Adjusting Multi-Factor Models for Basel II-consistent Economic Capital

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Summary. Understanding and analytically measuring concentration risk in credit portfolios is one of the major challenges in recent research. The measurement is necessary for the determination of regulatory capital under pillar 2 of Basel II as well as for managing the portfolio and allocating economic capital. For this task, the Asymptotic Single Risk Factor (ASRF) framework has to be left to capture these risks. This particularly refers to the correlation structure and granularity. There exist some models in the literature that in-depth deal with concentration risk such as name and sector concentrations. But there are several shortcomings within the literature. To achieve meaningful results the approaches should be determined and applied consistent to the pillar 1 capital what is often not fully satisfied. For this reason we adjust the multi-factor models in a way that they deliver Basel II-consistent results. Therefore, we determine an implied intra-sector concentration formula. Furthermore, we show that the mostly used Value at Risk is problematic when leaving the ASRF framework. Thus, we perform a matching procedure to make sure that the results using the coherent Expected Shortfall do not lead to an overestimation of risks compared to the Basel approach. We apply these modifications to some multi-factor approaches and perform an extensive numerical study similar to Cespedes et al. (2006) to get a closed form approximation formula. We test the impact of sector concentrations on several portfolios and compare the results within the different models. Finally we carry out a simulation study to compare the accuracy of the models in a more general manner. It is shown that the Pykhtin model mostly provides a higher and more stable accuracy than the Cespedes model but the latter has advantages for ad-hoc and sensitivity analyses. In principle both approaches are suitable for measuring Basel II-consistent economic capital when applied in the proposed way.

Keywords: Concentration Risk; Basel II; Multi-Factor Models; Coherency; Intra-Sector Correlation; Inter-Sector Correlation

JEL classification: G21, G28

1 Introduction

In recent years there have been significant improvements in understanding and measuring concentration risk in credit portfolios such as undiversified idiosyncratic risk and industry or country risk. The measurement of these risks is important against the background of regulatory capital needs as well as for computing the economic capital. Unfortunately, the existing approaches are mostly not fully consistent with the new capital adequacy framework (Basel II) – sometimes within the derivation and sometimes within the implementation – so that the benefit of these approaches is restricted. Furthermore, comparative analyses on these models are scarce. Against this background we address the following questions:

- How can the existing approaches be modified and adjusted to be consistent with the Basel framework? How can we deal with the problems that arise when leaving the assumptions of the Basel framework?
- Which methods are capable to measure concentration risk and how good do they perform in comparison? What are the advantages and disadvantages of these methods?

For answering these questions, we firstly investigate the assumptions underlying the Basel framework. The Basel II formula for measuring the Value at Risk of credit portfolios is based upon the so-called asymptotic single risk factor (ASRF) framework as explained in Gordy (2003). In this framework it is assumed that

- the portfolio is infinitely fine grained and thus it consists of a nearly infinite number of credits with small exposures, and
- only one systematic risk factor influences the default risk of all loans in the portfolio.

The first assumption implies that there are no name concentrations within the portfolio, thus all idiosyncratic risk is diversified completely. The second assumption implicates that there are no sector concentrations such as industry- or country-specific risk concentrations. These are idealizations that can be problematic for real world portfolios.

The Basel Committee on Banking Supervision (BCBS) already recognized the high importance of credit risk concentrations in the Basel framework: "Risk concentrations are arguably the single most important cause of major problems in banks."[1] Since it is difficult to incorporate credit risk concentrations in analytic approaches, in Basel II there is no quantitative approach mentioned how to deal with risk concentrations. Instead, it is only qualitatively demanded in pillar 2 of Basel II that "Banks should have in place effective internal policies, systems and controls to identify, measure, monitor, and control their credit risk concentra-

[1] See BCBS (2005) §770.

tions."[2] Thus it is each bank's task how to concretely meet these requirements. But of course the measurement and management of risk concentrations is not only important for determining the regulatory capital but also for determining the "true" portfolio risk. The capital needs regarding this "true" risk will be denoted as economic capital in the following.

From the mentioned types of concentration risk, name concentrations are better understood than sector concentrations. The theoretical derivation of the so-called granularity adjustment that accounts for name concentrations was done by Wilde (2001) and improved by Pykhtin and Dev (2002) and Gordy (2003). This can be called "portfolio name concentration" because the approach refers to the finite number of credits in the portfolio. The adjustment formulas are derived in a more straightforward approach by Martin and Wilde (2002), Rau-Bredow (2002) and Gordy (2004). Furthermore, the adjustment is extended and numerically analyzed in detail by Gürtler, Heithecker, and Hibbeln (2008). A related approach is the granularity adjustment from Gordy and Lütkebohmert (2007), whereas the semi-asymptotic approach from Emmer and Tasche (2005) refers to name concentrations due to a single name while the rest of the portfolio remains infinitely granular, so this can be called "single name concentration".

There also exist analytic and semi-analytic approaches that account for sector concentrations. One rigorous analytical approach is Pykhtin (2004) that is based on a similar principle as in Martin and Wilde (2002). An alternative is the semi-analytic model from Cespedes et al. (2006) that derives an approximation formula through a complex numerical mapping procedure. Another approach from Düllmann (2006) extends the binomial extension technique (BET) model from Moody's. Tasche (2006) suggest an ASRF-extension in an asymptotic multi-factor setting. Some numerical work on the performance of the Pykhtin model is done by Düllmann and Masschelein (2006). Furthermore, Düllmann (2008) presents a first comparison of different approaches on sector concentration risk. The problem is that the derivation and the application of the approaches is often inconsistently with the Basel II framework what is critical for the following reasons:

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[2] See BCBS (2005) § 773. Furthermore, because of the importance of this topic for the stability of the banking system, the Basel Committee launched the "Research Task Force Concentration Risk" that presented its final report in BCBS (2006). The Task Force collected information about the state of the art in current practice and academic literature, analyzed the impact of departures from the ASRF model and reviewed some methodologies to measure name and sector concentrations. An additional workstream focused on stress testing against the background of risk concentrations.

- Banks are demanded to measure concentration risks and "explicitly consider the extent of their credit risk concentrations in their assessment of capital adequacy under Pillar 2" of Basel II. Even if a bank uses a high-sophisticated multi-factor model, the results are not comparable with the Pillar 1 capital requirement if the results are not consistent to the Basel framework. Thus it remains unclear if or how much additional regulatory capital is needed regarding risk concentrations.
- Generally, it is not worthwhile to have a major gap between the regulatory and the "true" economic capital. A homogenization of these values was one goal of the new Capital Accord and would simplify the management of the credit portfolio.

For these reasons we demonstrate how multi-factor models can be used in a way that is consistent with the Basel framework and thereby avoid the problems that arise when leaving the ASRF framework. Furthermore we compare the capability of different multi-factor approaches in approximating the "true" portfolio risk through a simulation study.

The rest of the paper is outlined as follows. In section 2 we briefly describe the ASRF framework and the Basel formula. Moreover, we discuss the problems of the non-coherent Value at Risk in the context of concentration risk and present how the coherent Expected Shortfall can be used consistent with Basel II. In section 3 we introduce multi-factor models in general, and the Pykhtin as well as the Cespedes model in particular. There, we demonstrate how these approaches could be modified to achieve meaningful results. We compare the performance of the models with a simulation study in section 4. The paper concludes with section 5.

2 Coherent Concentration Risk Measurement in the Context of the Basel Framework

2.1 The ASRF Framework and the Basel II Formula

As mentioned before, the Basel II risk quantification formula is based upon the ASRF framework that assumes an infinitely granular portfolio and the existence of only one systematic risk factor. When these two assumptions are true, the relative portfolio loss $\tilde{L}$ in t = T almost surely equals the expected loss conditional on the realization of the systematic factor $\tilde{x}$

$$\tilde{L} - E\left(\tilde{L} \mid \tilde{x}\right) \to 0 \quad \text{a.s.}^{3} \tag{1}$$

[3] See Gordy (2003).

If the loss given default (LGD) is assumed to be deterministic, the conditional expectation can be written as

$$E\left(\tilde{L} \mid \tilde{x}\right)=\sum_{i=1}^{n} E\left(w_i \cdot LGD_i \cdot \tilde{I}_{Default,i} \mid \tilde{x}\right)=\sum_{i=1}^{n} w_i \cdot LGD_i \cdot E\left(\tilde{I}_{Default,i} \mid \tilde{x}\right), \tag{2}$$

where $\tilde{I}_{Default}$ represents the indicator function that is 1 in the event of default and 0 in case of survival of the obligor whereas w_i stands for the weight of credit i in the portfolio, with $i \in \{1, \ldots, n\}$. For the concrete application of formula (2), the conditional default expectation has to be determined. In the Basel II framework, the well known Vasicek model is used.[4] In this one-period one-factor model, the return of each obligor is driven by two components that realize at a future point in time T: a systematic part $\tilde{x}$ that influences all firms and a firm-specific (idiosyncratic) part $\tilde{\varepsilon}_i$.[5] Thus, the "normalized" asset returns[6] $\tilde{a}_i$ of each obligor i in t = T can be represented by the following model

$$\tilde{a}_i=\sqrt{\rho_i} \cdot \tilde{x}+\sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i, \tag{3}$$

in which $\tilde{x} \sim N(0,1)$ and $\tilde{\varepsilon}_i \sim N(0,1)$ are independently and identically normally distributed with mean zero and standard deviation one. In this model, the correlation structure of each firm i is represented by the firm-specific correlation $\sqrt{\rho_i}$ to the common factor. Hence, the correlation between two firms i, j can be expressed as $\sqrt{\rho_i} \cdot \sqrt{\rho_j}$ or simply as ρ for the case of a homogeneous correlation structure.

