PORTFOLIO OPTMIZATION

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MSc. Artificial Intelligence

COMP6212: Computational Finance, 2016/17 Assignment 1.

Abstract

Market index is a representation of the aggregate performance of a combination of several stocks or investments. Index values helps investors to track the market values over a long period of time. To get a return similar to that market index, a passive investor will have to buy all the stocks, which only a billionaire can buy. A brilliant way is to find a subset of the whole stocks that best approximate the return of the whole index. This work focus on how to optimize a portfolio so as to get the maximum return at a minimal risk. The work compared the performance of Markowitz optimal portfolio with the $\frac{1}{N}$ naïve equal weight portfolio. Also in this work, sparse index tracking was proven to be a better index tracking approach than greedy selection algorithm. Sharpe ratio is adopted as a measure of performance throughout the work. Furthermore, the task was extended to a study of how transaction costs can aid portfolio optimization adjustments.

Keywords Portfolio optimization, optimization adjustment, sparse portfolio, efficient frontier, greedy algorithm, index tracking, regularization.

1 Efficient Mean-Variance Frontier

In a market, each investment is associated with a given amount of capital called weight. A collection of these weight for all assets is known as a portfolio [1]. In the first part of this work, we have three securities with expected returns and corresponding covariance as:

$$\boldsymbol{m} = [0.10 \quad 0.20 \quad 0.15]^t, \quad \boldsymbol{C} = \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix}$$

To invest in this stocks, a 100 random portfolio were generated ensuring that the weight capital (π) for three assets sum up to one. Since this return is model from random Gaussian distribution, the probability density is localized around one area and spread about the mean. Then with the linear transformation property of multivariate Gaussian, it is easier to understanding what to learn and do with the data.

$$x \sim N(\mu, \varepsilon); \quad y = Ax \quad \rightarrow \quad y \sim N(A\mu, A\varepsilon A^t) \qquad (1)$$

Expected return and risk are calculated as the mean and variance of the distribution.

$$E(r) = \pi^t * \mathrm{m}\,; \qquad V = \pi * \mathrm{C} * \pi^t \qquad (2)$$

The efficient (E, V) combinations were computed and the E-V space scatter diagram presented in figure 1.

Figure 1 shows each portfolio expected return against allowable risks. Even though it appears that the portfolios with higher returns has higher risks, it can also observed that some portfolio has similar risk but different returns and vice versa. Hence it will be wise to choose carefully a portfolio with maximum return given a particular risk and a portfolio with minimum risk given a particular expected return. Hence portfolio selection should aim at picking few portfolios which give the desired combinations known as efficient combination [2]. This helps in understanding of investment behaviour in terms of mean and variance relationship which represents the desirable returns and undesirable risk in portfolio investment.

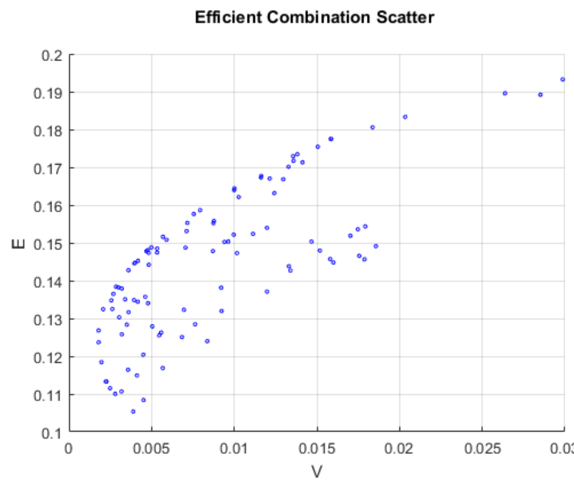

Figure 1. Scatter diagram in the E-V space.

According to [3] since the future is without uncertainties, a good investor makes provision for risk but maximize the return while minimizing the risk. [4] also encourage diversification in portfolio investment among securities with maximum returns and minimum risks and not on portfolio of large numbers since the combination of securities that has the same value is a good as any of them.

To understand portfolio selection better, efficient portfolio frontier was constructed for the given three-asset model as shown in figure 2. The efficient frontier curve plays two roles; firstly, as a benchmark for maximum return of an optimal portfolio for any given risk and secondly, as a benchmark for the minimum risk of an optimal portfolio for any given return. This means that no portfolio can be found above the curve and any portfolio below the curve is suboptimal, since the ones on the curve will provide a better return at that risk level.

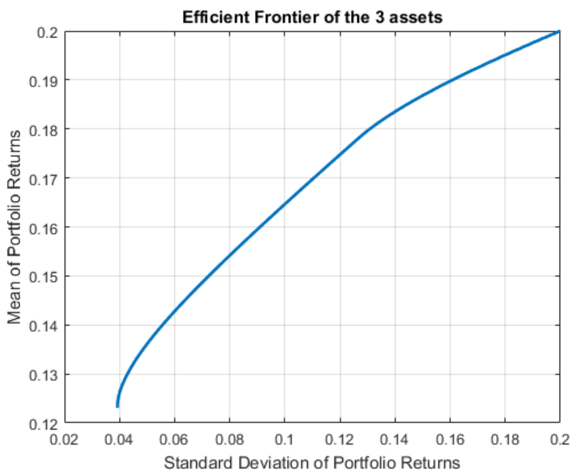

Figure 2. Efficient frontier of the three-asset portfolio

A step is taken further to evaluate the best combinations of two-asset from the three available stocks that best mimic the original three-asset model. This is done by taking the assets pair-wise in turn (their means and covariances) to estimate their efficient frontier. The efficient frontier graph and scatter points of the various pair-wise combinations are presented in figure 3 and 4 on the same scale for better comparisons.

