631)
	3-Period VaR	30.8130	0.3914	(30.0458 , 31.5802)
	ES	34.3547	0.4886	(33.397 , 35.3124)
(3 , 1)	VaR	-1.2702	0.1777	(-1.6185 , -0.9218)
	3-Period VaR	12.6120	0.1794	(12.2604 , 12.9636)
	ES	0.4140	0.2259	(-0.0287 , 0.8567)
(3 , 2)	VaR	17.1093	0.2618	(16.5962 , 17.6225)
	3-Period VaR	18.1488	0.2735	(17.6128 , 18.6848)
	ES	19.6134	0.3304	(18.9658 , 20.261)
(3 , 3)	VaR	37.1407	0.4382	(36.2819 , 37.9995)
	3-Period VaR	37.2867	0.4531	(36.3986 , 38.1748)
	ES	41.2184	0.5655	(40.11 , 42.3268)

Table 5.1: Simulated Natural VaR, 3-Period VaR and ES for Weak Markov - Normal, $\alpha = 1\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1,1)	VaR	-19.9461	0.0187	(-19.9828, -19.9094)
	3-Period VaR	-6.5824	0.0065	(-6.5952, -6.5696)
	ES	-16.2762	0.0993	(-16.4708, -16.0815)
(1,2)	VaR	0.0394	0.0894	(-0.1358, 0.2145)
	3-Period VaR	0.4216	0.0862	(0.2526, 0.5906)
	ES	1.8317	0.1063	(1.6234, 2.04)
(1,3)	VaR	19.3658	0.2233	(18.9282, 19.8035)
	3-Period VaR	19.4100	0.2265	(18.9661, 19.8539)
	ES	23.5059	0.2500	(23.0159, 23.9959)
(2,1)	VaR	-12.5668	0.0458	(-12.6565, -12.4771)
	3-Period VaR	1.2935	0.0451	(1.2051, 1.3819)
	ES	-11.6831	0.0530	(-11.787, -11.5791)
(2,2)	VaR	5.6925	0.1127	(5.4715, 5.9135)
	3-Period VaR	6.4613	0.0966	(6.272, 6.6506)
	ES	8.0603	0.1360	(7.7938, 8.3269)
(2,3)	VaR	23.5177	0.2157	(23.095, 23.9405)
	3-Period VaR	23.6298	0.2460	(23.1476, 24.112)
	ES	27.9709	0.2579	(27.4654, 28.4764)
(3,1)	VaR	-4.6912	0.1035	(-4.894, -4.4884)
	3-Period VaR	9.2273	0.1223	(8.9877, 9.4669)
	ES	-2.5922	0.1202	(-2.8278, -2.3566)
(3,2)	VaR	12.1052	0.1585	(11.7946, 12.4159)
	3-Period VaR	13.6600	0.1284	(13.4083, 13.9117)
	ES	15.1812	0.1785	(14.8315, 15.531)
(3,3)	VaR	28.8872	0.2578	(28.382, 29.3924)
	3-Period VaR	29.1151	0.2553	(28.6148, 29.6154)
	ES	33.9509	0.3048	(33.3535, 34.5484)

Table 5.2: Simulated Natural VaR, 3-Period VaR and ES for Weak Markov - Normal, $\alpha = 5\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1, 1)	VaR	-4.0663	0.1480	(-4.3564, -3.7762)
	3-Period VaR	0.5218	0.0632	(0.3979, 0.6457)
	ES	-2.7760	0.1716	(-3.1124, -2.4396)
(1, 2)	VaR	2.3788	0.0989	(2.185, 2.5726)
	3-Period VaR	2.6042	0.0906	(2.4266, 2.7818)
	ES	3.2467	0.1156	(3.0201, 3.4733)
(1, 3)	VaR	8.4814	0.0590	(8.3658, 8.597)
	3-Period VaR	8.4979	0.0644	(8.3716, 8.6242)
	ES	8.9620	0.0605	(8.8434, 9.0806)
(2, 1)	VaR	-2.0804	0.1266	(-2.3285, -1.8324)
	3-Period VaR	2.2461	0.0402	(2.1672, 2.325)
	ES	-0.9905	0.1431	(-1.271, -0.71)
(2, 2)	VaR	4.6868	0.0922	(4.506, 4.8676)
	3-Period VaR	4.7809	0.0760	(4.632, 4.9298)
	ES	5.5098	0.1057	(5.3027, 5.7169)
(2, 3)	VaR	10.2058	0.0391	(10.1292, 10.2825)
	3-Period VaR	10.2134	0.0362	(10.1424, 10.2844)
	ES	10.5168	0.0396	(10.4391, 10.5945)
(3, 1)	VaR	0.1834	0.1160	(-0.0439, 0.4107)
	3-Period VaR	3.6005	0.0169	(3.5674, 3.6336)
	ES	1.1668	0.1279	(0.9161, 1.4175)
(3, 2)	VaR	6.5748	0.0778	(6.4222, 6.7274)
	3-Period VaR	6.6163	0.0702	(6.4786, 6.754)
	ES	7.1952	0.0854	(7.0278, 7.3625)
(3, 3)	VaR	11.5602	0.0157	(11.5294, 11.591)
	3-Period VaR	11.5673	0.0190	(11.5301, 11.6045)
	ES	11.6765	0.0145	(11.6481, 11.7048)