Further, the probability of default of each obligor is exogenously given as PD_i.[7] Corresponding to formula (3), an obligor i defaults at t = T when its "normalized" return falls below a default threshold b_i which can be characterized by

$$\tilde{a}_i<b_i \quad \Leftrightarrow \quad \sqrt{\rho_i} \cdot \tilde{x}+\sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i<b_i. \tag{4}$$

Against this background the threshold b_i is determined by the exogenous specification of PD_i:[8]

$$PD_i=\operatorname{prob}\left(\tilde{a}_i<b_i\right)=N(b_i) \quad \Leftrightarrow \quad b_i=N^{-1}(PD_i). \tag{5}$$

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[4] See e.g. Vasicek (1987, 1991, 2002) and Finger (1999, 2001).

[5] To keep track of the model, stochastic variables are marked with a tilde "~".

[6] The returns are normalized by subtracting the expected return and dividing the resulting term by the standard deviation in order to get standard normally distributed variables.

[7] The probability of default could either be determined by the institution itself or by a rating agency.

[8] The term prob(A) stands for the probability of the occurrence of an uncertain event A. N(.) characterizes the cumulative standard normal distribution and N^{-1}(.) stands for the inverse of N(.).

Conditional on a realization of the systematic factor the probability of default of each obligor is[9]

$$\operatorname{prob}\left(\tilde{a}_i < b_i \mid \tilde{x}\right) = E\left(\tilde{I}_{\tilde{a}_i < b_i} \mid \tilde{x}\right) = N\left(\frac{N^{-1}(PD_i) - \sqrt{\rho_i} \cdot \tilde{x}}{\sqrt{1-\rho_i}}\right) =: p_i(\tilde{x}). \tag{6}$$

Applying formula (6) from the Vasicek model to formula (2) from the ASRF framework, the portfolio loss distribution can be computed. For quantification of the credit risk, the Value at Risk (VaR) on confidence level z can be used, that is the z-quantile q_z of the loss variable, in which $z \in (0,1)$ is the target solvency probability. Precisely, like Gordy (2004), we define the VaR as the loss that is only exceeded with the probability of at most 1–z, i.e.

$$\operatorname{VaR}_z\left(\tilde{L}\right) := q_z\left(\tilde{L}\right) := \inf\left(l:\ \operatorname{prob}\left(\tilde{L} \le l\right) \ge z\right). \tag{7}$$

In the context of the ASRF framework, the VaR can be computed similarly to formula (1) as

$$\operatorname{VaR}_z\left(\tilde{L}\right) - E\left(\tilde{L} \mid \tilde{x} = q_{1-z}(\tilde{x})\right) \to 0 \quad \text{a.s.}, \tag{8}$$

where $q_z(\tilde{x})$ stands for the z-quantile of the systematic factor. Recalling formula (2), (6), and the normality of the systematic factor, the VaR of the portfolio equals

$$\begin{aligned}\operatorname{VaR}_z^{(\text{Basel})}\left(\tilde{L}\right) &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i\left(q_{1-z}(\tilde{x})\right) \\ &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot N\left(\frac{N^{-1}(PD_i) + \sqrt{\rho_i} \cdot N^{-1}(0.999)}{\sqrt{1-\rho_i}}\right),\end{aligned} \tag{9}$$

if we insert the confidence level z = 0.999. This is the (well established) VaR formula used in Basel II. Obviously, the credit risk only relies on the systematic factor, since due to the infinite number of exposures the idiosyncratic risks associated with each individual obligor cancel out each other and are diversified completely.

2.2 Concentration Risk and Coherency

In recent years there is an extensive discussion about reasonable risk measures. Artzner et al. (1999) formulated four axioms that a risk measure should satisfy to be a coherent risk measure: translation invariance, subadditivity, positive homogeneity, and monotonicity. Unfortunately, the wide used VaR is not coherent because it is not necessarily subadditive. As long as we stay in the ASRF framework, this characteristic is not problematic because the VaR is exactly additive. This can be seen in formula (8) considering that the expectation operator is ad-

[9] In the following "E" stands for the expectation operator.

ditive. But if we leave the ASRF framework, this behavior is not guaranteed anymore. This is true for non-asymptotic portfolios as well as for multi-factor models. However, many contributions that deal with concentration risk in the context of the Basel framework use the VaR to quantify credit risk without questioning the risk measure, possibly because the Basel formula makes use of this risk measure, even if the subadditivity could get problematic in this context.[10] Thus, it could be beneficial to change the measure of risk, e.g. to use the coherent Expected Shortfall (ES), that is defined as[11]

$$\mathrm{ES}_z\left(\tilde{L}\right) = (1-z)^{-1} \cdot \left[\mathrm{E}\left(\tilde{L}\right) \cdot \tilde{I}_{(\tilde{L} \geq q_z)}\right] + q_z \cdot \left[(1-z) - \mathrm{prob}\left(\tilde{L} \geq q_z\right)\right] \tag{10}$$

with q_z for the VaR on confidence level z (see formula (7)), or simply as

$$\mathrm{ES}_z\left(\tilde{L}\right) = (1-z)^{-1} \cdot \left[\mathrm{E}\left(\tilde{L}\right) \cdot \tilde{I}_{(\tilde{L} \geq q_z)}\right] = \mathrm{E}\left(\tilde{L} \mid \tilde{L} \geq q_z\right) \tag{11}$$

for continuous distributions. But before we change the risk measure, we study the characteristics of the VaR for credit portfolios and analyze the need for using the ES.

For our analyses we start with removing the first assumption of the ASRF framework, so we have a finite number of loans. Therefore we use the binomial model of Vasicek, assuming homogeneous credits. If we recall the conditional probabilities of default from formula (6), we found the individual default events to be independent. Thus, the (conditional, still uncertain) number of defaults $\tilde{M} \mid x$ (and the gross loss rate) of the portfolio are binomial distributed with the probability $p(x)$, i.e.

$$\tilde{M} \mid x \sim B\left(n; p(x)\right). \tag{12}$$

With reference to Vasicek (1987), see also Gordy and Heitfield (2000), we are able to calculate the unconditional probability of having k defaults and we get

$$\mathrm{prob}\left(\tilde{L} = \frac{m}{n}\right) = \int_{-\infty}^{+\infty} \binom{n}{k} \cdot p(x)^m \cdot \left(1 - p(x)\right)^{n-m} \cdot dN(x) \tag{13}$$

where $\tilde{L}$ marks the (uncertain) portfolio gross loss rate. Finally, the VaR of the portfolio in the Vasicek model is

$$\mathrm{VaR}_z^{(\mathrm{Vasicek})}\left(\tilde{L}\right) = \inf\left(l:\ \mathrm{prob}\left(\tilde{L} \leq l\right) = \sum_{k=1}^{[l \cdot n]} \mathrm{prob}\left(\frac{m}{n}\right) \geq z\right).^{12} \tag{14}$$

[10] See e.g. Heitfield, Burton, and Chomsisengphet (2006), Cespedes et al. (2006), Düllmann (2006), as well as Düllmann and Masschelein (2006).

[11] See Acerbi and Tasche (2002).

[12] The symbolism $[l \cdot n]$ denotes the highest natural number that is smaller or equal to $(l \cdot n)$.

For an infinite number of credits the VaR of the Vasicek model converges towards the VaR of the ASRF framework.[13]

Now, we compute the VaR on confidence level z = 99.9% for non-asymptotic portfolios with PD = 0.5% and ρ = 20%. In Figure 1 we plot the VaR for the ASRF framework and for the Vasicek binomial model for n = 1 to n = 300 homogeneous credits. The VaR for an infinitely number of credits is 9.1%, for a finite number of credits the risk is higher because the unsystematic risk can not be diversified away. The problem is that the risk should be monotonously decreasing with a higher number of credits, but this behaviour is not reflected by the VaR as a risk measure. Although the subadditivity axiom is not violated in the example, it is obvious that the risk should rise at no time with a higher number of credits and thus a better diversification. It is also possible to construct superadditive examples in the context of credit risk but this example gives a clear demonstration that it is problematic to use the VaR if there is concentration risk such as name concentration.

- Figure 1 about here -

For comparison, we compute the ES for the same portfolio setting. For calculation of the ES for the Vasicek model, we have to apply formula (10) by using formula (14). The ES for the Basel model can be calculated with formula (11) and (9). With respect to Acerbi and Tasche (2002) and Pykhtin (2004) we get

$$\mathrm{ES}_z^{(\mathrm{Basel})}\left(\tilde{\mathrm{L}}\right)=\sum_{i=1}^{n}\frac{w_i\cdot \mathrm{LGD}_i}{1-z}\cdot \mathrm{N}_2\left(-\mathrm{N}^{-1}(z),\mathrm{N}^{-1}(\mathrm{PD}_i),\sqrt{\rho_i}\right), \tag{15}$$

where $\mathrm{N}_2(\cdot)$ stands for the bivariate cumulative normal distribution. As can be seen in Figure 2, the ES can satisfy the mentioned intuition of diversification. Further, our analyses show that the approximation formulas lead to better results if the ES is used instead of the VaR, especially if there is a high degree of concentration risk.[14] Thus, it is advisable to use the ES instead of the VaR if there is concentration risk within a portfolio. At this point the measured economic capital would be significantly higher by the use of ES, what is not the intended consequence of the change from VaR to ES. In our example even the ASRF solution rises from 9.1% to 11.81%. Instead, we would only like to use the appreciated properties for concentra-

[13] See Vasicek (2002) or Bluhm, Overbeck, and Wagner (2003).