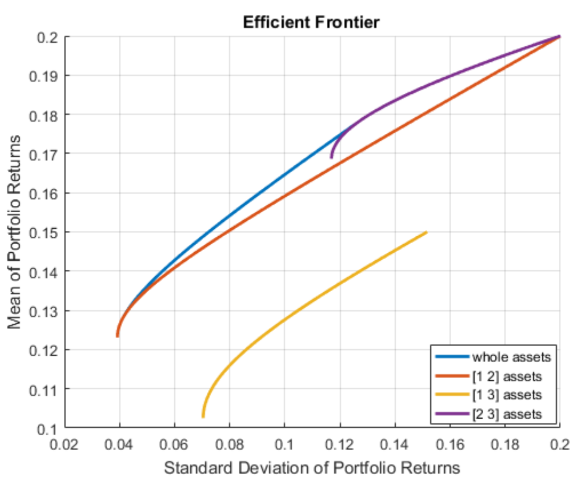

Figure 3. Comparison of the various efficient frontier portfolios

From figure 3 above, it can be seen that some combinations of the two-asset tries to approximate to the original three stocks. Assets [1 2] and [2 3] combinations try to mimic the overall stocks. Assets [2 3] best mimic the whole assets market behaviour but at high risk. Assets [1 2] can be selected for an investor who is interested in little or no risk, but it is not able to approximate the return at the middle. Assets [1 3] is highly suboptimal and that combination should be discouraged.

The scatter diagram in figure 4 also shows the Expected Return – Variance relationship of the pairwise combination of the assets and it could also be seen that asset [1 2] try to mimic the whole asset portfolio. Asset [2 3] can be chosen if the investor is willing to tolerate high risk rate. As for me, instead of buying all the assets, I will buy only assets [1 2] combinations and use the balance for some other things.

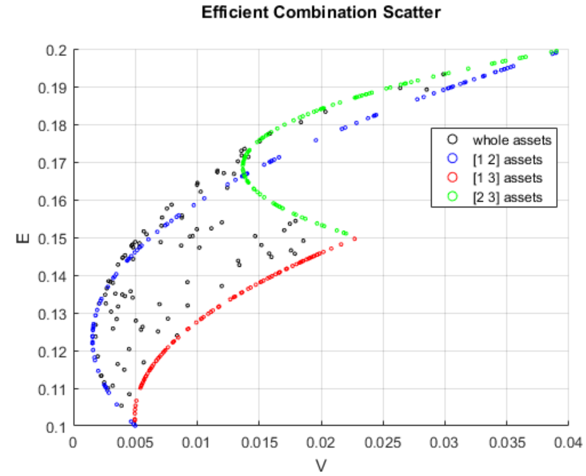

Figure 4. Comparison of the various (E, V) combinations scatter

1.1 Use of Linear Programming in NaiveMV Function for Computing Efficient Frontier

Linear programming (linprog) in MATLAB can only be used if the problem is entirely linear. Linprog is used in the NaiveMV method because the task is purely a scalar product which is linear and all the constraints that it's subjected to are all linear without any quadratic term. In the NaiveMV function, the task of estimating the efficient frontier is divided into three. The first part estimate the expected return without any reference to the variance, hence, estimate the maximum returns as high as possible. This part is a linear programming problem [5]. The second part estimates lowest variance without any reference to the expected return, hence, can get the lowest possible variance. This part is a quadratic programming problem [5]. The last part try to estimate expected return from a number of points within the interval calculated from the first two parts given any level of risk [5].

The use of linprog is to find that maximum expected return when risk is not put into considerations. In the NaiveMV function, we want to find out the maximum return portfolio unconstrained by any risk and the maximum return constrained by the minimum risk portfolio.

1.2 Use of CVX Convex Programming Toolbox

In this segment, the linprog and quodprog used in NaiveMV function were replaced by CVX and similar results were produced as shown in figure 5 (a-b). This is because CVX toolbox solves constrained optimization problem very efficiently as well.

This means that if an optimization problem is a convex problem, convex CVX optimization toolbox will solve it. According to [6], it has an advantage of allowing the optimization expression in a natural mathematical syntax.

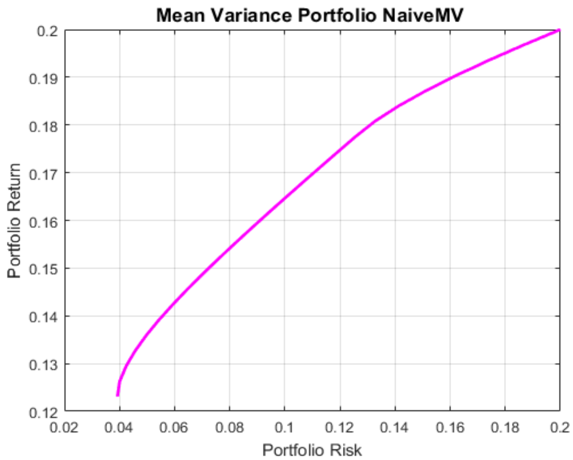

Figure 5(a). Using linprog and quadprog in solving constrained optimization problem

