Systemic Risk Contributions

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The paper puts forward a Merton-type multi-factor model for assessing banks' contributions to systemic risk. This model accounts for the major systemic risk drivers: the relative size of the banks, their individual default risk and the correlation of banks' assets as a proxy for interconnectedness. The systemic risk is measured in terms of the portfolio expected shortfall (ES). The individual systemic risk contributions are calculated based on the derivatives of the ES in order to ensure a full risk allocation among institutions. We compare the performance of a variance-reducing importance sampling algorithm and an analytical approximation. The latter is found to perform better for a higher granularity of exposures in the portfolio and higher asset correlations. We apply the model on a portfolio comprised of up to 86 major international banks for a 13-year time span. The results confirm that size is not a reliable proxy for the systemic importance of a financial institution in this framework. Furthermore, we propose using a time-varying confidence level in order to smooth ES over time. Finally, we put forward a proposal how our approach can be used in a macro-prudential regulation framework in order to implement a bank-specific systemic capital charge or a stabilisation fee.

Keywords:
Systemic Risk Contributions, Systemic Capital Charge, Expected Shortfall, Importance Sampling, Granularity Adjustment

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1. Introduction

The failure of certain financial institutions such as Lehman, Northern Rock or HRE during the crisis of 2007-2009 highlighted the significant adverse impact that a failure of a single firm can have on the financial system as a whole. Therefore, a firm-specific or microprudential approach is not sufficient to promote financial stability. Instead a careful assessment of a financial firm's *contribution* to the system-wide risk should be an important part of macro-prudential financial supervision.

The risk that refers to a financial system as a whole is often addressed as *systemic risk*. We define this term in the following as *the risk of a collapse of a financial system that entails a social welfare loss*. The task of addressing a systemic event and its negative externalities requires approaches for *measuring* the system-wide risk and decomposing it into the contributions of individual institutions. A macro-prudential approach would rely (i) on estimators of the *likelihood* of a systemic event, (ii) on measures of the *magnitude* of the potential loss or cost associated with the systemic event and (iii) on procedures for building up a sufficient capital basis in the financial system to bear (most of) this cost. As an important auxiliary condition, macro-prudential measures should contribute to reduce potential procyclical effects of regulation.

In this paper we focus on the subject of measuring and allocating systemic risk. For this purpose we propose a widely used credit risk model that treats the financial system of banks similar to a portfolio of assets. An important feature of this multi-factor model of institutions' asset returns is the allowance for interlinkages between banks through the asset correlations. Inferring these correlations appears to be especially attractive in light of difficulties measuring such an important driver of systemic risk as interconnectednes directly. Furthermore, the multi-factor correlation structure allows for a differentiated treatment of individual or certain groups of institutions. This reflects the fact that episodes of financial distress often arise from the exposure of groups of institutions to common risk factors.

In the portfolio context, a systemic event corresponds with the realisation of an extreme portfolio loss. The maximum systemic risk that is tolerated is defined as the *expected shortfall* (ES) at a confidence level q, i.e. the expected loss in the worst $100(1-q)\%$ of cases. The value of the confidence level q is set by the regulator depending on his risk tolerance. A macro-prudential tool based on the ES may generate procyclical effects because of cyclical risk components such as point-in-time default probabilities. Therefore, we consider also a time-variant confidence level $q(t)$ as a possible mitigant of procyclicality.

Two reasons motivate our selection of ES instead of the more commonly used *value at risk* (VaR). First, only ES is a coherent risk measure[1]. Second it is in line with the risk profile of a regulator who ultimately may have to bear losses that occur beyond the VaR-threshold.

A key advantage of modeling the financial system as a portfolio of assets is the availability of a rich literature on additive risk contributions for credit portfolios. We apply the *Euler allocation* or the marginal risk contribution based on the derivative of the risk measure with respect to the size of the individual position. This allocation principle has proved useful in portfolio-oriented risk management, particulary for the purpose of economic capital allocation, performance measurement,

[1] In the sense of Artzner et al. (1999). Frey and McNeil (2002) redefine the properties of a coherent risk measure in terms of credit risk.

portfolio optimisation or risk-sensitive pricing. According to Denault (2001) the Euler allocation can also be motivated by game theory as the partial derivatives correspond to the Aumann–Shapley value that lies in the core of a coalitional game. For more information on the concept of Euler contributions as well as related literature and economic motivation see Tasche (2008). For an axiomatic approach to coherent risk measures and capital allocation see also Kalkbrener (2005).

On the basis of the estimated system-wide tail risk and its decomposition into the individual institutions' contributions, a set of rule-based policy interventions, such as systemic capital charge or a stabilisation fee, can be designed.

In summary, we see the following three aspects as the main contribution of this paper:

1. We provide a full risk allocation across institutions based on the Euler allocation thereby adopting a methodology that is well-researched in the risk management literature for the assessment of systemic risk.

2. We compare the performance of an analytical approximation of the marginal risk contributions with a simulation-based importance sampling technique.

3. We use equity market information in order to gauge the market participants' collective evaluation of the otherwise difficult to quantify interlinkages that drive systemic risk.

4. We propose and empirically explore a time-varying confidence level of the ES as a method to mitigate procyclical effects of capital charges for systemic risk.

The remainder of the paper is organised as follows. Section 2 provides a brief review of selected literature. Section 3 presents the model set-up. Sections 4 and 5 give a detailed account of the estimation of the system-wide tail risk as well as tail risk contributions by means of an IS simulation and an analytical solution respectively. Section 6 reports the results of a simulation study carried out on a sample of large, internationally operating banks. In section 7 the risk drivers' impact on the systemic risk and on the institutions' systemic importance is analysed. Possible policy implications of the proposed methodology are presented in section 8. Finally, we summarize and draw conclusions in section 9.

2. Related literature

Many methods to assess the likelihood of a systemic event and to measure and allocate systemic risk have been discussed in the related literature. The IMF's Global Financial Stability report (IMF, 2009, pp 73-149) reviews the most recent approaches for detecting the tail risk of a financial system and assessing systemic risk by examining direct and indirect financial sector interlinkages.[2] De Nicolo and Kwast (2002) argue that the information contained in banks' equity returns can be used to measure the total (direct and indirect) dependence since stock prices reflect market participants' collective evaluation of the future prospects of the firm, including the total impact of its interactions

[2] De Nicolo and Kwast (2002) explain that direct interdependencies arise from inter-firm on and off-balance-sheet exposures: exposures arising from inter-bank loans, derivative contracts and repurchase agreements, from payment and settlement relationships. Indirect interdependencies arise from exposures to the same or similar assets through the loan participation market, loan concentrations to the same industry or otherwise highly correlated portfolios.

with other institutions. Therefore, equity returns and other market data are widely used to measure the fragility of financial institutions at individual and aggregate levels.

In the work conducted by Bartram et al. (2007), the default probabilities for a large sample of international banks are estimated from an observed time series of equity prices as well as from equity option prices on the assumptions of Merton's structural model (Merton, 1974). The banks under examination are divided into two groups: those which are immediately exposed to the adverse shocks in the forefront of a crises and those which are unexposed. Increasing default probabilities for the banks exposed to the crisis signals a systemic event. Increasing default probabilities for the unexposed banks instead captures the propagation of financial shocks throughout the banking system. Alternatively, Huang et al. (2009) deduce risk neutral default probabilities for major banks from their CDS spreads and asset return correlation from the co-movement of equity returns. Using these key parameters as input in a portfolio credit risk model, the authors suggest computing an indicator of systemic risk, namely the price of insurance against large default losses in the banking sector. The banking sector is represented by a hypothetical portfolio that consists of debt instruments issued by a pre-selected group of banks. The risk-neutral probability distribution of portfolio credit losses is then calculated by using Monte Carlo (MC) simulations. The theoretical insurance premium equals the risk-neutral expectation of portfolio credit losses given that the losses exceed some minimum share of the sector's total liabilities. The insurance premium rises when either the aggregated failure rate or the asset correlation or both increase.

Another application of the credit portfolio approach based on market data can be found in Segaviano and Goodhart (2009). The authors suggest, first, estimating the probabilities of distress of individual banks. This can be done on the basis of a structural model or by using CDS spreads or out-of-the-money option prices. The authors recommend using the nonparametric consistent information multivariate density optimising methodology in order to obtain the joint multivariate density of the banks' asset value movements. The joint density captures not only the distress correlation but also a possible tail dependence, thereby providing detailed information on the probability of the joint distress of two or more banks in the system. Based on this information, several indicators of banking stability can be constructed: (i) the joint probability of distress of all banks in the portfolio; (ii) a banking stability index that reflects the expected number of banks becoming distressed given at least one bank has become distressed; (iii) the conditional probabilities of distress for individual banks or specific groups of banks.

Lehar (2005) proposes a number of suitable methods for monitoring systemic risk, identifying systemically important banks and computing a bank-specific deposit insurance premium which takes into account systemic risk. Applying the maximum likelihood estimator developed by Duan (1994, 2000), the author extracts the market value of banks' assets from a time series of equity prices and book values of liabilities for a sample of 149 large banks. Then, in order to determine the joint dynamics of banks' asset portfolios, he estimates the asset correlations by means of an exponentially weighted moving-average model. By virtue of the joint probability distribution of banks' assets, he specifies the following indicators of systemic risk: (i) an asset-value-related systemic risk index by computing the probability that a group of banks with a total amount of assets greater than a certain fraction of all banks' assets goes bankrupt within a short period of time; (ii) a number-of-defaults-related systemic risk index by computing the probability that a certain number of banks go bankrupt within a short period of time; (iii) the value of a hypothetical deposit insurance (i.e. the value of a put option on a bank's assets under the assumption that all debt is insured), its volatility as well as the individual volatility contributions.

While most of the methods described above focus on the of systemic risk, Adrian and Brunnermeier (2009) suggest an approach for measuring the *contributions* that individual banks make to systemic risk. For this purpose the authors make use of the quantile regression technique. The general idea is, firstly, to choose an appropriate risk driver on the level of individual institutions, like returns on the market value of a bank's assets, and secondly, to run a q-quantile regression of this risk driver on a set of lagged state variables. The state variables are thereby supposed to capture the time variation in the conditional mean and the conditional volatility of the asset value returns. The predicted value from this regression indicates the value at risk for the financial institution under consideration, VaR_q^i. The system-wide value at risk, VaR_q^s, can be similarly obtained using a weighted sum of individual risk drivers at each point of time as the system-wide risk driver. By running a q-quantile regression of the system-wide risk driver on the lagged state variables augmented by the risk driver of one particular institution i, one obtains parameter estimates which can be subsequently used to determine the so-called conditional value at risk, CoVaR. $CoVaR_q^{s|i}$ denotes the value at risk of the financial system conditional on the institution i being in distress. It can be calculated as the predicted value from the last-mentioned regression conditioned on the risk driver of the institution i being equal to VaR_q^i. Then, the risk contribution of a particular institution is defined as the difference $\Delta CoVaR_q^i = CoVaR_q^{s|i} - VaR_q^s$. Eventually, the authors suggest predicting individual risk contributions $\Delta CoVaR_q^i$ on the basis of certain firm-specific characteristics like size, leverage and maturity mismatch.

As described in the IMF's Global Financial Stability Report (IMF, 2009, pp 86-90), the methodology suggested by Adrian and Brunnermeier (2009) can also be used to model the so-called conditional risk (CoRisk). The CoRisk measures the extent to which a default of one institution is likely to trigger other defaults in a financial system. Here, CDS spreads instead of asset returns act as risk drivers. With respect to CoVaR or CoRisk, it is worth noting that this kind of risk measure does not in general satisfy the full allocation principle, as the individual risk contributions do not necessarily add up to the system-wide risk.

In contrast, Tarashev et al. (2010) propose a method for decomposing a given measure of systemic risk into *additive* individual contributions. The authors adopt an allocation methodology from game theory called Shapley value. The role of a risk measure could, for instance, be taken on by the VaR of a portfolio comprising relevant financial institutions or, alternatively, by the ES. The above-mentioned portfolio represents the whole financial system. It incurs losses only if one or several institutions default, whereby the definition of a default is in line with Merton-type credit risk models. The system suffers losses equal to a pre-specified fraction of the defaulted firms' liabilities. Default events in the financial sector are positively correlated by assumption of a one-factor Vasicek model[3] for asset returns with a fixed, common correlation coefficient. Individual probabilities of default, (a fixed fraction of) the book value of liabilities and the chosen asset correlation coefficient entirely determine the probability distribution of portfolio losses and allow for the estimation of portfolio

[3]See Vasicek (1987).

risk, as measured by VaR or ES. Then, the risk at the portfolio level is attributed to the individual institutions by means of the Shapley value methodology. Thereby, for each specific institution, its contribution to the risk of all possible subportfolios in which this institution is present have to be computed. The contribution, an institution i makes to the risk of a subportfolio, is measured by the difference between the risk of this particular subportfolio including the institution i and its risk excluding the institution i. An institution's contribution to the systemic risk, or its Shapley value, is then the average of its contributions to all possible subportfolios. The authors suggest to use the Shapley value as a measure of an institution's systemic importance on whose basis macro-prudential policy interventions may be adjusted. Unfortunately, due to the rapidly increasing computational complexity, the Shapley value methodology can only be applied to very small portfolios or portfolios consisting of few homogeneous subportfolios.

The approaches described above only make use of publicly available data. Other researchers rely on detailed, proprietary information submitted to national regulators. Due to the fact that the focus of this paper is on the application of the credit portfolio methodology using market and balance sheet data, we refer to the IMF's GFSR (IMF, 2009) as well as the references therein for more information on network analysis and domino effect studies. Moreover, De Bandt and Hartmann (2000) provide a comprehensive survey on the theoretical and empirical literature on contagion in banking and financial markets as well as in payment and settlement systems. See also Nier et al. (2007) for further useful references. An example of an integrated systemic risk framework which combines standard techniques from market and credit risk management with a network model of a banking system is the OeNB's Systemic Risk Monitor, see Boss et al. (2006).

3. Model set-up

In the spirit of the asset value model that is widely used in risk management, we model the universe of n financial institutions by means of a portfolio whose loss distribution describes the risk of the entire financial system. Losses can only be induced by a distress of one or more institutions included in the portfolio. For the $i-$th institution, the exposure at distress, EAD_i, is defined as the book value of the institution's *liabilities* that are defined in this paper in nominal terms and after deducting capital. The relative exposure weight $w_i = EAD_i / \sum_{i=1}^{n} EAD_i$. The loss given distress, LGD_i, represents a fraction of the total liabilities which specifies the potential costs of the resolution or a bail-out of the distressed financial institution. An event of distress occurs at a predefined time horizon with the unconditional *distress probability* p_i. The event of distress is captured by the Bernoulli random variable $D_i \sim Be(p_i)$. In the spirit of the structural credit risk framework, we define distress as an event when the asset return of a financial institution hits or falls below its default threshold at a pre-specified time horizon. The default threshold specifies the point where the institution has to either enter resolution or be bailed out.