Table 5.3: Simulated Natural VaR, 3-Period VaR and ES for Weak Markov - Shifted Gamma, $\alpha = 1\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1 , 1)	VaR	-6.8289	0.0902	(-7.0057 , -6.6522)
	3-Period VaR	-0.6674	0.0440	(-0.7536 , -0.5812)
	ES	-5.1371	0.0997	(-5.3324 , -4.9417)
(1 , 2)	VaR	0.3688	0.0652	(0.2411 , 0.4966)
	3-Period VaR	1.0210	0.0501	(0.9227 , 1.1193)
	ES	1.5959	0.0686	(1.4615 , 1.7303)
(1 , 3)	VaR	7.2999	0.0423	(7.217 , 7.3829)
	3-Period VaR	7.3091	0.0396	(7.2315 , 7.3867)
	ES	8.0203	0.0413	(7.9394 , 8.1011)
(2 , 1)	VaR	-4.5342	0.0821	(-4.6951 , -4.3733)
	3-Period VaR	1.4089	0.0308	(1.3484 , 1.4694)
	ES	-3.0351	0.0887	(-3.2089 , -2.8612)
(2 , 2)	VaR	2.8471	0.0615	(2.7266 , 2.9676)
	3-Period VaR	3.2417	0.0470	(3.1496 , 3.3338)
	ES	3.9723	0.0640	(3.8469 , 4.0976)
(2 , 3)	VaR	9.3785	0.0289	(9.3218 , 9.4352)
	3-Period VaR	9.3860	0.0280	(9.3312 , 9.4408)
	ES	9.8810	0.0274	(9.8274 , 9.9346)
(3 , 1)	VaR	-2.0538	0.0700	(-2.191 , -1.9165)
	3-Period VaR	3.2042	0.0135	(3.1777 , 3.2307)
	ES	-0.6859	0.0741	(-0.8311 , -0.5407)
(3 , 2)	VaR	5.0796	0.0542	(4.9733 , 5.1859)
	3-Period VaR	5.3524	0.0393	(5.2753 , 5.4295)
	ES	5.9886	0.0562	(5.8784 , 6.0988)
(3 , 3)	VaR	11.1765	0.0148	(11.1476 , 11.2055)
	3-Period VaR	11.1839	0.0161	(11.1524 , 11.2154)
	ES	11.4090	0.0127	(11.3842 , 11.4339)

Table 5.4: Simulated Natural VaR, 3-Period VaR and ES for Weak Markov - Shifted Gamma, $\alpha = 5\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1,1)	VaR	1.0598	0.0183	(1.024, 1.0956)
	3-Period VaR	1.0602	0.0188	(1.0233, 1.097)
	ES	1.1885	0.0152	(1.1587, 1.2183)
(1,2)	VaR	1.4444	0.0021	(1.4402, 1.4486)
	3-Period VaR	1.4445	0.0024	(1.4398, 1.4492)
	ES	1.4595	0.0019	(1.4558, 1.4632)
(1,3)	VaR	1.4748	0.0010	(1.4728, 1.4769)
	3-Period VaR	1.4750	0.0010	(1.473, 1.477)
	ES	1.4817	0.0009	(1.4799, 1.4834)
(2,1)	VaR	1.2673	0.0089	(1.2498, 1.2848)
	3-Period VaR	1.2667	0.0101	(1.2468, 1.2865)
	ES	1.3341	0.0078	(1.3188, 1.3494)
(2,2)	VaR	1.4708	0.0012	(1.4685, 1.4731)
	3-Period VaR	1.4709	0.0012	(1.4686, 1.4732)
	ES	1.4785	0.0010	(1.4766, 1.4805)
(2,3)	VaR	1.4848	0.0006	(1.4837, 1.4859)
	3-Period VaR	1.4849	0.0005	(1.4838, 1.4859)
	ES	1.4888	0.0005	(1.4879, 1.4898)
(3,1)	VaR	1.3087	0.0093	(1.2906, 1.3269)
	3-Period VaR	1.3081	0.0095	(1.2895, 1.3268)
	ES	1.3666	0.0072	(1.3524, 1.3807)
(3,2)	VaR	1.4766	0.0009	(1.4749, 1.4783)
	3-Period VaR	1.4767	0.0008	(1.4751, 1.4784)
	ES	1.4828	0.0008	(1.4814, 1.4843)
(3,3)	VaR	1.4874	0.0005	(1.4865, 1.4883)
	3-Period VaR	1.4875	0.0004	(1.4867, 1.4884)
	ES	1.4908	0.0004	(1.4899, 1.4916)