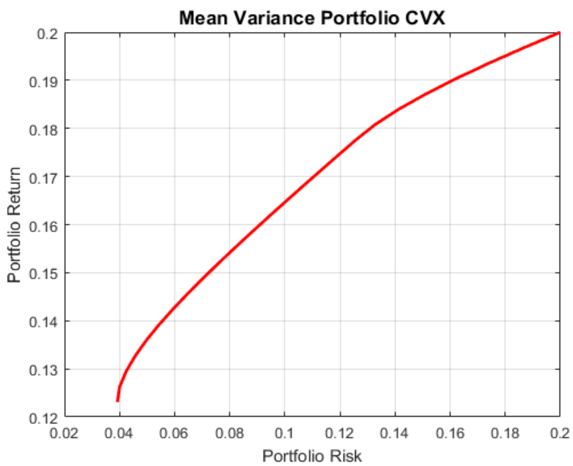

Figure 5(b). Using CVX toolbox for solving constrained optimization problem

2 Evaluation of Markowitz Optimal and $\frac{1}{N}$ Naïve Weight Portfolio

This section try to understand the effect of portfolio optimization in real data and also the varying effects of individual assets weight on the overall return. The real data were obtained from Google Finance. This comprises of daily FTSE 100 data and data for the prices of 30 companies in the FTSE index for the past three years (ranging from 2014-03-02 to 2017-03-02).

For portfolio selection, 3 out of 30 stocks were selected at random and built into a time series matrix of three stocks, then, split equally into training and test sets. Missing values in the time series were filled with next preceding values, hence their return for those days will be zeros.

For the return matrix of the three selected stocks, the daily return is computed using the formula:

$$\mathrm{R(t)} = \frac{S(t) - S(t-1)}{S(t-1)} = \frac{S(t)}{S(t-1)} - 1 \tag{3}$$

where R(t) = return at time t and S (t) = stock price at time t.

The expected Returns and Covariances were estimated from the training set of the time series.

2.1 Markowitz Optimal Weight Portfolio

The Markowitz portfolio is computed as those portfolios that has the minimal variance for a given expected return [1]. Hence the weight is optimized by minimizing the expected risk for any given expected return $p = \pi^t m$. The Markowitz portfolio is a constrained optimization problem solving the following problem:

$$\begin{cases} \pi = \min_{\pi} \| p 1_T - R\pi \|_2^2 \\ \text{subject to } \pi^T \mathrm{m} = p, \ \ \pi^T 1_T = 1 \end{cases} \tag{4}$$

The Markowitz portfolio weights for the three stocks were computed using CVX convex optimization toolbox. Figure 6 shows how the weights were distributed for the three stocks.

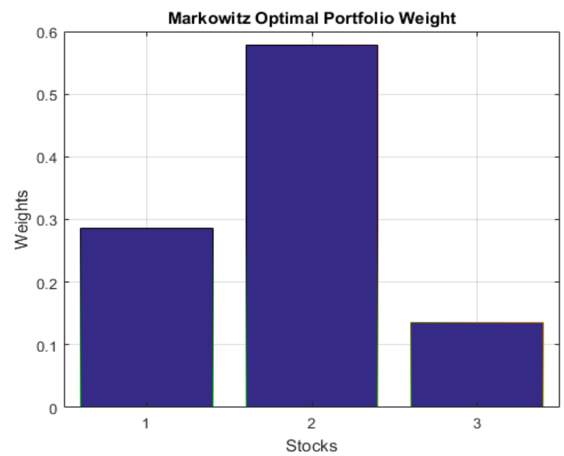

Figure 6. Markowitz optimal portfolio weights for the 3 stocks

The estimates were used to design efficient portfolio and it was observed that any portfolios pick at random from the efficient portfolio fall on the curve of the efficient frontier. This shows that the efficient frontier defines the maximum performance curve of a portfolio in market index when the efficient combination is made.

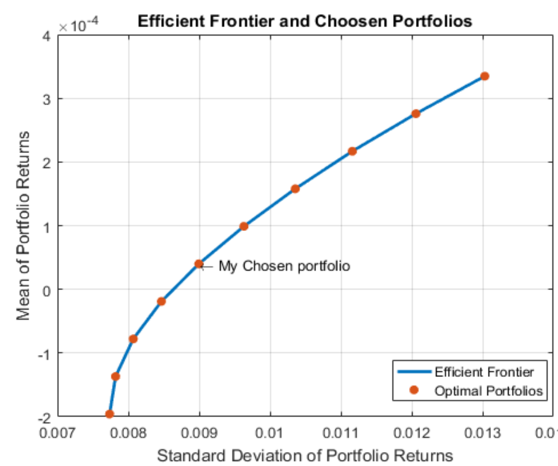

Figure 7. Markowitz optimal portfolio on efficient frontier.

In the graph (Figure 7), the portfolio marked as 'my chosen portfolio' is so due to the level of risk that I will be willing to take.

2.2 Simple $\frac{1}{N}$ Naïve Equal Weight Portfolio

The naïve simple $\frac{1}{N}$ portfolio assigns all N assets for investment equal $\frac{1}{N}$ weights [7]. For the three stocks in this experiment, each will receive $\frac{1}{3}$ of the capital mapped for investment. The weight distribution can be represented in the graph in figure 8.