To complete the asset value model, we further assume that the standardised asset returns $\{X_i\}_{i=1,\dots,n}$ are multivariate normally distributed with a full-rank correlation matrix. To explain where the linear dependence results from, we decompose $\{X_i\}$ into a systematic and an idiosyncratic component by means of a multi-factor model. Following Pykhtin (2004) we assume that the asset return of a finanicial institution i depends on a composite risk factor Y_i which is a convex combination of a set of systematic risk factors $\{Z_k\}_{k=1,...,m}$ with $m \ll n$. The systematic factors $\{Z_k\}$ are assumed to be independently standard normally distributed. The unsystematic part of the asset return variation is addressed by an independent idiosyncratic shock ϵ_i also being standard normal.

The model framework for the risk drivers $\{X_i\}_{i=1,...,n}$, the distress indicators $\{D_i\}_{i=1,...,n}$ and our target variable – the portfolio loss rate PL – can now be formally summarised as follows:

$$X_i = a_i Y_i + \sqrt{1 - a_i^2}\, \epsilon_i, \qquad a_i \in (0, 1) \tag{3.1}$$

$$Y_i = \sum_{k=1}^{m} \alpha_{ik} Z_k, \qquad \sum_{k=1}^{m} \alpha_{ik}^2 = 1 \tag{3.2}$$

$$Z_k, \epsilon_i \overset{iid}{\sim} N(0, 1) \text{ for all } k = 1, \ldots, m \text{ and } i = 1, \ldots, n$$

$$D_i = 1 \Leftrightarrow X_i \in \big(-\infty, \Phi^{-1}(p_i)\big] \tag{3.3}$$

$$PL = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i. \tag{3.4}$$

In the expressions above, the factor loading a_i specifies the sensitivity of the particular institution to the systematic risk, and the asset correlation between distinct institutions i and j is given by $\rho_{i,j} = a_i a_j \rho_{Y_i,Y_j}$, where $\rho_{Y_i,Y_j} := \sum_{k=1}^{m} \alpha_{ik} \alpha_{jk}$ denotes the correlation between the two composite factors.

As already mentioned in section 1, we are primarily interested in the ES at a confidence level q as a coherent measure of the portfolio tail risk. But for the sake of completeness, we also consider VaR as a widely used alternative risk measure. Let us denote the (discrete) cumulative distribution function of the portfolio loss rate by $F_{PL}(\cdot)$ and its quantile function by $F_{PL}^{-1}(\cdot)$. Then, VaR and ES can be defined as follows:

$$VaR_q(PL) := F_{PL}^{-1}(q) = \inf\big\{x \in [0, 1] \ : \ F_{PL}(x) \geqslant q\big\} \tag{3.5}$$

$$ES_q(PL) := \frac{1}{1 - q} \int_q^1 VaR_t(PL)dt. \tag{3.6}$$

As an alternative to (3.6), Kalkbrener (2005, p 434) considers an expression which turns out to be more instructive especially for simulation purposes later in this paper. As shown in Acerbi and Tasche (2002), if the distribution of portfolio loss were continuous, (3.6) would coincide with the tail conditional expectation (TCE) defined as

$$TCE_q(PL) = E\big[PL \mid PL \geqslant VaR_q(PL)\big]. \tag{3.7}$$

For a discrete loss distribution, however, the expression above has to be augmented with a correction term which adjusts the TCE measure upwards if the probability of the portfolio losses at the point

$VaR_q(PL)$ does not coincide with q:

$$ES_q(PL) = E\big[PL \mid PL \geqslant VaR_q(PL)\big] \\ + \frac{1}{1-q} VaR_q(PL)\big[F_{PL}\big(VaR_q(PL)\big) - q\big]. \tag{3.8}$$

For both measures of the overall portfolio risk (VaR and ES), we also seek to compute individual risk contributions satisfying the full allocation property, i.e. totalling the system-wide amount of risk. For this purpose we use marginal contributions that measure the impact of a small, marginal change in the exposure of a bank to the total tail risk of its portfolio. In the following sections we discuss the issue of assessing portfolio risk and risk contributions first by simulation and afterwards by an analytical approximation.

4. Measuring and allocating systemic risk by simulation

In this section we develop a simulation algorithm for the estimation of the portfolio tail risk as well as for the risk contributions of individual financial institutions. Initially, we touch on the issue of rare-event simulation by means of importance sampling (IS) in subsection 4.1 following which we present the IS estimates for the tail risk and risk constributions in subsection 4.2.

4.1. Importance sampling

Although it appears straightforward to compute an estimate of the portfolio tail risk by simulation, the brute-force MC technique may fail for such rare events as $PL \geqslant VaR_q(PL)$. To clarify this point, let us consider the issue of estimating the small probability of a rare event which is, concerning simulation efficiency, equivalent to estimating VaR. That probability can be represented in terms of the expectation:

$$\Pr\{PL > x_q\} = E\left[\mathbb{1}_{(x_q,1]}(PL)\right], \tag{4.1}$$

where x_q is close to 1 so that $\Pr\{PL > x_q\}$ is close to zero and $\mathbb{1}_A(Y)$ denotes an indicator function:

$$\mathbb{1}_A(Y) = \begin{cases} 1 & \text{for } Y \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Since the law of large numbers states that the sample mean converges to the expected value, the MC estimator seems to be a natural choice in the case of (4.1) when simulating PL.

Assume now that the small probability in our example coincides with $1-q$. Being the sample mean, the MC estimator of $1-q$ is unbiased and normally distributed with variance $q(1-q)/s$, where s denotes the number of simulation runs. This means, for instance, that for the estimation of $1-q = 0.001$ with at most 5% relative error at the 95% confidence level, more than 1.5×10^6 simulation runs are necessary. Due to the fact that only a small fraction of replications ($100(1-q)\%$ on average) produces portfolio losses equal to or exceeding VaR_q, draws like $D_i = 1 \mid PL = VaR_q(PL)$ would be even rarer. Against this background the estimation of VaR contributions by MC would involve either unacceptable runtimes or immense estimation errors. As has already been pointed out by

Merino and Nyfeler (2004) and Glasserman (2006) among others, a similar problem arises when estimating ES contributions. In order to reduce estimation errors, a plain MC simulation algorithm has to be modified, increasing the frequency of rare events while ensuring the estimator remains unbiased.

A promising technique for simulation of rare events and, therefore, for estimation of the tail risk as well as the risk contributions is importance sampling. For the Gaussian conditional independence framework, Glasserman and Li (2005) have already developed an appropriate two-stage IS algorithm leading to an asymptotically efficient estimator for a small probability like (4.1). Moreover, Glasserman (2006) provides further results on the IS estimation of VaR, ES and corresponding tail risk contributions. In appendix A we describe in detail an adoption of the aforementioned IS methodology to the model presented in section 3. In subsequent appendix B we provide the IS simulation algorithm for the portfolio loss distribution. On the basis of that simulated distribution, tail risk measures and corresponding risk contributions can be estimated, as described in the following subsection.

4.2. Estimating tail risk and risk contributions

To estimate the VaR at a confidence level q, as defined in (3.5), the following expression can be used:

$$\widehat{VaR}_q(PL) = \inf\{x \in [0,1] \,:\, \hat{F}_{PL}(x) \geqslant q\}. \tag{4.2}$$

For the ES, according to (3.8), we obtain the estimator:

$$\widehat{ES}_q(PL) = \frac{\sum_{k=1}^{s} PL^k \, \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)}{\sum_{k=1}^{s} \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)} + \frac{1}{1-q}\widehat{VaR}_q(PL)\big[\hat{F}_{PL}\big(\widehat{VaR}_q(PL)\big) - q\big]. \tag{4.3}$$

Moreover, the results on the additive contributions associated with quantile-based risk measures conducted by Tasche (2000) give us an idea of suitable IS estimators for the tail risk contributions. The author has proven that under certain continuity conditions imposed on the joint probability distribution of the individual loss variables $L_i := w_i \cdot LGD_i \cdot D_i$, the marginal contributions derived via differentiation of VaR and TCE can be represented in terms of the conditional expectation:

$$w_i \frac{\partial}{\partial w_i} VaR_q(PL) = E\big[L_i \mid PL = VaR_q(PL)\big] \tag{4.4}$$

and

$$w_i \frac{\partial}{\partial w_i} TCE_q(PL) = E\big[L_i \mid PL \geqslant VaR_q(PL)\big]. \tag{4.5}$$

Obviously, the risk contributions given above fulfil the full allocation condition. Thus, additionally taking a discontinuity correction for the ES into account, we are able to define the following importance sample estimators for the additive tail risk contributions:

$$\widehat{VaR}_q(L_i \mid PL) = \frac{\sum_{k=1}^{s} w_i \cdot LGD_i \cdot D_i^k \, \mathbb{1}_{\{\widehat{VaR}_q(PL)\}}\big(PL^k\big)\, l\big(PL^k\big)}{\sum_{k=1}^{s} \mathbb{1}_{\{\widehat{VaR}_q(PL)\}}\big(PL^k\big)\, l\big(PL^k\big)} \tag{4.6}$$

and

$$\widehat{ES}_q(L_i \mid PL) = \frac{\sum_{k=1}^{s} w_i \cdot LGD_i \cdot D_i^k \, \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)}{\sum_{k=1}^{s} \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)}$$

$$+ \frac{1}{1-q}\widehat{VaR}_q(L_i \mid PL)\big[\hat{F}_{PL}\big(\widehat{VaR}_q(PL)\big) - q\big]. \tag{4.7}$$

Applying the IS technique outlined above, instead of a plain MC simulation, can lead to substantial variance reduction when estimating VaR, ES and ES contributions. Note, nevertheless, that the problem concerning an efficient estimation of VaR contributions persists, since events like $D_i \mid PL = VaR_q(PL)$ are still relatively rare.

5. Measuring and allocating systemic risk using an analytical approximation

Although the number of simulation runs can be reduced considerably by using IS, the need for an approximative analytical solution has been accentuated repeatedly in the related literature. For the special case of a single-risk factor model and an (asymptotically) infinitely fine-grained portfolio, there exists an analytical solution for portfolio VaR (and ES) as well as for the VaR (ES) contributions (see Gordy (2003)). In this asymptotical setting, the idiosyncratic risk is diversified away and the risk contributions are portfolio-invariant. In order to mitigate the underestimation of VaR in finite portfolios, the closed-form expressions for a granularity adjustment have been derived by Wilde (2001) and Martin and Wilde (2002). Based on their results, Emmer and Tasche (2003) have determined contributions to the adjusted approximate portfolio VaR. These contributions are portfolio-dependent due to the existence of an undiversified idiosyncratic risk. The adjustment technique for VaR and ES has been extended more recently by Pykhtin (2004), who presented an analytical method for an approximative calculation of portfolio VaR and ES in the case of a *multi*-factor Merton framework.

In this section we derive analytical formulae for Euler tail risk contributions as partial derivatives of the approximate portfolio VaR and ES. In subsection 5.1 we briefly outline the model modification which is needed to permit an analytical solution. In subsections 5.2 and 5.3 we describe, for the sake of completeness, Pykhtin's analytical solution for VaR and ES. In the remaining subsections 5.4 and 5.5 we eventually derive analytical expressions for VaR and ES contributions.

5.1. The modified model

For the model presented in section 3, Pykhtin (2004) proposes an approach for assessing the portfolio tail risk analytically. Based on the asymptotic solution for the limiting, infinitely fine-grained portfolio along with the granularity adjustment in a one-factor framework, he derived another adjustment term to account for the multi-factor dependence structure. The approach is based on a comparable one-factor model whose implied portfolio loss distribution is similar to the original one. For this purpose a new "effective" systematic factor $\bar{Y}$ is introduced:

$$\bar{Y} = \sum_{k=1}^{m} \beta_k Z_k, \qquad \sum_{k=1}^{m} \beta_k^2 = 1. \tag{5.1}$$

The systematic factor $\bar{Y}$ is now the same for all institutions in the portfolio. Therefore, the model (3.1) can be rewritten in the following way

$$X_i = b_i \bar{Y} + \sqrt{1 - b_i^2}\, \varepsilon_i \tag{5.2}$$

where $\{\varepsilon_i\}_{i=1,\ldots,n}$ are independent standard normal variables, $b_i \equiv a_i \sum_{k=1}^{m} \alpha_{ik}\beta_k$ are the new factor loadings, and $\sum_{k=1}^{m} \alpha_{ik}\beta_k$ represents the correlation between Y_i and $\bar{Y}$.

The optimal choice of the coefficients $\{\beta_k\}$ is not obvious. Pykhtin suggested maximising the correlation between Y_i and $\bar{Y}$:

$$\max_{\{\beta_k\}} \left\{ \sum_{i=1}^{n} c_i \sum_{k=1}^{m} \alpha_{ik}\beta_k \right\} \qquad \text{w.r.t.} \ \sum_{k=1}^{m} \beta_k^2 = 1, \tag{5.3}$$

$$c_i = w_i \cdot LGD_i \cdot \Phi\left(\frac{\Phi^{-1}(p_i) + a_i \Phi^{-1}(q)}{\sqrt{1 - a_i^2}} \right). \tag{5.4}$$

Thereby, differentiating the Lagrange function

$$L\big(\{\beta_k\}, \lambda\big) = \sum_{i=1}^{n} c_i \sum_{k=1}^{m} \alpha_{ik}\beta_k - \lambda \left(\sum_{k=1}^{m} \beta_k^2 - 1 \right)$$

and putting the partial derivatives to zero yields

$$\beta_k = \frac{1}{2\lambda} \sum_{i=1}^{n} c_i \alpha_{ik}, \quad k = 1, \ldots, m,$$

$$\lambda = \frac{1}{2} \sqrt{\sum_{k=1}^{m} \left(\sum_{i=1}^{n} c_i \alpha_{ik} \right)^2} = \frac{1}{2} \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \rho_{Y_i,Y_j}}.$$

In doing so, we can eliminate $\{\alpha_{ik}\}$ from the equation:

$$b_i = \frac{a_i}{2\lambda} \sum_{k=1}^{m} \alpha_{ik} \sum_{j=1}^{n} c_j \alpha_{jk} = \frac{a_i}{2\lambda} \sum_{j=1}^{n} c_j \rho_{Y_i,Y_j}. \tag{5.5}$$

The factor loadings $\{b_i\}$ are all we need to know about the model representation (5.2) in order to carry on with the calculation of VaR and ES.

5.2. Value at Risk

Representation (5.2) has just the form of a one-factor model. In the case of the limiting portfolio, provided that $\sum_{i=1}^{n} w_i^2 \to 0$ while $n \to \infty$, the portfolio loss rate in a one-factor model is a function of the systematic risk factor

$$PL^{\infty}(\bar{Y}) := E[PL \mid \bar{Y}] = E\left[\sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i \mid \bar{Y}\right] = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(\bar{Y}), \tag{5.6}$$

and the corresponding asymptotic solution for the portfolio VaR is well known from Gordy (2003):

$$VaR_q^{\bar{Y}}(PL^{\infty}) \equiv PL^{\infty}(y_q) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(y_q), \tag{5.7}$$

where $p_i(y)$ is the probability of distress conditional on $\bar{Y} = y$:

$$p_i(y) = \Phi\left(\frac{\Phi^{-1}(p_i) - b_i y}{\sqrt{1 - b_i^2}}\right) \tag{5.8}$$

and y_q is a realisation of $\bar{Y}$ associated with the $(1-q)$ quantile of its probability distribution: $y_q = \Phi^{-1}(1-q)$.