[14] A numerical study can be requested from the authors.

tion risk without to be bound to increase the amount of economic capital. Therefore we will adjust the confidence level as described in the next section.

- Figure 2 about here -

2.3 Adjusting for Coherency in Concentrated Portfolios

If there are violations of the ASRF framework, the risk measure should be changed from VaR to ES. Our intention is to do this in a way that the results are consistent with the Basel II framework to get meaningful results for the additional capital requirement for concentration risk required under pillar 2. In addition, this establishes the possibility to bring the regulatory capital in line with economic capital measured in a more sophisticated way than the Basel II model. If a (hypothetical) portfolio fulfills the assumptions of the ASRF framework, the results should not differ, whether the risk is measured by the VaR or by the ES. Therefore, we examine the $\mathrm{VaR}_{99.9\%}$ on the given confidence level z = 99.9% for several (infinitely granular) bank portfolios of different quality. As a next step we determine the confidence level of the ES that is necessary to match the results for both risk measures. We define this ES-confidence level z (ES) implicitly as

$$\mathrm{ES}_z^{(\mathrm{Basel})}\left(\tilde{\mathrm{L}}\right) = \mathrm{VaR}_{0.999}^{(\mathrm{Basel})}\left(\tilde{\mathrm{L}}\right), \tag{16}$$

with $\mathrm{VaR}_{0.999}^{(\mathrm{Basel})}$ given by formula (9) and $\mathrm{ES}_z^{(\mathrm{Basel})}$ given by formula (15).

Firstly, we investigate the extreme cases that all creditors of a bank have a rating of (I) AAA or (VII) CCC.[15] As can be seen in Table 1, the ES-confidence level must be in a range between 99.67% and 99.74%. Using these values the economic capital is almost identical for VaR and ES.

- Table 1 about here -

Then, we use five portfolios with different credit quality distributions (very high, high, average, low, and very low) that are visualized in Figure 3.[16] All resulting confidence levels are

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[15] We used the idealized default rates from Standard & Poors, see Brand and Bahar (2001), ranging from 0.01% to 18.27%, but the results do not differ widely for different values.

[16] The portfolios with high, average, low, and very low quality are taken from Gordy (2000). We added a portfolio with very high quality.

between 99.71% and 99.73% with mean 99.72%. Therefore, an ES-confidence level of z = 99.72% seems to be accurate for most real world portfolios.

- Figure 3 about here -

3 Basel II-consistent Credit Risk Modeling in a Multi-Factor Setting

3.1 Multi-Factor Models in Credit Risk Modeling

To obtain a more realistic modeling of correlated defaults in a credit portfolio, we will introduce a typical multi-factor model. In this model the dependence structure between obligors is not driven by one global systematic risk factor but by sector specific risk factors. Additionally the obligors are divided into K sectors. Hereby a suitable sector assignment is important[17], this means asset correlations shall be high within a sector and low between different sectors. In contrast to the single factor model where the correlation structure for each firm i is completely described by ρ, we differentiate in the multi-factor model between an inter-sector correlation ρ_{Inter} and an intra-sector correlation ρ_{Intra}. The inter-sector correlation describes the correlation between the sector factors and the intra-sector correlation describes the sensitivity of the asset return to the corresponding sector factor. Thus, the asset return of obligor i in sector s can be represented by

$$\tilde{a}_{s,i} = \sqrt{\rho_{\text{Intra},i}} \cdot \tilde{x}_{s,i} + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \,, \tag{17}$$

where $\tilde{x}_{s,i}$ is the sector risk factor and $\tilde{\xi}_i$ is the idiosyncratic factor. $\tilde{x}_{s,i}$ and $\tilde{\xi}_i$ are standard-normal distributed variables that are independent of one another. The sector risk factor $\tilde{x}_{s,i}$ can be written as a combination of independent standard normally distributed factors $\tilde{z}_k$ (k = 1, …, K)

$$\tilde{x}_{s,i} = \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k \quad \text{with} \sum_{k=1}^{K} \alpha_{s,k}^2 = 1, \tag{18}$$

where the factor weights $\alpha_{s,k}$ are calculated via a Cholesky decomposition of the inter-sector correlation matrix, so that

$$\rho_{s,t}^{\text{Inter}} = \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} \,. \tag{19}$$

[17] As shown by Morinaga and Shiina (2005) an assignment of borrowers to the wrong sectors leads to a higher estimation error than a non-optimal sector definition.

Equation (17) and (18) imply that the asset correlation between two obligors is given as

$$\mathrm{Cor}(\tilde{x}_{s,i}, \tilde{x}_{t,j}) = \begin{cases} \sqrt{\rho_{\mathrm{Intra},i}} \cdot \sqrt{\rho_{\mathrm{Intra},j}} & ,\text{if } s = t, \\ \sqrt{\rho_{\mathrm{Intra},i}} \cdot \sqrt{\rho_{\mathrm{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} & ,\text{if } s \neq t. \end{cases} \tag{20}$$

Obligors in the same sector will be highly correlated with one another when their intra-sector correlation is high. The correlation of obligors in different sectors also depends on the factor weights, which are derived from the inter-sector correlation. Hence the dependence structure in the multi-factor model is completely described by the intra- and inter-sector correlations.

Taking formula (5) into account, the portfolio loss distribution can be written as

$$\tilde{L} = \sum_{s=1}^{K} \sum_{i=1}^{n_s} w_{s,i} \cdot \tilde{I}_{\tilde{a}_{s,i} < N^{-1}(PD_i)}, \tag{21}$$

where n_s is the number of obligors in sector s.

In the next three sections we will present different approaches to determine the distribution and tail expectations. Furthermore we will demonstrate how the models can be parameterized to be Basel II-consistent.

3.2 Monte-Carlo-Simulations and Parameterization through a Correlation Matching Procedure

A common approach to estimate the portfolio loss distribution is the use of Monte-Carlo-Simulations. In each simulation run the sector factors as well as the idiosyncratic factor of each obligor are randomly generated. Herewith the asset return is calculated according to (17). If $\tilde{x}_{s,i}$ is less then a threshold given by $N^{-1}(PD_i)$, the obligor i defaults. The portfolio loss is determined from formula (21) by summing up the exposures sizes $w_{s,i}$ multiplied by the LGD_i of each defaulted credit. To get a good approximation of the "true" loss distribution we choose 500,000 runs for our Monte-Carlo-Simulations. After running the simulation and sorting loss outcomes, we get the portfolio loss distribution. To obtain a tail expectation for a given confidence level z, in principle the mean for all loss realization equal or greater than q_z has to be calculated, whereas q_z is given by the $z \cdot 500{,}000\text{th}$ element of the simulated distribution.[18]

[18] The exact formulation is given in formula (10).

To calibrate the multi-factor model, most variables can be chosen identically to the single factor model. The only difference is the correlation structure that generally consists of inter- and intra-sector correlations as described above. The matrix of inter-sector correlations is usually derived from historical default rates or from equity correlations between industry sectors. In a full multi-factor model, the intra-sector correlations could be derived from historical default rates, too. The problem is that there are not always enough observations to get stable results. That is even more problematic when it is assumed that the correlation and the PD are interdependent like in Basel II. Furthermore, the results from the multi-factor model would normally not be consistent with Basel II because the correlation structure is completely different. Thus, it would not be possible to identify if there is need for additional regulatory capital under pillar 2. For both reasons the intra-sector correlations could be chosen by application of the Basel formula.

The easiest way to perform this is a direct employment of the Basel formula for the correlation that is

$$\rho_{\text{Basel}} = 0,12 \cdot \frac{1-e^{-50 \cdot \text{PD}}}{1-e^{-50}} + 0,24 \cdot \left(1 - \frac{1-e^{-50 \cdot \text{PD}}}{1-e^{-50}}\right) \tag{22}$$

for corporates. This is what Cespedes et al. (2006) did in their analyses. But this approach is critical for the following reason: In this approach it is assumed that the Basel framework is an upper barrier of the true risk and is only valid if there is only one sector or if all sectors are perfectly correlated. In all other cases there is an effect of sector diversification that leads to lower capital requirement compared to the Basel framework. In contrast to this assumption, the Basel correlation formula is not intended by the Basel committee to reflect the intra-sector correlation only. Instead, the framework is calibrated on well-diversified portfolios so that using the correlation formula in the single factor model should approximate the “true” risk using the full correlation structure in a multi-factor model.[19] Cespedes et al. (2006) already recognized this criticism and mentioned that it should be possible to use some scaling up for the intra-sector correlations and the resulting capital, respectively, but their calculations are based on the formula above.

Instead the intra-sector correlation could be chosen in a way that the regulatory capital can be matched with the economic capital that is simulated for a well-diversified portfolio within a multi-factor model. Therefore we define the “implicit intra-sector correlation” $\rho_{\text{Intra}}^{(\text{Implied})}$ with

[19] See Basel Committee on Banking Supervision (2006).

$$EC_{Multi}(\rho_{Inter}, \rho_{Intra}^{(Implied)}) = EC_{Single}(\rho_{Basel}) \,. \quad (23)$$

Unfortunately, the portfolios for which the calibration was done by the Basel Committee and the assumed inter-sector correlation are not publicly available. Thus, we first have to choose a concrete inter-sector correlation and determine the implicit intra-sector correlation for some hypothetical, well-diversified portfolios via Monte-Carlo-Simulations with several parameter trials. For the inter-sector correlation structure we use the matrix computed by Düllmann and Masschelein (2006) that is based on MSCI EMU industry indices (see Table 2).[20]

- Table 2 about here -

Our definition of a well-diversified portfolio is based on the overall sector concentration of the German banking system.[21] Even if it is theoretically possible to achieve lower capital requirements through different sector decomposition, this can only be done by a restricted number of banks because of the finite volume. The composition can be seen in Table 3. In addition, the total number of credits is assumed to be n = 5000 so that there is low granularity.