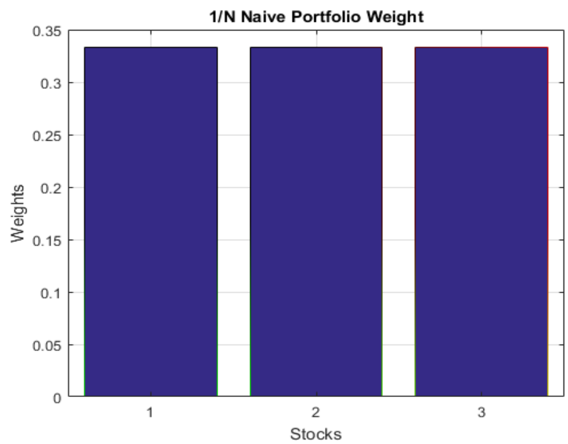

Figure 8. Simple $\frac{1}{N}$ portfolio weights for the 3 stocks

This is less complicated, simple, straightforward, very easy to implement. It does not require any optimization nor estimation of the moments of asset returns [7].

2.3 Performance on out-of-sample data

The performance of the Markowitz portfolio and that of $\frac{1}{N}$ Naïve portfolio were computed and compared on the testing data.

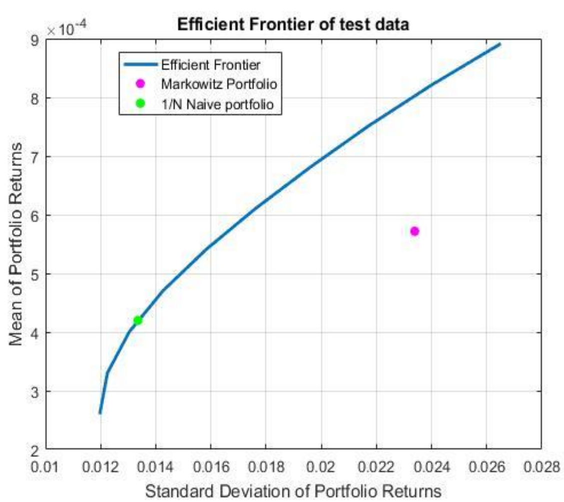

Figure 9. Performance on out-of-sample data

Figure 9 clearly show that Markowitz optimal portfolio does not perform better than $\frac{1}{N}$ Naïve portfolio on out-of-sample data. The $\frac{1}{N}$ equal weight portfolio lie on the efficient frontier. One may argue that the Markowitz portfolio shows a higher return but that is seen under a higher risk. To better evaluate the performance of both portfolio, Sharpe ratio approach was used in section 2.2.1.

2.3.1 Performance Measure using Sharpe Ratio

Sharpe ratio is the mean to standard deviation of portfolio return which analyzes a portfolio in terms of its return-to-risk value. [8]. A portfolio is said to outperform another portfolio if it maximizes its Sharpe ratio more than the other portfolio when compared on the same risk scale [8]. This means that maximum Sharpe ratio portfolios are located on the efficient frontier. The Sharpe ratio of a portfolio is computed using formula; $\boldsymbol{S} = \frac{\boldsymbol{m}-\boldsymbol{r}}{\boldsymbol{\sigma}}$, where r = riskless interest rate, m = expected return, σ = expected standard deviation. The risk free interest rate is not considered in this experiment, hence, it is set to zero and the formula becomes: $\boldsymbol{S} = \frac{\boldsymbol{m}}{\boldsymbol{\sigma}}$

The Markowitz optimal weight and Naïve equal weight rule portfolio were experimented on 3 random stocks severally and their portfolios evaluated based on Sharpe ratio. The results are presented in figure 10 (a-b).