Applying the granularity adjustment technique results in the second order Taylor approximation of $VaR_q(PL)$[4] and can be written as:

$$VaR_q(PL) \approx VaR_q^{\bar{Y}}(PL^{\infty}) + \Delta VaR_q(PL) \tag{5.9}$$

with an adjustment term given by

$$\begin{aligned} &\Delta VaR_q(PL) := \\ &-\frac{1}{2\big(PL^{\infty}(y_q)\big)'}\left[\big(\mathrm{var}(PL \mid \bar{Y} = y_q)\big)' - \mathrm{var}(PL \mid \bar{Y} = y_q)\left(\frac{\big(PL^{\infty}(y_q)\big)''}{\big(PL^{\infty}(y_q)\big)'} + y_q\right)\right]. \end{aligned} \tag{5.10}$$

The derivatives of the limiting portfolio used above can be found in appendix C, expressions (C.1) to (C.4).

So far there has been nothing special concerning the representation of the one-factor model (5.2) in terms of a convex combination of $\{Z_k\}$, as given by (5.1). However, in order to obtain a formula for $\mathrm{var}(PL \mid \bar{Y} = y_q)$ we need to take into account that asset returns are actually not independent given a realisation of the effective risk factor $\bar{Y}$. It can be seen from the following representation:

$$X_i = b_i\bar{Y} + \sum_{k=1}^{m}(a_i\alpha_{i,k} - b_i\beta_k)Z_k + \sqrt{1 - a_i^2}\,\epsilon_i.$$

[4] See proposition 2.2 in Emmer and Tasche (2003).

In fact, the conditional asset correlation between two distinct institutions i and j is given by

$$\rho_{i,j}^{\bar{Y}} = \frac{\rho_{i,j} - b_i b_j}{\sqrt{1-b_i^2}\sqrt{1-b_j^2}}. \tag{5.11}$$

Although meaningless as a correlation coefficient, expression (5.11) has to be extended to cover the case $j = i$, i.e. $\rho_{i,i}^{\bar{Y}} = (r_i^2 - b_i^2)/(1 - b_i^2)$.

Asset returns are only independent conditional on the whole set of systematic factors $\{Z_k\}$. Thus, according to the law of total variance, we may decompose $\mathrm{var}(PL \mid \bar{Y} = y)$ to separate the variance of the limiting portfolio loss $\mathrm{var}^{\infty}(\cdot)$ from the effect of granularity $\mathrm{var}^{GA}(\cdot)$:

$$\begin{aligned}\mathrm{var}(PL \mid \bar{Y} = y) &= \mathrm{var}^{\infty}(PL \mid \bar{Y} = y) + \mathrm{var}^{GA}(PL \mid \bar{Y} = y)\\ &\equiv \mathrm{var}\big(E[PL \mid \{Z_k\}] \mid \bar{Y} = y\big) + E\big[\mathrm{var}(PL \mid \{Z_k\}) \mid \bar{Y} = y\big]. \end{aligned}\tag{5.12}$$

Thereby, $E[PL \mid \{Z_k\}]$ corresponds to the limiting portfolio loss in the multi-factor setting (see (3.1) and (3.2)) given by:

$$\begin{aligned} PL^{\infty}(\{Z_k\}) := E[PL \mid \{Z_k\}] &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(\{Z_k\})\\ &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \Phi\left(\frac{\Phi^{-1}(p_i) - a_i \sum_{k=1}^{m} \alpha_{ik} Z_k}{\sqrt{1-a_i^2}}\right). \end{aligned}$$

Taking into account the conditional correlation parameter specified in equation (5.11), the multi-factor adjustment terms for the limiting case and for the effect of granularity can be given by

$$\begin{aligned} &\mathrm{var}^{\infty}(PL \mid \bar{Y} = y) \equiv \mathrm{var}\Big(E[PL \mid \{Z_k\}] \mid \bar{Y} = y\Big)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot \mathrm{cov}\Big(p_i(\{Z_k\}), p_j(\{Z_k\}) \mid \bar{Y} = y\Big)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \Big[C^{Gauss}\Big(p_i(y), p_j(y); \rho_{i,j}^{\bar{Y}}\Big) - p_i(y)p_j(y)\Big], \end{aligned}\tag{5.13}$$

$$\begin{aligned} &\mathrm{var}^{GA}(PL \mid \bar{Y} = y) \equiv E\big[\mathrm{var}(PL \mid \{Z_k\}) \mid \bar{Y} = y\big]\\ &= \sum_{i=1}^{n} w_i^2 \cdot LGD_i^2 \cdot E\Big[\Big(p_i(\{Z_k\}) - p_i(\{Z_k\})p_i(\{Z_k\})\Big) \mid \bar{Y} = y\Big]\\ &= \sum_{i=1}^{n} w_i^2 \cdot LGD_i^2 \Big[p_i(y) - C^{Gauss}\Big(p_i(y), p_i(y); \rho_{i,i}^{\bar{Y}}\Big)\Big], \end{aligned}\tag{5.14}$$

respectively. In the equations above, $C^{Gauss}(\cdot, \cdot; \rho)$ denotes the bivariate Gauss copula with the correlation parameter ρ. It assigns the conditional probability of a simultaneous distress of institutions i and j (extended to include the case $j = i$).

Due to the variance decomposition (5.12), the multi-factor adjustment for the portfolio VaR in equation (5.10) can also be represented as a sum of two terms: one correcting the VaR of the limiting portfolio for the systematic effect in the multi-factor setting, and another, addressing the granularity. Then, expression (5.9) turns out to be:

$$VaR_q(PL) \approx VaR_q^{appox}(PL) := VaR_q^{\bar{Y}}(PL^{\infty}) + \Delta VaR_q^{\infty}(PL) + \Delta VaR_q^{GA}(PL). \tag{5.15}$$

5.3. Expected shortfall

Pykhtin derived an analytical approximation of the ES using the integral representation (3.6) by setting:

$$\begin{aligned} ES_q(PL) &\approx \frac{1}{1-q}\int_q^1 \big(VaR_t^{\bar{Y}}(PL^{\infty}) + \Delta VaR_t(PL)\big)dt \\ &= ES_q^{\bar{Y}}(PL^{\infty}) + \Delta ES_q(PL). \end{aligned} \tag{5.16}$$

The first term in equation (5.16) represents ES in the case of the limiting portfolio within the one-factor framework:

$$ES_q^{\bar{Y}}(PL^{\infty}) = \frac{1}{1-q}\sum_{i=1}^{n} w_i \cdot LGD_i \cdot C^{Gauss}\big(p_i, 1-q; b_i\big). \tag{5.17}$$

The second term is the multi-factor adjustment defined as a linear function of the conditional variance:

$$\Delta ES_q(PL) := -\frac{\phi(y_q)}{2(1-q)} \frac{\mathrm{var}(PL \mid \bar{Y} = y_q)}{\big(PL^{\infty}(y_q)\big)'}. \tag{5.18}$$

Due to the additivity of the variance components (see equations (5.12), (5.13) and (5.14)), $\Delta ES_q(PL)$ can also be represented as a sum of its systematic and idiosyncratic parts. Therefore, the analytical approximation of the ES can finally be written as:

$$ES_q(PL) \approx ES_q^{approx}(PL) := ES_q^{\bar{Y}}(PL^{\infty}) + \Delta ES_q^{\infty}(PL) + \Delta ES_q^{GA}(PL). \tag{5.19}$$

In the case of large portfolios the systematic parts of VaR and ES, i.e. $VaR_q^{\bar{Y}}(PL^{\infty})+\Delta VaR_q^{\infty}(PL)$ and $ES_q^{\bar{Y}}(PL^{\infty}) + \Delta ES_q^{\infty}(PL)$, provide a reasonable approximation of the portfolio risk while the idiosyncratic parts, i.e. $\Delta VaR_q^{GA}(PL)$ and $\Delta ES_q^{GA}(PL)$, vanish. However, due to the fact that the portfolio under consideration could be relatively small and perhaps dominated by a few large exposures, the granularity adjustment terms could be nearly as large as the systematic part.

5.4. VaR contributions

The analytical formulae (5.15) and (5.18) suggested by Pykhtin can now be used to derive the additive contributions associated with the approximative portfolio tail risk measures.

Under the assumptions of an infinitely fine-grained portfolio and only one systematic risk factor, the contribution of an institution i to the VaR of the limiting portfolio, as defined by equation (5.7), would be completely portfolio-invariant because of the following result:

$$\frac{\partial}{\partial w_i} VaR_q^{\bar{Y}}(PL^\infty) = LGD_i \cdot p_i(y_q). \tag{5.20}$$

In addition to the stand-alone marginal risk contribution, a portfolio-dependent contribution arises according to equation (5.10) by reason of the following multi-factor granularity adjustment:

$$\begin{aligned}
\frac{\partial}{\partial w_i}\Delta VaR_q(PL) = &\left\{2\Big[\big(PL^\infty(y_q)\big)'\Big]^2\right\}^{-1} \\
\times &\left\{-\frac{\partial}{\partial w_i}\big(\mathrm{var}(PL \mid \bar{Y} = y_q)\big)'\big(PL^\infty(y_q)\big)' + \big(\mathrm{var}(PL \mid \bar{Y} = y_q)\big)'\frac{\partial}{\partial w_i}\big(PL^\infty(y_q)\big)'\right. \\
&+ \left[\frac{\partial}{\partial w_i}\big(\mathrm{var}(PL \mid \bar{Y} = y_q)\big)\big(PL^\infty(y_q)\big)' - \mathrm{var}(PL \mid \bar{Y} = y_q)\frac{\partial}{\partial w_i}\big(PL^\infty(y_q)\big)'\right] \\
&\times \left.\left(\frac{\big(PL^\infty(y_q)\big)''}{\big(PL^\infty(y_q)\big)'} + y_q\right) + \mathrm{var}(PL \mid \bar{Y} = y_q)\big(PL^\infty(y_q)\big)'\frac{\partial}{\partial w_i}\left(\frac{\big(PL^\infty(y_q)\big)''}{\big(PL^\infty(y_q)\big)'}\right)\right\}.
\end{aligned} \tag{5.21}$$

Again, we can make use of the variance decomposition (5.12). The derivatives on the right-hand side of equation (5.21) are given by expressions (C.5) to (C.13) in appendix C.

In order to calculate an approximation of the VaR contribution of the ith bank as a percentage of its own exposure, we just need to add up (5.20) and (5.21):

$$\frac{\partial}{\partial w_i} VaR_q(PL) \approx \frac{\partial}{\partial w_i} VaR_q^{approx}(PL) := \frac{\partial}{\partial w_i} VaR_q^{\bar{Y}}(PL^\infty) + \frac{\partial}{\partial w_i}\Delta VaR_q(PL). \tag{5.22}$$

The approximative VaR contributions defined as

$$VaR_q^{appox}(w_i \mid PL) := w_i \frac{\partial}{\partial w_i} VaR_q^{approx}(PL), \tag{5.23}$$

satisfy the full allocation property:

$$VaR_q^{appox}(PL) = \sum_{i=1}^{n} VaR_q^{appox}(w_i \mid PL).$$

5.5. ES contributions

Similar to the previous results, risk contributions based on $ES_q^{\bar{Y}}(PL^\infty)$ in (5.17) would be portfolio-invariant:

$$\frac{\partial}{\partial w_i} ES_q^{\bar{Y}}(PL^\infty) = \frac{LGD_i}{1-q} C^{Gauss}\big(p_i, 1-q; b_i\big). \tag{5.24}$$

The multi-factor adjustment term corrects for the undiversified idiosyncratic risk and can be obtained by the partial differentiation of equation (5.18) with respect to the exposure weights:

$$\frac{\partial}{\partial w_i}\Delta ES_q(PL) = -\frac{\phi(y_q)}{(1-q)}\left[2\big(PL^{\infty}(y_q)\big)'\right]^{-1}$$
$$\times\left[\frac{\partial}{\partial w_i}\big(\mathrm{var}(PL \mid \bar{Y}=y_q)\big)\big(PL^{\infty}(y_q)\big)' - \mathrm{var}(PL \mid \bar{Y}=y_q)\frac{\partial}{\partial w_i}\big(PL^{\infty}(y_q)\big)'\right]. \tag{5.25}$$

So we can approximate the ES contribution as a percentage of institution i's exposure

$$\frac{\partial}{\partial w_i}ES_q(PL) \approx \frac{\partial}{\partial w_i}ES_q^{approx}(PL) := \frac{\partial}{\partial w_i}ES_q^{\bar{Y}}(PL^{\infty}) + \frac{\partial}{\partial w_i}\Delta ES_q(PL) \tag{5.26}$$

or alternatively as a percentage of the total portfolio exposure

$$ES_q^{appox}(w_i \mid PL) := w_i\frac{\partial}{\partial w_i}ES_q^{approx}(PL). \tag{5.27}$$

The approximative ES contributions in (5.27) satisfy the full allocation property:

$$ES_q^{appox}(PL) = \sum_{i=1}^{n} ES_q^{appox}(w_i \mid PL).$$

6. A numerical example

In this section we conduct a numerical study for illustrative purposes. In subsection 6.1 the dataset and the input parameters used in the numerical analysis are described whereas subsection 6.2 contains the main results.

6.1. The data and the input parameters

In order to illustrate the implementation of the measurement approaches considered in the previous sections, we conduct a numerical study on a dataset containing a varying number of world's largest banks over a time span from January 1997 to January 2010. The number of banks varies between 54 and 86 depending on IPOs, mergers and data availability. The expected default frequency (EDF) – an estimate of the actual one-year probability of default – has been obtained from Moody's KMV CreditEdge on a monthly basis. The EDFs range from 0.01% to 19% with the median 0.07% before September 2008 and the median 0.32% afterwards. We set the EAD equal to the book value of the bank's liabilities, also obtained from CreditEdge on a yearly basis. We transform the yearly observations into monthly data by linear interpolation. We assume the LGD rate of 100% for all observations. The latter assumption is obviously very conservative, since neither the bail-out nor the resolution costs would be so high as the total liabilities of a distressed institution. However, to the best of our knowledge no reliable estimate of this quantity is presently available in the related literature.[5] Nevertheless, setting LGD to 100% serves the illustrative purposes of our

[5]Tarashev et al. (2010) set the LGD-rate to 55% without giving any reasons.

numerical study and can be thought as a proxy for the worst-case losses incurred by economic agents (depositors, bond investors etc.).

For the purposes of the multi-factor model (3.1), the banks have been grouped with respect to geographical region (Europe, North America, South America, Africa, Japan, Asia and Pacific excluding Japan) depending on the country where a bank is officially headquartered. See table 1 for summary statistics of the size distribution of banks in the sample. We have set the asset return correlation within the groups to the asset return correlation average of 42%, estimated for large banks on the basis of the Moody's KMV GCorr module, as reported by Tarashev et al. (2010, p 21). It implies homogenous factor loadings $a_i \equiv a = \sqrt{0.42}$. The heterogeneity in the dependence structure arises from the correlation of the composite factors. There are only six distinct composite factors in our example, one for each region, so that only the correlation coefficients $\rho_{Y_{reg(i)},Y_{reg(j)}}$ for distinct regions, i.e. for $reg(i) \neq reg(j)$, do not equal 1. Those correlation coefficients have been estimated from the time series of monthly returns on the DJTM total return indices[6] for the banking sector in the geographical regions under consideration obtained from Datastream. The estimated values are reported in Table 2.