- Table 3 about here -

If we assume a constant intra-sector correlation, the best match is achieved around $\rho_{Intra}^{(Implied)} = 25\%$[22] but the concrete results vary with the portfolio quality (see Table 4).[23] Thus, using a constant intra-sector correlation can lead to a significant underestimation of economic capital for high-quality portfolios and to an overestimation for low-quality portfolios.

- Table 4 about here -

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[20] The correlation structure based on the MSCI US is similar, see Düllmann and Masschelein (2006).

[21] Düllmann and Masschelein (2006) notice that the concentration is very similar to other countries like France, Belgium and Spain.

[22] This value results for both the VaR and the ES computed on the matched confidence level as described in section 2.3. The result is consistent with Düllmann and Masschelein (2006) who use a constant intra-sector correlation of 25% in their analysis.

[23] See Figure 3 for the portfolio characteristics.

To reduce the deviation, the intra-sector correlation should be decreasing in PD. We found that the following intra-sector correlation function leads to a good match for portfolios with different quality distributions:

$$\rho_{\text{Intra}}^{(\text{Implied})} = 0,19 \cdot \frac{1-e^{-50\cdot\text{PD}}}{1-e^{-50}} + 0,31 \cdot \left(1 - \frac{1-e^{-50\cdot\text{PD}}}{1-e^{-50}}\right) \tag{24}$$

Thus, in principle we use the correlation function from Basel but the correlation is centered on 25% instead of 18%.

Hence, all additional input data needed for typical multi-factor models, e.g. using Monte-Carlo-Simulations, are given with Table 2 and formula (24). Using these values, the multi-factor models should be consistent with the Basel framework. Thus, the measured economic capital is only lower than the regulatory capital if the portfolio is less concentrated than a typical, well-diversified portfolio and the needed economic capital will be above the capital requirement of the regulatory framework if there is more concentration risk in the credit portfolio.

3.3 Implementation for the Cespedes-Model

Cespedes et al. (2006) present a method to determine the economic capital in the multi-factor model via an estimating function DF( ) which depends on two parameters:

- the average sector concentration CDI and
- the average weighted inter-sector correlation $\overline{\beta}$.

Herewith the economic capital of a portfolio can be approximated as:

$$\text{EC}^{\text{mf}} \approx \text{DF} \cdot \text{EC}^{\text{sf}}, \tag{25}$$

thus, the economic capital in the multi-factor model EC^{mf} can be approximated by a well-defined diversification factor DF multiplied with the economic capital in the ASRF-model EC^{sf}. As mentioned before, Cespedes et al. assume that the Basel framework is an upper barrier of the true risk because no diversification effects between the sectors are considered, which implies that DF is always less or equal than one. In contrast, if we use our definition of the intra-sector correlation ρ_{Intra} from section 3.2, it is possible to obtain that $\text{EC}^{\text{mf}} > \text{EC}^{\text{sf}}$ as well as $\text{EC}^{\text{mf}} < \text{EC}^{\text{sf}}$, depending on the degree of diversification in comparison to the well-diversified portfolio defined in section 3.2. Hence, our later on calculated DF-function can be greater than one, that is, the DF-function measures not only the benefit from sector diversification but also the risk resulting from high sector concentration. Since we are interested in calculating the Expected Shortfall instead of using the Value at Risk, we modify this in the model. Thus (25) is substituted by:

$$ES_z^{mf} = DF \cdot \sum_{k=1}^{K} ES_z^k , \tag{26}$$

where ES_z^{mf} is the expected shortfall in the multi-factor model and ES_z^k is the expected shortfall in the ASRF-model for sector k. The latter can be determined numerically with formula (15). In principle, the approach works as follows: First, the ES_z^{mf} is calculated for a multitude of portfolios via Monte-Carlo-Simulations. For every simulated portfolio the diversification factor can be calculated according to formula (26). Finally, a regression is performed to get an approximation for DF as a function of the two parameters CDI and $\overline{\beta}$. If DF can capture the industry diversification effects, we are able to approximate ES_z^{mf} with formula (26) without additional Monte-Carlo-Simulations.

To derive the parameters which explain the effect of diversification and concentration in a multi-factor model, Cespedes et al. suggest to use the average inter-sector correlation $\overline{\beta}$. This can be interpreted as a scale of the dependence between the sectors. The formula for $\overline{\beta}$ is given as:

$$\overline{\beta} = \frac{\sum_{k=1}^{K} \sum_{j \neq k} \rho_{kj}^{inter} \cdot ES_z^k \cdot ES_z^j}{\sum_{k=1}^{K} \sum_{j \neq k} ES_z^k \cdot ES_z^j} . \tag{27}$$

The correlation is weighted by the expected shortfall in order to account for the contribution of each sector. The second suggested parameter is the capital diversification index denoted by CDI. It describes the sector concentration measured by the relative weight of each ES_z^k:[24]

$$CDI = \frac{\sum_{k=1}^{K} \left(ES_z^k \right)^2}{\left(\sum_{k=1}^{K} ES_z^k \right)^2} . \tag{28}$$

The CDI lies between the two extreme values:

- perfect sector diversification, that is $CDI = \frac{1}{n}$,
- perfect sector concentration, that is $CDI = 1$.

[24] This concentration measure is also known as the Herfindahl-Hirschmann-Index.

To avoid a too complex model Cespedes et al. neglect further potential input parameters to determine the DF-function. To approximate the multi-factor model, formula (26) can be re-written as:

$$ES_z^{mf}\left(CDI,\overline{\beta}\right) = DF\left(CDI,\overline{\beta}\right)\cdot\sum_{k=1}^{K} ES_z^k. \tag{29}$$

In the following, our procedure to estimate the DF-function is presented. To get a universally valid DF-factor as many portfolios as possible have to be generated and simulated. To reduce the necessary number of trials, the portfolios should be restricted to those with reasonable characteristics. Our portfolios are randomly generated using the following parameter setting. When we state several parameter values or a parameter range, the parameter is randomly drawn from this set.

For the intra-sector correlations we use the functional form of formula (24). The inter-sector correlation structure is taken from Table 2, so that all simulated portfolios are stemming from this sector definition. Every portfolio consists of {2,… , 11} sectors that are randomly drawn from the different industries. The sector weights are [0, 1]. The total number of credits is 5000, equally divided for every sector. Each sector consists of credits from the PD classes {AAA, AA, A, BBB, BB, B, CCC}. Instead of using equally distributed PD classes we draw the quality distribution from our predefined credit portfolio qualities {very high, high, average, low, very low} for every sector. We draw 22.000 portfolios and compute Monte-Carlo-Simulations with 100,000 trials for every portfolio.[25]

The results for DF can be seen in Figure 4.

- Figure 4 about here -

For determination of the functional form of DF we use a regression of the type[26]

$$DF = a_0 + a_1\cdot(1-CDI)\cdot(1-\overline{\beta}) + a_2\cdot(1-CDI)^2\cdot(1-\overline{\beta}) + a_3\cdot(1-CDI)\cdot(1-\overline{\beta})^2. \tag{30}$$

The final function

$$\begin{aligned} DF = {} & 1.2808 - 1.1451\times(1-CDI)\times(1-\overline{\beta}) \\ & +0.0765\times(1-CDI)^2\times(1-\overline{\beta}) + 0.2040\times(1-CDI)\times(1-\overline{\beta})^2 \end{aligned} \tag{31}$$

[25] The setting is similar to Cespedes et al. The main difference is the definition of the intra- and inter- sector correlations.

[26] We tried several different regressions but similar to Cespedes et al. this function worked best. In contrast to Cespedes et al. the first parameter a_0 does not equal 1 because our DF-factor is not bound by the single-factor-model.

with R^2 = 95,55% is plotted in Figure 5.[27] This function has to be used in formula (29) to get the approximation for the multi-factor model.

- Figure 4 about here -

3.4 Implementation for the Pykhtin-Model

In this section the multi-factor adjustment of Pykhtin (2004) is presented. It is an extension of the granularity adjustment, introduced by Gordy (2003), Wilde (2001) and Martin and Wilde (2002), for multi-factor models and provides an analytical method for calculating the VaR and ES of a credit portfolio.

The basic idea from Pykhtin is to approximate the portfolio loss $\tilde{L}$ in the multi-factor model with the loss $\tilde{\tilde{L}}$ for the same portfolio in an accurately adjusted single factor model. For this, the dependence structure is mapped into a single correlation factor to approximate the distribution of L optimally. This is done by defining a single risk factor $\tilde{\tilde{x}}$ subject to the original risk factors $\{\tilde{z}_k\}$ and solving an optimization problem so that the correlation between $\tilde{\tilde{x}}$ and the original sector factors $\{\tilde{x}_s\}$ is maximized.[28]

Via this approach it is possible, as shown by Wilde (2001), to calculate the z-quantile $q_z(\tilde{L})$ of the portfolio loss by a quadratic taylor series as

$$q_z(\tilde{L}) \approx q_z(\tilde{\tilde{L}}) + \left.\frac{dq_z(\tilde{\tilde{L}} + \varepsilon \cdot \tilde{U})}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2}\left.\frac{d^2 q_z(\tilde{\tilde{L}} + \varepsilon \cdot \tilde{U})}{d\varepsilon^2}\right|_{\varepsilon=0}, \tag{32}$$

where ε is the scale of the perturbation and U describes the approximation error between $\tilde{L}$ and $\tilde{\tilde{L}}$ that is $\tilde{U} = \tilde{L} - \tilde{\tilde{L}}$. The first term on the right-hand side is the z-quantile of loss $\tilde{\tilde{L}}$ for an infinitely fine grained portfolio in a single factor model. Since the conditions of the ASRF-model are satisfied, the loss distribution can be calculated by formula (9), so that

[27] The shape of the function is similar to Cespedes et al. but their range is from 0.1 to 1.0 whereas our function ranges from 0.4 to 1.3. Furthermore, we assume that they chose a higher number of trials for each simulation that result in a higher R^2 because of less simulation noise. Nevertheless, the simulation noise is unsystematic and should mostly cancel out each other.