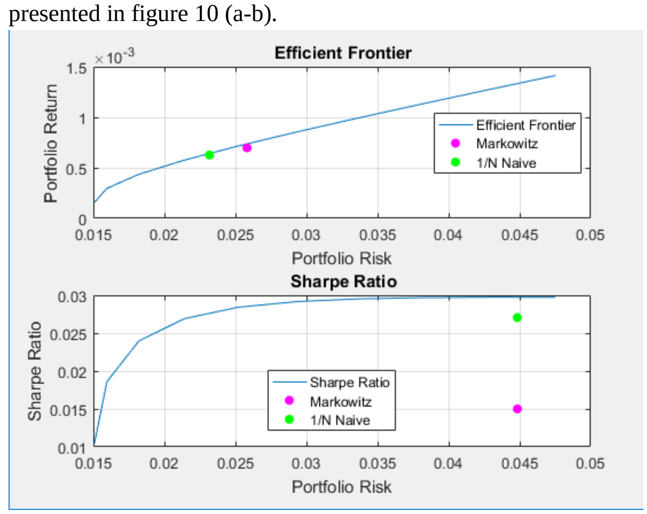

Figure 10(a). Performance on based on Sharpe ratio

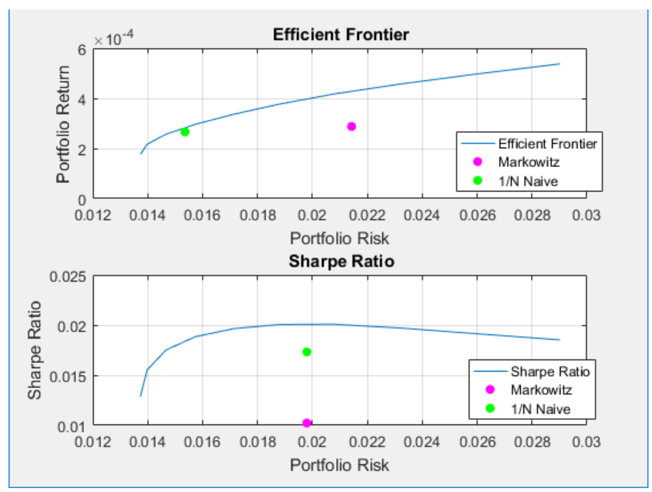

Figure 10(b). Performance on based on Sharpe ratio

It is observed from the Sharpe ratio evaluation that the Markowitz optimal portfolio does not perform better than the Naïve equal weight portfolio. Hence, I support the suggestion made by [7] that 1/N naïve rule should be used as a benchmark for assessing the performance of more sophisticated asset allocation rules.

3 Index Tracking

The goal of index tracking is to select the best subset asset combination that could best replicate the performance of the market index from time to time. FTSE100 market index shows the performance of the combination of whole 100 stocks. In this exercise, 30 constituents stocks were obtained independently. The task is to select a few best assets combinations that will correctly track the index of FTSE100.

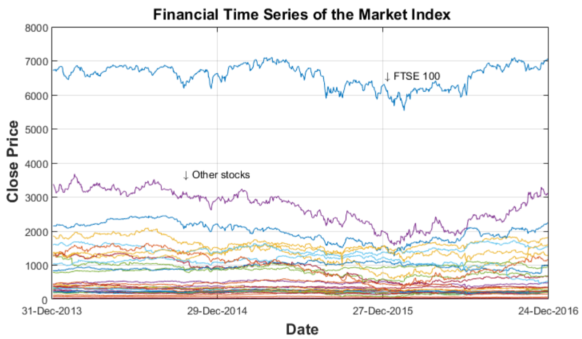

Figure 11. Time series of FTSE100 and constituent 30 stocks

In figure 11, FTSE100 is the topmost time series graph while the rest are the constituent 30 stocks. The goal now is to select and combine few of the constituent stock to mimic the FTSE100.

Greedy selection algorithm and sparse index tracking algorithm were implemented for optimal portfolio selection. The return matrix of the 30 stocks obtained in section 2 was built and the algorithms were constrained to select only a fifth of the whole available stocks that can best track the original index. There performances are evaluated on out-of-sample data.

3.1 Greedy Forward Selection Algorithm Method

Greedy forward selection algorithm uses a heuristics that enables it to select the first assets combinations that gives an optimal return. Its approach entails picking the first asset with maximum return and using it as a permanent member in the efficient combination portfolio and goes further to select the best two assets combination. These two assets becomes permanent member of the set to select three best combination until a desired number of assets combination is achieved. It is greedy because, it does not go back to check if an asset that does not make the list in the first few combination can actually combine with other assets and outperform the approximation function of already greedily selected assets. It does not have different N combinations to compare against each other, rather every selective combination is built around earlier selected combination.

The algorithm 1 provided by [9] was used to implement greedy selection of 6 best assets combination.

Algorithm 1: *Greedy Algorithm*

$initialize:\ S_o = \emptyset \ and \ k =$

$1. Set\ f^T(\pi^T_{S_o}) to\ a\ value\ large\ enough.$

$\mathbf{while}\ k \le C_o$

$\quad For\ all\ s \in N\backslash S_{k-1}, compute\ (\pi^T_{S_{k-1}} \cup \{s\})$

$\quad select\ S^*\ such\ that\ \min_{s \in N\backslash S_{k-1}} f^T(\pi^T_{S_{k-1}} \cup \{s\})$