6.2. Numerical results

First of all we compare how the proposed simulation and analytical techniques perform with regard to the calculation of the portfolio tail risk and marginal risk contributions. On the one hand, we run 100 plain MC and IS simulation scenarios, each scenario comprising $10,000$ independent replications of the portfolio loss variable PL. This enables us to compute pointwise empirical confidence intervals for the quantities under consideration. On the other hand, we approximate the tail risk and risk contributions analytically and check whether the approximated values fall into the simulated confidence intervals.

Figures 1 and 2 exemplify the considerable gain in precision compared to a plain MC simulation when estimating the loss distribution and the portfolio tail risk by means of IS.

Figure 3 compares the performance of the MC and IS estimators for the ES contributions. The box-and-whiskers plots clearly show a substantial reduction in variability using IS. The same is true for the estimators of the VaR contributions, although in this case a meaningful estimate via MC simulation would require much more replications due to the conditioning on a rare event. Please refer to Glasserman (2006) for further numerical examples on the performance of the IS algorithm concerning the problem of estimating the tail risk contributions for stylised credit portfolios.

The analytical method performs reasonably well for the calculation of portfolio VaR and ES with a tendency to uderestimation of the tail risk. The results we obtained for the ES using the Pykhtin method are illustrated in figure 4. The approximated ES value lies in more than 50% of cases within the 90% empirical confidence interval of the IS estimator. In the remaining cases the approximated value is close to this interval. The results for VaR are similar and we therefore omit the illustration. Additionally, figure 5 shows the analytically approximated ES in comparison with the IS-based estimate along with the relative difference between them. The Pykhtin's formula exhibits the poorest performance in the period of a relatively low system-wide risk. The best performance

[6]DJTM denotes the family of Dow Jones Total Market indices.

Table 1: Liability (LBS) size distribution of all banks within the sample at the beginning of 2008, aggregated by country.

Region	Country	Number of banks	Aggregate LBS	
			billion USD	% of total
EU	Austria	1	265	0.49
	Belgium	2	1,286	2.39
	Denmark	1	606	1.12
	France	3	5,571	10.33
	Germany	4	4,155	7.71
	Greece	1	111	0.21
	Iceland	1	64	0.12
	Italy	3	2,146	3.98
	Netherland	2	3,179	5.90
	Norway	1	244	0.45
	Russia	1	146	0.27
	Spain	3	1,988	3.69
	Sweden	3	1,122	2.08
	Switzerland	2	3,079	5.71
	United Kingdom	6	8,758	16.24
AMN	Canada	5	2,093	3.88
	USA	11	7,274	13.49
AMS	Brazil	3	352	0.65
AFR	South Africa	3	322	0.60
JP	Japan	5	4,577	8.49
AS&P	Australia	5	1,589	2.95
	China	10	3,456	6.41
	Hong Kong	2	212	0.39
	India	2	305	0.57
	Singapore	3	353	0.65
	South Korea	3	654	1.21
Total		86*	53,907	100

* These banks account for about 2/3 of the worldwide banking industry assets in 2007/2008 (approximated by assets of the largest 1,000 banks as reported by IFSL (2010)).

Table 2: The matrix of estimated Pearson's correlation coefficients for the composite factors, $\rho_{Y_{reg(i)},Y_{reg(j)}}$. All p-values are less then 1%

	EU	AMN	AMS	AFR	JP	AS
EU	1.00	0.80	0.65	0.63	0.44	0.85
AMN	0.80	1.00	0.42	0.44	0.39	0.73
AMS	0.65	0.42	1.00	0.50	0.46	0.68
AFR	0.63	0.44	0.50	1.00	0.32	0.62
JP	0.44	0.39	0.46	0.32	1.00	0.45
AS	0.85	0.73	0.68	0.62	0.45	1.00

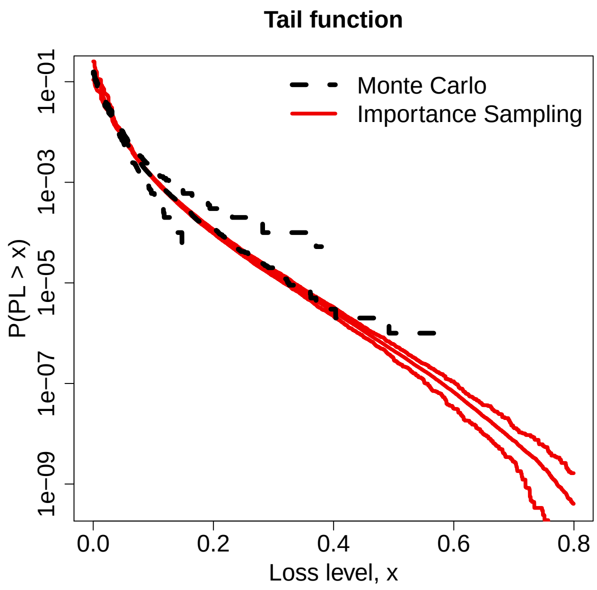

Figure 1: Log-lin graph of the portfolio loss tail function in September 2008 estimated via Monte Carlo simulation and importance sampling. In each case, the three curves show the mean and a pointwise 95% confidence interval computed on the basis of 100 independent scenarios.

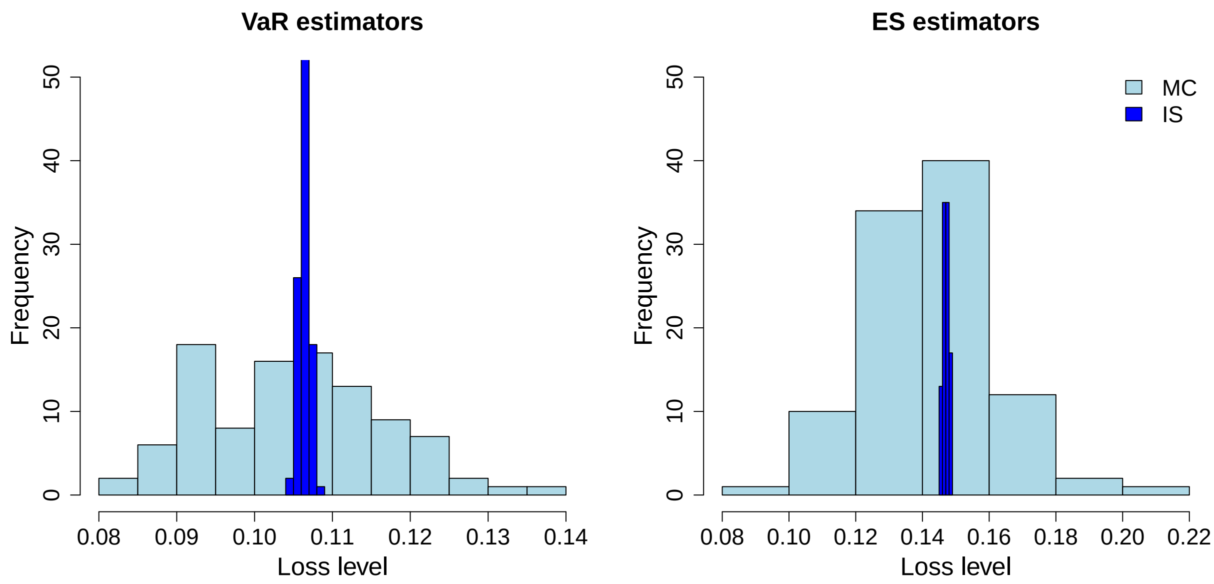

Figure 2: Comparison of Monte Carlo (MC) and importance sampling (IS) estimates for value at risk (VaR) and expected shortfall (ES) in September 2008. The histograms are computed on the basis of 100 independent scenarios.

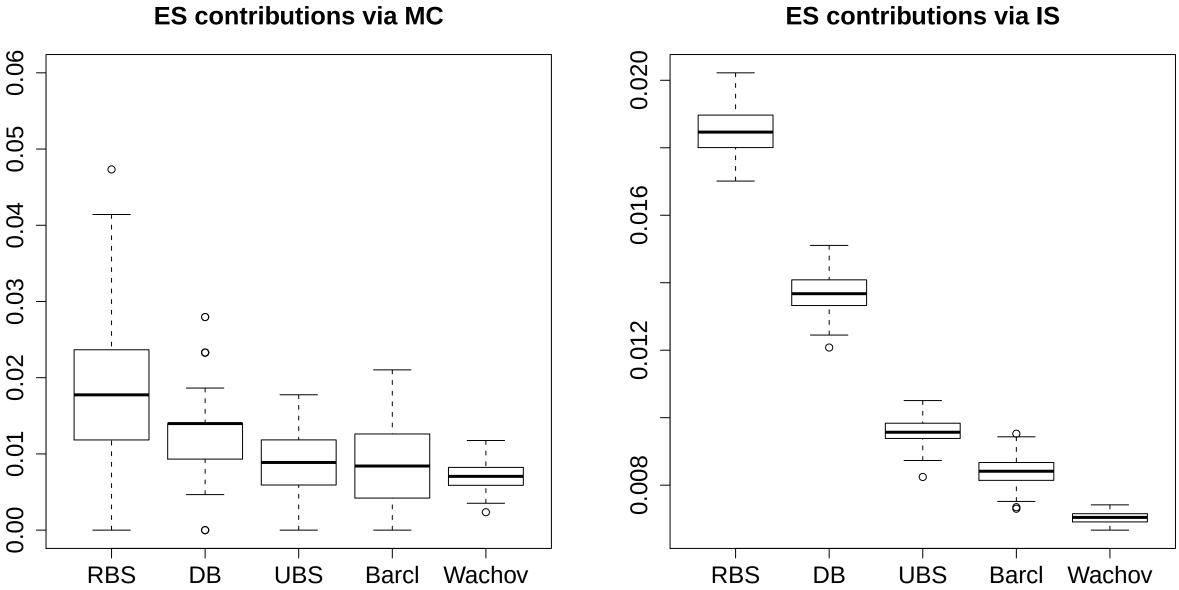

Figure 3: Comparison of the summary statistics for expected shortfall (ES) contributions in September 2008, estimated via Monte Carlo (MC) and importance sampling (IS). The box-and-whiskers plots are based on 100 independent scenarios. The five largest contributions are given as a fraction of the total liabilities of 84 banks in the portfolio.

is on the contrary at the peak of the crises. A more detailed performance test for the multi-factor-adjustment technique was carried out by Pykhtin (2004). Among other things, Pykhtin shows that the accuracy of the approximation improves when the risk factor correlation increases and portfolio granularity decreases.

With respect to the individual risk contributions approximated by the analytical formulae (5.23) and (5.27), we report in figure 6 results on the relative difference between the IS-estimated and analytically approximated ES contributions for some of those time periods when the relative difference at the portfolio level was less than 1%. The results are again better for higher values of the portfolio tail risk.

Whereas the analytical approximation of the portfolio tail measures performs well, the results of the analytical approximation of individual risk contributions should be interpreted with caution. While the precision of the IS estimator can be improved by simply performing more simulation runs, the analytical results strongly depend on the portfolio granularity and the correlation structure. Due to the changes in the correlation structure in the course of the model transformation from the multi-factor (3.1) to the one-factor (5.1) setting, the impact of the largest exposures in the portfolio may be overvalued. This effect would then be compensated by reduced and possibly even negative contributions of small-sized exposures in order to satisfy the additivity property. Thus, the analytical approximation for the risk contributions can, for instance, be used in preliminary studies on the topic when numerous calculations have to be accomplished quickly but the precision of the results is not crucial.

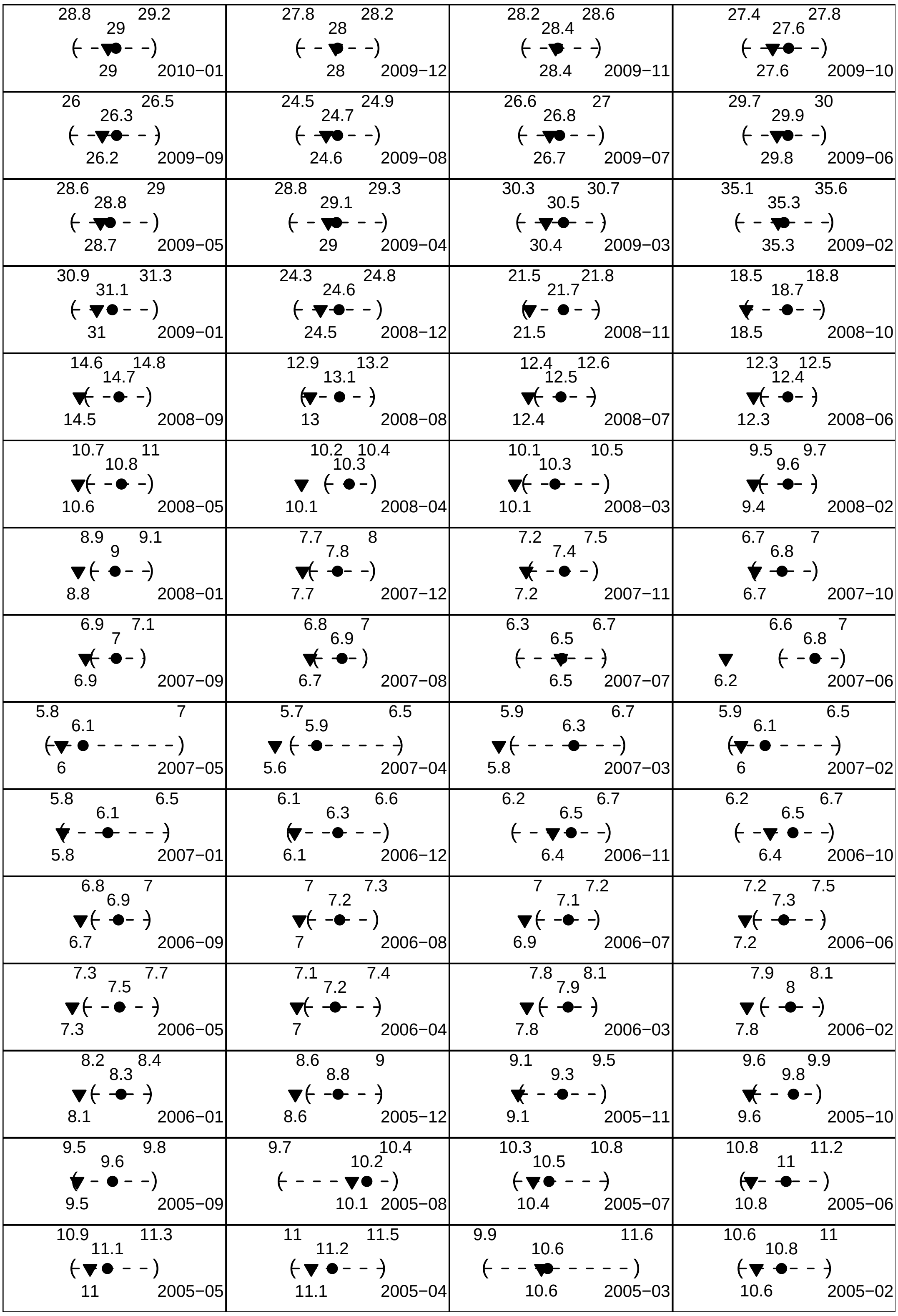

Figure 4: Comparison of expected shortfall (ES) values estimated via importance sampling with those approximated analytically. ES is given as a percentage of the total portfolio liabilities. For each month the analytically approximated portfolio ES is indicated by a triangle pointing down to its numerical value, whereas the patterns enclose 90% of all 100 sampled ES values with the mean indicated by a circle. Only the 60 latest observations are shown.