[28] The sets $\{\tilde{z}_k\}$ and $\{\tilde{x}_s\}$ are given as in section 3.1.

$$\tilde{\bar{L}} = l(\tilde{\bar{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot N\left[\frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\bar{x}}}{\sqrt{1-c_i}}\right], \tag{33}$$

where c_i is the correlation between the systematic risk factor $\tilde{\bar{x}}$ and the asset return $a_{s,i}$. Instead of using ρ as it is done in the ASRF-model, the new correlation parameter c_i is used to match the correlation structure in the multi-factor model. The loss quantile $q_z(\tilde{\bar{L}})$ is given by $l(N^{-1}(1-z))$. The derivation of c_i to obtain the maximum correlation between $\tilde{\bar{x}}$ and $\{\tilde{x}_s\}$ can be found in Appendix 1. As can be seen in formula (43), both the intra- and inter- sector correlations are needed to determine c_i, which can be taken from section 3.2.

It can be shown that the first derivative in formula (32) is equal to zero, since $\tilde{\bar{L}} = E\left[\tilde{L} \mid \tilde{\bar{x}}\right]$.[29] Hence, the so-called multi-factor adjustment Δq_z is described completely by the second derivative. According to Pykhtin Δq_z can be written as

$$\Delta q_z = q_z(L) - q_z(\bar{L}) \approx -\frac{1}{2 \cdot l'(\bar{x})} \cdot \left[v'(\bar{x}) - v(\bar{x}) \cdot \left(\frac{l''(\bar{x})}{l'(\bar{x})} + \bar{x} \right) \right]\Bigg|_{\bar{x}=N^{-1}(1-z)}, \tag{34}$$

where $l'(\bar{x})$ and $l''(\bar{x})$ are the first and second derivative of formula (33) and $v(\bar{x})$ is the conditional variance of U. The required formulas can be found in Appendix 2. As also shown in the Appendix, $v(\bar{x})$ can be decomposed into two terms, $v_\infty(\bar{x})$ and $v_{GA}(\bar{x})$. The term $v_\infty(\bar{x})$ describes the systematic risk adjustment, which is given by the difference between the multi-factor and single-factor loss distribution. The other term $v_{GA}(\bar{x})$ is the granularity adjustment, which measures the influence of single-name concentration. With these terms the multi-factor adjustment can be written as

$$\Delta q_z = \Delta q_z^{\infty} + \Delta q_z^{GA} \tag{35}$$

Finally, the approximation of a loss quantile $q_z(\tilde{L})$ in (32) is given by (33) and the multifactor adjustment:

$$q_z(\tilde{L}) = q_z(\tilde{\bar{L}}) + \Delta q_z^{\infty} + \Delta q_z^{GA}. \tag{36}$$

To determine the ES instead of a quantile in a multi-factor model, formula (11) can be rewritten as

$$ES_z(\tilde{L}) = ES_z(\tilde{\bar{L}}) + \frac{1}{1-z} \cdot \int_z^1 \Delta t_s(\tilde{L})\, ds. \tag{37}$$

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[29] That means the conditional expected obligor risk vanishes.

For this the quantile $q_z(\tilde{L})$ is substituted by the approximation (31). The first term of the right-hand site describes the ES for the single factor portfolio and the second term denoted by $\Delta ES_z(\tilde{L})$ is the multi-factor adjustment.

As shown by Pykhtin (2004) $ES_z(\tilde{\tilde{L}})$ and $\Delta ES_z(\tilde{L})$ can be calculated as

$$ES_z(\tilde{\tilde{L}}) = \frac{1}{1-z}\sum_{i=1}^{M} w_i \cdot LGD_i \cdot N_2\left[N^{-1}(PD_i), N^{-1}(1-z), c_i\right], \tag{38}$$

and

$$\Delta ES_z(\tilde{L}) = -\frac{1}{2\cdot(1-z)}\cdot n\left[N^{-1}(1-z)\frac{v\left[N^{-1}(1-z)\right]}{l'\left[N^{-1}(1-z)\right]}\right], \tag{39}$$

with $n(\cdot)$ denoting the standard normal density function. Again, the multifactor adjustment can be decomposed into a systematic and an idiosyncratic part by decomposing the conditional variance. Hence the ES for a portfolio in a multi-factor model is given as

$$ES_z(\tilde{L}) = ES_z(\tilde{\tilde{L}}) + \Delta ES_z^{\infty}(\tilde{L}) + \Delta ES_z^{GA}(\tilde{L}). \tag{40}$$

In principle it is straightforward to implement the Pykhtin model. For calculating the ES we have to compute formula (39) with the conditional variance from (51) and (54) by using (52), (46) and (48). If applied to large portfolios, its computation can be extremely time-consuming. The reason is that the double sum from formula (51) requires n^2-times the computation of the conditional asset correlation (52), with n being the number of credits, so that the computation time can easily exceed the time needed for Monte-Carlo-Simulations. An alternative performed by Düllmann and Masschelein (2006) is to neglect the multi-factor adjustment and using the term (33) only and/or to aggregate all credits for each sector and thus using the formulas on sector and not on borrower level.[30] In contrast, we propose to built PD- and EAD-classes for each of the sectors, so that the computation time is predominated by

$$Loops = (N_{PD-classes} \cdot N_{EAD-classes} \cdot S_{Sectors})^2, \tag{41}$$

where N_{PD}, N_{EAD} and S denote the number of PD-classes, EAD-classes and sectors. As the number of loops will not grow with bigger portfolios, it is possible to perform the approach on bucket level within reasonable time.

[30] This is proposed by Düllmann and Masschelein (2006).

4 Performance of the Concentration Risk Models

4.1 Analysis for Deterministic Portfolios

To determine the quality of the presented models we start our analysis with calculating the expected shortfall for five deterministic portfolios of different quality. We generate well-diversified portfolios consisting of 5000 credits, thus, we have neither high name nor high sector concentration risk. For this we choose the sectors and their weights as given in Table 3. The inter-sector correlation is given in Table 2 whereas the intra-sector correlation is calculated with formula (24). The five portfolios differ in their PD distribution which is provided in Figure 3. Portfolio 1 is the portfolio with the highest and Portfolio 5 is the one with the lowest credit quality distribution.

Table 5 shows that the calculated values of the Pykhtin model are very good approximations of the "real" ES given by Monte Carlo Simulations in almost all cases.[31] The outcomes of the Cespedes model are more imprecise. With decreasing credit quality the estimation error is increasing, which leads to an underestimation of risk in the portfolios above.

- Table 5 about here -

As a next step, we change the portfolio structure towards high sector concentration. Therefore, we increase the sector weights of two sectors. We assume that 45% of the creditors – in terms of their exposure – belong to the Information Technology sector and an equal amount belongs to the Telecommunication Services sector. The remaining 10% of exposure are equally assigned to the miscellaneous sectors.

As showed in Table 6, the risk materially increases for all types of portfolio quality. Especially, the Basel formula underestimates the risk by about 13% for all portfolio qualities. This is the (relative) amount that should be considered in the assessment of capital adequacy under pillar 2. The approximation formula of Pykhtin can capture this concentration risk with a negligible error in all cases. The Cespedes model underestimates the risk by about 6%. Thus, the sector concentration risk is not fully captured.

- Table 6 about here -

[31] In our analyses the number of simulation runs is 500,000.

Furthermore, we built credit portfolios with low sector concentration. For this purpose, we use the concept of naïve diversification so that every sector has an equal weight of 1/11. As can be seen in Table 7, the economic capital is significantly below the regulatory capital.[32] This also shows that it is easy to construct portfolios, which are better diversified than the overall credit market. Again, the Pykhtin model leads to good approximations, especially for "normal" credit qualities (Portfolio 2, 3, and 4). The Cespedes model underestimates the risk for low quality portfolio (Portfolio 4 and 5), whereas for good quality portfolios (Portfolio 1 and 2) the risk is slightly overestimated.

- Table 7 about here -

4.2 Simulation Study for Homogeneous and Heterogeneous Portfolios

To achieve more general results we test the models for different, randomly generated portfolios. For this we implement four simulation studies. In these studies we analyze the accuracy for homogeneous as well as for heterogeneous portfolios w.r.t. PD and EAD. In each simulation run we generate a portfolio and determine its ES by the three models. After 100 runs we calculate the root mean squared error for the outcomes of the Pykhtin model and the Cespedes model to quantify its performance in comparison to Monte-Carlo-Simulations using 500,000 trials. In the following we describe the four simulation settings.

Simulation I: In this scenario we generate portfolios with homogenous exposure sizes and homogenous PDs, that is, $w_i = 1/5000$ and $PD_i = PD = \text{constant}$ for each credit. To test the accuracy for different portfolio qualities a PD is drawn from a uniformly distribution between 0 and 10% before each new run. The sector structure and correlation is the same as in section 4.1.