$\quad\quad and\ set\ \pi^T_{S_k} \leftarrow \pi^T_{S_{k-1}} \cup \{s\}$

$\quad set\ S_k \leftarrow S_{k-1} \cup \{S^*\}\ and\ k \leftarrow k+1.$

$\mathbf{end\ while}$

$\pi^T_G \leftarrow \pi^T_{S_{Co}}\ and\ S^T_G \leftarrow S_{C_o}$

$\mathbf{return}\ \pi^T_G\ and\ S^T_G$

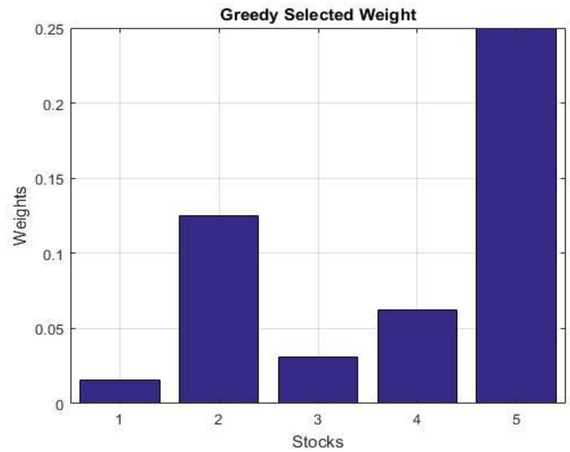

Figure 12(a). Greedy selected portfolio

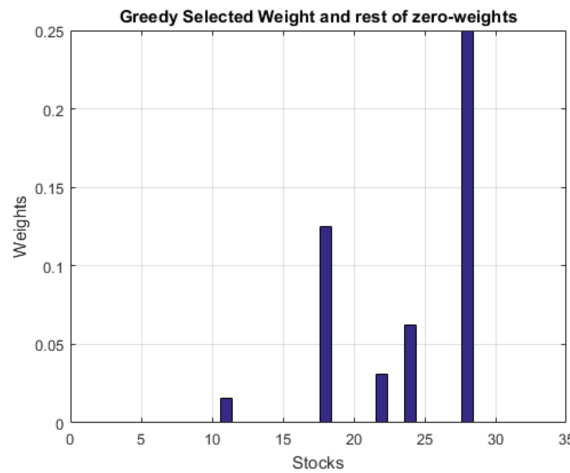

Figure 12(b). All 30 stocks showing the effect of greedy selection

Figure 12(a) shows the level of weight allocation to the greedy selected portfolios only. Figure 12(b) shows the level of weight allocation to the greedy selected portfolios amidst the other non-selected assets with zero weights.

3.2 Sparse Index Tracking Method

Sparse Index Tracking enable us to track the few asset combination that best approximate the performance of the FTSE market index by checking all possible combinatorial solution. In this approach, I used the CVX convex optimization to select the six assets that best mimic the market index. I added a regularization term as discussed in [1] following the equation (5):

$$\pi^T = \arg\min_{\pi} \ [\, ||y - R\pi||_2^2 + \mathcal{T}||\pi||_1 \,] \tag{5}$$

The regularization term $\mathcal{T}$ is added and tune to penalize the regression process. The aim of the tuning is to have only a fifth of the assets that are non-zero weights which will be the most optimal combination forming stable and robust portfolio. Setting $\mathcal{T} = 0$ means that all assets will be used in the efficient combination.

It was not easy to get only 6 stable portfolio of non-zero weights. Bootstrapping technique was used on the underlying data under 20 simulations and the most optimal was picked based on Sharpe ratio.

Short-selling played a vital role in index tracking optimization. Figure 13a and 13b shows the sparse index tracking when short selling is allowed and when it is not allowed respectively. In this, zero-weight thresholding becomes necessary especially in a situation where short selling of assets is allowed since some assets will have negative weights.

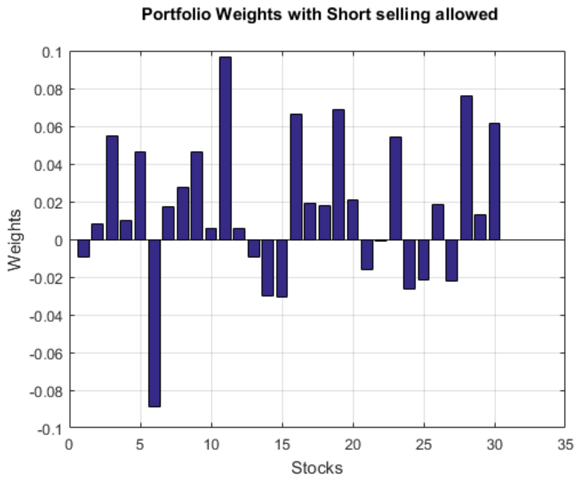

Figure 13(a): Sparse index tracking with short selling

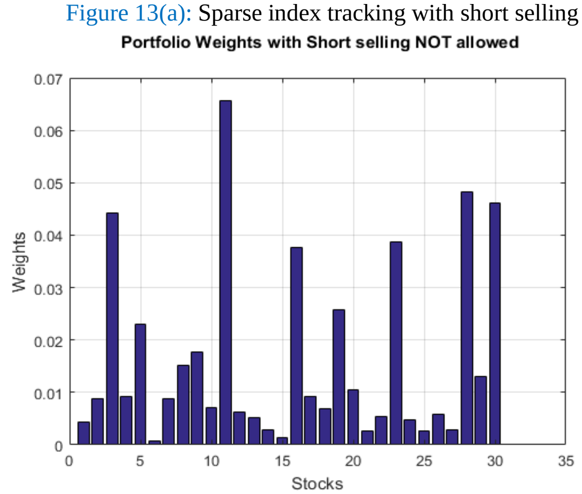

Figure 13(b): Sparse index tracking without short selling

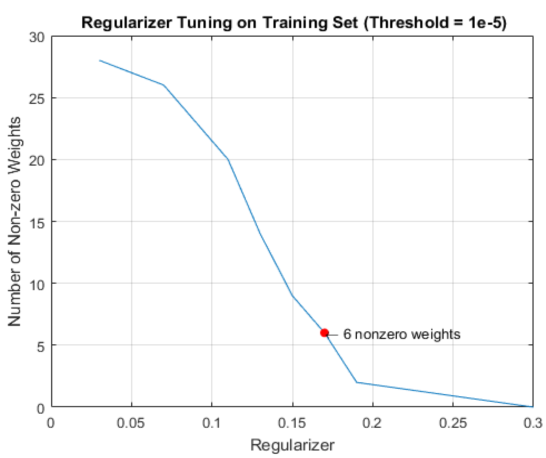

Figure 14: Plot of l_1 term vs non-zero weight

During the tuning process, a threshold of 1e-5 was fixed and any weight below a threshold is considered as a zero weight. The penalty term $\mathcal{T}$ is tuned from 0.03 to 0.3 and the graph in figure 14 shows the number of non-zeros weights as a function of the regularization term.

The sparse index tracking weight produces 1/5 assets combination using a regularization term of 0.3 at the threshold of 1e-5 as shown in figure 15 below.