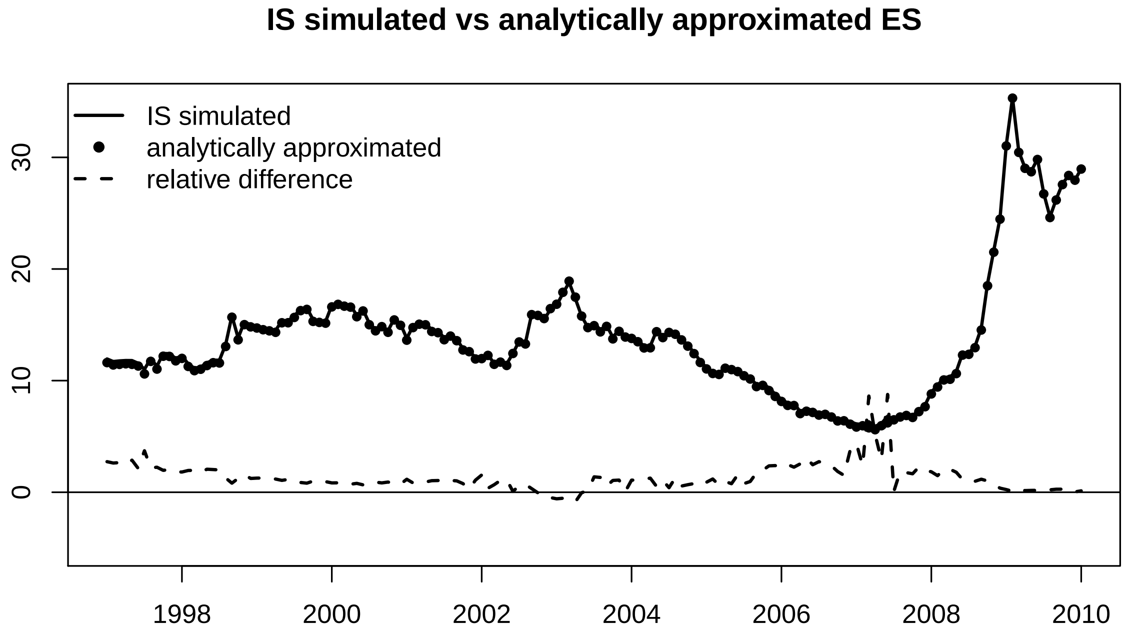

Figure 5: Comparison of expected shortfall (ES) values estimated via importance sampling with those approximated analytically. ES is given as a percentage of the total portfolio liabilities. Also plotted is the relative difference between the two estimates.

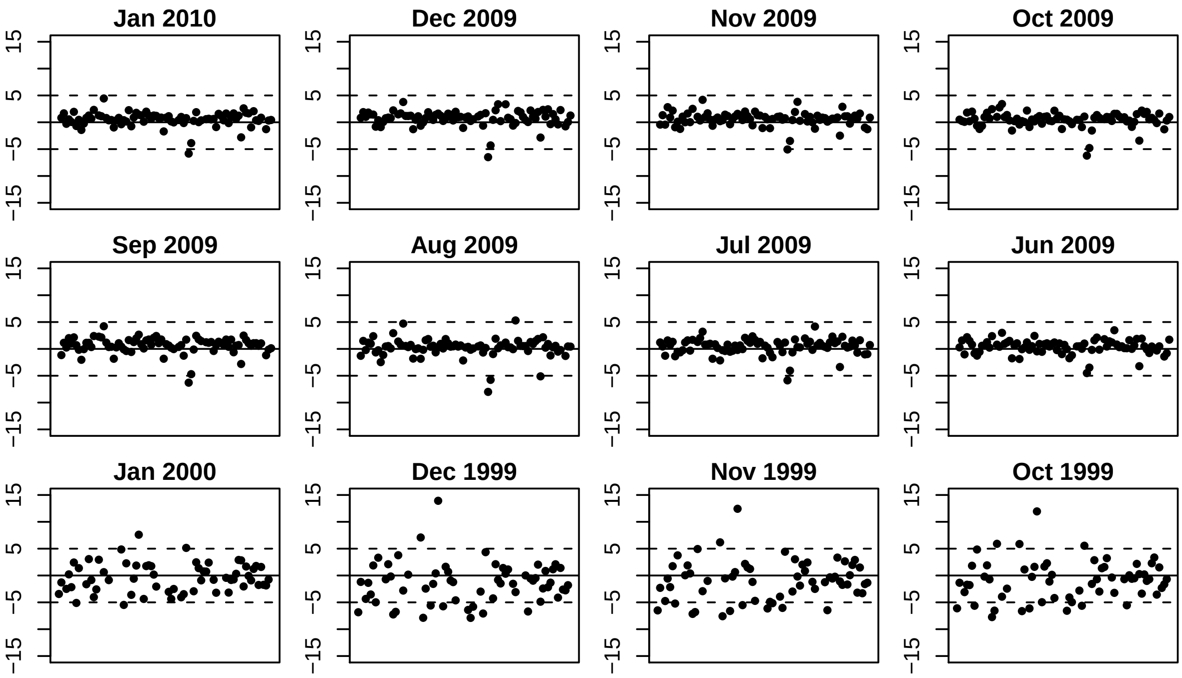

Figure 6: The relative difference (in percent) between the contributions to the expected shortfall estimated via importance sampling and those approximated analytically

7. Drivers of systemic risk and systemic importance

The impact of the risk drivers – the number of the exposures and their relative size (portfolio lumpiness), the individual probabilities of default, the asset correlations – on the portfolio tail risk measures have been analysed in great detail in the literature on market risk and credit risk. Additionally, Tarashev et al. (2010) presented some stylised examples for hypothetical financial systems in order to examine the characteristic properties of the system-wide ES. Thereby the authors applied the one-factor Merton/Vasicek framework with common factor loadings. The key messages from their work were:

- The level of systemic risk increases with the individual probabilities of default.
- Higher exposure to the common factors (captured by the asset correlation) increases the likelihood of joint failures and raises the tail risk.
- Greater lumpiness of the financial system, caused either by the increasing disparity of the relative size of banks or by their decreasing number, raises systemic risk.

Since we have made use of the time-invariant asset correlations in our numerical study, the evolution of the tail risk measures over the sample period is mainly driven by the evolution of the individual banks' default risk captured by means of one-year EDF figures. This is illustrated in figure 7 for the ES expressed both in USD (upper chart) and as a percentage of the total liabilities (lower chart). ES ranges from 5.61% to 35.31% of total LBS. The span of ES variation would become narrower if we took the EDF's time-variation out of the model. In this case, everything else being equal, the ES would match closely the pattern of aggregated banks' liabilities, which had been more or less steadily increasing until mid-2008, starting at $7,121bn in 1997 and reaching $57,950bn at the end of 2009.

To investigate the impact of the specific risk drivers on the systemic importance of individual institutions, let us consider a special case of the one-factor model for a stylised banking system. The system is populated by 66 banks which all share the same probability of default. All the banks can be separated into two groups, each accounting for 50% of the overall liabilities.

First, we would like to isolate the impact of the relative size for different levels of default probability. So we define one group of 62 equally-sized small banks and another group of 4 equally-sized big banks. To keep the exposures to the single systemic factor constant across the system, we set the pairwise asset correlation to 42%. The results for this stylised financial system are presented in the left-hand panel of figure 8. Logically, a higher probability of default has a positive impact on the systemic importance. Furthermore, notwithstanding the fact that both groups are equally sized, the group of big banks accounts for more than 50% of the overall ES according to its greater lumpiness. This effect is even more distinctive for small probabilities of default (below $1 - 2\%$) which are usual for the banking sector. Hence, among financially sound institutions the banks with larger exposures at distress affect the overall tail risk disproportionately heavily.

Second, in order to assess the impact of the asset correlation on the systemic importance, we divide our financial system into two homogeneous groups comprising 33 equally-sized banks each. One group is only moderately exposed to the systematic factor with the pairwise within-group asset

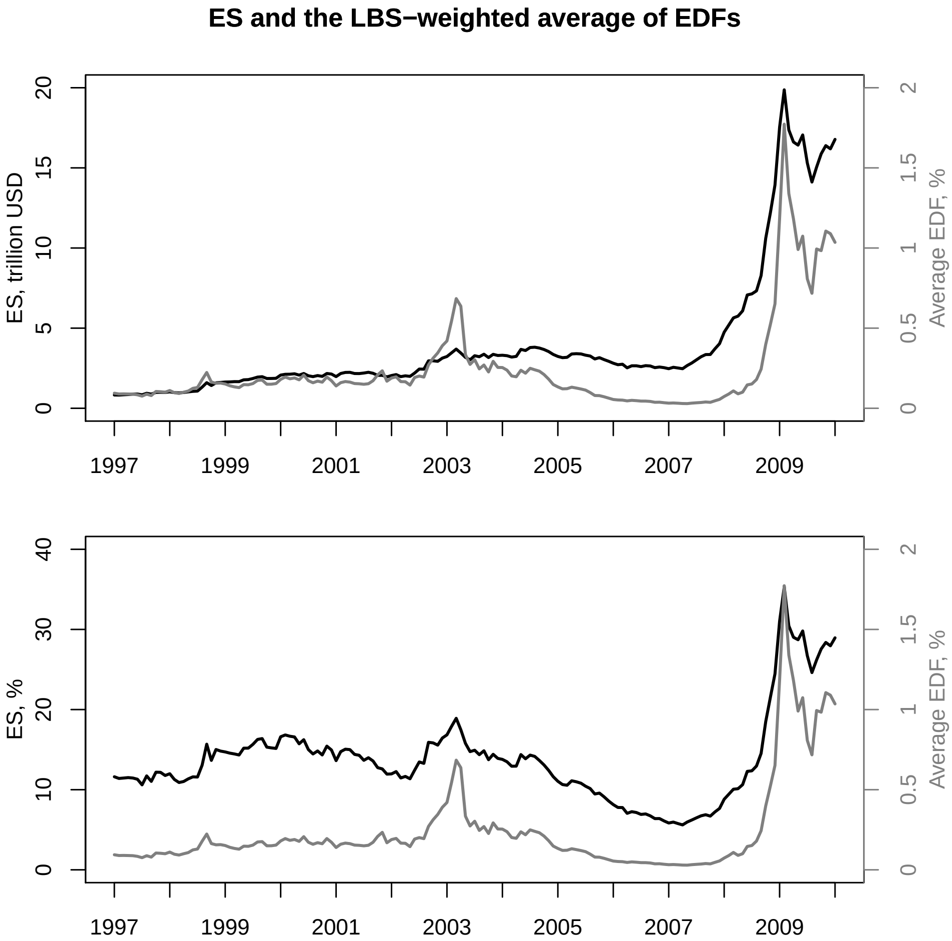

Figure 7: Evolution of the portfolio Expected shortfall (ES, black lines, left axes). Also plotted is the weighted average of EDFs (gray lines, right axes); the weights are the shares of individual banks in the total portfolio liabilities (LBS).

correlation of 20%. The banks assigned to the second group are highly correlated with a coefficient of 60%. The right-hand panel of figure 8 shows the intuitive result that a higher exposure to the systemic factor is linked with a higher systemic importance, since the probability of joint failures rises. This leads to a higher level of the tail risk. It is also worth noting that the tail risk contribution of the group of highly exposed banks increases faster within the range of small default probabilities (below 2%) than is the case for the other group.

It should be stressed that the systemic importance of a bank, if measured as a share in the systemic risk, is not only affected by its individual risk characteristics but also by the composition of the portfolio. Nevertheless, the cross-sectional Spearman's correlation between the (relative) size of an institution and its share in the portfolio ES,[7] which ranges from 79% to 94% with a median observation of 90%, reveals a tight relationship between the size and the systemic risk contribution.

[7]Note, that the Spearman's correlation increases in magnitude as the two variables become closer to being perfect monotone (possibly non-linear) functions of each other.

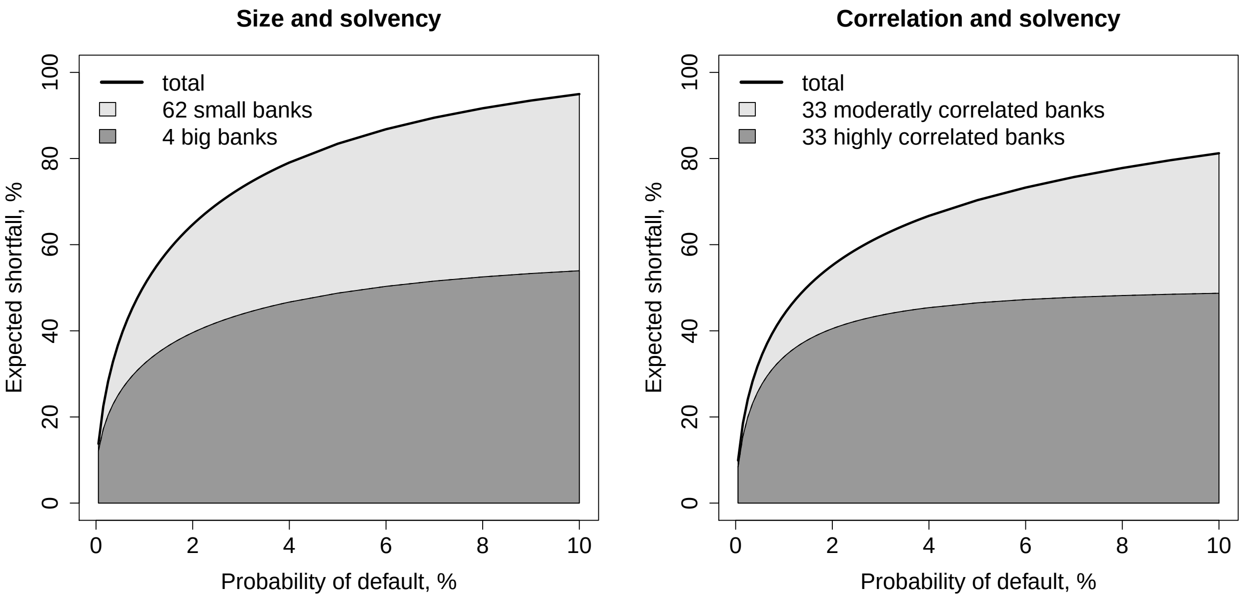

Figure 8: Systemic importance of two groups of banks with different sizes (left plot) and different exposures to the single systematic factor (right plot). Each of the two groups accounts for half of the total portfolio exposure.

The cross-sectional Spearman's correlation between the individual probability of default and the relative ES contribution is considerably lower: it ranges from -0.03% to 51% with a median observation of 26%. Thereby, the estimated negative values of the correlation coefficient are insignificant and from July 2002 on we observe only significant positive correlation (at the 95% level).