Simulation II: We generate portfolios with homogenous exposure sizes but heterogeneous PDs. For each credit we determine a PD from the distribution of a typical bank for small and medium-sized enterprises. The exposure size remains as in Simulation I. Again, the sector structure and correlation is taken from section 4.1.

[32] In the case of these negative deviations in comparison to the regulatory capital, it is not allowed – at least at present – to reduce the regulatory capital.

Simulation III: We generate portfolios with homogenous PDs as in Simulation I but with heterogeneous exposure sizes. First, we choose the number of sectors randomly between 4 and 11. Then we apply a uniform distribution between 0 and 1 for the weight of every sector and scale this so that the weights sum up to one. The weights for the credits in each sector are determined in the same manner. The correlations remain unchanged.

Simulation IV: In this setting the PDs as well as the exposure sizes of the generated portfolios are heterogeneous. The PDs are determined as in Simulation II and the exposure sizes as in Simulation III.

In each simulation we calculate the intra-sector correlations with formula (24) and choose 5,000 credits. These portfolios contain a relatively low amount of name concentration. Instead we focus on sector concentration. The reason is that the identical methodology for measuring name concentrations, the granularity adjustment, can be used within both approaches. Thus, we prefer to avoid name concentrations to be able to separately analyze the effect of sector concentrations.

The results of our analyses can be found in Table 6.

- Table 6 about here -

Again, the outcomes of the Pykhtin model are a very good approximation of the "true" result from the Monte-Carlo Simulations. Independent from the chosen simulating setting the estimation error is very low. On the basis of our results the performance of the Cespedes model cannot be clearly assessed. In two of the simulation studies we obtain good estimations that are even better than the results of the Pykhtin model. The results of simulation I and III are systematically overestimated. Interestingly, the Cespedes model performs much better when the PDs are heterogeneous.

5 Conclusion

In this paper we proposed a methodology to perform multi-factor models that are able to measure concentration risk in credit portfolios in terms of economic capital but still deliver results that are consistent with Basel II. Furthermore we applied this to different multi-factor approaches and compared their performance. It could be shown that it is possible to achieve good approximations in reasonable time when the approaches are adjusted in the proposed way.

We showed that it is problematic to use the Value at Risk if there is concentration risk in the portfolio what is often disregarded in the literature. Instead it is advisable to use a coherent risk measure like the Expected Shortfall. As the ES is by definition higher than the VaR, we perform a mapping procedure that determines the confidence level that should be used to get reasonable values. We find that a confidence level of $z = 99.72\%$ should be used for the ES. This assures to get results for the economic capital that are consistent to Basel II and simultaneously avoids the problems of the VaR when leaving the ASRF framework.

Furthermore, we chose the input parameters, especially the inter- and intra-sector correlations, in a way that the results are comparable with the regulatory pillar 1 capital. Thus, we do not follow some approaches that assume a pure diversification effect compared with the Basel formula. Instead, we relate the results to a well-diversified portfolio as assumed when calibrating the Basel formula and determine a function for the implied intra-sector correlation. Hence, it is possible to directly consider the extent of credit risk concentrations in the assessment of capital adequacy under Pillar 2. Using these modifications, we performed an extensive numerical study as in Cespedes et al. (2006) to get a closed form approximation formula. In addition we suggest building PD- and EAD-classes to compute the Pykhtin formula in reasonable time. Without this the model needs more computation time than Monte-Carlo-Simulations for large portfolios so that the approximation formula would not be helpful.

Having assured a Basel consistent capital requirement, we analyzed the impact of credit concentration risk and carried out a simulation study to compare the performance of the (modified) models from Cespedes et al. (2006) and Pykhtin (2004). We find that the Pykhtin model leads to very good results for homogeneous as well as heterogeneous PDs and EADs. The results of the Cespedes model have a varying accuracy. Interestingly, the approach works significantly better for heterogeneous PDs, In general, both models can be used for approximating the economic capital in a multi-factor setting when adjusted in the proposed way. The Pykhtin approach has some advantages in the accuracy of the results whereas the approach of Cespedes et al. (2006) has the advantage that it is allows for ad-hoc analyses including sensitivity analyses when the non-recurring extensive numerical work is progressed.

In further analyses it would be interesting to adjust the approach of Cespedes et al. (2006) to a specific portfolio. Under the (plausible) assumption that a bank's portfolio will only be faced to minor changes for a finite period, it should be possible to get a higher accuracy for this bandwidth of scenarios. Moreover, it would be helpful to know how much numerical work is necessary when the parameters are highly restricted to these realistic cases to achieve stable results because the extensive computation time is still a major challenge.[33]

[33] In a Matlab environment the computation took about one month on 3 PCs each with 3 GHz CPUs.

Appendix A

To relate $\tilde{\tilde{L}}$ to L the systematic factor $\tilde{\tilde{x}}$ is defined as

$$\tilde{\tilde{x}} = \sum_{k=1}^{K} b_k \cdot \tilde{z}_k, \qquad \text{where} \ \sum_{k=1}^{K} b_k^2 = 1. \tag{42}$$

On condition that $\tilde{\tilde{L}} = E\left[\tilde{L} \mid \tilde{\tilde{x}}\right]$ Pykhtin (2004) shows that c_i can be calculated as

$$c_i = \rho_{\text{Intra},i} \cdot \overline{\rho}_i = \rho_{\text{Intra},i} \cdot \sum_{k=1}^{K} \alpha_{i,k} \cdot b_k, \tag{43}$$

where $\overline{\rho}_i$ is the correlation between $\tilde{\tilde{x}}$ and $\tilde{x}$.

Since there is no unique method to determine the coefficients $\{b_k\}$, we use the approach presented by Pykhtin (2004). Via maximization of $\overline{\rho}_i$ solutions of $\{b_k\}$ are given as

$$b_k = \sum_{i=1}^{n} \frac{d_i \cdot \alpha_{ik}}{\lambda}, \tag{44}$$

where λ is the Lagrange multiplier so that $\{b_k\}$ satisfy $\sum_{k=1}^{K} b_k^2 = 1$. The weighting factor d_i is defined as

$$d_i = w_i \cdot LGD_i \cdot N\left[\frac{N^{-1}(PD_i) + \sqrt{\rho_{\text{Intra},i}} \cdot N^{-1}(q)}{\sqrt{1-\rho_{\text{Intra},i}}}\right]. \tag{45}$$

Appendix B

The derivatives of (33) are calculated by

$$l'(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i'(\tilde{\tilde{x}}) \tag{46}$$

and

$$l''(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i''(\tilde{\tilde{x}}). \tag{47}$$

The derivatives $p_i'(\tilde{\tilde{x}})$ and $p_i''(\tilde{\tilde{x}})$ of the conditional default probability are calculated by differentiation of equation (6) as

$$p_i'(\tilde{\tilde{x}}) = -\frac{\sqrt{c_i}}{\sqrt{1-c_i}} \cdot n\left[\frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i}}\right] \tag{48}$$

and

$$p_i''(\tilde{\tilde{x}}) = -\frac{c_i}{1-c_i} \cdot \frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i}} \cdot n\left[\frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i}}\right]. \tag{49}$$

Since $\tilde{\tilde{L}}$ is deterministic for given $\tilde{\tilde{x}}$, $v(\tilde{\tilde{x}})$ equals the conditional variance of $\tilde{L}$, this means $v(\tilde{\tilde{x}}) = var(\tilde{L} - \tilde{\tilde{L}} \mid \tilde{\tilde{x}}) = var(\tilde{L} \mid \tilde{\tilde{x}})$. To calculate $v(\tilde{\tilde{x}})$ the conditional variance can be decomposed as the sum of systematic and idiosyncratic parts:

$$v(\tilde{\tilde{x}}) = \underbrace{var[E(L \mid \{\tilde{z}_k\}) \mid \tilde{\tilde{x}}]}_{v_\infty(\tilde{\tilde{x}})} + \underbrace{E[var(L \mid \{\tilde{z}_k\}) \mid \tilde{\tilde{x}}]}_{v_{GA}(\tilde{\tilde{x}})}. \tag{50}$$

The first summand of Formula (50) $v_\infty(\tilde{\tilde{x}})$ can be calculated as

$$\begin{aligned} v_\infty(\tilde{\tilde{x}}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot \\ \left[N_2\left(N^{-1}\left[p_i(\tilde{\tilde{x}})\right], N^{-1}\left[p_j(\tilde{\tilde{x}})\right], \rho_{ij}^{\tilde{\tilde{x}}}\right) - p_i(\tilde{\tilde{x}}) \cdot p_j(\tilde{\tilde{x}})\right], \end{aligned} \tag{51}$$

where $\rho_{ij}^{\tilde{\tilde{x}}}$ describes the conditional asset correlation

$$\sqrt{\rho_{ij}^{\tilde{\tilde{x}}}} = \frac{\rho_{Intra,i} \cdot \rho_{Intra,j} \cdot \sum_{k=1}^{K} \alpha_{ik} \cdot \alpha_{jk} - \sqrt{c_i \cdot c_j}}{\sqrt{(1-c_i) \cdot (1-c_j)}}. \tag{52}$$

The first derivative of $v_\infty(\tilde{\tilde{x}})$ is given by:

$$\begin{aligned} v_\infty'(\tilde{\tilde{x}}) = 2 \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot p_i'(\tilde{\tilde{x}}) \\ \cdot \left[N\left(\frac{N^{-1}\left[p_j(\tilde{\tilde{x}})\right] - \sqrt{\rho_{ij}^{\tilde{\tilde{x}}}} \cdot N^{-1} \cdot \left[p_i(\tilde{\tilde{x}})\right]}{\sqrt{1-\rho_{ij}^{\tilde{\tilde{x}}}}}\right) - p_j(\tilde{\tilde{x}})\right]. \end{aligned} \tag{53}$$