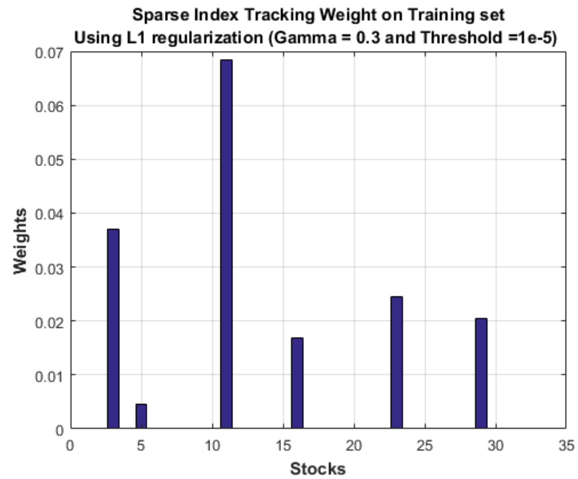

Figure 15: Sparse index tracking portfolio

3.3 Comparison of Greedy Forward Selection Algorithm and Sparse Index Tracking

The weights computed by each tracking approach were used to estimate the portfolio returns on the out-of-sample data. Tracking error were calculated using equation (6) and tabulated in table 1.

$$\text{Tracking error,} \quad T_{err} = \sum_{k=1}^{n} (\mathrm{y} - R_k \pi) \qquad (6)$$

where y = FTSE100 return on test half, R = return of the test half of 30 stock, π = portfolio weight of each tracking method.

Table 1. Tracking Error

Approach	Error (T_{err})
Greedy Forward Selection	9.364580877015413
Sparse Index Tracking	4.296096181136993

Based on tracking error, the Sparse index tracking tracks the FTSE100 with lesser error compare with the Greedy Forward selection approach.

The two approaches were also compared in terms of Sharpe ratio and the result (figure 16) also prove that sparse index tracking has an optimal tracking ability to compare to Greedy counterpart.

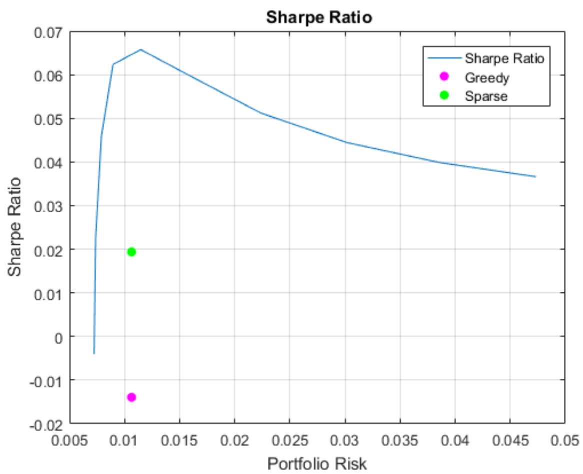

Figure 16: Sharpe ratio measure of greedy forward and sparse index tracking portfolio

The figure 17 and 18 show how each method is able to track the market index. It can be seen that sparse index tracking is a better option than greedy selection.

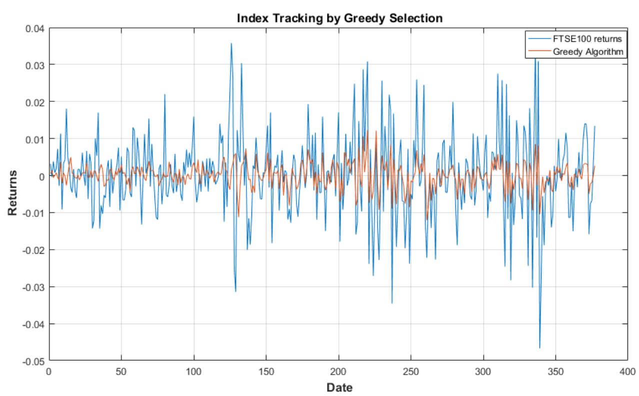

Figure 17: Greedy selection index tracking of FTSE100

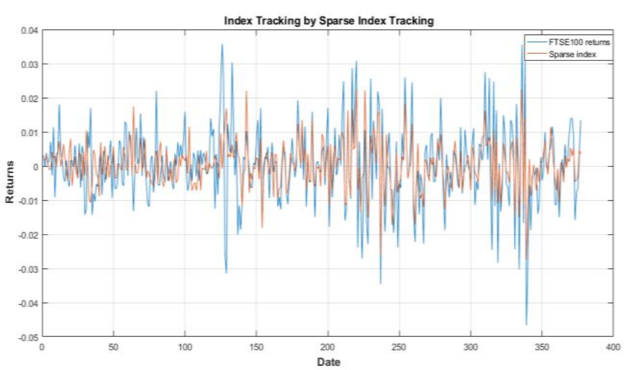

Figure 18: Sparse index tracking of FTSE100

4 Transaction Cost and Portfolio Adjustments

Having understood how one could select a subset of the portfolio that gives the optimal expected return, it will be interesting to consider the amount of wealth gained if some of the portfolio are purchased at a particular time and held for some period of time before selling. From my point of view, I may be have an optimal subset portfolio containing 10 assets that I will like to buy but I am financially buoyant to buy only 6 assets. I can buy 6 optimal subset portfolio and over some period in time, make some changes by selling off some assets and buying some from the initial 10 subset to replace the sold assets in the 6 subset portfolio. If this adjustment is made to the portfolio at a value of x and considering that some transaction costs is associated with the adjustment, Lobo et al. in [10] was considering how to optimize such adjusted portfolio. The new portfolio weight which used to be π, will now be $\pi + x$ and the wealth will be $\boldsymbol{R}^T(\pi + x)$ instead of $\boldsymbol{R}^T\pi$ without adjustment.