As to the time dimension, no definite link between the size of a financial institution and the amount of risk it contributes could be found. Figure 9 shows the dynamics of the banks' relative ES contributions for the 15 banks with historically largest risk contribution within the sample period. The development of the EDFs and shares in aggregated liabilities over time is plotted as well. As can be seen, the relative ES contribution of an individual bank often exceed its LBS share considerably, indicating a bank's overproportional contribution to the risk of the whole system. The link between the individual solvency risk and the tail risk contribution is more distinctive here. Apart from one bank with a significant negative coefficient and 6 banks with insignificant coefficients, time-related Spearman's correlation ranges from a minimum of 18% to a maximum of 96% with the median observation of 65%. It should be noted that in the example under consideration we keep the asset correlation constant over the whole sample period, thus reducing the overall model complexity.

From the results above we cannot deduce an obvious, definite link between the size of a financial institution and its systemic risk contribution. This is due to a complex interplay of risk drivers in both dimensions: cross-sectional and time-related. Hence, tailoring macro-prudential instruments simply to the size of a financial institution would be incoherent. It would miss the point of addressing the actual risk posed to the real economy and society by financial firms.

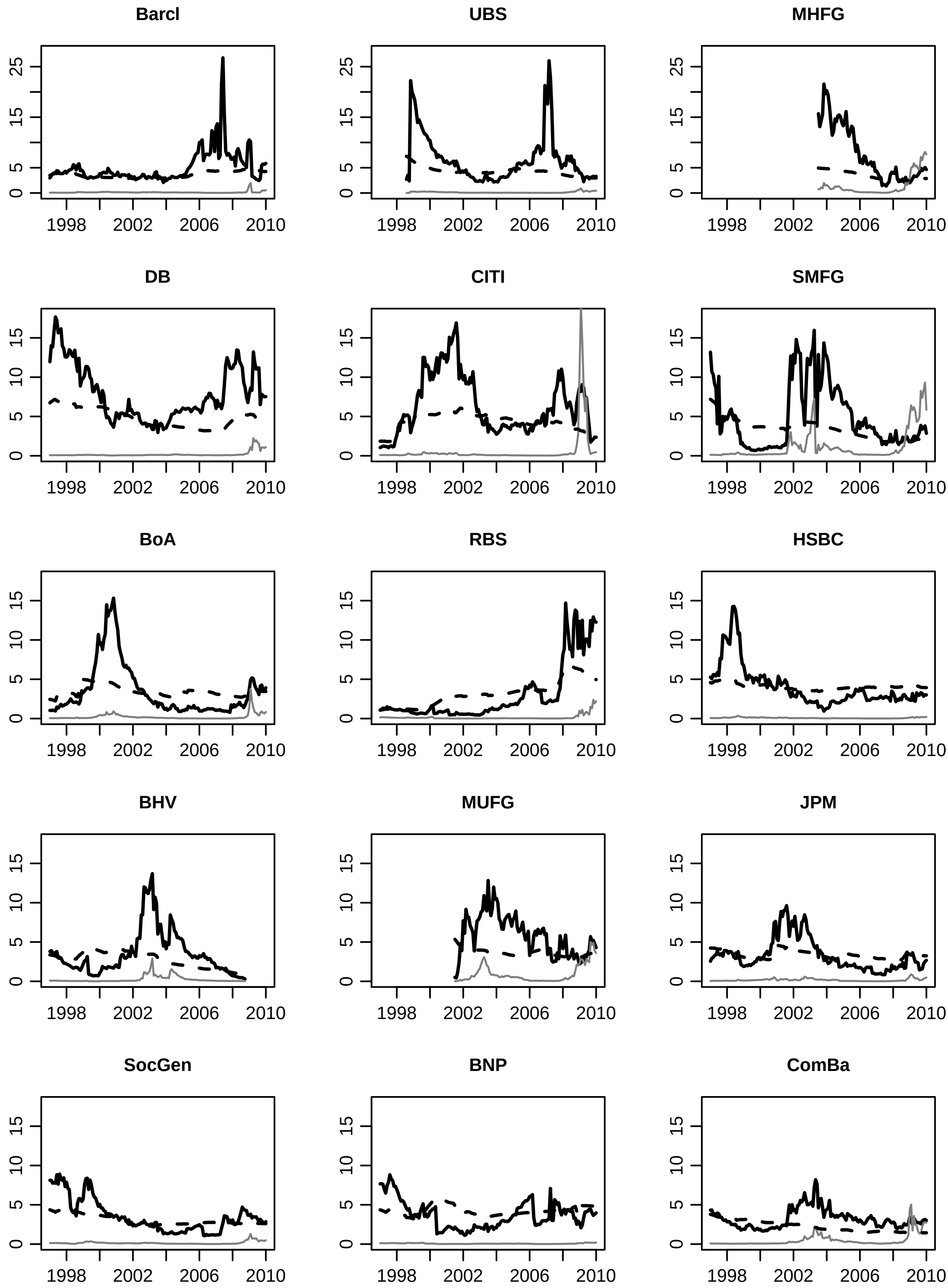

Figure 9: Dynamics of the banks' individual shares in the portfolio expected shortfall (solid black lines) in comparison with the EDFs (solid gray lines) and individual shares in the total portfolio liabilities (dashed lines).

8. Policy tools

A macro-prudential regulation should address both dimensions of the systemic risk, as is underlined by Borio (2009) among others:

The *cross-sectional dimension*, addressed in subsection 8.1, relates to the distribution of the aggregate risk in a financial system at a given point in time. The corresponding policy issue consists in the calibration of prudential instruments according to the level of the overall risk in the system and according to the contributions of individual institutions to the system-wide tail risk.

The *time dimension*, addressed in subsection 8.2, covers the evolution of the aggregate risk over time. The corresponding policy issue is to find a way to reduce the possible procyclicality of regulatory tools based on a measure of the system-wide financial risk.

8.1. Cross-sectional implementation

For implementing a systemic risk charge in the cross-section, a key challenge is how to internalize the negative externalities caused by financial institutions. This goal is achieved by using the institutions' contributions to the systemic risk as the building block. In this section we put forward a stylised example illustrating how a capital surcharge for systemic-risk can help the regulator to reduce the tail risk amount.

We consider the situation arising in January 2009 as an example. At this point of time, the portfolio comprises of 80 banks from the sample in Section ssec:data. The ES of the portfolio amounts to 31.38% of the system-wide liabilities or \$17,447bn. Individual ES contributions vary between 0.03‰ and 4% or \$1.8bn and \$2,230bn.

As the measurement and allocation of systemic risk involve model uncertainty and estimation errors, it may be advisable not to require a bank-specific surcharge on a continuous scale. We prefer a less granular approach by dividing the institutions into three different categories A, B and C defined by *systemic risk ratings*.[8] We apply a simple k-means clustering procedure on the ES contributions in order to define these categories.[9] The results are illustrated in the left-hand panel of figure 10.

Let us categorise the four banks belonging to the first group and indicated by diamonds as A-rated, "highly systemically important" institutions. This group holds 20% of the total assets of the system and contributes 38.8% to the overall ES. The individual ES contribution of every bank in this group equals or exceeds 2% of the portfolio exposure. The squares mark the second category comprises 16 B-rated, "moderately systemically important" banks. They individually contribute between 0.5% and 1.5% to the portfolio exposure. This second group of banks holds 47% of total assets and shares 43.5% of the overall tail risk. The remaining 60 banks are indicated by solid circles. They share 34% of total assets and 17.7% of the portfolio ES. Those banks account for risk

[8]The IMF (2010, Chapter 2) presents an approach under which regulators assign systemic risk ratings to each institution based on the amount of system-wide capital impairment that a hypothetical default of each institution would bring to bear on the financial system. Institutions with a higher systemic risk rating would be assessed as having higher capital surcharges. The level of capital surcharges would be predetermined – perhaps having to be agreed upon in international forums.

[9]The k-means method aims to partition the dataset into k groups. The grouping is done by minimising the sum of squares of distances between the data points and the clusters' centroids.

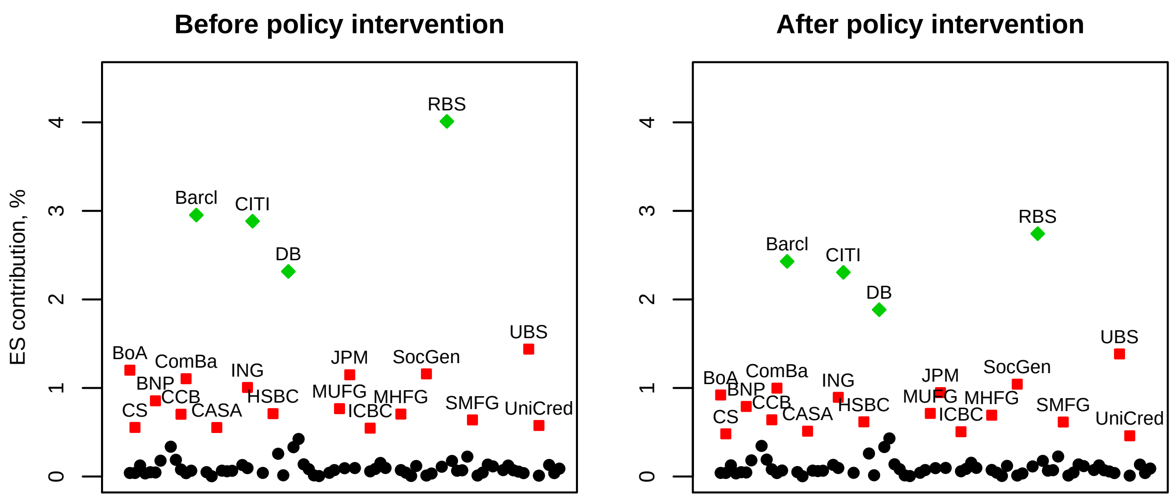

Figure 10: The landscape of systemically important banks before and after the policy intervention. 3 groups of banks have been identified by the k-means clustering method according to their contributions to the portfolio ES in January 2009.

contributions of less than 0.5% each and will be denoted as C-rated, "systemically less relevant" institutions.

We assume that the capital held by banks equals the amount of capital required by the regulator. Therefore, any capital charge for systemic risk will require an increase of capital and cannot be drawn from an existing "free" capital buffer on top of the regulatory minimum requirements. Furthermore, we assume that the systemic risk charge does not affect neither the size of a bank's balance sheet nor its exposure to the systematic factors (or asset correlations). The new capital requirements only affect the debt-to-equity ratio as the banks substitute their (short-term) debt by capital. In this case, the rising capital charge would leave the asset value of banks unchanged according to the Modigliani-Miller capital structure irrelevance principle.

Within Merton's framework (Merton, 1974), the following functional link between default probability and leverage ratio applies

$$p_i = \Phi\left(\frac{\ln(DPT_i/AVL_i) - \mu_{AVL_i}}{\sigma_{AVL_i}}\right),$$

where DPT denotes the default point, AVL – the market value of assets, μ_{AVL} – the expected asset return and σ_{AVL} – the volatility of asset value. DPT_i is defined in such a way, that a drop in the market value of bank i's assets below DPT_i triggers the default of the bank. Moody's KMV model, which builds on Merton's framework, calibrates DPT as a weighted average of long-term and short-term liabilities. The model operates with the so called distance to default (DtD) :

$$DtD_i := -\frac{\ln(DPT_i/AVL_i)}{\sigma_{AVL_i}}. \tag{8.1}$$

Using the CreditEdge data on EDFs and DtD, we can approximate the mapping function between

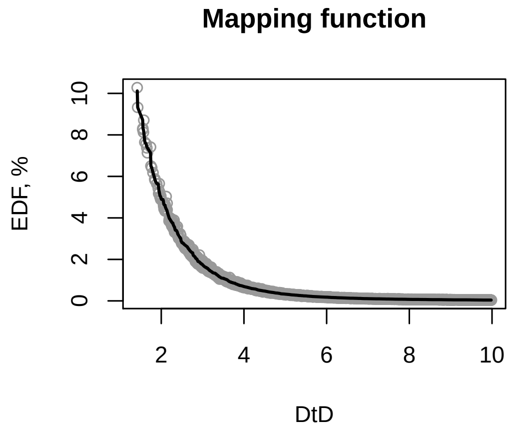

Figure 11: Mapping the distance to default into the EDF for a one-year time horizon.

DtD and EDF as shown in figure 11.

We further assume that for each rating category, national regulators have agreed upon a certain level of capital surcharges. In our simplified example, additional capital requirements are set to 50% of the current microprudential capital requirements for "highly systemically important" institutions, to 25% for "moderately systemically important" institutions and to nil for "systemically less relevant" banks.

According to the assumptions we made, the policy intervention results in a modified capital structure of the systemically important banks, reducing their short-term debt, as well as the default point, exactly by the amount of the additionally raised capital. By inserting the new DPT into (8.1) we find the corresponding distance to default and map it into the EDF. We use the new set of EDFs to run the analytical approximation of the portfolio ES and risk contributions after the policy intervention.

The new input parameters change the overall view of the systemic risk landscape, as demonstrated in the right-hand panel of figure 10. While the total capital in the system rises by 17% and the liabilities decrease by 0.75%, the system-wide ES undergoes a reduction of 13.93% from \$17,447bn to \$15,016bn. In nominal terms, the amount of capital is increased by \$418bn with the effect that the ES of the financial system is reduced by \$2,431bn. In other words the systemic risk charge reduces the system-wide risk by a factor of about six. The marginal, USD-denominated tail risk contributions of the banks with ratings A or B decrease. The total risk contribution of the A-rated banks declines by 30.91% from \$6,762bn to \$5,165bn and the contribution of the B-rated banks by 12.71% from \$7,598bn to \$6,741bn. In response to the changes in the portfolio structure, the total USD-denominated risk contribution of the 60 "systemically less relevant" banks increases slightly by 0.70% from \$3,088bn to \$3,109bn.

The empirical example relies on a relatively coarse differentiation between three groups of banks depending on their systemic risk contribution. The capital surcharge has been set arbitrarily since this example does not offer a methodology to determine a continuous, bank-specific *systemic capital charge (SCC)*. In the remaining of this section we present an approach that translates a bank's contribution to the ES of the financial system into a firm-specific capital charge.

We consider the i-th institution in the year t that is subject to minimum capital requirements (MCR). The key idea is to charge the difference between a "pure" systemic risk contribution and the original regulatory minimum capital requirement. If the micro-prudential regulatory capital requirement exceeds the systemic risk contribution of a bank, then no add-on for systemic risk is charged. The following equation summarizes this definition of an ES-based SCC:

$$SCC_i(PL,t) = \max\left\{EAD_i(t)\frac{\partial}{\partial w_i(t)}ES_{q(t)}(PL,t) - MCR_i(t), 0\right\}. \tag{8.2}$$

According to the figures on the total regulatory capital holding by the banks, for which we could obtain the corresponding data from Bankscope, in 2006/2007 73% out of 63 banks were well capitalised in the sense that they reported capital exceeding $MRS_i(t) + SCC_i(PL,t)$ as defined in (8.2). In 2008/2009 the same was only true for 32% out of 72 banks.[10]

Additional capital requirements as in (8.2) are generally in line with the FSB's recommendations to strengthen the loss absorbency of systemically important banks (see FSB, 2010). However, as pointed out by Gauthier et al. (2010), computing macro-prudential capital requirements is more complex than computing risk contributions itself. The simple formula (8.2) suggests setting the capital surcharges according to the currently observed capital levels and does not take into account the subsequent changes in the overall systemic risk landscape. Once new capital requirements are implemented, the banks' probabilities of default (and potentially also the asset correlations) decline resulting in lower tail risk and changed absolute and relative risk contributions. For this reason Gauthier et al. (2010) suggest an interative procedure to solve for the fixed point at which the capital allocation in the system is consistent with the banks' risk contributions. Such reallocation of the capital not only means that the undercapitalised banks raise capital or de-leverage, but also that the overcapitalised banks increase their leverage. A superior approach not simply based on the *re*allocation of the given total capital, would require the knowledge of the optimal total level of capital required in the banking system to withstand a predefined shock.