The second summand of Formula (50) $v_{GA}(\tilde{\tilde{x}})$ and its derivative $v_{GA}'(\tilde{\tilde{x}})$ are

$$v_{GA}(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i^2 \cdot \left(LGD_i^2\left[p_i(\tilde{\tilde{x}}) - N_2\left(N^{-1}\left[p_i(\tilde{\tilde{x}})\right], N^{-1}\left[p_j(\tilde{\tilde{x}})\right], \sqrt{\rho_{ii}^{\tilde{\tilde{x}}}}\right)\right]\right) \tag{54}$$

and

$$v_{GA}'(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i^2 \cdot p_i'(\tilde{\tilde{x}}) \cdot \left(LGD_i^2\left[1 - 2 \cdot N\left(\frac{N^{-1}\left[p_i(\tilde{\tilde{x}})\right] - \sqrt{\rho_{ii}^{\tilde{\tilde{x}}}} \cdot N^{-1}\left[p_i(\tilde{\tilde{x}})\right]}{\sqrt{1-\rho_{ii}^{\tilde{\tilde{x}}}}}\right)\right]\right). \tag{55}$$

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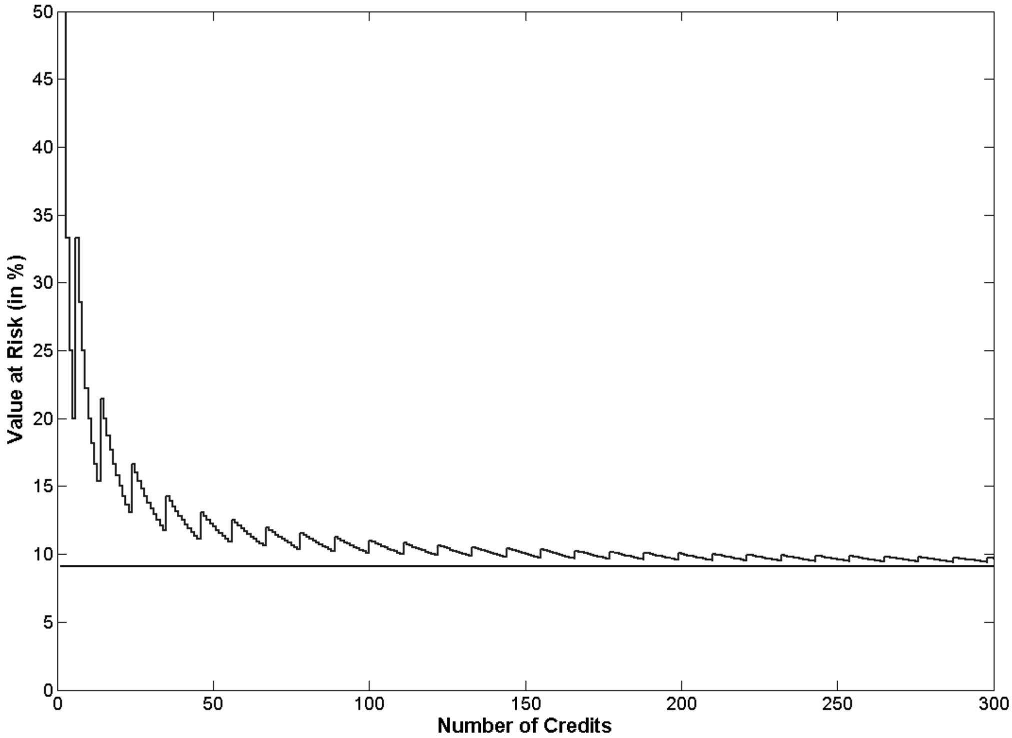