Table 5.5: Simulated Natural VaR, 3-Period VaR and ES for Weak Markov - Shifted Pareto, $\alpha = 1\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1 , 1)	VaR	0.5568	0.0209	(0.5158 , 0.5979)
	3-Period VaR	0.7275	0.0113	(0.7054 , 0.7497)
	ES	0.8636	0.0157	(0.8327 , 0.8944)
(1 , 2)	VaR	1.3909	0.0022	(1.3866 , 1.3953)
	3-Period VaR	1.3913	0.0022	(1.387 , 1.3955)
	ES	1.4234	0.0017	(1.42 , 1.4267)
(1 , 3)	VaR	1.4507	0.0010	(1.4488 , 1.4526)
	3-Period VaR	1.4508	0.0009	(1.449 , 1.4526)
	ES	1.4654	0.0008	(1.4638 , 1.467)
(2 , 1)	VaR	1.0051	0.0109	(0.9838 , 1.0265)
	3-Period VaR	1.0059	0.0101	(0.9861 , 1.0257)
	ES	1.1656	0.0079	(1.1501 , 1.1811)
(2 , 2)	VaR	1.4450	0.0010	(1.443 , 1.4469)
	3-Period VaR	1.4450	0.0009	(1.4432 , 1.4468)
	ES	1.4606	0.0009	(1.4589 , 1.4623)
(2 , 3)	VaR	1.4714	0.0005	(1.4703 , 1.4724)
	3-Period VaR	1.4714	0.0005	(1.4704 , 1.4725)
	ES	1.4795	0.0004	(1.4787 , 1.4804)
(3 , 1)	VaR	1.0645	0.0114	(1.0422 , 1.0867)
	3-Period VaR	1.0634	0.0114	(1.041 , 1.0858)
	ES	1.2143	0.0084	(1.1978 , 1.2309)
(3 , 2)	VaR	1.4558	0.0008	(1.4543 , 1.4573)
	3-Period VaR	1.4559	0.0006	(1.4546 , 1.4571)
	ES	1.4684	0.0007	(1.4671 , 1.4697)
(3 , 3)	VaR	1.4764	0.0004	(1.4756 , 1.4773)
	3-Period VaR	1.4766	0.0004	(1.4758 , 1.4773)
	ES	1.4831	0.0004	(1.4823 , 1.4838)

Table 5.6: Simulated Natural VaR, 3-Period VaR and ES for Weak Markov - Shifted Pareto, $\alpha = 5\%$, $n = 3$

Initial Credit State	Mean	Standard Deviation
(1 , 1)	7	0.25
(1 , 2)	2	2
(1 , 3)	-1	5
(2 , 1)	5	0.5
(2 , 2)	1	3
(2 , 3)	-2	6
(3 , 1)	3	1
(3 , 2)	0	4
(3 , 3)	-3	7

Table 5.7: Reassigned Parameters for Weak Markov - Normal Distribution Scenario

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1 , 1)	VaR	-9.3678	0.3191	(-9.9933 , -8.7424)
	ES	-6.4480	0.3264	(-7.0876 , -5.8084)
(1 , 2)	VaR	7.0522	0.1770	(6.7052 , 7.3991)
	ES	8.8234	0.2342	(8.3644 , 9.2823)
(1 , 3)	VaR	32.6852	0.4368	(31.829 , 33.5413)
	ES	36.5112	0.5282	(35.4759 , 37.5465)
(2 , 1)	VaR	-13.7567	0.2023	(-14.1533 , -13.3602)
	ES	-12.1014	0.2138	(-12.5205 , -11.6822)
(2 , 2)	VaR	9.7580	0.2285	(9.31 , 10.2059)
	ES	12.6357	0.3655	(11.9194 , 13.352)
(2 , 3)	VaR	34.9023	0.4733	(33.9747 , 35.8299)
	ES	38.7501	0.4953	(37.7794 , 39.7209)
(3 , 1)	VaR	-11.6679	0.1774	(-12.0155 , -11.3202)
	ES	-9.9140	0.1981	(-10.3023 , -9.5257)
(3 , 2)	VaR	11.6846	0.2230	(11.2475 , 12.1218)
	ES	13.7550	0.2493	(13.2664 , 14.2436)
(3 , 3)	VaR	37.0541	0.4090	(36.2524 , 37.8558)
	ES	41.1526	0.5365	(40.1012 , 42.2041)