An alternative to a capital charge for system-wide risk can be regular payments by the systemically important banks into a bank stabilisation fund. Although, this policy measure does not promote strengthening of the banks' capital basis, it has the advantage that the money paid into the fund would be available in a crisis situation without the need to tap the taxpayer's purse, e.g. for financing certain bridge banks. A yearly payable amount could be defined by emploing the banks' contributions to the tail risk of the whole banking sector. Further refinements could be contemplated. For example, in order to relieve the strain on savings banks and other mostly deposit-taking institutions, exposures could be reduced by the amount of ensured deposits, which could be achieved by setting $LGD_i < 1$ accordingly.

Both, a capital charge for system-wide risk and a stabilisation fee would reduce the competitive advantages to become systemically important. The latter statement provides an incentive for the systemically important institutions to reduce their share in the system-wide tail risk which is a

[10] If we used $q(t) = 0.999 \quad \forall t$ in (8.2), it would be 86% in 2006/2007 and 15% in 2008/2009 because of lower capital surcharges in good times and higher ones in bad times when q kept fixed.

desirable effect.

Against this background, the overall amount of the capital held in a financial sector is not necessarily to cover systemic risk completely, since the tail risk in the system can be far too high to be fully backed with capital.[11] Therefore, the level of the total capital requirements could be on average lower than the amount of the ES. This means that a certain fraction of the systemic risk will still be borne by the public.

8.2. Smoothing the path of the tail risk measure over time

Within the presented framework the evolution of systemic risk over time is mainly driven by the co-movement of the probabilities of default in the banking sector. It is economically meaningful to assume a close link between the actual default conditions in the banking sector and the likelihood for wide-spread losses to materialise. To take this into account, we suggest to use a time-varying probability of a systemic event which would depend on average default probabilities in the banking sector.

A nice side effect of this approach is the mitigation of possible procyclical effects of regulatory tools based on the proposed measure of systemic risk. In figure 7 we have seen how the use of point-in-time estimates of the default probability based on market prices can induce procyclicality in the tail risk measure. Market-based measures suggest that the system is strongest in times when market volatility is below average and market participants accumulate large amounts of risk. During an economic downturn or turbulent markets, probabilities of default (and asset correlations) increase and the tail risk measure increases. The described effect by itself is not a problem when considering the portfolio expected shortfall as a systemic risk indicator for the banking sector. Although in this regard the utilization of more forward-looking estimates of default probabilities would be an advantage. However, in order to establish such macro-prudential tools as systemic capital surcharges a procyclical pattern of the underlying risk measure may deem undesirable.

Against this background, using a time-varying probability of a systemic event which depends on the estimated average default risk in the banking sector is not only an economically meaningful way of dealing with the unknown likelihood of crises, but also a transparent method for mitigating the procyclical effect (if required).

Please note, that using a time-varying probability of a systemic event $1 - q$ is equivalent to using a time-varying confidence level of the tail risk measure q. Hitherto, q has been fixed somewhat arbitrarily at 99.9% – the confidence level for VaR required within the scope of the IRB approach under the Basel II regime. This value of q is used in the Basel II framework to ensure that the one-year default probability of a bank does not exceed 0.1%. In our settings, however, we can also allow q to evolve over time depending on the actual average default risk of the financial intermediaries. The logic behind our proposal is that in an expanding economy when banks exhibit low default probabilities, the anticipated likelihood of a systemic event declines and we can set the confidence level to $q \geqslant 0.999$. During an economic contraction, however, default probabilities rise throughout the

[11] During the time period under consideration the system-wide exposure (i.e. $\sum_{i=1}^{n} LBS_i(t)$) increased from 23 to 100 percent of the global GDP whereas the amount of the tail risk was varying between 6.8 and 29 percent of the global GDP according to $ES_{q=0.999}$ or between 3.7 and 16 percent according to $ES_{q(t)}$. The IMF's figures on the world GDP were taken.

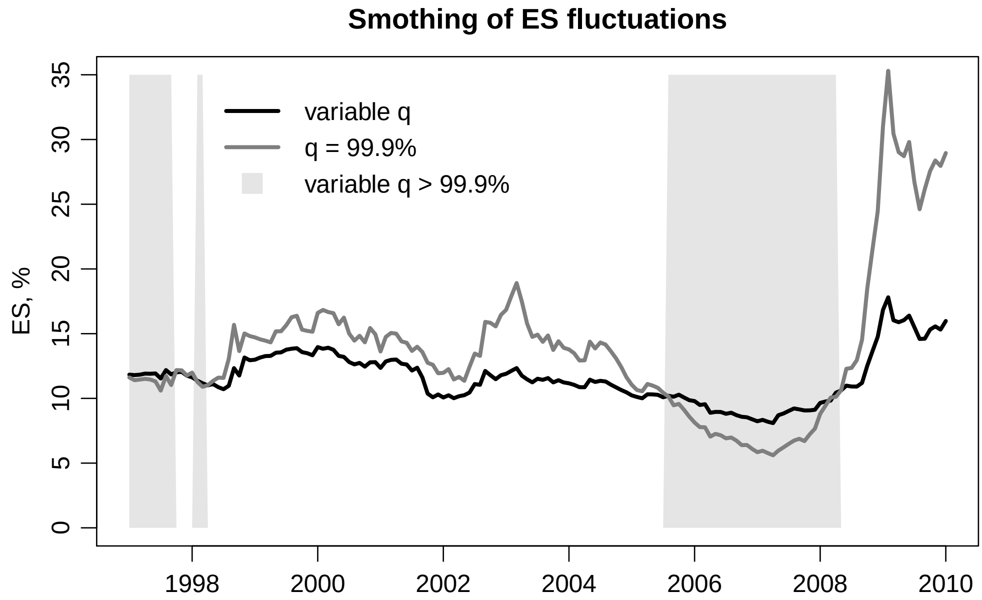

Figure 12: Evolution of the portfolio ES calculated according to the time-varying confidence levels $q(t)$ (black line) versus ES at the constant confidence level $q = 99.9$ (gray line).

system and we reduce the confidence level to $q < 0.999$ in anticipation of the increasing probability of widespread losses. Due to the fact that the asset correlations in our example do not vary over time, we basically tailor q to the cross-sectional exposure-weighted average of the banks' default probabilities:

$$q(t) = 1 - \sum_{i=1}^{n} \frac{EAD_i(t)}{\sum_{j=1}^{n} EAD_j(t)} p_i(t).$$

Please note also that for $LGD_i(t) = 1 \forall i = 1, \ldots, n$ the expression $\sum_{i=1}^{n} \frac{EAD_i(t)}{\sum_{j=1}^{n} EAD_j(t)} p_i(t)$ represents the expected portfolio loss at time t. The confidence level for the portfolio under consideration ranges from 98.23% to 99.97% with a median of 99.85%. By means of the time-varying confidence level, we achieve significant reductions in the variability of the system's ES, as figure 12 illustrates. Using joint default probabilities instead of individual ones and allowing for varying (default) correlations would amplify the observed effect even further.

9. Summary and conclusions

Addressing the system-wide risk of a financial system by macro-prudential regulation requires an approach that internalizes the potential costs of a systemic failure. We develop such an approach by assessing the systemic risk of the financial system and by allocating this risk to individual banks while the emphasis is on the second step. We employ for this macro-prudential allocation the Euler allocation principle that is widely used in the risk management of financial institutions.

The portfolio approach used in this paper has the additional advantage that the modelling requirements are based on standard techniques in the risk management literature and the basic data requirements are similar to those under the internal ratings based (IRB) approach of Basel II. For the major financial institutions, the method provides an assessment tool of systemic importance, based on publicly available information including market prices. Moreover, the model can be applied to smaller, not publicly traded institutions as well, provided that their probabilities of default and their exposures to common risk factors can be estimated based on available information.

In this paper a financial system is modeled as a portfolio of credit-risky financial institutions. The credit exposure is given by the liabilities of the firm and in this context defined by the nominal balance sheet amount less capital. From a public purse perspective we model systemic risk in terms of the expected shortfall (ES) of this portfolio consisting of those banks in the global financial system which may be deemed systemically relevant. The expected losses conditional on exceeding a given confidence level reflect the potential costs posed to society in the low-probability event when in the course of a systemic crisis several institutions become distressed and draw on the (explicit or implicit) guarantees given on their debt. The confidence level is therefore a policy parameter that could be selected in order to balance the risk preference of the public purse with the economic costs that higher capital charges or fees may entail for credit supply and ultimately for economic growth. This optimisation problem offers a new avenue of research not included in this paper.

After the systemic risk of the whole financial system has been quantified by means of the system-wide ES, it is allocated to individual banks by their marginal risk contribution to the system wide risk. An important advantage of this method is the full allocation property, which means that the systemic risk portions allocated to individual institutions equal the system-wide risk in the aggregate. For the purpose of simulation of the portfolio loss function, upon which the calculation of the portfolio ES and the risk contributions is based, we adopt a two-stage importance sampling (IS) method. The main advantage of this variance reduction technique over the plain Monte Carlo method is a considerable gain in efficiency when simulating such rare events as large portfolio losses. Therefore the proposed IS algorithm makes the estimation of the portfolio tail risk stable and the assessment of the tail risk contributions feasible. We also derive an analytical solution for a fast approximation of risk contributions based on a formula for the tail risk of a limiting portfolio with multi-factor and granularity adjustments.

We apply the proposed simulation and analytical approximation to a sample of large, internationally operating banks. The analytical solution performs well at the portfolio level, while the approximative results at the level of individual risk contributions depend on the portfolio composition and the correlation structure. Thus, the fast analytical solution could, for example, be used for the purposes of a preliminary analysis, whereas IS serves best to derive the final results on the risk allocation.

Having conducted a numerical study based on a proposed portfolio, we find that in the cross-sectional dimension the systemic importance of a financial institution is indeed tightly linked to the institution's relative size. But since the formal linkage is non-linear and portfolio-dependent, size alone should not be considered as a reliable proxy of systemic importance. Other risk drivers, such as institutions' probabilities of default and their exposures to common risk factors, have to be taken into consideration when assessing systemic importance within a portfolio framework.

As to the assessment of financial firms' systemic importance, we can abstain from the binary approach, whereby some firms would be considered of systemic importance and others would not, which would leave room for regulatory arbitrage. By means of individual tail risk contributions, the binary concept can be refined to a desirable degree either by introducing several systemic rating categories or by the utilisation of a direct functional link between an institution's marginal contribution to the systemic risk and its degree of systemic importance.

Relying on the marginal ES contributions as a measure of the institutions' systemic importance, policy tools can be adjusted accordingly. A possible capital-related policy option would be to impose a systemic capital charge as the amount of the systemic risk contribution not covered by the minimum capital requirements. Increasing overall risk-based capital requirements would reduce the probability of systemically important banks becoming distressed. An alterative non-capital based policy option involves charging a stabilisation fee that flows into a systemic risk fund. This would cover the externalities in a systemic crisis and dampen the incentives of financial institutions to become more systemically important. Thereby a total yearly amount that has to be paid into the fund can be defined at the system level in a counter-cyclical manner. It will then be allocated among the institutions according to their shares in the system-wide ES.

Regarding the time dimension of the systemic risk, we suggest utilization of a time-varying probability of the systemic event. We define this probability economically meaningful as a function of the exposure-weighted average of the institutions' individual default probabilities. So it is lower during an expansion phase when widespread losses are unlikely and higher in times of an economic downturn. A nice side effect of this approach is a considerable mitigation of the variability in the time path of the ES. The latter finding can help to mitigate possible procyclical effects of regulatory tools based on the proposed measure of systemic risk.

On the whole, the portfolio approach, which we put forward for modelling a system of financial institutions, along with the suggested method for setting the probability of the systemic event can help to understand the complex nature of systemic risk regarding its cross-sectional dimension as well as its evolution over time.

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A. Importance sampling for the portfolio loss distribution

In general, there are many variance reduction techniques, such as (adaptive) importance sampling, antithetic sampling, stratified sampling, and use of a control variable, see Fishman (1996, pp 255-334) for further reading. But IS performs evidently better than other methods in the simulation of rare events. Hall (1992, p 308) presents an efficiency diagram of a distribution approximation by means of IS. The diagram shows an efficiency curve which is lowest at the center of the distribution and diverging to $+\infty$ in both tails. This observation demonstrates the considerable potential of IS for the simulation of tail events and hence for the estimation of tail probabilities. Since the asymptotic gain in performance obtained by using an efficient simulation algorithm is equivalent for both a small probability and the corresponding quantile (Hall, 1992, pp 306 f), IS is a promising simulation technique for the estimation of distribution quantiles, too. Furthermore, it makes the calculation of tail risk contributions feasible. As explained by Egloff et al. (2005), IS rests on the simple fact that the expectation under the original probability measure can be expressed as the expectation under a different measure, provided that the change of measures is compensated by a likelihood ratio. In this case, the likelihood ratio is given by the Radon-Nikodym derivative of the original measure with respect to the alternative measure. Thus, the basic idea of the technique is (i) to transform the distribution from which we draw samples in a way that rare events occur more frequently and (ii) to weight each replication by a likelihood ratio in order to correct for the accomplished change of the distribution.

In a two-stage IS algorithm by Glasserman and Li (2005), the multivariate Gaussian distribution of the systematic factors $\mathbf{Y} = (Y_1 \ldots Y_n)'$ should be initially transformed such that "bad" realisations (negative values, in our case) occur more frequently leading to more defaults in the portfolio. This first step is essential for a portfolio with a high positive default correlation, since defaults tend to occur simultaneously, driven by systematic factors. The subsequent shifting of the conditional loss distribution into the region $[x_q, 1]$ by increasing conditional default probabilities leads to a further variance reduction. We describe the transformation od the conditional loss distribution first.

A.1. Tilting the conditional loss distribution

By shifting its probability mass into the tail we can make rare, large outcomes more likely. Therefore, the objective is to transform the conditional loss distribution in a particular way, turning the threshold x_q into the distributional mean. In order to achieve this, exponential tilting can be applied, as explained in Merino and Nyfeler (2004) and Glasserman and Li (2005). To conduct exponential tilting, explicit formulae for the conditional loss probability function and its moment generating function have to be known. They can easily be derived due to the conditional independence of the individual loss variables.