FIGURE 1: Value at Risk in the ASRF and the Vasicek model

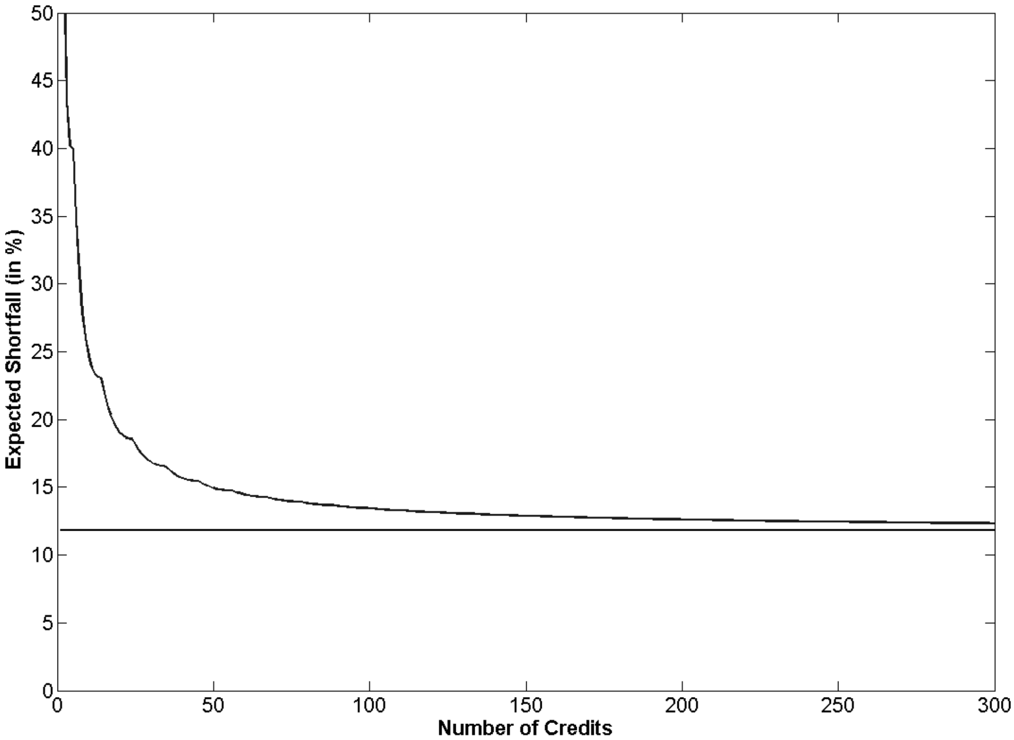

FIGURE 2: Expected Shortfall in the ASRF and the Vasicek model

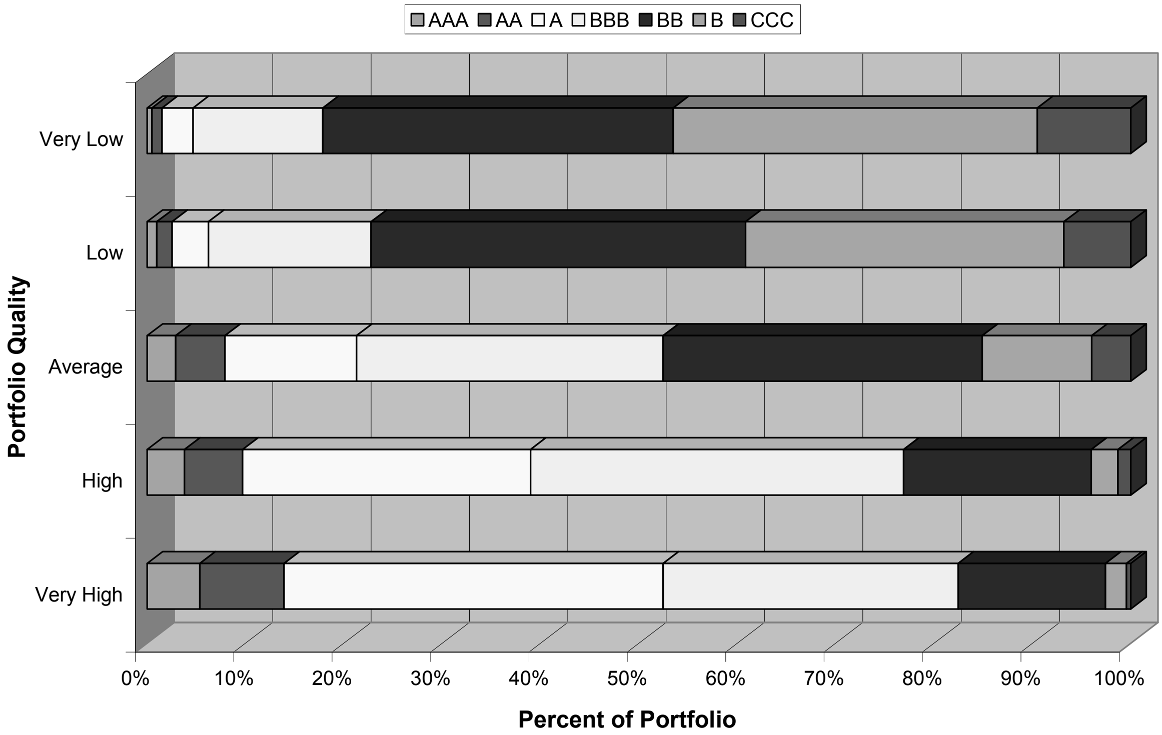

FIGURE 3: Portfolio quality distributions

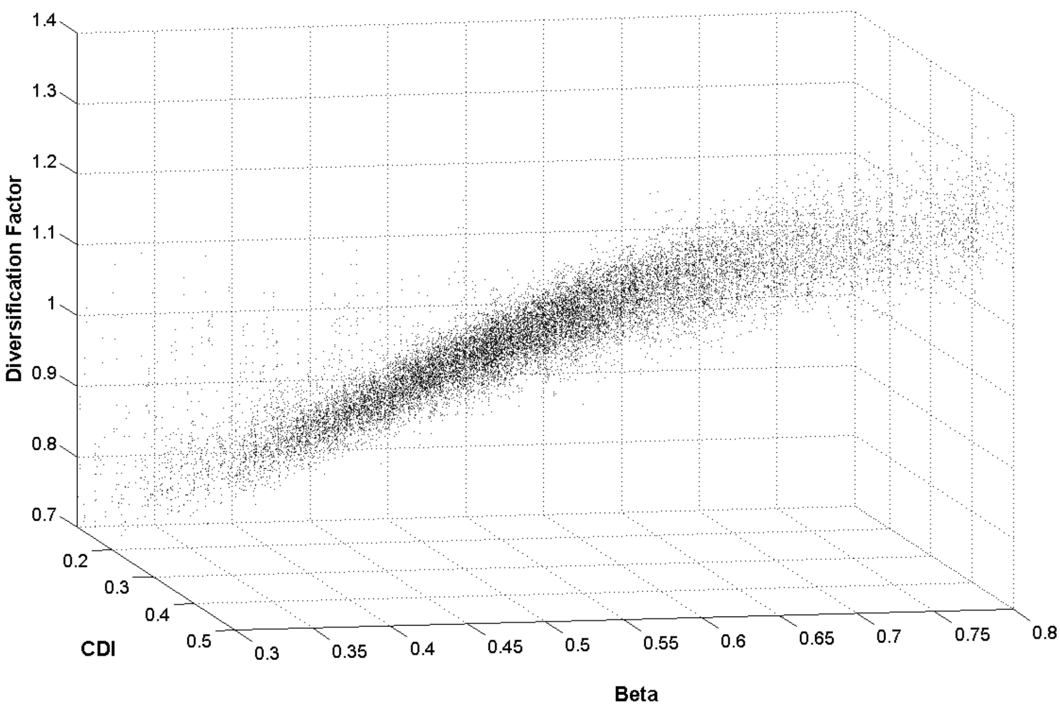

FIGURE 4: Diversification Factor of 22.000 simulations

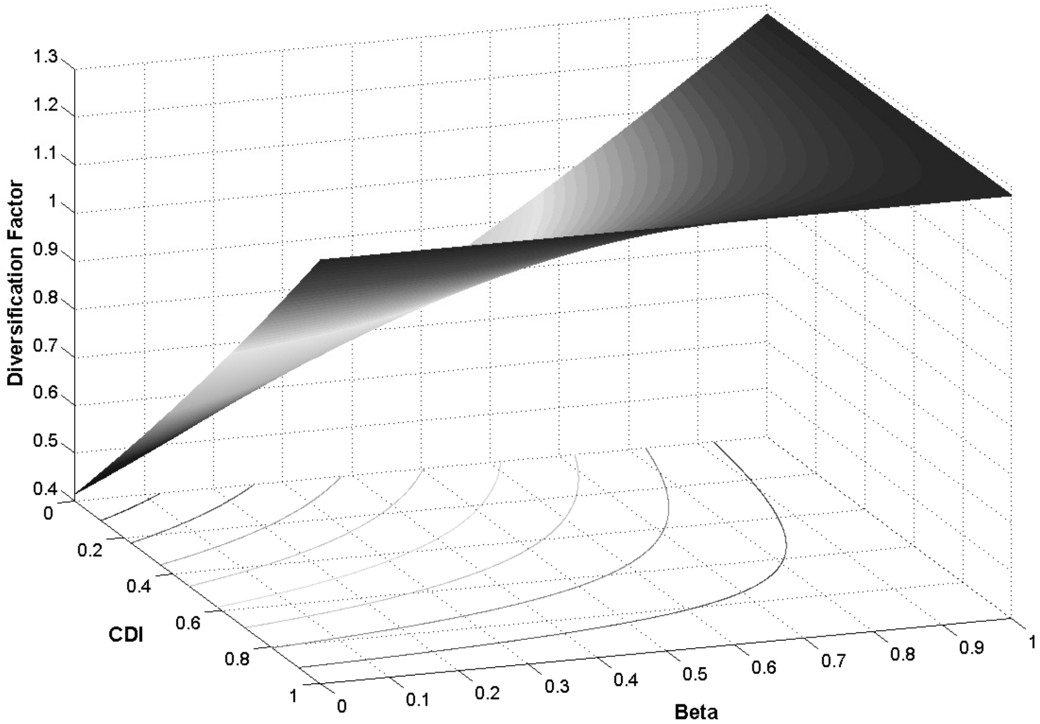

FIGURE 5: Surface plot of the DF-function

TABLE 1: Confidence level for the ES so that the ES is matched with the $VaR_{99.9\%}$ for portfolios of different quality

Portfolio Type / Quality	$VaR_{99,9\%}$ & ES_z	Confidence Level z (ES)
(I) AAA only	0.57%	99.672%
(II) Very High	6,12%	99.709%
(III) High	7.59%	99.711%
(IV) Average	12.94%	99.719%
(V) Low	20.89%	99.726%
(VI) Very Low	23.30%	99.727%
(VII) CCC only	57.00%	99.741%

TABLE 2: Inter-sector correlation structure based on MSCI industry indices

Sector	A	B	C1	C2	C3	D	E	F	H	I	J
A: Energy	100	50	42	34	45	46	57	34	10	31	69
B: Meterials		100	87	61	75	84	62	30	56	73	66
C1: Capital Goods			100	67	83	92	65	32	69	82	66
C2: Comm. svs & Supplies				100	58	68	40	8	50	60	37
C3: Transportation					100	83	68	27	58	77	67
D: Consumer Discretionary						100	76	21	69	81	66
E: Consumer Staples							100	33	46	56	66
F: Health Care								100	15	24	46
H: Information Technology									100	75	42
I: Telecommunication Services										100	62
J: Utilities											100

TABLE 3: Overall sector composition of the German banking system

Sector	Exposure Weight
A: Energy	0.18%
B: Meterials	6,01%
C1: Capital Goods	11,53%
C2: Comm. svs & Supplies	33,69%
C3: Transportation	7,14%
D: Consumer Discretionary	14,97%
E: Consumer Staples	6,48%
F: Health Care	9,09%
H: Information Technology	3,20%
I: Telecommunication Services	1,04%
J: Utilities	6,67%

TABLE 4: Implicit intra-sector correlations for different portfolio quality

Portfolio Type / Quality	Implicit Intra-Sector Correlation
(I) Very High	30%
(II) High	28%
(III) Average	25%
(IV) Low	23%
(V) Very Low	21%

TABLE 5: Comparison of the models for 5 different portfolios

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	5,83%	7,13%	12,48%	20,52%	23,15%
Pykhtin	ES	5,79%	7,19%	12,49%	20,54%	23,02%
	Absolute Error	-0,0004	0,0006	0,0001	0,0002	-0,0013
	Relative Error	-0,17%	0,89%	0,06%	0,11%	-0,56%
Cespedes	ES	5,70%	7,05%	11,91%	19,15%	21,36%
	Absolute Error	-0,0012	-0,0008	-0,0057	-0,0137	-0,0179
	Relative Error	-2,13%	-1,17%	-4,55%	-6,66%	-7,72%

TABLE 6: Comparison of the models for 5 high concentrated portfolios

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	7,12%	8,73%	14,95%	24,33%	26,77%
Basel II	ES	6,15%	7,54%	12,95%	20,81%	23,21%
	Absolute Error	0,0097	0,0119	0,02	0,0352	0,0356
	Relative Error	13,62%	13,63%	13,38%	14,47%	13,30%
Pykhtin	ES	7,11%	8,69%	14,90%	24,05%	26,78%
	Absolute Error	-0,0001	-0,0004	-0,0005	-0,0028	0,0001
	Relative Error	-0,14%	-0,46%	-0,69%	-0,01%	0,00%
Cespedes	ES	6,67%	8,18%	14,04%	22,58%	25,18%
	Absolute Error	-0,0045	-0,0055	-0,0091	-0,0175	-0,0159
	Relative Error	-6,32%	-6,30%	-6,09%	-7,19%	-5,93%

TABLE 7: Comparison of the models for 5 low concentrated portfolios

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	5,35%	6,58%	11,68%	19,59%	22,28%
Basel II	ES	6,15%	7,54%	12,95%	20,81%	23,21%
	Absolute Error	-0,0080	-0,0096	-0,0127	-0,0122	-0,0093
	Relative Error	-14,95%	-14,59%	-10,87%	-6,23%	-4,17%
Pykhtin	ES	5,29%	6,56%	11,76%	19,51%	21,92%
	Absolute Error	-0,0006	-0,0002	0,0008	-0,0008	-0,0036
	Relative Error	-1,12%	-0,3%	0,69%	-0,41%	-1,62%
Cespedes	ES	5,42%	6,64%	11,41%	18,35%	20,46%
	Absolute Error	0,0007	0,0006	-0,0027	-0,0124	-0,0182
	Relative Error	1,31%	0,91%	-2,31%	-6,33%	-8,17%

TABLE 8: Comparison of the models resulting from simulation studies with different parameter settings

	Pykhtin-Model		**Cespedes-Model**	
	Relative Error	Absolute Error	Relative Error	Absolute Error
Simulation I	0,78%	0,0015	7,71%	0,0237
Simulation II	2,89%	-0,0030	0,84%	0,0002
Simulation III	3,79%	-0,0004	7,71%	0,0210
Simulation IV	2,74%	-0,0029	2,38%	0,0018