Table 5.8: Re-Simulated Natural VaR and ES for Weak Markov - Normal, $\alpha = 1\%$, $n = 3$

Initial Credit State	Risk Measure	Average	S.D.	95% CI
(1 , 1)	VaR	-19.9265	0.0207	(-19.9671 , -19.886)
	ES	-13.6110	0.1183	(-13.8428 , -13.3793)
(1 , 2)	VaR	3.7800	0.0951	(3.5937 , 3.9664)
	ES	5.8172	0.1157	(5.5905 , 6.0439)
(1 , 3)	VaR	25.1363	0.2163	(24.7123 , 25.5602)
	ES	29.7909	0.2630	(29.2755 , 30.3063)
(2 , 1)	VaR	-17.8378	0.0153	(-17.8678 , -17.8079)
	ES	-15.9698	0.0746	(-16.1159 , -15.8237)
(2 , 2)	VaR	5.7845	0.1134	(5.5622 , 6.0068)
	ES	8.3738	0.1458	(8.088 , 8.6596)
(2 , 3)	VaR	26.9969	0.2361	(26.5341 , 27.4596)
	ES	31.8354	0.2930	(31.2612 , 32.4097)
(3 , 1)	VaR	-15.0125	0.0244	(-15.0603 , -14.9647)
	ES	-13.3528	0.0677	(-13.4856 , -13.2201)
(3 , 2)	VaR	7.6403	0.1223	(7.4007 , 7.88)
	ES	10.1364	0.1438	(9.8545 , 10.4183)
(3 , 3)	VaR	28.7893	0.2623	(28.2752 , 29.3034)
	ES	33.8539	0.2903	(33.285 , 34.4228)

Table 5.9: Re-Simulated Natural VaR and ES for Weak Markov - Normal, $\alpha = 5\%$, $n = 3$

Chapter 6

Concluding Remarks

The intent of this thesis is to present a model which takes care both market and credit risks. We have first proposed a surplus process in which the credit risk is modelled by a Markov chain on Standard and Poor's credit rating system. With this Markovian regime switching model, various risk measures have been considered. Risk measures including natural value at risk and expected shortfall were reviewed and adopted. We have also proposed a risk measure named n-period value at risk. Recursive equations were developed for these risk measures.

In the proposed surplus process, we modelled the market risk by assuming various return distribution scenarios. A total of three scenarios were assumed, including normal, shifted gamma and shifted pareto distribution. Monte Carlo simulations were carried for each scenario. Estimates of the three risk measures under each scenario were obtained. The results were found consistent with our expectation and other researches. The newly suggested and conservative risk measure, n-period value at risk works particularly well in normal return scenario.

In order to deal with some kind of dependency of the credit risks, we have attempted to adopt a weak Markov chain to model the credit rating dynamics in our surplus process. In particular, we considered a second order Markov model for the sake of simplicity. Second order transition probabilities, transition matrix and transition states have been re-estimated and restated to cope with the dependency structure. The estimation of transition probabilities was difficult due to insufficient credit dependency data. In order to simplify the estimation, credit state had been regrouped which in fact reduced the explaining power of the model. The surplus process and recursive equations for risk measures have also been rederived. Simulations were carried out again for the same scenarios but with different parameters set. Estimates of risk measures were adopted. The results were generally consistent with the use of first order Markov model but with some discrepancies between two particular credit state combinations. As a remedy, we have reassigned the simulation parameters with a different assumption and simulated again. The same discrepancies were found. We concluded that the estimated second order transition matrix and the length of observation period would be the cause and further investigation was required to solve out the problem.

We believed that the risk measures for portfolio with both market and credit risks are practically important and theoretically interesting. This thesis provides a simple model. We hope that our work may stimulate some interesting researches on this subject. Although the simulations are computationally intensive, hopefully this will not be a big problem with today's powerful computer.

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