The objective function is to maximize the expected return while keeping the cost unchanged amidst all the changes in the portfolio.

$$\text{maximize } \bar{\boldsymbol{R}}^T(\pi + x^+ - x^-) \qquad (7)$$

where the variables x^+ *and* x^- are related to buying and selling of assets respectively. The objective function will help to minimize error in the change in the portfolio returns before and after the adjustment and still keep the portfolio sparse. This will be to find the optimal return when buying or selling the assets. That is to say, that the investor is trying to find out what changes to the portfolio can give a maximum return at minimized risk. To achieve this, a lot of constrained as imposed to the optimization function [10]:

4.1 Budget Constraints:

Budget constraint has to do with the transaction cost constraints. An investor will want the amount of money lost in buying and selling of some assets to be rebalance with the amount of money gained in the portfolio adjustment. Hence, it is necessary to impose a limit to the allowable minimum transaction costs for the portfolio adjustment using the following constraint:

$$\mathbf{1}^T(x^+ - x^-) + \sum_{i=1}^{n}(\propto_i^+ x_i^+ + \propto_i^- x_i^-) \le 0 \qquad (8)$$

The constraint is impose to ensure that the transaction cost is within the budget limit and to ensure that the cost of buying and selling some assets does not exceed zero when balanced with the gain resulting from the adjustment, which means that the transaction cost is fully recovered [11]. Hence, it is said to be self-financing.

4.2 Short selling Constraints:

The second constraint is on short selling which means that the investor is not allowed to sell an asset he/she does not own.

$$x_i^+ \ge 0, \quad x_i^- \ge 0, \; i = 1, \dots, n \qquad (9)$$

4.3 Diversification Constraints:

This constraints placed a limit on the allowable amount that can be invested on any asset. The investor is constrained not to adjust the portfolio so high such that he might start losing in any sector. Hence the changes should be within some set of feasible solutions.

$$\pi_i + x_i^+ - x_i^- \ge \boldsymbol{s}_i, \quad i = 1, \dots, n \qquad (10)$$

In this constraint, $\boldsymbol{s}$ is used to ensure that the maximum amount of short selling allowed is bounded. In the course of adjustment, this constraint limit exposure on any subset portfolio selected for investment [10][12][13]. If this constraint is not put in place, some investors might sell off some assets without buying enough complimentary assets and the adjusted weight will fall below s_i which might result in a suboptimal portfolio.

4.4 Shortfall Risk Constraints:

This very powerful constraint places a minimum threshold, η, which the expected return should not fall below in the event of trying to minimize the risk, so that with the minimum cost of transaction, on average, the expected return should be bigger than the minimum return π^{low}.

$$\phi^{-1}(\eta_j) \left\| \Sigma^{\frac{1}{2}}(\pi_i + x_i^{\pm} x_i^{-}) \right\| \leq \bar{_a}T\,(\pi_i + x_i^{\pm} x_i^{-}) - \pi_j^{low}. \quad j = 1, 2. \tag{11}$$

Shortfall risk constraint ensures that the probability of a loss in wealth is below a threshold and the probability of the gaining a wealth bigger than η is above a boundary and this can be translated into constraints to be imposed on the portfolio adjustment optimization.

This constraint can be imposed on a bad return and on a disastrous return with different confidence intervals [10].

4.5 Extension to my Work:

To extend this to my work, I need to construct a set of feasible solution assets from which new assets can be picked from as I adjust my portfolio. This is to ensure that I don’t buy an asset that will add no value to my portfolio. Also, I need to fix a threshold for minimum return allowable for my portfolio as well as the maximum risk I can tolerate. I have to also determine the transaction costs of selling and buying any assets as well the expected gain associated with buying and selling of the assets in my portfolio. Then, following the objective function and associated constraints, since all the constraints are convex, inequality constraints, I can formulate a convex optimization problem which can be solved using CVX toolbox.

All these are to ensure that adjustment on the portfolio will not push the wealth out of the return matrix, $\boldsymbol{R}$ and with all these constraints put in place, stable and sparse portfolio will be maintained during portfolio adjustment.

5 Conclusions

In summary, we have performed both an experimental and theoretical study of portfolio optimization. This experiment was done on FTSE100 and some 30 constituents companies in UK market. It is of common knowledge that how the index grow is a measure of the performance of all the stocks in the market. This work was able to show that it is possible to reconstruct the original stock from an optimal combination of a smaller subset of the portfolio, provided none of stocks is exactly the same as another or a linear combination of each other. This efficient combination of few subset portfolio will be able to optimally mimic the overall stock (FTSE100). This efficient combination is gotten based on some algorithms that select the best portfolio weight for investment. Sparse index tracking and greedy forward selection algorithm were compared in this work and it was observed that greedy forward search for combinatorial complexity is suboptimal.

Nonetheless, it was surprising that Markowitz optimal portfolio was not able to consistently outperform the simple $\frac{1}{N}$ equal weight naïve portfolio. Finally, this work studied how some constraints can be added to improve optimization of portfolio adjustment when transaction costs is included.

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