The probability function of the loss distribution conditional on a realisation of the systematic factors is a product of n independent Bernoulli variables:

$$f_{PL|\mathbf{Y}}\left(\sum_{i=1}^{n} w_i \cdot LGD_i \cdot d_i\right) = \prod_{i=1}^{n} p_i(Y_i)^{d_i}[1 - p_i(Y_i)]^{1-d_i}, \qquad d_i \in \{0, 1\} \tag{A.1}$$

and the corresponding moment generating function, defined as $M_{PL|\mathbf{Y}}(\theta) = E\left[e^{\theta \cdot PL|\mathbf{Y}}\right]$, is:

$$M_{PL|\mathbf{Y}}(\theta) = \prod_{i=1}^{n} \left(1 - p_i(Y_i) + e^{w_i \cdot LGD_i \cdot \theta} p_i(Y_i)\right), \qquad \theta \in \mathbb{R}. \tag{A.2}$$

In the equations above, $p_i(Y_i)$ denotes the conditional default probability for an institution i given by:

$$p_i(Y_i) = \Phi\left(\frac{\Phi^{-1}(p_i) + a_i Y_i}{\sqrt{1 - a_i}}\right). \tag{A.3}$$

In general, an exponentially tilted sampling distribution $f^*(x;\theta)$ can be derived from the original density $f(x)$ and its moment generating function $M(\theta)$ as follows:

$$f^*(x;\theta) = \frac{e^{\theta x} f(x)}{M(\theta)} \tag{A.4}$$

which in our case implies an exponentially tilted sampling probability function $f^*_{PL|\mathbf{Y}}(\cdot;\theta)$ given by:

$$f^*_{PL|\mathbf{Y}}\left(\sum_{i=1}^{n} w_i \cdot LGD_i \cdot d_i; \theta\right) = e^{\theta \cdot \sum_{i=1}^{n} w_i \cdot LGD_i \cdot d_i} \prod_{i=1}^{n} \frac{p_i(Y_i)^{d_i}[1 - p_i(Y_i)]^{1-d_i}}{1 - p_i(Y_i) + e^{w_i \cdot LGD_i \cdot \theta} p_i(Y_i)}. \tag{A.5}$$

The likelihood ratio takes a simple form:

$$\begin{aligned} l\left(\sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i\right) &= \frac{f_{PL|\mathbf{Y}}\left(\sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i\right)}{f^*_{PL|\mathbf{Y}}\left(\sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i; \theta\right)} \\ &= e^{-\theta \cdot \sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i} \prod_{i=1}^{n} \left[1 - p_i(Y_i) + e^{w_i \cdot LGD_i \cdot \theta} p_i(Y_i)\right] \\ &\equiv \exp\bigl(-\theta PL + C_{PL|\mathbf{Y}}(\theta)\bigr), \end{aligned} \tag{A.6}$$

where $C_{PL|\mathbf{Y}}(\theta)$ denotes the conditional cumulant generating function of PL, defined as

$$C_{PL|\mathbf{Y}}(\theta) = \ln\bigl(M_{PL|\mathbf{Y}}(\theta)\bigr).$$

Taking the first derivative of the conditional cumulant generating function with respect to θ

$$\begin{aligned} \frac{\partial}{\partial\theta} C_{PL|\mathbf{Y}}(\theta) &= \sum\nolimits_{i=1}^{n} w_i \cdot LGD_i \cdot \frac{e^{\theta \cdot w_i \cdot LGD_i} p_i(Y_i)}{1 - p_i(Y_i) + e^{w_i \cdot LGD_i \cdot \theta} p_i(Y_i)} \\ &\equiv \sum\nolimits_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(Y_i;\theta), \end{aligned} \tag{A.7}$$

where $p_i(Y_i;\theta)$ denotes the "exponentially tilted" version of the conditional default probability (A.3), provides us with an expression for the mean of the conditional loss distribution. To make the conditional expected loss equal the threshold x_q we set $\theta \equiv \theta_{x_q}(\mathbf{y})$ according to:

$$\theta_{x_q}(\mathbf{y}) := \left\{\theta \,:\, \sum\nolimits_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(y_i;\theta) = x_q\right\}. \tag{A.8}$$

The theorem 1 in Sadowsky and Bucklew (1989, 508) states that $\theta_{x_q}(\mathbf{y})$ always exists and is unique.

If $x_q > E[PL \mid \mathbf{y}]$, then $\theta_{x_q}(\mathbf{y})$ is positive and the tilted default probabilities $p_i(y_i; \theta_{x_q}(\mathbf{y}))$ are greater than the original ones, leading to larger portfolio losses. Otherwise, $\theta_{x_q}(\mathbf{y})$ is negative and should be set to zero in order to estimate the tail risk, because there is no advantage in reducing $p_i(y_i)$. So the appropriate choice of the tilting parameter in our setting is:

$$\theta^+_{x_q}(\mathbf{y}) = \max\{0, \theta_{x_q}(\mathbf{y})\}. \tag{A.9}$$

A.2. Shifting the mean of the systematic factors

The original distribution of the systematic factors $\mathbf{Y}$ is multivariate standard-Gaussian with a correlation matrix Σ and its off-diagonal elements $\Sigma_{ij} = \rho_{Y_i,Y_j}$. For the purpose of IS, we accomplish the change of the probability measure simply by shifting the mean of $\mathbf{Y}$ from $\mathbf{0}$ to $\boldsymbol{\mu}$, leaving the correlation matrix Σ unchanged. The sample distribution is then multivariate Gaussian $N(\boldsymbol{\mu}, \Sigma)$. Under the new probability measure, equation (4.1) becomes:

$$\Pr\{PL > x_q\} = \tilde{E}\left[\mathbb{1}_{(x_q,1]}(PL)\, l(\mathbf{Y})\right], \tag{A.10}$$

where the expectation is defined with respect to the new sample distribution. The likelihood ratio $l(\cdot)$ is given by the ratio of the original to the new sample density:

$$l(\mathbf{Y}) = \frac{\exp\left(-\frac{1}{2}\mathbf{Y}'\Sigma^{-1}\mathbf{Y}\right)}{\exp\left(-\frac{1}{2}(\mathbf{Y}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right)} = \exp\left(\frac{1}{2}\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}'\Sigma^{-1}\mathbf{Y}\right). \tag{A.11}$$

A proper choice of $\boldsymbol{\mu}$ would lead to a significant variance reduction when estimating the expectation (A.10) by a sample mean. As it is not feasible to find the optimal distribution (it would require the knowledge of the quantity in question), a suboptimal choice has to be made. In a similar setting, Glasserman and Li (2005) have developed an asymptotically optimal IS estimator choosing $\boldsymbol{\mu}$ in a way that the mode of the sampling density coincides with the mode of the following function:

$$\mathbf{y} \mapsto \Pr\{PL > x_q \mid \mathbf{Y} = \mathbf{y}\} \exp\left(-\frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y}\right). \tag{A.12}$$

In the equation above, the issue of calculating a small probability is represented by the problem of integrating the conditional tail of PL over the distribution of the systematic factors. The unique value of $\mathbf{y}$, where the function (A.12) reaches its maximum, would serve as a good choice for $\boldsymbol{\mu}$. Unfortunately, it is not quite feasible to find the preferable solution, meaning we have to deal with a further approximation. Glasserman and Li (2005) suggest substituting the actual conditional tail probability in (A.12) with its upper bound according given on the right-hand side of next expression:

$$\Pr\{PL > x_q \mid \mathbf{Y} = \mathbf{y}\} \leqslant \exp\left[-\theta^+_{x_q}(\mathbf{y})x_q + C_{PL|\mathbf{Y}}\left(\theta^+_{x_q}(\mathbf{y})\right)\right]. \tag{A.13}$$

This eventually leaves us with the option of choosing the mean according to the solution of the following maximisation problem:

$$\boldsymbol{\mu}_{x_q} = \arg\max_{\mathbf{y}} \left\{ -\theta^{+}_{x_q}(\mathbf{y})x_q + C_{PL|\mathbf{Y}}\big(\theta^{+}_{x_q}(\mathbf{y})\big) - \frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y} \right\}. \tag{A.14}$$

B. The simulation algorithm

Now we outline the estimation procedure for the portfolio loss distribution function using the IS technique described in the previous appendix.

First and foremost, it is important to accentuate the fact that there is no need for a repetitive computation of shifting and tilting parameters for numerous different loss levels. Although the parameters μ_{x_q} and $\theta^{+}_{x_q}(\mathbf{y})$ depend on a particular loss quantile, it is sufficient for a practical implementation to choose only one value of x_q. This loss level should be located in the tail, close to $VaR_q(PL)$ and can be chosen on the basis of a short preliminary MC simulation run or the approximative analytical solution we will present later in the paper. The exact position of the loss threshold is not critical. For the chosen value of x_q the problem (A.14) needs to be solved numerically only once before starting the first simulation run. $\theta^{+}_{x_q}(\mathbf{y})$ has to be determined once for each realisation $\mathbf{y}$.

Furthermore, note that the total likelihood ratio for the two-stage IS algorithm is simply the product of the likelihood ratios (A.6) and (A.11).

Taking this information into account, we suggest the following IS simulation algorithm:

- Choose an appropriate loss level x_q.
- Find $\boldsymbol{\mu}_{x_q}$ by solving (A.14).
- For each replication $k = 1, \ldots, s$:
  - generate a realisation $\mathbf{y}$ from $N(\boldsymbol{\mu}_{x_q}, \Sigma)$;
  - calculate $p_i(y_i)$ as in (A.3) for $i = 1, \ldots, n$;
  - find $\theta^{+}_{x_q}(\mathbf{y})$ as in (A.9) by solving (A.8);
  - according to (3.3) generate Bernoulli default indicators either by simulating $D_i(y_i) \sim Be\big(p_i(y_i)\big)$ for $i = 1, \ldots, n$ directly or by means of X_i in (3.1);
  - calculate portfolio loss PL^k in the kth simulation run as in (3.4);
  - calculate the likelihood ratio

$$l\big(PL^k\big) = \exp\left[-\theta^{+}_{x_q}(\mathbf{y})PL^k + C_{PL|\mathbf{Y}}\big(\theta^{+}_{x_q}(\mathbf{y})\big) + \frac{1}{2}\boldsymbol{\mu}'_{x_q}\Sigma^{-1}\boldsymbol{\mu}_{x_q} - \boldsymbol{\mu}'_{x_q}\Sigma^{-1}\mathbf{y} \right]. \tag{B.1}$$

- Calculate the empirical cumulative distribution function for the portfolio loss according to

$$\hat{F}_{PL}(x) = 1 - \frac{1}{s}\sum\nolimits_{k=1}^{s} \mathbb{1}_{(x,1]}\big(PL^k\big)\, l\big(PL^k\big), \quad x \in [0,1]. \tag{B.2}$$

C. Derivatives used in the analytical approximation of tail risk and risk contributions

The first and second derivatives of the limiting portfolio loss with respect to y, initially used in expression (5.10), are as follows:

$$\big(PL^{\infty}(y)\big)' = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \big(p_i(y)\big)', \tag{C.1}$$

$$\big(PL^{\infty}(y)\big)'' = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \big(p_i(y)\big)'', \tag{C.2}$$

and the corresponding derivatives of the conditional probability of distress are:

$$p_i'(y) = -\frac{b_i}{\sqrt{1-b_i^2}}\phi\left(\frac{\Phi^{-1}(p_i)-b_i y}{\sqrt{1-b_i^2}}\right), \tag{C.3}$$

$$p_i''(y) = -\frac{b_i^2}{1-b_i^2}\frac{\Phi^{-1}(p_i)-b_i y}{\sqrt{1-b_i^2}}\phi\left(\frac{\Phi^{-1}(p_i)-b_i y}{\sqrt{1-b_i^2}}\right). \tag{C.4}$$

According to the variance representation as a sum of the limiting portfolio loss variance (5.13) and the granularity adjustment term (5.14), the first derivative of the conditional portfolio variance $\mathrm{var}(PL \mid \bar{Y} = y_q)$ can also be separated into two parts as follows:

$$\big(\mathrm{var}^{\infty}(PL \mid \bar{Y} = y)\big)' = 2\sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot p_i'(y)\,[Q_{ji}(y) - p_j(y)] \tag{C.5}$$

and

$$\big(\mathrm{var}^{GA}(PL \mid \bar{Y} = y)\big)' = \sum_{i=1}^{n} w_i^2 \cdot LGD_i^2 \cdot p_i'(y)\,[1 - 2Q_{ii}(y)] \tag{C.6}$$

with

$$Q_{ji}(y) = \Phi\left(\frac{\Phi^{-1}\big(p_j(y)\big) - \rho_{i,j}^{\bar{Y}}\Phi^{-1}\big(p_i(y)\big)}{\sqrt{1-\big(\rho_{i,j}^{\bar{Y}}\big)^2}}\right). \tag{C.7}$$

Then, the derivatives with respect to an individual exposure weight w_i, initially used to derive the multi-factor granularity adjustment for the VaR contributions in (5.21), can be obtained for both variance components:

$$\frac{\partial}{\partial w_i}\big(\mathrm{var}^{\infty}(PL \mid \bar{Y}=y)\big) \tag{C.8}$$
$$= 2\cdot LGD_i \sum_{j=1}^{n} w_j \cdot LGD_j \Big[C^{Gauss}\Big(p_i(y), p_j(y); \rho_{i,j}^{\bar{Y}}\Big) - p_i(y)p_j(y)\Big],$$

$$\frac{\partial}{\partial w_i}\big(\mathrm{var}^{GA}(PL \mid \bar{Y}=y)\big) \tag{C.9}$$
$$= 2w_i \cdot LGD_i^2\Big[p_i(y) - C^{Gauss}\big(p_i(y), p_j(y); \rho_{ij}^{\bar{Y}}\big)\Big].$$

The corresponding derivatives of (C.5) and (C.6) are:

$$\frac{\partial}{\partial w_i}\big(\mathrm{var}^{\infty}(PL \mid \bar{Y}=y)\big)' \tag{C.10}$$
$$= 2\cdot LGD_i \sum_{j=1}^{n} w_j \cdot LGD_j \cdot p_j{\prime}(y)\big[Q_{ij}(y) - p_i(y)\big]$$
$$+ 2\cdot LGD_i \cdot p_i'(y) \sum_{j=1}^{n} w_j \cdot LGD_j\big[Q_{ji}(y) - p_j(y)\big],$$

$$\frac{\partial}{\partial w_i}\big(\mathrm{var}^{GA}(PL \mid \bar{Y}=y)\big)' = 2w_i \cdot LGD_i^2 \cdot p_i'(y)\big(1-2Q_{ii}(y)\big). \tag{C.11}$$

Eventually, the last two expressions also used in the analytical approximation of the risk contributions are the following

$$\frac{\partial}{\partial w_i}\big(PL^{\infty}(y)\big)' = LGD_i \cdot p_i'(y), \tag{C.12}$$

$$\frac{\partial}{\partial w_i}\left(\frac{\big(PL^{\infty}(y)\big)''}{\big(PL^{\infty}(y)\big)'}\right) \tag{C.13}$$
$$= \frac{LGD_i \cdot p_i''(y)\sum_{j=1}^{n} w_j \cdot LGD_j \cdot p_j'(y) - LGD_i \cdot p_i'(y)\sum_{j=1}^{n} w_j \cdot LGD_j \cdot p_j''(y)}{\sum_{j=1}^{n} w_j \cdot LGD_j \cdot p_j'(y)}.$$